Classical Guitar Intonation and Compensation: The Well-Tempered Guitar

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Abstract

Inspired by the pioneering work of luthier Greg Byers in 1996, we build an intuitive model of classical guitar intonation that includes the effects of the resonant length of the fretted string, linear mass density, tension, and bending stiffness. We begin by deriving an expression for the vibration frequencies of a stiff string using boundary conditions that are pinned at the saddle but clamped at the fret. Adopting logarithmic frequency differences based on "cents" that decouple these physical effects, we introduce Taylor series expansions that exhibit clearly the origins of frequency shifts of fretted notes from the corresponding Twelve-Tone Equal Temperament (12-TET) values. We demonstrate a simple *in situ* technique for measurement of the changes in frequency of open strings arising from small adjustments in length, and we propose a simple procedure that any interested guitarist can use to estimate the corresponding shifts in frequency due to tension and bending stiffness for their own guitars and favorite string sets. Based on these results, we employ an RMS frequency error method to select values of saddle and nut setbacks that map fretted frequencies — for a particular string set on a particular guitar — almost perfectly onto their 12-TET targets. This exercise allows us to discuss a general approach to tempering an "off-the-shelf" guitar to further reduce the tonal errors inherent in any fretted musical instrument.

Contents

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1 Introduction and Background

Any musician who has wrestled with the temperament of a fretted stringed instrument is well aware of the challenges presented by tuning and pitch. In addition to the mathematical physics of musical scales [1, 2, 3], the mechanical specifications of the instrument and the strings themselves [4, 5, 6] require accommodation during both manufacturing [7, 8] and tuning to achieve harmonious results. We can gain an appreciation for this problem by analyzing the expression for the allowed vibration frequencies of an ideal string, given by [9, 10]

$$f_q = \frac{q}{2\,L_0}\sqrt{\frac{T_0}{\mu_0}}\,, \tag{1}$$

where $q \in \mathbb{N} = \{1, 2, \ldots\}$ identifies the "harmonic" of the fundamental frequency f_1, L_0 is the length of the free (unfretted) string from the saddle to the nut, T_0 is the tension in the free string, and $\mu_0 \equiv M/L_0$ is the linear mass density of a free string of mass M. The act of fretting the string changes its length, and therefore its frequency. For example, modern classical guitars are manufactured with frets placed along the fretboard using the Twelve-Tone Equal Temperament (12-TET) system, whereby the resonant length of a string pressed behind fret n ideally should be $2^{-n/12}\,L_0$, thereby producing a note with frequency $2^{n/12}\,f_1$. But this result can never be achieved perfectly in reality.

First, the string is elevated above the frets by the saddle and nut, so the fretted string is slightly elongated relative to the free string, and the resulting frequency is flattened in pitch. In principle, this effect could be accommodated by minute changes in the positions of the frets, but there are additional practical complications. For example, the string's tension is increased by the change in length, causing the frequency to sharpen by an amount that significantly exceeds the reduction caused by the increase in the resonant string length. In addition, the string is by no means ideal, and its intrinsic stiffness results in an additional increase in pitch that depends on its mechanical characteristics. These guitar intonation difficulties seem to preclude successful temperament, but remarkably the instrument can be *compensated* by moving the positions of the saddle and the nut by small distances during the manufacturing process [7, 8]. This compensation process helps temper the guitar so that it is *playable*. It must also be *tunable*, so that the guitar strings can be brought into compliance with 12-TET quickly and accurately. This requirement places significant constraints on the physical properties of manufactured strings. Our goal in this work is to build an intuitive understanding of these effects to aid in the compensation and subsequent tuning of the classical guitar, with an accessible approach to making measurements of string properties and deducing the corresponding effects of playability [11, 12, 13].

Throughout this work, we will use *cents* to describe small differences in pitch [14, 15, 16, 17, 8, 3]. One cent is one one-hundredth of a 12-TET half-step, so that there are 1200 cents per octave. An experienced guitar player can distinguish beat notes with a difference frequency of $\Delta f \approx 1$ Hz, which corresponds to 8 cents at A_3 ($f = 220$ Hz) or 5 cents at E_4 ($f = 329.63$ Hz). Using this approach, the difference in pitch between two frequencies f_1 and f_2 is defined as

$$\Delta\nu \equiv 1200 \log_2 \left(\frac{f_2}{f_1} \right) , \tag{2}$$

where $\log_2(x)$ is the logarithm base 2 of x. Let's choose the average frequency $f \equiv (f_1 + f_2)/2$ and the frequency difference $\Delta f \equiv f_2 - f_1$. Then

$$\Delta\nu = 1200 \log_2 \left(\frac{f + \Delta f/2}{f - \Delta f/2} \right) \approx \frac{1200}{\ln(2)} \frac{\Delta f}{f} \approx 1731 \frac{\Delta f}{f} , \tag{3}$$

where the approximation applies when $\Delta f \ll f$, and $\ln(x)$ is the natural logarithm function of x. As shown in Fig. 1, if the average frequency of the interval is used to compute $\Delta\nu$ — rather than the initial frequency f_1 — then the accuracy of Eq. (3) holds for almost an entire octave. In this plot, we chose $f_1 = A_3 = 220$ Hz, and allowed f_2 to vary from A_3 to $A_4 + 30$ Hz $= 470$ Hz. At the octave, the error in $\Delta\nu$ arising from Eq. (3) is only -46 cents, or -4%. The choice of a logarithmic frequency-difference scale *decouples* multiplicative factors that predict the frequency of a real fretted string, allowing us to build straightforward intuitive models of guitar intonation and compensation.

- We ignore polarization. Classical is elliptical.
- Impact on the boundary conditions is unclear.
- Frequency differences between the two linear polarizations aren't relevant; the amplitude decays uniformly, and there's no evidence of splitting at the fundamental or first several higher harmonics.

Ref for admittance. Don't worry about it.

We present the basics of our model of classical guitar strings in Section 2, following the pioneering work of G. Byers [7, 18]. We offer empirical reasons to doubt the need for a complicated model of string fretting that incorporates either the depth of fretting or the shape of the finger in Appendix A. Then, in Appendix B, we derive a new expression for the allowed vibration frequencies of a stiff string (although more complicated dynamical models are available [19]),

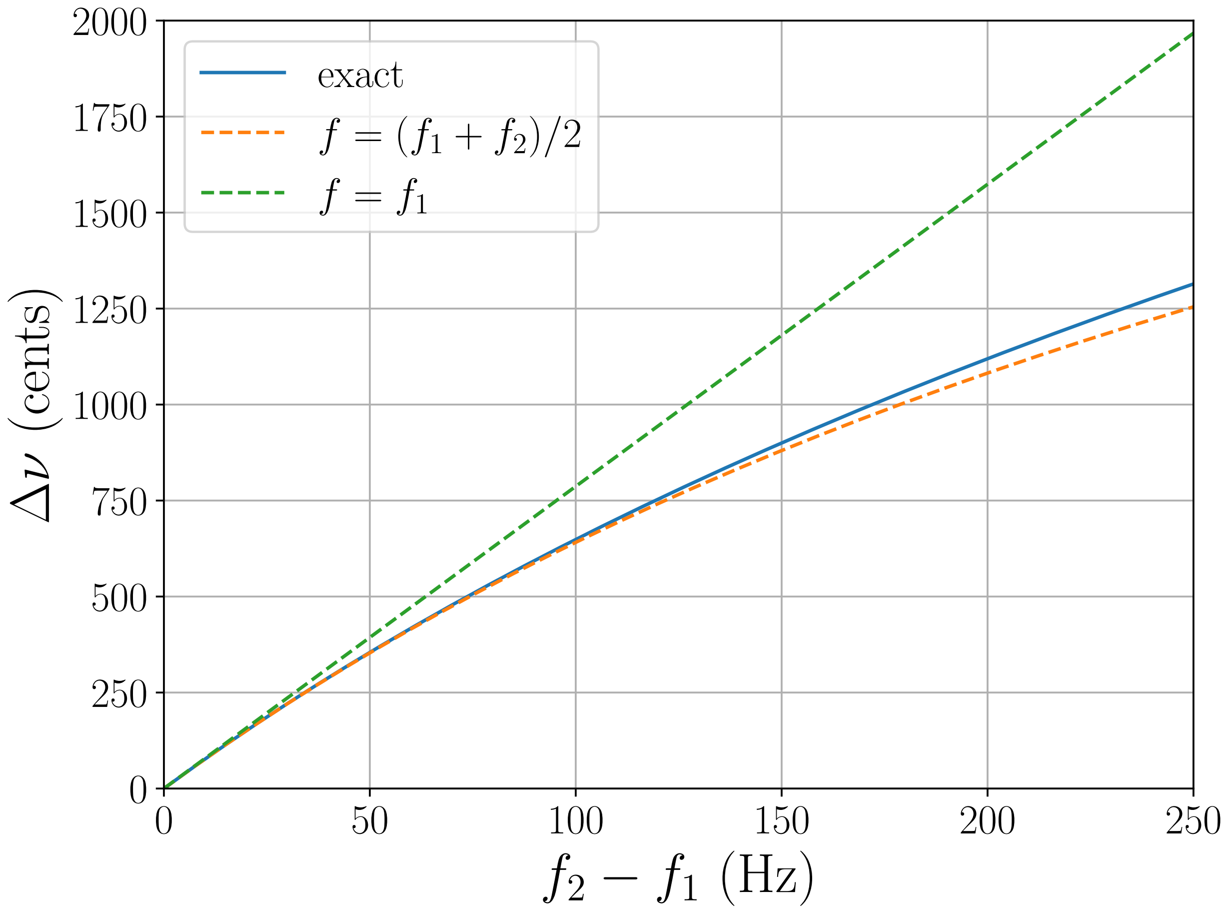


Figure 1: Plot of $\Delta\nu$ for $f_1 = A_3 = 220$ Hz and f_2 varying from A_3 to $A_4 + 30$ Hz = 450 Hz. We compare two different definitions of f in Eq. (3): the average of f_1 and f_2, and simply $f = f_1$. Using the average frequency leads to a significantly better approximation.

derived under the assumption that the boundary conditions at the saddle and the nut are not symmetric. Based on this result, we then discuss in detail the four contributions to frequency shifts and errors of non-ideal strings pressed behind a fret: the change in the resonant length of the vibrating string; a decrease in the linear mass density of the entire string; an increase in the tension of the entire string; and an increase in the mechanical stiffness of the resonating string. Our goal is to simplify the decoupled equations describing these effects through Taylor series expansions to allow an intuitive picture of the string's behavior to emerge. Nylon strings behave very differently than the metal strings used on acoustic guitars [20, 21]. They require time to "settle" and reach equilibrium after restringing, and it is unlikely that the uniform stiff rod approach used to develop equations for the resonant frequencies (including ours) apply to either the monofilament treble strings or the wound bass strings. Nevertheless, we are able to develop a phenomenological model of the mechanical characteristics of these strings that is consistent with measurements of frequency deviations on uncompensated classical guitars.

In Section 2.5, we collect all four of the effects mentioned above and develop a simple approximate expression for the total frequency shift of a fretted guitar string. We note that the two largest contributions to this deviation from 12-TET perfection are the increases in tension and bending stiffness, and that small changes in the positions of the nut and the saddle can largely compensate for these problems. In Section 3, we demonstrate a simple, straightforward experiment to measure *in situ* the response of a string's fundamental frequency to a change in length (and therefore tension), and we rely on a subsequent phenomenological approach to determining the open-string bending stiffness. Then, in Section 4, we use these values for a normal-tension nylon string set (as well as other string sets in Appendix D) to demonstrate a straightforward analytic approach to compensating for the tone errors in a guitar string, relying on a method — described in Appendix C — to minimize the root-mean-squared (RMS) frequency deviation at each fret. With these results in hand, in Section 5 we discuss a collaboration of guitar manufacturer and musician to temper the guitar using harmonic tuning to optimize it for a particular piece. Finally, in Section 6, we recapitulate our "recipes" for guitar compensation, including a simple method to include the effects of relief for large-amplitude string vibration.

This document — as well as the Python computer code needed to reproduce the figures — is available at GitHub [22].

2 Simple Model of Guitar Intonation

The starting point for prior efforts to understand guitar intonation and compensation [7, 8] is a formula for f_q, the transverse vibration frequency harmonic q of a stiff string, originally published by Morse in 1936 [23, 24, 25]:

$$f_q = \frac{q}{2\,L}\sqrt{\frac{T}{\mu}}\left[1 + 2B + 4\left(1 + \frac{\pi^2 q^2}{8}\right)B^2\right]. \tag{4}$$

Here L is the length of the string, T and μ are its tension and linear mass density, respectively, and B is a small "bending stiffness" coefficient to capture the relevant mechanical properties of the string. For a homogeneous string with a cylindrical cross-section, B is given by

$$B \equiv \sqrt{\frac{E\,\mathcal{A}\,s^2}{L^2 T}}\,, \tag{5}$$

where $\mathcal{A}$ and s are the cross-sectional area and the radius of gyration of the string, respectively, and E is Young's modulus (or the modulus of elasticity). But it's unlikely that Eq. (4) accurately describes the resonant frequencies of a nylon string on a classical guitar. First, it is derived by assuming that the vibration of the string is polarized vertically (perpendicular to the plane of the guitar top). This is true for a piano string, but not for a classical guitar string, which is polarized elliptically with the major axis parallel to the guitar top. Second, the factor of two in front of the bending stiffness arises from the assumption that the string is "clamped" at both ends, so that a particular set of symmetric mathematical boundary conditions must be applied to the partial differential equation (PDE) describing transverse vibrations of the string. However, measurements of the frequency of a stiff piano string showed that neither symmetric clamped nor "pinned" boundary conditions were completely correct [24]. In addition, Eq. (4) predicts values of saddle setbacks that are about twice as large as those used by experienced luthiers based on trial and error [26].

