

# Classical Guitar Intonation and Compensation: The Well-Tempered Guitar

M. B. Anderson\* and R. G. Beausoleil†

*Rosewood Guitar*

*8402 Greenwood Ave. N, Seattle, WA 98103*

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## Abstract

Inspired by the pioneering work of luthier Greg Byers in 1996, we build an intuitive model of classical guitar intonation that includes the effects of the resonant length of the fretted string, linear mass density, tension, and bending stiffness. We begin by deriving an expression for the vibration frequencies of a stiff string using boundary conditions that are pinned at the saddle but clamped at the fret. Adopting logarithmic frequency differences based on “cents” that decouple these physical effects, we introduce Taylor series expansions that exhibit clearly the origins of frequency shifts of fretted notes from the corresponding Twelve-Tone Equal Temperament (12-TET) values. We demonstrate a simple *in situ* technique for measurement of the changes in frequency of open strings arising from small adjustments in length, and we propose a simple procedure that any interested guitarist can use to estimate the corresponding shifts in frequency due to tension and bending stiffness for their own guitars and favorite string sets. Based on these results, we employ an RMS frequency error method to select values of saddle and nut setbacks that map fretted frequencies — for a particular string set on a particular guitar — almost perfectly onto their 12-TET targets. This exercise allows us to discuss a general approach to tempering an “off-the-shelf” guitar to further reduce the tonal errors inherent in any fretted musical instrument.

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\*[anderson.gtr@gmail.com](mailto:anderson.gtr@gmail.com)

†[ray.beausoleil@hpe.com](mailto:ray.beausoleil@hpe.com)

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## 1 Introduction and Background

Any musician who has wrestled with the temperament of a fretted stringed instrument is well aware of the challenges presented by tuning and pitch. In addition to the mathematical physics of musical scales [1, 2, 3], the mechanical specifications of the instrument and the strings themselves [4, 5, 6] require accommodation during both manufacturing [7, 8] and tuning to achieve harmonious results. We can gain an appreciation for this problem by analyzing the expression for the allowed vibration frequencies of an ideal string, given by [9, 10]

$$f_q = \frac{q}{2L_0} \sqrt{\frac{T_0}{\mu_0}}, \quad (1)$$

where  $q \in \mathbb{N} = \{1, 2, \dots\}$  identifies the “harmonic” of the fundamental frequency  $f_1$ ,  $L_0$  is the length of the free (unfretted) string from the saddle to the nut,  $T_0$  is the tension in the free string, and  $\mu_0 \equiv M/L_0$  is the linear mass density of a free string of mass  $M$ . The act of fretting the string changes its length, and therefore its frequency. For example, modern classical guitars are manufactured with frets placed along the fretboard using the Twelve-Tone Equal Temperament (12-TET) system, whereby the resonant length of a string pressed behind fret  $n$  ideally should be  $2^{-n/12} L_0$ , thereby producing a note with frequency  $2^{n/12} f_1$ . But this result can never be achieved perfectly in reality.

First, the string is elevated above the frets by the saddle and nut, so the fretted string is slightly elongated relative to the free string, and the resulting frequency is flattened in pitch. In principle, this effect could be accommodated by minute changes in the positions of the frets, but there are additional practical complications. For example, the string’s tension is increased by the change in length, causing the frequency to sharpen by an amount that significantly exceeds the reduction caused by the increase in the resonant string length. In addition,

the string is by no means ideal, and its intrinsic stiffness results in an additional increase in pitch that depends on its mechanical characteristics. These guitar intonation difficulties seem to preclude successful temperament, but remarkably the instrument can be *compensated* by moving the positions of the saddle and the nut by small distances during the manufacturing process [7, 8]. This compensation process helps temper the guitar so that it is *playable*. It must also be *tunable*, so that the guitar strings can be brought into compliance with 12-TET quickly and accurately. This requirement places significant constraints on the physical properties of manufactured strings. Our goal in this work is to build an intuitive understanding of these effects to aid in the compensation and subsequent tuning of the classical guitar, with an accessible approach to making measurements of string properties and deducing the corresponding effects of playability [11, 12, 13].

Throughout this work, we will use *cents* to describe small differences in pitch [14, 15, 16, 17, 8, 3]. One cent is one one-hundredth of a 12-TET half-step, so that there are 1200 cents per octave. An experienced guitar player can distinguish beat notes with a difference frequency of  $\Delta f \approx 1$  Hz, which corresponds to 8 cents at  $A_3$  ( $f = 220$  Hz) or 5 cents at  $E_4$  ( $f = 329.63$  Hz). Using this approach, the difference in pitch between two frequencies  $f_1$  and  $f_2$  is defined as

$$\Delta v \equiv 1200 \log_2 \left( \frac{f_2}{f_1} \right), \quad (2)$$

where  $\log_2(x)$  is the logarithm base 2 of  $x$ . Let's choose the average frequency  $f \equiv (f_1 + f_2)/2$  and the frequency difference  $\Delta f \equiv f_2 - f_1$ . Then

$$\Delta v = 1200 \log_2 \left( \frac{f + \Delta f/2}{f - \Delta f/2} \right) \approx \frac{1200}{\ln(2)} \frac{\Delta f}{f} = 1731 \frac{\Delta f}{f}, \quad (3)$$

where the approximation applies when  $\Delta f \ll f$ , and  $\ln(x)$  is the natural logarithm function of  $x$ . As shown in Fig. 1, if the average frequency of the interval is used to compute  $\Delta v$  — rather than the initial frequency  $f_1$  — then the accuracy of Eq. (3) holds for almost an entire octave. In this plot, we chose  $f_1 = A_3 = 220$  Hz, and allowed  $f_2$  to vary from  $A_3$  to  $A_4 + 30$  Hz = 470 Hz. At the octave, the error in  $\Delta v$  arising from Eq. (3) is only  $-46$  cents, or  $-4\%$ . The choice of a logarithmic frequency-difference scale *decouples* multiplicative factors that predict the frequency of a real fretted string, allowing us to build straightforward intuitive models of guitar intonation and compensation.

We present the basics of our model of classical guitar strings in Section 2, following the pioneering work of G. Byers [7, 18]. We offer empirical reasons to doubt the need for a complicated model of string fretting that incorporates either the depth of fretting or the shape of the finger in Appendix A. Then, in Appendix B, we derive a new expression for the allowed vibration frequencies of a stiff string (although more complicated dynamical models are available [19]), derived under the assumption that the boundary conditions at the saddle and the nut are not symmetric. Based on this result, we then discuss in detail the four contributions to frequency shifts and errors of non-ideal strings pressed behind a fret: the change in the resonant length of the vibrating string; a decrease in the linear mass density of the entire string; an increase in the tension of the entire string; and an increase in the mechanical stiffness of the resonating string. Our goal is to simplify the decoupled equations describing these effects through Taylor series expansions to allow an intuitive picture of the string's behavior to emerge. Nylon strings behave very differently than the metal strings used on acoustic guitars [20, 21]. They require time to “settle” and reach equilibrium after restringing, and it is unlikely that the uniform stiff



Figure 1: Plot of  $\Delta\nu$  for  $f_1 = A_3 = 220$  Hz and  $f_2$  varying from  $A_3$  to  $A_4 + 30$  Hz = 450 Hz. We compare two different definitions of  $f$  in Eq. (3): the average of  $f_1$  and  $f_2$ , and simply  $f = f_1$ . Using the average frequency leads to a significantly better approximation.

rod approach used to develop equations for the resonant frequencies (including ours) apply to either the monofilament treble strings or the wound bass strings. Nevertheless, we are able to develop a phenomenological model of the mechanical characteristics of these strings that is consistent with measurements of frequency deviations on uncompensated classical guitars.

In Section 2.5, we collect all four of the effects mentioned above and develop a simple approximate expression for the total frequency shift of a fretted guitar string. We note that the two largest contributions to this deviation from 12-TET perfection are the increases in tension and bending stiffness, and that small changes in the positions of the nut and the saddle can largely compensate for these problems. In Section 3, we demonstrate a simple, straightforward experiment to measure *in situ* the response of a string’s fundamental frequency to a change in length (and therefore tension), and we rely on a subsequent phenomenological approach to determining the open-string bending stiffness. Then, in Section 4, we use these values for a normal-tension nylon string set (as well as other string sets in Appendix D) to demonstrate a straightforward analytic approach to compensating for the tone errors in a guitar string, relying on a method — described in Appendix C — to minimize the root-mean-squared (RMS) frequency deviation at each fret. With these results in hand, in Section 5 we discuss a collaboration of guitar manufacturer and musician to temper the guitar using harmonic tuning to optimize it for a particular piece. Finally, in Section 6, we recapitulate our “recipes” for guitar compensation,

including a simple method to include the effects of relief for large-amplitude string vibration.

This document — as well as the Python computer code needed to reproduce the figures — is available at GitHub [22].

## 2 Simple Model of Guitar Intonation

The starting point for prior efforts to understand guitar intonation and compensation [7, 8] is a formula for  $f_q$ , the transverse vibration frequency harmonic  $q$  of a stiff string, originally published by Morse in 1936 [23, 24, 25]:

$$f_q = \frac{q}{2L} \sqrt{\frac{T}{\mu}} \left[ 1 + 2B + 4 \left( 1 + \frac{\pi^2 q^2}{8} \right) B^2 \right]. \quad (4)$$

Here  $L$  is the length of the string,  $T$  and  $\mu$  are its tension and linear mass density, respectively, and  $B$  is a small “bending stiffness” coefficient to capture the relevant mechanical properties of the string. For a homogeneous string with a cylindrical cross-section,  $B$  is given by

$$B \equiv \sqrt{\frac{\pi \rho^4 E}{4 T L^2}}, \quad (5)$$

where  $\rho$  is the radius of the string and  $E$  is Young’s modulus (or the modulus of elasticity). But it’s unlikely that Eq. (4) accurately describes the resonant frequencies of a nylon string on a classical guitar, because it assumes that the string is “clamped” at both ends, so that a particular set of symmetric boundary conditions must be applied to the partial differential equation (PDE) describing transverse vibrations of the string. We believe that this assumption is correct for the end of the string held at either the nut or the fret, but that the string is “pinned” (and not clamped) at the saddle. In Appendix B, we solve the PDE using these non-symmetric boundary conditions, and find

$$f_q = \frac{q}{2L} \sqrt{\frac{T}{\mu}} \left[ 1 + B + \left( 1 + \frac{1}{2} q^2 \pi^2 \right) B^2 \right]. \quad (6)$$

Note that this expression is valid only when  $B \ll 1$ . For a typical nylon guitar string with  $E \approx 5$  GPa,  $T \approx 60$  N,  $\rho \approx 0.5$  mm, and  $L \approx 650$  mm, we have  $B \approx 3 \times 10^{-3}$ . (In this case, the quadratic  $B$  term in Eq. (6) is only 2% as large as the linear term, and can generally be neglected. We will include it in our analysis below only for completeness.) We should use Eq. (6) with some caution, because the chemistry, materials science, and physics of nylon strings (particularly the wound bass strings) are quite complicated [20].

Our model is based on the schematic of the guitar shown in Fig. 2. The scale length of the guitar is  $X_0$ , but we allow the inside edges of both the saddle and the nut to be set back an additional distance  $\Delta S$  and  $\Delta N$ , respectively. The location on the  $x$ -axis of the center of the  $n^{\text{th}}$  fret is  $X_n$ . In the  $y$  direction,  $y = 0$  is taken as the surface of the fingerboard; the height of each fret is  $a$ , the height of the nut is  $a + b$ , and the height of the saddle is  $a + b + c$ . (For the time being, we are neglecting the art of *relief* practiced by expert luthiers that adjusts the value of  $b$  up the fretboard and strings. We discuss this effect in Section 4.)  $L_n$  is the *resonant length* of the string from the saddle to the center of fret  $n$ , and  $L'_n$  is the length of the string from the fret to the nut. The total length of the string is defined as  $\mathcal{L}_n \equiv L_n + L'_n$ . As discussed in more detail in Appendix A, we have chosen to include a line-segment intersection



Figure 2: A simple (side-view) schematic of the classical guitar used in this model. The scale length of the guitar is  $X_0$ , but we allow the edges of both the saddle and the nut to be set back an additional distance  $\Delta S$  and  $\Delta N$ , respectively. The location on the  $x$ -axis of the center of the  $n^{\text{th}}$  fret is  $X_n$ . (Note that the  $x$ -axis is directed toward the left in this figure.) In the  $y$  direction,  $y = 0$  is taken as the surface of the fingerboard; therefore the height of each fret above the fingerboard is  $a$ , the height of the nut is  $a + b$ , and the height of the saddle is  $a + b + c$ .  $L_n$  is the *resonant length* of the string from the saddle to the center of fret  $n$ , and  $L'_n$  is the length of the string from the fret to the nut. We have included a line-segment intersection at a distance  $d$  behind fret  $n$  to represent the slight increase in the distance  $L'_n$  caused by a finger.

at a distance  $d$  behind fret  $n$  to represent the slight increase in the distance  $L'_n$  caused by a finger. This differs from previous studies of guitar intonation and compensation [7, 8], but our approach is consistent with empirical observations for nylon strings.

We start with the form of the fundamental frequency of a fretted string given by Eq. (6) with  $q = 1$ , and apply it to the frequency of a string pressed just behind the  $n^{\text{th}}$  fret:

$$f_n = \frac{1}{2L_n} \sqrt{\frac{T_n}{\mu_n}} \left[ 1 + B_n + \left( 1 + \frac{\pi^2}{2} \right) B_n^2 \right], \quad (7)$$

where  $T_n$  and  $\mu_n$  are the modified tension and the linear mass density of the fretted string, and

$$B_n \equiv \sqrt{\frac{\pi \rho^4 E}{4 T_n L_n^2}}. \quad (8)$$

We note that  $T_n$  and  $\mu_n$  depend on  $L_n$ , the *total* length of the fretted string from the saddle to the nut. Ideally, in the 12-TET system [3],

$$f_n = \gamma_n f_0, \quad (12\text{-TET ideal}) \quad (9)$$

where  $f_0$  is the frequency of the open (unfretted) string, and

$$\gamma_n \equiv 2^{n/12}. \quad (10)$$

Therefore, the error interval — the difference between the fundamental frequency of the fretted string and the corresponding perfect 12-TET frequency — expressed in cents is given by

$$\begin{aligned} \Delta \nu_n &= 1200 \log_2 \left( \frac{f_n}{\gamma_n f_0} \right) \\ &= 1200 \log_2 \left( \frac{L_0}{\gamma_n L_n} \right) + 600 \log_2 \left( \frac{\mu_0}{\mu_n} \right) + 600 \log_2 \left( \frac{T_n}{T_0} \right) \\ &\quad + 1200 \log_2 \left[ \frac{1 + B_n + (1 + \pi^2/2) B_n^2}{1 + B_0 + (1 + \pi^2/2) B_0^2} \right], \end{aligned} \quad (11)$$

Table 1: Default specifications for the prototypical Classical Guitar modeled in this section. The default values of  $d$ ,  $\Delta S$ , and  $\Delta N$  can be either zero or the nonzero value listed in the table and discussed in the text.

| Parameter  | Description                        | Default Value (mm) |
|------------|------------------------------------|--------------------|
| $X_0$      | Scale length                       | 650                |
| $b$        | Height of the nut above fret 1     | 1                  |
| $c$        | Height of the saddle above the nut | 4                  |
| $d$        | Fretting distance                  | 0 or 10            |
| $\Delta S$ | Saddle setback                     | 0 or 1.9           |
| $\Delta N$ | Nut setback                        | 0 or $-0.37$       |

where  $\log_2$  is the (binary) logarithm function calculated with base 2.

The final form of Eq. (11) makes it clear that — for nylon guitar strings — there are four contributions to intonation:

1. *Resonant Length*: The first term represents the error caused by the increase in the length of the fretted string  $L_n$  compared to the ideal length  $X_n$ , which would be obtained if  $b = c = d = 0$  and  $\Delta S = \Delta N = 0$ .
2. *Linear Mass Density*: The second term is the error caused by the reduction of the linear mass density of the fretted string. This effect will depend on the *total* length of the string  $\mathcal{L}_n = L_n + L'_n$ .
3. *Tension*: The third term is the error caused by the *increase* of the tension in the string arising from the stress and strain applied to the string by fretting. This effect will also depend on the total length of the string  $\mathcal{L}_n$ .
4. *Bending Stiffness*: The fourth and final term is the error caused by the change in the bending stiffness coefficient arising from the decrease in the vibrating length of the string from  $L_0$  to  $L_n$ .

Note that the properties of the logarithm function have *decoupled* these physical effects by converting multiplication into addition. We will discuss each of these sources of error in turn below.

In the discussion that follows, we'll test our approximations for a prototypical Classical Guitar with the specifications listed in Table 1. Refer to Fig. 2 for a graphical representation of these parameters. In addition, as we develop models of the physical effects discussed above, we'll assume that the guitar string has the properties listed in Table 2. The string constant  $\kappa$  and the open-string bending stiffness  $B_0$  are introduced in Section 2.3 and Section 2.4 respectively. The linear frequency shift parameter  $R$  is discussed in Section 2.5, and a method for determining both  $\kappa$  and  $B_0$  in terms of  $R$  is discussed.

