# Classical Guitar Intonation and Compensation: The Well-Tempered Guitar

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**Abstract**

TBD.

## Contents



# 1 Introduction and Background

Discuss initial [1] and ongoing work by G. Byers [2], and recent studies of steel-string guitars [3].

Fundamental frequency of a string [4, 5]:

$$f_0 = \frac{1}{2\,L_0}\sqrt{\frac{T_0}{\mu_0}}\,, \tag{1}$$

where $L_0$ is the length of the free (unfretted) string from the saddle to the nut, $T_0$ is the tension in the free string, and $\mu_0 \equiv M/L_0$ is the linear mass density of a free string of mass $M$.

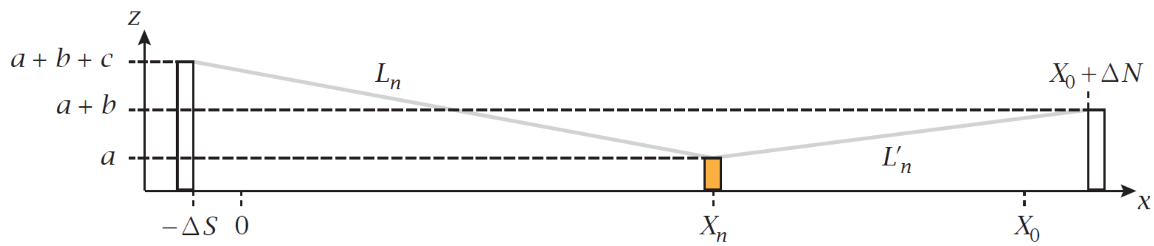



Figure 1: A simple (side-view) schematic of the classical guitar used in this model. The scale length of the guitar is $X_0$, but we allow the edges of both the saddle and the nut to be set back an additional distance $\Delta S$ and $\Delta N$, respectively. The location on the $x$-axis of the center of the $n^{\text{th}}$ fret is $X_n$. In the $z$ direction, $z = 0$ is taken as the surface of the fingerboard; therefore the height of each fret above the fingerboard is $a$, the height of the nut is $a + b$, and the height of the saddle is $a + b + c$. $L_n$ is the *resonant length* of the string from the saddle to the center of fret $n$, and $L'_n$ is the length of the string from the fret to the nut. Reflect the image through the $z$-axis.

## 2 Simple Model of Guitar Intonation

$$f_n = \frac{1}{2\,L_n}\sqrt{\frac{T_n}{\mu_n}}\left[1 + B_n + \left(1 + \frac{\pi^2}{8}\right)B_n^2\right], \tag{2}$$

where $L_n$ is the length of the vibrating string from the saddle to fret $n$, $T_n$ and $\mu_n$ are the tension and the linear mass density of the fretted string repectively. Here, we are using $B_n$ to represent a small string "bending stiffness" coefficient to capture the relevant mechanical properties of the transversely vibrating string. For a homogeneous string with a cylindrical cross section, $B_0$ is given by [6]

$$B_0 \equiv \sqrt{\frac{\pi\,\rho^4\,E}{T_0\,L_0^2}}, \tag{3}$$

where $\rho$ is the radius of the string and $E$ is Young's modulus (or the modulus of elasticity). Note that Eq. (2) is valid only when $B_n \ll 1$. For a typical nylon guitar string with $E \approx 4$ GPa, $T_0 \approx 60$ N, $\rho \approx 0.5$ mm, and $L_0 \approx 650$ mm, we have $B_0 \approx 6 \times 10^{-3}$. We'll use Eq. (2) and Eq. (3) with considerable caution, because it's unlikely that they describe the physics of nylon strings (particularly the wound base strings) with a high degree of accuracy [7].

