Classical Guitar Intonation and Compensation: The Well-Tempered Guitar

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Abstract

TBD.

Contents

1 Introduction and Background

Discuss initial [1] and ongoing [2] work by G. Byers, and recent studies of steel guitar strings [3]. General references on the physics of sound, music, and musical instruments are [4].

Fundamental frequency of a string [5]:

$$f_0 = \frac{1}{2\,L_0}\sqrt{\frac{T_0}{\mu_0}}\,, \tag{1}$$

where L_0 is the length of the free (unfretted) string from the saddle to the nut, T_0 is the tension in the free string, and $\mu_0 \equiv M/L_0$ is the linear mass density of a free string of mass M.

Throughout this work, we will use *cents* to describe small differences in pitch [6]. One cent is one one-hundredth of a Twelve-Tone Equal Temperament (12-TET) half step, so that there are 1200 cents per octave. Using this approach, the difference in pitch between frequencies f_1 and f_2 is defined as

$$\Delta\nu \equiv 1200\,\log_2\left(\frac{f_2}{f_1}\right)\,. \tag{2}$$

We define $f \equiv (f_1 + f_2)/2$ and $\Delta f \equiv f_2 - f_1$. Then

$$\Delta\nu = 1200\,\log_2\left(\frac{f + \Delta f/2}{f - \Delta f/2}\right) \approx \frac{1200}{\ln 2}\,\frac{\Delta f}{f}\,, \tag{3}$$

where the last approximation applies when $\Delta f \ll f$. An experienced guitar player can distinguish beat notes with a difference frequency of $\Delta f \approx 1$ Hz, which corresponds to 8 cents at A_2 ($f = 220$ Hz) or 5 cents at E_4 ($f = 329.63$ Hz).

2 Simple Model of Guitar Intonation

The starting point for prior efforts to understand guitar intonation and compensation [1, 3] is a formula for f_m, the transverse vibration frequency harmonic m of a stiff string, originally published by Morse in 1936 [7]:

$$f_m = \frac{m}{2\,L}\sqrt{\frac{T}{\mu}}\left[1 + 2B + 4\left(1 + \frac{\pi^2\,m^2}{8}\right)B^2\right]\,. \tag{4}$$

Here L is the length of the string, T and μ are its tension and the linear mass density, respectively, and B is a small "bending stiffness" coefficient to capture the relevant mechanical properties of the string. For a homogeneous string with a cylindrical cross section, B is given by

$$B \equiv \sqrt{\frac{\pi\,\rho^4\,E}{4\,T\,L^2}}\,, \tag{5}$$

where ρ is the radius of the string and E is Young's modulus (or the modulus of elasticity). But it's unlikely that Eq. (4) accurately describes the resonant frequencies of a nylon string on a classical guitar, because it assumes that the string is "clamped" at both ends, so that a particular set of symmetric boundary conditions must be applied to the partial differential equation (PDE) describing transverse vibrations of the string. We believe that this assumption is correct for the end of the string held at either the nut or the fret, but that the string is

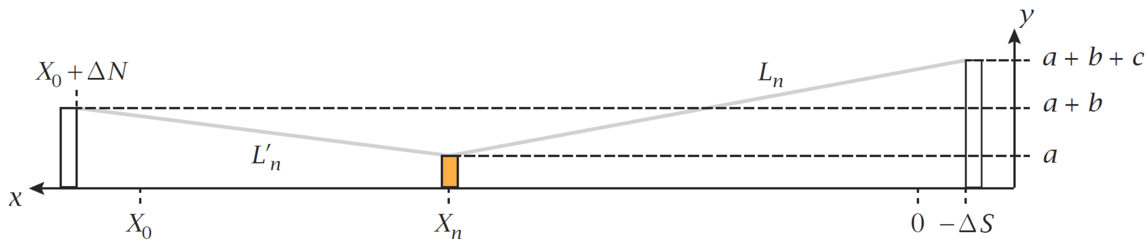


Figure 1: A simple (side-view) schematic of the classical guitar used in this model. The scale length of the guitar is X_0, but we allow the edges of both the saddle and the nut to be set back an additional distance ΔS and ΔN, respectively. The location on the x-axis of the center of the n^{th} fret is X_n. (Note that the x-axis is directed toward the left in this figure.) In the y direction, $y = 0$ is taken as the surface of the fingerboard; therefore the height of each fret above the fingerboard is a, the height of the nut is $a + b$, and the height of the saddle is $a + b + c$. L_n is the *resonant length* of the string from the saddle to the center of fret n, and L'_n is the length of the string from the fret to the nut.

"pinned" (and not clamped) at the saddle. In Appendix A, we solve the PDE using these non-symmetric boundary conditions, and find

$$f_m = \frac{m}{2L}\sqrt{\frac{T}{\mu}}\left[1 + B + \left(1 + \frac{1}{2}m^2\pi^2\right)B^2\right]. \tag{6}$$

Note that this expression is valid only when $B \ll 1$. For a typical nylon guitar string with $E \approx 5$ GPa, $T \approx 60$ N, $\rho \approx 0.5$ mm, and $L \approx 650$ mm, we have $B \approx 3 \times 10^{-3}$. (In this case, the quadratic term in B is only 2% as large as the linear term, and can generally be neglected. We will include it in our analysis below only for completeness.) We'll use Eq. (6) with some caution, because the physics of nylon strings (particularly the wound base strings) are quite complicated [8].

Our model is based on the schematic of the guitar shown in Fig. 1. The scale length of the guitar is X_0, but we allow the edges of both the saddle and the nut to be set back an additional distance ΔS and ΔN, respectively. The location on the x-axis of the center of the n^{th} fret is X_n. In the y direction, $y = 0$ is taken as the surface of the fingerboard; the height of each fret is a, the height of the nut is $a + b$, and the height of the saddle is $a + b + c$. L_n is the *resonant length* of the string from the saddle to the center of fret n, and L'_n is the length of the string from the fret to the nut. The total length of the string is defined as $\mathcal{L}_n \equiv L_n + L'_n$.

Previous studies of guitar intonation and compensation have chosen to include the apparent increase in length of the string caused by both the fretting depth and the shape of the fretted string under the finger [1, 3]. As the string is initially pressed to the fret, the total length $\mathcal{L}_n$ increases and causes the tension in the string — which is clamped at the saddle and the nut — to increase. As the string is pressed further, does the additional deformation of the string increase its tension (throughout the resonant length L_n)? There are at least two purely empirical reasons to doubt this hypothesis. First, we can mark a string (with a fine-point felt pen) above a particular fret and then observe the mark with a magnifying glass. As the string is pressed all the way to the finger board, the mark does not move perceptibly — it has become effectively *clamped* on the fret. Second, we can use either our ears or a simple tool to measure frequencies [9] to listen for a shift as we use different fingers and vary the fretted depth of a string. The apparent modulation is far less than would be obtained by classical vibrato (±15 cents), so we assume that once the string is minimally fretted the length(s) can be regarded as fixed. (If this were not the case, then fretting by different people or with different fingers, at a single string or with a barre, would cause additional, varying frequency shifts that would be audible and difficult to compensate.) We discuss this issue in more detail in Appendix B.

