Risk parity : choosing a risk-based approach

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2. Seeyond is a brand of Natixis Asset Management.

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SUMMARY

The risk parity approach offers investors solutions in constructing portfolios more resistant to crises and also able to capitalize on bullish markets. As its name suggests, this approach grants priority to risk budgeting, which is spread evenly between the different asset classes that make up the portfolio. This apparently simple process involves a number of potential difficulties, including:

- The method used to balance risks: Can capital be evenly weighted between all asset classes? Should weightings be set up inversely to the level of risk represented by each asset? Can the risk of each asset be taken into consideration and also its contribution to overall portfolio risk, given the correlations between assets?
- The crucial choice of risk measure, which will determine any further portfolio construction in terms of risk parity: Should asymmetric distribution be applied or solely loss-distribution? Do more advanced methods, which include an element of forecasting in addition to historical data, provide better results? Is it acceptable to use a simple and widespread measure such as historical volatility?

Past historical research includes many attempts to optimize the risk-return ratio, starting with the historic work of Markowitz and the efficient frontier concept **(Section 1)**.

However, the oversimplification underpinning this model, based on mean-variance concepts, fails to stand up to the harsh reality of the markets since normal distribution does not reflect real-life experience. It also fails to incorporate the asymmetry of returns, whereas risk-focused investors are more concerned about losses rather than gains. Furthermore, the suggested allocation models are often over-concentrated on a narrow range of assets offering the highest Sharpe ratios. This factor accentuates the fragility of the model which is highly dependent on the accuracy of estimated expected returns, variance and correlation. More precisely flawed expected-return parameters have a much greater impact on the quality of the model than inaccuracies in variance or correlation.

We therefore reject the precepts of the Markowitz theory, particularly as we have set a very broad investment universe as a prerequisite, in order to maximize potential portfolio diversification. However, it would be unreasonable to consider an exact estimate of expected returns from a very large number of assets.

As a result, we have examined several alternative portfolio construction methods which are not based on expected-return hypotheses and which present the advantage of minimizing concentration criteria for a given portfolio **(Section 2)**:

1. The Equal Weight method (EW) which allocates an identical weighting to each of the portfolio's assets;
2. Inverse Volatility method (IV), also known as the "naïve risk parity strategy", which weights assets inversely to their volatility;
3. The Equal Risk Contribution method (ERC) which takes into account the covariance between assets and reduces overall portfolio risk through diversification.

Out of these three methods, presented according to their advantages, disadvantages and relevance in turbulent markets, only one will be retained. The ERC method will show the most promising results in this paper, both in terms of volatility and cumulated rolling losses, as well as in terms of portfolio concentration.

Section 3 focuses on the choice of the risk measure, which we define as the chance, or increase in probability, of an event occurring which would make investors more risk averse. We are therefore seeking to detect a change in an asset's behavioral regime.

Among the range of risk parameters, we have analyzed widespread measures such as historical and implied volatility, Value at Risk (VaR) and Conditional Value at Risk (CVaR). However, the field of investigation was also widened to include other methods such as Lower Partial Moment (LPM), Exponential Volatility (EWMA or Exponentially Weighted Moving Average, a particular case of the GARCH* models) in which weightings allocated to past observations decrease exponentially over time. These methods are distinguished by whether they integrate predictive and asymmetric data or include projected future returns.

As we established in Section 1, approaches involving estimations of future returns such as VaR or CVaR will not be selected. This drawback does not apply to the LPM method, which also has the advantage of being a more realistic risk model, being based solely on loss distribution. Ultimately, despite their divergent approaches, the three risk parameters (Historical Volatility, Lower Partial Moment and Exponential Volatility) do not produce radically different results. Selection criteria will include complexity of implementation, relative portfolio stability and rate of turnover.

All of the results obtained using the various simulations are presented in **Section 4**.

In conclusion, this paper highlights the superiority of the ERC method for portfolio risk parity construction and gives investors a broad choice of risk measures to use, given that a pragmatic and simple approach based on historical volatility appears to work well. This study therefore proposes ways of implementing risk parity solutions and highlights the range of possible variants according to the selected construction choices.

Finally, it should be stipulated that this document covers portfolio construction methods in the context of a risk parity strategy. A second phase, taking market anticipations into account, will be the subject of a forthcoming research paper.

**GARCH: General Autoregressive Conditionally Heteroskedastic*

INTRODUCTION

Recent crises have made risk control a priority for investors and asset management companies. The research produced in their wake has changed the way diversified portfolios are constructed. Market upheaval has revealed that the risk for portfolios presented as balanced was in fact overly concentrated on the riskiest assets, a situation accentuated by poor control over interactions between assets within the portfolio.

Because it can take a very long time to recover from losses suffered during a crisis, risk control is now considered a key source of return through capital preservation. Minimizing losses has become a top priority in the quest for long-term return through efforts to reduce the magnitude of these losses and the frequency with which they occur.

A number of publications have recently highlighted new portfolio allocation methods that are no longer focused solely on expected returns but also take into account the risk incurred by investors, with the aim of maximizing not just return but rather risk-adjusted return.

The risk parity approach is a portfolio allocation method that addresses the issue by constructing portfolios where risks are evenly divided among different asset classes. The objective is to build portfolios better able to resist market downturns while taking advantage of any upturns.

The success of this method has led to the development of a number of offshoots, seen first in the United States and then in Europe. There are many ways to implement a risk budget-based allocation. Each solution entails choosing the right portfolio construction method and selecting the most relevant risk measure for each asset class and for the portfolio as a whole, to achieve the best combination of these different asset classes. The approaches selected naturally result in very different risk parity portfolio profiles.

Investors therefore need to properly understand the mechanisms involved, the underlying management objectives and the inherent limitations of these methods.

This report reflects our effort to provide investors with some “keys” to help them choose the right solution for their needs.

In this paper, we will analyze both the benefits and the pitfalls associated with the choice of risk-based portfolio construction method, the estimates and assumptions that underlie the models used and the risk measures considered.

The topics discussed therefore concern long-term strategic allocation. The role of tactical allocation, particularly from the perspective of the quest for return, will be addressed in a subsequent report. Lastly, we assume the portfolio is constructed from the broadest possible investment universe to benefit from the maximum potential for diversification.

1 RISK-RETURN TRADE-OFF IN OPTIMIZATION

Portfolio construction methods have for many years been influenced by the seminal work of Markowitz [1]. The relationship between risk and return has significantly shaped the nature of the construction algorithms used by asset management companies.
Is this theory, which seeks to optimize a portfolio based on expected return and variance, an acceptable solution for constructing a robust portfolio over time from a broad investment universe?

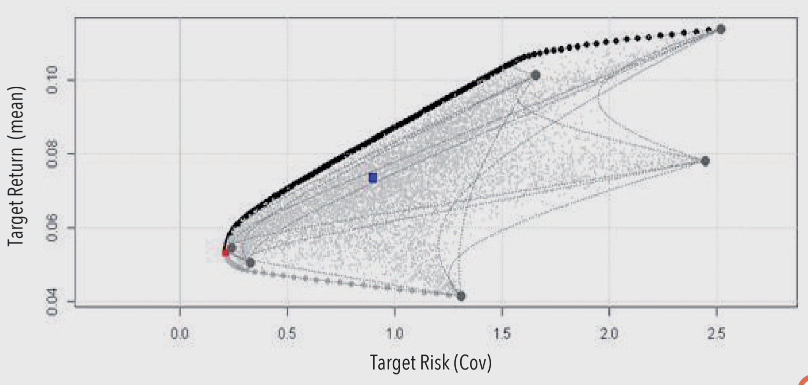
A. PRINCIPLES

In Markowitz's research, portfolios are constructed as an optimization:

- Maximizing the expected return for a given level of ex-ante risk;
- Minimizing the level of ex-ante risk incurred for a given level of return.

This then gives us what is known as the efficient frontier curve, which reflects the results obtained for different risk-return levels.
Each level of return is associated with a portfolio that minimizes risk (measured by variance). Conversely, each level of risk may be associated with a portfolio that maximizes expected return. The set of these optimal portfolios is called the efficient frontier or Markowitz frontier.

Efficient Frontier



This chart illustrates the concept of efficient frontier. Each asset in the portfolio is represented by a dark gray dot. The blue square pinpoints the risk-return profile for an equal capital-weighted portfolio. The red dot represents the so-called Minimum Variance allocation, that is, the allocation for which the ex-ante variance is the lowest.
The light gray dots correspond to the possible portfolios as a linear combination of eligible assets and give an understanding of the possible risk-return profiles.
The efficient frontier is depicted in black; it shows the allocations that maximize the return for a given level of risk. Those are the allocations sought by investors.

B. LIMITATIONS

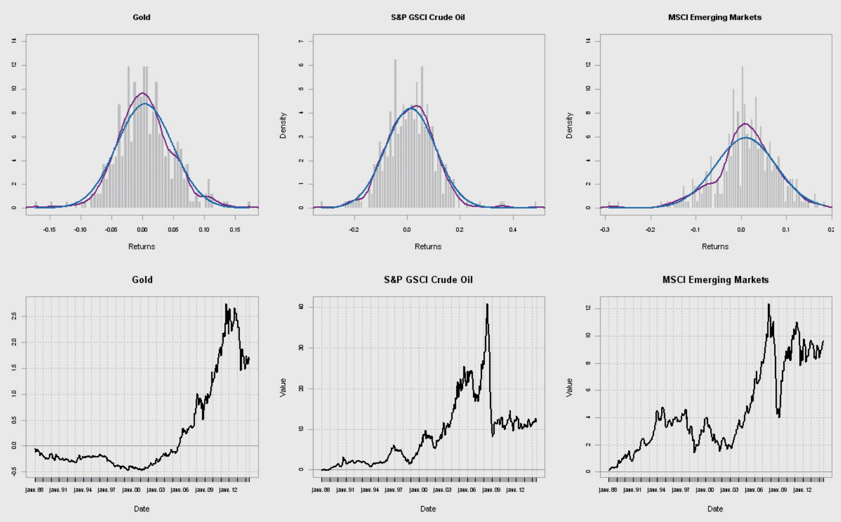
Despite the appeal of this approach, a number of limitations quickly arise:

1. First, in terms of assumptions:

- Returns from normally distributed assets:

Markowitz's Mean-Variance model is based on a normal distribution of returns, which is only a standard but imperfect approximation of the actual behavior of financial assets. This oversimplification does not pass the market reality test, as illustrated by the charts that show the return distribution of different assets for a given period and frequency (the blue curve corresponds to the normal distribution).

Actual behavior of financial assets



- Risk measured by variance:

The choice of variance as a risk measure is understandable if asset returns all follow a normal distribution. In this particular context, the loss distribution is defined entirely by the expectation and variance of the returns.
However, in practice, these first two moments are not enough to characterize the return distribution. Indeed, if two distributions with the same variance do not have the same loss profiles, they will necessarily present a different risk.
We can therefore express a second reservation about the approach proposed by the Markowitz model: positive and negative variations in returns around the mean cannot be taken into account in the same way. Variance does not incorporate the skewness of the returns when an investor is more concerned about downside risk.

2. But also in terms of optimization:

- Portfolio concentration:

Allocations are often limited to the few assets in the investment universe that have the best Sharpe ratios (that is, the best ratio of excess return over the risk-free rate and volatility). But incorrectly estimating these parameters could have a material impact for the investor, as the resulting portfolio would have a very high concentration of assets whose performance differs from the performance expected. Two major risks can therefore be identified when constructing a portfolio that targets assets with the best risk-return profiles:

➜ The period selected to define the expected return and its underlying risk does not necessarily present a profile that will be consistent with future events.
➜ The risk measure selected may be relevant only for some of the instruments in the selected universe.

