Classical Guitar Intonation and Compensation: The Well-Tempered Guitar

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September 26, 2020

Abstract

TBD.

Contents

1 Introduction and Background

Discuss initial [1] and ongoing work by G. Byers [2], and recent studies of steel-string guitars [3].

2 Simple Model of Guitar Intonation

Fundamental frequency of a string [4, 5]:

$$f_0 = \frac{1}{2\,L_0}\sqrt{\frac{T_0}{\mu_0}}\,, \tag{1}$$

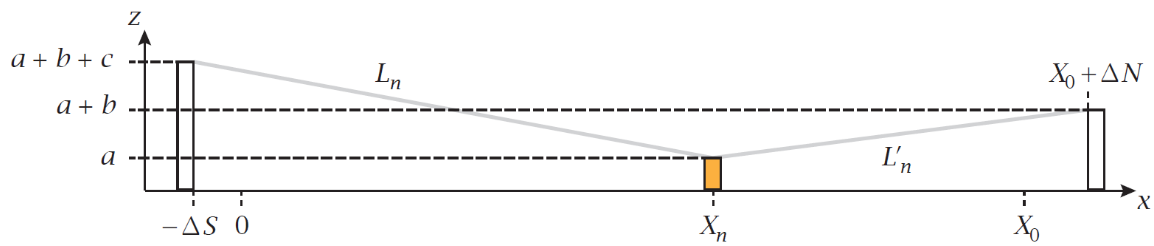


Figure 1: A simple (side-view) schematic of the classical guitar used in this model. The scale length of the guitar is X_0, but we allow the edges of both the saddle and the nut to be set back an additional distance ΔS and ΔN, respectively. The location on the x-axis of the center of the n^{th} fret is X_n. In the z direction, $z = 0$ is taken as the surface of the fingerboard; therefore the height of each fret above the fingerboard is a, the height of the nut is $a + b$, and the height of the saddle is $a + b + c$. L_n is the *resonant length* of the string from the saddle to the center of fret n, and L'_n is the length of the string from the fret to the nut.

where L_0 is the length of the free (unfretted) string from the saddle to the nut, T_0 is the tension in the free string, and $\mu_0 \equiv M/L_0$ is the linear mass density of a free string of mass M.

$$f_0 = \frac{1}{2\,L_0}\sqrt{\frac{T_0}{\mu_0}}\left[1 + B_0 + \left(1 + \frac{\pi^2}{8}\right)B_0^2\right], \tag{2}$$

where B_0 is a "string stiffness parameter." For a uniform string with a cylindrical cross section, B_0 given by [6]

$$B_0 \equiv \sqrt{\frac{\pi\,R^4 E}{T_0\,L_0^2}}, \tag{3}$$

where R is the radius of the string and E is Young's modulus (or the modulus of elasticity). For a typical nylon guitar string with $E \approx 2 - 4$ GPa, $T_0 \approx 50 - 70$ N, $R \approx 0.35 - 0.51$ mm, and $L_0 \approx 650$ mm, we have $B_0 \approx 0.007 - 0.026$, indicating that the corrections in Eq. (2) are not significant.

Throughout this work, we will use *cents* to describe small differences in pitch [7]. One cent is one one-hundredth of a 12-TET half step, so that there are 1200 cents per octave. The difference in pitch between frequencies f_1 and f_2 is therefore defined as

$$\Delta\nu \equiv 1200\,\log_2\left(\frac{f_2}{f_1}\right). \tag{4}$$

We define $f \equiv (f_1 + f_2)/2$ and $\Delta f \equiv f_2 - f_1$. Then

$$\Delta\nu = 1200\,\log_2\left(\frac{f + \Delta f/2}{f - \Delta f/2}\right) \approx \frac{1200}{\ln 2}\,\frac{\Delta f}{f}, \tag{5}$$

where the last approximation applies when $\Delta f \ll f$. An experienced guitar player can distinguish beat notes with a difference frequency of $\Delta f \approx 1$ Hz, which corresponds to 8 cents at A_2 ($f = 220$ Hz) or 5 cents at E_4 ($f = 329.63$ Hz).