Throughout this work, we will use *cents* to describe small differences in pitch [8]. One cent is one one-hundredth of a Twelve-Tone Equal Temperament (12-TET) half step, so that there are 1200 cents per octave. The difference in pitch between frequencies $f_1$ and $f_2$ is therefore defined as

$$\Delta\nu \equiv 1200\,\log_2\left(\frac{f_2}{f_1}\right). \tag{4}$$

We define $f \equiv (f_1 + f_2)/2$ and $\Delta f \equiv f_2 - f_1$. Then

$$\Delta\nu = 1200\,\log_2\left(\frac{f + \Delta f/2}{f - \Delta f/2}\right) \approx \frac{1200}{\ln 2}\,\frac{\Delta f}{f}, \tag{5}$$

where the last approximation applies when $\Delta f \ll f$. An experienced guitar player can distinguish beat notes with a difference frequency of $\Delta f \approx 1$ Hz, which corresponds to 8 cents at $A_2$ ($f = 220$ Hz) or 5 cents at $E_4$ ($f = 329.63$ Hz).

Our model begins with the schematic of the guitar shown in Fig. 1. The scale length of the guitar is $X_0$, but we allow the edges of both the saddle and the nut to be set back an additional distance $\Delta S$ and $\Delta N$, respectively. The location on the $x$-axis of the center of the $n^{\text{th}}$ fret is $X_n$. In the $z$ direction, $z = 0$ is taken as the surface of the fingerboard; the height of each fret is $a$, the height of the nut is $a + b$, and the height of the saddle is $a + b + c$. $L_n$ is the *resonant length* of the string from the saddle to the center of fret $n$, and $L'_n$ is the length of the string from the fret to the nut. The total length of the string is defined as $\mathcal{L}_n \equiv L_n + L'_n$. For reasons discussed below, we have not adopted a more complicated fretting model [2, 3]. We start with form of the fundamental frequency of a fretted string given by Eq. (2), and apply it to the frequency of a string pressed just behind the $n^{\text{th}}$ fret:

$$f_n = \frac{1}{2\,L_n}\sqrt{\frac{T_n}{\mu_n}}\,(1 + B_n)\ , \tag{6}$$

where we have neglected the smallest stiffness correction term proportional to $B_n^2$. We note that $T_n$ and $\mu_n$ depend on $\mathcal{L}_n$, the *total* length of the fretted string from the saddle to the nut. Ideally, in the 12-TET system [8],

$$f_n = \gamma_n\, f_0\,, \qquad \text{(12-TET ideal)} \tag{7}$$

where

$$\gamma_n \equiv 2^{n/12}\,. \tag{8}$$

Therefore, the error interval expressed in cents is given by

$$\begin{aligned}\Delta\nu_n &= 1200\,\log_2\left(\frac{f_n}{\gamma_n\, f_0}\right)\\ &= 1200\,\log_2\left(\frac{L_0}{\gamma_n\, L_n}\sqrt{\frac{\mu_0}{\mu_n}\,\frac{T_n}{T_0}}\,\frac{1 + B_n}{1 + B_0}\right)\\ &= 1200\,\log_2\left(\frac{L_0}{\gamma_n\, L_n}\right) + 600\,\log_2\left(\frac{\mu_0}{\mu_n}\right) + 600\,\log_2\left(\frac{T_n}{T_0}\right) + 1200\,\log_2\left(\frac{1 + B_n}{1 + B_0}\right)\,.\end{aligned} \tag{9}$$

The final form of Eq. (9) makes it clear that — for nylon guitar strings — there are four contributions to intonation:

1. *Resonant Length*: The first term represents the error caused by the increase in the length of the fretted string $L_n$ compared to the ideal length $X_n$, which would be obtained if $b = c = 0$ and $\Delta S = \Delta N = 0$.

2. *Linear Mass Density*: The second term is the error caused by the reduction of the linear mass density of the fretted string. This effect will depend on the *total* length of the string, given by $\mathcal{L}_n = L_n + L'_n$.

3. *Tension*: The third term is the error caused by the *increase* of the tension in the string arising from the stress and strain applied to the string by fretting. This effect will also depend on the total length of the string $\mathcal{L}_n$.