This method, when permitted, also tends to produce portfolios with levels of leverage that are unrealistic, given asset management companies' usual constraints, and inconsistent with an approach that aims to control risk.

- Lack of robustness/stability:

Allocation weights are not guaranteed to remain stable and a minor change in input data (expected return, variance and correlation) can significantly alter the results of the optimization. Several authors have proven that this construction algorithm tends to compound errors made in the initial assumptions and, as such, have coined the phrase "Error Maximizing Portfolios" [2].

This last point can be extremely costly even if return expectations are close to actual returns in the future period.

C. IMPACT OF ERRORS

Two researchers, Chopra and Ziemba [3], showed that minor changes in input parameters can result in major changes in the composition of the optimal portfolio in a calibration that uses a mean-variance approach.
They then took their work to the next level and quantified the relative impact of an inaccurate (or even erroneous) estimate based on the nature of the parameter used.
Their conclusions are as follows:

- The impact of an error in returns is 11 times greater than that of an error in variances
- The impact of an error in variances is twice as great as that of an error in correlations

These orders of magnitude increase with the investor's risk tolerance: the higher the percentage of risky assets, the greater the magnitude and, ultimately, the more substantial the errors.
The emphasis should therefore be primarily on the quality of the estimate of expected returns; volatility estimates should be highly accurate while correlations are the least sensitive parameter. Estimates (or views) expressed on expected returns therefore have the greatest influence on the allocation composition and, in the event of inaccuracy, the most adverse effect on portfolio optimization.
As a result, it is important to determine market parameters/information for which it is reasonable to express views regarding assets accurately, systematically and quantifiably and beyond the mere expression of market trends.
Example: Having a positive view on eurozone equities is not enough to accurately express an expected level of return over the coming months.

Beyond these intrinsic characteristics, it is therefore crucial to select an allocation method that is consistent with available market expectations.

For a very broad investment universe, we therefore believe it is not reasonable to model a portfolio by integrating return expectations for each component of the universe.

D. HOW DO WE BUILD AN ALLOCATION FROM A VERY BROAD INVESTMENT UNIVERSE?

Our objective is to build a portfolio from the broadest possible universe to maximize diversification potential. However, it is virtually impossible in the context of global asset management to express accurate views on all assets. The manager will therefore be forced to distort the allocation for just one asset segment, the one which he or she strongly believes will perform well in the future.
In the remainder of this paper, we will work within this framework and look only for allocation methods that meet the following criteria:

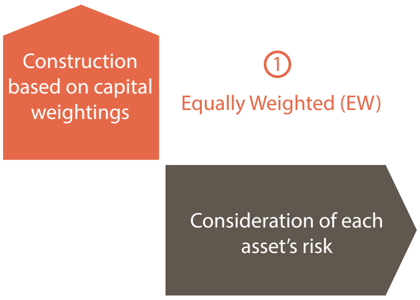
- It is not necessary to express return expectations for all of the assets in the investment universe. Consequently, few data need to be estimated.
- The method is robust; weighting variations are limited relative to the input assumptions.
- The method minimizes at least one of the usual portfolio concentration criteria (and not a risk measure directly). This condition enables us to avoid highly concentrated portfolios that have high turnover rates and that take advantage of only a small portion of their investment universe.

2 POSSIBLE ALTERNATIVES FOR GLOBAL ASSET MANAGEMENT

To illustrate our point, we will present several approaches that conform to the principles stated above and enable us to define a systematic allocation that can serve as the framework for a global portfolio. We should specify that, at this juncture, we will not consider any market view (a topic that will be discussed in a future paper) and will focus primarily on finding the right balance within a portfolio.
Consequently, only methodologies that satisfy the following conditions will be applied:

- ➜ The method requires no expected return assumption.
- ➜ The method seeks to minimize portfolio concentration for a given criterion.
- ➜ The method only has one solution. There is therefore no need to impose additional criteria to guarantee convergence.

A. EQUALLY WEIGHTED (EW)



This portfolio construction method is the easiest to implement. It simply involves attributing the same capital weightings to each asset class in the investment universe. The portfolio manager buys equal quantities of all available assets, with no consideration of the risk that each asset represents.
Stated more formally:

$$\forall (i,j) \quad \omega_i = \omega_j = 1/N$$

Where ω_i represents the weight of asset i in the portfolio and N represents the number of assets that make up that portfolio.

1. What are the benefits of this approach?

It requires no market expectations and no other assumptions about the assets in the portfolio.
This is the method that minimizes the portfolio's weight concentration (see Herfindahl-Hirschman index, Appendix 2).

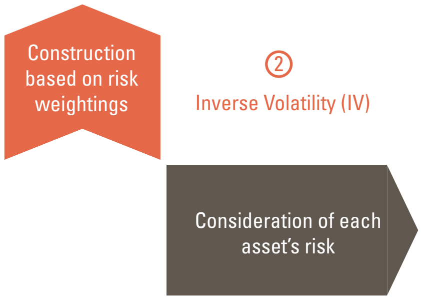
2. What are the drawbacks of this approach?

There is no risk management in this approach, which allocates the same amount of capital to risky assets and very low-risk assets.

3. Performance in periods of market turbulence

This construction method is referred to as "contrarian": it decreases the weight of assets that have advanced and increases the weight of assets that have declined at each rebalancing to maintain the portfolio's equal weighting. Capital is therefore allocated against recent trends, hence the use of the term "contrarian". It is therefore effective in the event of a market downturn.

B. INVERSE VOLATILITY (IV)



In this approach, asset weightings are inversely proportional to their volatility. We therefore have an allocation in which each asset's marginal risks are equal.

The formula for calculating the weightings is therefore:

$$\forall i \quad \omega_i = \frac{\frac{1}{\sigma_i}}{\Sigma_{j=1}^{N} \frac{1}{\sigma_j}}$$

Where ω_i represents the weight of asset i in the portfolio, σ_i the volatility of asset i and N the number of assets that make up this portfolio.

1. What are the benefits of this approach?

It requires only asset volatility. It minimizes the marginal risk concentration.

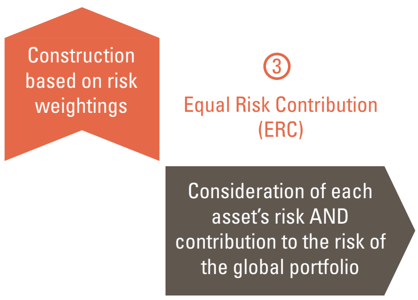
2. What are the drawbacks of this approach?

No consideration is given to the interactions between different assets and therefore to the diversification potential. This method is highly dependent on the investment universe.

3. Performance in periods of market turbulence

This method allows for an equitable allocation of risks across all assets in the portfolio if the correlations between assets are all the same and provided this risk is symmetrical. In the event of a financial crisis in which all assets recorrelate, this assumption is of course confirmed and this equally-weighted volatility portfolio construction method will result in a gradual deallocation of the most volatile assets in favor of the least volatile assets.

C. EQUAL RISK CONTRIBUTION (ERC)



This involves consideration of the interactions that exist between different assets likely to be included in the portfolio.
The standard deviation of a security's return includes both diversifiable (idiosyncratic) risk and non-diversifiable (systemic) risk. Our method needs to factor in the reduction in risks associated with allocation diversification. To do so, covariance rather than variance should be used to characterize exposure to allocation risk. The following quantities need to be defined:

Allocation volatility: $$\sigma_P(\omega)=\sqrt{\omega^T.\Omega.\omega}$$

Marginal risk: $$\forall i \quad MR_i = \frac{\partial \sigma_P}{\partial \omega_i} = \frac{(\Omega.\omega)_i}{\sigma_P(\omega)}$$

Risk contribution:
(on an absolute basis)

$$\forall i \quad CR_i = \omega_i \frac{\partial \sigma_P}{\partial \omega_i} = \omega_i \frac{(\Omega.\omega)_i}{\sigma_P(\omega)}$$

(as a percentage) $$\forall i \quad PCR_i = \frac{\omega_i}{\sigma_P(\omega)} \frac{\partial \sigma_P}{\partial \omega_i} = \frac{\omega_i}{\sigma_P(\omega)} \frac{(\Omega.\omega)_i}{\sigma_P(\omega)}$$

The ERC allocation method involves calibrating weights so that each asset has the same risk contribution.

This can be expressed as: $$\forall (i,j) \quad CR_i = CR_j = 1/N$$

1. What are the benefits of this approach?

It considers not only the volatility of each asset but also their correlation (the covariance matrix is derived from these two indicators). Unlike the previous techniques, the ERC approach offers the advantage of optimization at the global portfolio level and uses the decorrelation potential of each asset included in the portfolio.

It minimizes the portfolio's risk concentration (see Gini index, Appendix 2).
This technique also makes direct use of the covariance matrix and not its reverse. It therefore meets all our expectations in terms of stability.

- It considers not only the volatility of each asset but also their correlation.
- It minimizes the portfolio's risk concentration.
- It also makes direct use of the covariance matrix and not its reverse inverse.

2. What are the drawbacks of this approach?

Unlike the methods described above, weights need to be calibrated using a dynamic programming algorithm, making its implementation more knotty.

3. Performance in the event of a crisis

If the same crisis were to hit some or all of a portfolio constructed through an equally weighted contribution to this portfolio's risk, the allocation method would play its part in terms of risk diversification, by favoring assets with a lower volatility regime in proportion to assets with a higher volatility regime.

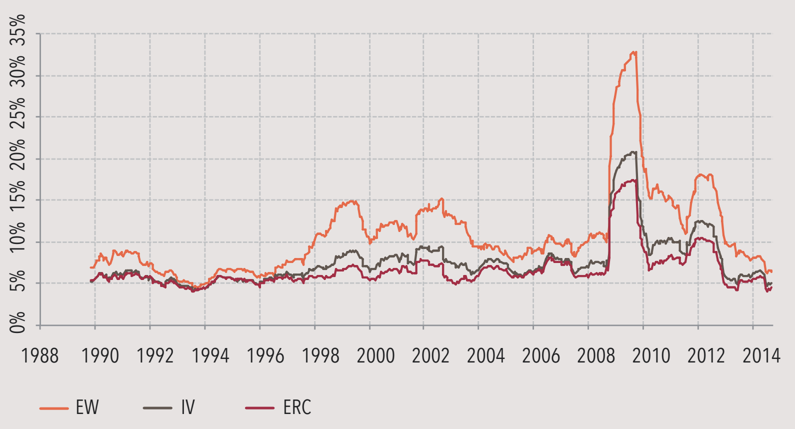
Unlike the Inverse Volatility method, this approach can be applied to more diverse investment universes, where only some of the assets see a change in volatility regime and correlation, since the construction method can effectively consider the relationships between assets without placing any conditions on the correlation differences.
The characteristics of this methodology are detailed in the very comprehensive report by Maillard, Roncalli and Teiletche [4].