As a compromise, we assume that the string is clamped at the nut but pinned at the saddle, and we neglect the impact of the polarization of the vibrating string. In Appendix B, we solve the PDE using these non-symmetric boundary conditions, and find

$$f_q = \frac{q}{2\,L}\sqrt{\frac{T}{\mu}}\left[1 + B + \left(1 + \frac{1}{2}q^2\pi^2\right)B^2\right]. \tag{6}$$

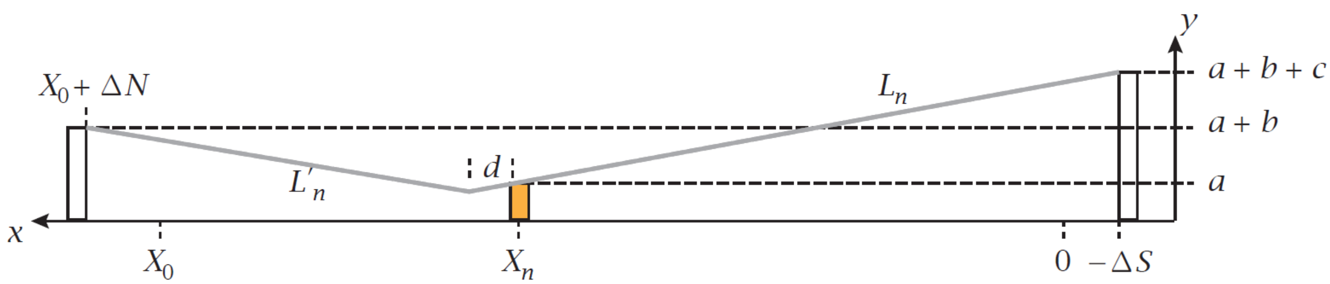


Figure 2: A simple (side-view) schematic of the classical guitar used in this model.

Note that this expression is valid only when $B \ll 1$. We'll see that a typical nylon guitar string has $B \approx 2 - 3 \times 10^{-3}$. In this case, the quadratic B term in Eq. (6) is only 2% as large as the linear term, and can generally be neglected. (We will include it in our numerical computations for completeness.) We should use Eq. (6) with some caution, because the chemistry, materials science, and physics of nylon strings (particularly the wound bass strings) are quite complicated [20]. With this in mind, we check the validity of this equation for the nylon strings we measure in Section 3.

Our model is based on the schematic of the guitar shown in Fig. 2. The scale length of the guitar is X_0, but we allow the inside edges of both the saddle and the nut to be set back an additional distance ΔS and ΔN, respectively. The location on the x-axis of the center of the n^{th} fret is X_n. In the y direction, $y = 0$ is taken as the surface of the fingerboard; the height of each fret is a, the height of the nut (i.e., the distance between the fingerboard and the bottom of the string) is $a+b$, and the height of the saddle is $a+b+c$. (For the moment, we are neglecting the art of *relief* practiced by expert luthiers that adjusts the value of b up the fretboard and strings. We discuss this effect below in Section 2.3.) L_n is the *resonant length* of the string from the saddle to the center of fret n, and L'_n is the length of the string from the fret to the nut. The total length of the string is defined as $\mathcal{L}_n \equiv L_n + L'_n$. As discussed in more detail in Appendix A, we have chosen to include a line-segment intersection at a distance d behind fret n to represent the slight increase in the distance L'_n caused by a finger. This differs from previous studies of guitar intonation and compensation [7, 8], but our approach is consistent with empirical observations for nylon strings.

We start with the form of the fundamental frequency of a fretted string given by Eq. (6) with $q = 1$, and apply it to the frequency of a string pressed just behind the n^{th} fret:

$$f_n = \frac{1}{2\,L_n}\sqrt{\frac{T_n}{\mu_n}}\left[1 + B_n + \left(1 + \frac{\pi^2}{2}\right)B_n^2\right], \tag{7}$$

where T_n and μ_n are the modified tension and the linear mass density of the fretted string, and

$$B_n \equiv \sqrt{\frac{E\,\mathcal{A}\,s^2}{4\,T_n\,L_n^2}}\,. \tag{8}$$

We note that T_n and μ_n depend on $\mathcal{L}_n$, the *total* length of the fretted string from the saddle to the nut. Ideally, in the 12-TET system [3],

$$f_n = \gamma_n\,f_0\,, \qquad \text{(12-TET ideal)} \tag{9}$$

where f_0 is the frequency of the open (unfretted) string, and

$$\gamma_n \equiv 2^{n/12}\,. \tag{10}$$

Therefore, the error interval — the difference between the fundamental frequency of the fretted string and the corresponding perfect 12-TET frequency — expressed in cents is given by

$$\begin{aligned}\Delta\nu_n &= 1200\,\log_2\left(\frac{f_n}{\gamma_n\,f_0}\right)\\ &= 1200\,\log_2\left(\frac{L_0}{\gamma_n\,L_n}\right) + 600\,\log_2\left(\frac{\mu_0}{\mu_n}\right) + 600\,\log_2\left(\frac{T_n}{T_0}\right)\\ &\quad + 1200\,\log_2\left[\frac{1 + B_n + (1+\pi^2/2)\,B_n^2}{1 + B_0 + (1+\pi^2/2)\,B_0^2}\right],\end{aligned} \tag{11}$$

where $\log_2$ is the (binary) logarithm function calculated with base 2.

The final form of Eq. (11) makes it clear that — for nylon guitar strings — there are four contributions to intonation:

1. *Resonant Length*: The first term represents the error caused by the increase in the length of the fretted string L_n compared to the ideal length X_n, which would be obtained if $b = c = d = 0$ and $\Delta S = \Delta N = 0$.

2. *Linear Mass Density*: The second term is the error caused by the reduction of the linear mass density of the fretted string. This effect will depend on the *total* length of the string $\mathcal{L}_n = L_n + L'_n$.

3. *Tension*: The third term is the error caused by the *increase* of the tension in the string arising from the stress and strain applied to the string by fretting. This effect will also depend on the total length of the string $\mathcal{L}_n$.

4. *Bending Stiffness*: The fourth and final term is the error caused by the change in the bending stiffness coefficient arising from the decrease in the vibrating length of the string from L_0 to L_n.

Note that the properties of the logarithm function have *decoupled* these physical effects by converting multiplication into addition. We will discuss each of these sources of error in turn below.

In the discussion that follows, we'll test our approximations for a prototypical Classical Guitar with the specifications listed in Table 1. Refer to Fig. 2 for a graphical representation of these parameters. In addition, as we develop models of the physical effects discussed above, we'll assume that the guitar string has the properties listed in Table 2. The string constant κ and the open-string bending stiffness B_0 are introduced in Section 2.3 and Section 2.4 respectively. The linear frequency shift parameter R is discussed in Section 2.5, and a method for determining both κ and B_0 in terms of R is discussed.

2.1 Resonant Length

The length L_0 of the open (unfretted) guitar string can be calculated quickly by referring to Fig. 2. We find:

$$L_0 = \sqrt{(X_0 + \Delta S + \Delta N)^2 + c^2} \approx X_0 + \Delta S + \Delta N + \frac{c^2}{2\,X_0}\,, \tag{12}$$

Table 1: Default specifications for the prototypical Classical Guitar modeled in this section. The default values of d, ΔS, and ΔN can be either zero or the nonzero value listed in the table and discussed in the text.

Parameter	Description	Default Value (mm)
X_0	Scale length	650
b	Height of the nut above fret 1	1
c	Height of the saddle above the nut	4
d	Fretting distance	0 or 10
ΔS	Saddle setback	0 or 1.8
ΔN	Nut setback	0 or -0.38

Table 2: Default specifications for a prototypical guitar string. The string constant κ and the open-string bending stiffness B_0 are introduced in Section 2.3 and Section 2.4 respectively, and the linear frequency shift parameter R is discussed in Section 2.5.

Parameter	Description	Default Value
ρ	String radius in mm	0.43
R	Linear frequency shift parameter	25
κ	String tension constant	51
B_0	Open-string bending stiffness	0.00236

where the approximation arising from the Taylor series is valid to second order in all small distances since $\{\Delta S, \Delta N, b, c\} \ll X_0^2$. Similarly, the resonant length L_n is given by

$$L_n = \sqrt{(X_n + \Delta S)^2 + (b+c)^2} \approx X_n + \Delta S + \frac{(b+c)^2}{2\,X_n}\,. \tag{13}$$

Then — if the guitar has been manufactured such that $X_n = X_0/\gamma_n$ — the resonant length error determined by the first term in the last line of Eq. (11) is approximately

$$\begin{aligned}1200\,\log_2\left(\frac{L_0}{\gamma_n\,L_n}\right) \approx \frac{1200}{\ln(2)}\Bigg[&\frac{\Delta N - (\gamma_n - 1)\,\Delta S}{X_0} - \frac{(\Delta N + \Delta S)^2 - \gamma_n^2\,\Delta S^2}{2\,X_0^2} \\ &- \frac{\gamma_n^2(b+c)^2 - c^2}{2\,X_0^2}\Bigg]\,.\end{aligned} \tag{14}$$

If the guitar is uncompensated, so that $\Delta S = \Delta N = 0$, the magnitude of this error on our Classical Guitar can be neglected in approximate treatments. However, we'll see that choosing $\Delta S > 0$ and $\Delta N < 0$ will allow us to substantially compensate for frequency shift contributions from other effects. In this case, we note that the three terms inside the bracket on the right-hand side of Eq. (14) are $\{-3.5\times10^{-3}, 1.4\times10^{-5}, -1.0\times10^{-4}\}$, respectively, for the parameter values given by Table 1, corresponding to frequency shifts of $\{-6.05, 0.02, -0.17\}$ cents. Therefore, the second two terms are negligible compared to the first, and we can approximate the resonant length error — for the purposes of estimating setbacks in Section 4 — as

$$1200\,\log_2\left(\frac{L_0}{\gamma_n\,L_n}\right) \approx \frac{1200}{\ln(2)}\left[\frac{\Delta N - (\gamma_n - 1)\,\Delta S}{X_0}\right]. \tag{15}$$

We'll include the term in Eq. (14) that is quadratic in b and c in our computation of setbacks detailed in Appendix C, and we'll use Eq. (12) and Eq. (13) when computing frequency errors.

2.2 Linear Mass Density

As discussed above, the linear mass density μ_0 of an open (unfretted) string is simply the total mass M of the string clamped between the saddle and the nut divided by the length L_0. Similarly, the mass density μ_n of a string held onto fret N is $M/\mathcal{L}_n$. Therefore

$$\frac{\mu_0}{\mu_n} = \frac{\mathcal{L}_n}{L_0} \equiv 1 + Q_n \,, \tag{16}$$

where we have followed Byers and defined the normalized relative displacement [7, 8]

$$Q_n \equiv \frac{\mathcal{L}_n - L_0}{L_0} \,, \tag{17}$$

where $\mathcal{L}_n = L_n + L'_n$. After judicious use of similar triangles and the Pythagorean Theorem we calculate L'_n for $n \geq 1$ as

$$L'_n = \frac{L_n}{X_n + \Delta S}\, d + \sqrt{(X_0 - X_n + \Delta N - d)^2 + \left(b + \frac{b + c}{X_n + \Delta S}\, d\right)^2} \,. \tag{18}$$

When $d \ll X_0$, we can expand Q_n to third order in all small distances and find

$$Q_n \approx \frac{[\gamma_n\, b + (\gamma_n - 1)\, c]^2}{2\, (\gamma_n - 1)\, X_0^2} \left(1 + \frac{\gamma_n^2}{\gamma_n - 1} \frac{d}{X_0}\right) \,, \tag{19}$$

Although it is arguable whether this approximation is simpler than the exact expression given by Eq. (17), it is quite clear that Q_n does not depend significantly on the setbacks ΔS or ΔN. For a guitar with the specifications listed in Table 1, Q_n falls in the range $25 - 45 \times 10^{-6}$ for $d \leq 10$ mm, corresponding to a net stretch of the string less than 0.03 mm. For the same parameters, when $d = 10$ mm, we find that $\Delta\nu_1 \approx 0.04$ cents, and is smaller at all other frets. Therefore, in general the shift due to linear mass density can be neglected without significant loss of accuracy in the approximate setback solutions we derive in Section 4.

2.3 Tension

Counterintuitively, nylon classical guitar strings have very different physical properties than those of steel strings [27, 28, 29], with completely different stress-strain curves. When fresh nylon strings are brought up to the required tension for the first time, they are stretched by a macroscopic distance ΔL that varies from 7 cm (for the first E_4 string) to 2 cm (for the sixth E_2 string). After only a few minutes, the string must be re-tensioned because it has experienced nonlinear viscoelastic relaxation and has gone flat by at least a half-step [30, 20, 21]. This process continues for several hours until the strings begin to "settle" and remain properly tuned for longer periods; after about 10 hours, they will respond at the correct frequencies for more than an hour provided that the temperature in the room doesn't change significantly [30, 20]. A string removed from the guitar after this stage has been reached will not relax back to its original "out-of-the-box" length — it has been permanently deformed. The frequency of most settled nylon strings string can be "dropped" one whole step and returned to the initial value, but attempts to increase tension further will reach a nonlinear stage where the frequency increases much less quickly and will often result in a broken string.

We will focus on the response of a settled string to a differential longitudinal strain, and neglect the transverse stress that causes insignificant changes in the radius of the string [20]. As we show in Section 3, in the settled regime we can infer an infinitesimal change δL of a nylon string with length L will result in a linear change in the tension by an amount [31]

$$\delta T = \mathcal{A}\, E_{\text{eff}}\, \frac{\delta L}{L}\,, \tag{20}$$

where E_{eff} is an effective linear modulus of elasticity representing the ratio of the differential stress $\delta T/\mathcal{A}$ to the differential strain $\delta L/L$. Therefore, we write the change in tension of a string stretched by touching fret n as

$$\delta T_n = \mathcal{A}\, E_{\text{eff}}\, Q_n\,, \tag{21}$$

where Q_n is the normalized infinitesimal displacement — here acting as the differential strain — defined by Eq. (17). Note that we are using the length L_0 of the unfretted string between the saddle and the nut as our reference length. In the case of steel strings, it may be appropriate to include the length of the string between the nut and the tuning peg if the tension on either side of the nut is the same [29]. But a nylon string on a classical guitar emerges from the outside of the nut at a sharp angle, and is held so tightly within the nut that δL of a settled string is twice as large between the nut and the tuners than that of the vibrating string.

Based on these considerations, we write the tension in a settled string clamped to fret n as

$$T_n = T_0 + \delta T_n = T_0\left(1 + \kappa\, Q_n\right)\,, \tag{22}$$

where we have defined the dimensionless linear "string constant"

$$\kappa \equiv \frac{\mathcal{A} E_{\text{eff}}}{T_0}\,. \tag{23}$$

The corresponding frequency shift due to the increase in tension caused by fretting is therefore given by the third term in the final line of Eq. (11) as

$$600\,\log_2\left(\frac{T_n}{T_0}\right) = 600\,\log_2\left(1 + \kappa\, Q_n\right)\,. \tag{24}$$

If we assume that $\kappa\, Q_n \ll 1$, then we can approximate this expression as

$$600\,\log_2\left(\frac{T_n}{T_0}\right) \approx \frac{600}{\ln(2)}\,\kappa\, Q_n\,, \tag{25}$$

where now Q_n is given by Eq. (19). In this form, it is clear that this frequency shift is larger than that caused by the linear mass density by a factor of κ.

