1. Introduction

Implementation of the stick-slip "Hybrid method" as proposed in [1]

2. Kinematics

2.1. Brief overview of the corotational formulation and finite element implementation

The derivations of the co-rotational formulation are described in [2] (see Figure 1 for details). We will just summarize the important parts. We start by defining the global displacement and rotations $\boldsymbol{d}$ and their variation $\delta\boldsymbol{d}$ as

$$\boldsymbol{d} = \begin{bmatrix} \boldsymbol{u}_1^T & \boldsymbol{\theta}_1^T & \boldsymbol{u}_2^T & \boldsymbol{\theta}_2^T \end{bmatrix}^T; \quad \delta\boldsymbol{d} = \begin{bmatrix} \delta\boldsymbol{u}_1^T & \delta\boldsymbol{w}_1^T & \delta\boldsymbol{u}_2^T & \delta\boldsymbol{w}_2^T \end{bmatrix}^T. \tag{1}$$

Additionally, we define set of local nodal quantities and their variation, as required by the corotational formulation, as

$$\bar{\boldsymbol{\mathcal{D}}} = \begin{bmatrix} \bar{u} & \bar{\boldsymbol{\Theta}}_1^T & \bar{\boldsymbol{\Theta}}_2^T \end{bmatrix}^T; \quad \delta\bar{\boldsymbol{\mathcal{D}}} = \begin{bmatrix} \delta\bar{u} & \delta\bar{\boldsymbol{\Theta}}_1^T & \delta\bar{\boldsymbol{\Theta}}_2^T \end{bmatrix}^T. \tag{2}$$

Using the above definitions and finite element interpolation, the centerline displacement in the current configuration is given by

$$\boldsymbol{u}_0^h(\xi) = \boldsymbol{u}_1 N_1(\xi) + \boldsymbol{u}_2 N_2(\xi) + \boldsymbol{R}_e \boldsymbol{u}^t(\xi). \tag{3}$$

with $\boldsymbol{R}_e$ being the rigid body rotation of the beam axes and $\boldsymbol{u}^t$ the transversal displacement of the beam centerline, defined as

$$\boldsymbol{u}^t(\xi) = \begin{bmatrix} 0 \\ u_2(\xi) \\ u_3(\xi) \end{bmatrix} = \boldsymbol{P}_1(\xi) \begin{bmatrix} \bar{\boldsymbol{\Theta}}_1 \\ \bar{\boldsymbol{\Theta}}_2 \end{bmatrix}. \tag{4}$$

Overall, this allows defining the variation of the centerline displacement, as well as its first and second time derivatives

$$\delta\boldsymbol{u}_0^h = \boldsymbol{R}_e \boldsymbol{H}_1 \boldsymbol{E}^T \delta\boldsymbol{d}, \tag{5}$$
$$\dot{\boldsymbol{u}}_0^h = \boldsymbol{R}_e \boldsymbol{H}_1 \boldsymbol{E}^T \dot{\boldsymbol{d}}, \tag{6}$$
$$\ddot{\boldsymbol{u}}_0^h = \boldsymbol{R}_e \boldsymbol{H}_1 \boldsymbol{E}^T \ddot{\boldsymbol{d}} + \boldsymbol{R}_e \boldsymbol{C}_1 \boldsymbol{E}^T \dot{\boldsymbol{d}}. \tag{7}$$

And same for the rotational variables, i.e.

$$\delta\boldsymbol{w}_0^h = \boldsymbol{R}_e \boldsymbol{H}_2 \boldsymbol{E}^T \delta\boldsymbol{d}, \tag{8}$$
$$\dot{\boldsymbol{w}}_0^h = \boldsymbol{R}_e \boldsymbol{H}_2 \boldsymbol{E}^T \dot{\boldsymbol{d}}, \tag{9}$$
$$\ddot{\boldsymbol{w}}_0^h = \boldsymbol{R}_e \boldsymbol{H}_2 \boldsymbol{E}^T \ddot{\boldsymbol{d}} + \boldsymbol{R}_e \boldsymbol{C}_2 \boldsymbol{E}^T \dot{\boldsymbol{d}}. \tag{10}$$

Upon assembly, this yields the following system of equations:

$$\mathbf{R}(\mathbf{D}, \dot{\mathbf{D}}, \ddot{\mathbf{D}}, t) = \mathbf{T}_{int}(\mathbf{D}, t) + \mathbf{T}_k(\mathbf{D}, \dot{\mathbf{D}}, \ddot{\mathbf{D}}, t) - \mathbf{T}_c(\mathbf{D}, \dot{\mathbf{D}}, t) - \mathbf{F}_{ext}(\mathbf{D}, t) = \mathbf{0} \tag{11}$$