D. CONCLUSION REGARDING THE CHOICE OF THE PORTFOLIO CONSTRUCTION METHOD

We have endeavored to assess the relevance of the different portfolio construction methods (Mean-Variance Optimization, Equal Weight (EW), Inverse Volatility (IV), Equal Risk Contribution (ERC)). Without wishing to anticipate the results presented in section 4, we can already see that the construction methods based on risk equalization within a portfolio are not all created equal. The simplest approaches, Equal Weight and Inverse Volatility (often referred to as "naïve risk parity") do not consider the interactions between assets and hence the decorrelation gains that could be achieved with a diversified portfolio.

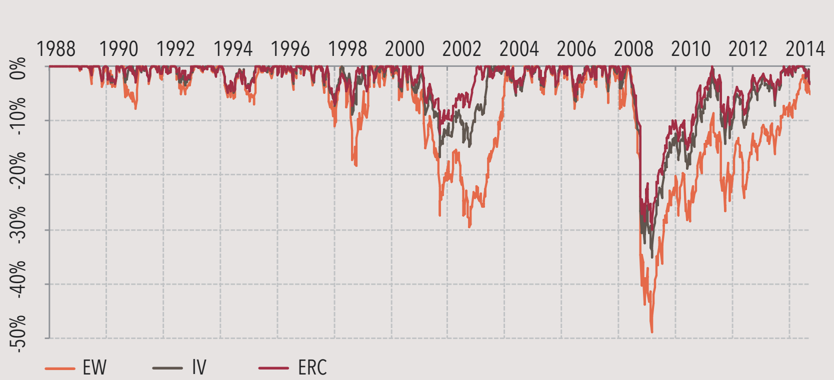
Results for the volatility of each method (measured here over 12 months) show that ERC, which considers the correlation between assets, generates the lowest volatility, regardless of market configuration.

Volatility (12M)



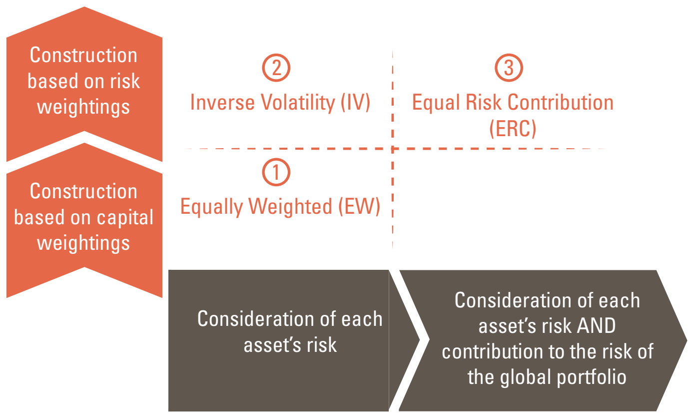
In terms of reducing drawdowns, ERC suffers the least in a market downturn. This is as expected, given the results for variability of returns for our three allocations. In contrast, EW, owing to its complete lack of risk management, bears the brunt of crises.

Drawdown



In the rest of this report, ERC will be the only method retained as a risk parity approach: it enables us to construct the most robust portfolios in various market configurations. It is admittedly impossible to eliminate risks related to asset volatility and problems associated with the correlation between assets within a portfolio, but the ERC method enables us to manage them as best we can and even to derive a source of value-added.

We still need to tackle the question of the risk measure to be paired with this method, which we will address in section 3 below.



3 CHOICE OF RISK MEASURE

The objective is to identify a measure indicating the imminent occurrence — or at least the increase in the likelihood — of an event to which the investor is averse. We therefore need to identify any regime change in an asset's performance.

A. REQUIREMENTS

A risk measure, as it could be used in risk parity strategies for example, needs to meet the requirements of the algorithm that we are using. As we saw above, it is essential to be able to calculate the contributions of each asset in the portfolio relative to this measure, and the sum of these contributions needs to be the same as the portfolio's overall risk. To this end, we simply need to satisfy Euler's identity. If the risk measure is (positively) homogeneous of degree n, i.e., if it verifies the following relationship:

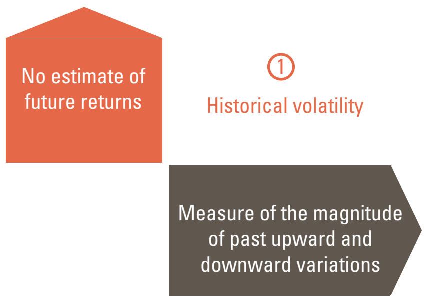
$$\forall \lambda > 0 \quad f(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda^n f(x_1, x_2, \dots, x_n)$$

Then it also satisfies Euler's identity:

$$\forall x = (x_1, x_2, \dots, x_n),\ f(x) = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x)$$

This equation ensures that the sum of the contributions will in fact be equal to total risk.

B. THE "MARKET STANDARD": HISTORICAL VOLATILITY

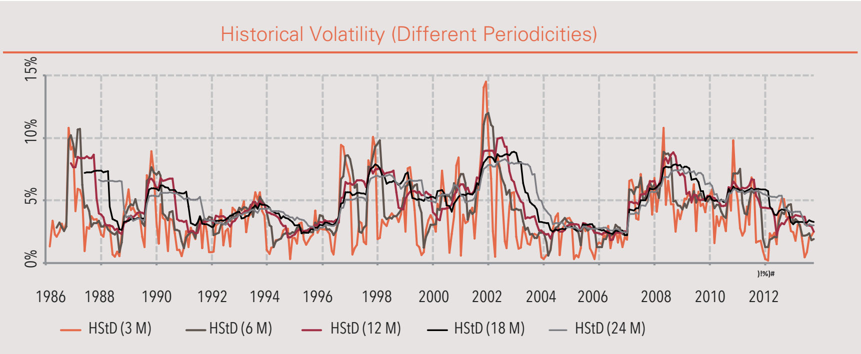


The most common risk measure associated with a financial asset or portfolio is the historical volatility. Although volatility is a fundamental principle in finance, it can be difficult to grasp. However, if we understand how it works we can optimize our strategy, as we will see below.
The volatility of a portfolio or, more generally, a financial asset, reflects the magnitude of the upward and downward variations in this asset around its mean. In practice, it refers to the annualized standard deviation of past variations during a given period.

More specifically, it can be expressed as:

$$\sigma^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \bar{r})^2$$

Where r_t is the asset's return at time t, $\bar{r}$ is the mean return for the entire period, and T is the length of the period.
The chart below shows how the choice of length of time affects not only the level but particularly the variability of volatility.



The shorter the observation period, the more volatile the risk parameter used. While no one method is better than another, the right calibration will be the one that most closely aligns with the investor's desired portfolio profile.

C. LOSSES ARE ALL THAT MATTER

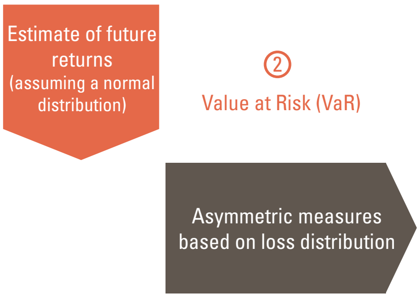
We saw above that we had to account for the non-normality of the distribution of asset returns that we wanted to model. In normal distribution-based approaches, loss distribution is by definition symmetrical to gain distribution.
Otherwise, all of the investor's attention and his or her risk aversion are focused only on loss distribution or on returns below the risk-free rate.
There are many ways to understand this skewness. We will focus only on the following measures, which all have a financial interpretation that we will define:

- Value at Risk (VaR)
- Conditional Value at Risk (CVaR)
- Lower Partial Moment (LPM)

This concept of skewness, not applicable using the Gaussian method (robustness of the measure over time and homogeneity of the measure for each asset), now presents two key problems:

- Issue of the variability of skewness over time for a given asset which seeks to precisely model the loss distribution and which requires the modeling to be an adequate measure of future risk.
- Issue of the variability of skewness for each asset where the skewness observed for an asset cannot be projected on to another asset whose return distribution is bound to be different.

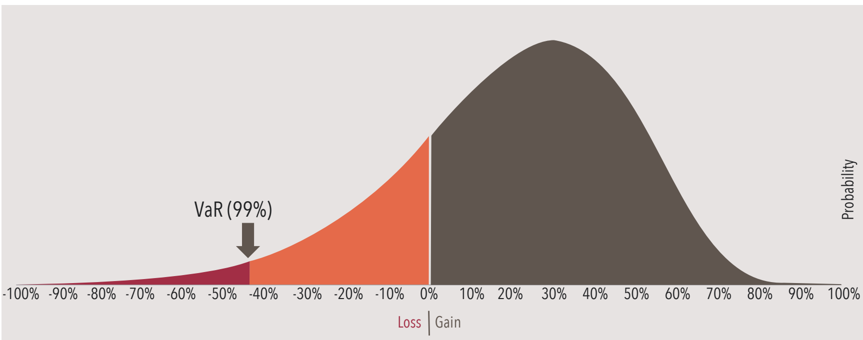
1. Value at Risk (VaR)



Value at Risk (VaR) is a general risk measure that can be used for a portfolio regardless of its return distribution. It represents an amount of loss not to be exceeded, with a certain cumulative probability (expressed as α) in a given time frame.
In the most commonly-used Gaussian framework, VaR is expressed directly on the basis of expected return, to which a constant multiple of the standard deviation is added.
In a non-Gaussian framework, it can be approximated analytically with the Cornish-Fisher expansion (Mina and Ulmer (1999) [5]) as follows:

$$\begin{cases} VaR_\alpha(\omega) = w^t\mu + \tilde{z}_\alpha \sigma(\omega) \\ \tilde{z}_\alpha \approx z_\alpha + \frac{1}{6}(z_\alpha^2 - 1)s + \frac{1}{24}(z_\alpha^3 - 3z_\alpha)k - \frac{1}{36}(2z_\alpha^3 - 5z_\alpha)s^2 \end{cases}$$

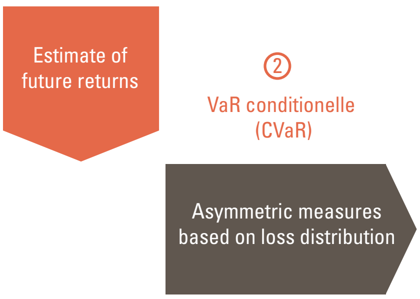
$\tilde{z}_\alpha$ describes the approximation of percentile α of a distribution with mean μ, standard deviation σ, skewness s and kurtosis k.



Because it is based on a return distribution, VaR requires an estimate of expected returns (the first moment of the distribution).

We have ruled out the possibility of estimating expected returns for each asset in our risk-based approach as we are using a highly diversified global portfolio. We have therefore elected not to use this risk measure.

2. Conditional Value at Risk (CVaR)



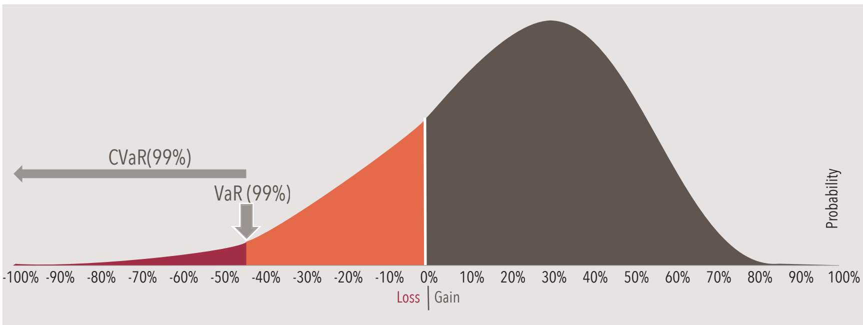
The following two criticisms have been leveled against VaR, resulting in the emergence of CVaR as an additional risk measure:

- It does not satisfy one of the axioms, which specifies that a so-called coherent risk measure must comply with: subadditivity (Artzner, 1999) [6]. In concrete terms, the VaR of the sum of two portfolios can be greater than the sum of the VaRs of each portfolio. This characteristic runs counter to the diversification principle in finance.
- It gives no indication of the extent of the portfolio loss if this loss were to exceed the set percentile.