We start with the form of the fundamental frequency of a fretted string given by Eq. (6) with $m = 1$, and apply it to the frequency of a string pressed just behind the n^{th} fret:

$$f_n = \frac{1}{2\,L_n}\sqrt{\frac{T_n}{\mu_n}}\left[1 + B_n + \left(1 + \frac{\pi^2}{2}\right)B_n^2\right], \tag{7}$$

where T_n and μ_n are the modified tension and the linear mass density of the fretted string, and

$$B_n \equiv \sqrt{\frac{\pi\,\rho^4\,E}{4\,T_n\,L_n^2}}\,. \tag{8}$$

We note that T_n and μ_n depend on $\mathcal{L}_n$, the *total* length of the fretted string from the saddle to the nut. Ideally, in the 12-TET system [6],

$$f_n = \gamma_n\,f_0\,, \qquad \text{(12-TET ideal)} \tag{9}$$

where f_0 is the frequency of the open (unfretted) string, and

$$\gamma_n \equiv 2^{n/12}\,. \tag{10}$$

Therefore, the error interval — the difference between the fundamental frequency of the fretted string and the corresponding perfect 12-TET frequency — expressed in cents is given by

$$\begin{aligned}\Delta\nu_n &= 1200\,\log_2\left(\frac{f_n}{\gamma_n\,f_0}\right) \\ &= 1200\,\log_2\left(\frac{L_0}{\gamma_n\,L_n}\right) + 600\,\log_2\left(\frac{\mu_0}{\mu_n}\right) + 600\,\log_2\left(\frac{T_n}{T_0}\right) \\ &\quad + 1200\,\log_2\left[\frac{1 + B_n + (1 + \pi^2/2)\,B_n^2}{1 + B_0 + (1 + \pi^2/2)\,B_0^2}\right].\end{aligned} \tag{11}$$

The final form of Eq. (11) makes it clear that — for nylon guitar strings — there are four contributions to intonation:

1. *Resonant Length*: The first term represents the error caused by the increase in the length of the fretted string L_n compared to the ideal length X_n, which would be obtained if $b = c = 0$ and $\Delta S = \Delta N = 0$.

2. *Linear Mass Density*: The second term is the error caused by the reduction of the linear mass density of the fretted string. This effect will depend on the *total* length of the string, given by $\mathcal{L}_n = L_n + L'_n$.

3. *Tension*: The third term is the error caused by the *increase* of the tension in the string arising from the stress and strain applied to the string by fretting. This effect will also depend on the total length of the string $\mathcal{L}_n$.

4. *Bending Stiffness*: The fourth and final term is the error caused by the change in the bending stiffness coefficient caused by changing the vibrating length of the string from L_0 to L_n.

Note that the properties of the logarithm function have *decoupled* these physical effects by converting multiplication into addition. We will discuss each of these sources of error in turn below.

2.1 Resonant Length

We can estimate the first term in the last line of Eq. (11) by referring to Fig. 1 and computing the resonant length L_n. We find:

$$L_n = \begin{cases} \sqrt{(X_0 + \Delta S + \Delta N)^2 + c^2} & n = 0\,, \\ \sqrt{(X_n + \Delta S)^2 + (b + c)^2} & n \geq 1\,. \end{cases} \tag{12}$$

When $b + c \ll X_0$, we can use a Taylor series to approximate L_n by

$$L_n \approx \begin{cases} X_0 + \Delta S + \Delta N + c^2/2\,X_0 & n = 0\,, \\ X_n + \Delta S + (b + c)^2/2\,X_n & n \geq 1\,. \end{cases} \tag{13}$$

Then — when the guitar has been manufactured such that $X_n = X_0/\gamma_n$ — the resonant length error is approximately

$$1200\,\log_2\left(\frac{L_0}{\gamma_n\,L_n}\right) \approx \frac{1200}{\ln(2)}\left[\frac{\Delta N - (\gamma_n - 1)\,\Delta S}{X_0} - \frac{\gamma_n^2(b + c)^2 - c^2}{2\,X_0^2}\right] \tag{14}$$

If the guitar is uncompensated, so that $\Delta S = \Delta N = 0$, this error is typically less than 0.25 cents. But, with $\Delta S > 0$ and $\Delta N < 0$, we can significantly *increase* the magnitude of this "error" and cause the frequency to shift lower. We'll see that this is our primary method of compensation.

2.2 Linear Mass Density

As discussed above, the linear mass density μ_0 of an open (unfretted) string is simply the total mass M of the string clamped between the saddle and the nut divided by the length L_0. Similarly, the mass density μ_n of a string held onto fret N is $M/\mathcal{L}_n$. Therefore

$$\frac{\mu_0}{\mu_n} = \frac{\mathcal{L}_n}{L_0} \equiv 1 + Q_n\,, \tag{15}$$

where we have followed Byers and defined [1, 3]

$$Q_n \equiv \frac{\mathcal{L}_n - L_0}{L_0}\,. \tag{16}$$

Since we expect that $Q_n \ll 1$, we can approximate the second term in the final line of Eq. (11) as

$$600\,\log_2\left(\frac{\mu_0}{\mu_n}\right) \approx \frac{600}{\ln(2)}\,Q_n\,. \tag{17}$$

Referring to Fig. 1, we see that $\mathcal{L}_n = L_n + L_n'$, and we calculate L_n' for $n \geq 1$ as

$$L_n' = \sqrt{(X_0 - X_n + \Delta N)^2 + b^2} \approx X_0 - X_n + \Delta N + \frac{b^2}{2\,(X_0 - X_n)}\,, \tag{18}$$

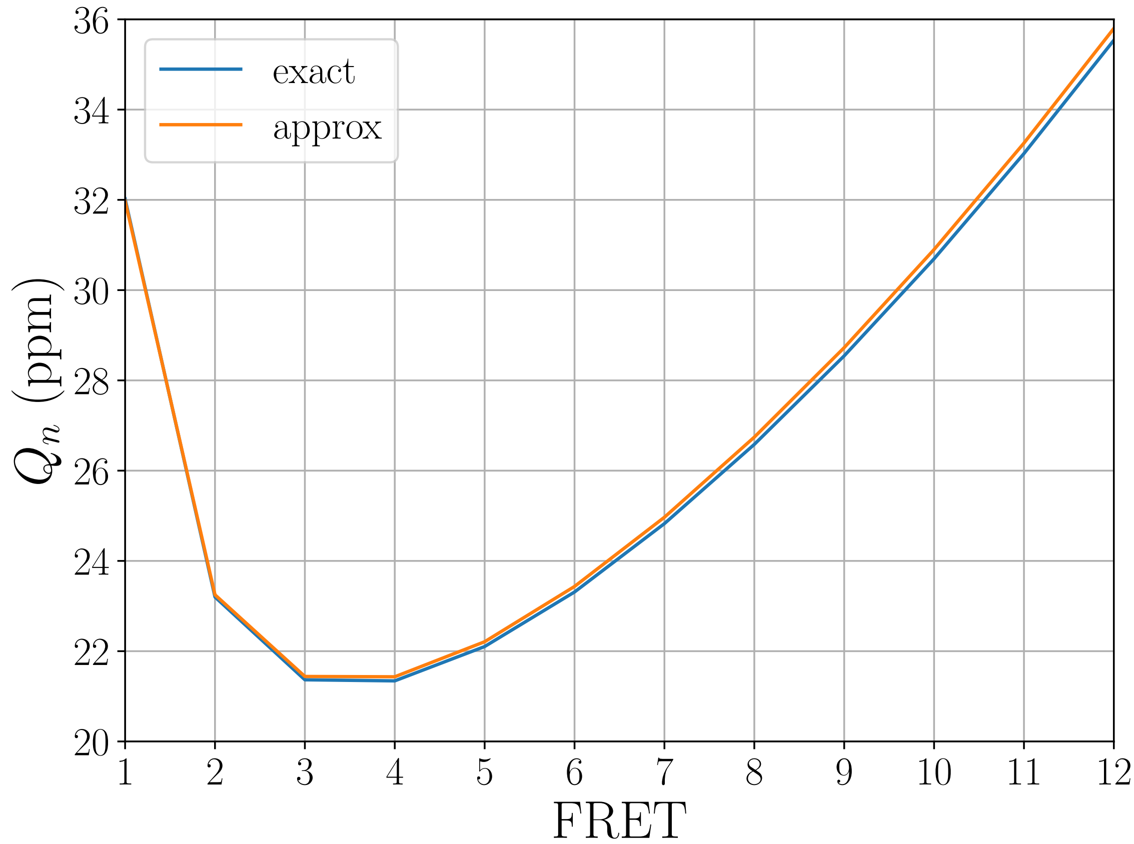


Figure 2: Comparison of the exact expression for the normalized displacement Q_n as a function of the fret number given by Eq. (16) with the approximate expression given by Eq. (20). Here the guitar has $b = 1.0$ mm, $c = 3.5$ mm, $\Delta S = 1.5$ mm, $\Delta N = -0.25$ mm, and a scale length of 650 mm.