## 2.1 Resonant Length

The length  $L_0$  of the open (unfretted) guitar string can be calculated quickly by referring to Fig. 2. We find:

$$L_0 = \sqrt{(X_0 + \Delta S + \Delta N)^2 + c^2} \approx X_0 + \Delta S + \Delta N + \frac{c^2}{2X_0}, \quad (12)$$



Table 2: Default specifications for a prototypical guitar string. The string constant  $\kappa$  and the open-string bending stiffness  $B_0$  are introduced in Section 2.3 and Section 2.4 respectively, and the linear frequency shift parameter  $R$  is discussed in Section 2.5.

| Parameter | Description                      | Default Value |
|-----------|----------------------------------|---------------|
| $\rho$    | String radius in mm              | 0.43          |
| $R$       | Linear frequency shift parameter | 25            |
| $\kappa$  | String tension constant          | 51            |
| $B_0$     | Open-string bending stiffness    | 0.00236       |

where the approximation arising from the Taylor series is valid to second order in all small distances since  $\{\Delta S, \Delta N, b, c\} \ll X_0^2$ . Similarly, the resonant length  $L_n$  is given by

$$L_n = \sqrt{(X_n + \Delta S)^2 + (b + c)^2} \approx X_n + \Delta S + \frac{(b + c)^2}{2 X_n}. \quad (13)$$

Then — if the guitar has been manufactured such that  $X_n = X_0/y_n$  — the resonant length error determined by the first term in the last line of Eq. (11) is approximately

$$1200 \log_2 \left( \frac{L_0}{y_n L_n} \right) \approx \frac{1200}{\ln(2)} \left[ \frac{\Delta N - (y_n - 1) \Delta S}{X_0} - \frac{(\Delta N + \Delta S)^2 - y_n^2 \Delta S^2}{2 X_0^2} - \frac{y_n^2 (b + c)^2 - c^2}{2 X_0^2} \right]. \quad (14)$$

If the guitar is uncompensated, so that  $\Delta S = \Delta N = 0$ , the magnitude of this error on our Classical Guitar can be neglected. However, we'll see that choosing  $\Delta S > 0$  and  $\Delta N < 0$  will allow us to substantially compensate for frequency shift contributions from other effects. In this case, we note that the three terms inside the bracket on the right-hand side of Eq. (14) are  $\{-3.5 \times 10^{-3}, 1.4 \times 10^{-5}, -1.0 \times 10^{-4}\}$ , respectively, for the parameter values given by Table 1, corresponding to frequency shifts of  $\{-6.05, 0.02, -0.17\}$  cents. Therefore, the second two terms are negligible compared to the first, and we can approximate the resonant length error — for the purposes of estimating setbacks — as

$$1200 \log_2 \left( \frac{L_0}{y_n L_n} \right) \approx \frac{1200}{\ln(2)} \left[ \frac{\Delta N - (y_n - 1) \Delta S}{X_0} \right]. \quad (15)$$

## 2.2 Linear Mass Density

As discussed above, the linear mass density  $\mu_0$  of an open (unfretted) string is simply the total mass  $M$  of the string clamped between the saddle and the nut divided by the length  $L_0$ . Similarly, the mass density  $\mu_n$  of a string held onto fret  $N$  is  $M/\mathcal{L}_n$ . Therefore

$$\frac{\mu_0}{\mu_n} = \frac{\mathcal{L}_n}{L_0} \equiv 1 + Q_n, \quad (16)$$

where we have followed Byers and defined the normalized relative displacement [7, 8]

$$Q_n \equiv \frac{\mathcal{L}_n - L_0}{L_0}, \quad (17)$$



where  $\mathcal{L}_n = L_n + L'_n$ . After judicious use of similar triangles and the Pythagorean Theorem we calculate  $L'_n$  for  $n \geq 1$  as

$$L'_n = \frac{L_n}{X_n + \Delta S} d + \sqrt{(X_0 - X_n + \Delta N - d)^2 + \left(b + \frac{b+c}{X_n + \Delta S} d\right)^2}. \quad (18)$$

When  $d \ll X_0$ , we can expand  $Q_n$  to third order in all small distances and find

$$Q_n \approx \frac{[\gamma_n b + (\gamma_n - 1) c]^2}{2 (\gamma_n - 1) X_0^2} \left(1 + \frac{\gamma_n^2}{\gamma_n - 1} \frac{d}{X_0}\right), \quad (19)$$

Although it is arguable whether this approximation is simpler than the exact expression given by Eq. (17), it is quite clear that  $Q_n$  does not depend significantly on the setbacks  $\Delta S$  or  $\Delta N$ .

Many luthiers provide “relief” to enlarge the effective height of the string (particularly for the wound bass strings) as the fret number grows to provide clearance for vibration amplitude at higher volume. In practice, this is accomplished by pivoting the fret board shown in Fig. 2 clockwise about  $x = X_0$ , increasing the height of the string above fret  $n$  by an amount

$$\begin{aligned} \Delta \gamma_n &= m (X_0 - X_n) \\ &= \frac{\gamma_n - 1}{\gamma_n} m X_0 \\ &= \frac{\gamma_n - 1}{\gamma_n} 2 \Delta \gamma_{12}, \end{aligned} \quad (20)$$

where  $\Delta \gamma_{12}$  is the relief at the twelfth fret and  $m = 2 \Delta \gamma_{12} / X_0 \geq 0$  is the downward slope of the fret board. If we update Eq. (13) and Eq. (18) (with  $d = 0$ ), then we obtain

$$L_n = \sqrt{(X_n + \Delta S)^2 + (b + \Delta \gamma_n + c)^2}, \text{ and} \quad (21a)$$

$$L'_n = \sqrt{(X_0 - X_n + \Delta N)^2 + (b + \Delta \gamma_n)^2}. \quad (21b)$$

These equations indicate that we could modify the approximation for  $Q_n$  given by Eq. (19) by replacing  $b \rightarrow b + \Delta \gamma_n$ , which results in the numerator

$$\gamma_n b + (\gamma_n - 1) c \rightarrow \gamma_n b + (\gamma_n - 1) (c + 2 \Delta \gamma_{12}), \quad (22)$$

indicating that the intuitive substitution  $c \rightarrow c + 2 \Delta \gamma_{12}$  captures the effect of relief. (Note that this should *not* be done when computing the length  $L_0$  of the open string!)

In Fig. 3, we plot a comparison between the exact expression for the normalized displacement  $Q_n$  given by Eq. (17) with the approximate expression given by Eq. (19) for two values of  $d$ . Here the guitar has the specifications listed in Table 1: the exact curves use  $\Delta S = 1.9$  mm and  $\Delta N = -0.37$  mm, and the approximate curves ignore the setbacks entirely. The slight difference between the exact and approximate values of  $Q_n$  for  $d = 10$  mm at the first fret can be eliminated if we include a term quadratic in  $d$  in Eq. (19). As predicted above, we see that the dependence of  $Q_n$  on the setback values is minimal. We have scaled the relative displacement by a factor of  $10^6$  for convenience. For example, at the first fret with  $d = 10$  mm,  $Q_1 \approx 47 \times 10^{-6}$ . This value corresponds to a net stretch of the string at the first fret of  $Q_1 L_0 = 0.03$  mm. Ignoring the setbacks at the 12<sup>th</sup> fret increases  $Q_{12}$  by 1%, a completely negligible amount. For the same parameters, when  $d = 10$  mm,  $\Delta v_1 \approx 0.04$  cents, and is smaller at all other frets. Therefore, in general the shift due to linear mass density can be neglected without significant loss of accuracy.

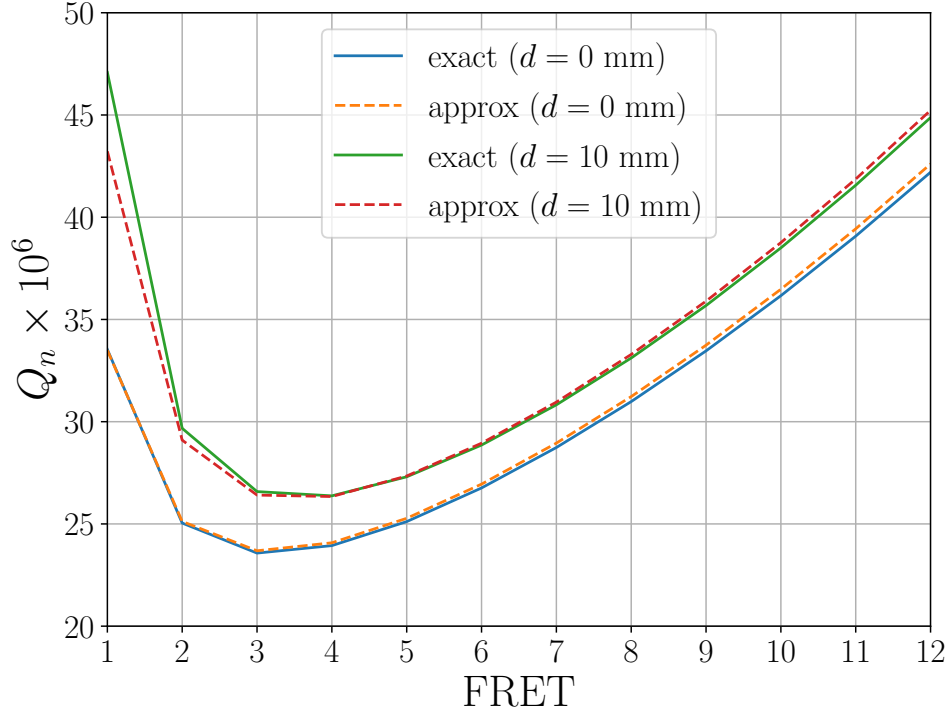


Figure 3: Comparison of the exact expression for the normalized displacement  $Q_n$  given by Eq. (17) with the approximate expression given by Eq. (19).

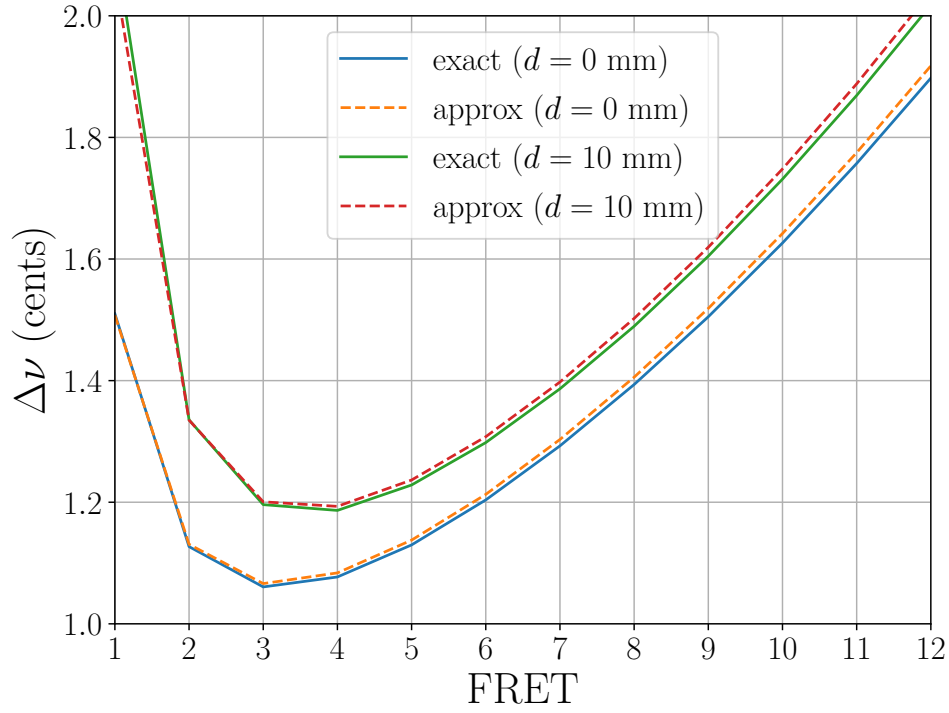


Figure 4: Comparison of the exact expression for the frequency shift due to tension increases given by the left-hand side of Eq. (26) with the approximate expression given by the right-hand side.

## 2.3 Tension

We've seen above that as a guitar string is fretted its length increases, thereby increasing its tension through linear elastic deformation [26]. We will focus on the response of the string to a longitudinal strain, and neglect the transverse stress that causes negligible changes in the radius of the string [20]. In this case, we can write the change in tension of a string experiencing an infinitesimal change in length from  $L$  to  $\mathcal{L}$  as

$$\Delta T = \pi \rho^2 E \frac{\mathcal{L} - L}{L}. \quad (23)$$

Therefore, the tension in a string clamped to fret  $n$  is

$$T_n = T_0 + \Delta T_n = T_0 (1 + \kappa Q_n), \quad (24)$$

where we have used Eq. (17) and defined the dimensionless linear “string constant”

$$\kappa \equiv \frac{\pi \rho^2 E}{T_0}. \quad (25)$$

If we assume that  $\kappa Q_n \ll 1$ , then we can approximate the third term in the final line of Eq. (11) as

$$600 \log_2 \left( \frac{T_n}{T_0} \right) \approx \frac{600}{\ln(2)} \kappa Q_n. \quad (26)$$

This frequency shift is larger than that caused by the linear mass density by a factor of  $\kappa$ .

In Fig. 4, we compare the exact expression for the frequency shift due to tension increases as a function of the fret number given by the left-hand side of Eq. (26) with the approximate expression given by the right-hand side. The exact curves for both  $d = 0$  mm and  $d = 10$  mm used the corresponding exact expressions for  $\mathcal{L}_n$  and  $L_0$ , while the approximate expressions relied only on Eq. (19). Here we have chosen  $\kappa = 51$  from Table 2 and the guitar has the same parameters as in Fig. 3.

## 2.4 Bending Stiffness

The bending stiffness of a string clamped at the  $n^{\text{th}}$  fret is given by Eq. (8), Eq. (13), and Eq. (24) as

$$B_n = \sqrt{\frac{\pi \rho^4 E}{4 T_n L_n^2}} = \sqrt{1 + \kappa Q_n} \frac{L_0}{L_n} \sqrt{\frac{\pi \rho^4 E}{4 T_0 L_0^2}} \approx \gamma_n B_0, \quad (27)$$

where the approximation applies when  $B_0 \ll 1$  and the largest contribution arises from the shortened length of the fretted string compared to that of the open string. This expression confirms our intuitive expectation that the stiffness of the string should increase as the length becomes shorter. Therefore, the fourth term in the final line of Eq. (11) can be approximated as

$$1200 \log_2 \left[ \frac{1 + B_n + (1 + \pi^2/2) B_n^2}{1 + B_0 + (1 + \pi^2/2) B_0^2} \right] \approx \frac{1200}{\ln(2)} \left[ (\gamma_n - 1) B_0 + \frac{1}{2} (\gamma_n^2 - 1) (1 + \pi^2) B_0^2 \right]. \quad (28)$$

In Fig. 5, we use Eq. (27) and Eq. (28) to compare the exact and approximate expressions for bending stiffness and the corresponding frequency shift based on Table 1 and Table 2. Note that we show the approximate frequencies with and without the quadratic terms, and we see

that the 2<sup>nd</sup>-order contribution is about 0.2 cents at the 12<sup>th</sup> fret. Once again, it is clear that  $B_n$  does not depend significantly on either  $\Delta S$  or  $\Delta N$ . In other words, the bending stiffness error does not depend on the tiny changes to the linear mass density or the tension that arise due to string fretting. Instead, it is an intrinsic mechanical property of the string: the stiffness increases as the length of the vibrating string becomes shorter.

## 2.5 Total Frequency Shift

Let's guide our intuition and prepare for the development of approximate expressions for  $\Delta S$  and  $\Delta N$  by relying on Taylor series expansions for all the effects described above. First, we'll ignore all quadratic terms in the resonant length error and adopt Eq. (15). Next, we'll neglect the small reduction in linear mass density caused by fretting, and then rely on the approximation to the frequency shift caused by tension increases given by Eq. (26). Finally, we'll describe the effects of bending stiffness using Eq. (28), *neglecting* the term proportional to  $B_0^2$ . Incorporating all of these terms, we find that the total frequency shift is given approximately by

$$\Delta \nu_n \approx \frac{1200}{\ln(2)} \left[ (\gamma_n - 1) \left( B_0 - \frac{\Delta S}{X_0} \right) + \frac{\Delta N}{X_0} + \frac{1}{2} \kappa Q_n \right]. \quad (29)$$

But how do we determine the bending stiffness  $B_0$  given by Eq. (5) and the spring constant  $\kappa$  given by Eq. (25) for a particular string?

