Classical Guitar Intonation and Compensation: The Well-Tempered Guitar

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Abstract

Inspired by the pioneering work of luthier Greg Byers in 1996, we build an intuitive model of classical guitar intonation that includes the effects of the resonant length of the fretted string, linear mass density, tension, and bending stiffness. We begin by deriving an expression for the vibration frequencies of a stiff string using boundary conditions that are pinned at the saddle but clamped at the fret. We adopt logarithmic frequency differences based on "cents" that decouple these physical effects, and then we introduce Taylor series expansions that exhibit clearly the origins of frequency shifts of fretted notes from the corresponding Twelve-Tone Equal Temperament (12-TET) values. Although we have postponed careful measurements of the changes in frequency of open strings arising from small changes in length, we propose a simple experimental method that any interested guitarist can use to estimate these parameters for their own guitars and their favorite string sets. Based on these results, we employ an RMS frequency error method to select values of saddle and nut setbacks that map fretted frequencies — for a particular string set on a particular guitar — almost perfectly onto their 12-TET targets. This exercise allow us to discuss a general approach to tempering an "off-the-shelf" guitar to further reduce the tonal errors inherent in any fretted musical instrument.

Contents

1 Introduction and Background

Any musician who has wrestled with the temperament of a fretted stringed instrument is well aware of the challenges presented by tuning and pitch. In addition to the mathematical physics of musical scales [1], the mechanical specifications of the instrument and the strings themselves [2, 3] require accommodation during both manufacturing [4, 5] and tuning to achieve harmonious results. We can gain an appreciation for this problem by analyzing the expression for the allowed vibration frequencies of an ideal string, given by [6, 7]

$$f_q = \frac{q}{2\,L_0}\sqrt{\frac{T_0}{\mu_0}}\,, \tag{1}$$

where $q \in \mathbb{N} = \{1, 2, \ldots\}$, L_0 is the length of the free (unfretted) string from the saddle to the nut, T_0 is the tension in the free string, and $\mu_0 \equiv M/L_0$ is the linear mass density of a free string of mass M. The act of fretting the string changes its length, and therefore its frequency. For example, modern classical guitars are manufactured with frets placed along the fretboard using the Twelve-Tone Equal Temperament (12-TET) system, whereby the resonant length of a string pressed behind fret n ideally should be $L_0 2^{-n/12}$, thereby producing a note with frequency $f_1 2^{n/12}$. But this result can never be achieved perfectly in reality. First, the string is elevated above the frets by the saddle and nut, so the fretted string is slightly elongated relative to the free string, and the resulting frequency is flattened in pitch. In principle, this effect could be accommodated by minute changes in the positions of the frets, but there are additional practical complications. For example, the string's tension and density are altered by the change in length, causing the frequency to sharpen by an amount that significantly exceeds the reduction caused by the increase in the resonant string length. In addition, the string is by no means ideal, and its intrinsic stiffness results in an additional increase in pitch that depends on its mechanical characteristics. These guitar intonation difficulties seem to preclude successful temperament, but remarkably the instrument can be *compensated* by moving the positions of the saddle and the nut by small distances during the manufacturing process. Our goal in this work is to build an intuitive understanding of these effects to aid in the compensation and subsequent tuning of the classical guitar.

Throughout this work, we will use *cents* to describe small differences in pitch [1]. One cent is one one-hundredth of a 12-TET half-step, so that there are 1200 cents per octave. An

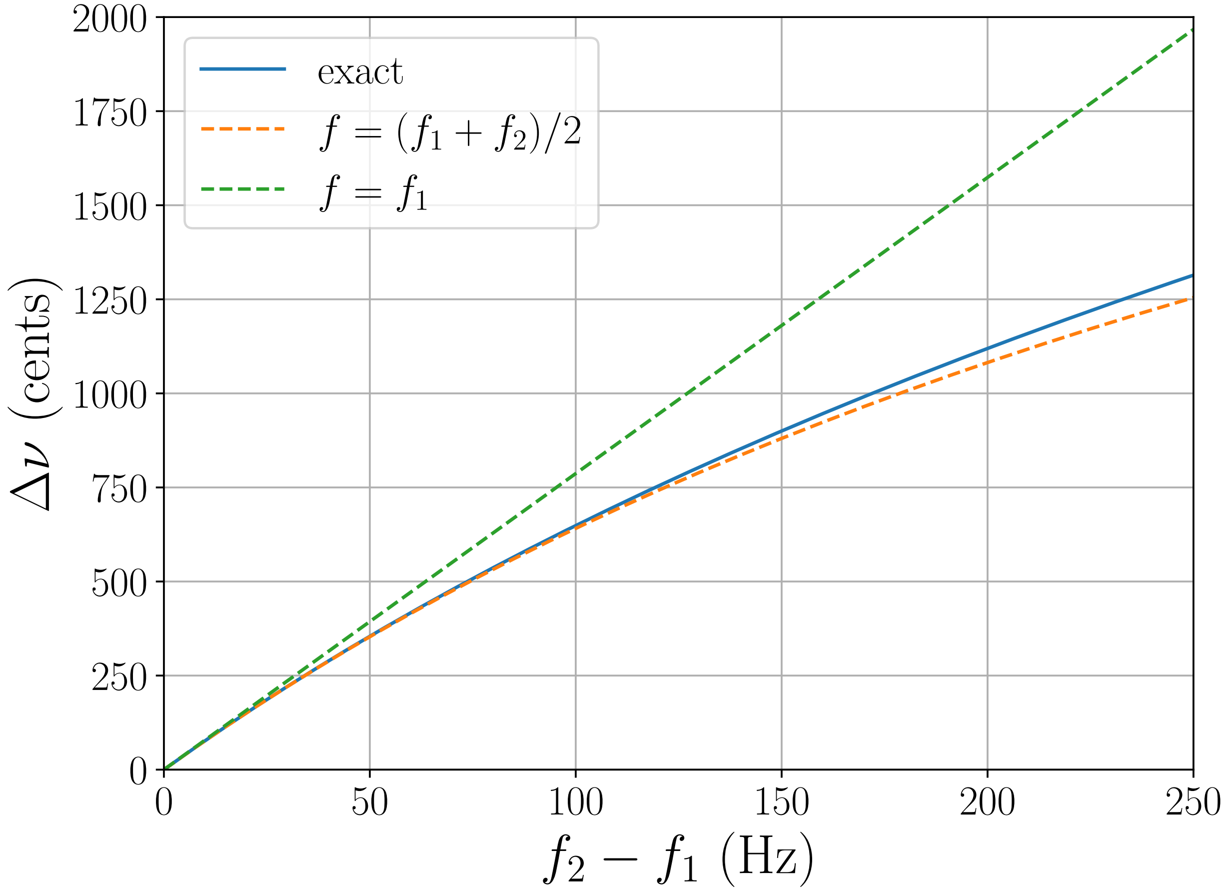


Figure 1: Plot of $\Delta\nu$ for $f_1 = A_3 = 220$ Hz and f_2 varying from A_3 to $A_4 + 30$ Hz $= 450$ Hz. We compare two different definitions of f in Eq. (3): the average of f_1 and f_2, and simply $f = f_1$. Using the average frequency leads to a significantly better approximation.

experienced guitar player can distinguish beat notes with a difference frequency of $\Delta f \approx 1$ Hz, which corresponds to 8 cents at A_3 ($f = 220$ Hz) or 5 cents at E_4 ($f = 329.63$ Hz). Using this approach, the difference in pitch between two frequencies f_1 and f_2 is defined as

$$\Delta\nu \equiv 1200 \log_2 \left(\frac{f_2}{f_1} \right) . \tag{2}$$

We define the average frequency $f \equiv (f_1 + f_2)/2$ and the frequency difference $\Delta f \equiv f_2 - f_1$. Then

$$\Delta\nu = 1200 \log_2 \left(\frac{f + \Delta f/2}{f - \Delta f/2} \right) \approx \frac{1200}{\ln 2} \frac{\Delta f}{f} , \tag{3}$$

where the last approximation applies when $\Delta f \ll f$. As shown in Fig. 1, if the average frequency of the interval is used to compute f — rather than the initial frequency f_1 — then the accuracy of Eq. (3) holds for almost an entire octave. In this plot, we chose $f_1 = A_3 = 220$ Hz, and allowed f_2 to vary from A_3 to $A_4 + 30$ Hz $= 450$ Hz. At the octave, the error in $\Delta\nu$ arising from Eq. (3) is -46 cents, or -4%. We compare two different definitions of f in Eq. (3): the average of f_1 and f_2, and simply $f = f_1$. Using the average frequency leads to a significantly better approximation.

We present the basics of our model of classical guitar strings in Section 2, following the pioneering work of G. Byers [4, 8]. We begin with a new expression for the allowed vibration frequencies of a stiff string, derived in Appendix A under the assumption that the boundary conditions at the saddle and the nut are not symmetric. We then include a discussion of the four contributions to frequency shifts and errors of non-ideal strings pressed behind a fret: the change in the resonant length of the string; a decrease in the linear mass density and an increase in the tension of the string; and the mechanical stiffness of the resonating string. Our goal is to simplify the equations through Taylor series expansions to allow an intuitive picture of the string's behavior to emerge. We offer an empirical reason to doubt the need for a complicated model of string fretting, and we explore this argument in greater detail in Appendix B. In Section 3, we suggest a simple experiment to estimate the response of the string's tension to the change in length caused by fretting, and we demonstrate the idea using a normal-tension string set on an Alhambra 8P guitar (as well as other string sets in Appendix D). Then, in Section 4, we use these estimates to demonstrate a straightforward analytic approach to compensating the errors in a guitar string, relying on a method — described in Appendix C — to minimize the root-mean-squared (RMS) frequency deviation at each fret. Finally, in Section 5 we discuss a collaboration of guitar manufacturer and musician to temper the guitar using harmonic tuning and optimize it for a particular piece.

This document – as well as the Python computer code needed to reproduce the figures – is available at GitHub [9].

2 Simple Model of Guitar Intonation

The starting point for prior efforts to understand guitar intonation and compensation [4, 5] is a formula for f_q, the transverse vibration frequency harmonic q of a stiff string, originally published by Morse in 1936 [10, 11, 12]:

$$f_q = \frac{q}{2\,L}\sqrt{\frac{T}{\mu}}\left[1 + 2B + 4\left(1 + \frac{\pi^2\,q^2}{8}\right)B^2\right]. \tag{4}$$

Here L is the length of the string, T and μ are its tension and linear mass density, respectively, and B is a small "bending stiffness" coefficient to capture the relevant mechanical properties of the string. For a homogeneous string with a cylindrical cross-section, B is given by

$$B \equiv \sqrt{\frac{\pi\,\rho^4\,E}{4\,T\,L^2}}, \tag{5}$$

where ρ is the radius of the string and E is Young's modulus (or the modulus of elasticity). But it's unlikely that Eq. (4) accurately describes the resonant frequencies of a nylon string on a classical guitar, because it assumes that the string is "clamped" at both ends, so that a particular set of symmetric boundary conditions must be applied to the partial differential equation (PDE) describing transverse vibrations of the string. We believe that this assumption is correct for the end of the string held at either the nut or the fret, but that the string is "pinned" (and not clamped) at the saddle. In Appendix A, we solve the PDE using these non-symmetric boundary conditions, and find

$$f_q = \frac{q}{2\,L}\sqrt{\frac{T}{\mu}}\left[1 + B + \left(1 + \frac{1}{2}q^2\pi^2\right)B^2\right]. \tag{6}$$

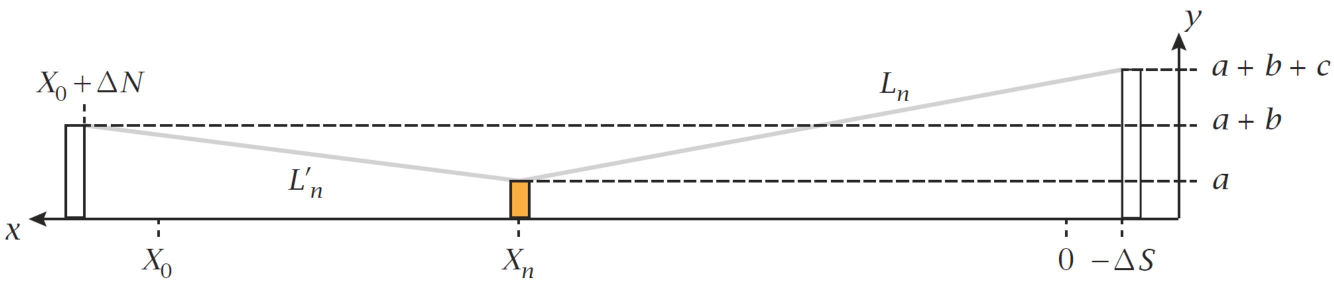


Figure 2: A simple (side-view) schematic of the classical guitar used in this model. The scale length of the guitar is X_0, but we allow the edges of both the saddle and the nut to be set back an additional distance ΔS and ΔN, respectively. The location on the x-axis of the center of the n^{th} fret is X_n. (Note that the x-axis is directed toward the left in this figure.) In the y direction, $y = 0$ is taken as the surface of the fingerboard; therefore the height of each fret above the fingerboard is a, the height of the nut is $a + b$, and the height of the saddle is $a + b + c$. L_n is the *resonant length* of the string from the saddle to the center of fret n, and L'_n is the length of the string from the fret to the nut.