where the approximation applies when $b^2 \ll X_0 - X_n$. Therefore, using Eq. (13), we have

$$\mathcal{L}_n = L_n + L'_n \approx X_0 + \Delta S + \Delta N + \frac{(b+c)^2}{2\,X_n} + \frac{b^2}{2\,(X_0 - X_n)}\,, \tag{19}$$

and

$$\begin{aligned} Q_n &\approx \frac{1}{2\,X_0}\left[\frac{(b+c)^2}{X_n} + \frac{b^2}{X_0 - X_n} - \frac{c^2}{X_0}\right] \\ &= \frac{\gamma_n}{2\,X_0^2}\left[(b+c)^2 + \frac{b^2}{\gamma_n - 1} - \frac{c^2}{\gamma_n}\right]. \end{aligned} \tag{20}$$

In Fig. 2, we plot a comparison between the exact expression for the normalized displacement Q_n given by Eq. (16) with the approximate expression given by Eq. (20). Here the guitar has $b = 1.0$ mm, $c = 3.5$ mm, $\Delta S = 1.5$ mm, $\Delta N = -0.25$ mm, and $X_0 = 650$ mm. We see that our Taylor series estimate is quite accurate even though it neglects both setbacks, indicating that Q_n is determined primarily by the values of b and c. For the same parameters, $\Delta\nu_{12} \approx 0.03$ cents, and will be even smaller for $n < 12$. In general, the shift due to linear mass density can be neglected without much loss of accuracy.

2.3 Tension

Elasticity properties [10]

$$\Delta T_n = E\,A\,\frac{\mathcal{L}_n - L_0}{L_0} = A\,E\,Q_n\,, \tag{21}$$

where we have used Eq. (16). Therefore, the tension of the fretted string is

$$T_n = T_0 + \Delta T_n = T_0\,(1 + \kappa\,Q_n)\ , \tag{22}$$

where we have defined the dimensionless "string constant"

$$\kappa \equiv \frac{A\,E}{T_0} = \frac{\pi\rho^2 E}{T_0}\,. \tag{23}$$

In this case, we assume that $\kappa Q_n \ll 1$, so that we can approximate the third term in the final line of Eq. (11) as

$$600\,\log_2\left(\frac{T_n}{T_0}\right) \approx \frac{600}{\ln(2)}\,\kappa\,Q_n\,. \tag{24}$$

This frequency shift is larger than that caused by the linear mass density error by a factor of κ.

2.4 Bending Stiffness

Since B_n is already relatively small, we only need to consider the largest contribution arising from the shortened length of the fretted string compared to that of the open string. We see from Eq. (12) that $L_n \approx L_0/\gamma_n$, so from Eq. (8) we have

$$B_n = \sqrt{\frac{\pi\,\rho^4\,E}{4\,T_n\,L_n^2}} \approx \frac{L_0}{L_n}\sqrt{\frac{\pi\,\rho^4\,E}{4\,T_0\,L_0^2}} = \gamma_n\,B_0\,. \tag{25}$$

Therefore, the fourth term in the final line of Eq. (11) can be approximated as

$$1200\,\log_2\left[\frac{1 + B_n + (1 + \pi^2/2)\,B_n^2}{1 + B_0 + (1 + \pi^2/2)\,B_0^2}\right] \approx \frac{1200}{\ln(2)}\left[(\gamma_n - 1)\,B_0 + \frac{1}{2}\,(\gamma_n^2 - 1)\,(1 + \pi^2)\,B_0^2\right]. \tag{26}$$

When $n = 12$ and $B_0 = 3 \times 10^{-3}$, the corresponding shift is approximately 5.5 cents, with the quadratic term contributing about 0.25 cents. Note that (to zero order in Q_n) this bending stiffness error does not depend on the tiny changes to the linear mass density or the tension that arises due to string fretting. Instead, it is an intrinsic property of the string.

Incorporating all of these effects — and neglecting the term proportional to B_0^2 — we find that the total frequency shift is given by

$$\Delta\nu_n \approx \frac{1200}{\ln(2)}\left[(\gamma_n - 1)\left(B_0 - \frac{\Delta S}{X_0}\right) + \frac{\Delta N}{X_0} + \frac{1}{2}\,\kappa\,Q_n\right]. \tag{27}$$

In this form, we see that the bending stiffness and the increase in string tension due to fretting sharpen the pitch, but that we can flatten it with a positive saddle setback and negative nut setback. In fact, a simple compensation strategy would be to choose $\Delta S = B_0\,X_0$ to compensate for stiffness, and then select $\Delta N = -\kappa\,X_0\,\overline{Q}/2$, where $\overline{Q}$ is the displacement averaged over a particular set of frets. We'll choose a more flexible method for compensation in Section 4, but we'll obtain results that are similar to this basic approach.

3 Experimental Estimate of the String Constant

How do we determine the spring constant κ given by Eq. (23)? Ideally, we could conduct an experiment that measures the change in the frequency of an open string — pinned at one end, and clamped at the other — as we make slight changes to its length [1, 3]. From Eq. (6), the derivative of the fundamental frequency of an open string is

$$\begin{aligned} \frac{df}{dL} &= \frac{f}{L}\left(-1 + \frac{L}{2T}\frac{dT}{dL} - \frac{L}{2\mu}\frac{d\mu}{dL} + \frac{L}{1+B}\frac{dB}{dL}\right) \\ &= \frac{f}{L}\left(-1 + \frac{1}{2}\kappa + \frac{1}{2} - \frac{B}{1+B}\right) \\ &\approx \frac{f}{L}\times\frac{1}{2}(\kappa - 1)\,, \end{aligned} \tag{28}$$

where we have used the analysis in Section 2 to determine that

$$\frac{dT}{dL} = \frac{T}{L}\kappa\,, \tag{29a}$$

$$\frac{d\mu}{dL} = -\frac{\mu}{L}\,, \text{ and} \tag{29b}$$

$$\frac{dB}{dL} = -\frac{B}{L}\,. \tag{29c}$$

$$\tag{29d}$$

Therefore, following Byers [1, 3], we define the parameter R to be

$$R \equiv \frac{d\,\ln(f)}{d\,\ln(L)} = \frac{L}{f}\frac{df}{dL} = \frac{1}{2}(\kappa - 1)\,, \tag{30}$$

which gives

$$\kappa = 2R + 1\,. \tag{31}$$

With this measurement of κ, we can estimate the bending stiffness coefficient by comparing Eq. (23) and Eq. (25), and writing B_0 as

$$B_0 = \sqrt{\kappa}\,\frac{\rho}{2L_0}\,. \tag{32}$$

It is relatively easy to estimate the values of κ and R for any guitar string with the aid of a simple device that can measure frequency shifts in cents [9]. First, we tune the open string to the correct 12-TET frequency. Then, we select a fret n, and measure the frequency deviation $\Delta\nu_n$ of the fretted string. If we substitute Eq. (32) into Eq. (27), then we find

$$\alpha\,\kappa + \beta\sqrt{\kappa} - \xi = 0\,, \tag{33}$$

where we have included the quadratic bending stiffness term, and defined the coefficients

$$\alpha \equiv \frac{1}{2}\left[Q_n + (\gamma_n^2 - 1)\,(1 + \pi^2)\left(\frac{\rho}{2L_0}\right)^2\right]\,, \tag{34a}$$

$$\beta \equiv (\gamma_n - 1)\,\frac{\rho}{2L_0}\,, \text{ and} \tag{34b}$$

$$\xi \equiv \frac{\ln(2)}{1200}\Delta\nu_n + \frac{(\gamma_n - 1)\,\Delta S - \Delta N}{X_0}\,. \tag{34c}$$

Equation (33) is quadratic in $\sqrt{\kappa}$, with the solution

$$\sqrt{\kappa} = \frac{-\beta + \sqrt{\beta^2 + 4\,\alpha\,\xi}}{2\,\alpha}\,, \tag{35}$$

where we have chosen the positive root.