It's impractical to assume that the modulus of elasticity of a particular string can be derived from published values of bulk nylon (particularly in the case of a wound string). Instead, let's assume that we know the value of  $\kappa$ , and then estimate the bending stiffness coefficient by comparing Eq. (5) and Eq. (25), and writing  $B_0$  as

$$B_0 = \sqrt{\kappa} \frac{\rho}{2 L_0} \approx \sqrt{\kappa} \frac{\rho}{2 X_0}. \quad (30)$$

To measure  $\kappa$ , in Section 3 we will conduct an experiment that measures the change in the frequency of an open string as we make slight changes to its length [7, 8]. From Eq. (6), the derivative of the fundamental frequency of an open string is

$$\begin{aligned} \frac{df}{dL} &= \frac{f}{L} \left( -1 + \frac{L}{2T} \frac{dT}{dL} - \frac{L}{2\mu} \frac{d\mu}{dL} + \frac{L}{1+B} \frac{dB}{dL} \right) \\ &= \frac{f}{L} \left( -1 + \frac{1}{2} \kappa + \frac{1}{2} - \frac{B}{1+B} \right) \\ &\approx \frac{f}{L} \times \frac{1}{2} (\kappa - 1), \end{aligned} \quad (31)$$

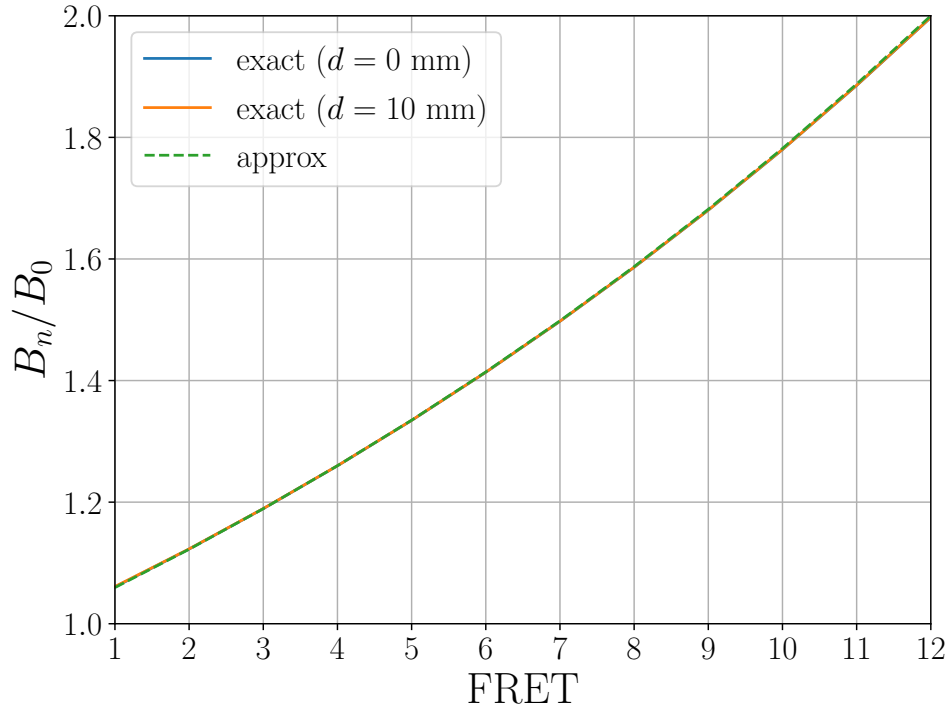
where we have used the analyses above to determine that

$$\frac{dT}{dL} = \frac{T}{L} \kappa, \quad (32a)$$

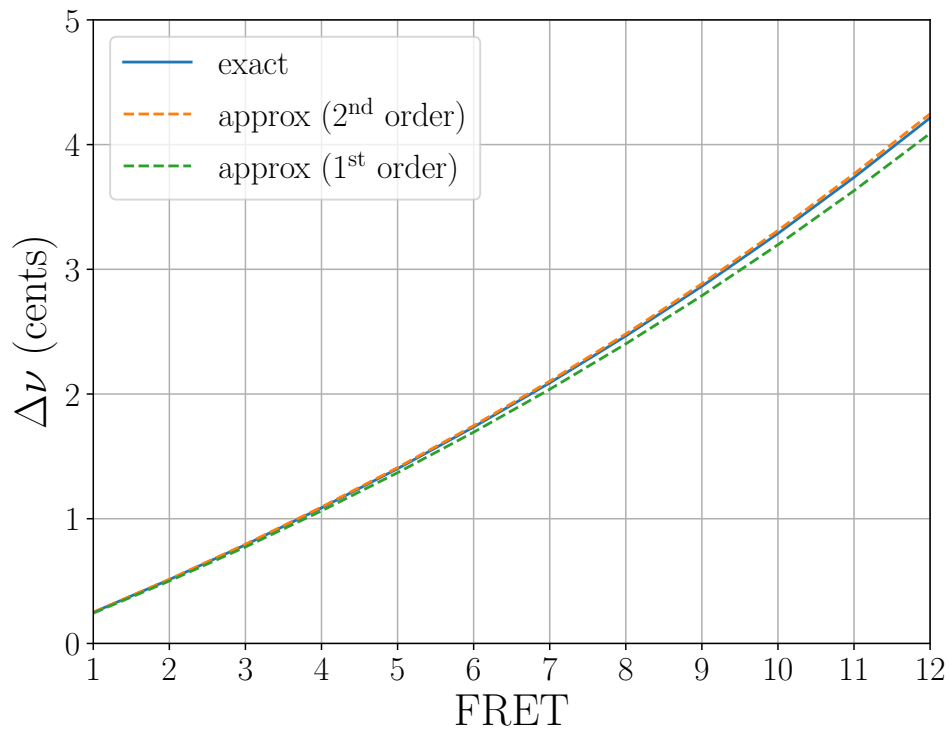
$$\frac{d\mu}{dL} = -\frac{\mu}{L}, \text{ and} \quad (32b)$$

$$\frac{dB}{dL} = -\frac{B}{L}, \quad (32c)$$

$$(32d)$$



(a) Bending stiffness



(b) Frequency shifts due to stiffness

Figure 5: A comparison of exact and approximate expressions for (a) bending stiffness, given by Eq. (27), and (b) the corresponding frequency shift, given by Eq. (28).

Table 3: String specifications for the D’Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 650 mm.

| String | Note           | Radius (mm) | Density (mg/mm) | Tension (N) |
|--------|----------------|-------------|-----------------|-------------|
| J4501  | E <sub>4</sub> | 0.356       | 0.374           | 68.6        |
| J4502  | B <sub>3</sub> | 0.409       | 0.505           | 52.0        |
| J4503  | G <sub>3</sub> | 0.512       | 0.836           | 54.2        |
| J4504  | D <sub>3</sub> | 0.368       | 1.920           | 70.0        |
| J4505  | A <sub>2</sub> | 0.445       | 3.289           | 67.3        |
| J4506  | E <sub>2</sub> | 0.546       | 5.470           | 62.8        |

and we have again assumed that  $B_0 \ll 1$ . Therefore, following Byers [7, 8], we define the parameter  $R$  to be

$$R \equiv \frac{L}{f} \frac{df}{dL} = \frac{1}{2} (\kappa - 1), \quad (33)$$

which gives

$$\kappa = 2R + 1. \quad (34)$$

We can anticipate the typical value of  $R$  for a nylon classical guitar string through a simple observation. On a classical guitar with a scale length of 650 mm, we can usually tune an open string a full step by winding the tuner/machine head three half turns. As we shall see below, this increases or decreases the string length above the first fret by about 3 mm, where the corresponding length of the open string from the saddle to the first fret is about 614 mm. Since a full step is (by definition) 200 cents, Eq. (3) tells us that

$$\frac{\Delta f}{f} \approx \frac{\ln(2)}{1200} \Delta \nu = \frac{200}{1731} = 0.116. \quad (35)$$

In this case, we estimate  $R$  to be

$$R \approx \frac{614}{3} 0.116 = 24, \quad (36)$$

giving  $\kappa \approx 49$ , close to the values listed in Table 2. The corresponding open-string bending stiffness is computed using Eq. (30) with the scale length given in Table 1. In Fig. 6, we compare the total frequency shifts predicted by Eq. (11) and Eq. (29) for this string on the Classical Guitar with  $\Delta S = \Delta N = 0$  mm at two different values of  $d$ . Note that the string is sharp at every fret, but even a large nonzero value of  $d$  is only important at the first fret. The bending stiffness is negligible at the first fret, but accounts for 65% of the shift at the 12<sup>th</sup> fret. The close agreement between the exact and approximate expressions for the frequency shifts gives us confidence that the equations we derive for the setbacks in Section 4 will be useful.

### 3 Experimental Estimate of the String Constant

It is relatively easy to measure *in situ* the value of  $R$  (and therefore  $\kappa$  and  $B_0$ ) for any guitar string with the aid of a device that can measure frequency [27], a simple ruler with fine markings (e.g., a string depth gauge), a magnifying glass or camera with a macro mode, and white correction fluid. For example, in Fig. 7 we show photographs of the nylon normal-tension first string

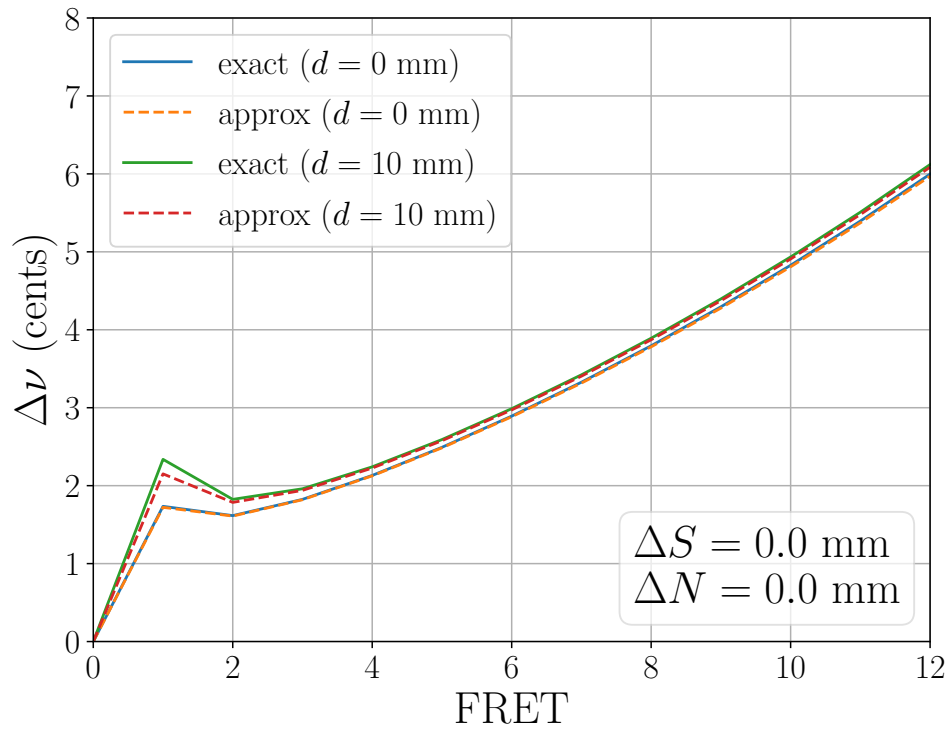
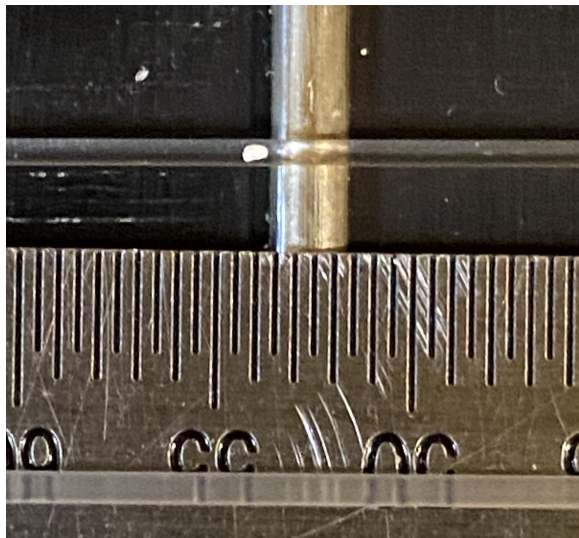
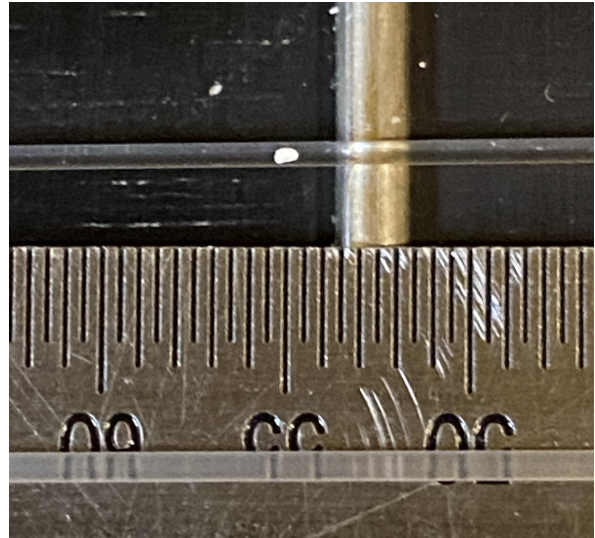


Figure 6: The total frequency shifts predicted by Eq. (11) and Eq. (29) for the Classical Guitar specified by Table 1 with  $\Delta S = \Delta N = 0$  mm, and a string with the parameters listed in Table 2.



(a)  $\Delta x_1$



(b)  $\Delta x_2$

Figure 7: Two examples of displacement measurements of a small deposit of white correction fluid relative to a D'Addario string-depth gauge marked in half-millimeter increments.



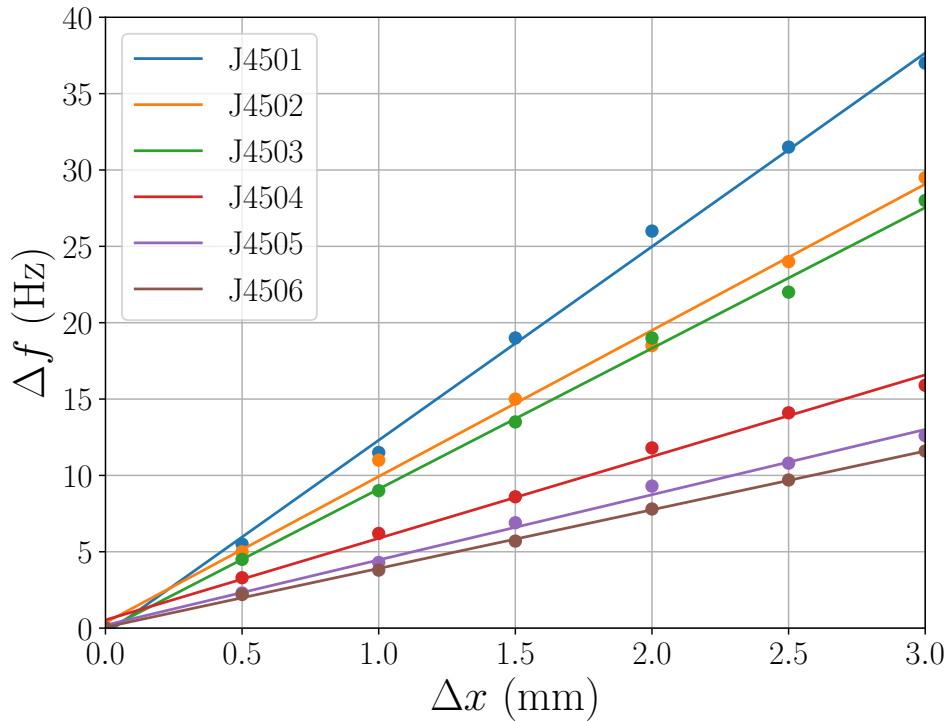


Figure 8: Results of experiments to measure  $R$  for each string in the D’Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45) set. The points represent the measurement data, while the lines are the results of linear least-squares fits to that data.

on an Alhambra 8P classical guitar. By depositing a small sample of white correction fluid on the string, we can measure small displacements within 0.125 mm against a string-depth gauge marked in half-millimeter increments. Then we can pluck the open string and measure its vibration frequency. We found that significantly stretching a string that had settled into equilibrium could result in a nonlinear reduction in  $\Delta f / \Delta L$ , so prior to our measurements we tuned each string down one whole step. The string stretches uniformly along its length, so at any position  $x$  the relative displacement  $\Delta x / x$  should be invariant. We chose to work near the first fret, which is located about 614 mm from the saddle, and we typically made seven measurements of displacement over a 3 mm range, as well as the corresponding frequencies.

For example, we began with a normal-tension nylon classical string set [28] with the specifications listed in Table 3 using metric units.<sup>1</sup> In Fig. 8, we plot our measurements of  $\Delta f$  as a function of the displacement  $\Delta x$  relative to the frequency of the string when  $\Delta x = 0$ . We then performed a least-squares fit to a straight line [29] (also shown in Fig. 8), determined the derivative  $\Delta f / \Delta L$ , and then computed  $R$  using Eq. (33) with  $L = 614$  mm and  $f$  defined as the average frequency over the range. The results are shown in Table 4. Here  $\sigma$  is the covariant (diagonal) uncertainty in  $R$  (so that, for example, the first string in the table has  $R = 23.6 \pm 0.5$ ), and  $\kappa = 2R + 1$ . We also estimate an effective modulus of elasticity  $E$  from Eq. (25), expressed in units of gigapascals (1 GPa =  $10^9$  N/m<sup>2</sup>). Similar measurements and results for other string sets are provided in Appendix D. Note — as predicted in Section 2.5 and shown in Fig. 9 — the expectation that the guitar will be *tunable* results in  $R$  values of manufactured strings that are

<sup>1</sup>Note that the correct unit of force in the metric system is Newtons (N), rather than kilograms, which is a unit of mass. In the British Imperial measurement system, the common units of mass are known as the “slug” and the “blob.”

Table 4: Derived physical properties of the D’Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 650 mm.

| String | $R$  | $\sigma$ | $\kappa$ | $B_0$   | $E$ (GPa) |
|--------|------|----------|----------|---------|-----------|
| J4501  | 23.6 | 0.5      | 48.2     | 0.00190 | 8.32      |
| J4502  | 23.8 | 0.7      | 48.6     | 0.00219 | 4.81      |
| J4503  | 28.8 | 0.7      | 58.7     | 0.00302 | 3.87      |
| J4504  | 22.4 | 0.8      | 45.7     | 0.00192 | 7.51      |
| J4505  | 23.8 | 0.8      | 48.6     | 0.00238 | 5.27      |
| J4506  | 28.6 | 0.4      | 58.2     | 0.00321 | 3.90      |

in the range 20 – 30. The still more important requirement that the guitar be *playable* leads us to the discussion of compensation in the next section.

Adopting these physical properties of the normal string set and applying them to a computation of the frequency deviations for our standard classical guitar, we obtain the predictions shown in Fig. 10 using Eq. (11). Anticipating our treatment of exact compensation in Section 4 and Appendix C, we compute the root-mean-squared (RMS) average of the frequency deviations for each string. This mean (over the first 12 frets) can be computed by squaring the frequency deviations shown in Fig. 10, averaging those values, and then taking the square root of the result.

Recall that we recalculated the expected frequency shift of a classical guitar string with asymmetric boundary conditions in Appendix B, and found an expression for  $f_q$  given by Eq. (56) that reduces — by about a factor of 2 — the impact of the bending stiffness relative to the symmetric (clamped) case in Eq. (4). Furthermore, we have not used more sophisticated techniques to calculate the bending stiffness for either monofilament nylon strings or wound nylon strings, opting instead for the phenomenological model given by Eq. (30). But is this approach valid? As a test, we compared our frequency shift estimates based on Section 2 with experimental measurements made using five different guitars

result in very large frequency deviations at all frets. As shown in Fig. ??, we measured the shifts at the first fret and found that they fell into the range of 4.5 – 5.75 cents, consistent with our predictions. At the 12<sup>th</sup> fret, we measured  $\Delta f = 18.5 - 19.5$  cents for the third and sixth strings, and  $\Delta f = 15.75 - 17.5$  cents for the other strings, in reasonably close agreement with our predictions. By contrast, the corresponding equation arising from symmetric (clamped) boundary conditions — Eq. (4) — predicts shifts that are 30 – 45% higher, and setbacks that are more than a factor of two larger. We conclude that Eq. (56) and Eq. (30) can be used to reliably predict setbacks for classical guitars.