In Fig. 3, we plot a comparison between the exact and approximate expressions for the frequency error resulting from the tension increase given by Eq. (24) and Eq. (25) for two values of d. The normalized displacement Q_n is computed using Eq. (17) in the exact curves and Eq. (19) in the approximate curves. Here the guitar has the specifications listed in Table 1: the exact curves use $\Delta S = 1.8$ mm and $\Delta N = -0.38$ mm, and the approximate curves ignore the setbacks entirely. The slight difference between the exact and approximate shifts for

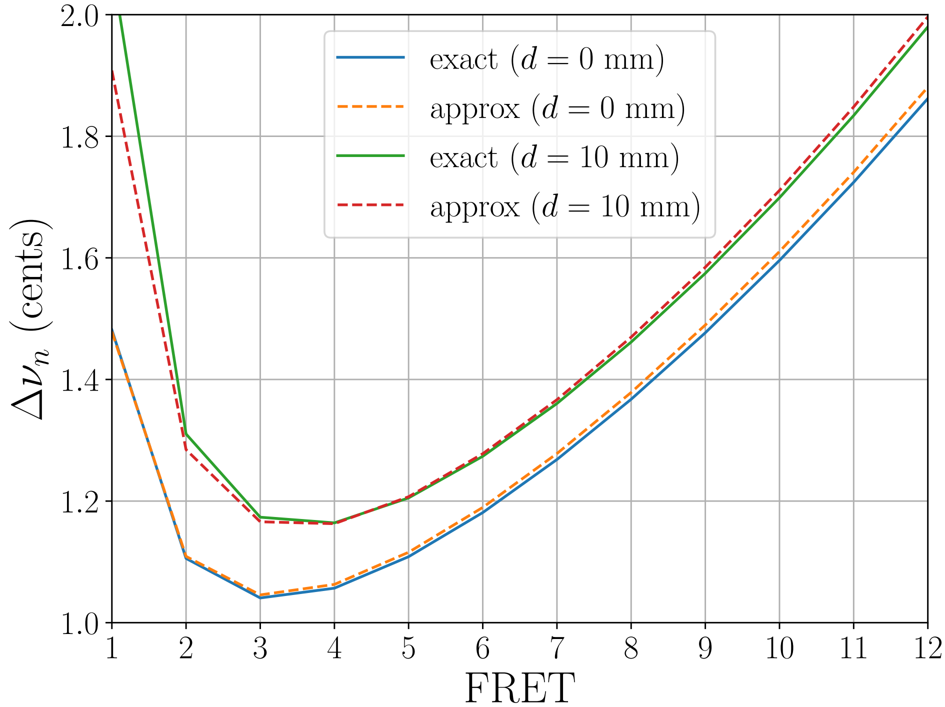


Figure 3: Comparison of the exact expression for the frequency shift caused by tension given by Eq. (24) with the approximate expression given by Eq. (25).

$d = 10$ mm at the first fret can be eliminated if we include a term quadratic in d in Eq. (19). As predicted above, we see that the dependence of Q_n — and therefore the tension shift — on the setback values is minimal.

Many luthiers provide "relief" to enlarge the effective height of the string (particularly for the wound bass strings) as the fret number grows to provide clearance for vibration amplitude at higher volume. In practice, this is accomplished by pivoting the fret board shown in Fig. 2 clockwise about $x = X_0$, increasing the height of the string above fret n by an amount

$$\begin{aligned}\Delta y_n &= m\,(X_0 - X_n) \\ &= \frac{\gamma_n - 1}{\gamma_n}\, m\, X_0 \\ &= \frac{\gamma_n - 1}{\gamma_n}\, 2\,\Delta y_{12}\,,\end{aligned} \tag{26}$$

where Δy_{12} is the relief at the twelfth fret and $m = 2\,\Delta y_{12}/X_0 \geq 0$ is the downward slope of the fret board. If we update Eq. (13) and Eq. (18) (with $d = 0$), then we obtain

$$L_n = \sqrt{(X_n + \Delta S)^2 + (b + \Delta y_n + c)^2}\,, \text{ and} \tag{27a}$$

$$L'_n = \sqrt{(X_0 - X_n + \Delta N)^2 + (b + \Delta y_n)^2}\,. \tag{27b}$$

These equations indicate that we could modify the approximation for Q_n given by Eq. (19) by replacing $b \longrightarrow b + \Delta y_n$, which results in the numerator

$$\gamma_n\, b + (\gamma_n - 1)\, c \longrightarrow \gamma_n\, b + (\gamma_n - 1)\, (c + 2\,\Delta y_{12})\,, \tag{28}$$

indicating that the intuitive substitution $c \longrightarrow c + 2\,\Delta y_{12}$ captures the effect of relief. (Note that this should *not* be done when computing the length L_0 of the open string!)

2.4 Bending Stiffness

The bending stiffness of a string clamped at the n^{th} fret is given by Eq. (8), Eq. (13), and Eq. (22) as

$$B_n = \sqrt{\frac{E\,\mathcal{A}\,s^2}{4\,T_n\,L_n^2}} = \sqrt{1+\kappa\,Q_n}\,\frac{L_0}{L_n}\sqrt{\frac{E\,\mathcal{A}\,s^2}{4\,T_0\,L_0^2}} \approx \gamma_n\,B_0\,, \tag{29}$$

where the approximation applies when $B_0 \ll 1$ and the largest contribution arises from the shortened length of the fretted string compared to that of the open string. This expression confirms our intuitive expectation that the stiffness of the string should increase as the length becomes shorter. Therefore, the fourth term in the final line of Eq. (11) can be approximated as

$$1200\,\log_2\left[\frac{1+B_n+(1+\pi^2/2)\,B_n^2}{1+B_0+(1+\pi^2/2)\,B_0^2}\right] \approx \frac{1200}{\ln(2)}\left[(\gamma_n-1)\,B_0+\frac{1}{2}\,(\gamma_n^2-1)\,(1+\pi^2)\,B_0^2\right]. \tag{30}$$

In Fig. 4, we use Eq. (30) to compare the exact and approximate expressions for frequency shifts due to bending stiffness based on Table 1 and Table 2. Note that we show the approximate frequencies with and without the quadratic terms, and we see that the 2$^{\text{nd}}$-order contribution is about 0.2 cents at the 12$^{\text{th}}$ fret. Once again, it is clear that B_n does not depend significantly on either ΔS or ΔN. In other words, the bending stiffness error does not depend on the tiny changes to the linear mass density or the tension that arise due to string fretting. Instead, it is an intrinsic mechanical property of the string: the stiffness increases as the length of the vibrating string becomes shorter. Comparing Fig. 3 and Fig. 4, we see that at the 12$^{\text{th}}$ fret the frequency error due to bending stiffness is about twice as large as that caused by the increase in tension.

2.5 Total Frequency Shift

Let's guide our intuition and prepare for the development of approximate expressions for ΔS and ΔN by relying on Taylor series expansions for all the effects described above. First, we'll ignore all quadratic terms in the resonant length error and adopt Eq. (15). Next, we'll neglect the small reduction in linear mass density caused by fretting, and then rely on the approximation to the frequency shift caused by tension increases given by Eq. (24). Finally, we'll describe the effects of bending stiffness using Eq. (30), *neglecting* the term proportional to B_0^2. Incorporating all of these terms, we find that the total frequency shift is given approximately by

$$\Delta\nu_n \approx \frac{1200}{\ln(2)}\left[(\gamma_n-1)\left(B_0-\frac{\Delta S}{X_0}\right)+\frac{\Delta N}{X_0}+\frac{1}{2}\,\kappa\,Q_n\right]. \tag{31}$$

But how do we determine the bending stiffness B_0 given by Eq. (5) and the spring constant κ given by Eq. (23) for a particular string?

To measure κ, in Section 3 we will conduct an experiment that measures the change in the frequency of an open string as we make slight changes to its length [7, 8]. From Eq. (6), the change δf of the fundamental frequency of an open string due to a small change in length δL

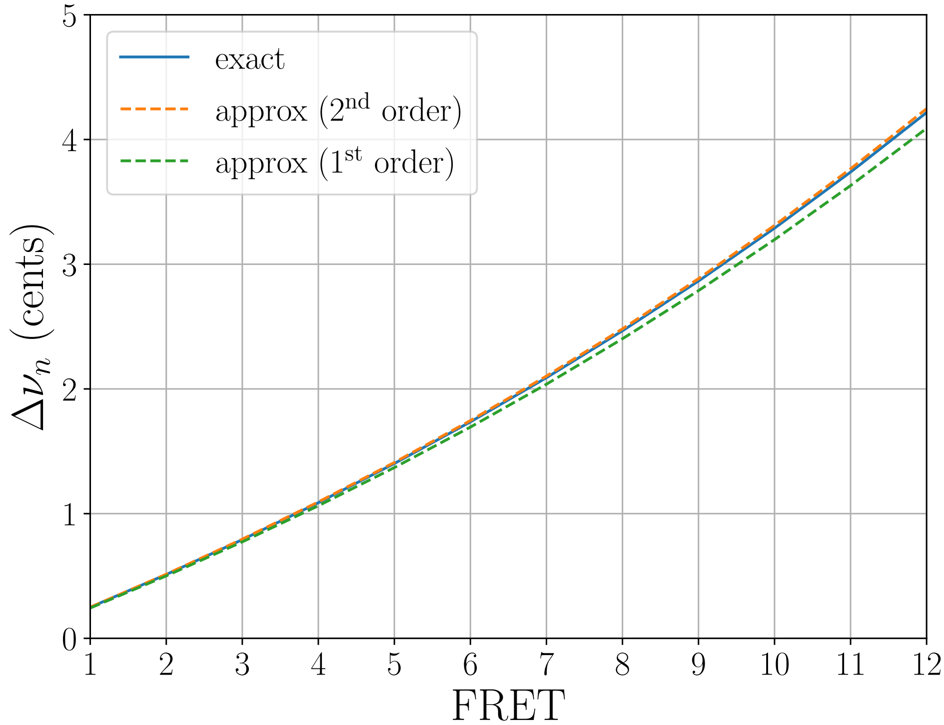


Figure 4: A comparison of exact and approximate expressions for the frequency shift due to bending stiffness, given by Eq. (30).

is

$$\begin{aligned}\frac{\delta f}{\delta L} &= \frac{f}{L}\left(-1 + \frac{L}{2\,T}\,\frac{\delta T}{\delta L} - \frac{L}{2\,\mu}\,\frac{\delta \mu}{\delta L} + \frac{L}{1+B}\,\frac{\delta B}{\delta L}\right)\\ &= \frac{f}{L}\left(-1 + \frac{1}{2}\,\kappa + \frac{1}{2} - \frac{B}{1+B}\right)\\ &\approx \frac{f}{L} \times \frac{1}{2}\,(\kappa - 1)\,,\end{aligned} \tag{32}$$

where we have used the analyses above to determine that

$$\frac{\delta T}{\delta L} = \frac{T}{L}\,\kappa\,, \tag{33a}$$

$$\frac{\delta \mu}{\delta L} = -\frac{\mu}{L}\,, \text{ and} \tag{33b}$$

$$\frac{\delta B}{\delta L} = -\frac{B}{L}\,, \tag{33c}$$

$$\tag{33d}$$

and we have again assumed that $B_0 \ll 1$. Therefore, following Byers [7, 8], we define the parameter R to be

$$R \equiv \frac{L}{f}\,\frac{\delta f}{\delta L} = \frac{1}{2}\,(\kappa - 1)\,, \tag{34}$$

which gives

$$\kappa = 2\,R + 1\,. \tag{35}$$

We can anticipate the typical value of R for a nylon classical guitar string through a simple observation. On a classical guitar with a scale length of 650 mm, we can usually tune an open string down a full step by winding the tuner/machine head five half turns down and then two half turns back up to re-tension the string. As we shall see in Section 3, this decreases the effective string length by 3 mm. Since a full step is (by definition) 200 cents, Eq. (3) tells us that

$$\frac{\Delta f}{f} \approx \frac{\ln(2)}{1200}\,\Delta\nu = \frac{200}{1731} = 0.116\,. \tag{36}$$

In this case, we estimate R to be

$$R \approx \frac{650}{3}\,0.116 = 25\,, \tag{37}$$

giving $\kappa \approx 51$, which are the values listed in Table 2.

It's impractical to assume that the effective (differential) modulus of elasticity of a particular string can be derived from published values of bulk nylon (particularly in the case of a wound string). Instead, let's assume that we know the value of κ, and then estimate the bending stiffness coefficient by comparing Eq. (5) and Eq. (23), and writing B_0 as

$$B_0 = \sqrt{\kappa}\,\frac{s}{L_0}\,, \tag{38}$$

As discussed in Appendix B, for a uniform cylindrical string/wire with radius ρ, $s = \rho/2$. This choice is valid for monofilament nylon strings [32]; if we provisionally accept it for wound nylon strings as well, then we have

$$B_0 = \sqrt{\kappa}\,\frac{\rho}{2\,L_0} \approx \sqrt{\kappa}\,\frac{\rho}{2\,X_0}\,. \tag{39}$$

We'll test this phenomenological ansatz in Section 3. We note that in the case of wound steel strings (often wrapped in metals like nickel or phosphor bronze), the bending stiffness depends on the radius of the nonuniform core alone [24, 29]. But the bass strings on a classical guitar are twisted and/or braided multifilament nylon strands wrapped in silver-plated copper, so a first-principles calculation of s is a nontrivial undertaking.

In Fig. 5, we compare the total frequency shifts predicted by Eq. (11) and Eq. (31) for the Classical Guitar specified by Table 1 with $\Delta S = \Delta N = 0$ mm, and a string with the parameters listed in Table 2 at two different values of d. Note that the string is sharp at every fret, but even a large nonzero value of d is only important at the first fret. The bending stiffness is negligible at the first fret, but accounts for 65% of the shift at the 12$^{\text{th}}$ fret. The close agreement between the exact and approximate expressions for the frequency shifts gives us confidence that the equations we derive for the setbacks in Section 4 will be useful.