We've used this approach to estimate κ and R of different string sets on the Alhambra 8P classical guitar, which has $X_0 = 650$ mm, $c = 3.5$ mm, $\Delta S = 1.5$ mm, and $\Delta N = 0.0$ mm. At the nut, $b = 1.0$ mm, but the height of the fret board decreases roughly linearly further toward the saddle, effectively increasing b. We estimate that $db/dx \approx -0.0034$, so that b has increased to 2.0 mm at the 12$^{\text{th}}$ fret. We show the specifications of a normal-tension classical string set in Table 1 using metric units[1]. In Table 2, we list the frequency deviation from 12-TET at the 12$^{\text{th}}$ fret for each string in this set, as well as the corresponding estimates of κ, R, E, and B_0, computed from Eq. (35), Eq. (30), Eq. (23), and Eq. (32), respectively. The units used for the elastic modulus are gigapascals (1 GPa = 10^9 N/m.). Similar measurements and results for other string sets are provided in Appendix D.

Table 1: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 650 mm.

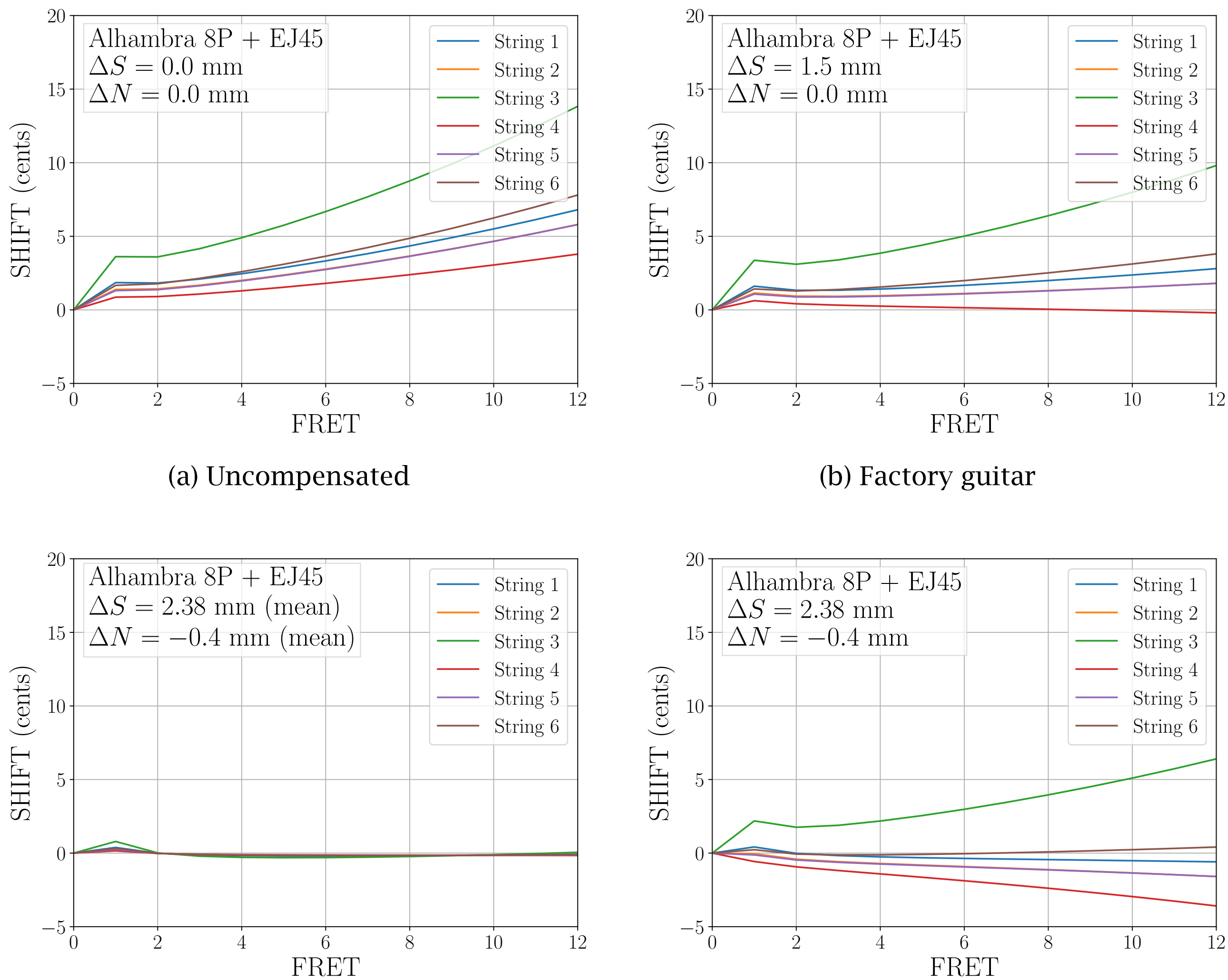
String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4501	E_4	0.36	3.74	68.62
J4502	B_3	0.41	5.05	52.04
J4503	G_3	0.51	8.36	54.26
J4504	D_3	0.37	19.21	69.99
J4505	A_2	0.44	32.90	67.27
J4506	E_2	0.55	54.72	62.80

Table 2: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 650 mm.

String	$\Delta\nu_{12}$ (cents)	R	κ	E (GPa)	B_0 ($\times 10^{-3}$)
J4501	3	28.3	57.5	9.9	2.1
J4502	2	20.3	41.5	4.1	2.0
J4503	10	55.0	110.9	7.3	4.1
J4504	0	12.3	25.6	4.2	1.4
J4505	2	18.8	38.5	4.2	2.1
J4506	4	23.4	47.9	3.2	2.9

Table 3: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45) on the Alhambra 8P classical guitar.

String	ΔS (mm)	ΔN (mm)
J4501	2.16	-0.43
J4502	1.92	-0.31
J4503	4.36	-0.82
J4504	1.30	-0.19
J4505	1.94	-0.28
J4506	2.62	-0.35



(a) Uncompensated

(b) Factory guitar

(c) Full compensation

(d) Mean compensation

Figure 3: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). Four different strategies of saddle and nut compensation are illustrated.

4 Classical Guitar Compensation

5 Tempering the Classical Guitar

> Temperament: A compromise between the acoustic purity of theoretically exact intervals, and the harmonic discrepancies arising from their practical employment.
> — Dr. Theo. Baker [11]

Shown in Fig. 4a, the factory guitar tuned to 12-TET shows the third string having the greatest error in tuning across the fretboard. Tuning the factory guitar to 12-TET exacts a perfect-fifth in the third string while playing a C major chord in first position. This results in the third string being too sharp (+7 cents) for the other common chords of E major (G#), A major and D major (A). A way to reduce this error is by lowering the pitch of the third string 7 cents below 12-TET with an electronic tuner. Another more comprehensive system is to tune all the strings harmonically to the fifth string, which lowers the third string by 7 cents as well as tempering the remaining strings.

In this particular case, the "Harmonic Tuning Method" can be followed using these steps:

1. Begin by tuning the fifth string to $A_2 = 110$ Hz, resulting in a fifth-fret harmonic of $A_4 = 440$ Hz. (This can also be tuned by ear using an A_4 tuning fork).

2. Tune that harmonic to the seventh fret harmonic of the fourth string, which is also $A_4 = 440$ Hz.

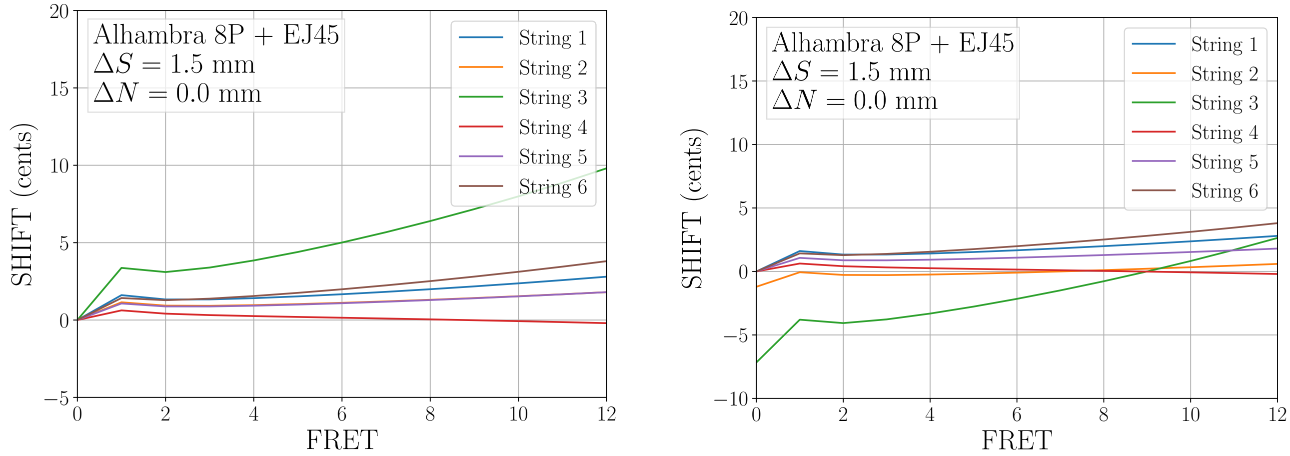
3. Tune the seventh fret harmonic on the fifth string (330 Hz, or 0.37 Hz sharper than 12-TET E_4 to the fifth-fret harmonic of the sixth string.

