# 1. Complementarity of Money and Consumption

Suppose the utility function in our classical monetary model is now

$$U(X_t, L_t) = \frac{X_t^{1-\gamma} - 1}{1-\gamma} - \chi \frac{N_t^{1+\varphi}}{1+\varphi}$$

where $X_t$ is a composite of consumption and money,

$$X_t = \left[(1-\vartheta)C_t^{1-\nu} + \vartheta\left(\frac{M_t}{P_t}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}}$$

(a) Derive the first order conditions for this economy.

HH maximize $\max_{\{C_{t+s}, N_{t+s}, B_{t+s}, \frac{M_{t+s}}{P_{t+s}}\}} E_t\left\{\sum_{s=0}^{\infty} \beta^s \left(\frac{X_{t+s}^{1-\gamma} - 1}{1-\gamma} + \chi \frac{N_{t+s}^{1+\varphi}}{1+\varphi}\right)\right\}$

$X_{t+s} = \left[(1-\theta)C_{t+s}^{1-\nu} + \theta\left(\frac{M_{t+s}}{P_{t+s}}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}}$

t all t+s s.t. $C_t = \frac{W_t}{P_t}N_t - \frac{B_t - Q_{t-1}B_{t-1}}{P_t} - \frac{M_t - M_{t-1}}{P_t} + TR_t + PR_t$ (real budget constraint)

$\mathcal{L}$ : $E_t\left\{\sum_{s=0}^{\infty}\left[\beta^s u(X_{t+s}[C_{t+s}, M_{t+s}], N_{t+s}) + \lambda_{t+s}\left(\frac{W_{t+s}}{P_{t+s}}N_{t+s} - \frac{B_{t+s} - Q_{t-1+s}B_{t-1+s}}{P_{t+s}} - \frac{M_{t+s} - M_{t-1+s}}{P_{t+s}} + TR_{t+s} + PR_{t+s} - C_{t+s}\right)\right]\right\}$

$\{C_{t+s}\}$ $\beta^s \underbrace{U_{X_{t+s}} X_{C_{t+s}}}_{U_{C_{t+1}}} = \lambda_{t+s}$

$\{N_{t+s}\}$ $\beta^s U_{N_{t+s}} + \lambda_{t+s}\frac{W_{t+s}}{P_{t+s}} = 0$

$\{B_{t+s}\}$ $\frac{\lambda_{t+s}}{P_{t+s}} = \frac{\lambda_{t+s+1}}{P_{t+s+1}} Q_{t+s} \Rightarrow \lambda_{t+s} = \frac{P_{t+s}}{P_{t+s+1}}\lambda_{t+s+1} Q_{t+s}$

$\{M_{t+s}\}$ $\beta^s U_{X_{t+s}} X_{M_{t+s}} + \frac{\lambda_{t+s+1}}{P_{t+s+1}} = \frac{\lambda_{t+s}}{P_{t+s}} \Rightarrow \lambda_{t+s} = P_{t+s}\beta^s U_{X_{t+s}} X_{M_{t+s}} + \frac{P_{t+s}}{P_{t+s+1}}\lambda_{t+s+1}$

$\Rightarrow$ Labor : $\beta^s U_{N_{t+s}} + \beta^s U_{X_{t+s}} X_{C_{t+s}} \frac{W_{t+s}}{P_{t+s}} = 0 \Rightarrow U_{N_{t+s}} + \underbrace{U_{X_{t+s}} X_{C_{t+s}}}_{U_{C_{t+s}}} \frac{W_{t+s}}{P_{t+s}} = 0$ : $U_{N_{t+s}} + U_{C_{t+s}} \frac{W_{t+s}}{N_{t+s}} = 0$

$\rightarrow$ Bonds : $\lambda_{t+s} = \frac{P_{t+s}}{P_{t+s+1}}\lambda_{t+s+1} Q_{t+s} \Rightarrow \beta^s U_{C_{t+s}} = \frac{P_{t+s}}{P_{t+s+1}}\beta^{s+1} U_{C_{t+s+1}} Q_{t+s}$

$U_{C_{t+s}} = \frac{P_{t+s}}{P_{t+s+1}}\beta U_{C_{t+s+1}} Q_{t+s}$ in $s=0$ : $U_{C_t} = \beta Q_t E_t\left(\frac{P_t}{P_{t+1}} U_{C_{t+1}}\right)$ Equiv to class result.

$\Rightarrow$ Money: $\beta^s U_{C_{t+s}} = P_{t+s}\beta^s \underbrace{U_{X_{t+s}} X_{M_{t+s}}}_{U_{M_{t+s}}} + \frac{P_{t+s}}{P_{t+s+1}}\beta^{s+1} U_{C_{t+s+1}}$

$U_{C_{t+s}} = P_{t+s} U_{M_{t+s}} + \frac{P_{t+s}}{P_{t+s+1}}\beta U_{C_{t+s+1}}$ in $s=0$ : $U_{C_t} = \beta E_t\left(\frac{P_t}{P_{t+1}} U_{C_{t+1}}\right) + U_{\frac{M_t}{P_t}}$

With $U_{N_{t+s}} = -\chi N_{t+s}^{\varphi}$

$$U_{C_{t+s}} = (1-\theta)\, X_{t+s}^{\nu-\gamma}\, C_{t+s}^{-\nu}$$

$$U_{M_{t+s}} = \theta\, X_{t+s}^{\nu-\gamma} \left(\frac{M_{t+s}}{P_{t+s}}\right)^{-\nu}$$

$\Rightarrow$ Labor: $U_{N_{t+s}} + U_{C_{t+s}} \frac{W_{t+s}}{P_{t+s}} = 0$

$$(1)\quad -\chi N_{t+s}^{\varphi} + (1-\theta) X_{t+s}^{\nu-\gamma} C_{t+s}^{-\nu} \frac{W_{t+s}}{P_{t+s}} = 0$$

$\Rightarrow$ Bonds: $U_{C_{t+s}} = \frac{P_{t+s}}{P_{t+s+1}} \beta\, U_{C_{t+s+1}}\, Q_{t+s}$

- $\frac{U_{C_{t+s+1}}}{U_{C_{t+s}}} = \frac{X_{t+s+1}^{\nu-\gamma} C_{t+s+1}^{-\nu}}{X_{t+s}^{\nu-\gamma} C_{t+s}^{-\nu}}$

$$X_{t+s}^{\nu-\gamma} C_{t+s}^{-\nu} = \frac{P_{t+s}}{P_{t+s+1}} \beta\, X_{t+s+1}^{\nu-\gamma} C_{t+s+1}^{-\nu} Q_{t+s}$$

$$(2)\quad C_{t+s}^{-\nu} = \frac{P_{t+s}}{P_{t+s+1}} \beta \left(\frac{X_{t+s+1}}{X_{t+s}}\right)^{\nu-\gamma} C_{t+s+1}^{-\nu}\, Q_{t+s}$$