3 Experimental Estimate of the String Constant

It is relatively easy to measure *in situ* the value of R (and therefore infer κ and B_0) for any guitar string with the aid of a device that can measure frequency [33], a simple ruler with fine markings (e.g., a string depth gauge), a magnifying glass or camera with a macro mode, and white correction fluid. For example, in Fig. 6 we show photographs of the nylon normal-tension first string on an Alhambra 8P classical guitar. By depositing a small sample of correction fluid on the string, we can measure small displacements against a gauge marked in half-millimeter

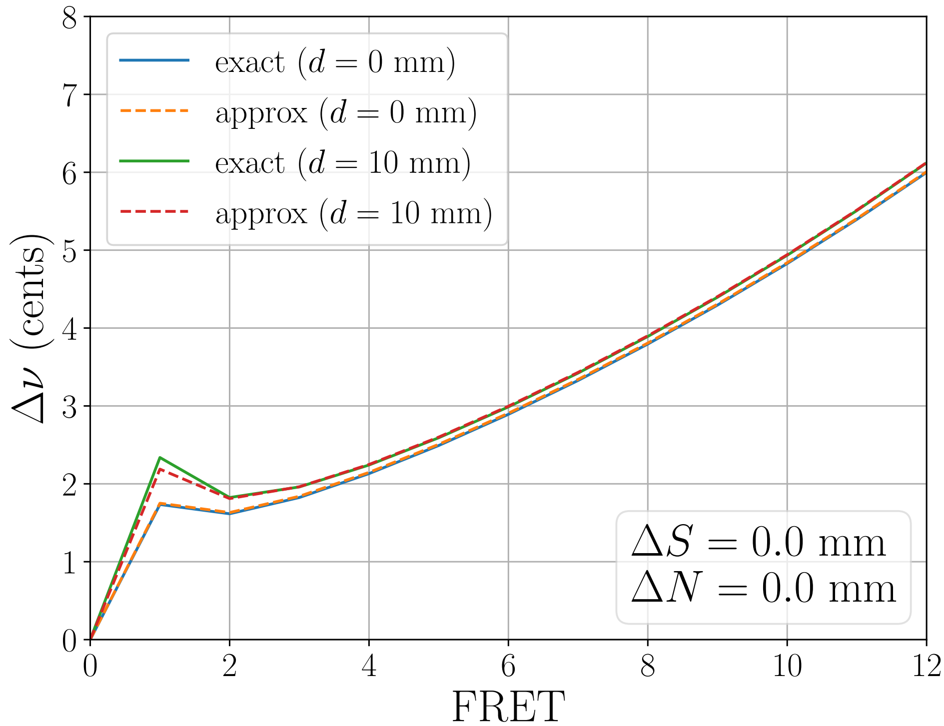
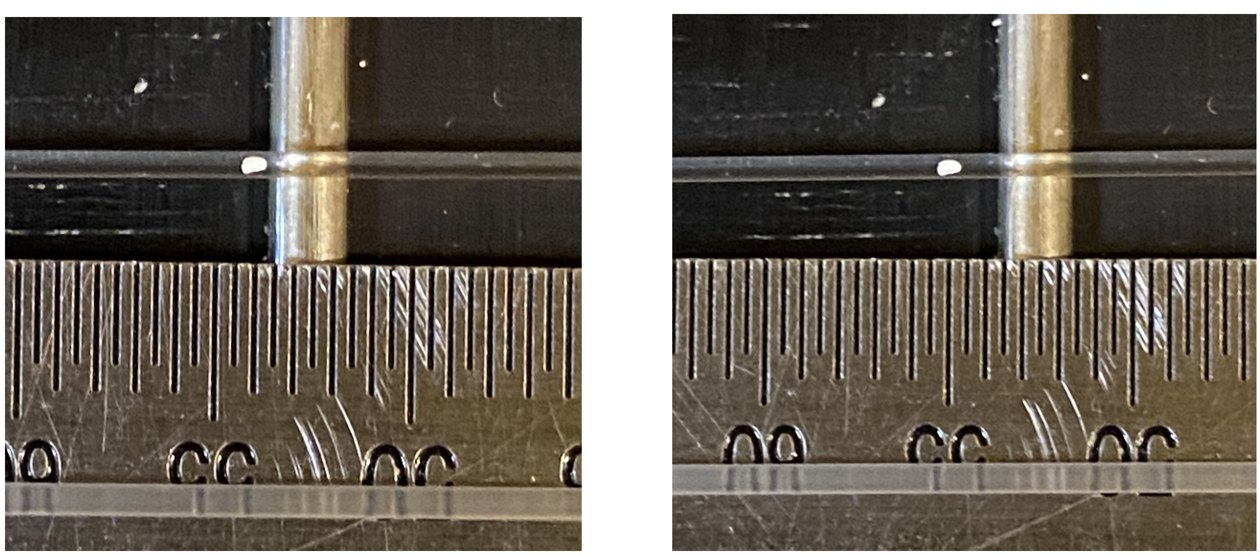


Figure 5: The total frequency shifts predicted by Eq. (11) and Eq. (31) for the Classical Guitar specified by Table 1 with $\Delta S = \Delta N = 0$ mm, and a string with the parameters listed in Table 2.



(a) Δx_1 (b) Δx_2

Figure 6: Two examples of displacement measurements of a small deposit of white correction fluid relative to a D'Addario string-depth gauge marked in half-millimeter increments.

Table 3: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 650 mm.

String	Note	ρ (mm)	μ (mg/mm)	T_0 (N)
J4501	E_4	0.356	0.374	68.6
J4502	B_3	0.409	0.505	52.0
J4503	G_3	0.512	0.836	54.2
J4504	D_3	0.368	1.920	70.0
J4505	A_2	0.445	3.289	67.3
J4506	E_2	0.546	5.470	62.8

increments. Then we can pluck the open string and measure its vibration frequency. All of our measurements were made with strings that had settled into equilibrium after at least ten hours of use, and we completed each set of measurements of a string in less than 10 minutes so that it did not have time to relax further [30, 20]. We found that significantly stretching a string that had settled into equilibrium resulted in a nonlinear frequency shift Δf as a function of ΔL (and occasionally broke the string). Therefore, prior to our measurements we tuned each string down one whole step by turning the tuning machine down five half-turns, stretching the string vertically to pull it through the nut, and then re-tensioning the string with two half-turns. The string stretches uniformly along its length, so at any position x the relative displacement $\Delta x/x$ should be invariant. For convenience, we therefore chose to work near the first fret as a visual marker, which is located 614 mm from the saddle on a guitar with a 650 mm scale length. We made seven measurements of displacement over a 3 mm range (100 times the stretch that results from normal fretting), as well as the corresponding frequencies.

For example, we began with a normal-tension nylon classical string set [34] with the specifications listed in Table 3 using metric units.[1] In Fig. 7, we plot our measurements of Δf as a function of the displacement Δx relative to the frequency of the string when $\Delta x = 0$. The error bars (which arise primarily because of imperfect measurements of Δx) represent the standard deviation of 10 independent measurements. We then performed a least-squares fit to a straight line [35] (also shown in Fig. 7), determined the derivative $\Delta f/\Delta L$, and then computed R using Eq. (34) with $L = 614$ mm and f defined as the average frequency over the range. The results are shown in Table 4. Here σ is the covariant (diagonal) uncertainty in R (so that, for example, the first string in the table has $R = 23.6 \pm 0.5$), and $\kappa = 2R + 1$. We compute the open-string bending stiffness B_0 using Eq. (39), and we also estimate an effective (differential) modulus of elasticity E_{eff} from Eq. (23), expressed in units of gigapascals (1 GPa = 10^9 N/m^2). Similar measurements and results for other string sets are provided in Appendix D. Note — as predicted in Section 2.5 and shown in Fig. 8 for all strings sets *except* the light tension set — the expectation that the guitar will be *tunable* results in R values of manufactured strings that are in the range 20 – 30. (It is unclear why the light strings appear to have much higher R values. Their volume densities are within a few percent of those of the normal tension strings, so perhaps there's a significant difference in the corresponding manufacturing process.) The still more important requirement that the guitar be *playable* leads us to the discussion of compensation in the next section.

[1] Note that the correct unit of force in the metric system is Newtons (N), rather than kilograms, which is a unit of mass. In the British Imperial measurement system, the common units of mass are known as the "slug" and the "blob."

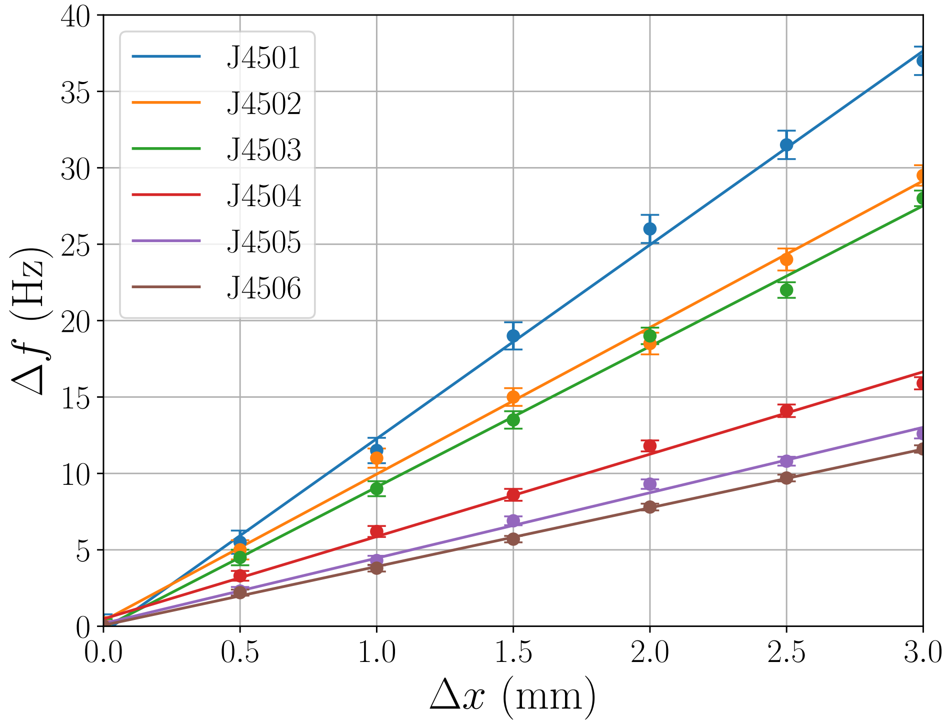


Figure 7: Results of experiments to measure R for each string in the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45) set. The points represent the measurement data, while the lines are the results of linear least-squares fits to that data.

Table 4: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 650 mm.

String	R	σ	κ	B_0	E_{eff} (GPa)
J4501	23.6	0.5	48.2	0.00190	8.33
J4502	23.8	0.7	48.7	0.00220	4.82
J4503	28.8	0.6	58.6	0.00301	3.87
J4504	22.5	0.8	46.0	0.00192	7.56
J4505	23.9	0.7	48.7	0.00239	5.28
J4506	28.6	0.4	58.2	0.00321	3.90

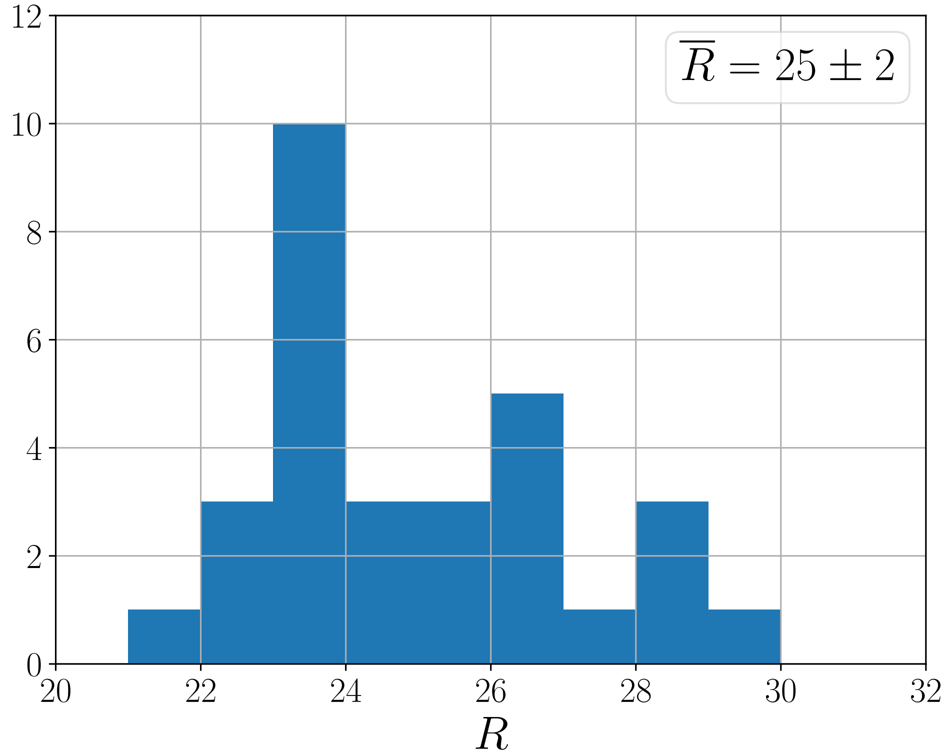


Figure 8: A histogram of the parameter R for all strings *except* those in the nylon light tension set presented in Appendix D.1, which seem to have anomalously high values.

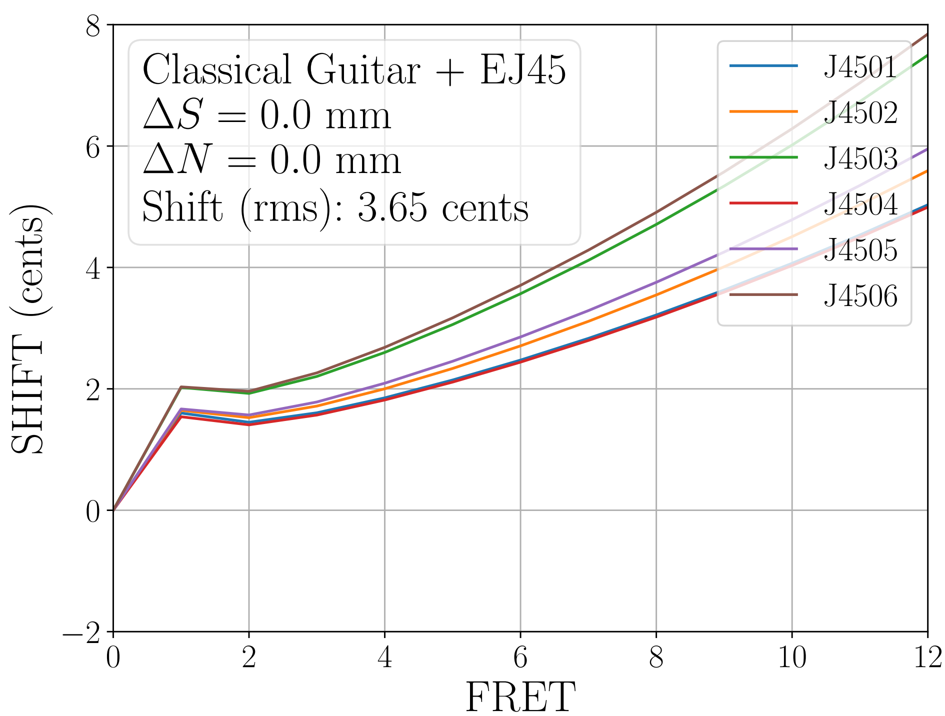


Figure 9: Frequency errors for an uncompensated Classical Guitar with normal tension nylon strings (D'Addario EJ45).