Note that this expression is valid only when $B \ll 1$. For a typical nylon guitar string with $E \approx 5$ GPa, $T \approx 60$ N, $\rho \approx 0.5$ mm, and $L \approx 650$ mm, we have $B \approx 3 \times 10^{-3}$. (In this case, the quadratic B term in Eq. (6) is only 2% as large as the linear term, and can generally be neglected. We will include it in our analysis below only for completeness.) We should use Eq. (6) with some caution, because the chemistry, materials science, and physics of nylon strings (particularly the wound bass strings) are quite complicated [13].

Our model is based on the schematic of the guitar shown in Fig. 2. The scale length of the guitar is X_0, but we allow the edges of both the saddle and the nut to be set back an additional distance ΔS and ΔN, respectively. The location on the x-axis of the center of the n^{th} fret is X_n. In the y direction, $y = 0$ is taken as the surface of the fingerboard; the height of each fret is a, the height of the nut is $a+b$, and the height of the saddle is $a+b+c$. (For the time being, we are neglecting the art of *relief* practiced by expert luthiers that adjusts the value of b up the fretboard and strings.) L_n is the *resonant length* of the string from the saddle to the center of fret n, and L'_n is the length of the string from the fret to the nut. The total length of the string is defined as $\mathcal{L}_n \equiv L_n + L'_n$. As discussed in more detail in Appendix B, we have chosen to ignore the apparent increase in length of the string caused by both the fretting depth and the shape of the fretted string under the finger that have been included in previous studies of guitar intonation and compensation [4, 5] because their effects are very small and generally inaudible.

We start with the form of the fundamental frequency of a fretted string given by Eq. (6) with $q = 1$, and apply it to the frequency of a string pressed just behind the n^{th} fret:

$$f_n = \frac{1}{2\,L_n}\sqrt{\frac{T_n}{\mu_n}}\left[1 + B_n + \left(1 + \frac{\pi^2}{2}\right)B_n^2\right], \tag{7}$$

where T_n and μ_n are the modified tension and the linear mass density of the fretted string, and

$$B_n \equiv \sqrt{\frac{\pi\,\rho^4\,E}{4\,T_n\,L_n^2}}\,. \tag{8}$$

We note that T_n and μ_n depend on $\mathcal{L}_n$, the *total* length of the fretted string from the saddle to the nut. Ideally, in the 12-TET system [1],

$$f_n = \gamma_n f_0 \,, \qquad \text{(12-TET ideal)} \tag{9}$$

where f_0 is the frequency of the open (unfretted) string, and

$$\gamma_n \equiv 2^{n/12} \,. \tag{10}$$

Therefore, the error interval — the difference between the fundamental frequency of the fretted string and the corresponding perfect 12-TET frequency — expressed in cents is given by

$$\begin{aligned} \Delta\nu_n &= 1200 \log_2 \left(\frac{f_n}{\gamma_n f_0} \right) \\ &= 1200 \log_2 \left(\frac{L_0}{\gamma_n L_n} \right) + 600 \log_2 \left(\frac{\mu_0}{\mu_n} \right) + 600 \log_2 \left(\frac{T_n}{T_0} \right) \\ &\quad + 1200 \log_2 \left[\frac{1 + B_n + (1 + \pi^2/2)\, B_n^2}{1 + B_0 + (1 + \pi^2/2)\, B_0^2} \right] , \end{aligned} \tag{11}$$

where $\log_2$ is the (binary) logarithm function calculated with base 2.

The final form of Eq. (11) makes it clear that — for nylon guitar strings — there are four contributions to intonation:

1. *Resonant Length*: The first term represents the error caused by the increase in the length of the fretted string L_n compared to the ideal length X_n, which would be obtained if $b = c = 0$ and $\Delta S = \Delta N = 0$.

2. *Linear Mass Density*: The second term is the error caused by the reduction of the linear mass density of the fretted string. This effect will depend on the *total* length of the string $\mathcal{L}_n = L_n + L'_n$.

3. *Tension*: The third term is the error caused by the *increase* of the tension in the string arising from the stress and strain applied to the string by fretting. This effect will also depend on the total length of the string $\mathcal{L}_n$.

4. *Bending Stiffness*: The fourth and final term is the error caused by the change in the bending stiffness coefficient arising from the decrease in the vibrating length of the string from L_0 to L_n.

Note that the properties of the logarithm function have *decoupled* these physical effects by converting multiplication into addition. We will discuss each of these sources of error in turn below.

2.1 Resonant Length

The length L_0 of the open (unfretted) guitar string can be calculated quickly by referring to Fig. 2. We find:

$$L_0 = \sqrt{(X_0 + \Delta S + \Delta N)^2 + c^2} \approx X_0 + \Delta S + \Delta N + \frac{c^2}{2\, X_0} \,, \tag{12}$$

where the approximation arises from the Taylor series that applies since $c^2 \ll X_0^2$. Similarly, since $(b+c)^2 \ll X_0^2$, the resonant length L_n is given by

$$L_n = \sqrt{(X_n + \Delta S)^2 + (b+c)^2} \approx X_n + \Delta S + \frac{(b+c)^2}{2\,X_n}\,. \tag{13}$$

Then — if the guitar has been manufactured such that $X_n = X_0/\gamma_n$ — the resonant length error determined by the first term in the last line of Eq. (11) is approximately

$$1200\,\log_2\left(\frac{L_0}{\gamma_n\,L_n}\right) \approx \frac{1200}{\ln(2)}\left[\frac{\Delta N - (\gamma_n - 1)\,\Delta S}{X_0} - \frac{\gamma_n^2 (b+c)^2 - c^2}{2\,X_0^2}\right], \tag{14}$$

where ln is the natural logarithm function. If the guitar is uncompensated, so that $\Delta S = \Delta N = 0$, the magnitude of this error is typically less than 0.25 cents, and can be neglected. However, with $\Delta S > 0$ and $\Delta N < 0$, we can significantly flatten the frequency shift — for example, with $\Delta S = 2.0$ mm and $\Delta N = -0.25$ mm, the contribution of this term to the shift at the 12$^{\text{th}}$ fret is -6 cents. We'll see that this is our primary method of compensation.

2.2 Linear Mass Density

As discussed above, the linear mass density μ_0 of an open (unfretted) string is simply the total mass M of the string clamped between the saddle and the nut divided by the length L_0. Similarly, the mass density μ_n of a string held onto fret N is $M/\mathcal{L}_n$. Therefore

$$\frac{\mu_0}{\mu_n} = \frac{\mathcal{L}_n}{L_0} \equiv 1 + Q_n\,, \tag{15}$$

where we have followed Byers and defined [4, 5]

$$Q_n \equiv \frac{\mathcal{L}_n - L_0}{L_0}\,. \tag{16}$$

Since we expect that $Q_n \ll 1$, we can approximate the second term in the final line of Eq. (11) as

$$600\,\log_2\left(\frac{\mu_0}{\mu_n}\right) \approx \frac{600}{\ln(2)}\,Q_n\,. \tag{17}$$

Referring to Fig. 2, we see that $\mathcal{L}_n = L_n + L'_n$, and we calculate L'_n for $n \geq 1$ as

$$L'_n = \sqrt{(X_0 - X_n + \Delta N)^2 + b^2} \approx X_0 - X_n + \Delta N + \frac{b^2}{2\,(X_0 - X_n)}\,, \tag{18}$$

where the approximation applies when $b^2 \ll (X_0 - X_n)^2$. Therefore, using Eq. (13), we have

$$\mathcal{L}_n = L_n + L'_n \approx X_0 + \Delta S + \Delta N + \frac{(b+c)^2}{2\,X_n} + \frac{b^2}{2\,(X_0 - X_n)}\,, \tag{19}$$

and

$$\begin{aligned} Q_n &\approx \frac{1}{2\,X_0}\left[\frac{(b+c)^2}{X_n} + \frac{b^2}{X_0 - X_n} - \frac{c^2}{X_0}\right] \\ &= \frac{\gamma_n}{2\,X_0^2}\left[(b+c)^2 + \frac{b^2}{\gamma_n - 1} - \frac{c^2}{\gamma_n}\right] \\ &= \frac{\gamma_n - 1}{2\,X_0^2}\left(\frac{\gamma_n}{\gamma_n - 1}\,b + c\right)^2. \end{aligned} \tag{20}$$

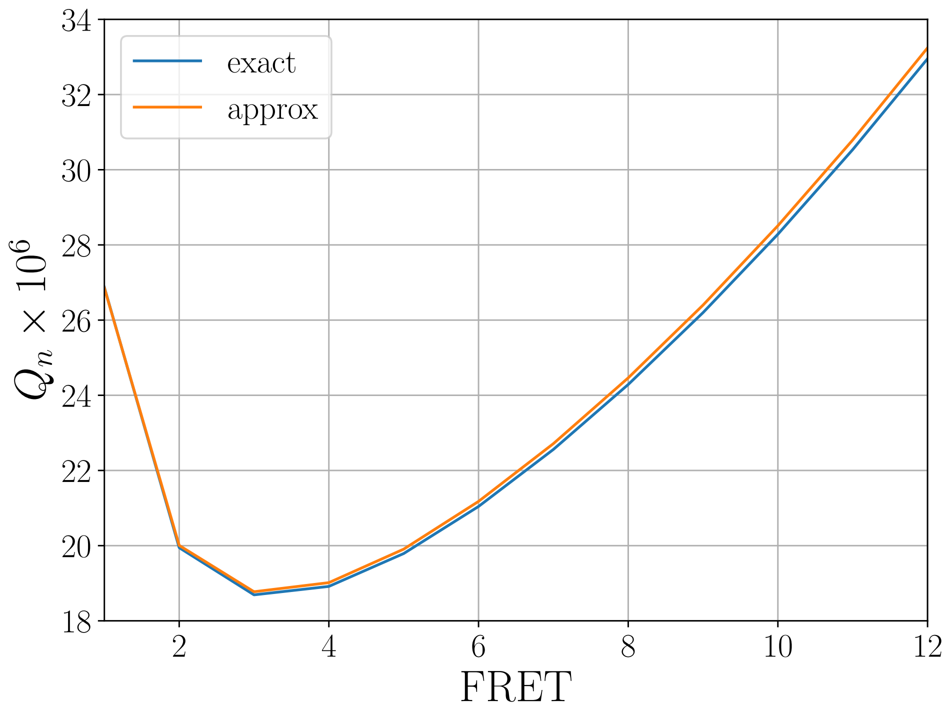


Figure 3: Comparison of the exact expression for the normalized displacement Q_n as a function of the fret number given by Eq. (16) with the approximate expression given by Eq. (20). For example, at the fifth fret, $Q_n \approx 20 \times 10^{-6}$. Here the guitar has $b = 1.0$ mm, $c = 3.5$ mm, $\Delta S = 1.75$ mm, $\Delta N = -0.25$ mm, and a scale length of 650 mm.