$\Rightarrow$ Money: $U_{C_{t+s}} = P_{t+s} U_{M_{t+s}} + \frac{P_{t+s}}{P_{t+s+1}} \beta\, U_{C_{t+s+1}}$

- $\frac{U_{M_{t+s}}}{U_{C_{t+s}}} = \frac{\theta X_{t+s}^{\nu-\gamma} \frac{M_{t+s}^{-\nu}}{P_{t+s}^{-\nu}}}{(1-\theta) X_{t+s}^{\nu-\gamma} C_{t+s}^{-\nu}} = \frac{\theta}{(1-\theta)} \frac{\frac{M_{t+s}^{-\nu}}{P_{t+s}^{-\nu}}}{C_{t+s}^{-\nu}}$

$$\Rightarrow 1 = \frac{\theta}{1-\theta} \left(\frac{M_{t+s}}{C_{t+s} P_{t+s}}\right)^{-\nu} + \frac{P_{t+s}}{P_{t+s+1}} \beta \left(\frac{X_{t+s+1}}{X_{t+s}}\right)^{\nu-\gamma} \left(\frac{C_{t+s+1}}{C_{t+s}}\right)^{-\nu}$$

$$(3)\quad C_{t+s}^{-\nu} = \frac{\theta}{1-\theta} \left(\frac{M_{t+s}}{P_{t+s}}\right)^{-\nu} + \frac{P_{t+s}}{P_{t+s+1}} \beta \left(\frac{X_{t+s+1}}{X_{t+s}}\right)^{\nu-\gamma} C_{t+s+1}^{-\nu}$$

In $s=0$ and considering expectations at $t$

(1) Labor: $\chi N_t^{\varphi} = (1-\theta) X_t^{\nu-\gamma} C_t^{-\nu} \frac{W_t}{P_t}$

(2) Bonds: $C_t^{-\nu} = \beta\, E_t \left( \frac{P_t}{P_{t+1}} C_{t+1}^{-\nu} \left(\frac{X_{t+1}}{X_t}\right)^{\nu-\gamma} Q_t \right)$

(3) Money: $C_t^{-\nu} = \frac{\theta}{1-\theta} \left(\frac{M_t}{P_t}\right)^{-\nu} + \beta\, E_t \left( \frac{P_t}{P_{t+1}} C_{t+1}^{-\nu} \left(\frac{X_{t+1}}{X_t}\right)^{\nu-\gamma} \right)$

(b) Under what conditions does this economy predict that money is neutral? Explain why.

For money neutrality we would need the labor supply not to depend on money demand, which it does now.
If it didnt we would have a 4 unknowns 4 equations subsystem of equations within the system of eq. that determines equilibrium.

(1) $Y_t = A_t N_t$ : production fn

(2) $\frac{W_t}{P_t} = A_t$ : Firm's FOC determines labor demand

(3) $\frac{W_t}{P_t} = f(N_t, C_t)$ : labor supply comes from hh FOC

(4) $Y_t = C_t$ : good's market clearing.

Labor demand here:

$$\frac{W_t}{P_t} = \frac{\chi N_t^{\varphi}}{C_t^{-\nu}} \cdot \frac{1}{(1-\theta) X_t^{\nu-\gamma}}$$

In the model with CRRA utility $\frac{W_t}{P_t} = \frac{\chi N_t^{\varphi}}{C_t^{-\nu}}$

So if $\frac{1}{(1-\theta) X_t^{\nu-\gamma}}$ doesn't depend on $M_t$ and $P_t$ (or more generally, nominal variables) we can still get money neutrality / classical dichotomy result.

That would be the case if $\nu = \gamma$, so $X_t^0 = 1$.

## Firms

$$Y_t = A_t N_t$$

$$\max_{N_t} \quad A_t N_t - \frac{W_t}{P_t} N_t$$

$$\text{FOC}: \quad A_t = \frac{W_t}{P_t}$$

## Government

$$\frac{B_t}{P_t} + \frac{M_t}{P_t} = Tr_t + \frac{Q_{t-1} B_{t-1}}{P_t} + \frac{M_{t-1}}{P_{t-1}}$$

## Market clearing

Labor: $N_t^{hh} = N_t^{firms}$

Bond: $B_t^{gov} = B_t^{hh}$

Money: $M_t^{gov} = M_t^{hh}$

Output: $C_t = Y_t$

# Money demand

2) Bonds: $$C_t^{-v} = \beta E_t \left( \frac{P_t}{P_{t+1}} C_{t+1}^{-v} \left( \frac{X_{t+1}}{X_t} \right)^{v-\gamma} Q_t \right)$$

3) Money: $$C_t^{-v} = \frac{\theta}{1-\theta} \left( \frac{M_t}{P_t} \right)^{-v} + \beta E_t \left( \frac{P_t}{P_{t+1}} C_{t+1}^{-v} \left( \frac{X_{t+1}}{X_t} \right)^{v-\gamma} \right)$$

$$\frac{(3)}{(2)} \qquad 1 = \frac{\theta}{1-\theta} \left( \frac{M_t}{P_t} \right)^{-v} C_t^{v} + \frac{1}{Q_t}$$

$$1 - \frac{1}{Q_t} = \frac{\theta}{1-\theta} \left( \frac{M_t}{P_t} \right)^{-v} C_t^{v} \qquad / \cdot (\,)^{\frac{1}{-v}}$$

$$\left( 1 - \frac{1}{Q_t} \right)^{-\frac{1}{v}} \left( \frac{1-\theta}{\theta} \right)^{-\frac{1}{v}} = \frac{M_t}{P_t} C_t^{-1}$$

$$\frac{M_t}{P_t} = \left( 1 - \frac{1}{Q_t} \right)^{-\frac{1}{v}} \left( \frac{1-\theta}{\theta} \right)^{-\frac{1}{v}} C_t$$

(c) Solve analytically for the steady state of the model (as far as you can), assuming $A = 1$.

$C_t = C, \quad N_t = N, \quad Y_t = Y, \quad A = 1$

1. $Y = N = C$

2. $\frac{W_t}{P_t} = 1$

3. (1) Labor: $\chi N_t^{\varphi} = (1-\theta) X_t^{\nu-\gamma} C_t^{-\nu} \frac{W_t}{P_t} \Rightarrow \chi N^{\varphi} = (1-\theta) X_t^{\nu-\gamma} C^{-\nu}$ : so $X_t$ will also be in S.S.