Adopting these physical properties of the normal string set and applying them to a computation of the frequency deviations for our standard classical guitar, we obtain the predictions shown in Fig. 9 using Eq. (11). Anticipating our treatment of exact compensation in Section 4 and Appendix C, we compute the root-mean-squared (RMS) average of the frequency deviations for each string. This mean (over the first 12 frets) can be computed by squaring the frequency deviations shown in Fig. 9, averaging those values, and then taking the square root of the result.

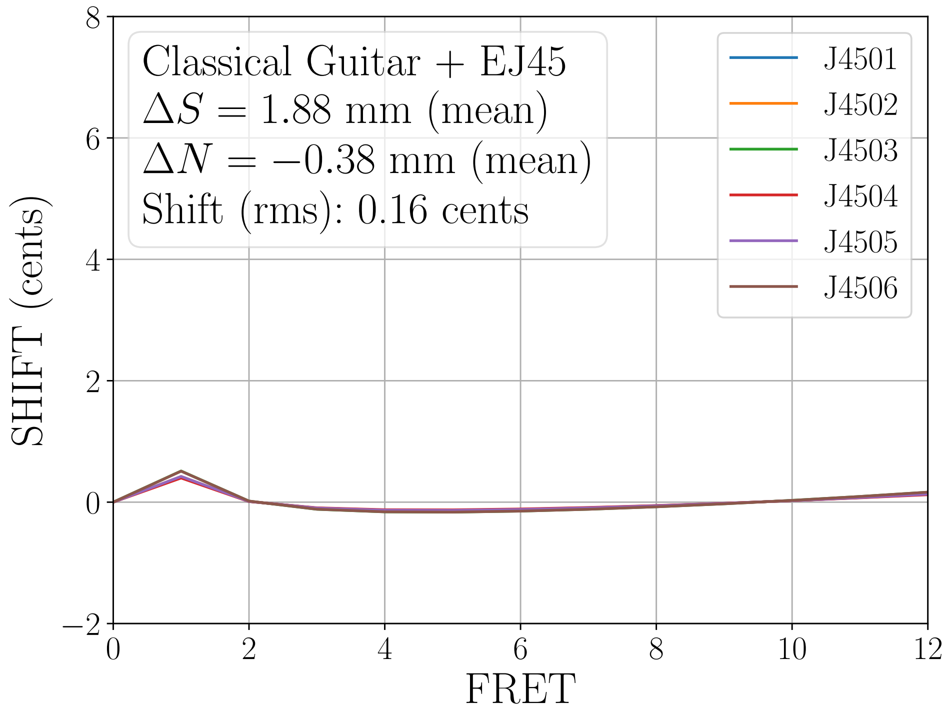
Recall that we recalculated the expected frequency shift of a classical guitar string with asymmetric boundary conditions in Appendix B, and found an expression for f_q given by Eq. (59) that reduces — by about a factor of 2 — the impact of the bending stiffness relative to the symmetric (clamped) case in Eq. (4). Furthermore, we have not used more sophisticated techniques to calculate the bending stiffness for either monofilament nylon strings or wound nylon strings, opting instead for the phenomenological model given by Eq. (39). But is this approach valid? As a test, we compared our frequency shift estimates based on Section 2 with experimental measurements made using five different guitars

result in very large frequency deviations at all frets. As shown in Fig. **??**, we measured the shifts at the first fret and found that they fell into the range of 4.5 – 5.75 cents, consistent with our predictions. At the 12$^{\text{th}}$ fret, we measured $\Delta f = 18.5 - 19.5$ cents for the third and sixth strings, and $\Delta f = 15.75 - 17.5$ cents for the other strings, in reasonably close agreement with our predictions. By contrast, the corresponding equation arising from symmetric (clamped) boundary conditions — Eq. (4) — predicts shifts that are 30 – 45% higher, and setbacks that are more than a factor of two larger. We conclude that Eq. (59) and Eq. (39) can be used to reliably predict setbacks for classical guitars.

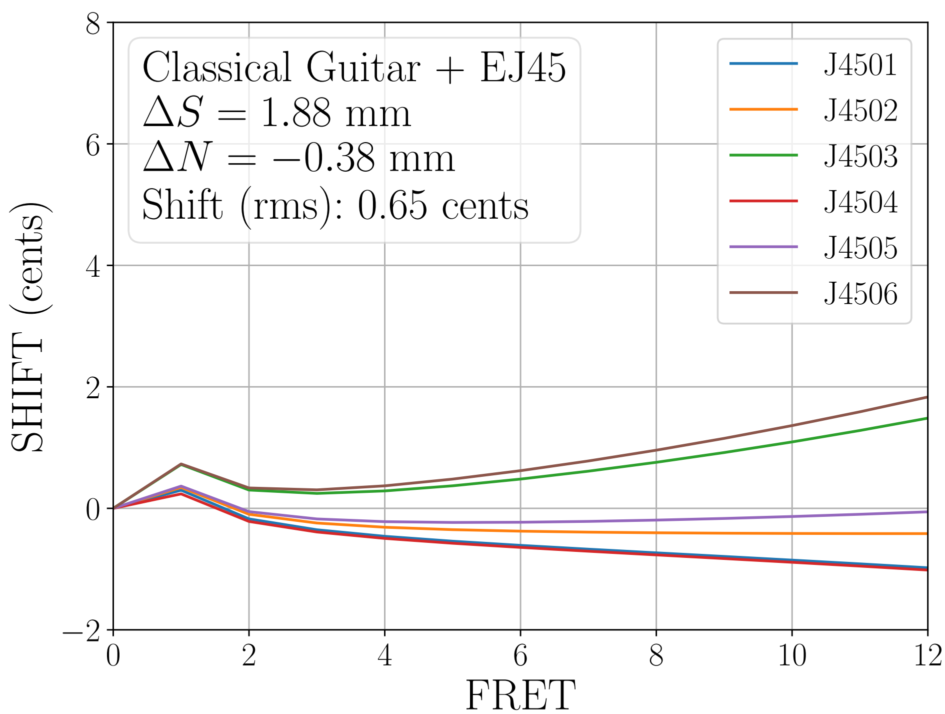
4 Classical Guitar Compensation

In Section 2, we noted that the bending stiffness and the increase in string tension due to fretting sharpen the pitch, but that we can flatten it with a positive saddle setback and negative nut setback. In Appendix C, we develop a method ("RMS Minimize") that numerically solves Eq. (11) for the values of ΔS and ΔN that minimize the root-mean-square of the frequency errors of a string over a particular set of frets. In the case of our Classical Guitar with normal-tension nylon strings — shown in Fig. 9 for the case of zero setbacks — we use this method with $d = 0$ to obtain the nonzero setbacks listed in Table 5. The corresponding frequency deviations are shown in Fig. 10a (assuming that all other aspects of the guitar remain unchanged). Of course, manufacturing a guitar with unique saddle and nut setbacks for each string (of a particular tension) can be challenging, so we also plot in Fig. 10b the shifts obtained by setting each of the values of ΔS and ΔN to the mean of the corresponding column in Table 5. In both cases, the RMS error is significantly smaller than that of the uncompensated guitar shown in Fig. 9.

This purely numerical approach using Eq. (11) is accurate but not illuminating. Let's build an intuitive understanding of how guitar compensation works, and then calculate approximate formulas for ΔS and ΔN that will help us appreciate the impact of particular choices of (for example) b, c, and d. As in Fig. 5, let's choose the guitar and string properties listed in Table 1 and Table 2, and then vary ΔS and ΔN and then Eq. (11) to determine the pitch of the string at each fret. Figure 11 shows that increasing the saddle setback tends to rotate the pitch curve clockwise, and increasing the magnitude of the negative nut setback displaces the pitch curve



(a) Full compensation



(b) Mean compensation

Figure 10: Frequency shifts (in cents) for the Classical Guitar with normal tension nylon strings (D'Addario EJ45). In (a) we use the individual values for each string that are listed in Table 5. In (b), we set ΔS and ΔN to the mean of the corresponding column in that table.

Table 5: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45) on the Classical Guitar.

String	ΔS (mm)	ΔN (mm)	$\overline{\Delta\nu}_{\rm rms}$ (cents)
J4501	1.48	-0.36	0.15
J4502	1.69	-0.36	0.15
J4503	2.33	-0.43	0.19
J4504	1.49	-0.34	0.14
J4505	1.83	-0.36	0.15
J4506	2.46	-0.43	0.19

almost uniformly downward. It appears that we can compensate the guitar by finding a value of ΔS that results in values of $\Delta\nu_n$ that are equal for, say, frets with $n \geq 3$, and then calculating a value of ΔN that sets $\Delta\nu_{12} = 0$.

We'll use Eq. (31) and the approximation for Q_n given by Eq. (19). Let's set $d = 0$, and then treat γ_n as a continuous variable. If we set $d\,\Delta\nu_n / d\,\gamma_n = 0$, then we obtain

$$B_0 - \frac{\Delta S}{X_0} + \frac{\kappa}{4\,X_0^2}\left[(b+c)^2 - \frac{b^2}{(\gamma_n - 1)^2}\right] = 0\,. \tag{40}$$

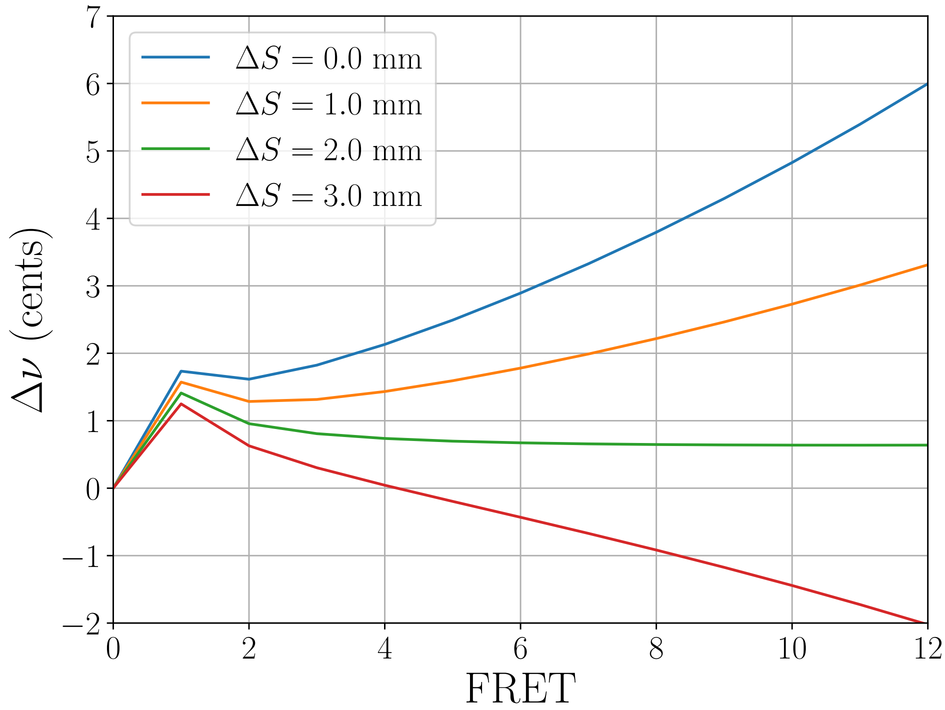
The average of $(\gamma_n - 1)^{-2}$ over the 3$^{\rm rd}$ through 12$^{\rm th}$ frets is about 7. Adopting this value, we substitute the resulting solution for ΔS back into Eq. (31) with $n = 12$ and $d \neq 0$, and solve for ΔN. We finally obtain

$$\Delta S = B_0\,X_0 + \frac{\kappa}{4\,X_0}\left[(b+c)^2 - 7\,b^2\right] - \frac{2\,\kappa}{X_0^2}\,(2\,b+c)^2\,d\,, \text{ and} \tag{41a}$$

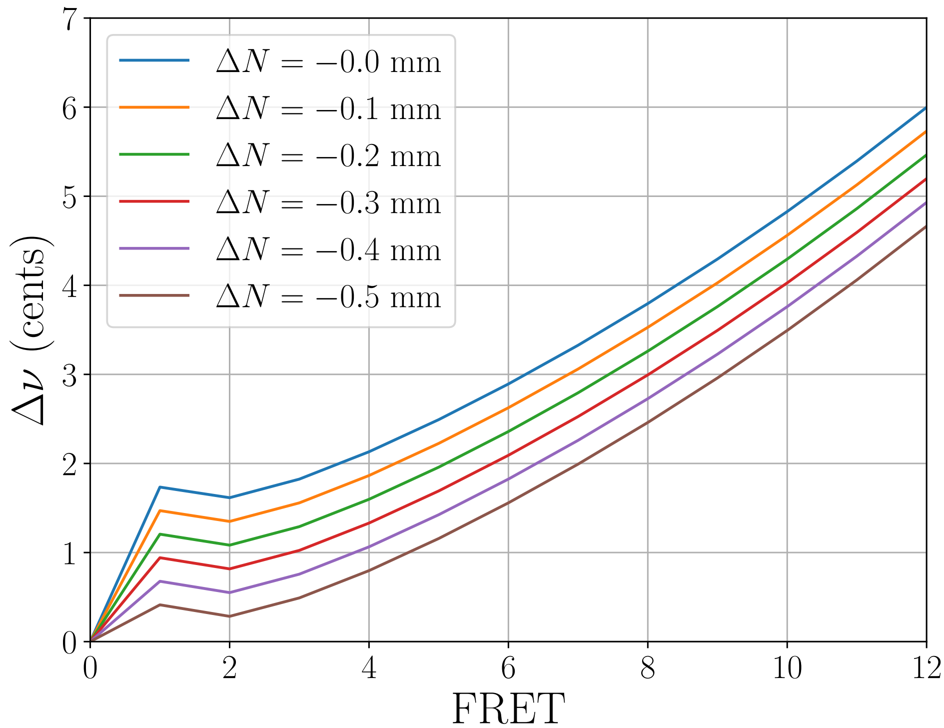
$$\Delta N = -\frac{\kappa}{2\,X_0}\,(5\,b+c)\,b - \frac{2\,\kappa}{X_0^2}\,(2\,b+c)^2\,d\,. \tag{41b}$$

Note that we have subtracted the same correction for $d \neq 0$ in ΔN from ΔS, because in our numerical studies using Eq. (11) and the RMS + BFGS method we found that $\Delta S - \Delta N$ had a constant value of approximately $B_0\,X_0 + (\kappa/4\,X_0)\,(2b + c)^2$ across all of our string sets for $0 \leq d \leq 10$ mm. These approximations are remarkably accurate given their origin: even when $d = 10$ mm, they predict values of the setbacks that increase the residual RMS frequency errors by only 2 – 3%. We note that the largest contribution to the saddle setback is the product of the bending stiffness and the scale length, and that the nut setback can be reduced significantly by choosing a relatively small value of b. For example, luthiers often build the nut so that the center of each string is held at the same value above the fret board (e.g., 62.5 mils), so that $b = 1.6\text{ mm} - \rho - a$. When a is 1 mm, the thicker strings have quite small values of b.