Solution is carried out by means of the HHT-α method, yielding

$$\mathbf{R}^{n+\alpha} = (1+\alpha)(\mathbf{T}_{int}^{n+1} - \mathbf{T}_c^{n+1} - \mathbf{F}_{ext}^{n+1}) - \alpha(\mathbf{T}_{int}^{n} - \mathbf{T}_c^{n} - \mathbf{F}_{ext}^{n}) + \mathbf{T}_k^{n+1} = \mathbf{0}, \tag{12}$$
$$\tilde{\mathbf{K}}\Delta\mathbf{D} = -\mathbf{R}^{n+\alpha}, \tag{13}$$

with

$$\tilde{\mathbf{K}} = (1+\alpha)\left(\mathbf{K}_{int} - \mathbf{K}_c - \mathbf{K}_{ext}\right) + \frac{1}{\beta\Delta t^2}\mathbf{M} + \frac{\gamma}{\beta\Delta t}\left(\mathbf{C}_k - (1+\alpha)\mathbf{C}_c\right). \tag{14}$$

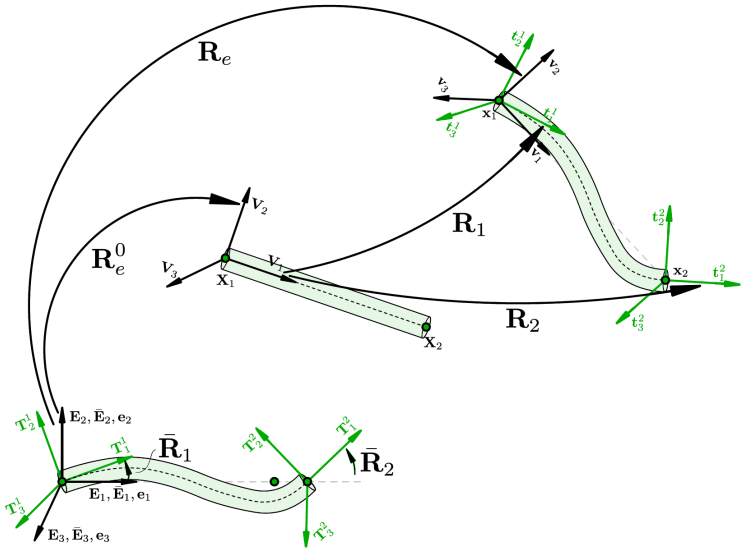

Figure 1: Co-rotational element

2.2. Regularized contact formulation

The contact formulation presented in [2], is briefly presented in this section. In order to consider the contact against a rigid surface, the weak form is extended using the variation of a penalty potential δW_c as

$$\delta W = \delta W_{int} + \delta W_{kin} - \delta W_{ext} - \delta W_c = 0. \tag{15}$$

Following [3], a penalty potential is defined. This potential takes into account the normal and frictional contributions of the beam contact against a rigid master surface Γ_s

$$\delta W_c = \int_0^{l_0} \left(p_N \delta g_N + \boldsymbol{t}_T \cdot \delta \boldsymbol{g}_T \right) ds, \tag{16}$$

with g_N being the normal gap respect the master surface, $\boldsymbol{g}_T$ the relative displacement in the tangential direction, p_N the contact pressure and $\boldsymbol{t}_T$ the frictional contact traction. All values in (16) are evaluated at the beam centerline by assuming that the beam radius is small enough to disregard the couples generated by the frictional force [4]. Given the fact that circular beams are used, g_N is defined as

$$g_N(s) = d_s(s) - r, \tag{17}$$

with $d_s(s)$ being the distance of a point on the beam centerline to the surface and r the radius of the beam, which is assumed constant. The variation of g_N is given by

$$\delta g_N = \frac{\partial g_N}{\partial \boldsymbol{x}} \cdot \delta \boldsymbol{x}_0 = \frac{\partial d_{\Gamma_s}}{\partial \boldsymbol{x}} \cdot \delta \boldsymbol{u}_0 = \boldsymbol{n} \cdot \delta \boldsymbol{u}_0, \tag{18}$$

with $\boldsymbol{n}$ being the surface normal. In order to obtain the variation of $\boldsymbol{g}_T$, the centerline displacement is decomposed into its normal and tangential components, respect to the master surface [5], i.e.

$$\delta \boldsymbol{u}_0 = (\delta \boldsymbol{u}_0 \cdot \boldsymbol{n})\, \boldsymbol{n} + \delta \boldsymbol{u}_T = \delta g_N \frac{\partial g_N}{\partial \boldsymbol{x}} + \delta \boldsymbol{g}_T. \tag{19}$$