$\chi C^{\varphi} = (1-\theta) X_t^{\nu-\gamma} C^{-\nu}$ (*)

4. (2) Bonds: $C_t^{-\nu} = \beta E_t\left(\frac{P_{t+1}}{P_t} C_{t+1}^{-\nu} Q_t\right) \Rightarrow C^{-\nu} = \beta E_t\left(\frac{P_{t+1}}{P_t}\right) C^{-\nu} Q_t \Rightarrow 1 = \beta E_t\left(\frac{P_{t+1}}{P_t}\right) Q_t$

5. (3) Money: $C_t^{-\nu} = \frac{\theta}{1-\theta}\left(\frac{M_t}{P_t}\right)^{-\nu} + \beta E_t\left(\frac{P_t}{P_{t+1}} C_{t+1}^{-\nu} \left(\frac{X_{t+1}}{X_t}\right)^{\nu-\gamma}\right)$

$C^{-\nu} = \frac{\theta}{1-\theta}\left(\frac{M_t}{P_t}\right)^{-\nu} + \beta E_t\left(\frac{P_t}{P_{t+1}} C^{-\nu}\right) \quad / : C^{-\nu} = \cdot C^{\nu}$

$1 = \frac{\theta}{1-\theta} C^{\nu} \left(\frac{M_t}{P_t}\right)^{-\nu} + \beta E_t\left(\frac{P_t}{P_{t+1}}\right) \Rightarrow 1 - \beta E_t\left(\frac{P_t}{P_{t+1}}\right) = \frac{\theta}{1-\theta} C^{\nu} \left(\frac{M_t}{P_t}\right)^{-\nu}$

$\left(\frac{M_t}{P_t}\right)^{-\nu} = \frac{1 - \beta E_t\left(\frac{P_t}{P_{t+1}}\right)}{\frac{\theta}{1-\theta} C^{\nu}} \quad / ( )^{-\frac{1}{\nu}}$

$\frac{M_t}{P_t} = \left(1 - \beta E_t\left(\frac{P_t}{P_{t+1}}\right)\right)^{-\frac{1}{\nu}} \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{\nu}} C$

(*) $\chi C^{\varphi+\nu} = (1-\theta) X_t^{\nu-\gamma}$

$C^{\varphi+\nu} = \frac{(1-\theta) X_t^{\nu-\gamma}}{\chi}$

$$X_t = \left[(1-\vartheta)C_t^{1-\nu} + \vartheta\left(\frac{M_t}{P_t}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}}$$

assume $P_t = P_{t+1}$ ; $\frac{M_t}{P_t} = (1-\beta)^{-\frac{1}{\nu}} \left(\frac{1-\theta}{\theta}\right)^{\frac{1}{\nu}} C$

$X_t = \left[(1-\theta) C^{1-\nu} + \theta \left[(1-\beta)^{-\frac{1-\nu}{\nu}} \left(\frac{1-\theta}{\theta}\right)^{\frac{1-\nu}{\nu}} C^{1-\nu}\right]\right]^{\frac{1}{1-\nu}}$

$= \left(C^{1-\nu}\left[(1-\theta) + \theta\left(\frac{1-\theta}{\theta}\right)^{\frac{1-\nu}{\nu}} (1-\beta)^{-\frac{1-\nu}{\nu}}\right]\right)^{\frac{1}{1-\nu}}$

$1 - \frac{1-\nu}{\nu}$

$\frac{\nu - 1 - \nu}{\nu} = -\frac{1}{\nu}$

$= C \left[(1-\theta) + \theta^{-\frac{1}{\nu}} (1-\theta)^{\frac{1-\nu}{\nu}} (1-\beta)^{-\frac{1-\nu}{\nu}}\right]^{\frac{1}{1-\nu}}$

$C^{\varphi+\nu} = \frac{(1-\theta)\, C^{\nu-\gamma} \left[(1-\theta) + \theta^{-\frac{1}{\nu}}(1-\theta)^{\frac{1-\nu}{\nu}} (1-\beta)^{-\frac{1-\nu}{\nu}}\right]^{\frac{\nu-\gamma}{1-\nu}}}{\chi} \quad / : C^{\nu-\gamma}$

$C^{\varphi+\gamma} = (1-\theta) \left[(1-\theta) + \theta^{-\frac{1}{\nu}}(1-\theta)^{\frac{1-\nu}{\nu}} (1-\beta)^{-\frac{1-\nu}{\nu}}\right]^{\frac{\nu-\gamma}{1-\nu}} \chi^{-1}$

$C^{\varphi+\gamma} = (1-\theta) \quad (1-\theta)^{\frac{\nu-\gamma}{1-\nu}} \left[1 + \theta^{-\frac{1}{\nu}} (1-\theta)^{\frac{1-2\nu}{\nu}} (1-\beta)^{-\frac{1-\nu}{\nu}}\right]^{\frac{\nu-\gamma}{1-\nu}} \chi^{-1}$

$C = (1-\theta)^{\frac{1-\gamma}{\varphi+\gamma}} \left[1 + \theta^{-\frac{1}{\nu}} (1-\theta)^{\frac{1-2\nu}{\nu}} (1-\beta)^{-\frac{1-\nu}{\nu}}\right]^{\frac{\nu-\gamma}{1-\nu}\cdot\frac{1}{\varphi+\gamma}} \chi^{-\frac{1}{\varphi+\gamma}}$

(d) Based on your steady state equations describe an algorithm for how to solve for the steady state.

① Define steady state: $C_t = C$, $N_t = N$, $Y_t = Y$, $A = 1$

② Plug in market clearing: $C_t = Y_t \Leftrightarrow C = Y$

③ Plug in firm side

(1) $Y = N$

(2) $\frac{W}{P} = 1$

④ Plug in household side

(1) labor demand → arrive to a fn of C on X(M)

(2) Bonds → arrive to a condition on Q

(3) Money → arrive to a second condition of C on X (M)

solve for (1) and (3)

(e) How would you calibrate $\vartheta$ given knowledge of $\nu$? (I.e., what moments of the data would you use and how?)

$$X_t = \left[(1-\vartheta)C_t^{1-\nu} + \vartheta\left(\frac{M_t}{P_t}\right)^{1-\nu}\right]^{\frac{1}{1-\nu}}$$

v fixed; log lin version of $\hat{X}_t$:

$$\hat{X}_t = (1-\theta)\left(\frac{C}{X}\right)^{1-V} \hat{C}_t + \theta \left(\frac{M/P}{X}\right)^{1-V} (\hat{M}_t - \hat{P}_t)$$

fo $(1-\theta)$ and $(\theta)$ have a moment condition which is consumption/ money demand share.