## 4 Classical Guitar Compensation

In Section 2, we noted that the bending stiffness and the increase in string tension due to fretting sharpen the pitch, but that we can flatten it with a positive saddle setback and negative nut setback. In Appendix C, we develop a method (“RMS + BFGS”) that numerically solves Eq. (11) for the values of  $\Delta S$  and  $\Delta N$  that minimize the root-mean-square of the frequency errors of a string over a particular set of frets. In the case of our Classical Guitar with normal-tension nylon strings — shown in Fig. 10 for the case of zero setbacks — we use this method with  $d = 0$

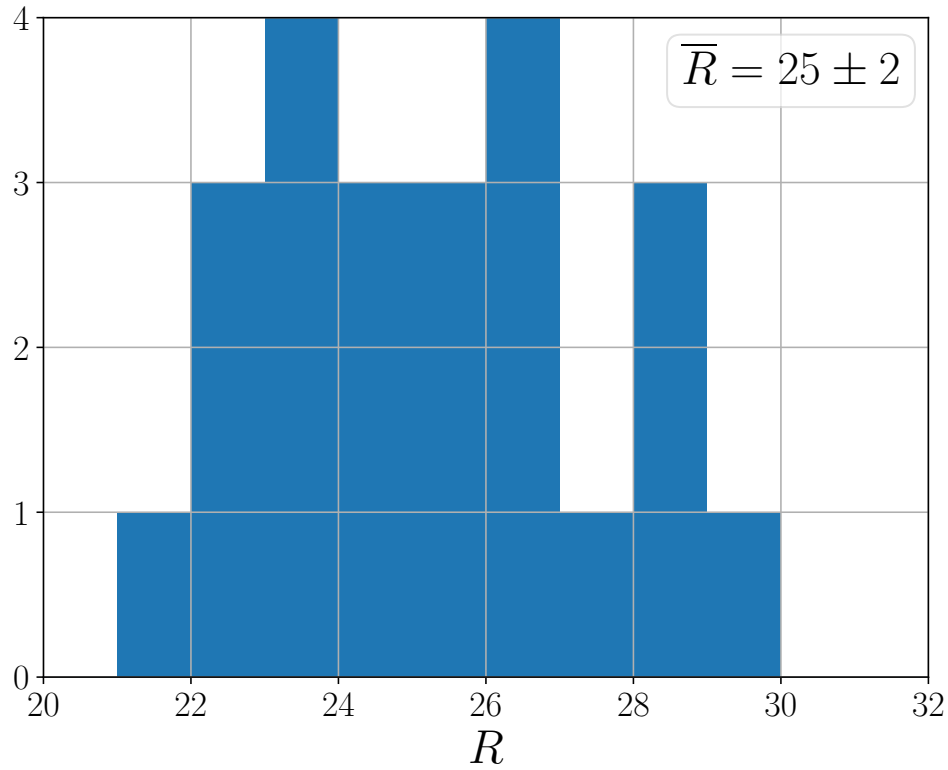


Figure 9: A histogram of the parameter  $R$  for all strings *except* those in the nylon light tension set presented in Appendix D.1, which seem to have anomalously high values.

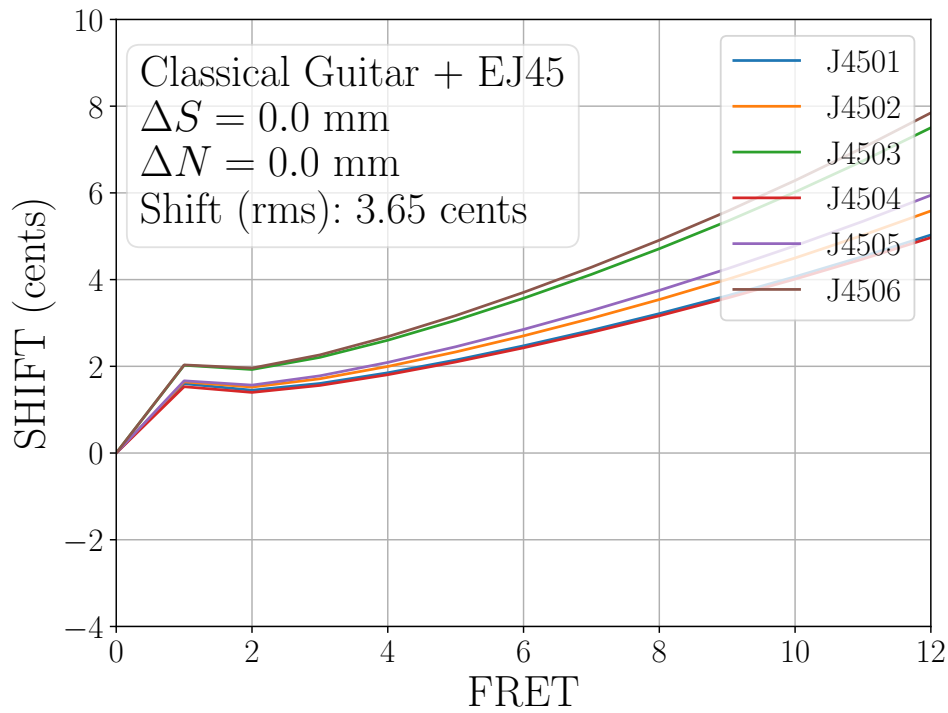


Figure 10: Frequency errors for an uncompensated Classical Guitar with normal tension nylon strings (D'Addario EJ45).

to obtain the nonzero setbacks listed in Table 5. The corresponding frequency deviations are shown in Fig. 11a (assuming that all other aspects of the guitar remain unchanged). Of course, manufacturing a guitar with unique saddle and nut setbacks for each string (of a particular tension) can be challenging, so we also plot in Fig. 11b the shifts obtained by setting each of the values of  $\Delta S$  and  $\Delta N$  to the mean of the corresponding column in Table 5. In both cases, the RMS error is significantly smaller than that of the uncompensated guitar shown in Fig. 10.

Table 5: Predicted setbacks for the D’Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45) on the Classical Guitar.

| String | $\Delta S$ (mm) | $\Delta N$ (mm) | $\overline{\Delta \nu}_{\text{rms}}$ (cents) |
|--------|-----------------|-----------------|--|
| J4501  | 1.48            | -0.36           | 0.15   |
| J4502  | 1.69            | -0.36           | 0.15   |
| J4503  | 2.33            | -0.43           | 0.19   |
| J4504  | 1.48            | -0.34           | 0.14   |
| J4505  | 1.82            | -0.36           | 0.15   |
| J4506  | 2.46            | -0.43           | 0.19   |

This purely numerical approach using Eq. (11) is accurate but not illuminating. Let’s build an intuitive understanding of how guitar compensation works, and then calculate approximate formulas for  $\Delta S$  and  $\Delta N$  that will help us appreciate the impact of particular choices of (for example)  $b$ ,  $c$ , and  $d$ . As in Fig. 6, let’s choose the guitar and string properties listed in Table 1 and Table 2, and then vary  $\Delta S$  and  $\Delta N$  and then Eq. (11) to determine the pitch of the string at each fret. Figure 12 shows that increasing the saddle setback tends to rotate the pitch curve clockwise, and increasing the magnitude of the negative nut setback displaces the pitch curve almost uniformly downward. It appears that we can compensate the guitar by finding a value of  $\Delta S$  that results in values of  $\Delta \nu_n$  that are equal for, say, frets with  $n \geq 3$ , and then calculating a value of  $\Delta N$  that sets  $\Delta \nu_{12} = 0$ .

We’ll use Eq. (29) and the approximation for  $Q_n$  given by Eq. (19). Let’s set  $d = 0$ , and then treat  $\gamma_n$  as a continuous variable. If we set  $d \Delta \nu_n / d \nu = 0$ , then we obtain

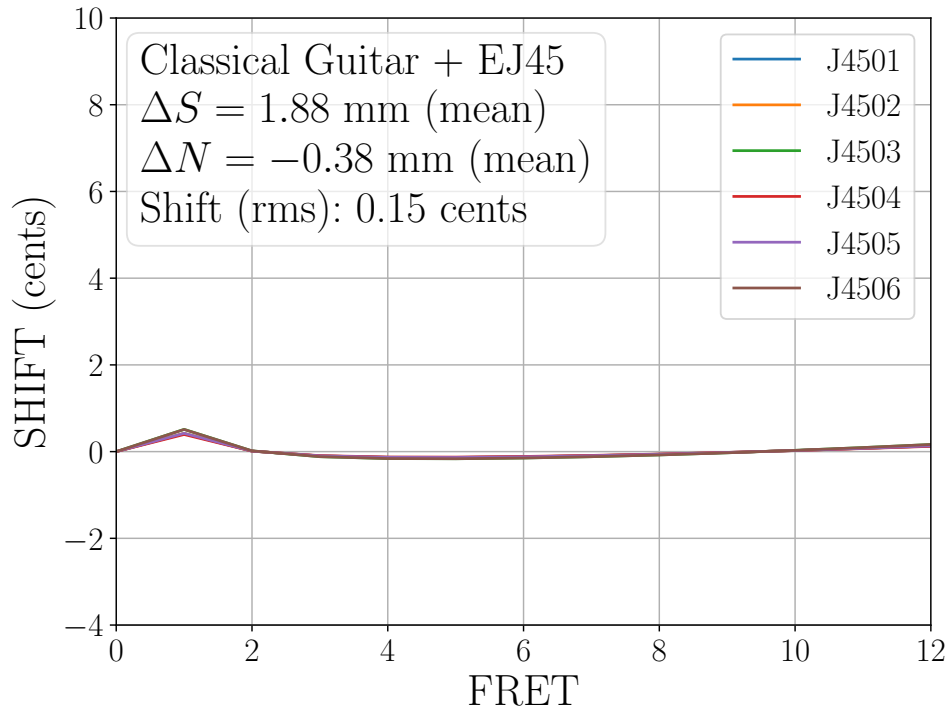
$$B_0 - \frac{\Delta S}{X_0} + \frac{\kappa}{4 X_0^2} \left[ (b + c)^2 - \frac{b^2}{(\gamma_n - 1)^2} \right] = 0. \quad (37)$$

The average value of  $(\gamma_n - 1)^{-2}$  over the 3<sup>rd</sup> through 12<sup>th</sup> frets is about 7. Adopting this value, we substitute the resulting solution for  $\Delta S$  back into Eq. (29) with  $n = 12$  and  $d \neq 0$ , and solve for  $\Delta N$ . We finally obtain

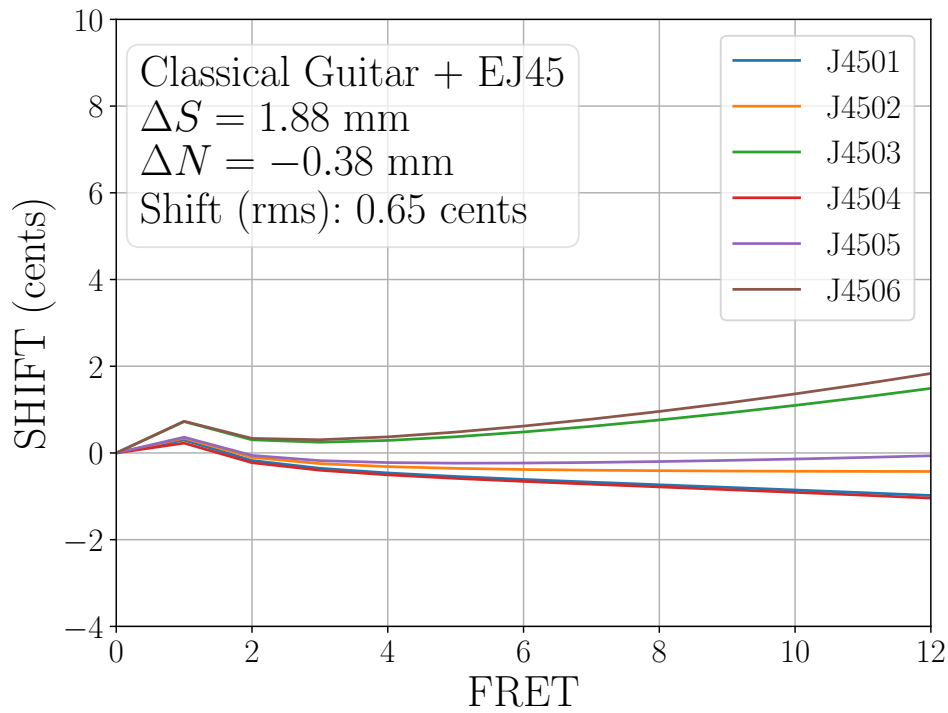
$$\Delta S = B_0 X_0 + \frac{\kappa}{4 X_0} [(b + c)^2 - 7 b^2] - \frac{2 \kappa}{X_0^2} (2 b + c)^2 d, \text{ and} \quad (38a)$$

$$\Delta N = -\frac{\kappa}{2 X_0} (5 b + c) b - \frac{2 \kappa}{X_0^2} (2 b + c)^2 d. \quad (38b)$$

Note that we have subtracted the same correction for  $d \neq 0$  in  $\Delta N$  to  $\Delta S$ , because in our numerical studies using Eq. (11) and the RMS + BFGS method we found that  $\Delta S - \Delta N$  had the constant value  $B_0 X_0 + (\kappa/4 X_0) (2b + c)^2$  across all of our string sets for  $0 \leq d \leq 10$  mm. These approximations are remarkably accurate given their origin: when  $d = 0$  mm, the values of the



(a) Full compensation



(b) Mean compensation

Figure 11: Frequency shifts (in cents) for the Classical Guitar with normal tension nylon strings (D’Addario EJ45). In (a) we use the individual values for each string that are listed in Table 5. In (b), we set  $\Delta S$  and  $\Delta N$  to the mean of the corresponding column in that table.

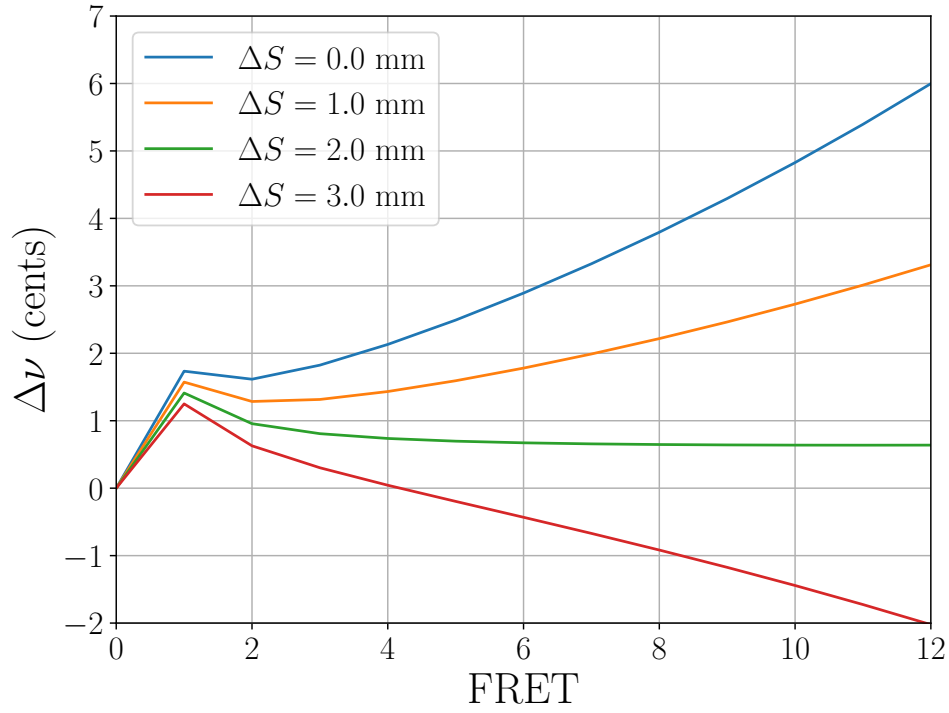
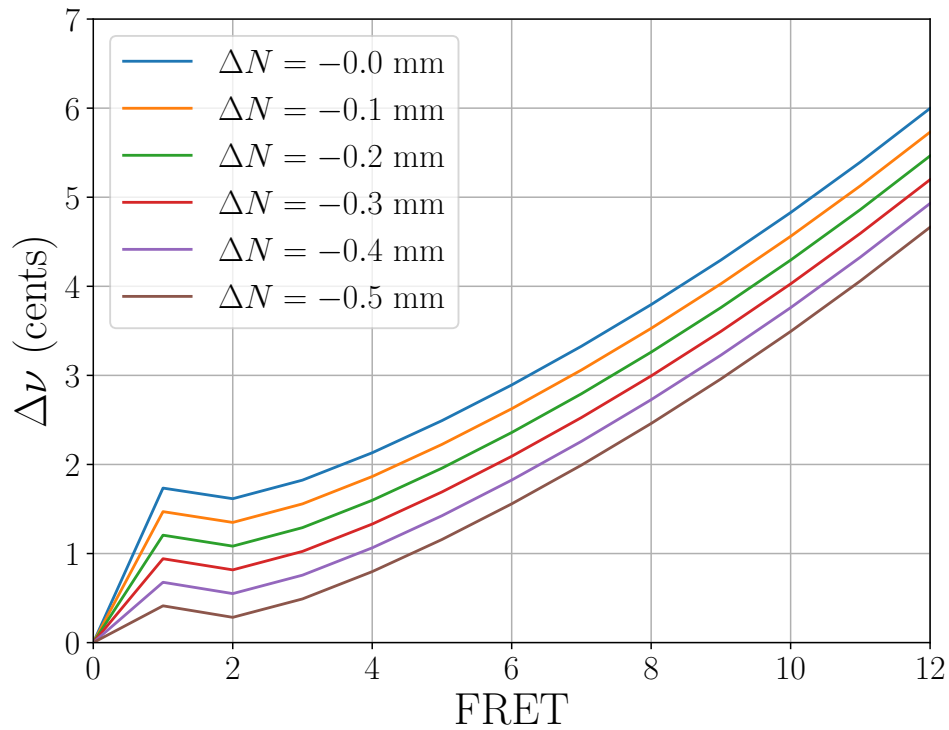
(a) Frequency shifts ( $\Delta N = 0$ )(b) Frequency shifts ( $\Delta S = 0$ )

Figure 12: In (a), we plot the frequency shifts for our classical guitar for several saddle setbacks with  $\Delta N = 0$ . Here we use the string parameters listed in Table 2. In (b), we set  $\Delta S = 0$  and plot the frequency shifts for several nut “setbacks.”

setbacks they predict are within 1% of the numerical predictions, and when  $d = 10$  mm they are within 2%.