4. The seventh fret harmonic on the fifth string can tune the remaining fretted strings: the ninth fret on the third string, the fifth fret on the second string, and the open first string.

Table 4: Harmonic tuning methodology based on A_4 and E_4. The asterisk indicates a harmonic with a null at the designated fret.

Reference String/Fret	Target String/Fret
A*/5 (A_4)	D*/7
A*/7 (E_4)	E*/5
A*/7 (E_4)	G/9
A*/7 (E_4)	B/5
A*/7 (E_4)	E/0

[1]Note that the correct unit of force in the metric system is Newtons (N), rather than kilograms, which is a unit of mass.



(a) Factory guitar — 12-TET tuned

(b) Factory guitar — harmonically tuned

Figure 4: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). Here we compare the factory guitar tuned to 12-TET with the same guitar harmonically tuned.

6 Conclusion

A Vibration Frequencies of a Stiff String

Here we outline the calculation of the normal mode frequencies of a vibrating stiff string with non-symmetric boundary conditions. We begin with the wave equation [12]

$$\mu \frac{\partial^2}{\partial t^2} y(x) = T \frac{\partial^2}{\partial x^2} y(x) - E\, S\, \mathcal{K}^2 \frac{\partial^4}{\partial x^4} y(x)\,, \tag{36}$$

where μ and T are respectively the linear mass density and the tension of the string, E is its Young's modulus (or the modulus of elasticity), S is the cross-sectional area, and $\mathcal{K}$ is the radius of gyration of the string. (For a uniform cylindrical string/wire with radius ρ, $S = \pi \rho^2$ and $\mathcal{K} = \rho/2$.) If we scale x by the length L of the string, and t by $1/\omega_0 \equiv (L/\pi)\sqrt{\mu/T}$, then we obtain the dimensionless wave equation

$$\pi^2 \frac{\partial^2}{\partial t^2} y(x) = \frac{\partial^2}{\partial x^2} y(x) - B^2 \frac{\partial^4}{\partial x^2} y(x)\,, \tag{37}$$

where B is the "bending stiffness parameter" given by

$$B \equiv \sqrt{\frac{E\, S\, \mathcal{K}^2}{L^2 T}}\,. \tag{38}$$

We assume that $y(x)$ is a sum of terms of the form

$$y(x) = C\, e^{k x - i \omega t}\,, \tag{39}$$

requiring that k and ω satisfy the expression

$$B^2 k^4 - k^2 - (\pi\, \omega)^2 = 0\,, \tag{40}$$

or

$$k^2 = \frac{1 \pm \sqrt{1 + (2\,\pi\,B\,\omega)^2}}{2\,B^2}\,. \tag{41}$$

Therefore, given ω, we have four possible choices for k: $\pm k_1$, or $\pm i k_2$, where

$$k_1^2 = \frac{\sqrt{1 + (2\,\pi\,B\,\omega)^2} + 1}{2\,B^2}\,,\ \text{and} \tag{42a}$$
$$k_2^2 = \frac{\sqrt{1 + (2\,\pi\,B\,\omega)^2} - 1}{2\,B^2}\,. \tag{42b}$$

The corresponding general solution to Eq. (37) has the form

$$y(x) = e^{-i\omega t}\left(C_1^+ e^{k_1 x} + C_1^- e^{-k_1 x} + C_2^+ e^{i k_2 x} + C_2^- e^{-i k_2 x}\right)\,. \tag{43}$$

As discussed in Section 2, the boundary conditions for the case of a classical guitar string are not symmetric. At $x = 0$ (the saddle), the string is pinned (but not clamped), so that $y = 0$ and $\partial^2 y/\partial x^2 = 0$. However, at $x = 1$ (the fret) the string is clamped, so that $y = 0$ and $\partial y/\partial x = 0$. Applying these constraints to Eq. (43), we obtain

$$0 = C_1^+ + C_1^- + C_2^+ + C_2^-\,, \tag{44a}$$
$$0 = k_1^2\left(C_1^+ + C_1^-\right) - k_2^2\left(C_2^+ + C_2^-\right)\,, \tag{44b}$$
$$0 = C_1^+ e^{k_1} + C_1^- e^{-k_1} + C_2^+ e^{i k_2} + C_2^- e^{-i k_2}\,,\ \text{and} \tag{44c}$$
$$0 = k_1\left(C_1^+ e^{k_1} - C_1^- e^{-k_1}\right) + i k_2\left(C_2^+ e^{i k_2} - C_2^- e^{-i k_2}\right)\,. \tag{44d}$$

Since $k_1^2 + k_2^2 \neq 0$, the first two of these equations tell us that $C_1^- = -C_1^+ \equiv -C_1$, and $C_2^- = -C_2^+ \equiv -C_2$. Therefore, the second two equations become

$$C_1\,\sinh(k_1) = -i\,C_2\,\sin(k_2)\,,\ \text{and} \tag{45a}$$
$$k_1\,C_1\,\cosh(k_1) = -i\,k_2\,C_2\,\cos(k_2)\,. \tag{45b}$$

Dividing the first of these equations by the second, we find

$$\tan(k_2) = \frac{k_2}{k_1}\,\tanh k_1\,. \tag{46}$$

From Eq. (42), we see that $k_1^2 - k_2^2 = 1/B^2$, so that

$$k_1 = \frac{1}{B}\sqrt{1 + (B\,k_2)^2}\,. \tag{47}$$

In the case of a classical guitar, we expect that $B \ll 1$, so $k_1 \approx 1/B \gg 1$, and therefore $\tanh k_1 \longrightarrow 1$. Substituting Eq. (47) into Eq. (46), we obtain

$$\tan(k_2) = \frac{B\,k_2}{\sqrt{1 + (B\,k_2)^2}}\,. \tag{48}$$

We expect that $B\,k_2 \ll 1$, so we assume that $k_2 = m\pi(1+\epsilon)$, where m is an integer greater than or equal to 1, and $\epsilon \ll 1$. Therefore, to second order in ϵ, we have $\tan(k_2) \approx m\pi\epsilon$, and

$$\epsilon = \frac{B\,(1+\epsilon)}{\sqrt{1 + [m\,\pi\,B\,(1+\epsilon)]^2}}\,. \tag{49}$$

The denominator of the right-hand side of this equation has a Taylor expansion given by $1-\frac{1}{2}\,[m\,\pi\,B\,(1+\epsilon)]^2$, indicating that it will not contribute to ϵ to second order in B. Therefore, to this order,

$$\epsilon \approx \frac{B}{1-B} \approx B + B^2\,. \tag{50}$$

We substitute $k = \pm i k_2$ into Eq. (40) with $k_2 = m\pi/(1-B)$ to obtain

$$\begin{aligned} \omega &= \frac{k_2}{\pi}\sqrt{1+(B\,k_2)^2} \\ &= \frac{m}{1-B}\sqrt{1+m^2\pi^2\left(\frac{B}{1-B}\right)^2} \\ &\approx m\left[1+B+\left(1+\frac{1}{2}m^2\pi^2\right)B^2\right]. \end{aligned} \tag{51}$$

Restoring the time scaling by $1/\omega_0$, and defining the frequency (in cycles/second) $f = \omega/2\pi$, we finally have

$$f_m = \frac{m}{2\,L}\sqrt{\frac{T}{\mu}}\left[1+B+\left(1+\frac{1}{2}m^2\pi^2\right)B^2\right]. \tag{52}$$

We use this result to build our model in Section 2.