Conditional Value at Risk (CVaR) is an additional risk measure used to define the portfolio's expected loss when this loss exceeds VaR. VaR is thus concerned with the (1 – α) percentiles of the distribution, while CVaR focuses on the tail of the loss distribution: the α percentiles for which neither the expectation nor the distribution are known.
CVaR is a coherent risk measure that fully satisfies Artzner's axioms (see Appendix 1). It is calculated as the average of the expected returns considering that these returns are lower than the portfolio's return percentile at confidence level c:

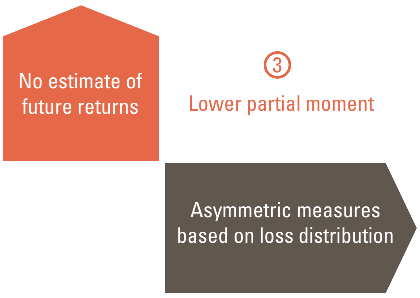
$$CVaR_{\alpha}(\omega) = \mathrm{E}[-w^t r \mid w^t r > Q_{w^t r}c\,]$$

$Q_{w^t r}c$ describes the portfolio's return percentile at confidence level c (or the portfolio's VaR at confidence level c).



CVaR, like VaR, is based on a return distribution and therefore also requires an estimate of expected returns. Consequently, and despite the appeal of CVaR versus VaR, we will not be using this risk measure.

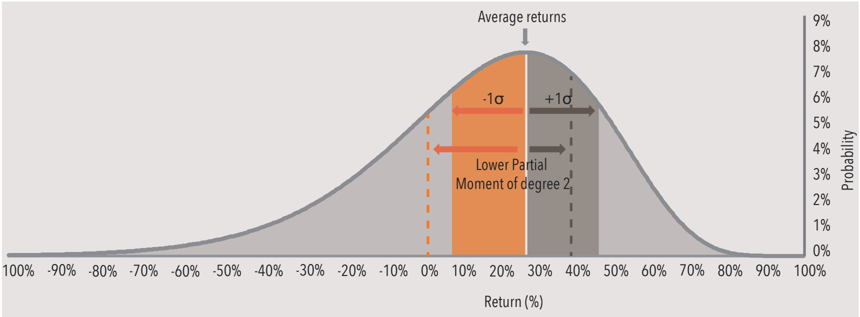
3. Lower Partial Moment (LPM) of degree k



The Lower Partial Moment of degree k method offers an additional loss distribution-based solution. This approach was primarily introduced into modern portfolio theory by Bawa (1975), Fishburn (1977) and Nawrocki (1991, 1992 and 1999). They presented a new family of risk measures defined by τ (often referred to as the risk-free rate or, for simplicity's sake, 0) and k (the degree of the moment that an investor can specify to reflect his or her risk aversion. A large k will penalize large deviations and therefore risk aversion increases with k).
Discretized LPM of degree k is defined as follows:

$$MPI_{k}(\tau,R_{i})\ \ \frac{1}{T-1}\sum_{t=1}^{T}\left(max\ (0,\tau-R_{i})\right)^{k}$$

Semivariance relative to the risk-free rate is a special case of the LPM for k=2 denoting the risk for the investor by expressing the variance of the portfolio's excess return relative to the risk-free rate provided that this excess return is negative.

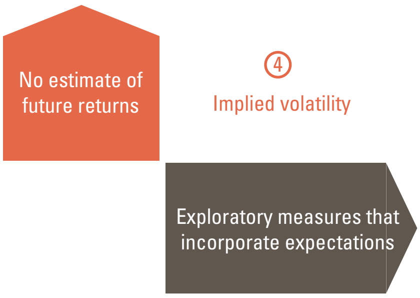


Semivariance captures the impact of skewness on investment decisions and also allows for more realistic risk modeling. We use this method in the remainder of our report as it is compatible with the constraints we have defined.

D. FROM PAST TO FUTURE

The risk measures we have reviewed rely on historical data. However, an investor may decide to incorporate an element of anticipation to calibrate his or her estimate of the immediate risk.

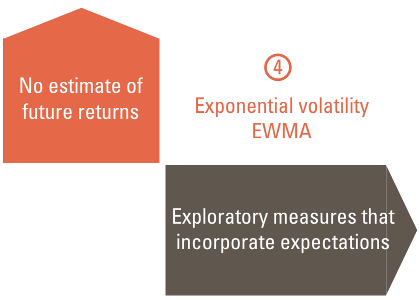
1. Implied Volatility



The "standard" measure of volatility discussed above is a risk indicator calculated on the basis of prices observed over an earlier period. It therefore describes the past. Implied Volatility is derived from the prices of listed (or standardized) derivatives and reflects market participants' expectations of the magnitude of future changes for a given time frame (limited to the market standard) and expectations of certain market parameters (such as dividends, rates, credit, repos, etc.). The greater the uncertainty about an asset, the higher its implied volatility. If we know the implied volatility, we are better able to anticipate the possibility of changes in the underlying asset.

Implied volatility is only available for a limited number of financial assets for which a liquid derivative listed market exists. As our approach utilizes the broadest possible investment universe, we cannot select Implied Volatility as the risk measure for our portfolio construction algorithm.

2. Exponentially Weighted Moving Average (EWMA) volatility



Some modern methods give more weight to recent information to better capture developing trends. One well-known method provides an example of this approach: the general autoregressive conditional heteroskedastic (GARCH) modeling.

The terms that make up this acronym are defined as follows:

- Autoregressive: variance (or volatility) is a function of its previous values
- Conditional: future volatility (tomorrow's volatility) depends on today's volatility
- Heteroskedastic: variance is not constant but varies over time

This terminology covers a family of models and is more accurately written GARCH(p,q), p and q corresponding respectively to the number of returns and the number of variances that will be used to estimate the next value. To simplify the notations in the discussion that follows, we will assume that p=q=1.
More formally, we can summarize the model as:

$$\sigma_{t+1}^2 = a + b.r_t^2 + c.\sigma_t^2$$

where r represents return and a, b and c are constants to be calibrated.

The Exponentially Weighted Moving Average (EWMA) model, popularized by JP Morgan and used in its RiskMetrics tool, follows this principle and in fact corresponds to a specific case of GARCH(1,1). The major difference between the two models is that GARCH(1,1) takes mean reversion into account while the EWMA model does not. It is possible to get from GARCH(1,1) to EWMA by changing parameter a to 0 and with the sum of parameters b and c equal to 1:

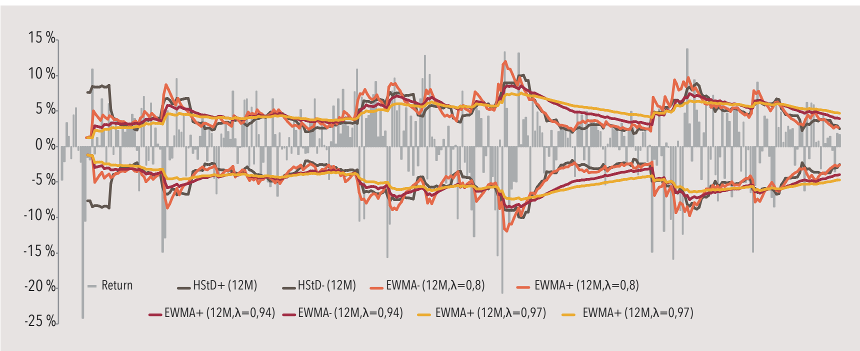
$GARCH(1,1) = \sigma_{t+1}^2 = a + b.r_t^{\,2} + c.\sigma_t^2$ then becomes $GARCH(1,1) = \sigma_{t+1}^2 = b.r_t^{\,2} + (1-b).\sigma_t^2$
which is equivalent to the formula of the model $EWMA = \sigma_{t+1}^2 = \lambda.\sigma_t^2 + (1-\lambda).r_t^{\,2}$

In the EWMA model, the weights assigned to past observations decrease exponentially the further back we go.
λ is a constant smoothing factor between 0 and 1. The closer it is to 0, the lower the weight this estimator assigns to recent returns and the smoother the curve.

We believe that the EWMA method is a risk measure that can be selected to construct a portfolio using a risk-based approach such as risk parity.

The chart below illustrates:

- The behavior of EWMA volatility compared with that of historical volatility (orange curves), in particular the magnitude of the changes in EWMA in the event of a sharp movement.
- The impact of the choice of λ parameter on the responsiveness of EWMA volatility (grey curves), allowing an accurate definition of the dynamics we would like to capture.



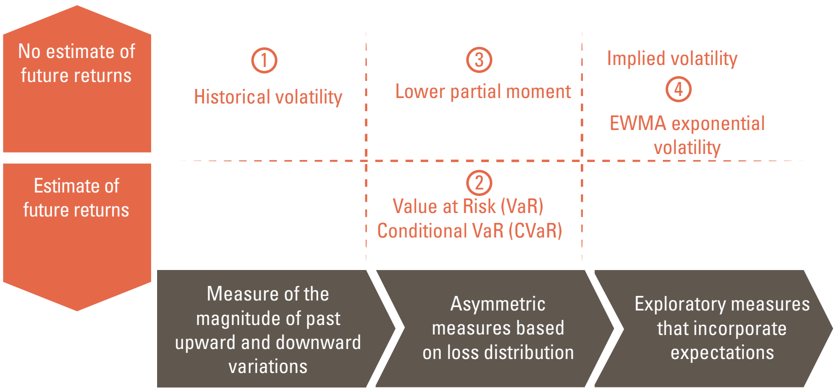
In our study, we will limit ourselves to the EWMA method as a forward-looking measure of future variance. Calibrating GARCH parameters is in fact a complicated

exercise and would require implementation of methodologies that are outside the scope of this report.
To avoid introducing calibration bias, we will use the parameters recommended by RiskMetrics™:

λ = 0.94 for daily data, for a one-day forecast;
λ = 0.97 for monthly forecasts.

E. CONCLUSION REGARDING THE CHOICE OF THE RISK MEASURE

As it was the case with the construction method, we do not believe approaches involving the use of estimates of future returns such as VaR and its avatar CVaR are relevant.
From that standpoint, the Lower Partial Moment (LPM) method differs from the others in that it adds another dimension, as in theory it allows for more realistic risk modeling by considering only loss distribution.
The choice of risk parameter depends on the expectations of the investor, who can select relatively stable parameters. Historical Volatility remains a very widely used measurement tool, but can lead to highly volatile results or, conversely, can suffer from a high degree of inertia, depending on the selected time span.
However, if we rank these methods in terms of robustness — which can be estimated based on volatility (over 12 months) and drawdowns, for example — we find few differences in the results of our empirical study. The only area where our approximate methods differ remains the measure of the portfolio's concentration/diversification and the frequency of the portfolio's turnover, which we will discuss in the next section.