In Fig. 3, we plot a comparison between the exact expression for the normalized displacement Q_n given by Eq. (16) with the approximate expression given by Eq. (20). Here the guitar has $b = 1.0$ mm, $c = 3.5$ mm, $\Delta S = 1.5$ mm, $\Delta N = -0.25$ mm, and $X_0 = 650$ mm. We see that our Taylor series estimate is quite accurate even though it neglects both setbacks, indicating that Q_n is determined primarily by the values of b and c. For the same parameters, $\Delta\nu_{12} \approx$ 0.03 cents, and will be even smaller for $n < 12$. In general, the shift due to linear mass density can be neglected without significant loss of accuracy.

2.3 Tension

As a guitar string is fretted, its length increases, thereby increasing its tension through linear elastic deformation [14]. We will focus on the response of the string to a longitudinal strain, and neglect the transverse stress that causes negligible changes in the diameter of the string [13]. In this case, we can write the change in tension of a string experiencing a change in length from L to $\mathcal{L}$ as

$$\Delta T = \pi\,\rho^2\,E\,\frac{\mathcal{L} - L}{L}\,. \tag{21}$$

Therefore, the tension in a string clamped to fret n is

$$T_n = T_0 + \Delta T_n = T_0\,(1 + \kappa\,Q_n)\;, \tag{22}$$

where we have used Eq. (16) and defined the dimensionless "string constant"

$$\kappa \equiv \frac{\pi\rho^2 E}{T_0}\,. \tag{23}$$

In this case, we assume that $\kappa\, Q_n \ll 1$, so that we can approximate the third term in the final line of Eq. (11) as

$$600\,\log_2\left(\frac{T_n}{T_0}\right) \approx \frac{600}{\ln(2)}\,\kappa\, Q_n\,. \tag{24}$$

This frequency shift is larger than that caused by the linear mass density error by a factor of κ.

2.4 Bending Stiffness

Since B_n is already relatively small, we only need to consider the largest contribution arising from the shortened length of the fretted string compared to that of the open string. We see from Eq. (13) that $L_n \approx L_0/\gamma_n$, so from Eq. (8) we have

$$B_n = \sqrt{\frac{\pi\,\rho^4\,E}{4\,T_n\,L_n^2}} \approx \frac{L_0}{L_n}\sqrt{\frac{\pi\,\rho^4\,E}{4\,T_0\,L_0^2}} = \gamma_n\,B_0\,. \tag{25}$$

Therefore, the fourth term in the final line of Eq. (11) can be approximated as

$$1200\,\log_2\left[\frac{1+B_n+(1+\pi^2/2)\,B_n^2}{1+B_0+(1+\pi^2/2)\,B_0^2}\right] \approx \frac{1200}{\ln(2)}\left[(\gamma_n-1)\,B_0+\frac{1}{2}\,(\gamma_n^2-1)\,(1+\pi^2)\,B_0^2\right]. \tag{26}$$

When $n = 12$ and $B_0 = 3\times 10^{-3}$, the corresponding shift is approximately 5.5 cents, with the quadratic term contributing about 0.25 cents. Note that (to zero order in Q_n) this bending stiffness error does not depend on the tiny changes to the linear mass density or the tension that arises due to string fretting. Instead, it is an intrinsic mechanical property of the string.

Incorporating all of these effects — and *temporarily* neglecting the term proportional to B_0^2 — we find that the total frequency shift is given by

$$\Delta\nu_n \approx \frac{1200}{\ln(2)}\left[(\gamma_n-1)\left(B_0-\frac{\Delta S}{X_0}\right)+\frac{\Delta N}{X_0}+\frac{1}{2}\,\kappa\,Q_n\right]. \tag{27}$$

In this form, we see that the bending stiffness and the increase in string tension due to fretting sharpen the pitch, but that we can flatten it with a positive saddle setback and negative nut setback. In fact, a simple compensation strategy would be to choose $\Delta S = B_0\,X_0$ to compensate for stiffness, and then select $\Delta N = -\kappa\,X_0\,\overline{Q}/2$, where $\overline{Q}$ is the displacement averaged over a particular set of frets. We'll rely on a more flexible method for compensation in Section 4, but we'll obtain results that are similar to those found using this basic approach.

3 Experimental Estimate of the String Constant

How do we determine the spring constant κ given by Eq. (23)? Ideally, we could conduct an experiment that measures the change in the frequency of an open string — pinned at one end, and clamped at the other — as we make slight changes to its length [4, 5]. From Eq. (6), the derivative of the fundamental frequency of an open string is

$$\begin{aligned}\frac{d\,f}{d\,L} &= \frac{f}{L}\left(-1+\frac{L}{2\,T}\,\frac{d\,T}{d\,L}-\frac{L}{2\,\mu}\,\frac{d\,\mu}{d\,L}+\frac{L}{1+B}\,\frac{d\,B}{d\,L}\right)\\ &= \frac{f}{L}\left(-1+\frac{1}{2}\,\kappa+\frac{1}{2}-\frac{B}{1+B}\right)\\ &\approx \frac{f}{L}\times\frac{1}{2}\,(\kappa-1)\,,\end{aligned} \tag{28}$$

where we have used the analysis in Section 2 to determine that

$$\frac{d\,T}{d\,L} = \frac{T}{L}\,\kappa\,, \tag{29a}$$

$$\frac{d\,\mu}{d\,L} = -\frac{\mu}{L}\,, \text{ and} \tag{29b}$$

$$\frac{d\,B}{d\,L} = -\frac{B}{L}\,. \tag{29c}$$

$$\tag{29d}$$

Therefore, following Byers [4, 5], we define the parameter R to be

$$R \equiv \frac{d\,\ln(f)}{d\,\ln(L)} = \frac{L}{f}\,\frac{d\,f}{d\,L} = \frac{1}{2}\,(\kappa - 1)\,, \tag{30}$$

which gives

$$\kappa = 2\,R + 1\,. \tag{31}$$

With this measurement of κ, we can estimate the bending stiffness coefficient by comparing Eq. (23) and Eq. (25), and writing B_0 as

$$B_0 = \sqrt{\kappa}\,\frac{\rho}{2\,L_0}\,. \tag{32}$$

We can estimate the typical value of R for nylon classical guitar strings through a simple observation. On a classical guitar with a scale length of 650 mm, we can usually tune an open string a full step by winding the tuner/machine head three half turns. As we shall see below, this increases or decreases the string length above the first fret by about 3 mm, where the corresponding length of the open string is about 614 mm. Since a full step is (by definition) 200 cents, Eq. (3) tells us that

$$\frac{\Delta f}{f} \approx \frac{\ln(2)}{1200}\,\Delta\nu = \frac{200}{1731} = 0.116\,. \tag{33}$$

In this case, we estimate R to be

$$R \approx \frac{614}{3}\,0.116 = 24\,, \tag{34}$$

giving $\kappa \approx 49$ and $\sqrt{\kappa} \approx 7$.

It is relatively easy to estimate the values of κ and R for any guitar string with the aid of a simple device that can measure frequency shifts in cents [15]. First, we tune the open string to the correct 12-TET frequency. Then, we select a fret n, and measure the frequency deviation $\Delta\nu_n$ of the fretted string while muting the other strings to eliminate sympathetic vibrations. If we solve Eq. (32) for κ and then substitute the result into Eq. (27), we find

$$\alpha\,B_0^2 + \beta\,B_0 - \xi = 0\,, \tag{35}$$

where we have included the quadratic bending stiffness term, and defined the coefficients

$$\alpha \equiv \frac{1}{2}\left[\left(\frac{2\,L_0}{\rho}\right)^2 Q_n + (\gamma_n^2 - 1)\,(1 + \pi^2)\right]\,, \tag{36a}$$

$$\beta \equiv \gamma_n - 1\,, \text{ and} \tag{36b}$$

$$\xi \equiv \frac{\ln(2)}{1200}\,\Delta\nu_n + \frac{(\gamma_n - 1)\,\Delta S - \Delta N}{X_0}\,. \tag{36c}$$

Table 1: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 650 mm.

String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4501	E_4	0.356	3.74	68.6
J4502	B_3	0.409	5.05	52.0
J4503	G_3	0.512	8.36	54.3
J4504	D_3	0.368	19.21	70.0
J4505	A_2	0.445	32.90	67.3
J4506	E_2	0.546	54.72	62.8

Table 2: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 650 mm.

String	R	σ	κ	B_0	E (GPa)
J4501	23.6	0.5	48.2	0.00190	8.33
J4502	23.8	0.7	48.6	0.00219	4.81
J4503	28.8	0.7	58.7	0.00302	3.87
J4504	22.4	0.8	45.7	0.00192	7.51
J4505	23.8	0.8	48.6	0.00238	5.27
J4506	28.6	0.4	58.2	0.00321	3.90

Equation (35) is quadratic in B_0, with the solution

$$B_0 = \frac{\sqrt{\beta^2 + 4\,\alpha\,\xi} - \beta}{2\,\alpha} \approx \frac{\xi}{\beta}\,, \tag{37}$$

where we have chosen the positive root to ensure that $B_0 > 0$, and the final approximate expression on the right-hand side applies when $\beta^2 \gg 4\,\alpha\,\xi$.

We've used this approach to estimate κ and R of different string sets on the Alhambra 8P classical guitar, which has $X_0 = 650$ mm, $c = 3.5$ mm, $\Delta S = 1.5$ mm, and $\Delta N = 0.0$ mm. At the nut, $b = 1.0$ mm, but the height of the fret board decreases roughly linearly further toward the saddle, effectively increasing b. We estimate that $db/dx \approx -0.0034$, so that b has increased to 2.0 mm at the 12$^{\text{th}}$ fret. We show the specifications of a normal-tension classical string set in Table 1 using metric units[1]. In Table 2, we list the frequency deviation from 12-TET at the 12$^{\text{th}}$ fret for each string in this set, as well as the corresponding estimates of κ, R, E, and B_0, computed from Eq. (37), Eq. (30), Eq. (23), and Eq. (32), respectively. The units used for the elastic modulus are gigapascals (1 GPa = 10^9 N/m^2.). Similar measurements and results for other string sets are provided in Appendix D.

[1]Note that the correct unit of force in the metric system is Newtons (N), rather than kilograms, which is a unit of mass.