4. *Bending Stiffness*: The fourth and final term is the error caused by the change in the bending stiffness coefficient caused by changing the length of the string from $L_0$ to $L_n$.

We will discuss each of these three sources of error in turn below.

## 2.1 Resonant Length

We can estimate the first term in the last line of Eq. (9) by referring to Fig. 1 and computing the resonant length $L_n$. We find:

$$L_n = \begin{cases} \sqrt{(X_0 + \Delta S + \Delta N)^2 + c^2}\,, & n = 0 \\ \sqrt{(X_n + \Delta S)^2 + (b + c)^2}\,. & n \geq 1 \end{cases} \tag{10}$$

When $b + c \ll X_0$, we can approximate $L_n$ by

$$L_n \approx \begin{cases} X_0 + \Delta S + \Delta N + c^2/2\,X_0 & n = 0\,, \\ X_n + \Delta S + (b + c)^2/2\,X_n & n \geq 1\,. \end{cases} \tag{11}$$

Then — when the guitar has been manufactured such that $X_n = X_0/\gamma_n$ — the resonant length error is approximately

$$1200 \log_2\left(\frac{L_0}{\gamma_n L_n}\right) \approx -\frac{1200}{\ln(2)}\left[\frac{(\gamma_n - 1)\,\Delta S - \Delta N}{X_0} + \frac{\gamma_n^2 (b + c)^2 - c^2}{2\,X_0^2}\right] \tag{12}$$

If the guitar is uncompensated, so that $\Delta S = \Delta N = 0$, this error is typically less than 0.25 cents. But, with $\Delta S > 0$ and $\Delta N < 0$, we can significantly *increase* the magnitude of this "error" and cause the frequency to shift lower. We'll see that this is our primary method of compensation. TBD: Add a figure here?

Previous studies of guitar intonation and compensation have chosen to include the apparent increase in length of the string caused by both the fretting depth and the shape of the fretted string under the finger [2, 3]. As the string is initially pressed to the fret, the total length $\mathcal{L}_n$ increases and causes the tension in the string — which is clamped at the saddle and the nut — to increase. As the string is pressed further, does the additional deformation of the string increase its tension (throughout the resonant length $L_n$)? There are at least two purely empirical reasons to doubt this hypothesis. First, we can mark a string (with a fine-point felt pen) above a particular fret and then observe the mark with a magnifying glass. As the string is pressed all the way to the finger board, the mark does not move perceptibly — it has become effectively *clamped* on the fret. Second, we can use either our ears or a simple tool to measure frequencies [9] to listen for a shift as we use different fingers and vary the fretted depth of a string. The apparent modulation is far less than would be obtained by classical vibrato ($\pm$15 cents), so we assume that once the string is minimally fretted the length(s) can be regarded as fixed. (If this were not the case, then fretting by different people or with different fingers, at a single string or with a barre, would cause additional, varying frequency shifts that would be audible and difficult to compensate.)

## 2.2 Linear Mass Density

As discussed above, the linear mass density $\mu_0$ of an open (unfretted) string is simply the total mass $M$ of the string clamped between the saddle and the nut divided by the length $L_0$. Similarly, the mass density $\mu_n$ of a string held onto fret $N$ is $M/\mathcal{L}_n$. Therefore

$$\frac{\mu_0}{\mu_n} = \frac{\mathcal{L}_n}{L_0} \equiv 1 + Q_n\,, \tag{13}$$

where we have followed Byers and defined [2, 3]

$$Q_n \equiv \frac{\mathcal{L}_n - L_0}{L_0}\,. \tag{14}$$

Since we expect that $Q_n \ll 1$, we can approximate the second term in the final line of Eq. (9) as

$$600\,\log_2\left(\frac{\mu_0}{\mu_n}\right) \approx \frac{600}{\ln(2)}\,Q_n\,. \tag{15}$$

Referring to Fig. 1, we see that $\mathcal{L}_n = L_n + L'_n$, and we calculate $L'_n$ as

$$L'_n = \begin{cases} 0\,, & n = 0 \\ \sqrt{(X_0 - X_n + \Delta N)^2 + b^2}\,. & n \geq 1 \end{cases} \tag{16}$$