That moment can be use to calibrate for $\theta$.

(f) Given knowledge of other parameters, how would you set $M$ such that $P = 1$ in steady state?

Money demand

$$\frac{M_t}{P_t} = C_t \left( \left(\frac{1-\theta}{\theta}\right)\left(1 - \frac{1}{Q_t}\right)\right)^{-\frac{1}{v}} \iff M_t = P_t C_t \left( \left(\frac{1-\theta}{\theta}\right)\left(1 - \frac{1}{Q_t}\right)\right)^{-\frac{1}{v}}$$

Money supply : $M_t^S$

In Equilibrium $M_t^S = M_t^D$

$$M_t^S = P_t C_t \left( \left(\frac{1-\theta}{\theta}\right)\left(1 - \frac{1}{Q_t}\right)\right)^{-\frac{1}{v}}$$

$P_t = 1$ if $M_t^S = C_t \left( \left(\frac{1-\theta}{\theta}\right)\left(1 - \frac{1}{Q_t}\right)\right)^{-\frac{1}{v}}$

ss $\quad M^S = C \left( \left(\frac{1-\theta}{\theta}\right)\left(1 - \beta\right)\right)^{-\frac{1}{v}}$

with

$$C = (1-\theta)^{\frac{1-\gamma}{\varphi+\gamma}} \left[ 1 + \theta^{-\frac{1}{v}} (1-\theta)^{\frac{1-2v}{v}} (1-\beta)^{-\frac{1-v}{v}} \right]^{\frac{v-\gamma}{1-v}\cdot\frac{1}{\varphi+\gamma}} \chi^{-\frac{1}{\varphi+\gamma}}$$

(g) Derive the log-linearized model.

Standard model

(1) $Y_t = A_t N_t$ : Production fn

(2) $\frac{W_t}{P_t} = A_t$ : Labor demand

(3) $\frac{W_t}{P_t} = \frac{\chi N_t^{\varphi}}{C_t^{-\nu}} \cdot \frac{1}{(1-\theta) X_t^{\nu-\gamma}}$ : Labor supply

(4) $Y_t = C_t$ : mkt clearing goods market

(5) $\frac{M_t}{P_t} = C_t \left(\frac{1-\theta}{\theta}\left(1-\frac{1}{Q_t}\right)\right)^{-\frac{1}{\nu}}$ : money demand

(6) $1 = \beta E_t \left(Q_t \frac{P_t}{P_{t+1}} \left(\frac{X_{t+1}}{X_t}\right)^{\nu-\gamma} \frac{C_{t+1}^{-\nu}}{C_t^{-\nu}}\right)$ : Bonds demand

loglinearization

$$g(z) = f(x,y)$$
$$g'(z) z \hat{z}_t = f_1(x,y) x \hat{x}_t + f_2(x,y) y \hat{y}_t$$

(1) $Y_t = A_t N_t \Rightarrow \hat{Y}_t = \hat{A}_t + \hat{N}_t$

(2) $\frac{W_t}{P_t} = A_t \Rightarrow \hat{W}_t - \hat{P}_t = \hat{A}_t$

(3) $\frac{W_t}{P_t} = \frac{\chi N_t^{\varphi}}{C_t^{-\nu}} \cdot \frac{1}{(1-\theta) X_t^{\nu-\gamma}}$

SS : $\frac{W}{P} = \frac{\chi N^{\varphi} C^{\nu}}{(1-\theta) X^{\nu-\gamma}}$

$$\log W_t - \log P_t = \log \chi + \varphi \log N_t + \nu \log C_t - \log(1-\theta) - (\nu-\gamma) \log X_t$$

SS : $\log W - \log P = \log \chi + \varphi \log N + \nu \log C - \log(1-\theta) - (\nu-\gamma) \log X$

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$$\hat{W}_t - \hat{P}_t = \varphi \hat{N}_t + \nu \hat{C}_t - (\nu-\gamma) \hat{X}_t$$

(4) $Y_t = C_t \Rightarrow \hat{Y}_t = \hat{C}_t$

(5) $$\frac{M_t}{P_t} = C_t \left(\frac{1-\theta}{\theta}\left(1-\frac{1}{Q_t}\right)\right)^{-\frac{1}{\nu}}$$

$$\underbrace{M_t}_{g(z)} = \underbrace{P_t C_t \left(\frac{\theta}{1-\theta}\right)^{\frac{1}{\nu}} \left(1-\frac{1}{Q_t}\right)^{-\frac{1}{\nu}}}_{f(P,C,Q)} \qquad SS: \quad M = PC\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{\nu}}\left(1-\frac{1}{Q}\right)^{-\frac{1}{\nu}}$$

$$g'(z) z \hat{z}_t = f_P(P,C,Q) P \hat{P}_t + f_C(P,C,Q) C \hat{C}_t + f_Q Q \hat{Q}_t$$

$M_t$ : $M\hat{M}_t$

$P_t$ : $C\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{\nu}}\left(1-\frac{1}{Q}\right)^{-\frac{1}{\nu}} P\hat{P}_t = \frac{M}{P} P \hat{P}_t = M\hat{P}_t$

$C_t$ : $P\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{\nu}}\left(1-\frac{1}{Q}\right)^{-\frac{1}{\nu}} C\hat{C}_t = \frac{M}{C} C\hat{C}_t = M\hat{C}_t$

$Q_t$ : $-\frac{1}{\nu}\frac{M}{\left(1-\frac{1}{Q}\right)} \cdot \frac{1}{Q^2} Q\hat{Q}_t = -\frac{1}{\nu}\frac{M}{\left(1-\frac{1}{Q}\right)Q}\hat{Q}_t$

Putting together

$$M\hat{M}_t = M\hat{P}_t + M\hat{C}_t - \frac{1}{\nu}\frac{M}{\left(1-\frac{1}{Q}\right)Q}\hat{Q}_t$$

$$\hat{M}_t = \hat{P}_t + \hat{C}_t - \frac{1}{\nu}\frac{1}{Q\left(1-\frac{1}{Q}\right)}\hat{Q}_t$$

$$\hat{M}_t - \hat{P}_t = \hat{C}_t - \frac{1}{\nu(Q-1)}\hat{Q}_t$$

Recall in SS $1 = \beta Q$ $\quad \frac{1}{Q-1} = \frac{1}{\frac{1}{\beta}-1} = \frac{1}{\frac{1-\beta}{\beta}} = \frac{\beta}{1-\beta}$

$$\Leftrightarrow \quad \hat{M}_t - \hat{P}_t = \hat{C}_t - \frac{1}{\nu}\frac{\beta}{1-\beta}\hat{Q}_t$$