Our model begins with the schematic of the guitar shown in Fig. 1. The scale length of the guitar is X_0, but we allow the edges of both the saddle and the nut to be set back an additional distance ΔS and ΔN, respectively. The location on the x-axis of the center of the n^{th} fret is X_n. In the z direction, $z = 0$ is taken as the surface of the fingerboard; the height of each fret is a, the height of the nut is $a+b$, and the height of the saddle is $a+b+c$. L_n is the *resonant length* of the string from the saddle to the center of fret n, and L'_n is the length of the string from the fret to the nut. The total length of the string is defined as $\mathcal{L}_n \equiv L_n + L'_n$. For reasons discussed below, we have not adopted a more complicated fretting model [2, 3]. We start with the simple form of the fundamental frequency of a string given by Eq. (1), and apply it to the frequency of a string pressed just behind the n^{th} fret:

$$f_n = \frac{1}{2\,L_n}\sqrt{\frac{T_n}{\mu_n}}\,, \tag{6}$$

where T_n and μ_n are respectively the tension and linear mass density of the fretted string. We note that T_n and μ_n depend on $\mathcal{L}_n$, the *total* length of the fretted string from the saddle to the nut. Ideally, in the 12-TET equal-temperament system [7],

$$f_n = \gamma_n\, f_0\,, \qquad \text{(12-TET ideal)} \tag{7}$$

where

$$\gamma_n \equiv 2^{n/12}\,. \tag{8}$$

Therefore, the error interval expressed in cents is given by

$$\begin{aligned}
\Delta\nu_n &= 1200\,\log_2\left(\frac{f_n}{\gamma_n\, f_0}\right) \\
&= 1200\,\log_2\left(\frac{L_0}{\gamma_n\, L_n}\sqrt{\frac{\mu_0}{\mu_n}\,\frac{T_n}{T_0}}\right) \\
&= 1200\,\log_2\left(\frac{L_0}{\gamma_n\, L_n}\right) + 600\,\log_2\left(\frac{\mu_0}{\mu_n}\right) + 600\,\log_2\left(\frac{T_n}{T_0}\right)\,.
\end{aligned} \tag{9}$$

The final form of Eq. (9) makes it clear that — for nylon guitar strings — there are three contributions to intonation:

1. *Resonant Length*: The first term represents the error caused by the increase in the length of the fretted string L_n compared to the ideal length X_n, which would be obtained if $b = c = 0$ and $\Delta S = \Delta N = 0$.

2. *Linear Mass Density*: The second term is the error caused by the reduction of the linear mass density of the fretted string. This effect will depend on the *total* length of the string, given by $\mathcal{L}_n = L_n + L'_n$.

3. *Tension*: The third (and most complex) term is the error caused by the *increase* of the tension in the string caused by the stress and strain applied to the string by fretting. This effect will also depend on the total length of the string $\mathcal{L}_n$.

We will discuss each of these three sources of error in turn below.

2.1 Resonant Length Error

We can estimate the first term in the last line of Eq. (9) by referring to Fig. 1 and computing the resonant length L_n. We find:

$$L_n = \begin{cases} \sqrt{(X_0 + \Delta S + \Delta N)^2 + c^2}\,, & n = 0 \\ \sqrt{(X_n + \Delta S)^2 + (b+c)^2}\,. & n \ge 1 \end{cases} \tag{10}$$

When $b + c \ll X_0$, we can approximate L_n by

$$L_n \approx \begin{cases} X_0 + \Delta S + \Delta N + c^2/2\,X_0 & n = 0\,, \\ X_n + \Delta S + (b+c)^2/2\,X_n & n \geq 1\,. \end{cases} \tag{11}$$

Then — when the guitar has been manufactured such that $X_n = X_0/\gamma_n$ — the resonant length error is approximately

$$1200\,\log_2\left(\frac{L_0}{\gamma_n\,L_n}\right) \approx -\frac{1200}{\ln(2)}\left[\frac{(\gamma_n - 1)\,\Delta S - \Delta N}{X_0} + \frac{\gamma_n^2(b+c)^2 - c^2}{2\,X_0^2}\right] \tag{12}$$

If the guitar is uncompensated, so that $\Delta S = \Delta N = 0$, this error is typically less than 0.25 cents. But, with $\Delta S > 0$ and $\Delta N < 0$, we can significantly *increase* the magnitude of this "error" and cause the frequency to shift lower. We'll see that this is our primary method of compensation. TBD: Add a figure here?