B Fretting Model

$$L'_n = d + \sqrt{(X_0+\Delta N - X_n - d)^2 + b^2} \approx X_0 - X_n + \Delta N + \frac{b^2}{2\,(X_0+\Delta N - X_n - d)}\,, \tag{53}$$

C Compensation by Minimizing RMS Error

The root-mean-square (RMS) frequency error (in cents) averaged over the frets $n \in \{1, n_{\max}\}$ *of a particular string* is given by

$$\overline{\Delta\nu} \equiv \sqrt{\frac{\sum_{n=1}^{n_{\max}}\Delta\nu_n^2}{n_{\max}}}\,, \tag{54}$$

where $\Delta\nu_n$ is given by Eq. (27). Here we will vary both ΔS and ΔN to minimize $\overline{\Delta\nu}$. In this case, it is sufficient to minimize the quantity

$$\chi^2 = \sum_{n=1}^{n_{\max}}\left[\frac{\ln(2)}{1200}\,\Delta\nu_n\right]^2 \tag{55}$$

such that the gradient of χ^2 with respect to ΔS and ΔN vanishes. The components of this gradient are

$$\frac{\partial}{\partial\Delta S}\chi^2 = -\frac{2}{X_0}\sum_n(\gamma_n-1)\left[(\gamma_n-1)\left(B_0-\frac{\Delta S}{X_0}\right)+\frac{\Delta N}{X_0}+\frac{\kappa}{2}\,Q_n\right],\ \text{and} \tag{56a}$$

$$\frac{\partial}{\partial\Delta N}\chi^2 = \frac{2}{X_0}\sum_n\left[(\gamma_n-1)\left(B_0-\frac{\Delta S}{X_0}\right)+\frac{\Delta N}{X_0}+\frac{\kappa}{2}\,Q_n\right]. \tag{56b}$$

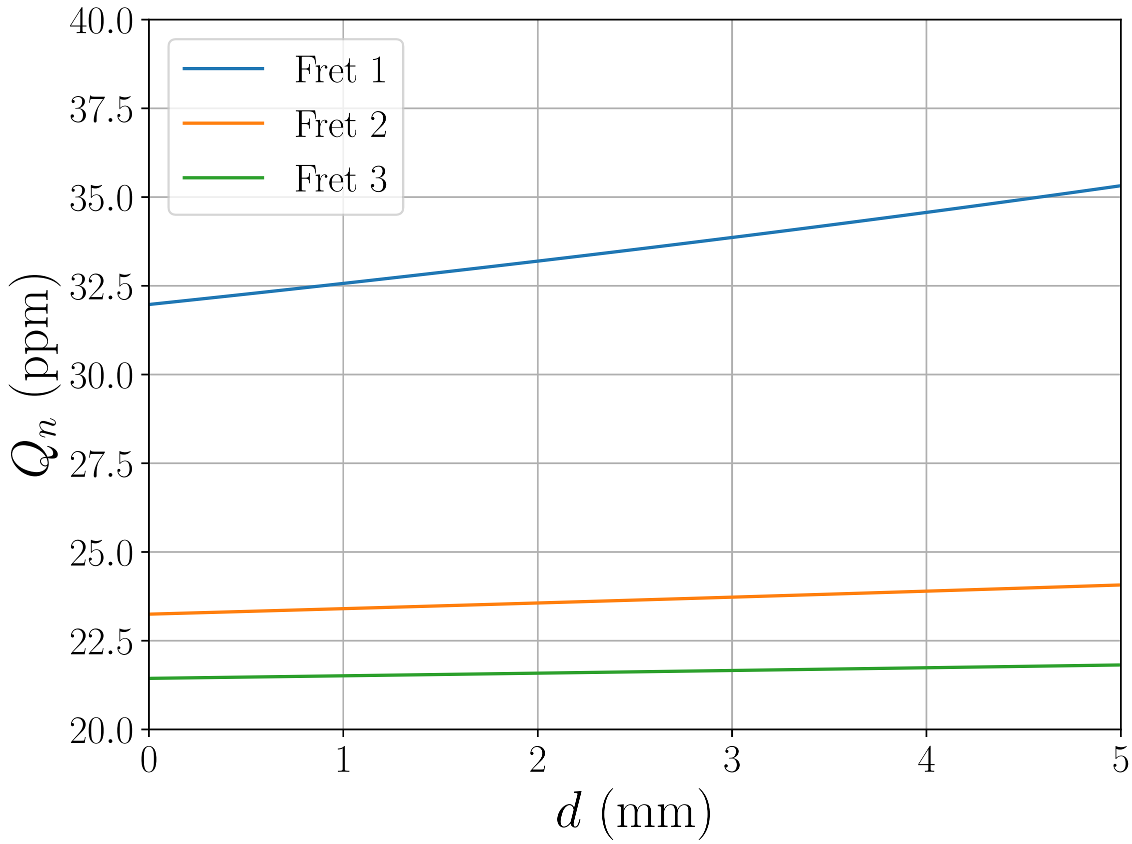


Figure 5: Plot of the normalized displacement Q_n as a function of the fret number for three different values of the parameter d. Here the guitar has b = 1.0 mm, c = 3.5 mm, no setbacks, and a scale length of 650 mm.

Setting both of these components to zero, we can rewrite them as the matrix equation

$$\begin{bmatrix} \sigma_2 & -\sigma_1 \\ \sigma_1 & -\sigma_0 \end{bmatrix} \begin{bmatrix} \Delta S \\ \Delta N \end{bmatrix} = X_0 \begin{bmatrix} \sigma_2 \, B_0 + \frac{1}{2}\,\kappa\,\overline{Q}_1 \\ \sigma_1 \, B_0 + \frac{1}{2}\,\kappa\,\overline{Q}_0 \end{bmatrix}, \tag{57}$$

where

$$\sigma_k \equiv \sum_{n=1}^{n_{\max}} (\gamma_n - 1)^k \;, \text{ and} \tag{58}$$

$$\overline{Q}_k \equiv \sum_{n=1}^{n_{\max}} (\gamma_n - 1)^k \; Q_n \,. \tag{59}$$

Equation (57) has the straightforward analytic solution

$$\begin{bmatrix} \Delta S \\ \Delta N \end{bmatrix} = \frac{X_0}{\sigma_1^2 - \sigma_0\,\sigma_2} \begin{bmatrix} -\sigma_0 & \sigma_1 \\ -\sigma_1 & \sigma_2 \end{bmatrix} \begin{bmatrix} \sigma_2 \, B_0 + \frac{1}{2}\,\kappa\,\overline{Q}_1 \\ \sigma_1 \, B_0 + \frac{1}{2}\,\kappa\,\overline{Q}_0 \end{bmatrix}. \tag{60}$$

For completeness, the corresponding solution when the quadratic stiffness term is included is given by

$$\begin{bmatrix} \Delta S \\ \Delta N \end{bmatrix} = \frac{X_0}{\sigma_1^2 - \sigma_0\,\sigma_2} \begin{bmatrix} -\sigma_0 & \sigma_1 \\ -\sigma_1 & \sigma_2 \end{bmatrix} \begin{bmatrix} \sigma_2 \, B_0 + \frac{1}{2}\,(1+\pi^2)\,(2\sigma_2 + \sigma_3)\,B_0^2 + \frac{1}{2}\,\kappa\,\overline{Q}_1 \\ \sigma_1 \, B_0 + \frac{1}{2}\,(1+\pi^2)\,(2\sigma_1 + \sigma_2)\,B_0^2 + \frac{1}{2}\,\kappa\,\overline{Q}_0 \end{bmatrix}. \tag{61}$$

The corresponding Hessian matrix for this problem is

$$H = \begin{bmatrix} \frac{\partial^2 \chi^2}{\partial \Delta S^2} & \frac{\partial^2 \chi^2}{\partial \Delta N \, \partial \Delta S} \\ \frac{\partial^2 \chi^2}{\partial \Delta S \, \partial \Delta N} & \frac{\partial^2 \chi^2}{\partial \Delta N^2} \end{bmatrix} = \frac{2}{X_0^2} \begin{bmatrix} \sigma_2 & -\sigma_1 \\ -\sigma_1 & \sigma_0 \end{bmatrix}. \tag{62}$$

The Hessian is positive definite if and only if all of its eigenvalues are positive, and in the case of a 2×2 real matrix, this holds when the determinant is greater than zero. It is easy to verify numerically (and with some effort algebraically) that $\mathrm{Det}(H) > 0$ for $n_{\max} > 1$. Therefore, the solution for ΔS and ΔN given by Eq. (60) minimizes the RMS frequency error.

