Classical Guitar Intonation and Compensation

M. B. Anderson and R. G. Beausoleil
Rosewood Guitar
8402 Greenwood Ave. N, Seattle, WA 98103

September 13, 2020

Abstract

TBD.

Contents

1 Introduction and Background

Initial [1] and ongoing work by G. Byers[2].

Recent studies of steel-string guitars [3]

2 Simple Model of Guitar Intonation

Fundamental frequency of a string [4, 5]:

$$f_0 = \frac{1}{2\,L_0}\sqrt{\frac{T_0}{\mu_0}}\,, \tag{1}$$

where L_0 is the length of the free (unfretted) string from the saddle to the nut, T_0 is the tension in the free string, and $\mu_0 \equiv M/L_0$ is the linear mass density of a free string of mass M.

$$f_0 = \frac{1}{2\,L_0}\sqrt{\frac{T_0}{\mu_0}}\left[1 + B_0 + \left(1 + \frac{\pi^2}{8}\right)B_0^2\right], \tag{2}$$

where B_0 is a "string stiffness parameter." For a uniform string with a cylindrical cross section, B_0 given by [6]

$$B_0 \equiv \sqrt{\frac{\pi\,R^4 E}{T_0\,L_0^2}}, \tag{3}$$

where R is the radius of the string and E is Young's modulus (or the modulus of elasticity). For a typical nylon guitar string with $E \approx 2 - 4$ GPa, $T_0 \approx 50 - 70$ N, $R \approx 0.35 - 0.51$ mm, and $L_0 \approx 650$ mm, we have $B_0 \approx 0.007 - 0.026$.

Throughout this work, we will use *cents* to describe small differences in pitch [7]. One cent is one one-hundredth of a 12-TET half step, so that there are 1200 cents per octave. The difference in pitch between frequencies f_1 and f_2 is therefore defined as

$$\Delta\nu \equiv 1200\,\log_2\left(\frac{f_2}{f_1}\right). \tag{4}$$

We define $f \equiv (f_1 + f_2)/2$ and $\Delta f \equiv f_2 - f_1$. Then

$$\Delta\nu = 1200\,\log_2\left(\frac{f + \Delta f/2}{f - \Delta f/2}\right) \approx \frac{1200}{\ln 2}\,\frac{\Delta f}{f}, \tag{5}$$

where the last approximation applies when $\Delta f \ll f$. An experienced guitar player can distinguish beat notes with a difference frequency of $\Delta f \approx 1$ Hz, which corresponds to 8 cents at A_2 ($f = 220$ Hz) or 5 cents at E_4 ($f = 329.63$ Hz).

Our model begins with the simple form of the fundamental frequency of a string given by Eq. (1), and applies it to the frequency of a string pressed just behind the n^{th} fret:

$$f_n = \frac{1}{2\,L_n}\sqrt{\frac{T_n}{\mu_n}}, \tag{6}$$

where (as shown in Fig. TBD) L_n is the *resonant length* of the string from the saddle to fret n, and T_n and μ_n are respectively the corresponding tension and linear mass density of the fretted string. We note that T_n and μ_n depend on $\mathcal{L}_n$, the *total* length of the fretted string from the saddle to the nut. Ideally, in the 12-TET equal-temperament system [7],

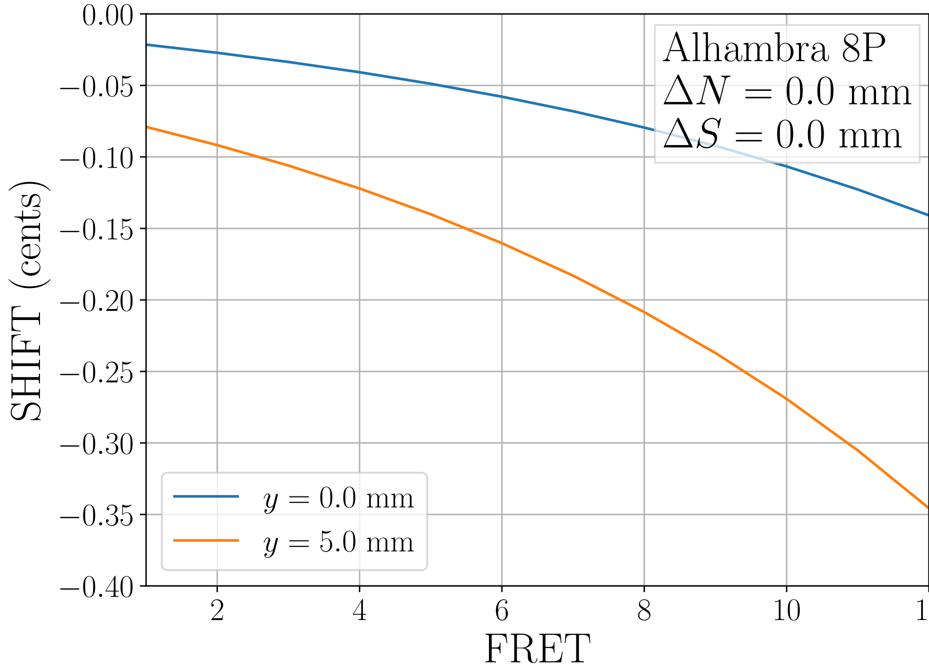
$$f_n = \gamma_n\,f_0\,, \qquad \text{(12-TET ideal)} \tag{7}$$