Previous studies of guitar intonation and compensation have chosen to include the apparent increase in length of the string caused by both the fretting depth (Matt: How does this sound?) and the shape of the fretted string under the finger [2, 3]. As the string is initially pressed to the fret, the total length $\mathcal{L}_n$ increases and causes the tension in the string — which is clamped at the saddle and the nut — to increase. As the string is pressed further, does the additional deformation of the string increase its tension (throughout the resonant length L_n)? There are at least two purely empirical reasons to doubt this hypothesis. First, we can mark a string (with a fine-point felt pen) above a particular fret and then observe the mark with a magnifying glass. As the string is pressed all the way to the finger board, the mark does not move perceptibly — it has become effectively *clamped* on the fret. Second, we can use either our ears or a simple tool to measure frequencies [8] to listen for a shift as we use different fingers and vary the fretted depth of a string. The apparent modulation is far less than would be obtained by classical vibrato (± 15 cents), so we assume that once the string is minimally fretted the length(s) can be regarded as fixed. (If this were not the case, then fretting by different people or with different fingers, at a single string or with a barre, would cause additional, varying frequency shifts that would be audible and difficult to compensate.)

2.2 Linear Mass Density Error

As discussed above, the linear mass density μ_0 of an unfretted string is simply the total mass M of the string clamped between the saddle and the nut divided by the length L_0. Similarly, the mass density μ_n of a string held onto fret N is $M/\mathcal{L}_n$. Therefore

$$\frac{\mu_0}{\mu_n} = \frac{\mathcal{L}_n}{L_0} \equiv 1 + \lambda_n\,, \tag{13}$$

where

$$\lambda_n \equiv \frac{\mathcal{L}_n - L_0}{L_0}\,. \tag{14}$$

Since we expect that $\lambda_n \ll 1$, we can approximate the second term in the final line of Eq. (9) as

$$600\,\log_2\left(\frac{\mu_0}{\mu_n}\right) \approx \frac{600}{\ln(2)}\,\lambda_n\,. \tag{15}$$

Referring to Fig. 1, we see that $\mathcal{L}_n = L_n + L'_n$, and we calculate L'_n as

$$L'_n = \begin{cases} 0\,, & n = 0 \\ \sqrt{(X_0 - X_n + \Delta N)^2 + b^2}\,. & n \geq 1 \end{cases} \tag{16}$$

Assuming that $b^2 \ll X_0 - X_n$, we expand the $n \neq 1$ expression to obtain

$$L'_n \approx X_0 - X_n + \Delta N + \frac{b^2}{2\,(X_0 - X_n)} \tag{17}$$

Therefore, using Eq. (11), we have for $n \neq 1$

$$\mathcal{L}_n = L_n + L'_n \approx X_0 + \Delta S + \Delta N + \frac{(b+c)^2}{2\,X_n} + \frac{b^2}{2\,(X_0 - X_n)}\,, \tag{18}$$

and

$$\begin{aligned} \lambda_n &\approx \frac{1}{2\,X_0}\left[\frac{(b+c)^2}{X_n} + \frac{b^2}{X_0 - X_n} - \frac{c^2}{X_0}\right] \\ &= \frac{\gamma_n}{2\,X_0^2}\left[(b+c)^2 + \frac{b^2}{\gamma_n - 1} - \frac{c^2}{\gamma_n}\right]. \end{aligned} \tag{19}$$

When we substitute this expression into Eq. (15), the resulting error is generally quite small. Suppose that $b = 1.0$ mm, $c = 3.5$ mm, $X_0 = 650$ mm, $n = 12$, and $X_{12} = X_0/2 = 325$ mm. Then the error is about 0.03 cents, and will be even smaller for $n < 12$. If we add this shift due to the linear mass density to the residual quadratic resonant length shift given by Eq. (12), then we find the total error

$$\Delta\nu_n = \frac{300}{\ln(2)}\,\frac{\gamma_n}{X_0^2}\left[\frac{b^2}{\gamma_n - 1} + \frac{c^2}{\gamma_n} - (2\gamma_n - 1)(b+c)^2\right]. \tag{20}$$

For the same parameters, $\Delta\nu_{12} = -0.11$ cents, and $|\Delta\nu_n| < |\Delta\nu_{12}|$ for $n < 12$.

2.3 Tension Error

Elasticity properties [9]

$$\Delta T_n = E\,A\,\frac{\mathcal{L}_n - L_0}{L_0} = A\,E\,\lambda_n\,, \tag{21}$$

where we have used Eq. (14). Therefore, the tension of the fretted string is

$$T_n = T_0 + T_n = T_0\,(1 + \kappa\,\lambda_n)\;, \tag{22}$$

where we have defined the dimensionless "string constant"

$$\kappa \equiv \frac{A\,E}{T_0} = \frac{\pi R^2 E}{T_0}\,. \tag{23}$$

In this case, we assume that $\kappa\lambda_n \ll 1$, so that we can approximate the third term in the final line of Eq. (9) as

$$600\,\log_2\left(\frac{T_n}{T_0}\right) \approx \frac{600}{\ln(2)}\,\kappa\,\lambda_n\,. \tag{24}$$

This frequency shift is larger than that caused by the linear mass density error by a factor of κ.