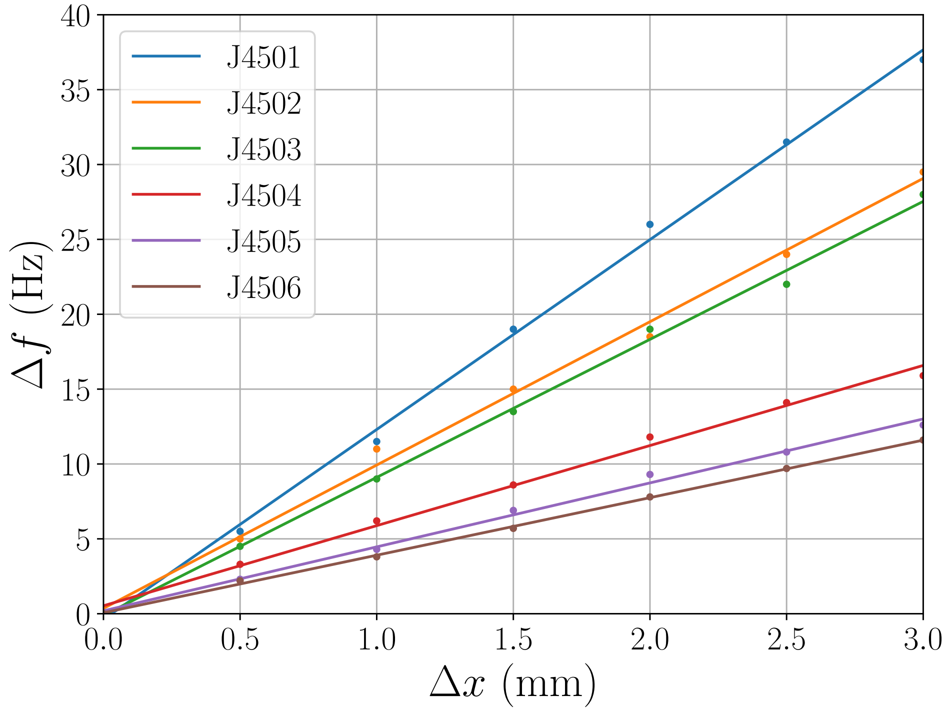
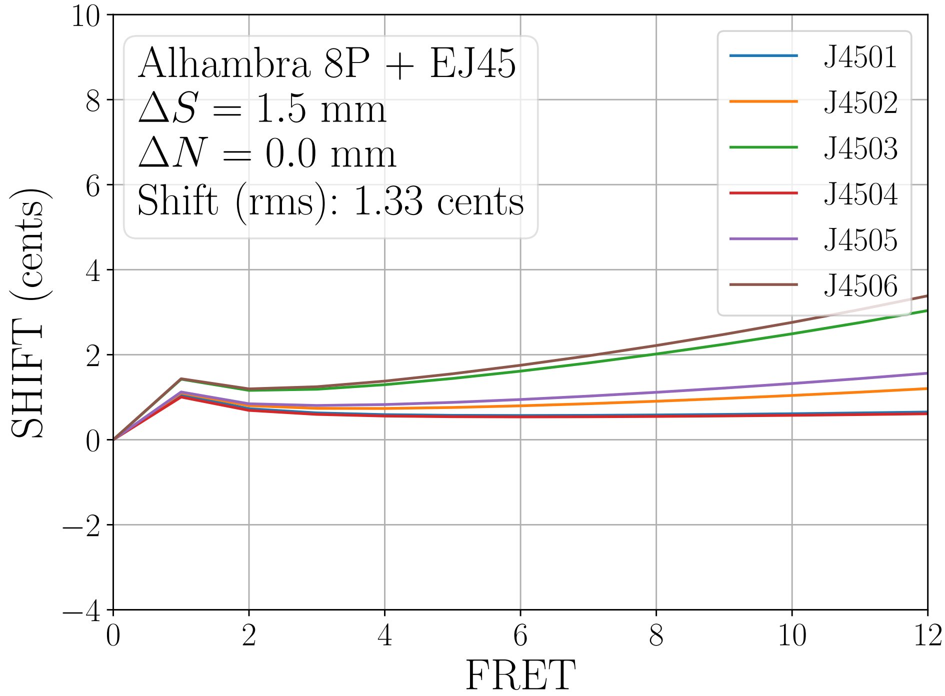


Figure 4: Results of experiments to measure R for each string in the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45) set. The points represent the measurement data, while the lines are the results of linear least-squares fits to that data.

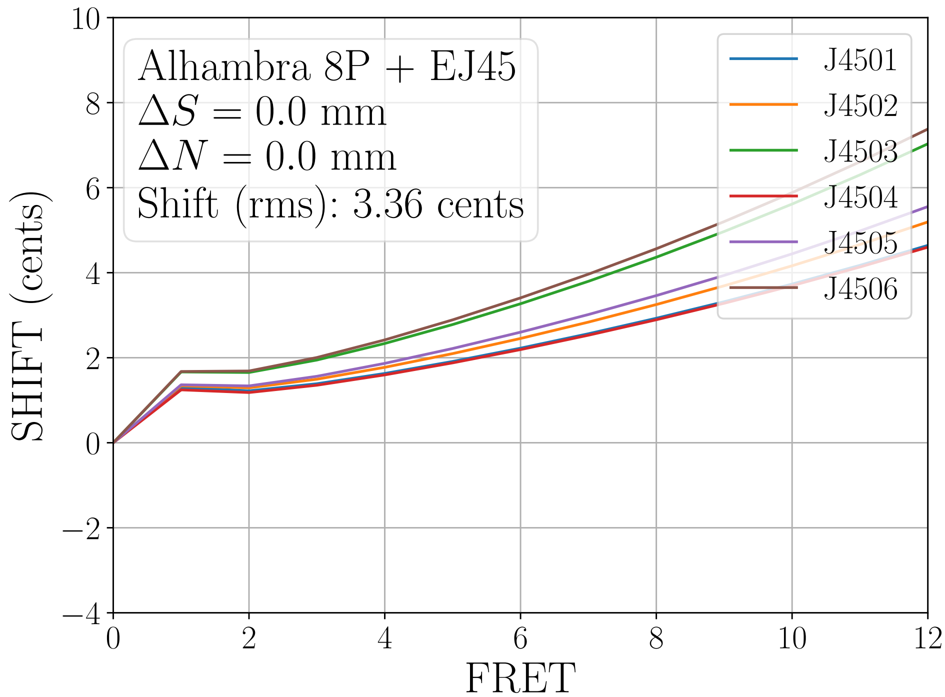
4 Classical Guitar Compensation

Using the data and results presented in Table 2, we can explore different approaches to compensating guitar strings for bending stiffness and string tension perturbations. For example, in Fig. 5a, using Eq. (11) we plot the frequency deviation (in cents) from ideal 12-TET for each string at each of the first 12 frets of an Alhambra 8P classical guitar, assuming that the open string has been perfectly tuned to the correct frequency. Recall that the Alhambra 8P is manufactured with a saddle setback of 1.5 mm, presumably to offset the effects of bending stiffness in the strings. For comparison, in Fig. 5b, we plot the same deviations for the case where $\Delta S = 0$, which increases the error of each string at the 12$^{\text{th}}$ fret by 3 – 4 cents. Recall from Section 2 that we could crudely predict the values of the saddle and nut setbacks by inspecting Eq. (27). For example, from Table 2, we estimate $\Delta S \approx B_0\, X_0 = 2.7$ mm and $\Delta N \approx -1.5$ mm for the third (G) string.

Instead of this simple approach, we adopt the method described in Appendix C and adjust the setbacks to minimize the root-mean-squared average of the frequency deviations for each string. This mean (over the first 12 frets) can be computed by squaring the frequency deviations shown in Fig. 5b, averaging those values, and then taking the square root of the result. The setbacks we obtain are listed in Table 3, and the corresponding frequency deviations — obtained with different setbacks for every string — are shown in Fig. 6a (assuming that all other aspects of the Alhambra 8P remain unchanged). Note that the saddle setbacks tend to be larger — and the nut setbacks smaller — than the simple estimates that we made above. This is easily understood by examining Eq. (27): the portion of ΔS that exceeds $B_0\, X_0$ scales with $\gamma_n - 1$, and helps to compensate for tension errors as n increases.

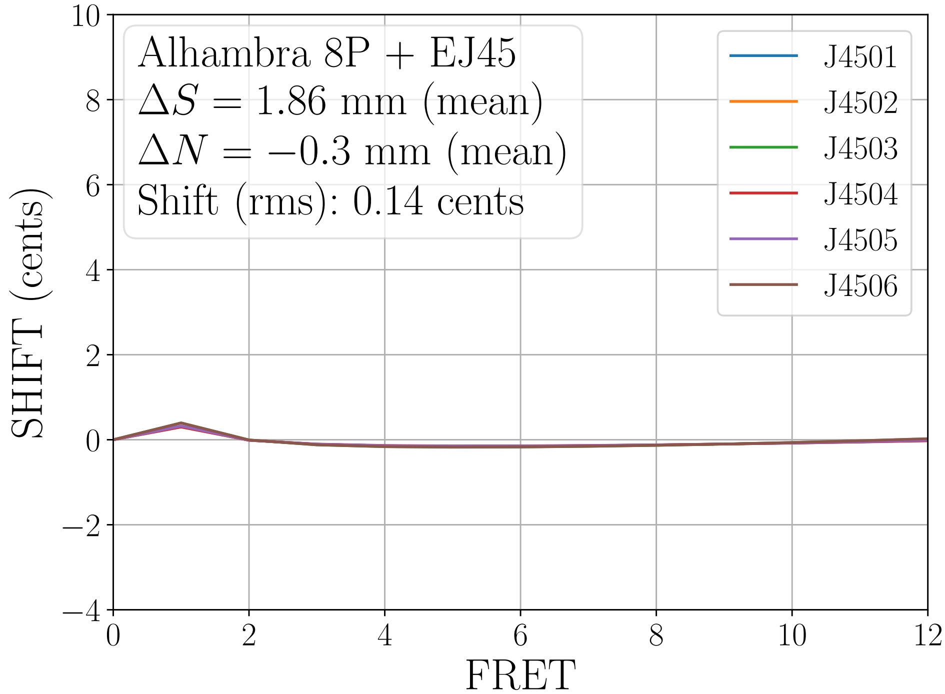


(a) Factory guitar

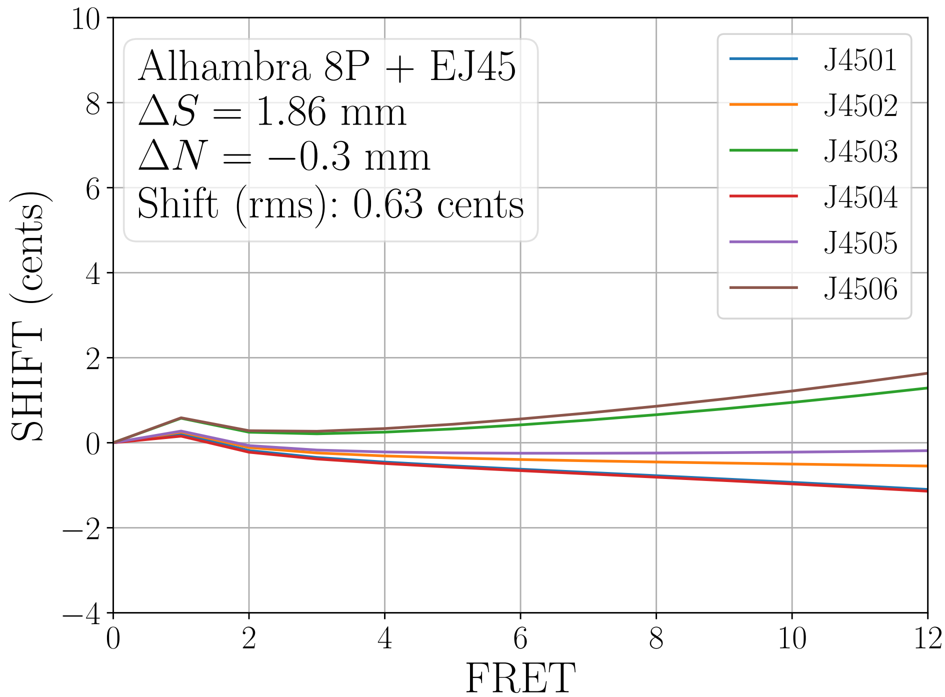


(b) Uncompensated

Figure 5: Frequency shifts (in cents) for an Alhambra 8P guitar with normal tension nylon strings (D'Addario EJ45). In (a), we show the deviations of the guitar as manufactured in the factory, completely consistent with our measurements. In (b), we show the same 12-TET errors that would arise if $\Delta S = 0$ for the same guitar.



(a) Full compensation



(b) Mean compensation

Figure 6: Frequency shifts (in cents) for an "Alhambra 8P" guitar with normal tension nylon strings (D'Addario EJ45). Instead of the factory setbacks, in (a) we use the individual values for each string that are listed in Table 3. In (b), we set ΔS and ΔN to the mean of the corresponding column in that table.

Table 3: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45) on the Alhambra 8P classical guitar.