where

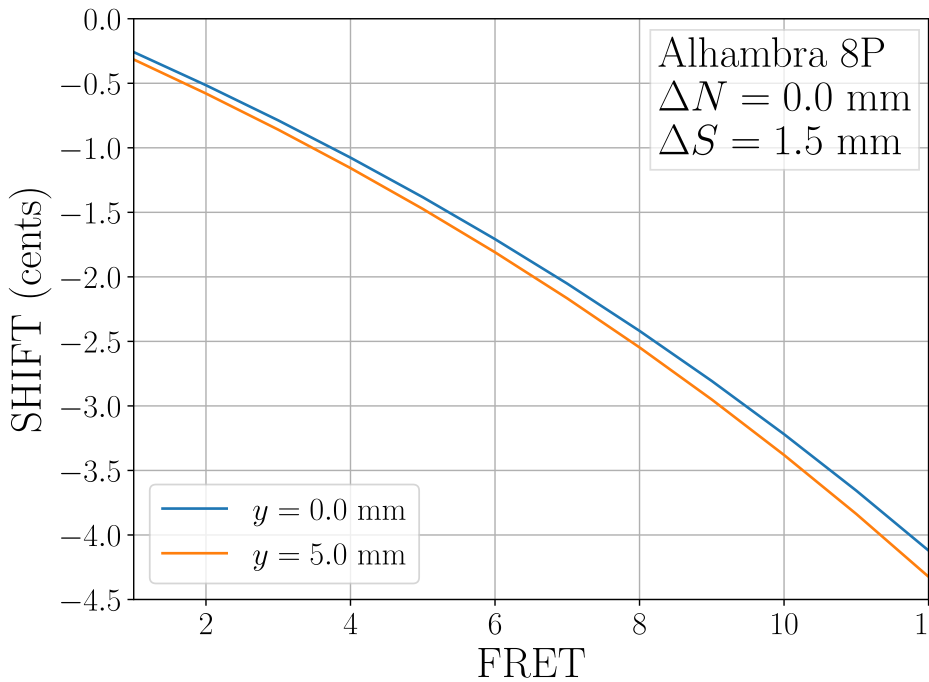
$$\gamma_n \equiv 2^{n/12}. \tag{8}$$

Therefore, the error interval expressed in cents is given by

$$\begin{aligned}
\Delta\nu_n &= 1200\,\log_2\left(\frac{f_n}{\gamma_n\,f_0}\right) \\
&= 1200\,\log_2\left(\frac{L_0}{\gamma_n\,L_n}\sqrt{\frac{\mu_0}{\mu_n}\,\frac{T_n}{T_0}}\right) \\
&= 1200\,\log_2\left(\frac{L_0}{\gamma_n\,L_n}\right) + 600\,\log_2\left(\frac{\mu_0}{\mu_n}\right) + 600\,\log_2\left(\frac{T_n}{T_0}\right).
\end{aligned} \tag{9}$$