3 Experimental Estimate of the String Constant

$$
\begin{aligned}
L_{n\geq 1}(y) &= \sqrt{(X_n + \Delta S)^2 + (b+c)^2 + y^2} \\
&\approx X_n + \Delta S + \frac{(b+c)^2 + y^2}{2\,X_n}\,.
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
L'_{n\geq 1}(y) &= \sqrt{(X_0 - X_n + \Delta N)^2 + b^2 + y^2} \\
&\approx X_0 - X_n + \Delta N + \frac{b^2 + y^2}{2\,(X_0 - X_n)}\,.
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\lambda_n(y) &\approx \frac{1}{2\,X_0}\left[\frac{(b+c)^2 + y^2}{X_n} + \frac{b^2 + y^2}{X_0 - X_n} - \frac{c^2}{X_0}\right] \\
&= \lambda_n(0) + \Delta\lambda_n(y)\,,
\end{aligned}
\tag{27}
$$

where $\lambda_n(0)$ is given by Eq. (19), and

$$
\Delta\lambda_n(y) \equiv \frac{1}{2\,(\gamma_n - 1)}\left(\frac{\gamma_n\, y}{X_0}\right)^2\,.
\tag{28}
$$

Following the same approach we used to derive Eq. (20), we can derive the change in the total shift due to both resonant length and linear mass density for a transverse displacement y. To second order in y, we find that

$$
\Delta\nu_n(y) \approx \frac{600}{\ln(2)}\,\frac{3 - 2\gamma_n}{2\,(\gamma_n - 1)}\left(\frac{\gamma_n\, y}{X_0}\right)^2\,.
\tag{29}
$$

We have plotted this expression for the first 12 frets and $y = 5$ mm in Fig. 2. This shift is quite small compared to the experimental errors we'll obtain in the shifts due to tension, and we ignore it in what follows.