4 EMPIRICAL COMPARISON OF THE SELECTED METHODS

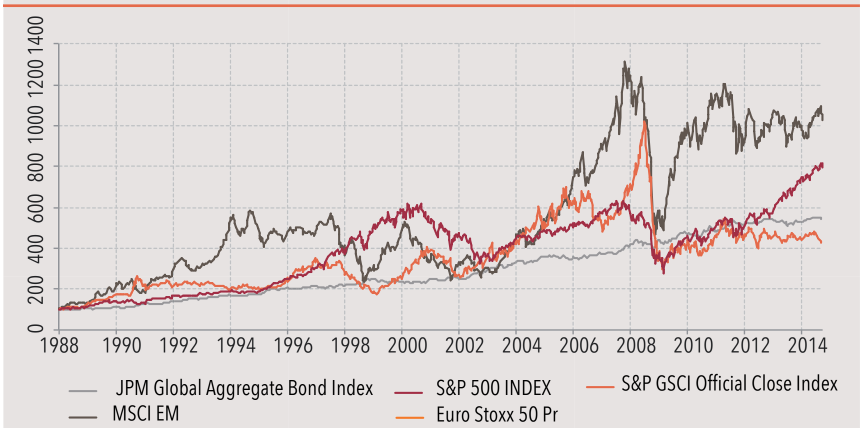
A. CHOICE OF UNIVERSE

To test the performance of each of the alternatives considered, we have selected five representative asset classes:

- Global aggregate bonds (index used: JPM Global Aggregate Bond Index)
- Eurozone equities (index used: Euro Stoxx 50 Pr)
- US equities (index used: S&P 500)
- Emerging market equities (index used: MSCI EM)
- Commodities (index used: S&P GSCI which contains industrial and precious metals, energy, agricultural commodities).

For the sake of clarity and simplicity, we have limited the number of indices used. The universe covered is nevertheless sufficiently broad and diversified to measure the relevance of each approach over a long period.

Investment universe



B. RESULTS OF THE COMPARATIVE STUDY OF CONSTRUCTION METHODS

In this section, we compare three portfolio construction methods:

- Equally Weighted (EW)
- Inverse Volatility (IV)
- Equally Risk Contribution (ERC)

At this point, our comparative study concerns four statistics enabling us to assess these three methods' robustness over time:

- Cumulative return
- Trailing return
- Volatility
- Drawdown

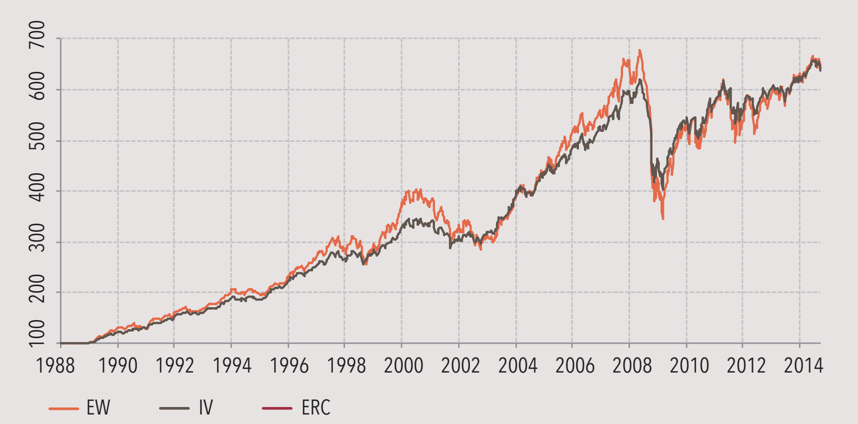
This initial comparative study of retained portfolio construction methods enabled us to create the following table, showing that the ERC method consistently differs from the others:

Portfolio construction method	Cumulative return	Trailing return	Volatility	Drawdown
Equally weighted (EW)		+	+	+
Inverse Volatility (IV)	Very similar	Extremum	Magnitude	Magnitude
Equally Risk Contribution (ERC)		-	-	-

1. Cumulative return

As this study focuses on the robust construction of portfolios in terms of risk and risk-adjusted return (and consequently is not focused solely on the hunt for the highest return), the three methods compared (Inverse Volatility, Equally Weighted and Equal Risk Contribution) have virtually the same return, at around 7% on an annualized basis over the cumulative period considered.

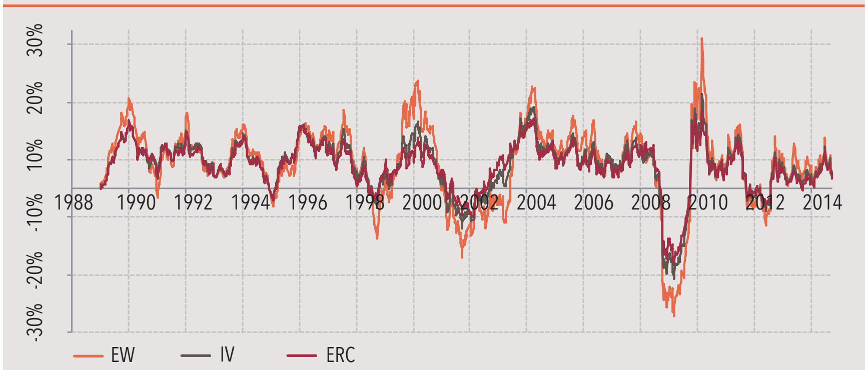
Cumulative performance



2. Annual trailing returns

The ERC method proved most consistent of the three in terms of annual return distribution, as the other two approaches produced more volatile returns. Because the final return was equivalent, investors would be best rewarded by selecting the ERC approach, which spares them the most volatility.

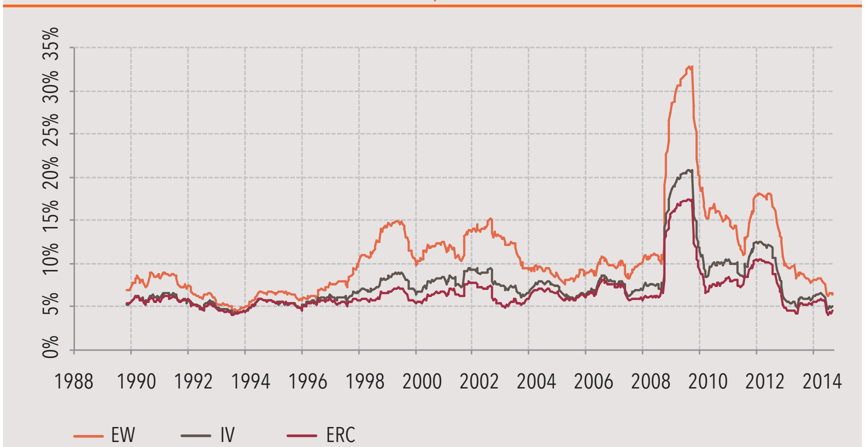
Trailing performances (12M)



3. Volatility (measured here over 12 months)

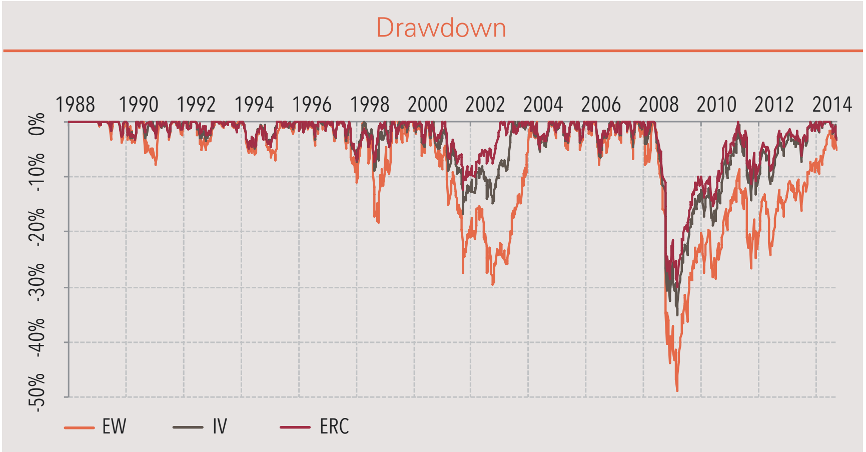
The results expressed here are fully consistent with the theory. Volatility in ERC, which takes into account the correlation between assets, is lower than in the IV allocation, regardless of market configuration.

Volatility (12M)



4. Drawdown

As we had expected, given the results for variability of returns for our three allocations, ERC suffers the least in a market downturn. In contrast, EW, owing to its complete lack of risk management, bears the brunt of crises.



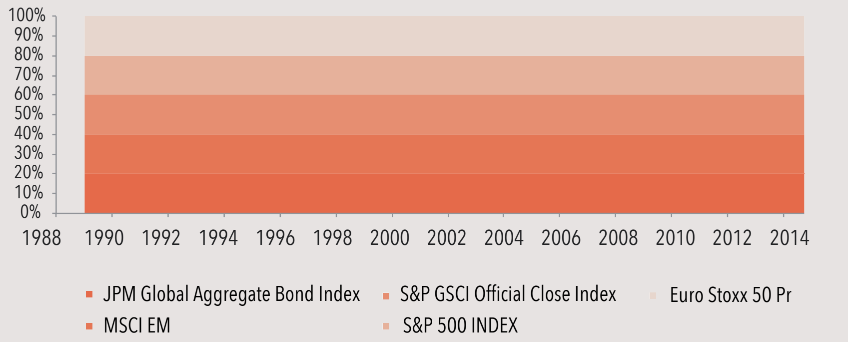
5. Statistical comparison summary table

	ERC	EW	IV
Annualized Return	7.00%	7.17%	7.15%
Annualized Std Dev	6.93%	11.77%	8.02%
Annualized Sharpe (Rf=0%)	1.0098	0.6095	0.8913
Annualized Downside Deviation	4.77%	8.35%	5.59%
Average Drawdown	1.72%	2.88%	1.89%
Maximum Drawdown	30.07%	48.88%	35.15%
Historical VaR (95%)	-1.30%	-2.33%	-1.49%
Historical CVaR (95%)	-2.14%	-3.92%	-2.55%
Skewness	-1.1757	-0.878	-1.1172
Excess Kurtosis	11.2399	9.3187	10.7112
Win Prob(%)	57.13%	55.75%	57.35%
Ann.Sharpe.2001	1.3899	0.9714	1.2963
Ann.Sharpe.2002	1.5591	-0.4521	0.4631
Ann.Sharpe.2008	0.1404	-0.0194	0.0866

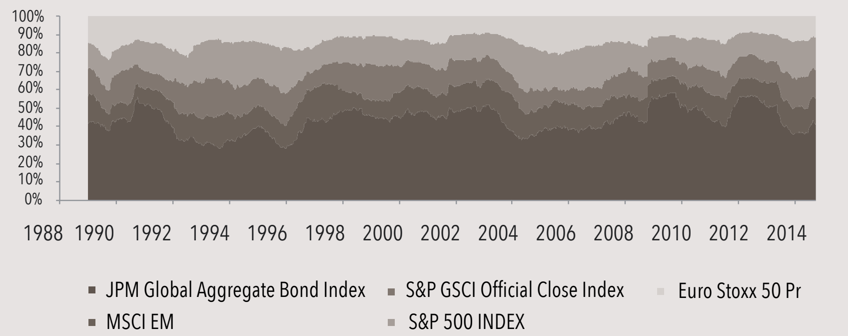
6. Change in the composition of the allocation over the period considered

The following graphical comparison gives some idea of the responsiveness of the different methods as well as the magnitude of the changes in allocations that they produce.