String	ΔS (mm)	ΔN (mm)	$\overline{\Delta\nu}_{\text{rms}}$ (cents)
J4501	1.47	-0.28	0.14
J4502	1.67	-0.29	0.14
J4503	2.30	-0.34	0.16
J4504	1.47	-0.27	0.13
J4505	1.81	-0.28	0.14
J4506	2.43	-0.34	0.16

We close this section with a few comments. First, all of the compensated frequency deviation plots shown in this version of our work are theoretical; in the future, we will present experimental results to determine the efficacy of this compensation approach. Second, it is nontrivial to manufacture a guitar with different setbacks for each string [4], and it is unlikely that the exact values listed in Table 3 are applicable to other string sets. We have measured frequency deviations at the 12^{th} fret for three other string sets, and in Appendix D we have reproduced the compensation procedure for them. There is enough variation in the setbacks across all strings that it would be difficult to find values that could work well for all string sets. Finally, following the analysis of Appendix C, it is possible to determine a single setback pair $\{\Delta S, \Delta N\}$ that minimizes the RMS frequency errors of an ensemble of strings over a collection of frets simply by computing the mean of the setbacks over all strings, and then using these mean values when manufacturing the guitar. We illustrate this approach in Fig. 6b; the frequency deviations are not as small as those in Fig. 6a, but they are generally less than 5 cents. In the next section, we discuss a method to temper the guitar to reduce these errors further.

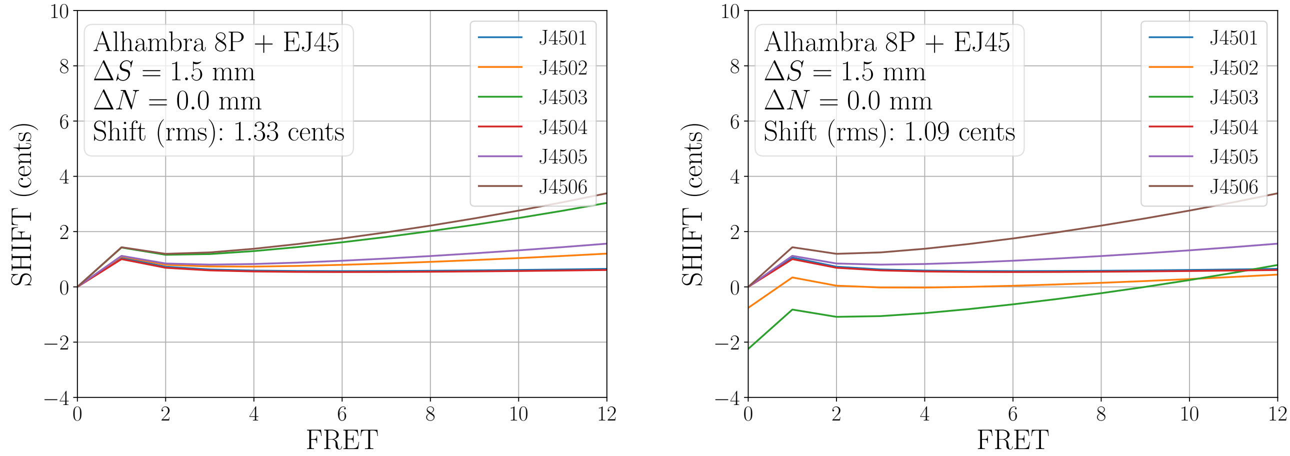
5 Tempering the Classical Guitar

> Temperament: A compromise between the acoustic purity of theoretically exact intervals, and the harmonic discrepancies arising from their practical employment.
> — Dr. Theo. Baker [16]

In Fig. 7a, the Alhambra 8P factory guitar with normal tension strings tuned to 12-TET shows the third string having the greatest error in tuning across the fretboard. Tuning the factory guitar to 12-TET exacts a perfect-fifth in the third string while playing a C major chord in first position. This results in the third string being too sharp for the other common chords of E major (G#), A major and D major (A), particularly when the guitar is played at a higher fret position. One way to reduce this error is by lowering the pitch of the third string below 12-TET with an electronic tuner. Another more comprehensive system is to tune all the strings harmonically to the fifth string, which lowers the third string by 7 cents as well as tempering the remaining strings.

In this particular case, the "Harmonic Tuning Method" can be followed using these steps:

1. Begin by tuning the fifth string to $A_2 = 110$ Hz, resulting in a fifth-fret harmonic of $A_4 = 440$ Hz. (This can also be tuned by ear using an A_4 tuning fork).



(a) Factory guitar — 12-TET tuned

(b) Factory guitar — harmonically tuned

Figure 7: Frequency shift (in cents) for an Alhambra 8P guitar with normal tension nylon strings (D'Addario EJ45). Here we compare (a) the factory guitar tuned to 12-TET with (b) the same guitar harmonically tuned using the approach outlined in Table 4.

2. Tune that harmonic to the seventh fret harmonic of the fourth string, which is also $A_4 = 440$ Hz.
3. Tune the seventh fret harmonic on the fifth string (330 Hz, or 0.37 Hz sharper than 12-TET E_4) to the fifth-fret harmonic of the sixth string.
4. The seventh fret harmonic on the fifth string can tune the remaining fretted strings: the ninth fret on the third string, the fifth fret on the second string, and the open first string.

Table 4: Harmonic tuning methodology based on A_4 and E_4. The asterisk indicates a harmonic with a null at the designated fret.

Reference String/Fret	Target String/Fret
A*/5 (A_4)	D*/7
A*/7 (E_4)	E*/5
A*/7 (E_4)	G/9
A*/7 (E_4)	B/5
A*/7 (E_4)	E/0

We have summarized these steps in Table 4, and in Figure 7b we show the same guitar tuned in this fashion. Note that other tuning choices can be made depending on the piece being played. For example, the third string could also be tuned at the second fret to $A_3 = 220$ Hz using the fifth-string harmonic at the 12^{th} fret, and/or the first string could be tuned at the fifth fret to A_4 using the fifth-fret harmonic of the fifth string. The flexibility of the harmonic tuning method — and its reliance on only an A_4 tuning fork — is a great asset for the classical guitarist.

6 Conclusion

In this work, we have constructed a simple model of classical guitar intonation that includes the effects of the resonant length of the fretted string, linear mass density, tension, and bending stiffness. We have described a simple experimental approach to estimating the increase in string tension arising from an increase in its length, and then the corresponding mechanical stiffness. This allows us to determine the saddle and nut positions needed to compensate the guitar for a particular string, and we propose a simple approach to find averages of these positions to accommodate a variety of strings. This "mean" method benefits further from temperament techniques — such as harmonic tuning — that can enhance the intonation of the classical guitar for particular musical pieces.

In the future, we intend to conduct more precise experiments to measure R for a reasonable set of guitar strings, and then to compare our theoretical predictions of compensated guitar/string frequency deviations with experimental results. We have placed the text of this manuscript (as well as the computational tools needed to reproduce our numerical results) online [9] so as to invite comment from and collaboration with interested musicians.

A Vibration Frequencies of a Stiff String

Here we outline the calculation of the normal mode frequencies of a vibrating stiff string with non-symmetric boundary conditions. We begin with the wave equation [11]

$$\mu \frac{\partial^2}{\partial t^2} y(x) = T \frac{\partial^2}{\partial x^2} y(x) - E\,S\,\mathcal{K}^2 \frac{\partial^4}{\partial x^4} y(x)\,, \tag{38}$$

where μ and T are respectively the linear mass density and the tension of the string, E is its Young's modulus (or the modulus of elasticity), S is the cross-sectional area, and $\mathcal{K}$ is the radius of gyration of the string. (For a uniform cylindrical string/wire with radius ρ, $S = \pi\rho^2$ and $\mathcal{K} = \rho/2$.) If we scale x by the length L of the string, and t by $1/\omega_0 \equiv (L/\pi)\sqrt{\mu/T}$, then we obtain the dimensionless wave equation

$$\pi^2 \frac{\partial^2}{\partial t^2} y(x) = \frac{\partial^2}{\partial x^2} y(x) - B^2 \frac{\partial^4}{\partial x^2} y(x)\,, \tag{39}$$

where B is the "bending stiffness parameter" given by

$$B \equiv \sqrt{\frac{E\,S\,\mathcal{K}^2}{L^2 T}}\,. \tag{40}$$

We assume that $y(x)$ is a sum of terms of the form

$$y(x) = C\,e^{k x - i\omega t}\,, \tag{41}$$

requiring that k and ω satisfy the expression

$$B^2 k^4 - k^2 - (\pi\,\omega)^2 = 0\,, \tag{42}$$

or

$$k^2 = \frac{1 \pm \sqrt{1 + (2\,\pi\,B\,\omega)^2}}{2\,B^2}\,. \tag{43}$$

Therefore, given ω, we have four possible choices for k: $\pm k_1$, or $\pm i k_2$, where

$$k_1^2 = \frac{\sqrt{1 + (2\,\pi\,B\,\omega)^2} + 1}{2\,B^2}\,, \text{ and} \tag{44a}$$

$$k_2^2 = \frac{\sqrt{1 + (2\,\pi\,B\,\omega)^2} - 1}{2\,B^2}\,. \tag{44b}$$

The corresponding general solution to Eq. (39) has the form

$$y(x) = e^{-i\omega t}\left(C_1^+ e^{k_1 x} + C_1^- e^{-k_1 x} + C_2^+ e^{i k_2 x} + C_2^- e^{-i k_2 x}\right)\,. \tag{45}$$

As discussed in Section 2, the boundary conditions for the case of a classical guitar string are not symmetric. At $x = 0$ (the saddle), the string is pinned (but not clamped), so that $y = 0$ and $\partial^2 y/\partial x^2 = 0$. However, at $x = 1$ (the fret) the string is clamped, so that $y = 0$ and $\partial y/\partial x = 0$. Applying these constraints to Eq. (45), we obtain

$$0 = C_1^+ + C_1^- + C_2^+ + C_2^-\,, \tag{46a}$$

$$0 = k_1^2\left(C_1^+ + C_1^-\right) - k_2^2\left(C_2^+ + C_2^-\right)\,, \tag{46b}$$

$$0 = C_1^+ e^{k_1} + C_1^- e^{-k_1} + C_2^+ e^{i k_2} + C_2^- e^{-i k_2}\,, \text{ and} \tag{46c}$$

$$0 = k_1\left(C_1^+ e^{k_1} - C_1^- e^{-k_1}\right) + i k_2\left(C_2^+ e^{i k_2} - C_2^- e^{-i k_2}\right)\,. \tag{46d}$$

Since $k_1^2 + k_2^2 \neq 0$, the first two of these equations tell us that $C_1^- = -C_1^+ \equiv -C_1$, and $C_2^- = -C_2^+ \equiv -C_2$. Therefore, the second two equations become

$$C_1\,\sinh(k_1) = -i\,C_2\,\sin(k_2)\,, \text{ and} \tag{47a}$$

$$k_1\,C_1\,\cosh(k_1) = -i\,k_2\,C_2\,\cos(k_2)\,. \tag{47b}$$

Dividing the first of these equations by the second, we find

$$\tan(k_2) = \frac{k_2}{k_1}\,\tanh k_1\,. \tag{48}$$

From Eq. (44), we see that $k_1^2 - k_2^2 = 1/B^2$, so that

$$k_1 = \frac{1}{B}\sqrt{1 + (B\,k_2)^2}\,. \tag{49}$$

In the case of a classical guitar, we expect that $B \ll 1$, so $k_1 \approx 1/B \gg 1$, and therefore $\tanh k_1 \longrightarrow 1$. Substituting Eq. (49) into Eq. (48), we obtain

$$\tan(k_2) = \frac{B\,k_2}{\sqrt{1 + (B\,k_2)^2}}\,. \tag{50}$$

We expect that $B\,k_2 \ll 1$, so we assume that $k_2 = q\pi(1+\epsilon)$, where $q \in \mathbb{N}$ is an integer greater than or equal to 1, and $\epsilon \ll 1$. Therefore, to second order in ϵ, we have $\tan(k_2) \approx q\pi\epsilon$, and

$$\epsilon = \frac{B\,(1+\epsilon)}{\sqrt{1 + \left[q\,\pi\,B\,(1+\epsilon)\right]^2}}\,. \tag{51}$$

The denominator of the right-hand side of this equation has a Taylor expansion given by $1-\frac{1}{2}\left[q\,\pi\,B\,(1+\epsilon)\right]^2$, indicating that it will not contribute to ϵ to second order in B. Therefore, to this order,

$$\epsilon \approx \frac{B}{1-B} \approx B + B^2 . \tag{52}$$

We substitute $k = \pm i k_2$ into Eq. (42) with $k_2 = q\pi/(1-B)$ to obtain

$$\begin{aligned} \omega &= \frac{k_2}{\pi}\sqrt{1+(B\,k_2)^2} \\ &= \frac{q}{1-B}\sqrt{1+q^2\pi^2\left(\frac{B}{1-B}\right)^2} \\ &\approx q\left[1+B+\left(1+\frac{1}{2}q^2\pi^2\right)B^2\right] . \end{aligned} \tag{53}$$

Restoring the time scaling by $1/\omega_0$, and defining the frequency (in cycles/second) $f = \omega/2\pi$, we finally have

$$f_q = \frac{q}{2\,L}\sqrt{\frac{T}{\mu}}\left[1+B+\left(1+\frac{1}{2}q^2\pi^2\right)B^2\right] . \tag{54}$$

We use this result to build our model in Section 2.