Assuming that $b^2 \ll X_0 - X_n$, we expand the $n \neq 1$ expression to obtain

$$L'_n \approx X_0 - X_n + \Delta N + \frac{b^2}{2\,(X_0 - X_n)} \tag{17}$$

Therefore, using Eq. (11), we have for $n \neq 1$

$$\mathcal{L}_n = L_n + L'_n \approx X_0 + \Delta S + \Delta N + \frac{(b+c)^2}{2\,X_n} + \frac{b^2}{2\,(X_0 - X_n)}\,, \tag{18}$$

and

$$\begin{aligned} Q_n &\approx \frac{1}{2\,X_0}\left[\frac{(b+c)^2}{X_n} + \frac{b^2}{X_0 - X_n} - \frac{c^2}{X_0}\right] \\ &= \frac{\gamma_n}{2\,X_0^2}\left[(b+c)^2 + \frac{b^2}{\gamma_n - 1} - \frac{c^2}{\gamma_n}\right]. \end{aligned} \tag{19}$$

When we substitute this expression into Eq. (15), the resulting error is generally quite small. Suppose that $b = 1.0$ mm, $c = 3.5$ mm, $X_0 = 650$ mm, $n = 12$, and $X_{12} = X_0/2 = 325$ mm. Then the error is about 0.03 cents, and will be even smaller for $n < 12$. If we add this shift due to the linear mass density to the residual quadratic resonant length shift given by Eq. (12), then we find the total error

$$\Delta\nu_n = \frac{300}{\ln(2)}\,\frac{\gamma_n}{X_0^2}\left[\frac{b^2}{\gamma_n - 1} + \frac{c^2}{\gamma_n} - (2\gamma_n - 1)(b+c)^2\right]. \tag{20}$$

For the same parameters, $\Delta\nu_{12} = -0.11$ cents, and $|\Delta\nu_n| < |\Delta\nu_{12}|$ for $n < 12$.

## 2.3 Tension

Elasticity properties [10]

$$\Delta T_n = E\,A\,\frac{\mathcal{L}_n - L_0}{L_0} = A\,E\,Q_n\,, \tag{21}$$

where we have used Eq. (14). Therefore, the tension of the fretted string is

$$T_n = T_0 + T_n = T_0\,(1 + \kappa\,Q_n)\,, \tag{22}$$

where we have defined the dimensionless "string constant"

$$\kappa \equiv \frac{A\,E}{T_0} = \frac{\pi r^2 E}{T_0}\,. \tag{23}$$

In this case, we assume that $\kappa Q_n \ll 1$, so that we can approximate the third term in the final line of Eq. (9) as

$$600\,\log_2\left(\frac{T_n}{T_0}\right) \approx \frac{600}{\ln(2)}\,\kappa\,Q_n\,. \tag{24}$$

This frequency shift is larger than that caused by the linear mass density error by a factor of $\kappa$.

### 2.4 Bending Stiffness

Since $B_n$ is already relatively small, we only need to consider the largest contribution arising from the shortened length of the fretted string compared to that of the open string. We see from Eq. (10) that $L_n \approx L_0/\gamma_n$, so from Eq. (3) we have

$$B_n = \sqrt{\frac{\pi\,\rho^4\,E}{T_n\,L_n^2}} \approx \frac{L_0}{L_n}\sqrt{\frac{\pi\,\rho^4\,E}{T_0\,L_0^2}} = \gamma_n\,B_0\,. \tag{25}$$

Therefore, the fourth term in the final line of Eq. (9) can be approximated as

$$1200\,\log_2\left(\frac{1+B_n}{1+B_0}\right) \approx \frac{1200}{\ln(2)}\,(\gamma_n - 1)\,B_0\,. \tag{26}$$

When $n = 12$ and $B_0 = 6 \times 10^{-3}$, the corresponding shift is approximately 10 cents. Note that (to zero order in $Q_n$) this bending stiffness error does not depend on the tiny changes to the linear mass density or the tension that arises due to string fretting. Instead, it is an intrinsic property of the string.