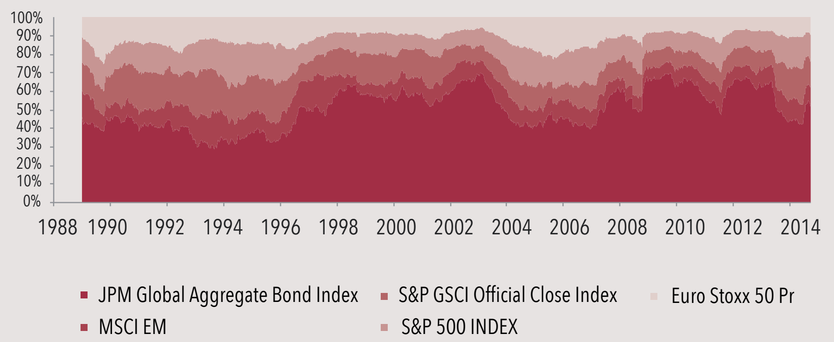
Capital weightings: EW



Capital weightings: IV

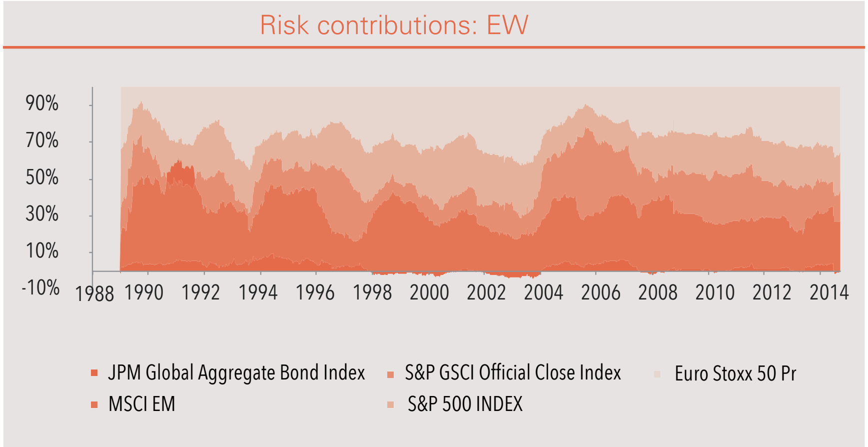


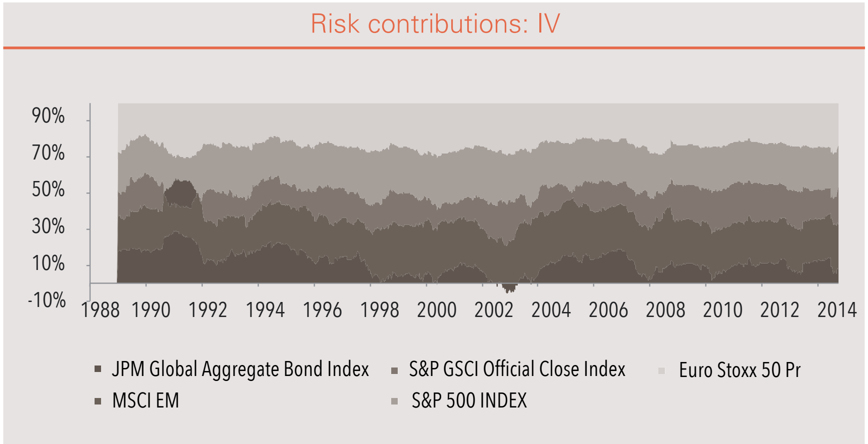
Capital weightings: ERC

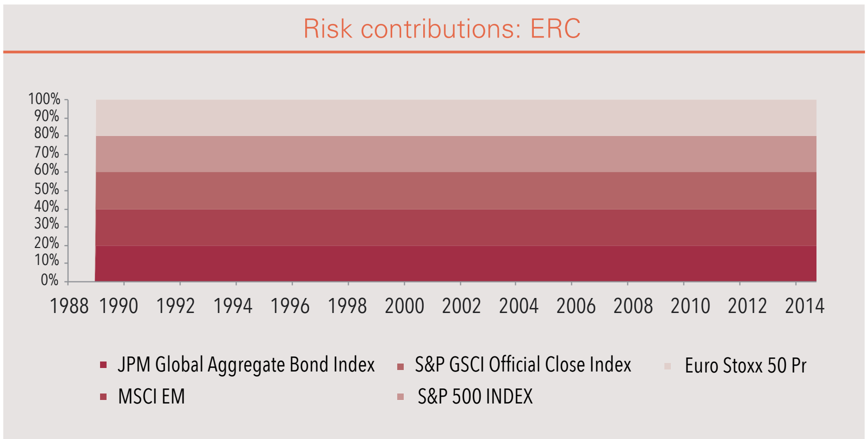


7. Change in the risk contributions of the allocations

A graphical comparison of risk contributions reveals the very significant disparities between the three approaches. For example, the weight and the role played by commodities, in the portfolio's risk allocation vary significantly depending on whether the IV or EW method is used. This comparison also highlights the weaknesses of these two methods in terms of achieving the right balance of risk in the portfolio.







C. RESULTS WITH CHOSEN RISK MEASURES

In this section, we now turn our focus to three risk measures that we have deemed relevant:

- Historical Standard Deviation (HSTD) = historical volatility
- Exponentially Weighted Moving Average (EWMA) = exponential volatility
- Lower Partial Moment (LPM)

Our comparative study now concerns six statistics that enable us to differentiate the contributions made by these three measures with respect to the characteristics expected by investors:

- Cumulative return
- Trailing return
- Volatility
- Drawdown
- Allocation concentration
- Turnover rate

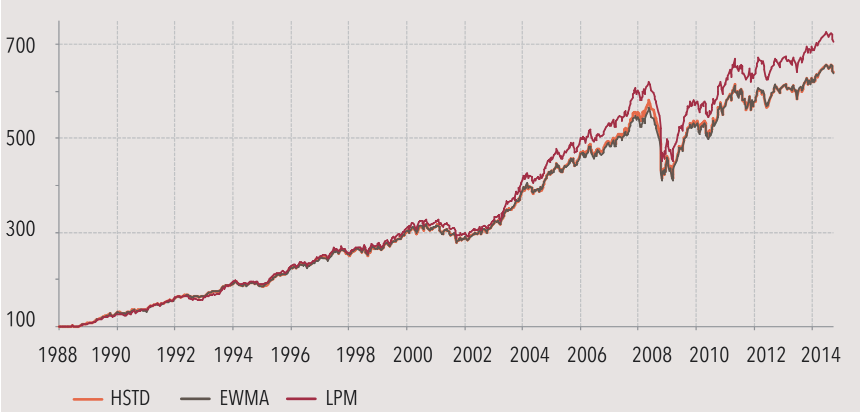
This time it is much more difficult to differentiate between the measures used with the previously selected criteria. We need to consider portfolio concentration and portfolio turnover rate to be able to fully understand the contribution of the risk measures selected and recognize the positive diversification characteristics of simple Historical Volatility compared with more sophisticated measures such as EWMA and LPM:

Risk measure	Cumulative return	Trailing return	Volatility	Drawdown	Concentration	Turnover rate
Lower Partial Moment (LPM)					+	+
Exponentially Weighted Moving Average (EWMA)	Very similar	Very similar	Very similar	Very similar	Variability	Magnitude
Historical Standart Deviation (HSTD)					-	-

1. Cumulative return

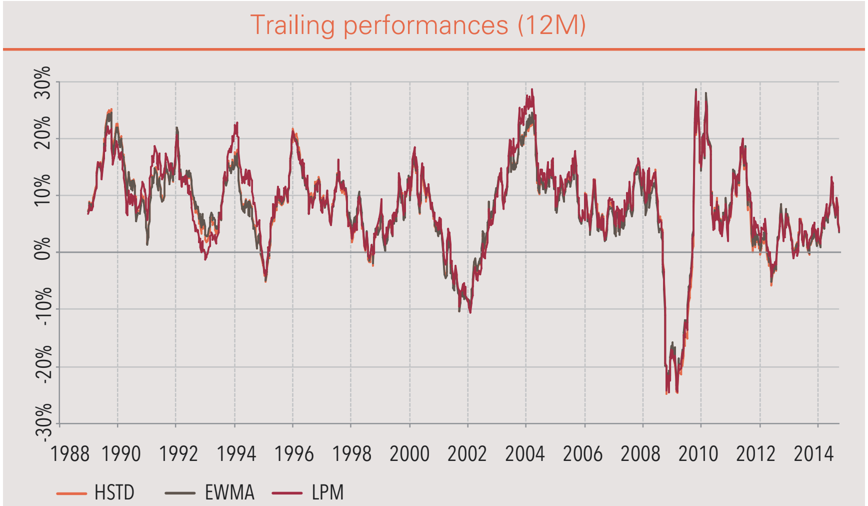
The three methods observed (Historical Volatility, EWMA, LPM) produce returns that are relatively similar to and correlated with one another, i.e. 7.16%, 7.15% and 7.54%, respectively.
Bear in mind that the point of this study is not to determine the method that maximizes a portfolio's return but that which indicates the immediate occurrence — or at least the increase in the likelihood — of an event to which the investor is averse and which might cause losses. Cumulative return, while inherently of interest to measure the performance that can be achieved over the period considered, is not sufficient to rank our three risk-based approaches.

Cumulative performance



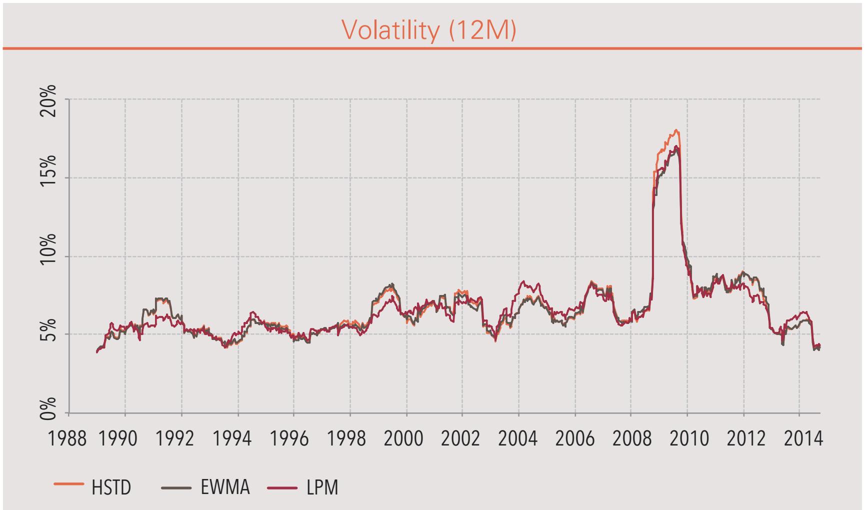
2. Annual trailing returns

The trailing returns for the three methods observed are also very close in level (with the LPM method having a slight edge) and closely correlated in their changes over the period considered. It is therefore impossible for us to discriminate the three risk-based approaches based on observations of trailing returns.