4 Classical Guitar Compensation

5 Conclusion

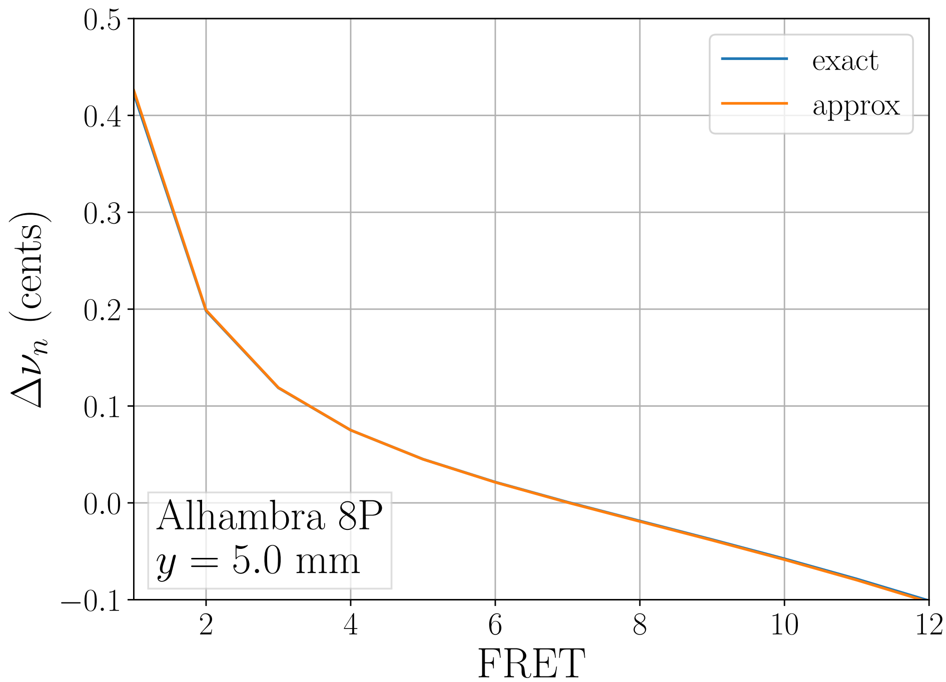
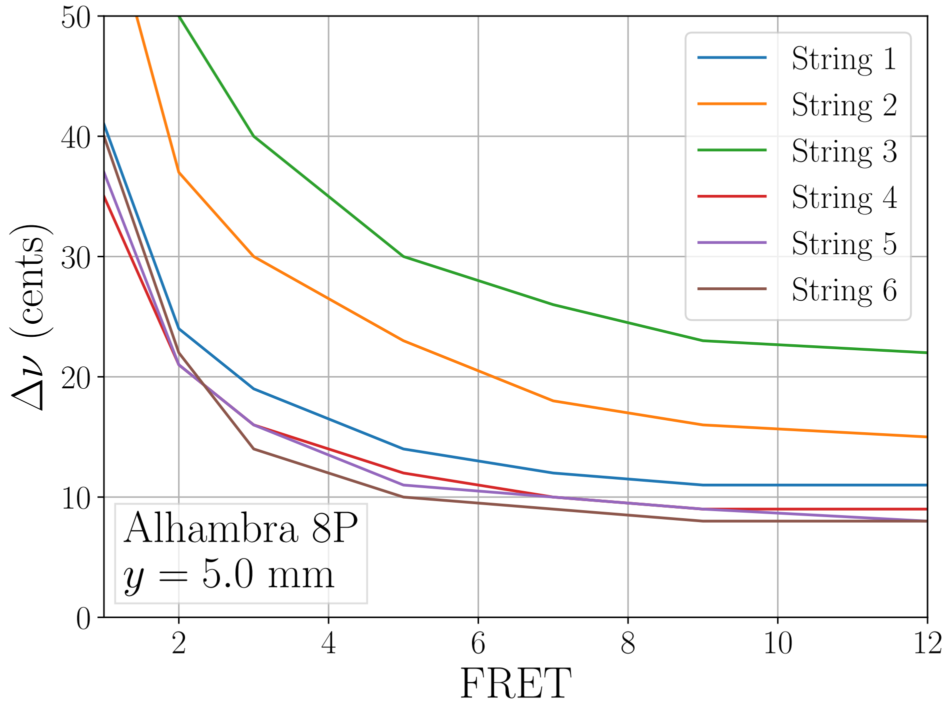
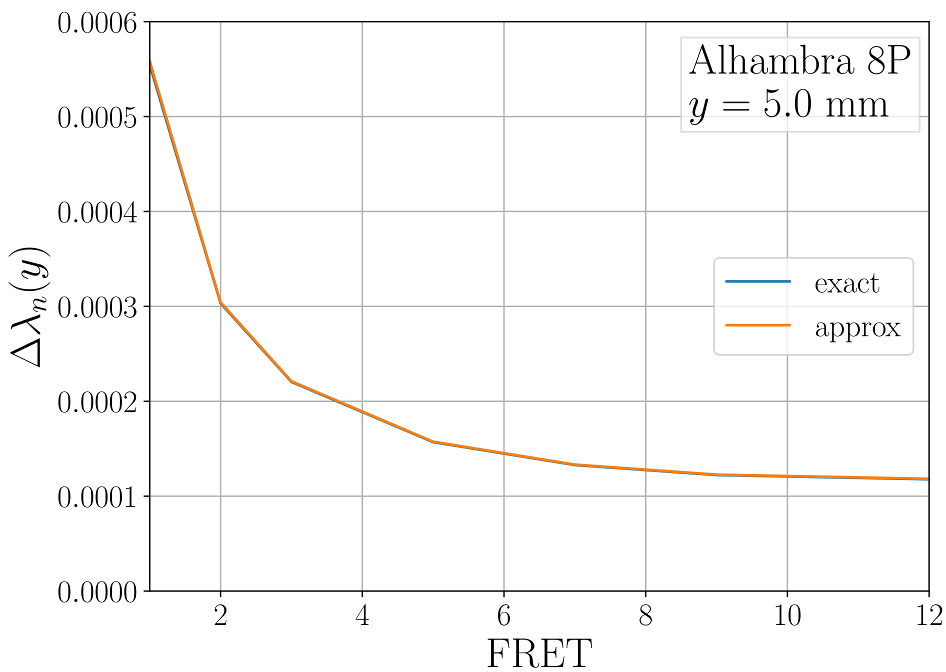


Figure 2: Total frequency shift (in cents) due to resonant length and linear mass density for a transverse displacement of $y = 5$ mm. This shift is identical for each string, and should be smaller than the experimental errors we'll accumulate using our transverse displacement approach.