As mentioned above, it is nontrivial to manufacture a guitar with different setbacks for each string [7], and it is unlikely that the exact values listed in Table 5 are applicable to other string sets. We have measured the values of R for five other string sets, and in Appendix D we have reproduced the exact compensation procedure for them that we performed above for normal tension strings (with $d = 0$). Although each set exhibits variation between strings (and with respect to other sets) in individual setbacks for each string, they are similar enough that we suspect that there is the potential for great simplification in guitar design. For example, following the analysis of Appendix C, it is possible to determine a single setback pair $\{\Delta S, \Delta N\}$



(a) Frequency shifts ($\Delta N = 0$)



(b) Frequency shifts ($\Delta S = 0$)

Figure 11: In (a), we plot the frequency shifts for our classical guitar for several saddle setbacks with $\Delta N = 0$. Here we use the string parameters listed in Table 2. In (b), we set $\Delta S = 0$ and plot the frequency shifts for several nut "setbacks."

that minimizes the RMS frequency errors of an ensemble of strings over a collection of frets simply by computing the mean of the setbacks over all strings, and then using these mean values when manufacturing the guitar. If we consider five of the six string sets we have measured here — neglecting the light tension strings because of their pathologically high values of R — we can plot the exact setback predictions shown in Fig. 12, and then use these results to predict the mean values. In Fig. 12a, we see that the saddle setbacks are reasonably well described in terms of the string radius ρ by the expression

$$\Delta S = (4.4 \pm 0.2)\,\rho\,. \tag{42}$$

This relationship remains true for $0 \le d \le 10$ mm. Therefore, we can either compute the mean of the saddle setbacks directly, or using the average value of the string lengths ($\overline{\rho} = 0.43 \pm 0.08$ mm). Either way, we obtain $\overline{\Delta S} = 1.8 \pm 0.4$ mm. Similarly, in Fig. 12b, we show a histogram of the values of the nut setbacks, and compute the mean value $\Delta N = -0.38 \pm 0.03$ mm, which we recall is proportional to the scale length X_0. Note that these results are remarkably similar to the values used in Fig. 10b; if we plot the frequency deviations of those five string sets with these particular mean setback values, then we find that the maximum error always occurs at the twelfth fret, and it is always less than 1 cent. In the next section, we discuss a method to temper the guitar to reduce these errors further.

5 Tempering the Classical Guitar

> Temperament: A compromise between the acoustic purity of theoretically exact intervals, and the harmonic discrepancies arising from their practical employment.
> — Dr. Theo. Baker [36]

In Fig. 10b, a uniformly compensated classical guitar with normal tension strings tuned to 12-TET shows (of the treble strings) the third string has the greatest error in tuning across the fretboard. Tuning this guitar to 12-TET exacts a perfect-fifth in the third string while playing a C major chord in first position. This results in the third string being too sharp for the other common chords of E major (G#), A major, and D major (A), particularly when the guitar is played at a higher fret position. One way to reduce this error is by lowering the pitch of the third string below 12-TET with an electronic tuner. Another more comprehensive system is to tune all the strings harmonically to the fifth string, which lowers the third string by 7 cents as well as tempering the remaining strings.

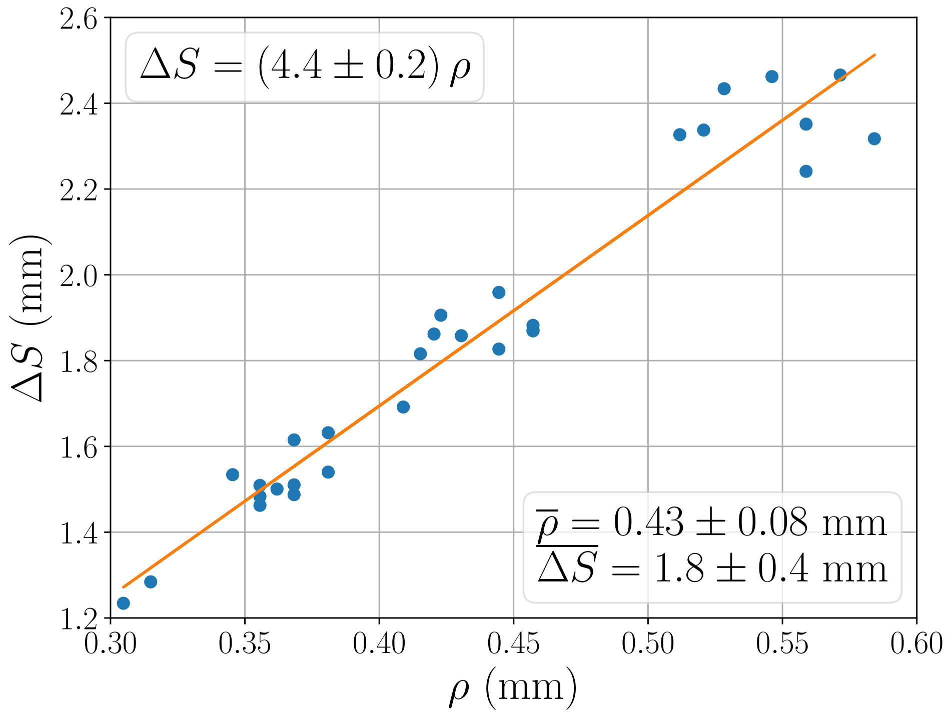
In this particular case, the "Harmonic Tuning Method" can be followed using these steps:

1. Begin by tuning the fifth string to $A_2 = 110$ Hz, resulting in a fifth-fret harmonic of $A_4 = 440$ Hz. (This can also be tuned by ear using an A_4 tuning fork).

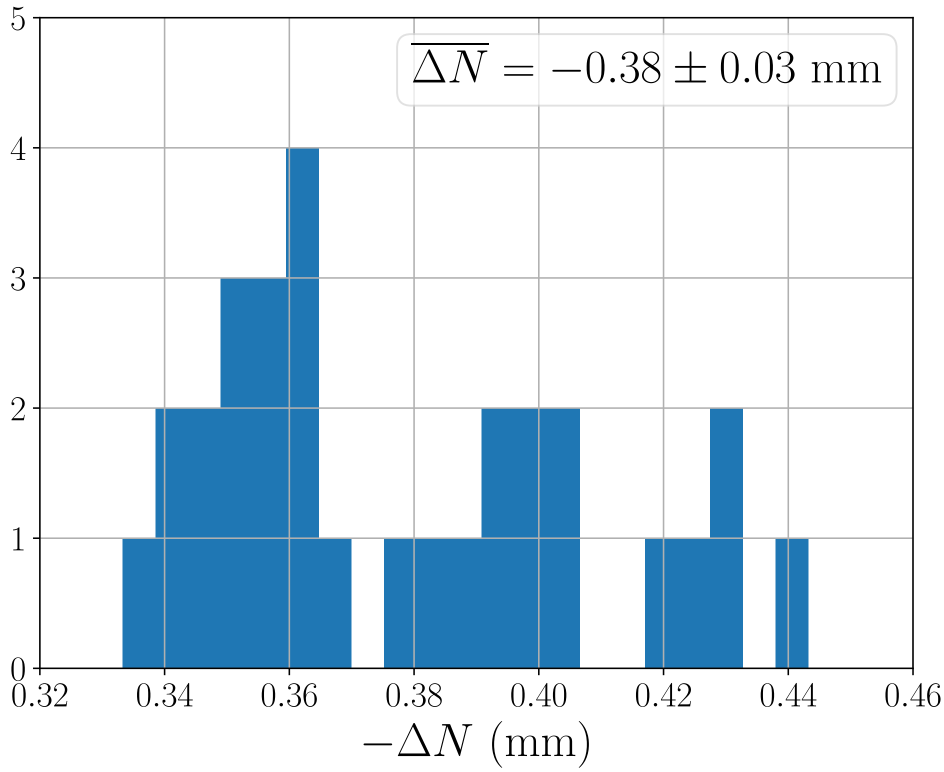
2. Tune that harmonic to the seventh fret harmonic of the fourth string, which is also $A_4 = 440$ Hz.

3. Tune the seventh fret harmonic on the fifth string (330 Hz, or 0.37 Hz sharper than 12-TET E_4) to the fifth-fret harmonic of the sixth string.

4. The seventh fret harmonic on the fifth string can tune the remaining fretted strings: the ninth fret on the third string, the fifth fret on the second string, and the open first string.



(a) Five-set saddle setback data with fit



(b) Five-set nut setback data with mean

Figure 12: Construction of mean saddle and nut setbacks over five selected string sets. In (a) we plot the saddle setback for each string as a function of the string radius, with the result of the best linear fit. In (b), we present a histogram of the nut setbacks and their mean.

Table 6: Harmonic tuning methodology based on A_4 and E_4. The asterisk indicates a harmonic with a null at the designated fret.

Reference String/Fret	Target String/Fret
A*/5 (A_4)	D*/7
A*/7 (E_4)	E*/5
A*/7 (E_4)	G/9
A*/7 (E_4)	B/5
A*/7 (E_4)	E/0

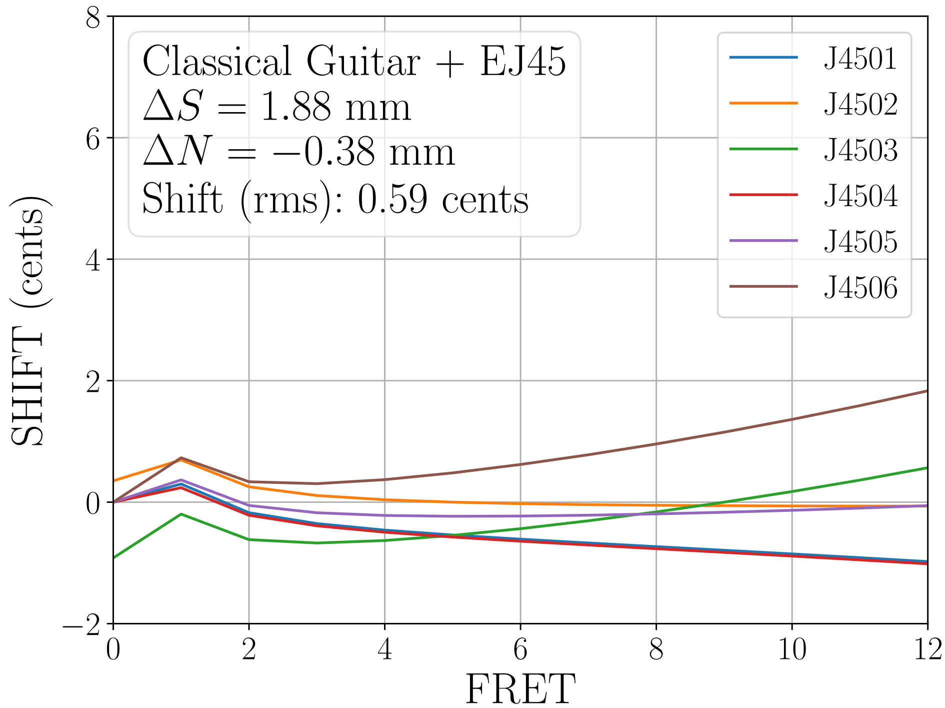


Figure 13: Frequency shift (in cents) for the mean compensated classical guitar with normal tension nylon strings (D'Addario EJ45) shown with 12-TET tuning in Fig. 10b. Here the same guitar has been harmonically tuned using the approach outlined in Table 6.

We have summarized these steps in Table 6, and in Fig. 13 we show the same guitar tuned in this fashion. Although the RMS shift over all strings is similar to that obtained by 12-TET tuning, the reduction in errors by strings 2 and 3 on the second and higher frets is significant. Note that other tuning choices can be made depending on the piece being played. For example, the third string could also be tuned at the second fret to $A_3 = 220$ Hz using the fifth-string harmonic at the 12$^{\text{th}}$ fret, and/or the first string could be tuned at the fifth fret to A_4 using the fifth-fret harmonic of the fifth string. The flexibility of the harmonic tuning method — and its reliance on only an A_4 tuning fork — is a great asset for the classical guitarist. Of course, how the guitar string is plucked has an impact on the resulting tone, but we defer a discussion of this effect to the literature [37, 38].

6 Conclusion: The Recipes

In this work, we have constructed a model of classical guitar intonation that includes the effects of the resonant length of the fretted string, linear mass density, tension, and bending stiffness. We have described a simple experimental approach to estimating the increase in string tension arising from an increase in its length, and then the corresponding mechanical stiffness. This allows us to determine the saddle and nut positions needed to compensate the guitar for a particular string, and we propose a simple approach to find averages of these positions to accommodate a variety of strings. This "mean" method benefits further from temperament techniques — such as harmonic tuning — that can enhance the intonation of the classical guitar for particular musical pieces.