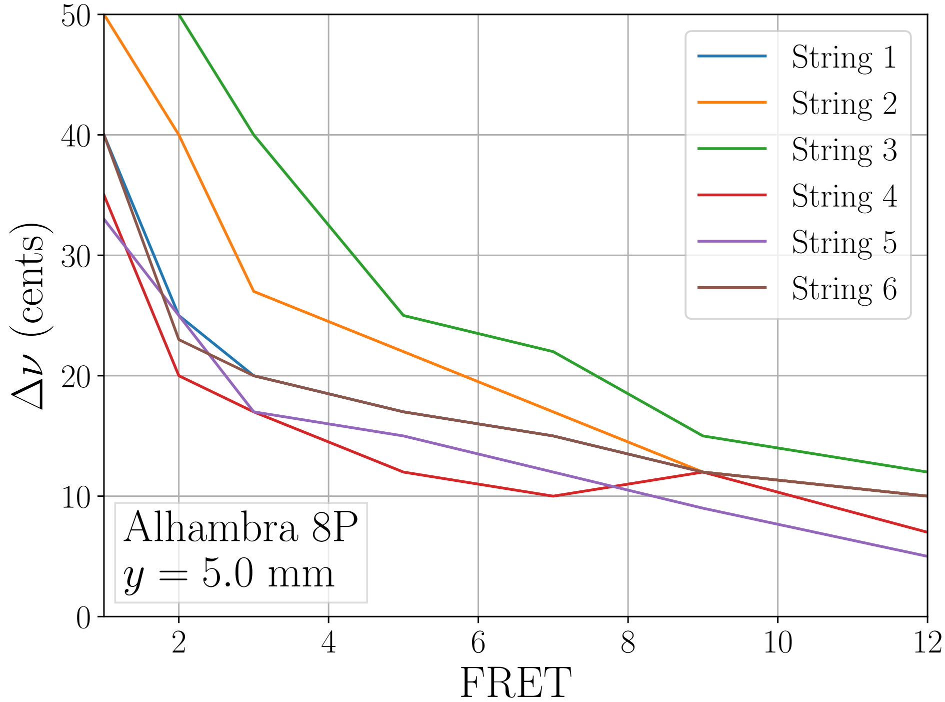


(a) Frequency shift for an uncompensated guitar

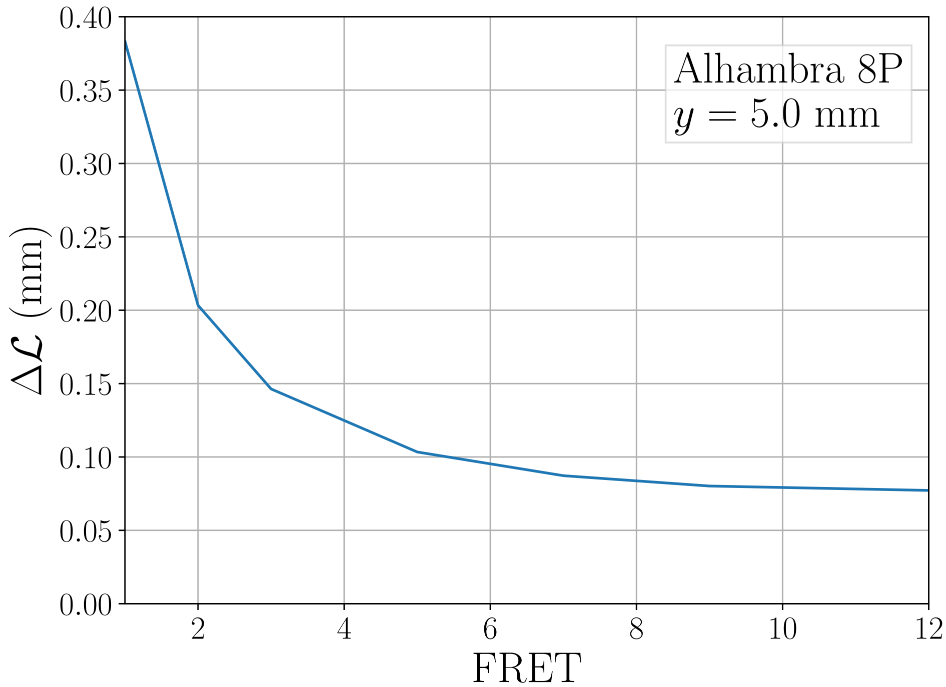


(b) Frequency shift for a factory guitar

Figure 1: Frequency shift (in cents) due to the fretted length L_n for an uncompensated (a) and factory (b) Alhambra 8P guitar, for both zero and nonzero lateral displacement y.