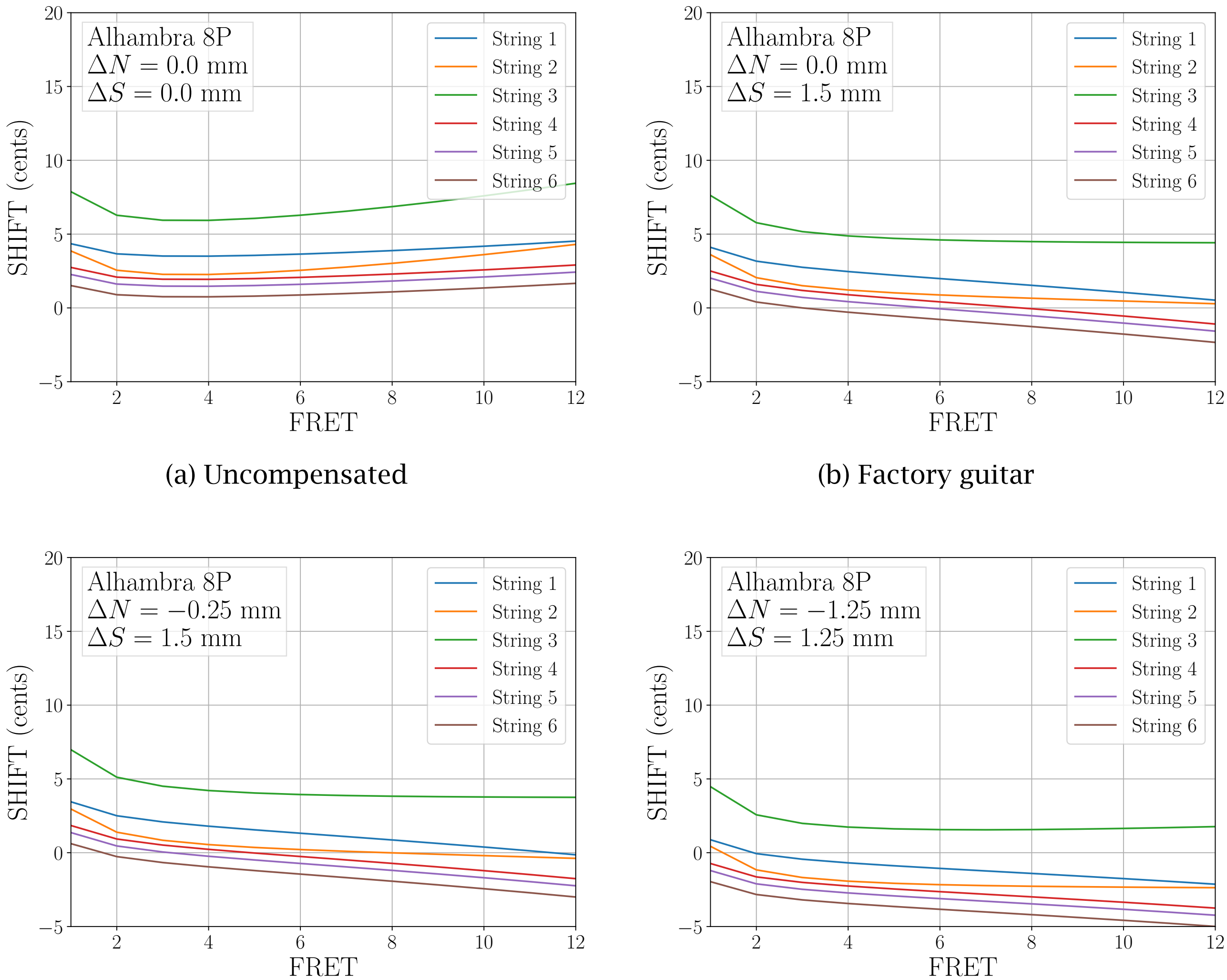


(a) Experimental data



(b) Calculated change in total string length $\mathcal{L}$

Figure 3: Frequency shift (in cents) (a) and change in total string length $\mathcal{L}$ (b) due to lateral displacement y.



(a) Uncompensated

(b) Factory guitar

(c) Saddle and nut

(d) Preserved scale length

Figure 4: Frequency shift (in cents) for four different strategies of saddle and nut compensation.

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