Our calculations have relied on Eq. (6), which was derived by compromising for empirical reasons on symmetric boundary conditions and assuming that the string was pinned to the saddle rather than clamped. We then separated the contributions to the frequency deviations from ideal values caused by fretting by expressing these differences using the definition of logarithmic "cents" given by Eq. (2), resulting in the analytically exact expression for nonideal frequency shifts given by Eq. (11). We have used this equation to plot frequency errors at each of the first twelve frets for a prototypical Classical Guitar with a variety of compensation strategies based on the RMS + BFGS method described in Appendix C. Because the height of each string above the frets is small compared to the scale length, there are Taylor series approximations of the terms in Eq. (11) that we used to derive Eq. (31) to guide our understanding of the underlying principles of guitar compensation. This intuition led us to approximate estimates of the ideal values of the saddle and nut setbacks given by Eq. (41).

From these results, we have been able to create two "recipes" — based on the RMS + BFGS method and the Taylor-series-based approximation — that predict saddle and nut setbacks that enable the guitar to compensate for the frequency effects of fretting. Applying any one of these algorithms to a particular guitar design always begins with the same five steps:

1. Determine the scale length of the guitar by doubling the distance between the inside edge of the nut and the center of the 12$^{\text{th}}$ fret.

2. Using Fig. 2 as a guide, carefully measure the values of b and c. It is possible that the luthier has selected a saddle with vertical curvature, resulting in different values of c for each string.

3. Estimate the relief Δy_{12} at the 12$^{\text{th}}$ fret for each string. Measure the action (height) y_{12}

of the string above fret 12; then $\Delta y_{12} = y_{12} - b - c/2$ and we rescale the height of the saddle above the nut to $\tilde{c} = c + 2\,\Delta y_{12}$.

4. Select a string set with values of κ and B_0 listed in one of the derived physical properties tables in this paper, or follow the procedure developed in Section 3 to determine these quantities for a different string set.

5. Referring to Fig. 2 as a guide, choose a preferred value of the fretting distance d to account for the size of the finger.

The most accurate algorithm then adds one more step:

6. Using Eq. (66) to provide input values to the BFGS minimization algorithm, determine the saddle and nut setbacks for the selected string set.

Alternatively, a promising approximate approach can be followed:

6. Use Eq. (41) to compute the saddle and nut setbacks for the selected string set.

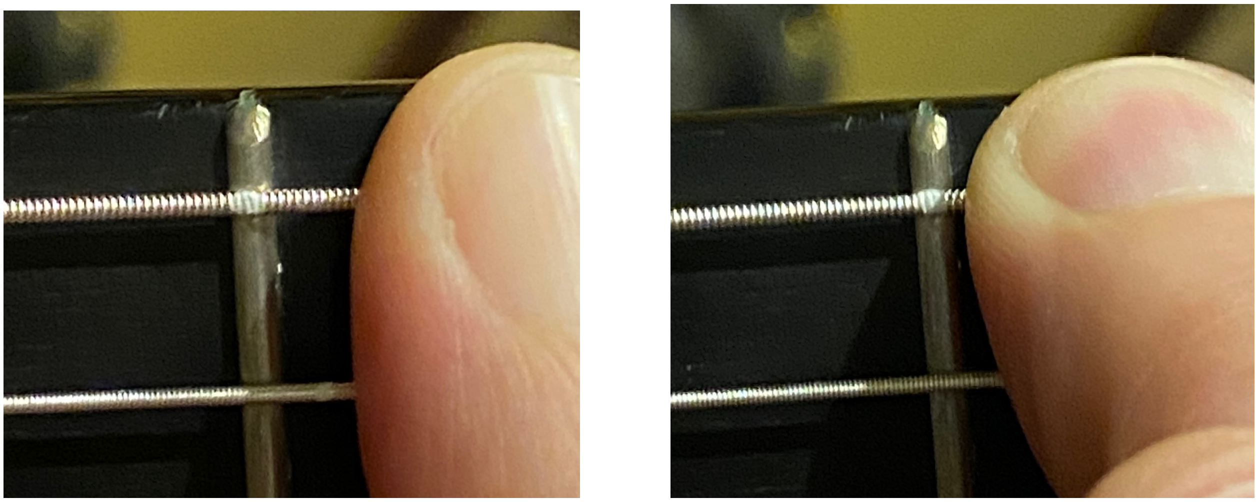
But perhaps the simplest reasonably accurate compensation approach is to adopt the results of Fig. 12: compute the saddle setback using $\Delta S = 4.4\,\overline{\rho}$, and select the nut setback to be $\Delta N = -0.38 \times (650\text{ mm}/X_0)$ mm, or about 15 mils for a guitar with a scale length of 650 mm.

These setback estimates can be averaged across the string set to design compensated nuts and saddles that should be relatively easy to fabricate. Nevertheless, we understand that high-end (concert) guitars that are likely to rely on one or two string sets (and the appropriate value of d for one guitar player) will benefit from the full, more accurate treatment of individual string setbacks.

We have placed the text of this manuscript (as well as the computational tools needed to reproduce our numerical results and all the graphs presented here) online [22] to invite comment and contributions from and collaboration with interested luthiers and musicians.

A Fretting Classical Guitar Strings

Previous studies of guitar intonation and compensation [7, 8] included a contribution to the incremental change in the length of the fretted string caused by both the depth and the shape of the string under the finger. As the string is initially pressed to the fret, the total length $\mathcal{L}_n$ increases and causes the tension in the string to increase. When the string is pressed further, does the additional deformation of the string increase its tension (throughout the resonant length L_n)? There are at least two purely empirical reasons to doubt this hypothesis. First, as shown in Fig. 14, we can mark a string (with a small deposit of white correction fluid) above a particular fret and then observe the mark with a magnifying glass. As the string is pressed flat on the finger board with two fingers, the mark does not move perceptibly — it has become *clamped* on the fret. Second, we can use either our ears or a simple tool to measure frequencies [33] to listen for a shift as we apply different fingers and vary the fretted depth of a string. The apparent modulation is far less than would be obtained by classical vibrato ($\pm$15 cents) — which causes the mark on the string to move visibly — so we assume that once the string is minimally fretted the length(s) can be regarded as fixed. (If this were not the case, then fretting by different people or with different fingers, at a single string or with a



(a) Before fretting

(b) After fretting

Figure 14: Location of a small marker of white correction fluid before and after fretting.

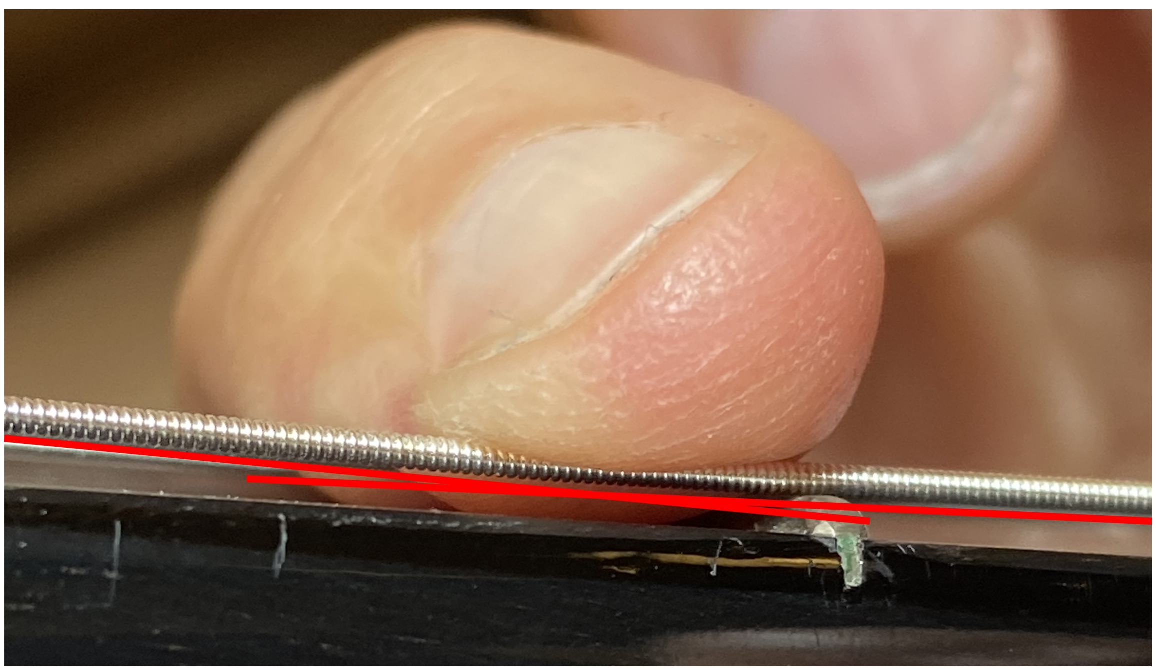


Figure 15: Photo of a wound nylon E_2 string clamped at the first fret of a classical guitar. The shape of the fretted string can be well approximated by two line segments intersecting about 5–6 mm behind the fret.

barre, would cause additional, varying frequency shifts that would be audible and difficult to compensate.)

In Section 2, we have included this concept in a simple way to determine the effect it will have on the frequency shift due to increased string tension. First, as shown in Fig. 15, as the string is pressed onto the fret, its shape is described quite well by two line segments intersecting behind the fret. Here it is clear that the finger is shaped by the string more than the string is shaped by the finger. We have taken this observation into consideration in Fig. 2 by introducing such an intersection point at a distance d behind fret n to represent the slight increase in the distance L'_n caused by a finger. The consequences of this choice are discussed in Section 2.2, and the impact it has on (for example) the frequency error due to tension is shown in Fig. 3.

B Vibration Frequencies of a Stiff String

Here we outline the calculation of the normal mode frequencies of a vibrating stiff string with non-symmetric boundary conditions. We begin with the wave equation [24]

$$\mu \frac{\partial^2}{\partial t^2} y(x) = T \frac{\partial^2}{\partial x^2} y(x) - E \mathcal{A} s^2 \frac{\partial^4}{\partial x^4} y(x) , \tag{43}$$

where μ and T are respectively the linear mass density and the tension of the string, E is its Young's modulus (or the modulus of elasticity), $\mathcal{A}$ is the cross-sectional area, and s is the radius of gyration of the string. (For a uniform cylindrical string/wire with radius ρ, $\mathcal{A} = \pi \rho^2$ and $s = \rho/2$.) If we scale x by the length L of the string, and t by $1/\omega_0 \equiv (L/\pi)\sqrt{\mu/T}$, then we obtain the dimensionless wave equation

$$\pi^2 \frac{\partial^2}{\partial t^2} y(x) = \frac{\partial^2}{\partial x^2} y(x) - B^2 \frac{\partial^4}{\partial x^2} y(x) , \tag{44}$$

where B is the "bending stiffness parameter" given by

$$B \equiv \sqrt{\frac{E \mathcal{A} s^2}{L^2 T}} . \tag{45}$$

We assume that $y(x)$ is a sum of terms of the form

$$y(x) = C\, e^{k x - i \omega t} , \tag{46}$$

requiring that k and ω satisfy the expression

$$B^2 k^4 - k^2 - (\pi\, \omega)^2 = 0 , \tag{47}$$

or

$$k^2 = \frac{1 \pm \sqrt{1 + (2\, \pi\, B\, \omega)^2}}{2\, B^2} . \tag{48}$$

Therefore, given ω, we have four possible choices for k: $\pm k_1$, or $\pm i k_2$, where

$$k_1^2 = \frac{\sqrt{1 + (2\, \pi\, B\, \omega)^2} + 1}{2\, B^2} , \text{ and} \tag{49a}$$

$$k_2^2 = \frac{\sqrt{1 + (2\, \pi\, B\, \omega)^2} - 1}{2\, B^2} . \tag{49b}$$

The corresponding general solution to Eq. (44) has the form

$$y(x) = e^{-i\omega t}\left(C_1^+ e^{k_1 x} + C_1^- e^{-k_1 x} + C_2^+ e^{ik_2 x} + C_2^- e^{-ik_2 x}\right) . \tag{50}$$

As discussed in Section 2, we have decided that the boundary conditions for the case of a classical guitar string are not likely to be symmetric. At $x = 0$ (the saddle), we assume that the string is pinned (but not clamped), so that $y = 0$ and $\partial^2 y/\partial x^2 = 0$. However, at $x = 1$ (the fret) we assume that the string is clamped, so that $y = 0$ and $\partial y/\partial x = 0$. Applying these constraints to Eq. (50), we obtain

$$0 = C_1^+ + C_1^- + C_2^+ + C_2^- , \tag{51a}$$
$$0 = k_1^2 \left(C_1^+ + C_1^-\right) - k_2^2 \left(C_2^+ + C_2^-\right) , \tag{51b}$$
$$0 = C_1^+ e^{k_1} + C_1^- e^{-k_1} + C_2^+ e^{ik_2} + C_2^- e^{-ik_2} , \text{ and} \tag{51c}$$
$$0 = k_1 \left(C_1^+ e^{k_1} - C_1^- e^{-k_1}\right) + ik_2 \left(C_2^+ e^{ik_2} - C_2^- e^{-ik_2}\right) . \tag{51d}$$

Since $k_1^2 + k_2^2 \neq 0$, the first two of these equations tell us that $C_1^- = -C_1^+ \equiv -C_1$, and $C_2^- = -C_2^+ \equiv -C_2$. Therefore, the second two equations become

$$C_1 \sinh(k_1) = -i\, C_2 \sin(k_2) , \text{ and} \tag{52a}$$
$$k_1\, C_1 \cosh(k_1) = -i\, k_2\, C_2 \cos(k_2) . \tag{52b}$$

Dividing the first of these equations by the second, we find

$$\tan(k_2) = \frac{k_2}{k_1} \tanh k_1 . \tag{53}$$

From Eq. (49), we see that $k_1^2 - k_2^2 = 1/B^2$, so that

$$k_1 = \frac{1}{B}\sqrt{1 + (B\, k_2)^2} . \tag{54}$$

In the case of a classical guitar, we expect that $B \ll 1$, so $k_1 \approx 1/B \gg 1$, and therefore $\tanh k_1 \longrightarrow 1$. Substituting Eq. (54) into Eq. (53), we obtain

$$\tan(k_2) = \frac{B\, k_2}{\sqrt{1 + (B\, k_2)^2}} . \tag{55}$$

We expect that $B\, k_2 \ll 1$, so we assume that $k_2 = q\pi(1 + \epsilon)$, where $q \in \mathbb{N}$ is an integer greater than or equal to 1, and $\epsilon \ll 1$. Therefore, to second order in ϵ, we have $\tan(k_2) \approx q\pi\epsilon$, and

$$\epsilon = \frac{B\,(1 + \epsilon)}{\sqrt{1 + \left[q\,\pi\, B\,(1 + \epsilon)\right]^2}} . \tag{56}$$

The denominator of the right-hand side of this equation has a Taylor expansion given by $1 - \frac{1}{2}\left[q\,\pi\, B\,(1 + \epsilon)\right]^2$, indicating that it will not contribute to ϵ to second order in B. Therefore, to this order,

$$\epsilon \approx \frac{B}{1 - B} \approx B + B^2 . \tag{57}$$

We substitute $k = \pm i k_2$ into Eq. (47) with $k_2 = q\pi/(1-B)$ to obtain

$$\begin{aligned} \omega &= \frac{k_2}{\pi}\sqrt{1 + (B\,k_2)^2} \\ &= \frac{q}{1-B}\sqrt{1 + q^2\pi^2\left(\frac{B}{1-B}\right)^2} \\ &\approx q\left[1 + B + \left(1 + \frac{1}{2}q^2\pi^2\right)B^2\right] . \end{aligned} \tag{58}$$

Restoring the time scaling by $1/\omega_0$, and defining the frequency (in cycles/second) $f = \omega/2\pi$, we finally have

$$f_q = \frac{q}{2\,L}\sqrt{\frac{T}{\mu}}\left[1 + B + \left(1 + \frac{1}{2}q^2\pi^2\right)B^2\right] . \tag{59}$$

We use this result to build our model in Section 2.