3. Volatility (measured here over 12 months)

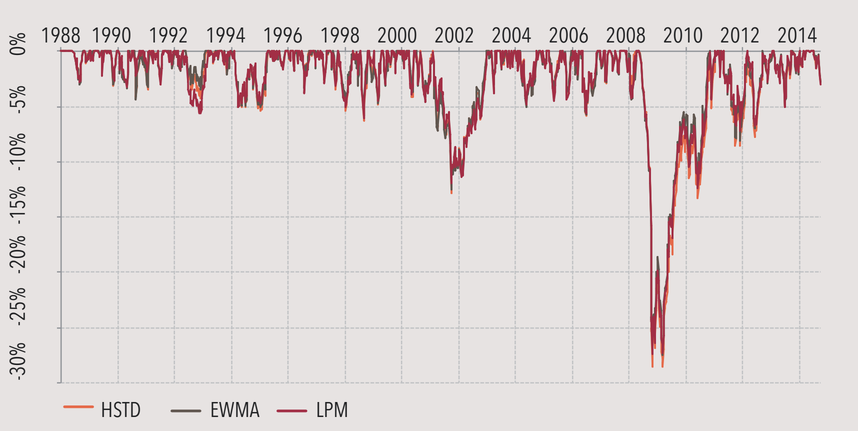
Volatility, along with drawdown, is one of the most important parameters in determining the relevance of the three methods considered. The results observed do not discredit any one method in particular, as their average volatilities over the period are very close to 7%. Even within the period considered we cannot differentiate between market regimes that would enable us to establish a ranking for these three methods.



4. Drawdown

Drawdown is the most important factor, along with volatility, when ranking the three methods observed. However, the results observed show few differences, as the troughs reached by the three methods are very close, at -27.38% for LPM, -27.48% for EWMA and -28.54% for standard volatility.

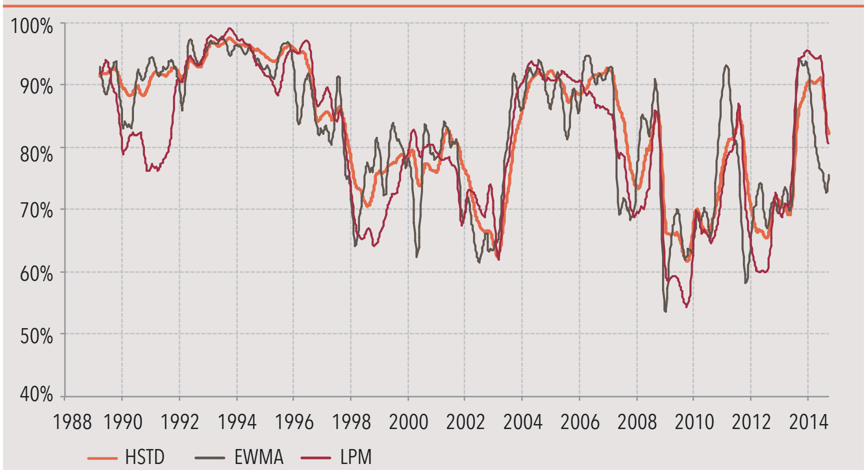
Drawdown



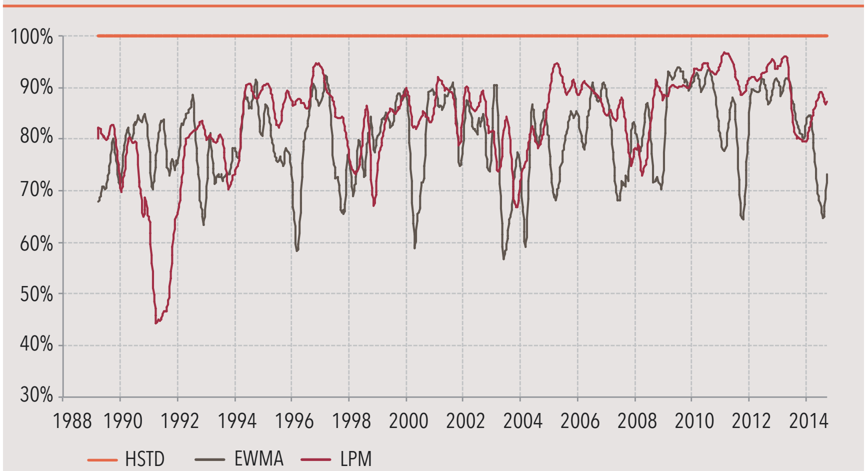
5. Weight concentration measure

We now introduce concentration indicators into our search for differentiating factors allowing us to distinguish between the different measures examined in our study. This will enable us to explore another facet of allocation robustness.
We will look first at capital concentration and then turn to risk contribution concentration (Gini index).

Capital concentration



Gini index



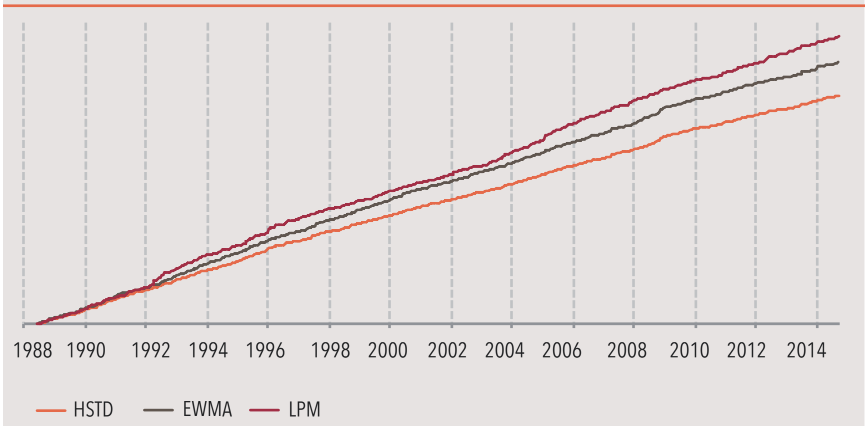
In both cases, the historical standard deviation stands out as the optimal risk measure in terms of consistency in capital allocation and, by construction, in the contribution to the portfolio's overall risk. LPM is the most inconsistent candidate in this study.

6. Asset turnover rate measure

In addition to the previous indicators, we will now introduce asset turnover rate. This is another important factor as it allows us to measure the potential cost associated with asset rebalancing.
Unsurprisingly, we find that the dynamics generated by the EWMA and LPM measures systematically result in additional costs, regardless of market regime.
Our preference is therefore for the HSTD method, which generates the fewest transaction costs.

Historical standard deviation stands out as the optimal risk measure in terms of consistency in capital allocation and, by construction, in the contribution to the portfolio's overall risk. LPM is the most inconsistent candidate in this study.
We find that the dynamics generated by the EWMA and LPM measures systematically result in additional costs, regardless of market regime.
Our preference is therefore for the HSTD method, which generates the fewest transaction costs.

Cumulative turnover rate



7. Statistical comparison summary table

This summary table shows, based on a portfolio constructed using the ERC method, a set of statistical measures for the portfolios that we constructed using the three risk measures selected. Our results show very few differences across all our statistical indicators, except for kurtosis (HSTD concentrates its median returns to a greater extent) and turnover rate (HSTD generates fewer transaction costs).

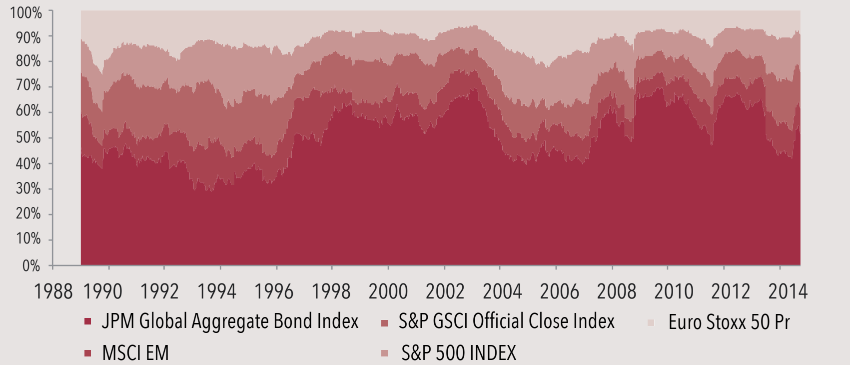
	HSTD	EWMA	LPM
Annualized Return	7.16%	7.15%	7.54%
Annualized Std Dev	7.03%	6.92%	6.92%
Annualized Sharpe (Rf=0%)	1.0176	1.0332	1.0906
Annualized Downside Deviation	4.86%	4.71%	4.69%
Average Drawdown	1.80%	1.76%	1.89%
Maximum Drawdown	28.54%	27.48%	27.38%
Historical VaR (95%)	-1.35%	-1.32%	-1.29%
Historical CVaR (95%)	-2.15%	-2.11%	-2.06%
Skewness	-1.3689	-1.0237	-1.1477
Excess Kurtosis	14.3031	9.6107	11.4284

	HSTD	EWMA	LPM
Win Prob(%)	58.15%	57.64%	58.73%
Omega	1.91	1.96	1.99
Avg TurnOver	1.93%	2.21%	2.43%
Ann.Sharpe.2001	1.32	1.3247	1.4026
Ann.Sharpe.2002	2.1554	2.1293	1.8626
Ann.Sharpe.2008	0.2244	0.259	0.2615

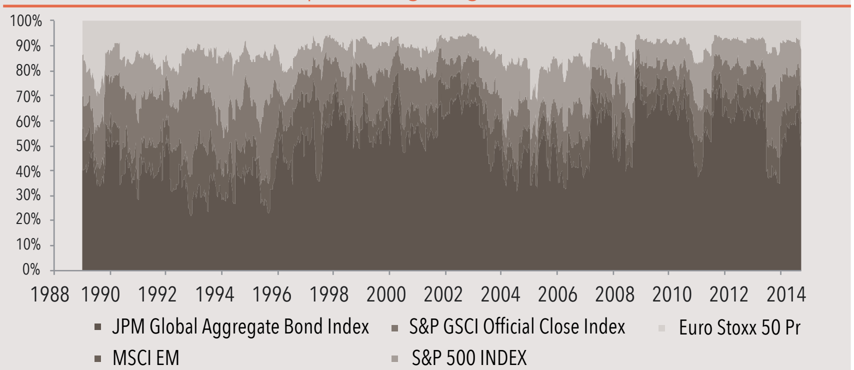
8. Change in the composition of the allocation over the period considered

A graphical comparison shows the responsiveness of the different methods as well as the magnitude of the changes in allocations that they produce. The results are as expected and are consistent with our previous conclusions. HSTD causes the least volatility in the definition of the weightings from one rebalancing to the next.