D Other Classical Guitar String Sets

D.1 Light Tension

Table 5: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4301	E_4	0.35	3.62	66.39
J4302	B_3	0.40	4.87	50.24
J4303	G_3	0.50	8.08	52.48
J4304	D_3	0.36	18.23	66.41
J4305	A_2	0.42	27.41	56.06
J4306	E_2	0.53	51.59	59.21

Table 6: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

String	$\Delta\nu_{12}$ (cents)	R	κ	E (GPa)	B_0 ($\times 10^{-3}$)
J4301	3	23.2	47.3	8.2	1.8
J4302	2	11.2	23.4	2.3	1.5
J4303	10	55.6	112.2	7.4	4.1
J4304	0	22.8	46.7	7.8	1.9
J4305	2	24.8	50.5	5.1	2.3
J4306	4	15.5	32.0	2.1	2.3

Table 7: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43) on the Alhambra 8P classical guitar.

String	ΔS (mm)	ΔN (mm)
J4301	1.87	-0.35
J4302	1.31	-0.17
J4303	4.35	-0.83
J4304	1.88	-0.35
J4305	2.22	-0.37
J4306	2.00	-0.24

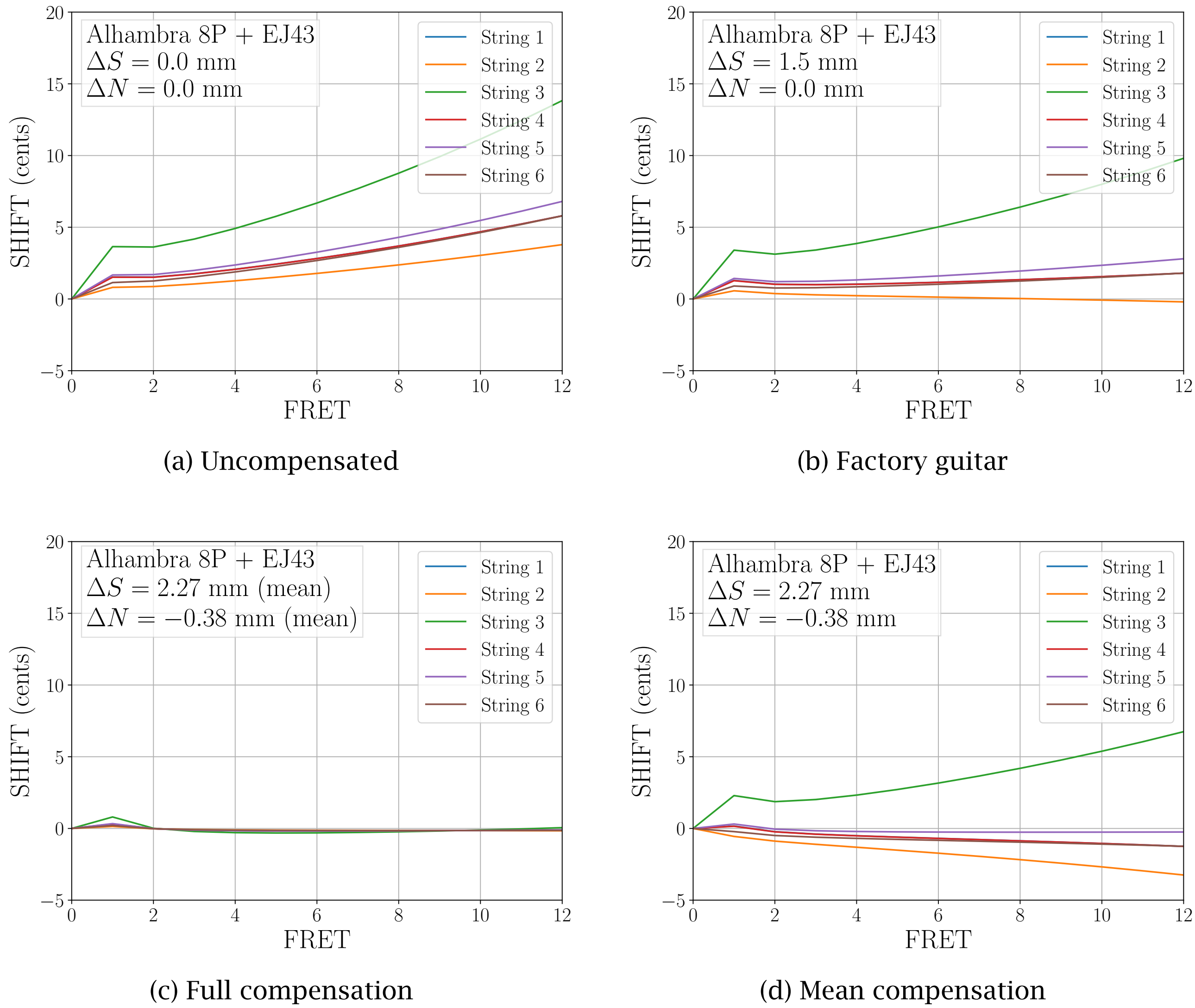


Figure 6: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). Four different strategies of saddle and nut compensation are illustrated.

D.2 Hard Tension

Table 8: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). The corresponding scale length is 650 mm.

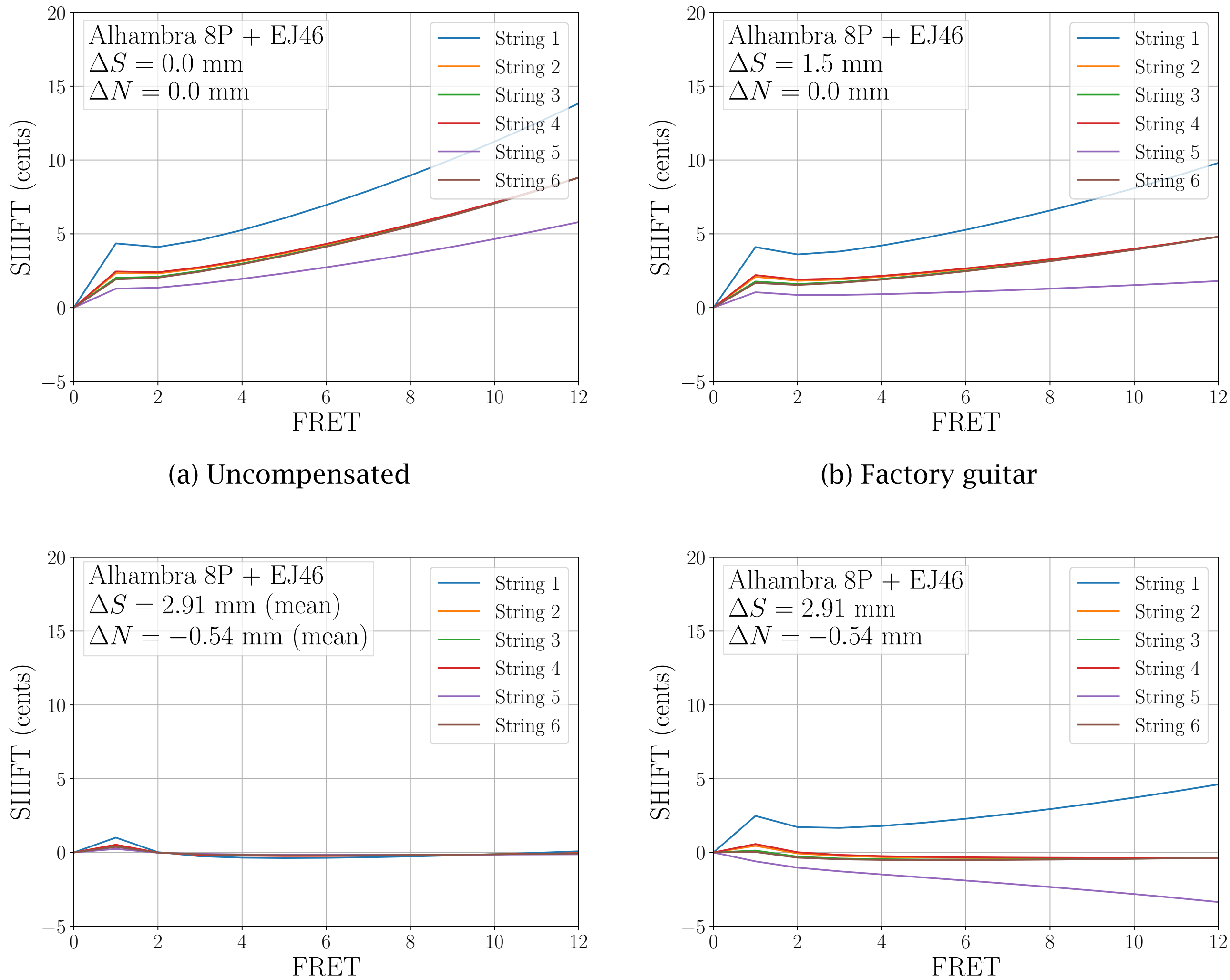
String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4601	E_4	0.36	3.86	70.88
J4602	B_3	0.42	5.22	53.83
J4603	G_3	0.52	8.57	55.61
J4604	D_3	0.38	20.07	73.14
J4605	A_2	0.46	34.87	71.31
J4606	E_2	0.56	56.67	65.04