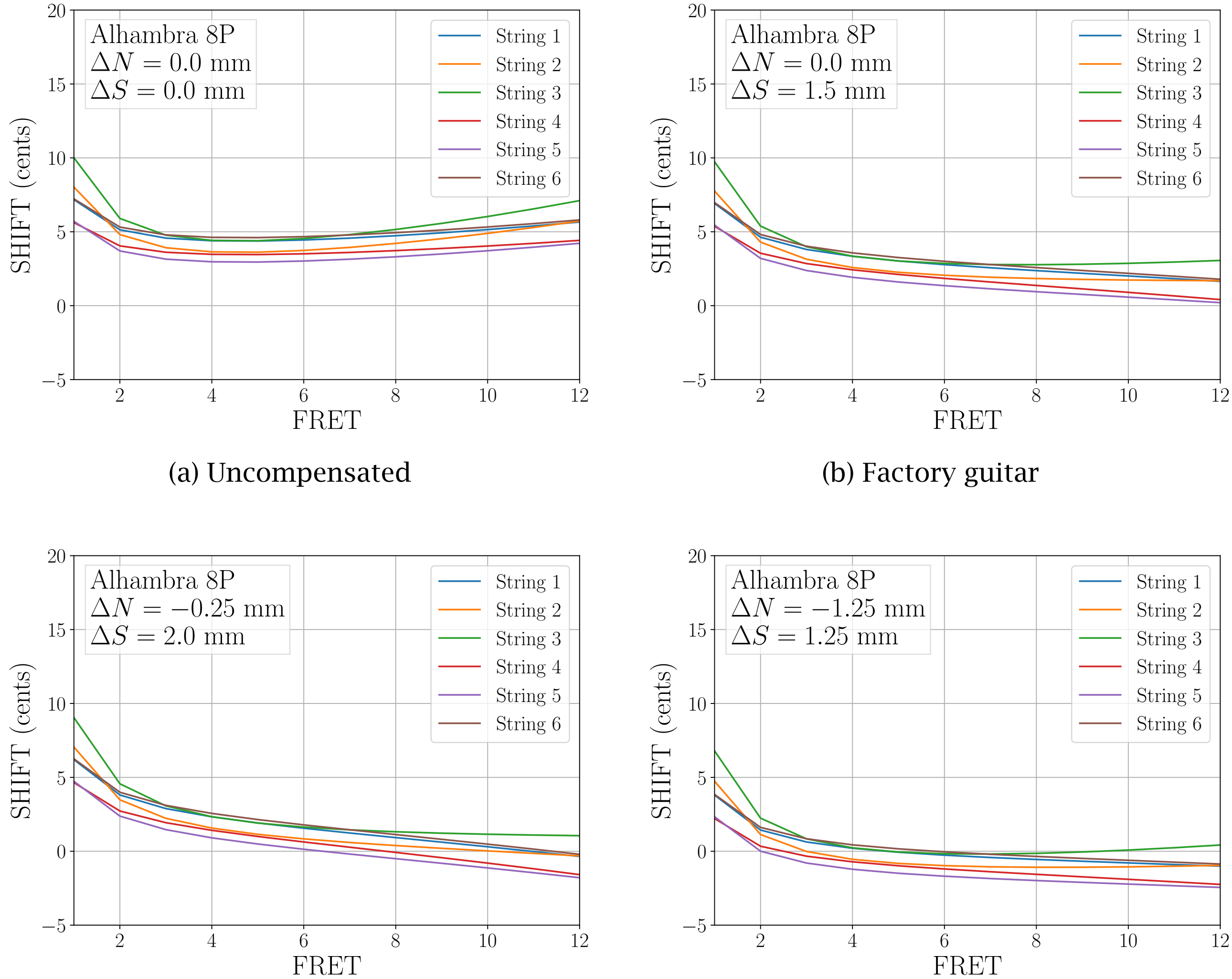


(a) Experimental data



(b) Calculated change in total string length $\mathcal{L}$

Figure 2: Frequency shift (in cents) (a) and change in total string length $\mathcal{L}$ (b) due to lateral displacement y.



(a) Uncompensated

(b) Factory guitar

(c) Saddle and nut

(d) Preserved scale length

Figure 3: Frequency shift (in cents) for four different strategies of saddle and nut compensation.

3 Experimental Estimate of the String Constant

4 Classical Guitar Compensation

5 Conclusion

References

[1] G. Byers, Guitars of Gregory Byers: Intonation (2020). See http://byersguitars.com/intonation and http://www.byersguitars.com/Research/Research.html.

[2] G. Byers, "Classical Guitar Intonation," American Lutherie **47**, 368 (1996).

[3] G. U. Varieschi and C. M. Gower, "Intonation and Compensation of Fretted String Instruments," Amer. J. Phys. **78**, 47 (2010).

[4] P. M. Morse, *Vibration and Sound* (Acoustical Society of America, New York, 1981).

[5] P. M. Morse, *Vibration and Sound*, pp. 84–85, in [4] (1981).

[6] P. M. Morse, *Vibration and Sound*, pp. 166–170, in [4] (1981).

[7] D. S. Durfee and J. S. Colton, "The physics of musical scales: Theory and experiment," Amer. J. Phys. **835–842**, 368 (2015).