Isolating $\delta \boldsymbol{g}_T$ in the above expression yields

$$\delta \boldsymbol{g}_T = \left(\boldsymbol{I} - \frac{\partial g_N}{\partial \boldsymbol{x}} \otimes \frac{\partial g_N}{\partial \boldsymbol{x}} \right) \delta \boldsymbol{u}_0, \tag{20}$$

$$\dot{\boldsymbol{g}}_T = \left(\boldsymbol{I} - \frac{\partial g_N}{\partial \boldsymbol{x}} \otimes \frac{\partial g_N}{\partial \boldsymbol{x}} \right) \dot{\boldsymbol{u}}_0. \tag{21}$$

In order to facilitate the convergence of the Newton Raphson algorithm, both the contact pressure p_N and frictional contact force $\boldsymbol{t}_T$ are regularized. The contact pressure is evaluated using the penalty regularization used in [6–9]

$$p_N^{reg} = \begin{cases} \bar{p}_N - \varepsilon_c g_N, & g \leq 0 \\ \dfrac{\varepsilon_c \bar{g}_N - \bar{p}_N}{\bar{g}_N^2} g_N^2 - \varepsilon_c g_N + \bar{p}_N, & 0 < g_N \leq \bar{g}_N \\ 0, & g > \bar{g}_N \end{cases} \tag{22}$$

with ε_c being the penalty parameter, $\bar{p}_N = \frac{1}{2}\varepsilon_c \bar{g}_N$ and $\bar{g}_N$ being the normal gap value at which the contact pressure starts increasing. As in [8], $\bar{g}_N$ is taken as 10% of the beam radius.

The discrete contact force vector is defined as

$$\boldsymbol{T}_c = \int_0^{l_0} \boldsymbol{E}\boldsymbol{H}_1^T \boldsymbol{R}_e^T \boldsymbol{f}_c \, ds, \tag{23}$$

with $\boldsymbol{f}_c$ being the contact force, given by

$$\boldsymbol{f}_c = \boldsymbol{f}_c^N + \boldsymbol{f}_c^T, \tag{24}$$

with

$$\boldsymbol{f}_c^N = p_N^{reg} \frac{\partial g_N}{\partial \boldsymbol{x}}, \tag{25}$$

$$\boldsymbol{f}_c^T = -\mu p_N^{reg} \frac{\dot{\boldsymbol{g}}_T^h}{\sqrt{\|\dot{\boldsymbol{g}}_T^h\|^2 + \varepsilon}}. \tag{26}$$

2.3. Hybrid stick-slip formulation as in [1]

We briefly describe the formulation presented in [1], but adapted to our notation. The normal and tangential components in 24 are given by

$$\boldsymbol{f}_c^N = p_N^{lin} \frac{\partial g_N}{\partial \boldsymbol{x}}, \tag{27}$$

$$\boldsymbol{f}_c^T = -\frac{\dot{\boldsymbol{g}}_T^h}{\|\dot{\boldsymbol{g}}_T^h\|} \min\left(\varepsilon_c^T \|\dot{\boldsymbol{g}}_T\|, \mu p_N^{lin}\right), \tag{28}$$

with

$$p_N^{lin} = \varepsilon_c^N \max(-g_N, 0). \tag{29}$$

Note: the regularized normal pressure is no longer used. They claim that in very few instances the Gauss Point falls into $g_N = 0$, which is the only point with a non-defined derivative.
Note: now two penalty parameters are defined, ε_c^N and ε_c^T, but ε_c^T is defined as $\varepsilon_c^T = \Delta t \varepsilon_c^N$ (see last paragraph of section 4.2 in [1]).
Note: restricting to the slip case, replacing p_N^{lin} by p_N^{reg} equations (27), (28) turn into equations (25), (26) for the specific case of $\varepsilon = 0$.

The use of the above equations would yield to a linearaly regularized friction law, which does not account for stick (see Figure 2), which does not avoid long-term sliding. The hybrid algorithm proposed in [1] is proposed to actually enforce stick. This is done by using the following algorithm:

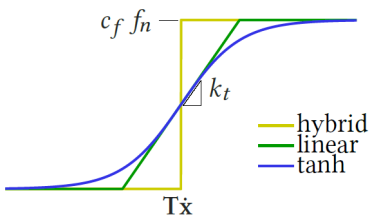

Figure 2: Friction law taken from [1]

- while $\|\boldsymbol{R}\| < \epsilon_R^{1/2}$, repeat at each iteration k
  - solve system (13) using definitions (27) and (28) for the contact force
- $\forall$ nodes compute $\boldsymbol{f}_c^T$ as defined in equation (28) and check Coulomb limit:
  - if $\|\boldsymbol{f}_c^T\| < \mu p_N^{lin} \rightarrow$ stick $\rightarrow$ apply fixed Dirichlet boundary conditions
  - if $\|\boldsymbol{f}_c^T\| = \mu p_N^{lin} \rightarrow$ slip $\rightarrow$ leave node free
- while $\|\boldsymbol{R}\| < \epsilon_R$, repeat at each iteration k
  - solve system (13) using definitions (27) and (28) for the contact force
    - $\forall$ **fixed nodes** compute $\boldsymbol{f}_c^T$ as defined in equation (28) and check Coulomb limit:
      - if $\left(\|\boldsymbol{f}_c^T\| > \mu p_N^{lin}\right)$ & $(\|\boldsymbol{R}\| < \epsilon_R^{1/2}) \rightarrow$ free that node