Table 9: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). The corresponding scale length is 650 mm.

String	$\Delta\nu_{12}$ (cents)	R	κ	E (GPa)	B_0 ($\times 10^{-3}$)
J4601	3	69.4	139.8	24.1	3.3
J4602	2	35.4	71.9	7.1	2.7
J4603	10	29.2	59.4	3.9	3.1
J4604	0	37.8	76.6	12.3	2.6
J4605	2	18.2	37.5	4.1	2.2
J4606	4	27.2	55.5	3.7	3.2

Table 10: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46) on the Alhambra 8P classical guitar.

String	ΔS (mm)	ΔN (mm)
J4601	4.13	-1.04
J4602	2.80	-0.53
J4603	2.90	-0.44
J4604	2.76	-0.57
J4605	1.95	-0.28
J4606	2.93	-0.41



(a) Uncompensated

(b) Factory guitar

(c) Full compensation

(d) Mean compensation

Figure 7: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). Four different strategies of saddle and nut compensation are illustrated.

D.3 Extra Hard Tension

Table 11: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). The corresponding scale length is 650 mm.

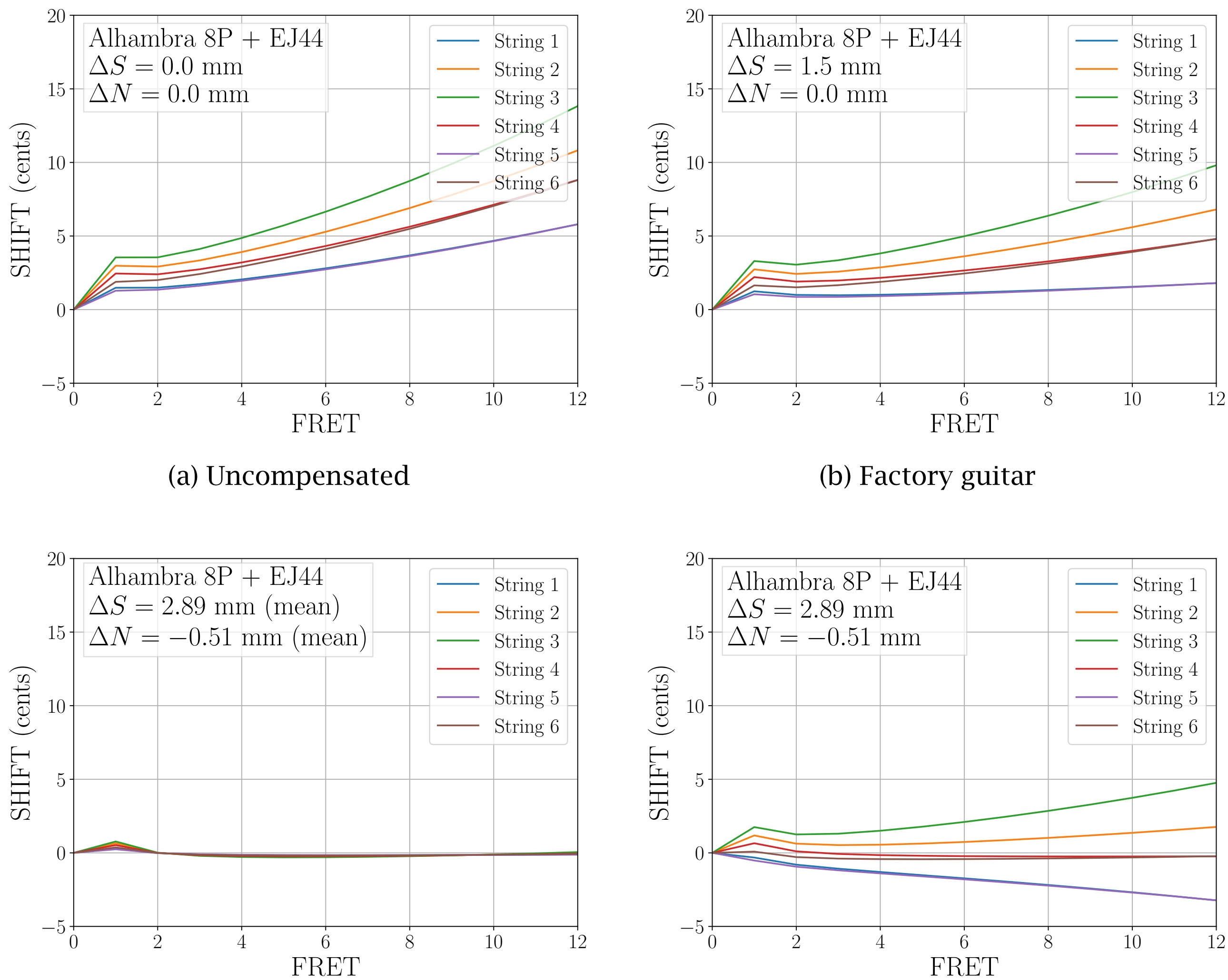
String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4401	E_4	0.37	4.01	73.57
J4402	B_3	0.42	5.44	56.07
J4403	G_3	0.53	8.91	57.86
J4404	D_3	0.38	20.07	73.14
J4405	A_2	0.46	34.87	71.31
J4406	E_2	0.57	61.36	70.42

Table 12: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). The corresponding scale length is 650 mm.

String	$\Delta\nu_{12}$ (cents)	R	κ	E (GPa)	B_0 ($\times 10^{-3}$)
J4401	3	22.2	45.4	7.8	1.9
J4402	2	45.9	92.7	9.3	3.1
J4403	10	53.6	108.2	7.1	4.2
J4404	0	37.8	76.6	12.3	2.6
J4405	2	18.2	37.5	4.1	2.2
J4406	4	26.6	54.3	3.7	3.2

Table 13: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44) on the Alhambra 8P classical guitar.

String	ΔS (mm)	ΔN (mm)
J4401	1.89	-0.34
J4402	3.38	-0.69
J4403	4.39	-0.80
J4404	2.76	-0.57
J4405	1.95	-0.28
J4406	2.94	-0.40



(a) Uncompensated

(b) Factory guitar

(c) Full compensation

(d) Mean compensation

Figure 8: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). Four different strategies of saddle and nut compensation are illustrated.

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