## 3 Experimental Estimate of the String Constant

How do we estimate the spring constant $\kappa$ given by Eq. (23)? Recall how we introduced this concept in Section 2.3: increasing the tension by a quantity $\Delta T$ causes a shift in the frequency of a guitar string by $\Delta\nu$ in cents. Although we were considering fretted strings when we derived Eq. (9), the terms in that equation can be generalized to describe the case of an open string that has been stretched longitudinally. Suppose that we continue to clamp the string at the saddle and the nut, but that we tighten the tuning gear to stretch that string's length by an amount $\Delta L$. The change in the string's frequency due to the change in the open resonant length is zero, because $L_0/\gamma_0 L_0 = 1$. The linear mass density of the string is smaller now because there is less material between the saddle and the nut, causing the frequency shift (in cents)

$$600\,\log_2\left(\frac{\mu}{\mu + \Delta\mu}\right) \approx \frac{600}{\ln(2)}\,\frac{\Delta L}{L}\,, \tag{27}$$

where $L$ is the initial length of the string. Finally, the tension in the string increases by $\Delta T$ due to the elastic properties of the string. Following the discussion in Section 2.3, the corresponding frequency shift is

$$600\,\log_2\left(\frac{T + \Delta T}{T}\right) \approx \frac{600}{\ln(2)}\,\frac{\Delta L}{L}\,\kappa\,, \tag{28}$$

Table 1: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

| String | $R$ | $\kappa$ | Modulus (GPa) | Stiffness |
|---|---|---|---|---|
| J4301 | $4.39 \times 10^4$ | 49.8 | 8.62 | $3.79 \times 10^{-3}$ |
| J4302 | $5.02 \times 10^4$ | 57.0 | 5.62 | $4.68 \times 10^{-3}$ |
| J4303 | $4.79 \times 10^4$ | 54.4 | 3.57 | $5.72 \times 10^{-3}$ |
| J4304 | $5.02 \times 10^4$ | 57.0 | 9.52 | $4.13 \times 10^{-3}$ |
| J4305 | $4.39 \times 10^4$ | 49.8 | 5.05 | $4.55 \times 10^{-3}$ |
| J4306 | $5.27 \times 10^4$ | 59.9 | 3.97 | $6.35 \times 10^{-3}$ |

where $T$ is the initial tension of the string. Therefore, the total frequency shift of the open string caused by a change $\Delta L$ in the string's length is

$$\Delta\nu \approx \frac{600}{\ln(2)} \frac{\Delta L}{L} (\kappa + 1) \,. \tag{29}$$

Solving this expression for the string constant, we find

$$\kappa = \frac{\ln(2)}{600} R - 1 \,, \tag{30}$$

where

$$R \equiv \frac{L}{\Delta L} \Delta\nu \tag{31}$$

is a parameter originally defined by Byers[1] [2, 3].

It is relatively easy to estimate the value of $R$ for any guitar string with the aid of a simple device that can measure frequency shifts in cents [9] and either calipers or a ruler with finely marked graduations. With a fine-point felt pen, make a small mark on the string (say, just past the nut), and measure its location relative to some convenient point. Then tighten that string's tuning gear to increase the frequency by a shift measured by the electronic tuner. (A particularly convenient shift is four half-steps, corresponding to $\Delta\nu = 400$ cents, which will require a stretch of approximate 5 – 6 mm.) Should we include photos here? Also, we're assuming that the string stretches uniformly (i.e., the nut isn't grabbing the string); we'll have to check this.