(6) steady state:

$$\frac{1}{Q} = \beta$$

$$\underbrace{C_t^{-\nu}}_{g(z)} = \beta\left(\underbrace{Q_t}_{1} \frac{\overset{2}{P_t}}{\underset{3}{P_{t+1}}}\left(\frac{\overset{4}{X_{t+1}}}{\underset{5}{X_t}}\right)^{\nu-\gamma} \underbrace{C_{t+1}^{-\nu}}_{6}\right)$$

$C_t^{-\nu}$ : $-\nu C^{-\nu-1} C\hat{C}_t = -\nu C^{-\nu}\hat{C}_t$

$Q_t$ : $\beta C^{-\nu} Q\hat{Q}_t$

$P_t$ : $\beta\frac{Q}{\cancel{P}} C^{-\nu} \cancel{P}\hat{P}_t = \beta Q C^{-\nu}\hat{P}_t$

$P_{t+1}$ : $-\beta Q\frac{\cancel{P}}{P^{\cancel{2}}} C^{-\nu} \cancel{P}\hat{P}_{t+1} = -\beta Q C^{-\nu}\hat{P}_{t+1}$

$X_{t+1}$ : $\beta Q(\nu-\gamma)\frac{1}{X} X \hat{X}_{t+1} C^{-\nu} = \beta Q\hat{X}_{t+1} C^{-\nu}$

$X_t$ : $-\beta Q(\nu-\gamma)\frac{X}{X^2} X\hat{X}_t C^{-\nu} = -\beta Q\hat{X}_t C^{-\nu}$

$C_{t+1}$ : $-\nu\beta Q C^{-\nu-1} C\hat{C}_{t+1} = -\nu\beta Q C^{-\nu}\hat{C}_{t+1}$

putting together:

$$-\nu C^{-\nu}\hat{C}_t = \beta C^{-\nu} Q\hat{Q}_t + \beta Q C^{-\nu}\hat{P}_t - \beta Q C^{-\nu}\hat{P}_{t+1} + \beta Q\hat{X}_{t+1}C^{-\nu} - \beta Q\hat{X}_t C^{-\nu} - \nu\beta Q C^{-\nu}\hat{C}_{t+1} \qquad / : C^{-\nu}$$

$$\nu(\hat{C}_{t+1}\beta Q - \hat{C}_t) = \beta Q\left[\hat{Q}_t + \hat{P}_t - \hat{P}_{t+1} + (\nu-\gamma)(\hat{X}_{t+1} - \hat{X}_t)\right]$$

SS : $\beta Q = 1$

$$\nu(\hat{C}_{t+1} - \hat{C}_t) = \hat{Q}_t + (\hat{P}_t - \hat{P}_{t+1}) + (\nu-\gamma)(\hat{X}_{t+1} - \hat{X}_t)$$

## Loglinearization of $X$

$$X_t = \Bigg[\underbrace{(1-\vartheta)C_t^{1-\nu} + \vartheta\left(\frac{M_t}{P_t}\right)^{1-\nu}}_{A}\Bigg]^{\frac{1}{1-\nu}}$$

SS: $X^{1-v} = \underbrace{(1-\theta)C^{1-v} + \theta\left(\frac{M}{P}\right)^{1-v}}_{A}$

$\Leftrightarrow X^{1-v} = A$

$$g(x) = f(c, M, P)$$

$$g'(x)\, x\, \hat{x}_t = f_c(c, M, P)\, x\, \hat{x}_t + f_M(c, M, P)\, M\, \hat{M}_t + f_P(c, M, P)\, P\, \hat{P}_t$$

$X_t$ : $x\,\hat{x}_t$

$C_t$ : $\frac{1}{1-v} A^{\frac{1}{1-v}-1} (1-\theta)(1-v) C^{-v}\, C\, \hat{c}_t = (1-\theta) A^{\frac{1}{1-v}-1} C^{1-v} \hat{c}_t$

$M_t$ : $\frac{1}{1-v} A^{\frac{1}{1-v}-1} \theta(1-v) \frac{M^{-v}}{P^{1-v}} M \hat{M}_t = \theta A^{\frac{1}{1-v}-1} \left(\frac{M}{P}\right)^{1-v} \hat{M}_t$

$P_t$ : $-\frac{1}{1-v} A^{\frac{1}{1-v}-1} \theta(1-v) \frac{M^{1-v}}{P^{2-v}} P \hat{P}_t = -\theta A^{\frac{1}{1-v}-1} \left(\frac{M}{P}\right)^{1-v} \hat{P}_t$

$$\Rightarrow \quad x\hat{x}_t = (1-\theta) A^{\frac{1}{1-v}-1} C^{1-v} \hat{c}_t + \theta A^{\frac{1}{1-v}-1} \left(\frac{M}{P}\right)^{1-v} (\hat{M}_t - \hat{P}_t) \quad \bigstar$$

$$X\hat{X}_t = (1-\theta) X^{v} C^{1-v} \hat{c}_t + \theta X^{v} \left(\frac{M}{P}\right)^{1-v} (\hat{M}_t - \hat{P}_t) \qquad /: X^{v}$$

$$X^{1-v} \hat{X}_t = (1-\theta) C^{1-v} \hat{c}_t + \theta \left(\frac{M}{P}\right)^{1-v} (\hat{M}_t - \hat{P}_t)$$

$$\hat{X}_t = (1-\theta)\left(\frac{C}{X}\right)^{1-v} \hat{c}_t + \theta \left(\frac{M/P}{X}\right)^{1-v} (\hat{M}_t - \hat{P}_t)$$

$\bigstar \quad \frac{1}{1-v} - 1 = \frac{1-1+v}{1-v} = \frac{v}{1-v}$

$A^{\frac{v}{1-v}} = (X^{1-v})^{\frac{v}{1-v}} = X^{v}$

$$\left((1-\theta)C^{1-v} + \theta\left(\frac{M}{P}\right)^{1-v}\right) \hat{X}_t = (1-\theta) C^{1-v} \hat{c}_t + \theta \left(\frac{M}{P}\right)^{1-v} (\hat{M}_t - \hat{P}_t)$$

$$(1-\theta) C^{1-v} \hat{X}_t + \theta\left(\frac{M}{P}\right)^{1-v} \hat{X}_t = (1-\theta) C^{1-v} \hat{c}_t + \theta \left(\frac{M}{P}\right)^{1-v} (\hat{M}_t - \hat{P}_t)$$

$$(1-\theta) C^{1-v} (\hat{X}_t - \hat{c}_t) = \theta \left(\frac{M}{P}\right)^{1-v} (\hat{M}_t - \hat{P}_t - \hat{X}_t)$$