Note: this algorithm has been written based on the description of section 4.3 in [1]. There are some parts that are not fully clear. Specifically: 1) if the fixed node is let free, can it be fixed again if the frictional force is below the Coulomb limit? My understanding is that everything is kept like this till the end of the simulation 2) using equation (28), the frictional force cannot be larger than μp_N^{lin}, therefore the $>$ symbol in the last check should be replaced by $=$? This would not make any difference, but I just wonder if I missed something in the description
Note: in our case, the frictional force is calculated at the Gauss points, whereas we can only fix nodes (not Gauss points). We should think of a way of transferring information from Gauss points to nodes or viceversa. Otherwise, we can check the Coulomb law at nodes and fix them if needed. The actual force we can keep using them at the
Note: notice that, as explained in [1], the last check can lead to some nodes using the penalty frictional law, therefore sliding below the Coulomb limit. Apparently, this is very rare and has minor effects.

2.4. Proposed hybrid stick-slip

Let's propose a very simple implementation where we keep almost all ingredients that we have, including the regularized laws for both normal contact and friction. I belive that we can still use them as they are just more general than the linear laws that they use, and what actually matters is that we check the Coulomb limit to enforce Dirichlet boundary conditions (in red modifications respect to formulation in [1]).

Initialize $\mathbf{f}_c^T$, $\mathbf{p}_N$ nodal vectors, of size $(3n_{nodes} \times 1)$ and $(n_{nodes} \times 1)$ respectively

- while $\|\boldsymbol{R}\| < \epsilon_R^{1/2}$, repeat at each iteration k
  - set $\mathbf{f}_c^T(:) = 0$, $\mathbf{f}_c^N(:) = 0$
  - solve system (13) using definitions (25) and (26) for the contact force at each Gauss Point, integrated then into $\boldsymbol{T}_c$ as in equation (23) and assembled into the global $\mathbf{T}_c$ vector.
  - compute $\boldsymbol{f}_{c,elem}^T = \frac{1}{l_0}\int_0^{l_0} \boldsymbol{f}_c^T\, ds$, $p_N^{elem} = \frac{1}{l_0}\int_0^{l_0} p_N^{reg}\, ds$ and assemble 1/2 of each term to the corresponding positions of nodes 1 and 2 in the nodal vectors $\mathbf{f}_c^T$, $\mathbf{p}_N$.
- $\forall$ node i and using $\mathbf{f}_c^T$, $\mathbf{p}_N$, compare $\boldsymbol{f}_{c,i}^T$ to $p_{N,i}$
  - if $\|\boldsymbol{f}_{c,i}^T\| < \mu p_{N,i}$ $\rightarrow$ stick $\rightarrow$ apply fixed Dirichlet boundary conditions
  - if $\|\boldsymbol{f}_{c,i}^T\| = \mu p_{N,i}$ $\rightarrow$ slip $\rightarrow$ leave node free
- while $\|\boldsymbol{R}\| < \epsilon_R$, repeat at each iteration k
  - set $\mathbf{f}_c^T(:) = 0$, $\mathbf{f}_c^N(:) = 0$
  - solve system (13) using definitions (25) and (26) for the contact force at each Gauss Point, integrated then into $\boldsymbol{T}_c$ as in equation (23) and assembled into the global $\mathbf{T}_c$ vector.
  - compute $\boldsymbol{f}_{c,elem}^T = \frac{1}{l_0}\int_0^{l_0} \boldsymbol{f}_c^T\, ds$, $p_N^{elem} = \frac{1}{l_0}\int_0^{l_0} p_N^{reg}\, ds$ and assemble 1/2 of each term to the corresponding positions of nodes 1 and 2 in the nodal vectors $\mathbf{f}_c^T$, $\mathbf{p}_N$.
    - $\forall$ **fixed nodes** i and using $\mathbf{f}_c^T$, $\mathbf{p}_N$, compare $\boldsymbol{f}_{c,i}^T$ to $p_{N,i}$
      - if $\left(\|\boldsymbol{f}_{c,i}^T\| > \mu p_{N,i}\right)$ & $\left(\|\boldsymbol{R}\| < \epsilon_R^{1/2}\right)$ $\rightarrow$ free that node

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