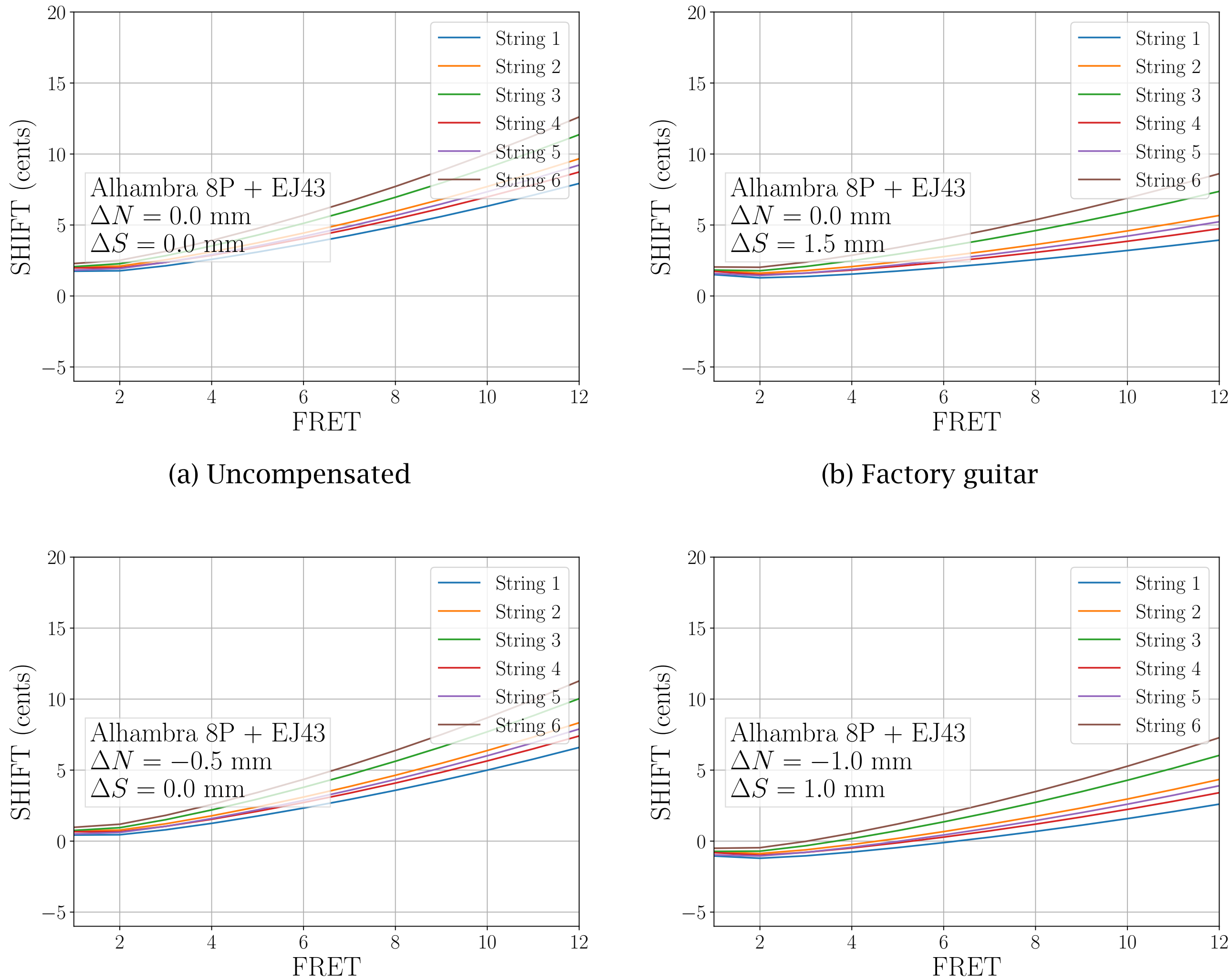
## 4 Classical Guitar Compensation

$$\Delta\nu_n \approx \frac{1200}{\ln(2)} \left[ \gamma_n \left( B_0 - \frac{\Delta S}{X_0} \right) + \frac{\Delta N}{X_0} + 2\,\kappa\, Q_n \right] . \tag{32}$$

Strategy: Use $\Delta S$ to compensate for bending stiffness, and $\Delta N$ to compensate for tension increases arising from fretting. Doing so predicts that $\Delta S \approx 3$ mm for the Alhambra 8P with light-tension strings, or about twice the saddle setback that is actually manufactured.