B Fretting Model

As discussed in Section 2, unlike previous studies of guitar intonation and compensation [4, 5] we have neglected to include a contribution to the incremental change in the length of the fretted string caused by both the depth and the shape of the string under the finger. As the string is initially pressed to the fret, the total length $\mathcal{L}_n$ increases and causes the tension in the string — which is pinned at the saddle and clamped at the nut — to increase. As the string is pressed further, does the additional deformation of the string increase its tension (throughout the resonant length L_n)? There are at least two purely empirical reasons to doubt this hypothesis. First, we can mark a string (with a fine-point felt pen) above a particular fret and then observe the mark with a magnifying glass. As the string is pressed all the way to the finger board, the mark does not move perceptibly — it has become *clamped* on the fret. Second, we can use either our ears or a simple tool to measure frequencies [15] to listen for a shift as we use different fingers and vary the fretted depth of a string. The apparent modulation is far less than would be obtained by classical vibrato (± 15 cents), so we assume that once the string is minimally fretted the length(s) can be regarded as fixed. (If this were not the case, then fretting by different people or with different fingers, at a single string or with a barre, would cause additional, varying frequency shifts that would be audible and difficult to compensate.)

Here we include this concept in a simple way to determine the effect it could have on the frequency shift due to increased string tension. In Fig. 8, we adopt the schematic of the guitar shown in Fig. 2, but we allow a small, horizontal linear section of string with length d to represent the action of the finger. In this case, the resonant length L_n is unaffected, but the remaining string length becomes

$$L'_n = d + \sqrt{(X_0+\Delta N - X_n - d)^2 + b^2} \approx X_0 - X_n + \Delta N + \frac{b^2}{2\,(X_0+\Delta N - X_n - d)} . \tag{55}$$

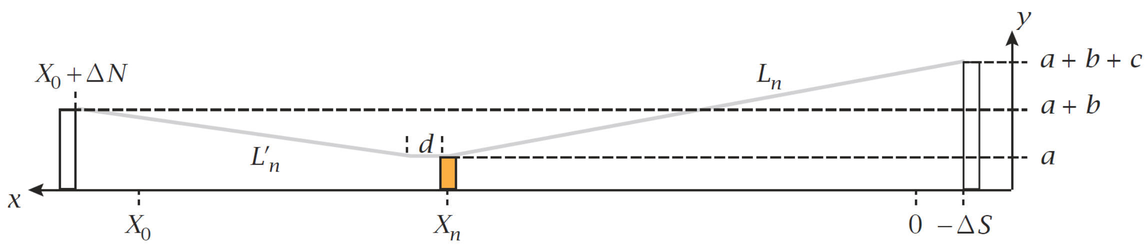


Figure 8: A recapitulation of Fig. 2 with the addition of a horizontal linear distance d at fret n to represent the slight increase in the distance L'_n caused by a finger.

Roughly speaking, the act of fretting increases the effective value of b^2 in L'_n by a factor of $1 + d/(X_0 - X_n)$. We can use this result and Eq. (20) to determine the increase in the relative displacement Q_n; we find

$$\Delta Q_n \approx \left(\frac{\gamma_n}{\gamma_n - 1}\right)^2 \frac{b^2\, d}{2\, X_0^3}\,. \tag{56}$$

The first factor on the right-hand side of this expression is well approximated as

$$\left(\frac{\gamma_n}{\gamma_n - 1}\right)^2 \approx \left[\frac{12}{n\,\ln(2)}\right]^2, \tag{57}$$

which is clearly largest at the first fret. When $d > 0$, the corresponding increase in the frequency shift given by Eq. (27) is

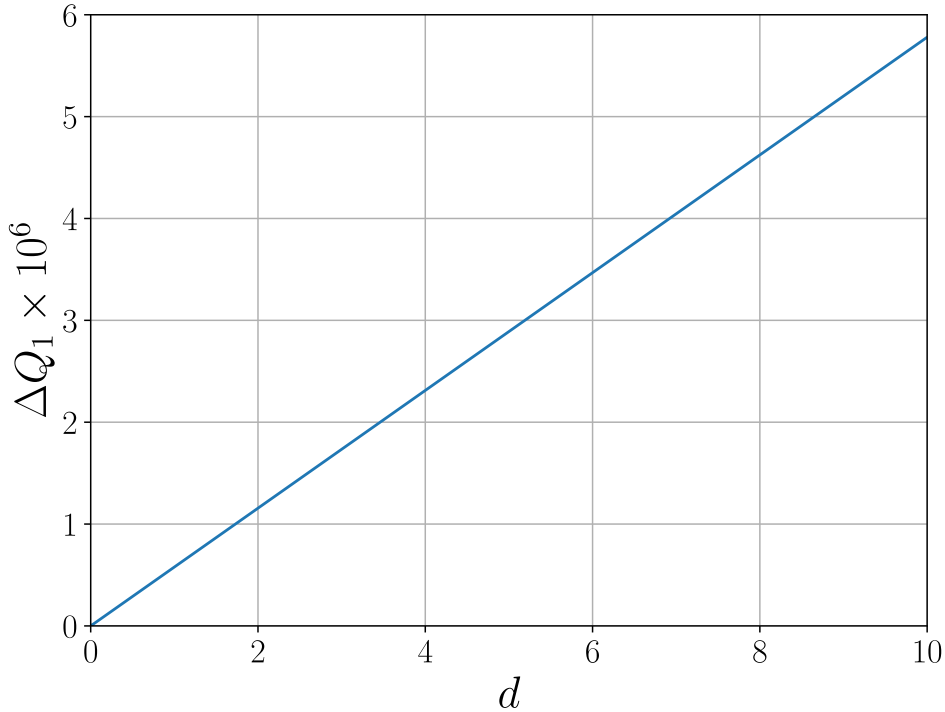
$$\Delta\nu_n(d) - \Delta\nu_n(0) = \frac{600}{\ln(2)} \left(\frac{\gamma_n}{\gamma_n - 1}\right)^2 \frac{\kappa\, b^2\, d}{2\, X_0^3}\,. \tag{58}$$

In Fig. 9a, we plot Q_1 for the first fret as a function of the distance parameter d. For this example, we have adopted the parameters of a "no-relief" Alhambra 8P guitar using normal tension strings. In the worst case, where $d = 1$ cm, Q_1 increases by almost 6×10^{-6}. For a string with $\kappa = 100$, in Fig. 9b we plot $\Delta\nu_n(d) - \Delta\nu_n(0)$ as a function of d. Despite this high value of κ and the prediction in Fig. 9b that the shift on the first fret could increase by as much as 0.5 cents, our measurements for $\Delta\nu_1$ using the Alhambra 8P guitar and normal tension strings is consistent with the theoretical values using $d = 0$ shown in Fig. 5a. Therefore, without a compelling reason to do so, we are reluctant to choose a value of d greater than 0. It is true that we can press the string so hard that it touches the fret board, but the increase in the frequency shift is still less than 1 cent, and this is likely the result of dragging the string over the fret (similar to vibrato) and causing a local change in tension that is inconsistent with the boundary conditions used to derive Eq. (6).

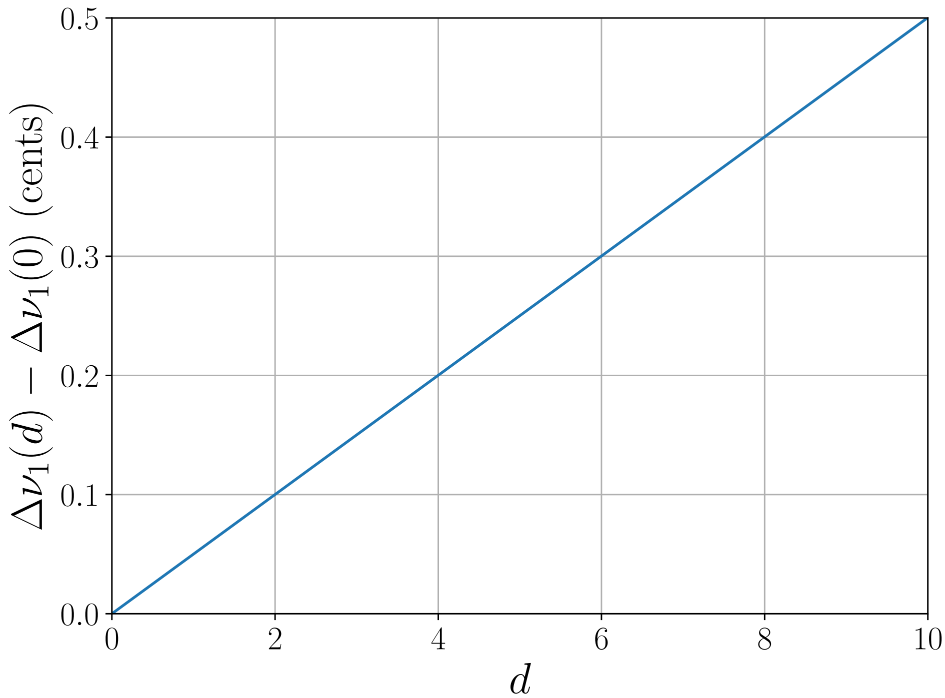
C Compensation by Minimizing RMS Error

The root-mean-square (RMS) frequency error (in cents) averaged over the frets $n \in \{1, n_{\max}\}$ (for $n_{\max} > 1$) *of a particular string* is given by

$$\overline{\Delta\nu}_{\rm rms} \equiv \sqrt{\frac{\sum_{n=1}^{n_{\max}} \Delta\nu_n^2}{n_{\max}}}\,, \tag{59}$$



(a) Relative displacement Q_1



(b) Additional frequency shift for $\kappa = 100$

Figure 9: In (a), we plot the relative displacement ΔQ_1 for the first fret as a function of the fretting distance parameter d using Eq. (56). Here the guitar has the same parameters as the "no-relief" Alhambra 8P, with normal tension strings. In (b), we show the additional frequency shift given by Eq. (58) at the first fret of a string with $\kappa = 100$ as d increases from 0. For $n > 1$, the relative displacement and additional shift are reduced by a factor of n^2.