2 options

(1) $\hat{X}_t = (1-\theta)\left(\frac{C}{X}\right)^{1-v} \hat{c}_t + \theta \left(\frac{M/P}{X}\right)^{1-v} (\hat{M}_t - \hat{P}_t)$

(2) $(1-\theta) C^{1-v} (\hat{X}_t - \hat{c}_t) = \theta \left(\frac{M}{P}\right)^{1-v} (\hat{M}_t - \hat{P}_t - \hat{X}_t)$

## Loglinearized system of equations:

(1) $\hat{y}_t = \hat{A}_t + \hat{N}_t$ : Production fn

(2) $\hat{W}_t - \hat{P}_t = \hat{A}_t$ : Labor demand

(3) $\hat{w}_t - \hat{p}_t = \varphi \hat{N}_t + \nu \hat{C}_t - (\nu - \gamma) \hat{x}_t$ : Labor supply

(4) $\hat{y}_t = \hat{C}_t$ : mkt clearing goods market

(5) $\hat{M}_t - \hat{P}_t = \hat{C}_t - \frac{1}{\nu} \frac{\beta}{1-\beta} \hat{Q}_t$ : money demand

(6) $\nu ( \hat{C}_{t+1} - \hat{C}_t) = \hat{Q}_t + (\hat{P}_t - \hat{P}_{t+1}) + (\nu - \gamma)(\hat{X}_{t+1} - \hat{X}_t)$ : Bonds demand

(7) $\hat{X}_t = (1-\theta) \left(\frac{C}{X}\right)^{1-\nu} \hat{C}_t + \theta \left(\frac{M/P}{X}\right)^{1-\nu} (\hat{M}_t - \hat{P}_t)$

## DAG

$U: \; N, P$

$Z: \; A(1)^{*}, \; m$ $\qquad$ * Since $A=1$ will be treated as a parameter

$Y: \; m-p, \; y, \; w-p, \; c, \; x, \; Q$

$$H(U,Z) = \begin{bmatrix} \hat{c}_t - \hat{y}_t \\ \hat{c}_t - \frac{1}{\nu}\frac{\beta}{1-\beta}\hat{Q}_t - (\hat{m}_t - \hat{p}_t) \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{matrix} \text{-gm-} \\ \text{-mm-} \end{matrix} \iff H_U \, dU + H_Z \, dZ = 0$$

$$\frac{dH}{dY} = \begin{bmatrix} \phi_{gmmp} & \phi_{gmy} & \phi_{gmwp} & \phi_{gmc} & \phi_{gmx} & \phi_{gmQ} \\ \phi_{mmmp} & \phi_{mmy} & \phi_{mmwp} & \phi_{mmc} & \phi_{mmx} & \phi_{mmQ} \end{bmatrix}_{2\times 6}$$

$$\frac{dY}{dU} = \begin{bmatrix} \phi_{mpn} & \phi_{mpp} \\ \phi_{yn} & \phi_{yp} \\ \phi_{wpn} & \phi_{wpp} \\ \phi_{cn} & \phi_{cp} \\ \phi_{xn} & \phi_{xp} \\ \phi_{Qn} & \phi_{Qp} \end{bmatrix}_{6\times 2} \qquad \frac{dY}{dZ} = \begin{bmatrix} \phi_{mpm} \\ \phi_{ym} \\ \phi_{wpm} \\ \phi_{cm} \\ \phi_{xm} \\ \phi_{Qm} \end{bmatrix}_{6\times 1}$$

$\phi$ defined as following:

**Money identity** $\quad$ (1) M-P: $\hat{m}_t - \hat{p}_t = \hat{m}_t - \hat{p}_t$ $\quad$ (identity to have the correct chain rule) * Appendix

$\phi_{mpn} = 0$

$\phi_{mpp} = -I$

$\phi_{mpm} = I$

**Firm Block** $\quad$ (2) $y$: $\hat{y}_t = \hat{N}_t$

$\phi_{yn} = I$

$\phi_{yp} = 0$

$\phi_{ym} = 0$

(3) wp: $\hat{w}_t - \hat{p}_t = \hat{A}_t$

$\phi_{wpn} = 0$

$\phi_{wpp} = 0$

$\phi_{wpm} = 0$

**Household Block** $\quad$ (4) C: 
$$\hat{c}_t = \underbrace{\left[\nu - (\nu-\gamma)(1-\theta)\left(\frac{c}{x}\right)^{1-\nu}\right]^{-1}}_{B} \left[(\hat{w}_t - \hat{p}_t) - \varphi \hat{N}_t + (\nu-\gamma)\theta\left(\frac{M/P}{x}\right)^{1-\nu}(\hat{m}_t - \hat{p}_t)\right]$$

$$\phi_{cn} = B(\phi_{wpn} - \varphi I)$$

$$\phi_{cp} = B\left(\phi_{wpp} - (\nu-\gamma)\theta\left(\frac{M/P}{x}\right)^{1-\nu} I\right)$$

$$\phi_{cm} = B(\nu-\gamma)\theta\left(\frac{M/P}{x}\right)^{1-\nu} I$$

(solving for x) (5) X: 
$$\hat{x}_t = (1-\theta)\left(\frac{c}{x}\right)^{1-\nu}\hat{c}_t + \theta\left(\frac{M/P}{x}\right)^{1-\nu}(\hat{m}_t - \hat{p}_t)$$

$$\phi_{xn} = (1-\theta)\left(\frac{c}{x}\right)^{1-\nu}\phi_{cn}$$

$$\phi_{xp} = (1-\theta)\left(\frac{c}{x}\right)^{1-\nu}\phi_{cp} - \theta\left(\frac{M/P}{x}\right)^{1-\nu} I$$

$$\phi_{xm} = (1-\theta)\left(\frac{c}{x}\right)^{1-\nu}\phi_{cm} + \theta\left(\frac{M/P}{x}\right)^{1-\nu} I$$