## 5 Conclusion

[1] Byers expressed this parameter in terms of the fractional frequency shift (in Hertz) as $R = (L/\Delta L)(\Delta f/f) \approx (\ln(2)/1200)(L/\Delta L)\Delta\nu$. Therefore, our dimensionless value of $R$ is larger than Byers' by a factor of about 1730.




(a) Uncompensated

(b) Factory guitar

(c) Saddle and nut

(d) Preserved scale length

Figure 2: Frequency shift (in cents) for four different strategies of saddle and nut compensation.

Table 2: String properties for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 25.5 inches.

| String | Note | Diameter (in) | Density (lb/in) | Tension (lb) |
|---|---|---|---|---|
| J4501 | $E_4$ | 0.0280 | $2.092 \times 10^{-5}$ | 15.3 |
| J4502 | $B_3$ | 0.0322 | $2.827 \times 10^{-5}$ | 11.6 |
| J4503 | $G_3$ | 0.0403 | $4.679 \times 10^{-5}$ | 12.1 |
| J4504 | $D_3$ | 0.0290 | $1.075 \times 10^{-4}$ | 15.6 |
| J4505 | $A_2$ | 0.0350 | $1.842 \times 10^{-4}$ | 15.0 |
| J4506 | $E_2$ | 0.0430 | $3.063 \times 10^{-4}$ | 14.0 |

Table 3: String properties for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 650 mm.

| String | Note | Radius (mm) | Density (kg/mm) | Tension (N) |
|---|---|---|---|---|
| J4501 | $E_4$ | 0.356 | $3.737 \times 10^{-7}$ | 68.6 |
| J4502 | $B_3$ | 0.409 | $5.050 \times 10^{-7}$ | 52.0 |
| J4503 | $G_3$ | 0.512 | $8.358 \times 10^{-7}$ | 54.3 |
| J4504 | $D_3$ | 0.368 | $1.921 \times 10^{-6}$ | 70.0 |
| J4505 | $A_2$ | 0.445 | $3.290 \times 10^{-6}$ | 67.3 |
| J4506 | $E_2$ | 0.546 | $5.472 \times 10^{-6}$ | 62.8 |

Table 4: String properties for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 25.5 inches.

| String | Note | Diameter (in) | Density (lb/in) | Tension (lb) |
|---|---|---|---|---|
| J4501 | $E_4$ | 0.0275 | $2.024 \times 10^{-5}$ | 14.8 |
| J4502 | $B_3$ | 0.0317 | $2.729 \times 10^{-5}$ | 11.2 |
| J4503 | $G_3$ | 0.0397 | $4.525 \times 10^{-5}$ | 11.7 |
| J4504 | $D_3$ | 0.0280 | $1.020 \times 10^{-4}$ | 14.8 |
| J4505 | $A_2$ | 0.0330 | $1.535 \times 10^{-4}$ | 12.5 |
| J4506 | $E_2$ | 0.0420 | $2.888 \times 10^{-4}$ | 13.2 |

Table 5: String properties for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

| String | Note | Radius (mm) | Density (kg/mm) | Tension (N) |
|---|---|---|---|---|
| J4501 | $E_4$ | 0.349 | $3.615 \times 10^{-7}$ | 66.4 |
| J4502 | $B_3$ | 0.403 | $4.875 \times 10^{-7}$ | 50.2 |
| J4503 | $G_3$ | 0.504 | $8.083 \times 10^{-7}$ | 52.5 |
| J4504 | $D_3$ | 0.356 | $1.823 \times 10^{-6}$ | 66.4 |
| J4505 | $A_2$ | 0.419 | $2.741 \times 10^{-6}$ | 56.1 |
| J4506 | $E_2$ | 0.533 | $5.159 \times 10^{-6}$ | 59.2 |

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