As mentioned above, it is nontrivial to manufacture a guitar with different setbacks for each string [7], and it is unlikely that the exact values listed in Table 5 are applicable to other string sets. We have measured the values of  $R$  for five other string sets, and in Appendix D we have reproduced the exact compensation procedure for them that we performed above for normal tension strings (with  $d = 0$ ). Although each set exhibits variation between strings (and with respect to other sets) in individual setbacks for each string, they are similar enough that we suspect that there is the potential for great simplification in guitar design. For example, following the analysis of Appendix C, it is possible to determine a single setback pair  $\{\Delta S, \Delta N\}$  that minimizes the RMS frequency errors of an ensemble of strings over a collection of frets simply by computing the mean of the setbacks over all strings, and then using these mean values when manufacturing the guitar. If we consider five of the six string sets we have measured here — neglecting the light tension strings because of their pathologically high values of  $R$  — we can plot the exact setback predictions shown in Fig. 13, and then use these results to predict the mean values. In Fig. 13a, we see that the saddle setbacks are reasonably well described in terms of the string radius  $\rho$  by the expression

$$\Delta S = (4.4 \pm 0.2) \rho. \quad (39)$$

Therefore, we can either compute the mean of the saddle setbacks directly, or using the average value of the string lengths ( $\bar{\rho} = 0.43 \pm 0.08$  mm). Either way, we obtain  $\overline{\Delta S} = 1.9 \pm 0.4$  mm. Similarly, in Fig. 13b, we show a histogram of the values of the nut setbacks, and compute the mean value  $\Delta N = -0.37 \pm 0.03$  mm, which we recall is proportional to the scale length  $X_0$ . Note that these results are remarkably similar to the values used in Fig. 11b; if we plot the frequency deviations of those five string sets with these particular mean setback values, then we find that the maximum error always occurs at the twelfth fret, and it is always less than 1 cent. In the next section, we discuss a method to temper the guitar to reduce these errors further.

## 5 Tempering the Classical Guitar

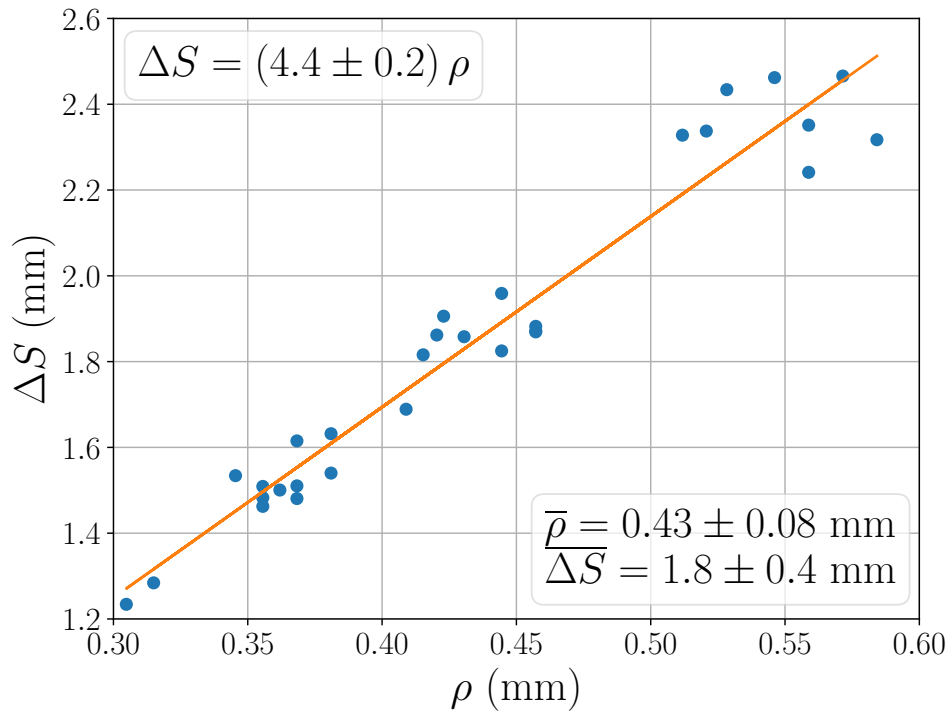
Temperament: A compromise between the acoustic purity of theoretically exact intervals, and the harmonic discrepancies arising from their practical employment.

— Dr. Theo. Baker [30]

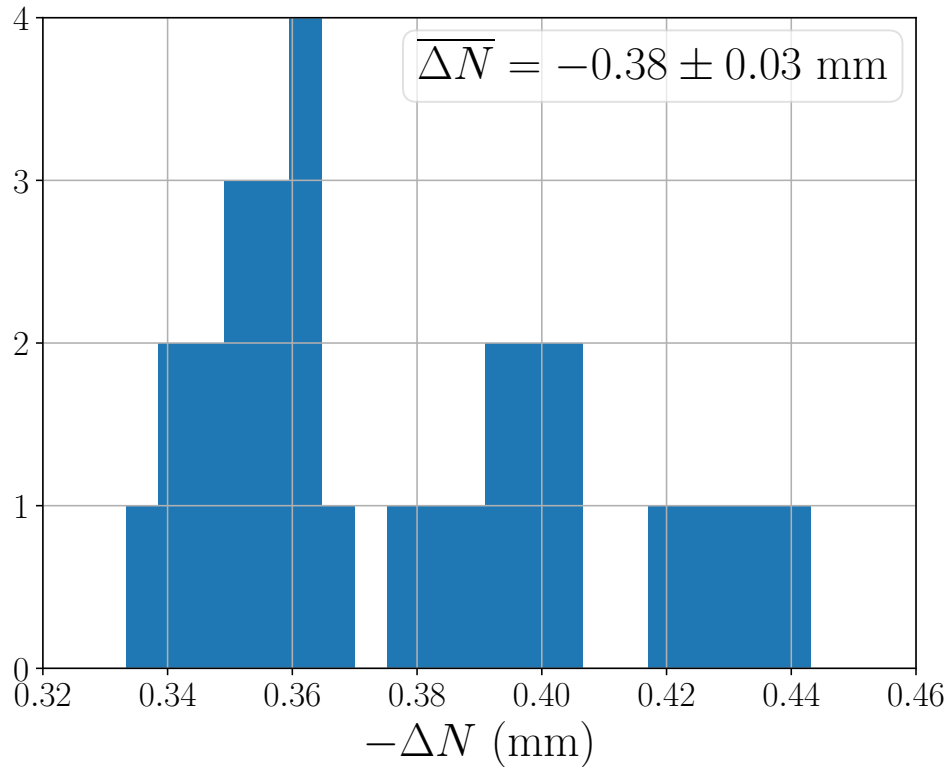
In Fig. 11b, a uniformly compensated classical guitar with normal tension strings tuned to 12-TET shows (of the treble strings) the third string has the greatest error in tuning across the fretboard. Tuning this guitar to 12-TET exacts a perfect-fifth in the third string while playing a C major chord in first position. This results in the third string being too sharp for the other common chords of E major (G#), A major, and D major (A), particularly when the guitar is played at a higher fret position. One way to reduce this error is by lowering the pitch of the third string below 12-TET with an electronic tuner. Another more comprehensive system is to tune all the strings harmonically to the fifth string, which lowers the third string by 7 cents as well as tempering the remaining strings.

In this particular case, the “Harmonic Tuning Method” can be followed using these steps:





(a) Five-set saddle setback data with fit



(b) Five-set nut setback data with mean

Figure 13: Construction of mean saddle and nut setbacks over five selected string sets. In (a) we plot the saddle setback for each string as a function of the string radius, with the result of the best linear fit. In (b), we present a histogram of the nut setbacks and their mean.

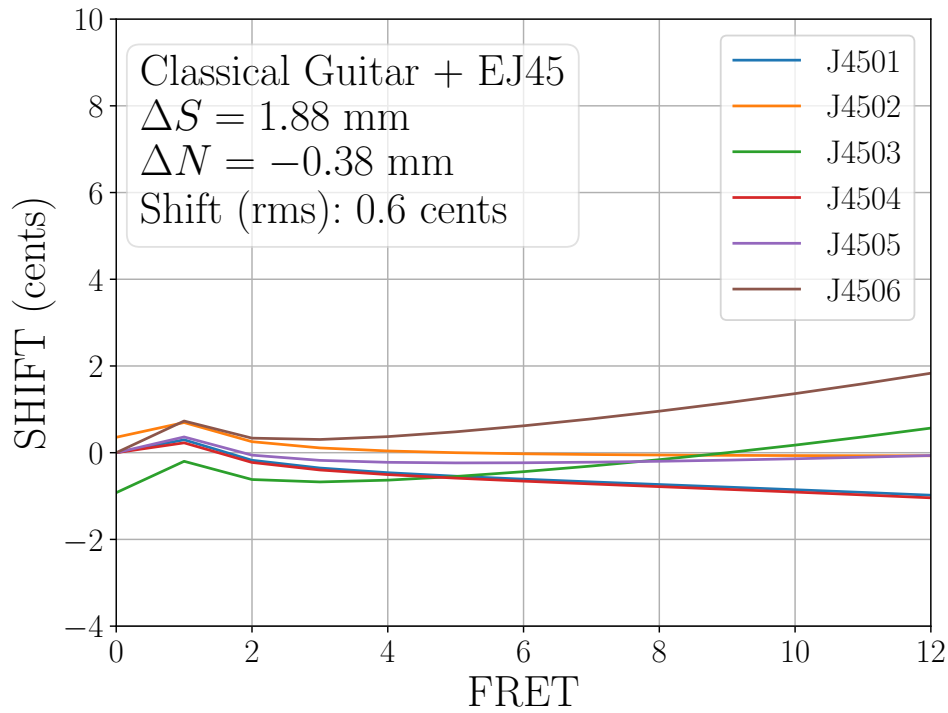


Figure 14: Frequency shift (in cents) for the mean compensated classical guitar with normal tension nylon strings (D’Addario EJ45) shown with 12-TET tuning in Fig. 11b. Here the same guitar has been harmonically tuned using the approach outlined in Table 6.

1. Begin by tuning the fifth string to  $A_2 = 110$  Hz, resulting in a fifth-fret harmonic of  $A_4 = 440$  Hz. (This can also be tuned by ear using an  $A_4$  tuning fork).
2. Tune that harmonic to the seventh fret harmonic of the fourth string, which is also  $A_4 = 440$  Hz.
3. Tune the seventh fret harmonic on the fifth string (330 Hz, or 0.37 Hz sharper than 12-TET  $E_4$ ) to the fifth-fret harmonic of the sixth string.
4. The seventh fret harmonic on the fifth string can tune the remaining fretted strings: the ninth fret on the third string, the fifth fret on the second string, and the open first string.

Table 6: Harmonic tuning methodology based on  $A_4$  and  $E_4$ . The asterisk indicates a harmonic with a null at the designated fret.

| Reference String/Fret | Target String/Fret |
|-----------------------|--------------------|
| $A^*/5$ ( $A_4$ )     | $D^*/7$            |
| $A^*/7$ ( $E_4$ )     | $E^*/5$            |
| $A^*/7$ ( $E_4$ )     | G/9                |
| $A^*/7$ ( $E_4$ )     | B/5                |
| $A^*/7$ ( $E_4$ )     | E/0                |

We have summarized these steps in Table 6, and in Fig. 14 we show the same guitar tuned in this fashion. Although the RMS shift over all strings is similar to that obtained by 12-TET tuning, the reduction in errors by strings 2 and 3 on the second and higher frets is significant. Note that other tuning choices can be made depending on the piece being played. For example, the third string could also be tuned at the second fret to  $A_3 = 220$  Hz using the fifth-string harmonic at the 12<sup>th</sup> fret, and/or the first string could be tuned at the fifth fret to  $A_4$  using the fifth-fret harmonic of the fifth string. The flexibility of the harmonic tuning method — and its reliance on only an  $A_4$  tuning fork — is a great asset for the classical guitarist. Of course, how the guitar string is plucked has an impact on the resulting tone, but we defer a discussion of this effect to the literature [31, 32].

## 6 Conclusion: The Recipes

In this work, we have constructed a model of classical guitar intonation that includes the effects of the resonant length of the fretted string, linear mass density, tension, and bending stiffness. We have described a simple experimental approach to estimating the increase in string tension arising from an increase in its length, and then the corresponding mechanical stiffness. This allows us to determine the saddle and nut positions needed to compensate the guitar for a particular string, and we propose a simple approach to find averages of these positions to accommodate a variety of strings. This “mean” method benefits further from temperament techniques — such as harmonic tuning — that can enhance the intonation of the classical guitar for particular musical pieces.

Our calculations have relied on Eq. (6), which was derived by noting that the string was pinned to the saddle rather than clamped. We then separated the contributions to the frequency deviations from ideal values caused by fretting by measuring these differences using the definition of logarithmic “cents” given by Eq. (2). This approach has allowed us to determine Taylor series approximations to each of these contributions that are valid when the height of a string above each fret is small compared to the scale length. From these approximations, we have been able to create two arithmetic “recipes” — the RMS method and a quadratic (in  $B_0$ ) approximation — that predict saddle and nut setbacks that enable the guitar to compensate for the frequency effects of fretting. Applying any one of these algorithms to a particular guitar design always begins with the same five steps:

1. Determine the scale length of the guitar by doubling the distance between the inside edge of the nut and the center of the 12<sup>th</sup> fret.
2. Using Fig. 2 as a guide, carefully measure the values of  $b$  and  $c$ . It is possible that the luthier has selected a saddle with vertical curvature, resulting in different values of  $c$  for each string.
3. Estimate the relief  $\Delta y_{12}$  at the 12<sup>th</sup> fret for each string. Measure the action (height)  $y_{12}$  of the string above fret 12; then  $\Delta y_{12} = y_{12} - b - c/2$  and we rescale the height of the saddle above the nut to  $\tilde{c} = c + 2\Delta y_{12}$ .
4. Select a string set with values of  $\kappa$  and  $B_0$  listed in one of the derived physical properties tables in this paper, or follow the procedure developed in Section 3 to determine these quantities for a different string set.

5. Referring to Fig. 2 as a guide, choose a preferred value of the fretting distance  $d$  to account for the size of the finger.

The most accurate algorithm then adds one more step:

6. Using Eq. (70) and the first line of Eq. (19) — with  $L_0$ ,  $L_n$ , and  $L'_n$  computed using the exact forms of Eq. (12), Eq. (13), and Eq. (18), respectively — determine the saddle and nut setbacks for the selected string set.

The quadratic approximation is based on Eq. (71) (which chooses a 12-fret RMS average) and is accessible using a hand calculator. In this case, Step 6 is updated and a Step 7 is added to adjust the setbacks for a nonzero value of  $d$ :

6. Compute the setbacks for  $d = 0$  mm using Eq. (71):

$$\Delta S \approx \left[ B_0 + \frac{3}{2} (1 + \pi^2) B_0^2 \right] X_0 + \frac{\kappa}{4 X_0} (-9.848 b^2 + 2 b \tilde{c} + \tilde{c}^2) , \text{ and}$$

$$\Delta N \approx B_0^2 X_0 - \frac{\kappa}{2 X_0} (5.633 b^2 + b \tilde{c}) .$$

7. Choose a fretting distance  $d$ . Then adjust the setbacks computed in the previous step using

$$\Delta N' \approx \Delta N [1 + 13 (d/X_0) + 280 (d/X_0)^2] , \text{ and}$$

$$\Delta S' \approx \Delta S + \Delta N [13 (d/X_0) + 280 (d/X_0)^2] .$$

As  $d$  increases, the magnitude of the negative nut setback increases, and the saddle setback decreases because  $\Delta N < 0$ .

Of course, this quadratic approximation can be simplified further. For example, the terms proportional to  $B_0^2$  can be neglected without much loss of accuracy (but without much gain in simplicity), yielding a corresponding linear approximation. Furthermore, the term in  $\Delta S$  that is proportional to  $\kappa$  can be neglected, but we recall from Section 4 that the resulting value of  $\Delta S$  will be about 15 – 20% too small. But perhaps the simplest reasonably accurate compensation approach is to adopt the results of Fig. 13: compute the saddle setback using  $\Delta S = 4.4 \bar{\rho}$ , and select the nut setback to be  $\Delta N = -0.37 \times (650 \text{ mm}/X_0)$  mm, or about 15 mils for a guitar with a scale length of 650 mm.

These setback estimates can be averaged across the string set to design compensated nuts and saddles that should be relatively easy to fabricate. Nevertheless, we understand that high-end (concert) guitars that are likely to rely on one or two string sets (and the appropriate value of  $d$  for one guitar player) will benefit from the full, more accurate treatment of individual string setbacks.

Throughout this work, it has become clear that the frequency errors at the first fret are difficult to eliminate. Increasing the magnitude of the nut setback will also increase the frequency errors at all other frets. The only obvious solution to this problem is also sacrilegious (and may result in string “buzzing”): raise the height of the first fret to meet the string. Perhaps a clever luthier will discover a more acceptable method for accomplishing this objective.

We have placed the text of this manuscript (as well as the computational tools needed to reproduce our numerical results and all the graphs presented here) online [22] to invite comment and contributions from and collaboration with interested luthiers and musicians.