**Bonds Block** $\quad$ (6) Q: 
$$Q_t = -\nu(\hat{c}_t - \hat{c}_{t+1}) - (\hat{p}_t - \hat{p}_{t+1}) + (\nu-\gamma)(\hat{x}_t - \hat{x}_{t+1})$$

$$\phi_{Qn} = -\nu\,\phi_{cn}(I - I_p) + (\nu-\gamma)\,\phi_{xn}(I - I_p)$$

$$\phi_{Qp} = -\nu\,\phi_{cp}(I - I_p) - (I - I_p) + (\nu-\gamma)\,\phi_{xp}(I - I_p)$$

$$\phi_{Qm} = -\nu\,\phi_{cm}(I - I_p) + (\nu-\gamma)\,\phi_{xm}(I - I_p)$$

# Appendix

**DAG** $U$: $N, P$

$Z$: $A, m$

$y$: $y$, $w-p$, $c$, $x$, $Q$

$$H(U,Z) = \begin{pmatrix} \text{-gm-} & \hat{c}_t - \hat{y}_t \\ \text{-mm-} & \hat{c}_t - \frac{1}{\nu}\frac{\beta}{1-\beta}\hat{Q}_t - (\hat{M}_t - \hat{P}_t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$H(U,Z) = 0 \Leftrightarrow H_U\, dU + H_Z\, dZ = 0$$

$$dU = -H_U(\bar{U},\bar{Z})^{-1} H_Z(\bar{U},\bar{Z})\, dZ$$

Note: mm in $H$ $\hat{c}_t - \frac{1}{\nu}\frac{\beta}{1-\beta}\hat{Q}_t - (\hat{M}_t - \hat{P}_t)$ does not only depend on $U$ and $Z$ through $y$ but also directly! ($M_t$ is on $Z$, $P_t$ on $U$)

$$H_U = \frac{dH}{dU} = \frac{dH}{dy}\cdot\frac{dy}{dU}$$

: equality breaks! $H$ depends on $U$ not only through $y$! also directly

$$H_Z = \frac{dH}{dZ} = \frac{dH}{dy}\cdot\frac{dy}{dZ}$$

The problem is fixed by adding $\frac{dH}{dZ}$ and $\frac{dH}{dU}$ directly:

$\left(= \frac{dgm}{dy}\cdot\frac{dy}{dn}\right)$ vector

$$\begin{bmatrix} \frac{dgm}{dn} & \frac{\partial gm}{\partial p} \\ \frac{d\,mm}{\partial n} & \frac{d\,mm}{\partial p} \end{bmatrix} \neq \begin{bmatrix} \frac{dgm}{dn} & \frac{\partial gm}{\partial p} \\ \frac{d\,mm}{\partial n} & \frac{d\,mm}{\partial p} - I \end{bmatrix} = H_U$$

$$\begin{bmatrix} \frac{dgm}{\partial m} \\ \frac{dmm}{\partial m} \end{bmatrix} \neq \begin{bmatrix} \frac{dgm}{\partial m} \\ \frac{dmm}{\partial m} + I \end{bmatrix} = H_Z$$

This was solved by including an artificial $y = M_t - P_t$ and have an identity condition so that the system captures how $(M_t - P_t)$ react in mm equation on $H$.