where $\Delta\nu_n$ is given by Eq. (27). Here we will vary both ΔS and ΔN to minimize $\overline{\Delta\nu}_{\rm rms}$. In this case, it is sufficient to minimize the quantity

$$\chi^2 = \sum_{n=1}^{n_{\max}} \left[\frac{\ln(2)}{1200}\,\Delta\nu_n\right]^2 \tag{60}$$

such that the gradient of χ^2 with respect to ΔS and ΔN vanishes. The components of this gradient are

$$\frac{\partial}{\partial\Delta S}\,\chi^2 = -\frac{2}{X_0}\sum_n(\gamma_n-1)\left[(\gamma_n-1)\left(B_0-\frac{\Delta S}{X_0}\right)+\frac{\Delta N}{X_0}+\frac{\kappa}{2}\,Q_n\right],\ \text{and} \tag{61a}$$

$$\frac{\partial}{\partial\Delta N}\,\chi^2 = \frac{2}{X_0}\sum_n\left[(\gamma_n-1)\left(B_0-\frac{\Delta S}{X_0}\right)+\frac{\Delta N}{X_0}+\frac{\kappa}{2}\,Q_n\right]. \tag{61b}$$

Setting both of these expressions to zero, we can rewrite them as the matrix equation

$$\begin{bmatrix}\sigma_2 & -\sigma_1\\ \sigma_1 & -\sigma_0\end{bmatrix}\begin{bmatrix}\Delta S\\ \Delta N\end{bmatrix} = X_0\begin{bmatrix}\sigma_2\,B_0+\frac{1}{2}\,\kappa\,\overline{Q}_1\\ \sigma_1\,B_0+\frac{1}{2}\,\kappa\,\overline{Q}_0\end{bmatrix}, \tag{62}$$

where

$$\sigma_k \equiv \sum_{n=1}^{n_{\max}}(\gamma_n-1)^k\ ,\ \text{and} \tag{63}$$

$$\overline{Q}_k \equiv \sum_{n=1}^{n_{\max}}(\gamma_n-1)^k\,Q_n\,. \tag{64}$$

We note that

$$g_k \equiv \sum_{n=1}^{n_{\max}}\gamma_n^k = \frac{\gamma_k\,(\gamma_{k n_{\max}}-1)}{\gamma_k-1}, \tag{65}$$

and therefore

$$\sigma_0 = n_{\max}\,, \tag{66a}$$

$$\sigma_1 = g_1-n_{\max}\,,\ \text{and} \tag{66b}$$

$$\sigma_2 = g_2-2g_1+n_{\max}\,. \tag{66c}$$

Equation (62) has the straightforward analytic solution

$$\begin{bmatrix}\Delta S\\ \Delta N\end{bmatrix} = \frac{X_0}{\sigma_0\,\sigma_2-\sigma_1^2}\begin{bmatrix}\sigma_0 & -\sigma_1\\ \sigma_1 & -\sigma_2\end{bmatrix}\begin{bmatrix}\sigma_2\,B_0+\frac{1}{2}\,\kappa\,\overline{Q}_1\\ \sigma_1\,B_0+\frac{1}{2}\,\kappa\,\overline{Q}_0\end{bmatrix}, \tag{67}$$

or

$$\Delta S = \left(B_0-\frac{\kappa}{2}\,\frac{\sigma_1\,\overline{Q}_0-\sigma_0\,\overline{Q}_1}{\sigma_0\,\sigma_2-\sigma_1^2}\right)X_0\,,\ \text{and} \tag{68a}$$

$$\Delta N = -\frac{\kappa}{2}\,\frac{\sigma_2\,\overline{Q}_0-\sigma_1\,\overline{Q}_1}{\sigma_0\,\sigma_2-\sigma_1^2}\,X_0\,. \tag{68b}$$

For completeness, the solution when the quadratic stiffness term is included is given by

$$\begin{bmatrix}\Delta S\\ \Delta N\end{bmatrix} = \frac{X_0}{\sigma_0\,\sigma_2-\sigma_1^2}\begin{bmatrix}\sigma_0 & -\sigma_1\\ \sigma_1 & -\sigma_2\end{bmatrix}\begin{bmatrix}\sigma_2\,B_0+\frac{1}{2}\,(1+\pi^2)\,(2\sigma_2+\sigma_3)\,B_0^2+\frac{1}{2}\,\kappa\,\overline{Q}_1\\ \sigma_1\,B_0+\frac{1}{2}\,(1+\pi^2)\,(2\sigma_1+\sigma_2)\,B_0^2+\frac{1}{2}\,\kappa\,\overline{Q}_0\end{bmatrix}. \tag{69}$$

The corresponding Hessian matrix for this problem is the symmetric matrix

$$H = \begin{bmatrix} \frac{\partial^2 \chi^2}{\partial \Delta S^2} & \frac{\partial^2 \chi^2}{\partial \Delta N \, \partial \Delta S} \\ \frac{\partial^2 \chi^2}{\partial \Delta S \, \partial \Delta N} & \frac{\partial^2 \chi^2}{\partial \Delta N^2} \end{bmatrix} = \frac{2}{X_0^2} \begin{bmatrix} \sigma_2 & -\sigma_1 \\ -\sigma_1 & \sigma_0 \end{bmatrix} . \tag{70}$$

We can apply the second partial derivative test to the Hessian to determine whether we've found an extremum of χ^2. If the determinant of the Hessian is positive, and (in the case of a 2×2 matrix) one of the diagonal elements is positive, then we have found a minimum. The second condition is satisfied by $\sigma_0 = n_{\max} > 0$ when $n_{\max} \geq 1$. The determinant is given by

$$\mathrm{Det}(H) = \frac{4}{X_0^4} \left(n_{\max}\, g_2 - g_1^2 \right) , \tag{71}$$

which is indeed greater than 0 for $n_{\max} > 1$ since the quantity

$$\frac{n_{\max}\, g_2}{g_1^2} = \frac{n_{\max}(\gamma_1 - 1)(\gamma_{n_{\max}} + 1)}{(\gamma_1 + 1)(\gamma_{n_{\max}} - 1)} \approx 1 + \frac{\ln^2(2)}{12^3} \left(n_{\max}^2 - 1 \right) > 1 . \tag{72}$$

Therefore, we can be confident that the solution for ΔS and ΔN given by Eq. (68) minimizes the RMS frequency error provided that we are averaging over at least the first two frets.

The setback solution given by Eq. (68) is valid for a single string, and results like those shown in Table 3 and Fig. 6a assume that the guitar is built such that each string — from a particular set of strings — has a unique saddle and nut setback. Suppose that we'd prefer to engineer a guitar with single, uniform values of both ΔS and ΔN that provide reasonable compensation across an entire string set (or an ensemble of strings from a variety of manufacturers). In this case, Eq. (59) becomes

$$\overline{\Delta\nu}_{\mathrm{rms}} \equiv \sqrt{\frac{\sum_{m=1}^{m_{\max}} \sum_{n=1}^{n_{\max}} \left[\Delta\nu_n^{(m)} \right]^2}{m_{\max}\, n_{\max}}} , \tag{73}$$

where m labels the strings in the set, and Eq. (27) has been updated to become

$$\Delta\nu_n^{(m)} \approx \frac{1200}{\ln(2)} \left\{ (\gamma_n - 1) \left[B_0^{(m)} - \frac{\Delta S}{X_0} \right] + \frac{\Delta N}{X_0} + \frac{1}{2}\, \kappa^{(m)}\, Q_n \right\} . \tag{74}$$

If we rigorously follow the same approach that we used to arrive at Eq. (68), in the multi-string case we obtain

$$\begin{bmatrix} \Delta S \\ \Delta N \end{bmatrix} = \frac{1}{m_{\max}} \begin{bmatrix} \sum_{m=1}^{m_{\max}} \Delta S^{(m)} \\ \sum_{m=1}^{m_{\max}} \Delta N^{(m)} \end{bmatrix} , \tag{75}$$

where

$$\begin{bmatrix} \Delta S^{(m)} \\ \Delta N^{(m)} \end{bmatrix} = \frac{X_0}{\sigma_1^2 - \sigma_0\, \sigma_2} \begin{bmatrix} -\sigma_0 & \sigma_1 \\ -\sigma_1 & \sigma_2 \end{bmatrix} \begin{bmatrix} \sigma_2\, B_0^{(m)} + \frac{1}{2}\, \kappa^{(m)}\, \overline{Q}_1 \\ \sigma_1\, B_0^{(m)} + \frac{1}{2}\, \kappa^{(m)}\, \overline{Q}_0 \end{bmatrix} . \tag{76}$$

In other words, we can find the optimum values for both ΔS and ΔN simply by averaging the corresponding setbacks over a commercially interesting collection of string sets.

D Other Classical Guitar String Sets

D.1 Light Tension

Table 5: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

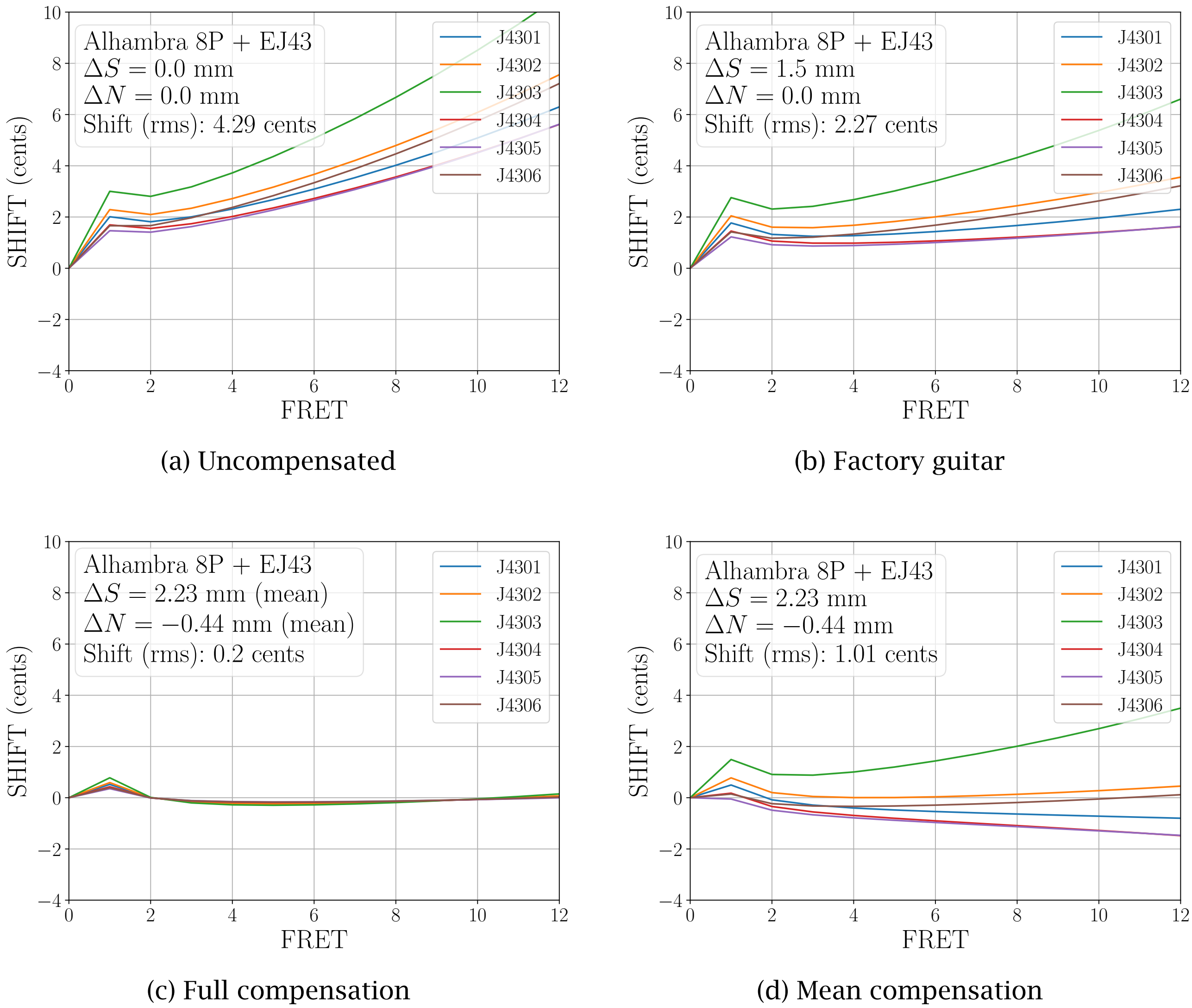
String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4301	E_4	0.349	3.62	66.4
J4302	B_3	0.403	4.87	50.2
J4303	G_3	0.504	8.08	52.5
J4304	D_3	0.356	18.23	66.4
J4305	A_2	0.419	27.41	56.1
J4306	E_2	0.533	51.59	59.2

Table 6: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

String	R	σ	κ	B_0	E (GPa)
J4301	37.8	0.5	76.6	0.00235	13.28
J4302	42.6	1.0	86.2	0.00287	8.50
J4303	55.0	0.4	111.1	0.00409	7.30
J4304	31.4	1.2	63.7	0.00218	10.65
J4305	26.1	0.5	53.2	0.00235	5.40
J4306	28.5	1.1	57.9	0.00312	3.84

Table 7: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43) on the Alhambra 8P classical guitar.