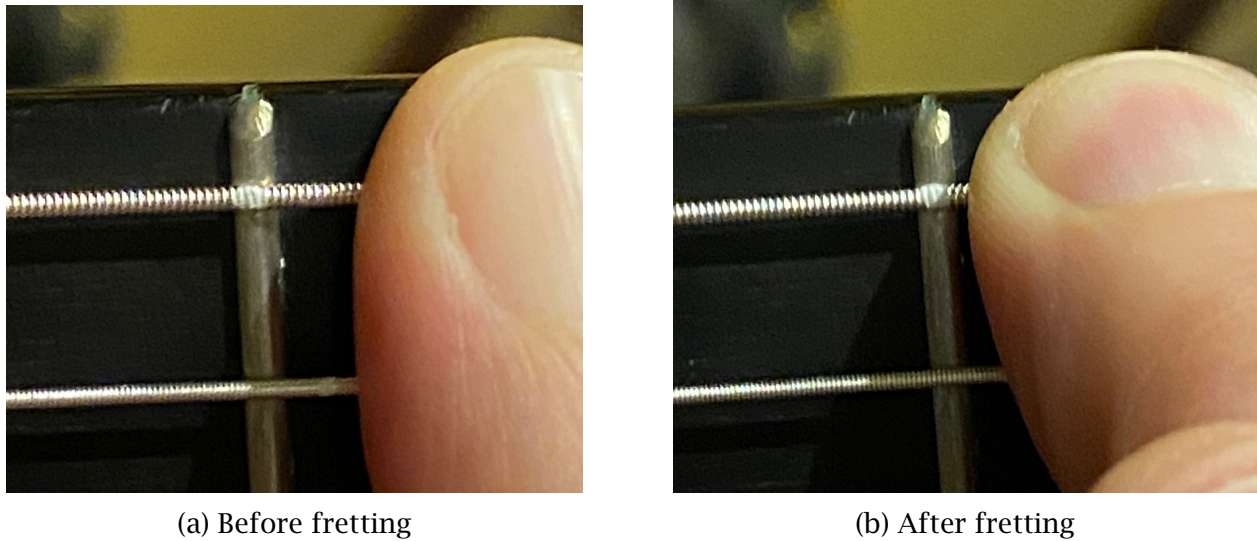


Figure 15: Location of a small marker of white correction fluid before and after fretting.

## A Fretting Classical Guitar Strings

Previous studies of guitar intonation and compensation [7, 8] included a contribution to the incremental change in the length of the fretted string caused by both the depth and the shape of the string under the finger. As the string is initially pressed to the fret, the total length  $\mathcal{L}_n$  increases and causes the tension in the string — which is pinned at the saddle and clamped at the nut — to increase. As the string is pressed further, does the additional deformation of the string increase its tension (throughout the resonant length  $L_n$ )? There are at least two purely empirical reasons to doubt this hypothesis. First, as shown in Fig. 15, we can mark a string (with a small deposit of white correction fluid) above a particular fret and then observe the mark with a magnifying glass. As the string is pressed flat on the finger board with two fingers, the mark does not move perceptibly — it has become *clamped* on the fret. Second, we can use either our ears or a simple tool to measure frequencies [27] to listen for a shift as we use different fingers and vary the fretted depth of a string. The apparent modulation is far less than would be obtained by classical vibrato ( $\pm 15$  cents), so we assume that once the string is minimally fretted the length(s) can be regarded as fixed. (If this were not the case, then fretting by different people or with different fingers, at a single string or with a barre, would cause additional, varying frequency shifts that would be audible and difficult to compensate.)

In Section 2, we have included this concept in a simple way to determine the effect it will have on the frequency shift due to increased string tension. First, as shown in Fig. 16, as the string is pressed onto the fret, its shape is described quite well by two line segments intersecting behind the fret. Here it is clear that the finger is shaped by the string more than the string is shaped by the finger. We have taken this observation into consideration in Fig. 2 by introducing such an intersection point at a distance  $d$  behind fret  $n$  to represent the slight increase in the distance  $L'_n$  caused by a finger. The consequences of this choice are discussed in Section 2.2, and the impact it has on (for example) the relative displacement  $Q_n$  is shown in Fig. 3.

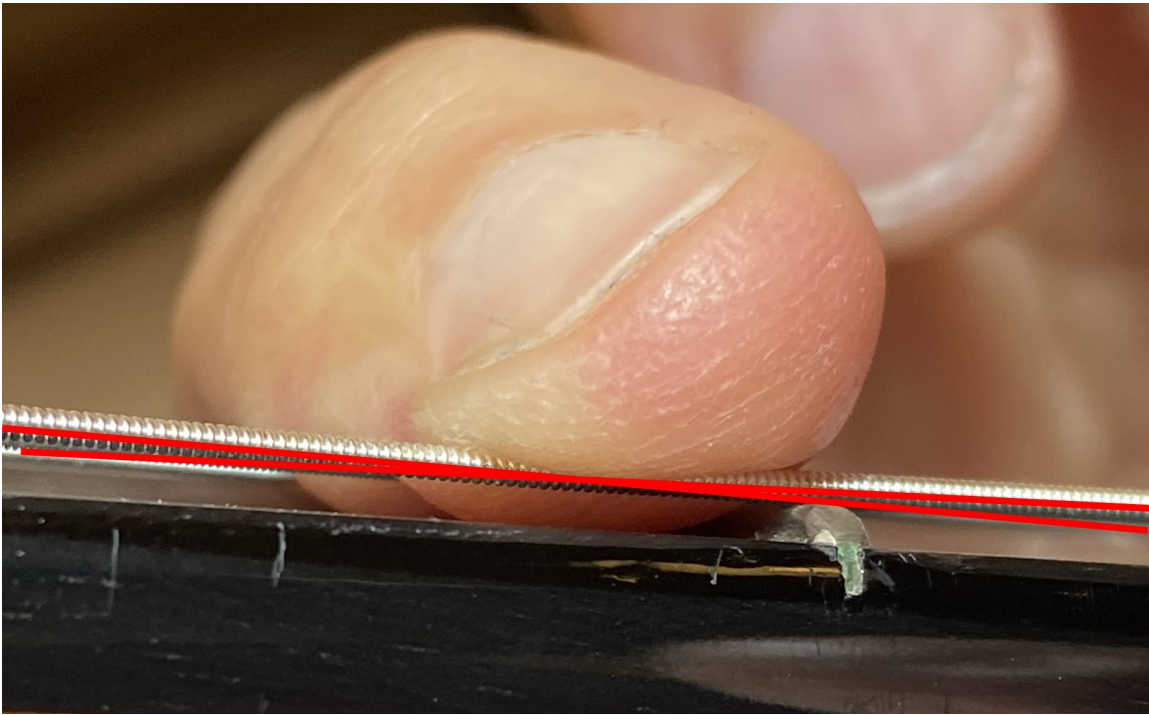


Figure 16: Photo of a wound nylon  $E_2$  string clamped at the first fret of a classical guitar. The shape of the fretted string can be well approximated by two line segments intersecting about 5–6 mm behind the fret.

## B Vibration Frequencies of a Stiff String

Here we outline the calculation of the normal mode frequencies of a vibrating stiff string with non-symmetric boundary conditions. We begin with the wave equation [24]

$$\mu \frac{\partial^2}{\partial t^2} y(x) = T \frac{\partial^2}{\partial x^2} y(x) - E S \mathcal{K}^2 \frac{\partial^4}{\partial x^4} y(x), \quad (40)$$

where  $\mu$  and  $T$  are respectively the linear mass density and the tension of the string,  $E$  is its Young's modulus (or the modulus of elasticity),  $S$  is the cross-sectional area, and  $\mathcal{K}$  is the radius of gyration of the string. (For a uniform cylindrical string/wire with radius  $\rho$ ,  $S = \pi \rho^2$  and  $\mathcal{K} = \rho/2$ .) If we scale  $x$  by the length  $L$  of the string, and  $t$  by  $1/\omega_0 \equiv (L/\pi)\sqrt{\mu/T}$ , then we obtain the dimensionless wave equation

$$\pi^2 \frac{\partial^2}{\partial t^2} y(x) = \frac{\partial^2}{\partial x^2} y(x) - B^2 \frac{\partial^4}{\partial x^4} y(x), \quad (41)$$

where  $B$  is the “bending stiffness parameter” given by

$$B \equiv \sqrt{\frac{E S \mathcal{K}^2}{L^2 T}}. \quad (42)$$

We assume that  $y(x)$  is a sum of terms of the form

$$y(x) = C e^{kx - i\omega t}, \quad (43)$$



requiring that  $k$  and  $\omega$  satisfy the expression

$$B^2 k^4 - k^2 - (\pi \omega)^2 = 0, \quad (44)$$

or

$$k^2 = \frac{1 \pm \sqrt{1 + (2 \pi B \omega)^2}}{2 B^2}. \quad (45)$$

Therefore, given  $\omega$ , we have four possible choices for  $k$ :  $\pm k_1$ , or  $\pm i k_2$ , where

$$k_1^2 = \frac{\sqrt{1 + (2 \pi B \omega)^2} + 1}{2 B^2}, \text{ and} \quad (46a)$$

$$k_2^2 = \frac{\sqrt{1 + (2 \pi B \omega)^2} - 1}{2 B^2}. \quad (46b)$$

The corresponding general solution to Eq. (41) has the form

$$y(x) = e^{-i\omega t} \left( C_1^+ e^{k_1 x} + C_1^- e^{-k_1 x} + C_2^+ e^{i k_2 x} + C_2^- e^{-i k_2 x} \right). \quad (47)$$

As discussed in Section 2, the boundary conditions for the case of a classical guitar string are not symmetric. At  $x = 0$  (the saddle), the string is pinned (but not clamped), so that  $y = 0$  and  $\partial^2 y / \partial x^2 = 0$ . However, at  $x = 1$  (the fret) the string is clamped, so that  $y = 0$  and  $\partial y / \partial x = 0$ . Applying these constraints to Eq. (47), we obtain

$$0 = C_1^+ + C_1^- + C_2^+ + C_2^-, \quad (48a)$$

$$0 = k_1^2 (C_1^+ + C_1^-) - k_2^2 (C_2^+ + C_2^-), \quad (48b)$$

$$0 = C_1^+ e^{k_1} + C_1^- e^{-k_1} + C_2^+ e^{i k_2} + C_2^- e^{-i k_2}, \text{ and} \quad (48c)$$

$$0 = k_1 (C_1^+ e^{k_1} - C_1^- e^{-k_1}) + i k_2 (C_2^+ e^{i k_2} - C_2^- e^{-i k_2}). \quad (48d)$$

Since  $k_1^2 + k_2^2 \neq 0$ , the first two of these equations tell us that  $C_1^- = -C_1^+ \equiv -C_1$ , and  $C_2^- = -C_2^+ \equiv -C_2$ . Therefore, the second two equations become

$$C_1 \sinh(k_1) = -i C_2 \sin(k_2), \text{ and} \quad (49a)$$

$$k_1 C_1 \cosh(k_1) = -i k_2 C_2 \cos(k_2). \quad (49b)$$

Dividing the first of these equations by the second, we find

$$\tan(k_2) = \frac{k_2}{k_1} \tanh k_1. \quad (50)$$

From Eq. (46), we see that  $k_1^2 - k_2^2 = 1/B^2$ , so that

$$k_1 = \frac{1}{B} \sqrt{1 + (B k_2)^2}. \quad (51)$$

In the case of a classical guitar, we expect that  $B \ll 1$ , so  $k_1 \approx 1/B \gg 1$ , and therefore  $\tanh k_1 \rightarrow 1$ . Substituting Eq. (51) into Eq. (50), we obtain

$$\tan(k_2) = \frac{B k_2}{\sqrt{1 + (B k_2)^2}}. \quad (52)$$



We expect that  $B k_2 \ll 1$ , so we assume that  $k_2 = q\pi(1 + \epsilon)$ , where  $q \in \mathbb{N}$  is an integer greater than or equal to 1, and  $\epsilon \ll 1$ . Therefore, to second order in  $\epsilon$ , we have  $\tan(k_2) \approx q\pi\epsilon$ , and

$$\epsilon = \frac{B(1 + \epsilon)}{\sqrt{1 + [q\pi B(1 + \epsilon)]^2}}. \quad (53)$$

The denominator of the right-hand side of this equation has a Taylor expansion given by  $1 - \frac{1}{2} [q\pi B(1 + \epsilon)]^2$ , indicating that it will not contribute to  $\epsilon$  to second order in  $B$ . Therefore, to this order,

$$\epsilon \approx \frac{B}{1 - B} \approx B + B^2. \quad (54)$$

We substitute  $k = \pm ik_2$  into Eq. (44) with  $k_2 = q\pi/(1 - B)$  to obtain

$$\begin{aligned} \omega &= \frac{k_2}{\pi} \sqrt{1 + (B k_2)^2} \\ &= \frac{q}{1 - B} \sqrt{1 + q^2 \pi^2 \left( \frac{B}{1 - B} \right)^2} \\ &\approx q \left[ 1 + B + \left( 1 + \frac{1}{2} q^2 \pi^2 \right) B^2 \right]. \end{aligned} \quad (55)$$

Restoring the time scaling by  $1/\omega_0$ , and defining the frequency (in cycles/second)  $f = \omega/2\pi$ , we finally have

$$f_q = \frac{q}{2L} \sqrt{\frac{T}{\mu}} \left[ 1 + B + \left( 1 + \frac{1}{2} q^2 \pi^2 \right) B^2 \right]. \quad (56)$$

We use this result to build our model in Section 2.

## C Compensation by Minimizing RMS Error

The root-mean-square (RMS) frequency error (in cents) averaged over the frets  $n \in \{1, n_{\max}\}$  (for  $n_{\max} > 1$ ) of a particular string is given by

$$\overline{\Delta v}_{\text{rms}} \equiv \sqrt{\frac{\sum_{n=1}^{n_{\max}} \Delta v_n^2}{n_{\max}}}, \quad (57)$$

where  $\Delta v_n$  is given by Eq. (11). Here we will vary both  $\Delta S$  and  $\Delta N$  to minimize  $\overline{\Delta v}_{\text{rms}}$ . In this case, it is sufficient to minimize the quantity

$$\chi^2 = \sum_{n=1}^{n_{\max}} \left[ \frac{\ln(2)}{1200} \Delta v_n \right]^2 \quad (58)$$

such that the gradient of  $\chi^2$  with respect to  $\Delta S$  and  $\Delta N$  vanishes. Let's rewrite Eq. (11) as

$$\frac{\ln(2)}{1200} \Delta v_n = W_n + Z_n, \quad (59)$$

where

$$W_n = \ln \left( \frac{L_0}{y_n L_n} \right), \text{ and} \quad (60a)$$

$$Z_n = \ln \left[ \sqrt{\frac{\mu_0}{\mu_n} \frac{T_n}{T_0}} \frac{1 + B_n + (1 + \pi^2/2) B_n^2}{1 + B_0 + (1 + \pi^2/2) B_0^2} \right]. \quad (60b)$$

In Section 2, we determined that — for the purposes of estimating the values of the setbacks —  $W_n$  could be represented accurately by

$$W_n \approx \frac{\Delta N - (y_n - 1) \Delta S}{X_0}. \quad (61)$$

Furthermore, we discovered that  $Z_n$  does not depend even to second order on either  $\Delta S$  or  $\Delta N$ . Therefore, the components of the gradient of  $\chi^2$  are

$$\frac{\partial}{\partial \Delta S} \chi^2 = 2 \sum_n (W_n + Z_n) \frac{\partial W_n}{\partial \Delta S} = -\frac{2}{X_0} \sum_n (y_n - 1) (W_n + Z_n), \text{ and} \quad (62a)$$

$$\frac{\partial}{\partial \Delta N} \chi^2 = 2 \sum_n (W_n + Z_n) \frac{\partial W_n}{\partial \Delta S} = \frac{2}{X_0} \sum_n (W_n + Z_n). \quad (62b)$$

Setting each of these expressions to zero and solving them for  $\Delta S$  and  $\Delta N$ , we obtain

$$\Delta S = \frac{\sigma_0 \bar{Z}_1 - \sigma_1 \bar{Z}_0}{\sigma_0 \sigma_2 - \sigma_1^2} X_0, \text{ and} \quad (63a)$$

$$\Delta N = -\frac{\sigma_2 \bar{Z}_0 - \sigma_1 \bar{Z}_1}{\sigma_0 \sigma_2 - \sigma_1^2} X_0, \quad (63b)$$

where

$$\sigma_k \equiv \sum_{n=1}^{n_{\max}} (y_n - 1)^k, \text{ and} \quad (64)$$

$$\bar{Z}_k \equiv \sum_{n=1}^{n_{\max}} (y_n - 1)^k Z_n. \quad (65)$$

The corresponding Hessian matrix for this problem is the symmetric matrix

$$H = \begin{bmatrix} \frac{\partial^2 \chi^2}{\partial \Delta S^2} & \frac{\partial^2 \chi^2}{\partial \Delta N \partial \Delta S} \\ \frac{\partial^2 \chi^2}{\partial \Delta S \partial \Delta N} & \frac{\partial^2 \chi^2}{\partial \Delta N^2} \end{bmatrix} = \frac{2}{X_0^2} \begin{bmatrix} \sigma_2 & -\sigma_1 \\ -\sigma_1 & \sigma_0 \end{bmatrix}. \quad (66)$$

We can apply the second partial derivative test to the Hessian to determine whether we've found an extremum of  $\chi^2$ . If the determinant of the Hessian is positive, and (in the case of a  $2 \times 2$  matrix) one of the diagonal elements is positive, then we have found a minimum. The determinant is greater than zero for  $n_{\max} \geq 2$ , and the second condition is satisfied by  $\sigma_0 = n_{\max} > 0$  when  $n_{\max} \geq 1$ . Therefore, we can be confident that the solution for  $\Delta S$  and  $\Delta N$  given by Eq. (63) minimizes the RMS frequency error provided that we are averaging over at least the first two frets.