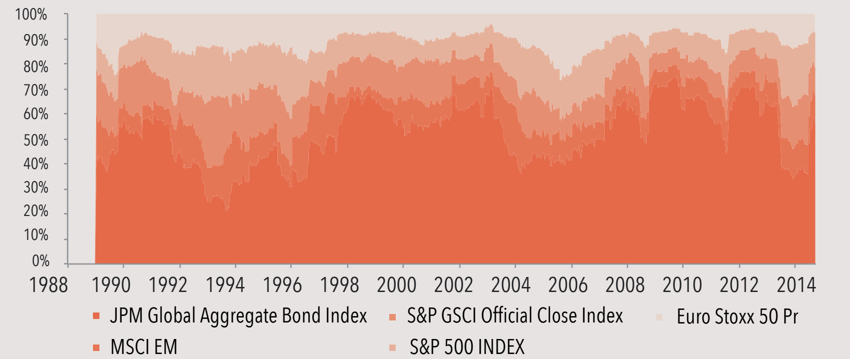
Capital weightings: HSTD



Capital weightings: EWMA



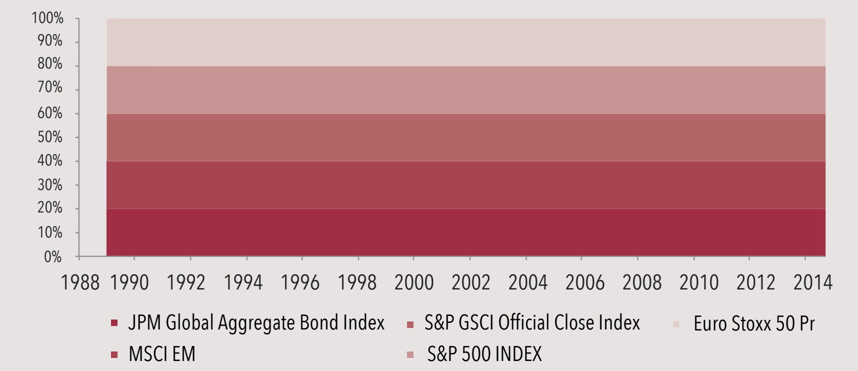
Capital weightings: LPM



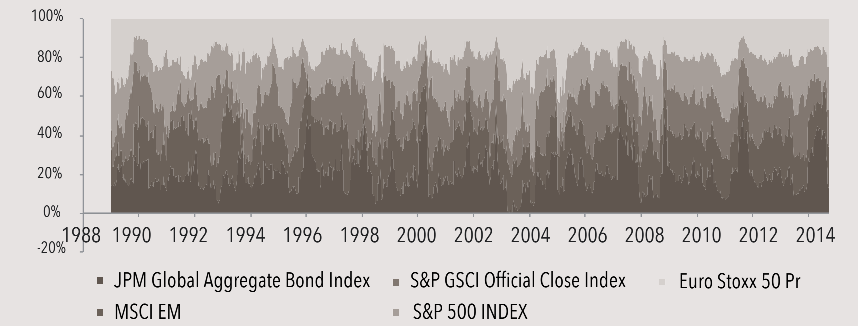
9. Change in the risk contributions of the allocations

Again, a graphical comparison of risk contributions shows the very significant disparities between the three risk measures selected, particularly from the standpoint of the dynamics and the magnitude of the variations.

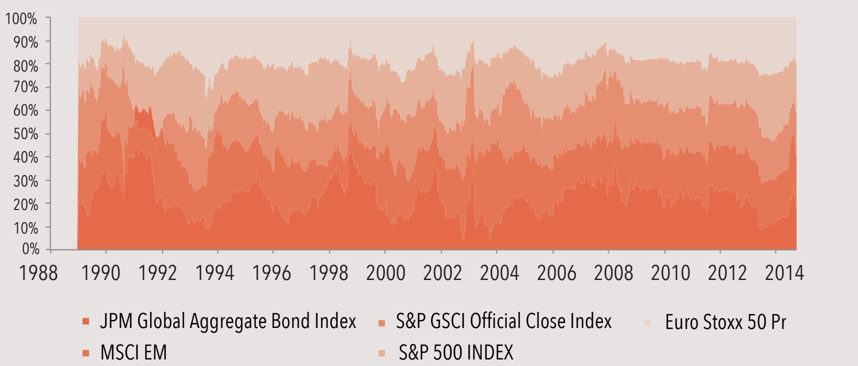
Risk contributions: HSTD



Risk contributions: EWMA



Risk contributions: LPM



CONCLUSION

In this paper, we have endeavored to examine the most relevant methods for constructing portfolios using a risk-based approach to ensure their robustness in a changing environment.

The first step was to choose a portfolio construction method that did not require an estimate of future returns, then to select the right risk modeling.

This paper shows that some approaches are less relevant than others. The constraint we set ourselves - the ability to invest in a global universe - rules out certain approaches that would be acceptable in theory, but that are not practicable when covering a very broad range of investments.

Approaches based on return expectations pose the dual problem of, firstly, the manager's actual ability to make assumptions that are both reliable and comprehensive and, secondly, the magnitude of the mistakes he or she would make if the expectation were incorrect. As such, there is some advantage in trying to estimate volatilities or correlation matrices rather than future returns.

The use of normal distributions in some of the methods creates imperfect approximations and can prove costly given the reality of the markets.

To summarize the results presented in detail in the historical simulations section, we note the following key observations:

- ➜ The **Equally Weighted (EW)** method is certainly easy to implement and yields a portfolio that minimizes weight concentration, but presents the major flaw in allocating the same amount of capital to a risky asset as it does to a low-risk asset.
- ➜ The **Inverse Volatility (IV)** method properly considers an asset's risk (as measured by its volatility) but does not consider the interplay between the assets that make up a portfolio; we are disinclined to use this method due to the absence of any benefit in terms of correlations between assets.
- ➜ Because it incorporates the correlation matrix, the **Equal Risk Contribution (ERC)** method offers a solution that considers both the risk each asset represents and its relative contribution to the global portfolio as a whole.

The results of the historical simulations for the period from January 1, 1989 to September 30, 2014 show that the ERC method produces more consistent results than the other methods analyzed. The volatility and magnitude of the drawdowns are reduced with the ERC method. One would expect that if the Sharpe ratio were used as an indicator of the risk-return profile, the ERC method would have the best outcome.

We also saw in section I.D that the method was required to minimize at least one of the usual portfolio concentration criteria. While it is not directly a risk measure, this condition avoids highly concentrated portfolios with high asset turnover rates which benefit from a small portion of their investment universe. The calculations presented in Section 4 show that the ERC allocation achieves the best Gini index (an index that measures a portfolio's variance concentration).

Regarding the choice of risk measure, a number of parameters are available to investors who would like to implement an allocation involving the ERC method. We do not believe that risk measures that require an estimate of future returns – particularly using a normal distribution such as VaR or CVaR – are appropriate as they involve too many uncertainties. As we saw above, an erroneous estimate of future returns has a very significant impact on results.

We therefore prefer methods releasing us from this constraint. From this short-list, we then aim to identify one which takes into account:

- The magnitude of past upward and downward variations (Historical Volatility)
- Loss distribution alone (Lower Partial Moment)
- Expected returns (implied or Exponential Volatility, such as the EWMA method).

When we look at the results for the period from January 1, 1989 to September 30, 2014, we find that, despite the significant disparities in the approaches, the robustness of the allocation is very similar regardless of the risk measure selected.

In the end, the methods observed show few differences and, despite their theoretical appeal, the most appropriate measures do not seem to improve the results.

Selection of the method will therefore place greater emphasis on criteria relating to the complexity of implementation or the turnover rate.

In conclusion, we can draw two important insights from this study:

- **The ERC method is the most relevant for constructing a portfolio using a risk management-based approach.**
- **Sophisticated risk measurement methods used to determine the risk parity allocation offer no differentiating value-added compared with simpler parameters such as the historical volatility.**

These two steps are crucial to constructing a risk parity allocation. They do not, however, include one last step, one that is just as crucial and that will be the subject of our next report: the role of tactical allocation, particularly from the perspective of the quest for return.

APPENDIX 1: ARTZNER'S AXIOMS (1999)

A risk measure ρ is considered coherent [7] if it satisfies axioms 1 to 4 below (Z_1 and Z_2 are random variables):

Axiom 1 - Monotonicity: if $Z_1 \leq Z_2$ then $\rho(Z_1) \geq \rho(Z_2)$
Axiom 2 - Subadditivity: $\rho(Z_1+Z_2) \leq \rho(Z_1) + \rho(Z_2)$
Axiom 3 - Positive homogeneity: if $\alpha \geq 0$ then $\rho(\alpha Z) = \alpha\rho(Z)$
Axiom 4 - Translation invariance: if $\alpha \in R$ then $\rho(Z+\alpha) = \rho(Z)-\alpha$

Föllmer and Schied (2002) [8] introduced a fifth axiom concerning another risk measure family: convex risk measures consider the fact that a large position in a financial instrument can result in a liquidity risk. Föllmer and Schied thus replace axioms 2 and 3 with the following axiom 5:

Axiom 5 - Convexity: $\forall\lambda \in (0,1)$ $\rho(\lambda Z_1+(1-\lambda)Z_2) \leq \lambda\rho(Z_1)+(1-\lambda)\rho(Z_2)$

APPENDIX 2: VARIOUS INDICATORS IN USE

A. CONCENTRATION MEASURES

In this paper, we seek to describe the different allocation methods we have defined based primarily on the concentration relative to a given criterion that the end investor can understand from a financial perspective.
By construction, these indicators will be normalized between 0 (maximum diversification) and 1 (extreme concentration in one single asset).

1. Weights: adaptation of the Herfindahl-Hirschman index

This index has historically been used to estimate market share concentration for a given industry. It is then used to determine the extent of competition between companies.
Here we use a normalized form between 0 and 1 to quantify each asset's share of capital within the allocation, expressed as:

$$HHI = \frac{\omega^T . \omega - 1/N}{1 - 1/N}$$

The allocation that will maximize this indicator is that which distributes the capital most evenly among the different assets. By construction, EW is the optimal method and corresponds to a level of 1.

2. Risk contributions: Gini index

The Gini index is a measure of dispersion using the Lorenz curve. It has been used extensively by economists to quantify income inequality (or the concentration of wealth) in a given geographic region. Maillard, Roncalli and Teiletche (2008) introduced it as a measure of the inequality of risk contributions within the allocation.
More formally, we will use the two-step approach of Chaves, Hsu and Li (2011) [9] in our calculations.
For a given date, we rank the risk contributions in ascending order, expressed as a percentage
We apply the following formula:

$$Gini = \frac{2}{N}\sum_{i=1}^{N} i.(PCTR_i - \overline{PCTR})$$

Where $PCTR_i$ represents the n[th] risk contribution expressed as a percentage, $\overline{PCTR}$,the average of the risk contributions expressed as a percentage, and N the number of assets in the portfolio.

B. OTHER MEASURES USED

1. Downside risk/upside potential: Omega ratio

This approach was introduced by Keating and Shadwick (2002) [10] as a "natural" way to analyze performance and takes all moments into account without requiring assumptions as to the form of the historical returns distribution. A return threshold is defined with which the investor is satisfied and the ratio between expected gains and expected losses is calculated (both relative to the stated threshold). More formally, the Omega ratio, relative to a returns threshold L, is expressed as:

$$\mathrm{Omega(L)} = \frac{\int_{L}^{+\infty}(1 - F(r))dr}{\int_{-\infty}^{L} F(r)dr}$$

Where F represents the variance-covariance matrix, σ_i et ω_i respectively represent the volatility and weight of asset i, and N represents the number of assets in the portfolio.
Intuitively, this indicator is expressed as the call-put ratio (option price ratio) for a strike equivalent to the returns threshold.

$$Omega(L) = \frac{Call(L)}{Put(L)}$$

2. Turnover

The portfolio's turnover was defined by DeMiguel et al. (2009) [11] as the average of the sum of the amounts reallocated at each change in allocation. In practice, the higher it is, the higher the portfolio implementation costs. It is expressed as follows:

$$TurnOver = \frac{1}{T-1}\sum_{t=1}^{T}\sum_{i=1}^{N}|\omega_{t+}^{i} - \omega_{t-}^{i}|$$

Where ω_{t-}^{i} is the weight of asset i before rebalancing at t, ω_{t+}^{i} is the weight of asset i after rebalancing at t, and T is the number of rebalancing periods.

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