String	ΔS (mm)	ΔN (mm)	$\overline{\Delta\nu}_{\rm rms}$ (cents)
J4301	1.90	-0.45	0.20
J4302	2.31	-0.51	0.23
J4303	3.29	-0.65	0.29
J4304	1.73	-0.38	0.17
J4305	1.80	-0.31	0.15
J4306	2.37	-0.34	0.16



(a) Uncompensated

(b) Factory guitar

(c) Full compensation

(d) Mean compensation

Figure 10: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). Four different strategies of saddle and nut compensation are illustrated.

D.2 Hard Tension

Table 8: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). The corresponding scale length is 650 mm.

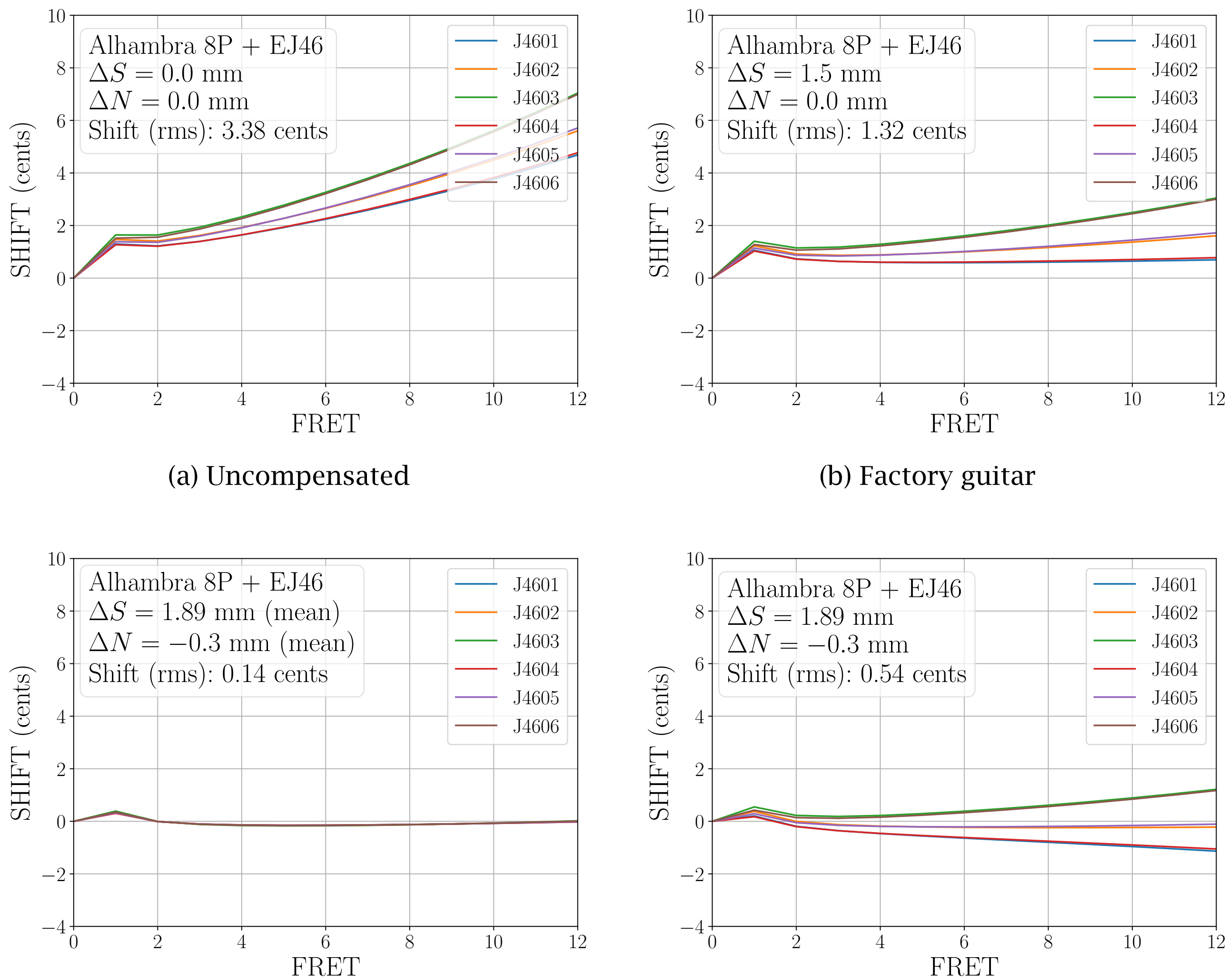
String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4601	E_4	0.362	3.86	70.9
J4602	B_3	0.415	5.22	53.8
J4603	G_3	0.521	8.57	55.6
J4604	D_3	0.381	20.07	73.1
J4605	A_2	0.457	34.87	71.3
J4606	E_2	0.559	56.67	65.0

Table 9: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46).

String	R	σ	κ	B_0	E (GPa)
J4601	23.5	0.5	47.9	0.00193	8.25
J4602	26.2	0.3	53.5	0.00234	5.31
J4603	28.3	1.0	57.5	0.00304	3.76
J4604	22.7	0.3	46.4	0.00200	7.44
J4605	24.0	0.2	49.0	0.00246	5.33
J4606	25.5	0.3	51.9	0.00310	3.44

Table 10: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46) on the Alhambra 8P classical guitar.

String	ΔS (mm)	ΔN (mm)	$\overline{\Delta\nu}_{\mathrm{rms}}$ (cents)
J4601	1.49	-0.28	0.14
J4602	1.79	-0.31	0.15
J4603	2.31	-0.34	0.16
J4604	1.53	-0.27	0.13
J4605	1.87	-0.29	0.14
J4606	2.33	-0.30	0.15



(a) Uncompensated

(b) Factory guitar

(c) Full compensation

(d) Mean compensation

Figure 11: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). Four different strategies of saddle and nut compensation are illustrated.

D.3 Extra Hard Tension

Table 11: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). The corresponding scale length is 650 mm.

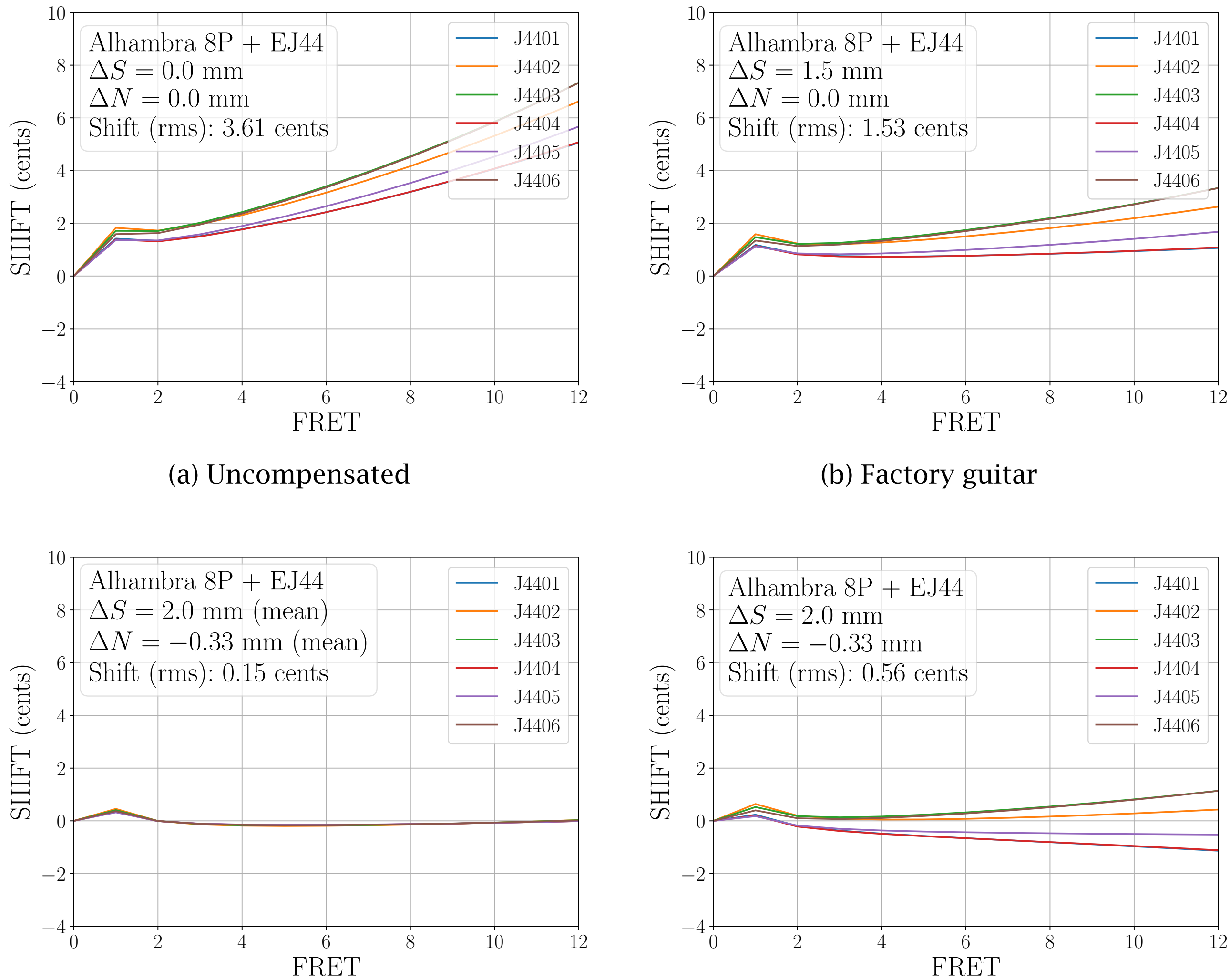
String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4401	E_4	0.368	4.01	73.6
J4402	B_3	0.423	5.44	56.1
J4403	G_3	0.528	8.91	57.9
J4404	D_3	0.381	20.07	73.1
J4405	A_2	0.457	34.87	71.3
J4406	E_2	0.571	61.36	70.4

Table 12: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). The corresponding scale length is 650 mm.

String	R	σ	κ	B_0	E (GPa)
J4401	25.8	0.4	52.6	0.00206	9.09
J4402	33.2	1.1	67.3	0.00267	6.72
J4403	29.6	0.6	60.2	0.00315	3.97
J4404	25.0	0.5	51.0	0.00209	8.17
J4405	23.7	0.2	48.5	0.00245	5.27
J4406	26.6	0.2	54.3	0.00324	3.73

Table 13: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44) on the Alhambra 8P classical guitar.

String	ΔS (mm)	ΔN (mm)	$\overline{\Delta\nu}_{\mathrm{rms}}$ (cents)
J4401	1.60	-0.31	0.15
J4402	2.09	-0.39	0.18
J4403	2.40	-0.35	0.17
J4404	1.61	-0.30	0.14
J4405	1.85	-0.28	0.14
J4406	2.44	-0.32	0.15



(a) Uncompensated

(b) Factory guitar

(c) Full compensation

(d) Mean compensation

Figure 12: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). Four different strategies of saddle and nut compensation are illustrated.

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