Again, to guide our intuition, we follow the logic used to develop Eq. (29) and approximate  $Z_n$  as

$$Z_n \approx \frac{1}{2} \kappa Q_n + (y_n - 1) B_0. \quad (67)$$

Then our estimates for the setbacks become

$$\Delta S = \left( B_0 + \frac{\kappa}{2} \frac{\sigma_0 \bar{Q}_1 - \sigma_1 \bar{Q}_0}{\sigma_0 \sigma_2 - \sigma_1^2} \right) X_0, \text{ and} \quad (68a)$$

$$\Delta N = -\frac{\kappa}{2} \frac{\sigma_2 \bar{Q}_0 - \sigma_1 \bar{Q}_1}{\sigma_0 \sigma_2 - \sigma_1^2} X_0, \quad (68b)$$

where

$$\overline{Q}_k \equiv \sum_{n=1}^{n_{\max}} (\gamma_n - 1)^k Q_n. \quad (69)$$

If we approximate  $\overline{Q}_k \approx \sigma_k \overline{Q}$ , where  $\overline{Q}$  is the relative displacement averaged over a particular set of frets, we obtain the estimates discussed in Section 2.5:  $\Delta S \approx B_0 X_0$ , and  $\Delta N \approx -\kappa \overline{Q} X_0 / 2$ . The corresponding solution when the quadratic stiffness term is included is given in matrix form by

$$\begin{bmatrix} \Delta S \\ \Delta N \end{bmatrix} = \frac{X_0}{\sigma_0 \sigma_2 - \sigma_1^2} \begin{bmatrix} \sigma_0 & -\sigma_1 \\ \sigma_1 & -\sigma_2 \end{bmatrix} \begin{bmatrix} \sigma_2 B_0 + \frac{1}{2} (1 + \pi^2) (2\sigma_2 + \sigma_3) B_0^2 + \frac{1}{2} \kappa \overline{Q}_1 \\ \sigma_1 B_0 + \frac{1}{2} (1 + \pi^2) (2\sigma_1 + \sigma_2) B_0^2 + \frac{1}{2} \kappa \overline{Q}_0 \end{bmatrix}. \quad (70)$$

If we choose  $n_{\max} = 12$ , then when  $d = 0$  we can compute  $\sigma_k$  and  $\overline{Q}_k$  in terms of powers of  $b$  and  $c$  to find

$$\Delta S = \left[ B_0 + \frac{3}{2} (1 + \pi^2) B_0^2 \right] X_0 + \frac{\kappa}{4 X_0} (-9.848 b^2 + 2 b c + c^2), \text{ and} \quad (71)$$

$$\Delta N = B_0^2 X_0 - \frac{\kappa}{2 X_0} (5.633 b^2 + b c). \quad (72)$$

The setback solution given by Eq. (63) is valid for a single string, and results like those shown in Table 5 and Fig. 11a assume that the guitar is built such that each string — from a particular set of strings — has a unique saddle and nut setback. Suppose that we'd prefer to engineer a guitar with single, uniform values of both  $\Delta S$  and  $\Delta N$  that provide reasonable compensation across an entire string set (or an ensemble of strings from a variety of manufacturers). In this case, Eq. (57) becomes

$$\overline{\Delta v}_{\text{rms}} \equiv \sqrt{\frac{\sum_{m=1}^{m_{\max}} \sum_{n=1}^{n_{\max}} [\Delta v_n^{(m)}]^2}{m_{\max} n_{\max}}}, \quad (73)$$

where  $m$  labels the strings in the set, and Eq. (59) has been updated to become

$$\frac{\ln(2)}{1200} \Delta v_n^{(m)} = W_n^{(m)} + Z_n^{(m)}, \quad (74)$$

If we rigorously follow the same approach that we used to arrive at Eq. (63), in the multi-string case we obtain

$$\Delta S = \frac{1}{m_{\max}} \sum_{m=1}^{m_{\max}} \Delta S^{(m)}, \text{ and} \quad (75a)$$

$$\Delta N = \frac{1}{m_{\max}} \sum_{m=1}^{m_{\max}} \Delta N^{(m)}, \quad (75b)$$

where

$$\Delta S^{(m)} = \frac{\sigma_0 \overline{Z}_1^{(m)} - \sigma_1 \overline{Z}_0^{(m)}}{\sigma_0 \sigma_2 - \sigma_1^2} X_0, \text{ and} \quad (76a)$$

$$\Delta N^{(m)} = -\frac{\sigma_2 \overline{Z}_0^{(m)} - \sigma_1 \overline{Z}_1^{(m)}}{\sigma_0 \sigma_2 - \sigma_1^2} X_0, \quad (76b)$$

reflecting the unique values of  $\kappa^{(m)}$  and  $B_0^{(m)}$  for each string in each set. In other words, we can find the optimum values for both  $\Delta S$  and  $\Delta N$  simply by averaging the corresponding setbacks over a commercially interesting collection of string sets.

## D Other Classical Guitar String Sets

### D.1 Light Tension – Nylon

Table 7: String specifications for the D’Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

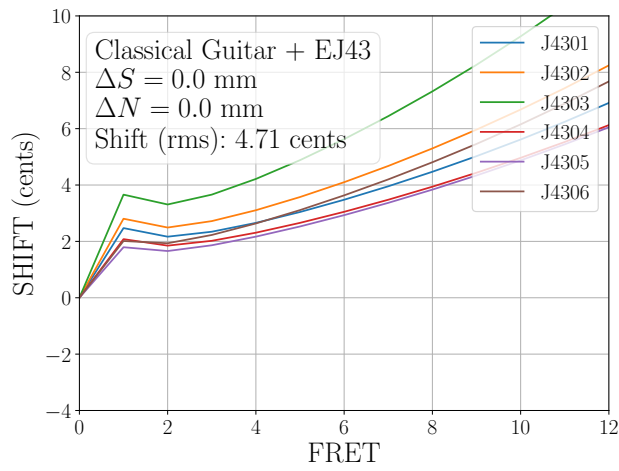
| String | Note           | Radius (mm) | Density (mg/mm) | Tension (N) |
|--------|----------------|-------------|-----------------|-------------|
| J4301  | E <sub>4</sub> | 0.349       | 0.361           | 66.4        |
| J4302  | B <sub>3</sub> | 0.403       | 0.487           | 50.2        |
| J4303  | G <sub>3</sub> | 0.504       | 0.808           | 52.5        |
| J4304  | D <sub>3</sub> | 0.356       | 1.822           | 66.4        |
| J4305  | A <sub>2</sub> | 0.419       | 2.741           | 56.0        |
| J4306  | E <sub>2</sub> | 0.533       | 5.158           | 59.2        |

Table 8: Derived physical properties of the D’Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

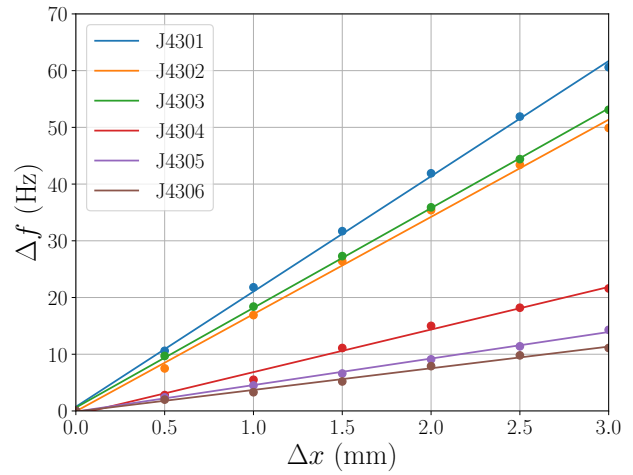
| String | $R$  | $\sigma$ | $\kappa$ | $B_0$   | $E$ (GPa) |
|--------|------|----------|----------|---------|-----------|
| J4301  | 37.8 | 0.5      | 76.6     | 0.00235 | 13.28     |
| J4302  | 42.6 | 1.0      | 86.2     | 0.00287 | 8.50      |
| J4303  | 55.0 | 0.4      | 111.1    | 0.00409 | 7.30      |
| J4304  | 31.4 | 1.2      | 63.7     | 0.00218 | 10.65     |
| J4305  | 26.1 | 0.5      | 53.2     | 0.00235 | 5.40      |
| J4306  | 28.5 | 1.1      | 57.9     | 0.00312 | 3.83      |

Table 9: Predicted setbacks for the D’Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43) on the Classical Guitar.

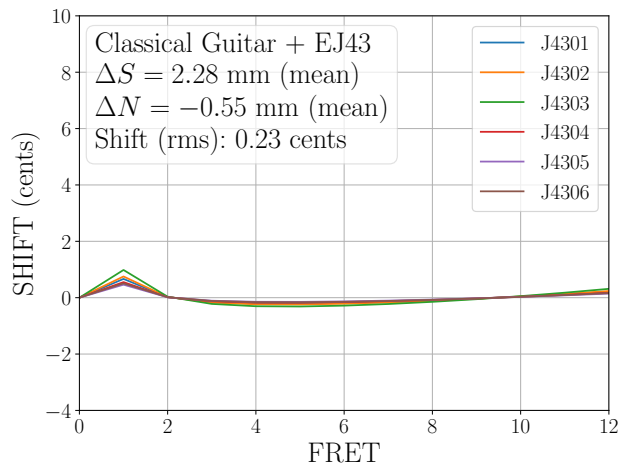
| String | $\Delta S$ (mm) | $\Delta N$ (mm) | $\overline{\Delta v}_{\text{rms}}$ (cents) |
|--------|-----------------|-----------------|--|
| J4301  | 1.95            | -0.57           | 0.24                                       |
| J4302  | 2.37            | -0.64           | 0.27                                       |
| J4303  | 3.39            | -0.82           | 0.36                                       |
| J4304  | 1.76            | -0.47           | 0.20                                       |
| J4305  | 1.83            | -0.39           | 0.17                                       |
| J4306  | 2.40            | -0.43           | 0.18                                       |



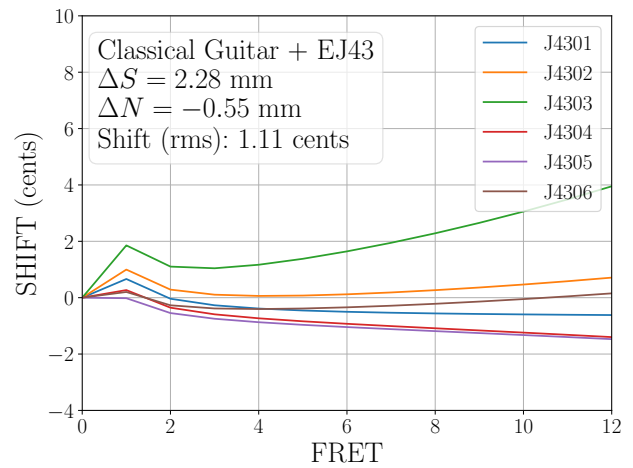
(a) Uncompensated



(b) Linear fits



(c) Full compensation



(d) Mean compensation

Figure 17: Frequency shift (in cents) for a classical guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings - Light Tension (EJ43). Four different strategies of saddle and nut compensation are illustrated.

## D.2 Hard Tension – Nylon

Table 10: String specifications for the D’Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). The corresponding scale length is 650 mm.

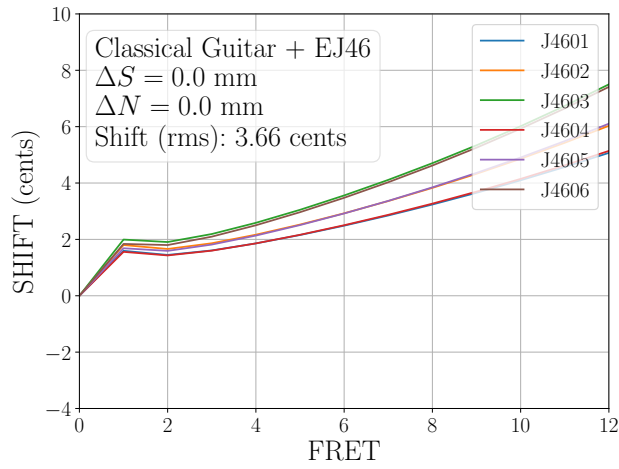
| String | Note           | Radius (mm) | Density (mg/mm) | Tension (N) |
|--------|----------------|-------------|-----------------|-------------|
| J4601  | E <sub>4</sub> | 0.362       | 0.386           | 70.9        |
| J4602  | B <sub>3</sub> | 0.415       | 0.522           | 53.8        |
| J4603  | G <sub>3</sub> | 0.521       | 0.856           | 55.6        |
| J4604  | D <sub>3</sub> | 0.381       | 2.007           | 73.1        |
| J4605  | A <sub>2</sub> | 0.457       | 3.486           | 71.3        |
| J4606  | E <sub>2</sub> | 0.559       | 5.666           | 65.0        |

Table 11: Derived physical properties of the D’Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46).

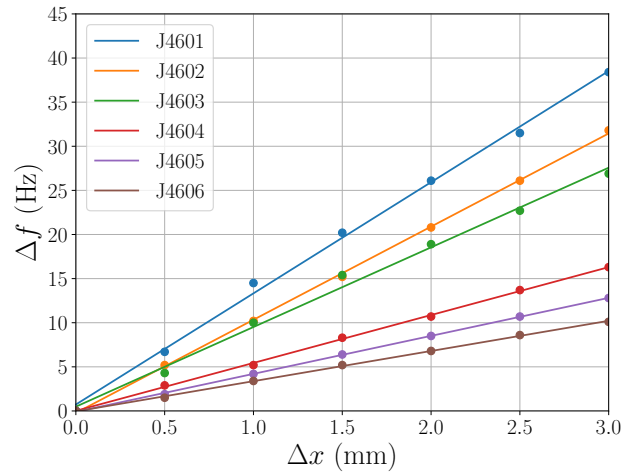
| String | $R$  | $\sigma$ | $\kappa$ | $B_0$   | $E$ (GPa) |
|--------|------|----------|----------|---------|-----------|
| J4601  | 23.5 | 0.5      | 47.9     | 0.00193 | 8.25      |
| J4602  | 26.2 | 0.3      | 53.5     | 0.00234 | 5.31      |
| J4603  | 28.3 | 1.0      | 57.5     | 0.00304 | 3.75      |
| J4604  | 22.7 | 0.3      | 46.4     | 0.00200 | 7.43      |
| J4605  | 24.0 | 0.2      | 49.0     | 0.00246 | 5.32      |
| J4606  | 25.5 | 0.3      | 51.9     | 0.00310 | 3.44      |

Table 12: Predicted setbacks for the D’Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46) on the Classical Guitar.

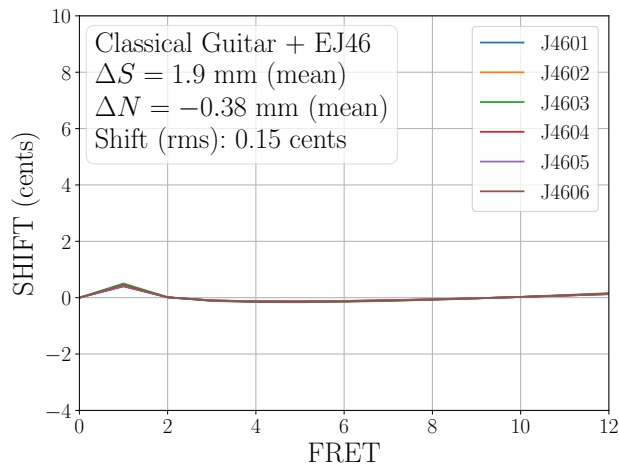
| String | $\Delta S$ (mm) | $\Delta N$ (mm) | $\overline{\Delta v_{\text{rms}}}$ (cents) |
|--------|-----------------|-----------------|--|
| J4601  | 1.50            | -0.36           | 0.15                                       |
| J4602  | 1.82            | -0.40           | 0.17                                       |
| J4603  | 2.34            | -0.42           | 0.18                                       |
| J4604  | 1.54            | -0.35           | 0.15                                       |
| J4605  | 1.88            | -0.36           | 0.15                                       |
| J4606  | 2.35            | -0.38           | 0.17                                       |



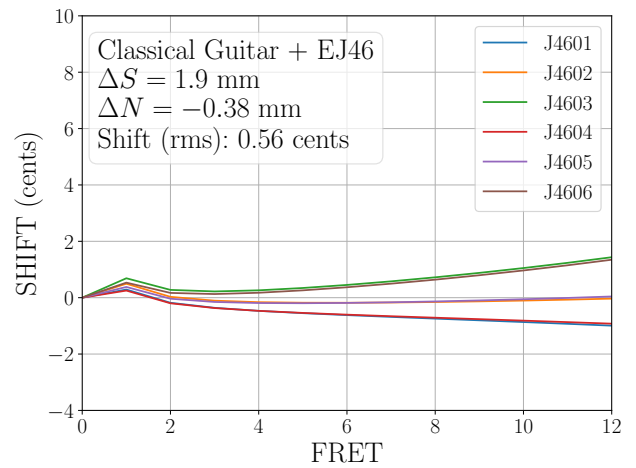
(a) Uncompensated



(b) Linear fits



(c) Full compensation



(d) Mean compensation

Figure 18: Frequency shift (in cents) for a classical guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings - Hard Tension (EJ46). Four different strategies of saddle and nut compensation are illustrated.



### D.3 Extra Hard Tension – Nylon

Table 13: String specifications for the D’Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). The corresponding scale length is 650 mm.

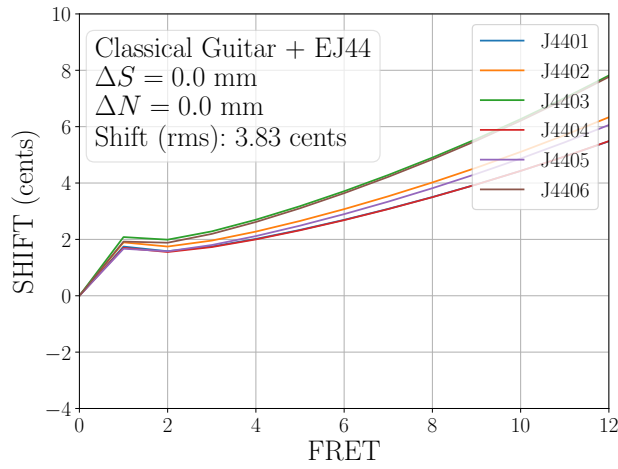
| String | Note           | Radius (mm) | Density (mg/mm) | Tension (N) |
|--------|----------------|-------------|-----------------|-------------|
| J4401  | E <sub>4</sub> | 0.368       | 0.401           | 73.6        |
| J4402  | B <sub>3</sub> | 0.423       | 0.544           | 56.1        |
| J4403  | G <sub>3</sub> | 0.528       | 0.891           | 57.8        |
| J4404  | D <sub>3</sub> | 0.381       | 2.007           | 73.1        |
| J4405  | A <sub>2</sub> | 0.457       | 3.486           | 71.3        |
| J4406  | E <sub>2</sub> | 0.571       | 6.134           | 70.4        |

Table 14: Derived physical properties of the D’Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). The corresponding scale length is 650 mm.