# Question (g)

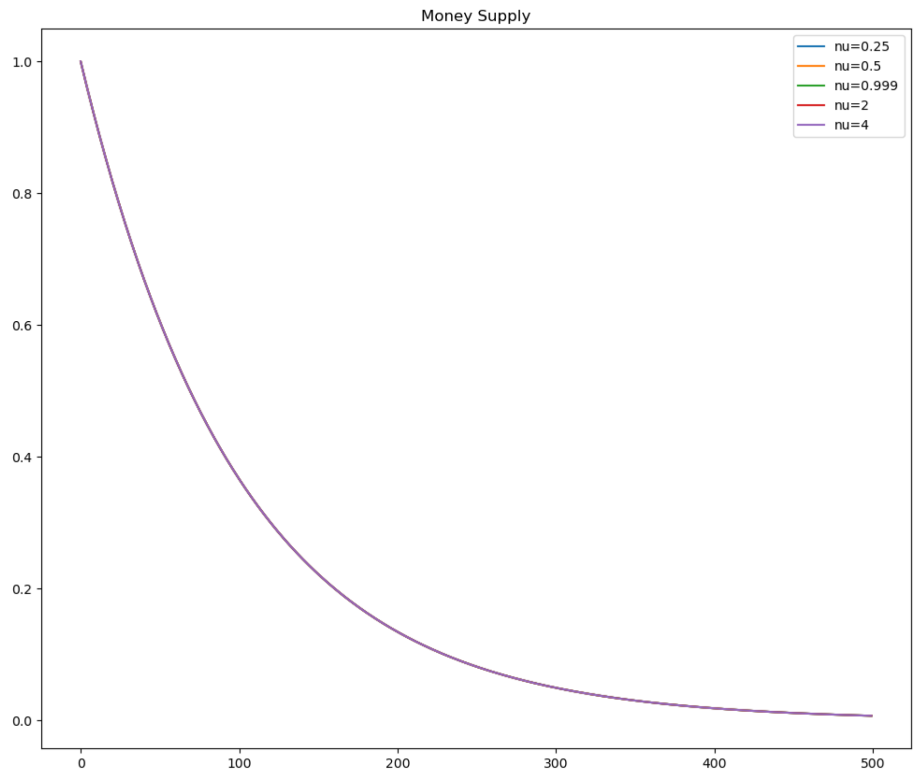


Figure 1: Shock in Money Supply

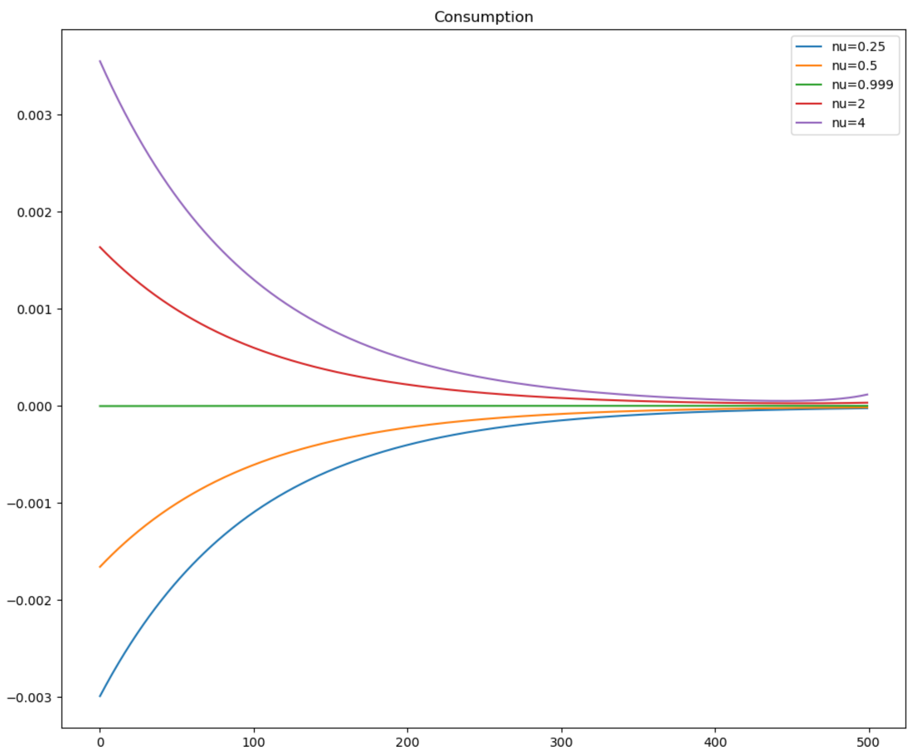


Figure 2: IRF of Consumption

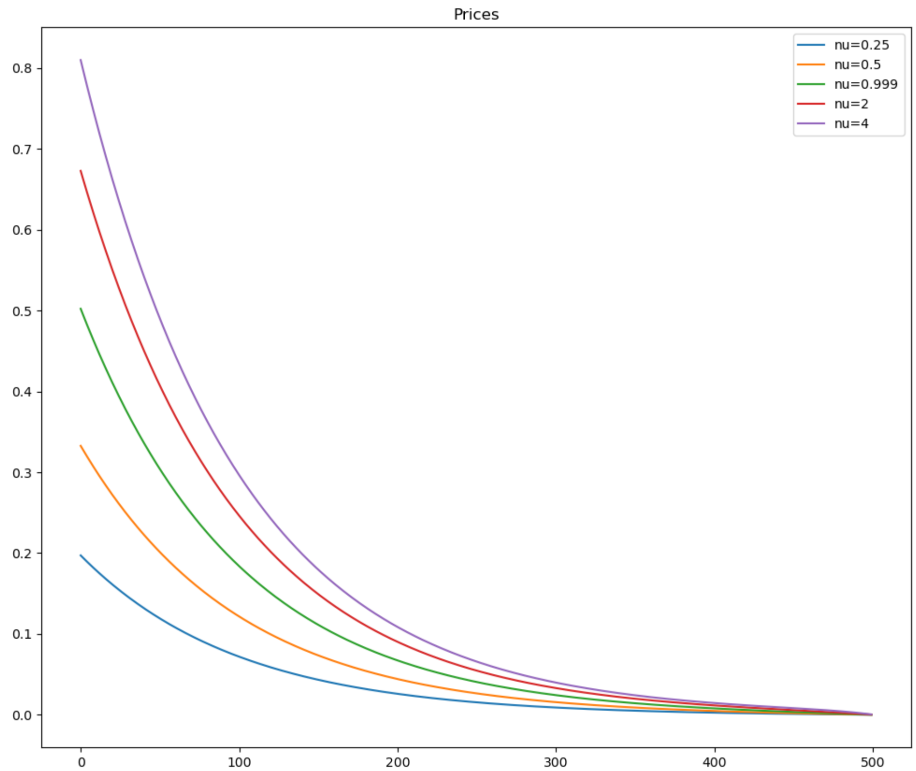


Figure 3: IRF of Prices

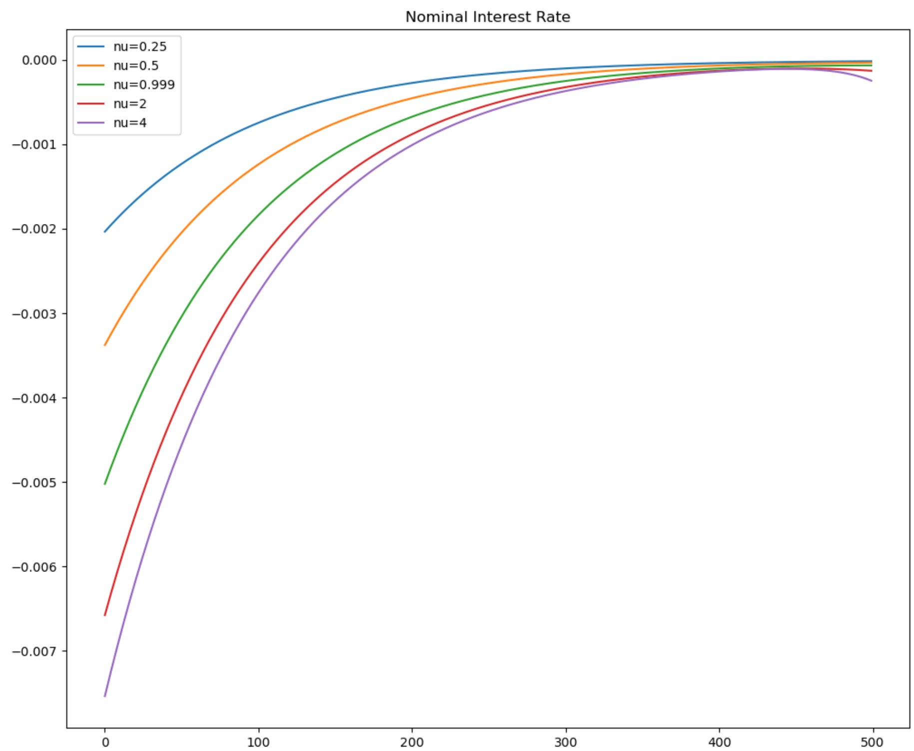


Figure 4: IRF of Nominal Interest Rate

## Question (h)

Figure 1 plots the shock to $M_t$. It goes to one in the instant of the shock and the effect diminishes over time.
Figure 2 exposes the fact that the reaction of Consumption to a Money Supply shock depends on the parameters of the utility function. In particular, we can confirm the analysis from question (b), when $\gamma = \nu$ (here, when both are equal to 1), then Consumption does not depend on $M_t$, and the $IRF$ is horizontal. More on this on question (i).
Figure 3 shows the reaction of the price level. We can see that the sign of the adjustment does not depend on parameter $nu$, however, the magnitude of the jump and also the steepness of the IRF does depend on $nu$. The smaller the $nu$, the higher the jump in prices at the moment of the money supply shock. We can see there exist a relationship between consumption adjustment and price adjustment to que shock. For the parameter space where consumption increases ($\nu = 4, 2$) with the shock, the price jump is the highest. Also, in the opposite scenario, where consumption jumps down with the shock ($\nu = 0.25, 0.5$), the price jump is smaller. The jump is 0.5 with money neutrality.

Finally, Figure 4 shows the IRF of the Nominal Interest Rate. Again, money neutrality scenario is the medium result, where we can see that for higher values of $\nu$, the effect in the nominal rate is higher, and for smaller values of $\nu$, the nominal rate impulse response function gets a smaller shock.

## Question (i)

If money supply increases consumption, we should rule out $\nu = 0.25, 0.5$. As we can see in Figure 2, for those values, consumption jumps down when the economy faces a positive money supply shock.

## Question (j)

See code in GitHub