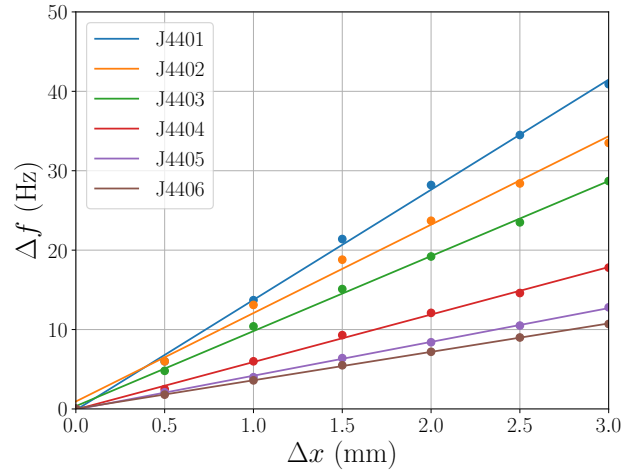
| String | $R$  | $\sigma$ | $\kappa$ | $B_0$   | $E$ (GPa) |
|--------|------|----------|----------|---------|-----------|
| J4401  | 25.8 | 0.4      | 52.6     | 0.00206 | 9.09      |
| J4402  | 27.7 | 0.9      | 56.3     | 0.00244 | 5.62      |
| J4403  | 29.6 | 0.6      | 60.2     | 0.00315 | 3.97      |
| J4404  | 25.0 | 0.5      | 51.0     | 0.00209 | 8.17      |
| J4405  | 23.7 | 0.2      | 48.5     | 0.00245 | 5.26      |
| J4406  | 26.6 | 0.2      | 54.3     | 0.00324 | 3.72      |

Table 15: Predicted setbacks for the D’Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44) on the Classical Guitar.

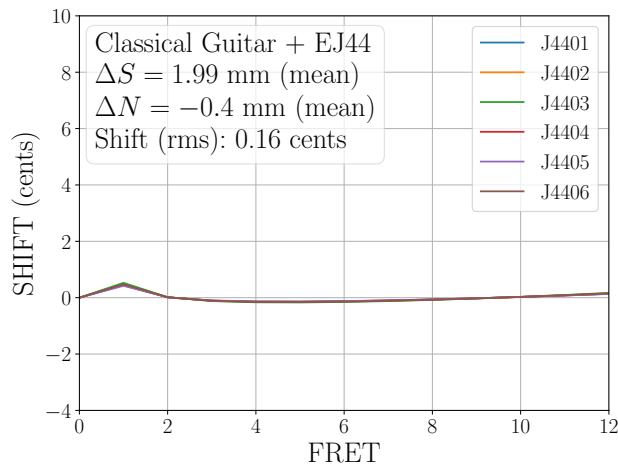
| String | $\Delta S$ (mm) | $\Delta N$ (mm) | $\overline{\Delta v_{\text{rms}}}$ (cents) |
|--------|-----------------|-----------------|--|
| J4401  | 1.62            | -0.39           | 0.17                                       |
| J4402  | 1.91            | -0.42           | 0.18                                       |
| J4403  | 2.43            | -0.44           | 0.19                                       |
| J4404  | 1.63            | -0.38           | 0.16                                       |
| J4405  | 1.87            | -0.36           | 0.15                                       |
| J4406  | 2.47            | -0.40           | 0.17                                       |



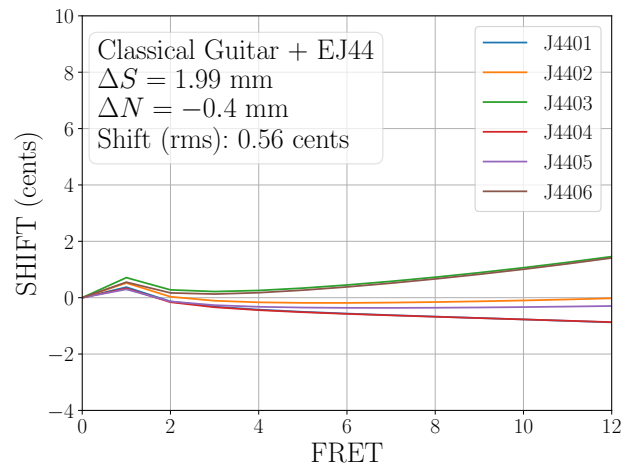
(a) Uncompensated



(b) Linear fits



(c) Full compensation



(d) Mean compensation

Figure 19: Frequency shift (in cents) for a classical guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings - Extra Hard Tension (EJ44). Four different strategies of saddle and nut compensation are illustrated.

## D.4 Normal Tension – Carbon

Table 16: String specifications for the D’Addario Pro-Arte Carbon Classical Guitar Strings – Normal Tension (EJ45FF). The corresponding scale length is 650 mm.

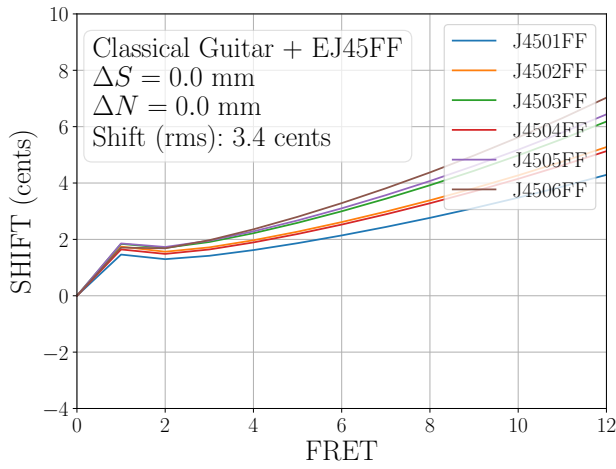
| String  | Note           | Radius (mm) | Density (mg/mm) | Tension (N) |
|---------|----------------|-------------|-----------------|-------------|
| J4501FF | E <sub>4</sub> | 0.305       | 0.464           | 85.3        |
| J4502FF | B <sub>3</sub> | 0.345       | 0.607           | 62.6        |
| J4503FF | G <sub>3</sub> | 0.420       | 0.893           | 58.0        |
| J4504FF | D <sub>3</sub> | 0.356       | 1.643           | 59.9        |
| J4505FF | A <sub>2</sub> | 0.445       | 3.089           | 63.2        |
| J4506FF | E <sub>2</sub> | 0.559       | 5.715           | 65.6        |

Table 17: Derived physical properties of the D’Addario Pro-Arte Carbon Classical Guitar Strings – Normal Tension (EJ45FF). The corresponding scale length is 650 mm.

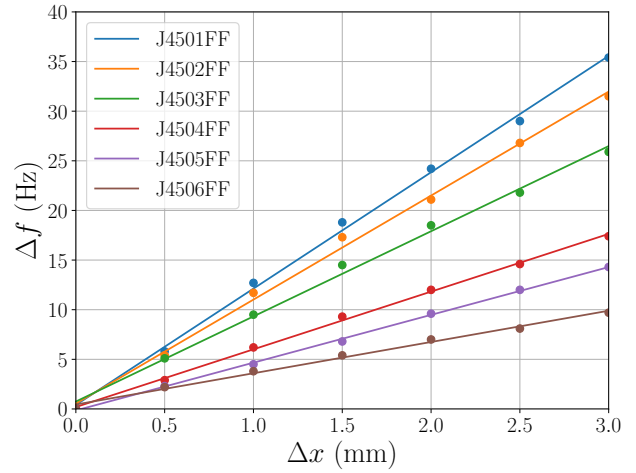
| String  | $R$  | $\sigma$ | $\kappa$ | $B_0$   | $E$ (GPa) |
|---------|------|----------|----------|---------|-----------|
| J4501FF | 21.8 | 0.4      | 44.6     | 0.00157 | 13.04     |
| J4502FF | 26.0 | 0.6      | 53.1     | 0.00194 | 8.86      |
| J4503FF | 26.9 | 0.8      | 54.7     | 0.00239 | 5.71      |
| J4504FF | 24.3 | 0.4      | 49.6     | 0.00193 | 7.47      |
| J4505FF | 26.9 | 0.4      | 54.7     | 0.00253 | 5.57      |
| J4506FF | 23.5 | 0.9      | 47.9     | 0.00298 | 3.20      |

Table 18: Predicted setbacks for the D’Addario Pro-Arte Carbon Classical Guitar Strings – Normal Tension (EJ45FF) on the Classical Guitar.

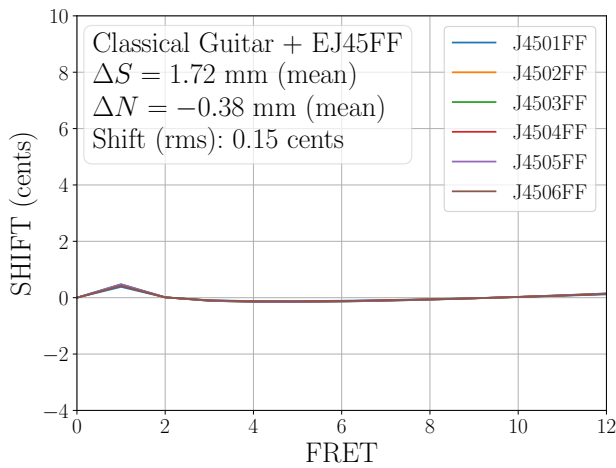
| String  | $\Delta S$ (mm) | $\Delta N$ (mm) | $\overline{\Delta v}_{\text{rms}}$ (cents) |
|---------|-----------------|-----------------|--|
| J4501FF | 1.23            | -0.33           | 0.14                                       |
| J4502FF | 1.53            | -0.40           | 0.17                                       |
| J4503FF | 1.86            | -0.41           | 0.17                                       |
| J4504FF | 1.51            | -0.37           | 0.16                                       |
| J4505FF | 1.96            | -0.41           | 0.17                                       |
| J4506FF | 2.24            | -0.35           | 0.15                                       |



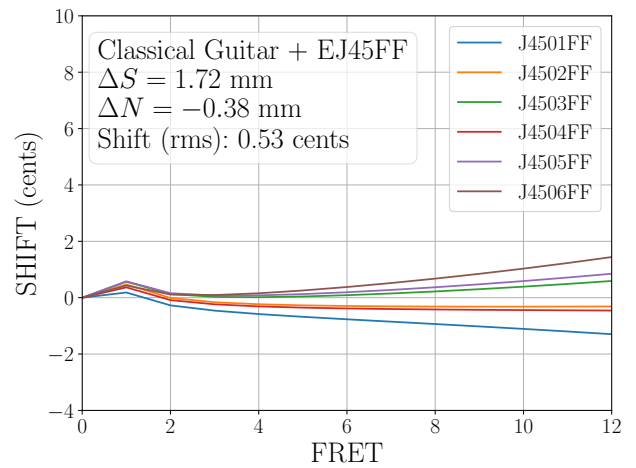
(a) Uncompensated



(b) Linear fits



(c) Full compensation



(d) Mean compensation

Figure 20: Frequency shift (in cents) for a classical guitar with D'Addario Pro-Arte Carbon Classical Guitar Strings – Normal Tension (EJ45FF). Four different strategies of saddle and nut compensation are illustrated.

## D.5 Hard Tension – Carbon

Table 19: String specifications for the D’Addario Pro-Arte Carbon Classical Guitar Strings – Hard Tension (EJ46FF). The corresponding scale length is 650 mm.

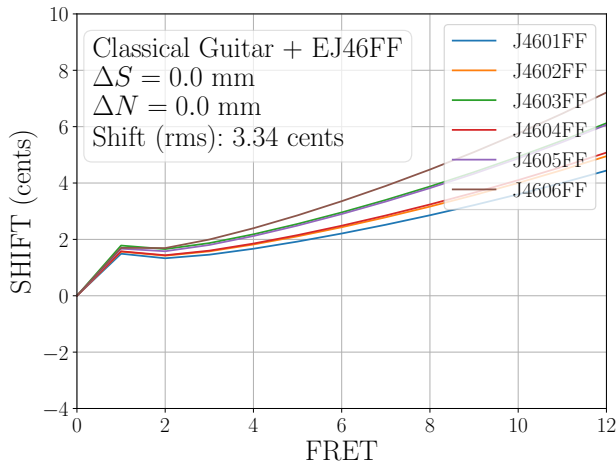
| String  | Note           | Radius (mm) | Density (mg/mm) | Tension (N) |
|---------|----------------|-------------|-----------------|-------------|
| J4601FF | E <sub>4</sub> | 0.315       | 0.500           | 91.8        |
| J4602FF | B <sub>3</sub> | 0.356       | 0.643           | 66.3        |
| J4603FF | G <sub>3</sub> | 0.431       | 0.946           | 61.4        |
| J4604FF | D <sub>3</sub> | 0.368       | 1.839           | 67.0        |
| J4605FF | A <sub>2</sub> | 0.457       | 3.554           | 72.7        |
| J4606FF | E <sub>2</sub> | 0.584       | 6.125           | 70.3        |

Table 20: Derived physical properties of the D’Addario Pro-Arte Carbon Classical Guitar Strings – Hard Tension (EJ46FF). The corresponding scale length is 650 mm.

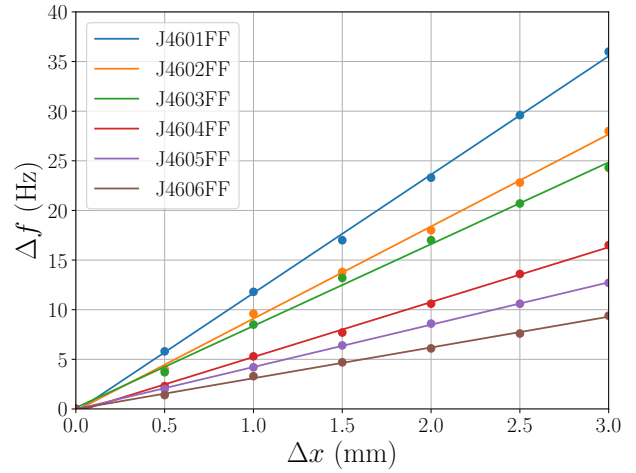
| String  | $R$  | $\sigma$ | $\kappa$ | $B_0$   | $E$ (GPa) |
|---------|------|----------|----------|---------|-----------|
| J4601FF | 22.2 | 0.3      | 45.4     | 0.00163 | 13.38     |
| J4602FF | 23.1 | 0.4      | 47.1     | 0.00188 | 7.86      |
| J4603FF | 25.8 | 0.6      | 52.6     | 0.00240 | 5.55      |
| J4604FF | 23.1 | 0.4      | 47.2     | 0.00195 | 7.42      |
| J4605FF | 23.8 | 0.2      | 48.6     | 0.00245 | 5.37      |
| J4606FF | 23.1 | 0.4      | 47.2     | 0.00309 | 3.09      |

Table 21: Predicted setbacks for the D’Addario Pro-Arte Carbon Classical Guitar Strings – Hard Tension (EJ46FF) on the Classical Guitar.

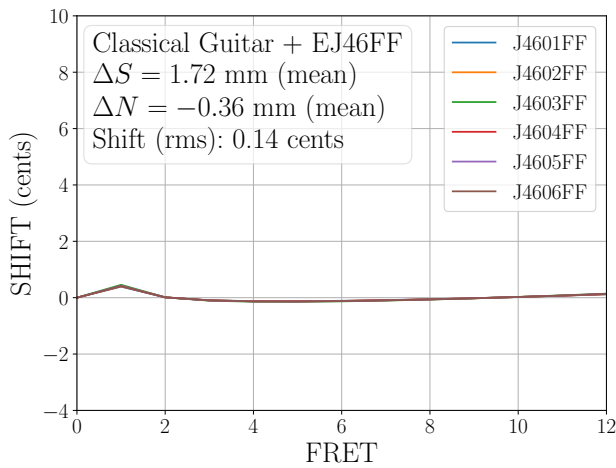
| String  | $\Delta S$ (mm) | $\Delta N$ (mm) | $\overline{\Delta v}_{\text{rms}}$ (cents) |
|---------|-----------------|-----------------|--|
| J4601FF | 1.28            | -0.34           | 0.14                                       |
| J4602FF | 1.46            | -0.35           | 0.15                                       |
| J4603FF | 1.86            | -0.39           | 0.17                                       |
| J4604FF | 1.51            | -0.35           | 0.15                                       |
| J4605FF | 1.87            | -0.36           | 0.15                                       |
| J4606FF | 2.32            | -0.35           | 0.15                                       |



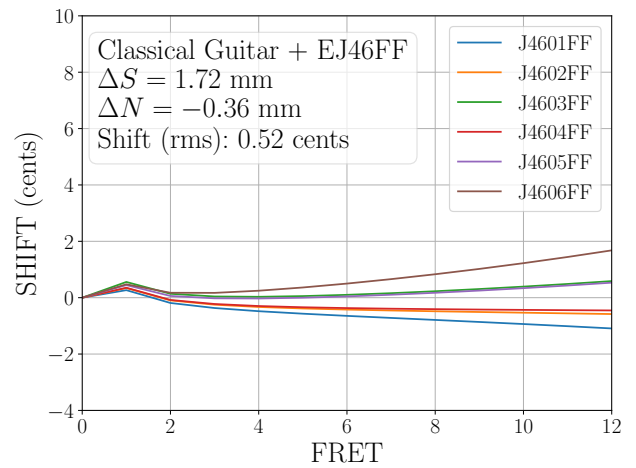
(a) Uncompensated



(b) Linear fits



(c) Full compensation



(d) Mean compensation

Figure 21: Frequency shift (in cents) for a classical guitar with D'Addario Pro-Arte Carbon Classical Guitar Strings - Hard Tension (EJ46FF). Four different strategies of saddle and nut compensation are illustrated.

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