C Compensation by Minimizing RMS Error

The root-mean-square (RMS) frequency error (in cents) averaged over the frets $n \in \{1, n_{\max}\}$ (for $n_{\max} > 1$) *of a particular string* is given by

$$\overline{\Delta\nu}_{\rm rms} \equiv \sqrt{\frac{\sum_{n=1}^{n_{\max}} \Delta\nu_n^2}{n_{\max}}} , \tag{60}$$

where $\Delta\nu_n$ is given by Eq. (11). Here we will vary both ΔS and ΔN to minimize $\overline{\Delta\nu}_{\rm rms}$. In this case, it is sufficient to minimize the quantity

$$\chi^2 = \sum_{n=1}^{n_{\max}} \left[\frac{\ln(2)}{1200}\,\Delta\nu_n\right]^2 \tag{61}$$

such that the gradient of χ^2 with respect to ΔS and ΔN vanishes. Let's rewrite Eq. (11) as

$$\frac{\ln(2)}{1200}\,\Delta\nu_n = W_n + Z_n , \tag{62}$$

where

$$W_n = \ln\left(\frac{L_0}{\gamma_n\,L_n}\right) , \text{ and} \tag{63a}$$

$$Z_n = \ln\left[\sqrt{\frac{\mu_0}{\mu_n}\,\frac{T_n}{T_0}}\;\frac{1 + B_n + (1 + \pi^2/2)\,B_n^2}{1 + B_0 + (1 + \pi^2/2)\,B_0^2}\right] . \tag{63b}$$

In Section 2, we determined that — for the purposes of estimating the values of the setbacks — W_n could be represented reasonably accurately by

$$W_n \approx \frac{\Delta N - (\gamma_n - 1)\,\Delta S}{X_0} , \tag{64}$$

but for completeness we'll add the term in Eq. (14) that is quadratic in b and c to Z_n. Furthermore, we discovered that Z_n does not depend to second order on either ΔS or ΔN. Therefore, the components of the gradient of χ^2 are

$$\frac{\partial}{\partial \Delta S}\chi^2 = 2\sum_n (W_n + Z_n)\frac{\partial W_n}{\partial \Delta S} = -\frac{2}{X_0}\sum_n (\gamma_n - 1)(W_n + Z_n) \;, \text{ and} \tag{65a}$$

$$\frac{\partial}{\partial \Delta N}\chi^2 = 2\sum_n (W_n + Z_n)\frac{\partial W_n}{\partial \Delta N} = \frac{2}{X_0}\sum_n (W_n + Z_n) \;. \tag{65b}$$

Setting each of these expressions to zero and solving them for ΔS and ΔN, we obtain

$$\Delta S = \frac{\sigma_0\,\overline{z}_1 - \sigma_1\,\overline{z}_0}{\sigma_0\,\sigma_2 - \sigma_1^2}\,X_0\,, \text{ and} \tag{66a}$$

$$\Delta N = -\frac{\sigma_2\,\overline{z}_0 - \sigma_1\,\overline{z}_1}{\sigma_0\,\sigma_2 - \sigma_1^2}\,X_0\,, \tag{66b}$$

where

$$\sigma_k \equiv \sum_{n=1}^{n_{\max}} (\gamma_n - 1)^k \;, \text{ and} \tag{67}$$

$$\overline{z}_k \equiv \sum_{n=1}^{n_{\max}} (\gamma_n - 1)^k \, Z_n\,. \tag{68}$$

The corresponding Hessian matrix for this problem is the symmetric matrix

$$H = \begin{bmatrix} \frac{\partial^2\chi^2}{\partial\Delta S^2} & \frac{\partial^2\chi^2}{\partial\Delta N\,\partial\Delta S} \\ \frac{\partial^2\chi^2}{\partial\Delta S\,\partial\Delta N} & \frac{\partial^2\chi^2}{\partial\Delta N^2} \end{bmatrix} = \frac{2}{X_0^2}\begin{bmatrix} \sigma_2 & -\sigma_1 \\ -\sigma_1 & \sigma_0 \end{bmatrix}. \tag{69}$$

We can apply the second partial derivative test to the Hessian to determine whether we've found an extremum of χ^2. If the determinant of the Hessian is positive, and (in the case of a 2×2 matrix) one of the diagonal elements is positive, then we have found a minimum. The determinant is greater than zero for $n_{\max} \geq 2$, and the second condition is satisfied by $\sigma_0 = n_{\max} > 0$ when $n_{\max} \geq 1$. Therefore, we can be confident that the solution for ΔS and ΔN given by Eq. (66) minimizes χ^2 accurately to first order in ΔS and ΔN provided that we are averaging over at least the first two frets. Note that the diagonal elements of the Hessian also allows to estimate the increase in the residual RMS frequency error caused by small changes δs and δn in the saddle and nut setbacks respectively; we obtain

$$\overline{\delta\nu}_{\text{rms}} = \frac{1}{n_{\max}\,\overline{\Delta\nu}_{\text{rms}}}\left[\frac{1200}{\ln(2)}\right]^2\left[\sigma_2\left(\frac{\delta s}{X_0}\right)^2 + \sigma_0\left(\frac{\delta n}{X_0}\right)^2\right] \tag{70}$$

We can further refine the predicted values of these setbacks to accommodate the small second-order terms in ΔS and ΔN neglected in the resonant length error approximation used in Eq. (64). Relying on Eq. (11) as the exact expression for the frequency error $\Delta\nu_n$, we can use Eq. (66) to provide initial values for a nonlinear minimization of $\sum_n \Delta\nu_n^2$ over the first 12 frets. We adopt the quasi-Newton algorithm of Broyden, Fletcher, Goldfarb, and Shanno [39], a second-order algorithm for numerical optimization. Typically, this additional step changes the setback values by only a fraction of a percent. We'll refer to this approach as the "RMS

Minimize" nethod, and we use it throughout this work to compute the setbacks for each string under study. Note that the approximate equations given by Eq. (41) also can be used to compute initial values for this final nonlinear minimization.

The setback solution given by Eq. (66) is valid for a single string, and results like those shown in Table 5 and Fig. 10a assume that the guitar is built such that each string — from a particular set of strings — has a unique saddle and nut setback. Suppose that we'd prefer to engineer a guitar with single, uniform values of both ΔS and ΔN that provide reasonable compensation across an entire string set (or an ensemble of strings from a variety of manufacturers). In this case, Eq. (60) becomes

$$\overline{\Delta\nu}_{\text{rms}} \equiv \sqrt{\frac{\sum_{m=1}^{m_{\max}} \sum_{n=1}^{n_{\max}} \left[\Delta\nu_n^{(m)}\right]^2}{m_{\max}\, n_{\max}}}\,, \tag{71}$$

where m labels the strings in the set, and Eq. (62) has been updated to become

$$\frac{\ln(2)}{1200}\,\Delta\nu_n^{(m)} = W_n^{(m)} + Z_n^{(m)}\,, \tag{72}$$

If we rigorously follow the same approach that we used to arrive at Eq. (66), in the multi-string case we obtain

$$\Delta S = \frac{1}{m_{\max}} \sum_{m=1}^{m_{\max}} \Delta S^{(m)}\,, \text{ and} \tag{73a}$$

$$\Delta N = \frac{1}{m_{\max}} \sum_{m=1}^{m_{\max}} \Delta N^{(m)}\,, \tag{73b}$$

where

$$\Delta S^{(m)} = \frac{\sigma_0\, \overline{Z}_1^{(m)} - \sigma_1\, \overline{Z}_0^{(m)}}{\sigma_0\,\sigma_2 - \sigma_1^2}\, X_0\,, \text{ and} \tag{74a}$$

$$\Delta N^{(m)} = -\frac{\sigma_2\, \overline{Z}_0^{(m)} - \sigma_1\, \overline{Z}_1^{(m)}}{\sigma_0\,\sigma_2 - \sigma_1^2}\, X_0\,, \tag{74b}$$

reflecting the unique values of $\kappa^{(m)}$ and $B_0^{(m)}$ for each string in each set. In other words, we can find the optimum values for both ΔS and ΔN simply by averaging the corresponding setbacks over a commercially interesting collection of string sets.

D Other Classical Guitar String Sets

D.1 Light Tension – Nylon

Table 7: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

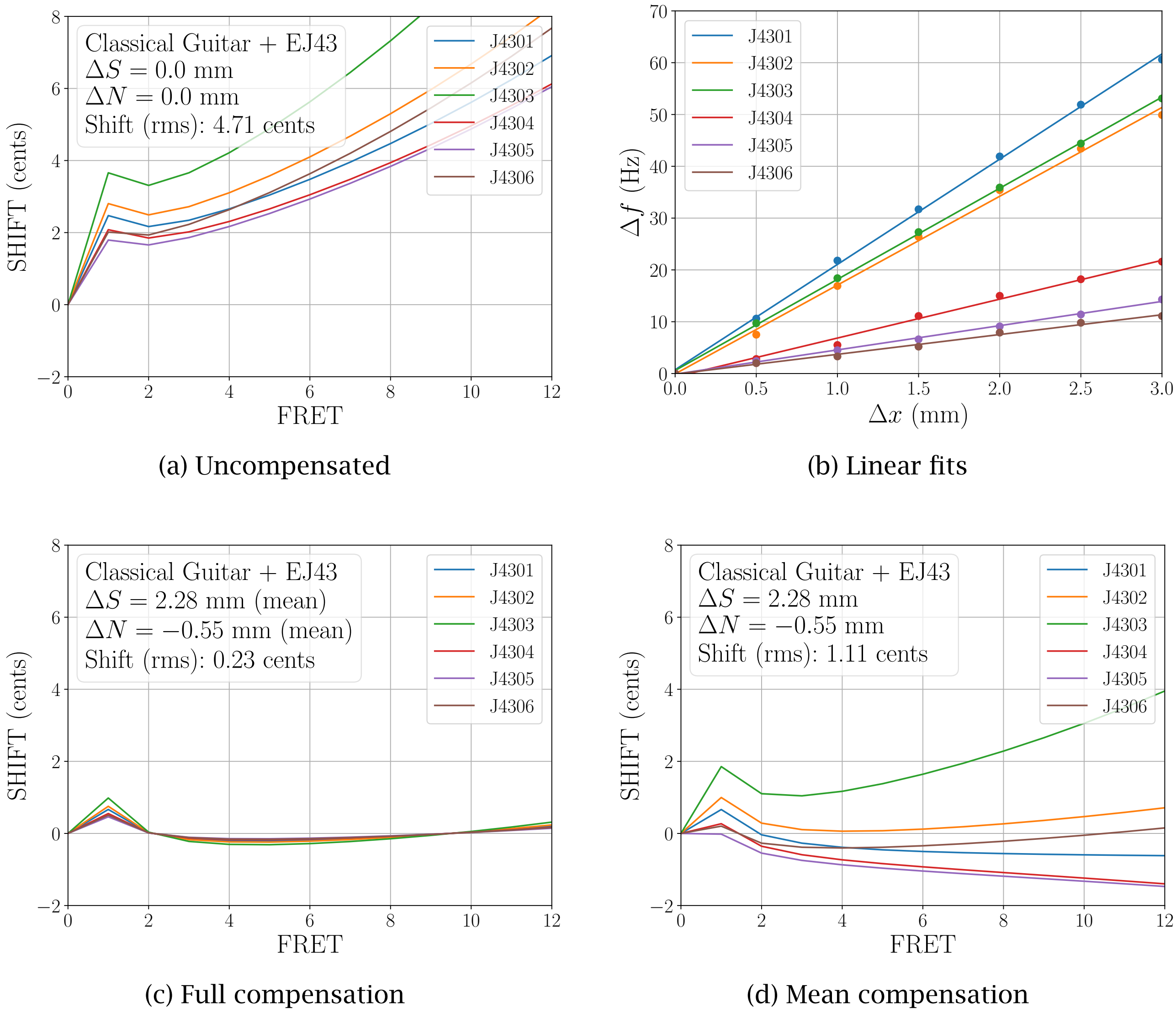
String	Note	ρ (mm)	μ (mg/mm)	T_0 (N)
J4301	E_4	0.349	0.361	66.4
J4302	B_3	0.403	0.487	50.2
J4303	G_3	0.504	0.808	52.5
J4304	D_3	0.356	1.822	66.4
J4305	A_2	0.419	2.741	56.0
J4306	E_2	0.533	5.158	59.2

Table 8: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

String	R	σ	κ	B_0	E_{eff} (GPa)
J4301	37.8	0.5	76.6	0.00235	13.28
J4302	42.6	1.0	86.2	0.00287	8.50
J4303	55.0	0.4	111.1	0.00409	7.30
J4304	31.4	1.2	63.7	0.00218	10.65
J4305	26.1	0.5	53.2	0.00235	5.40
J4306	28.5	1.1	57.9	0.00312	3.83

Table 9: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43) on the Classical Guitar.

String	ΔS (mm)	ΔN (mm)	$\overline{\Delta\nu}_{\text{rms}}$ (cents)
J4301	1.95	-0.57	0.24
J4302	2.37	-0.64	0.27
J4303	3.39	-0.82	0.36
J4304	1.76	-0.47	0.20
J4305	1.83	-0.39	0.17
J4306	2.40	-0.43	0.18



(a) Uncompensated

(b) Linear fits

(c) Full compensation

(d) Mean compensation

Figure 16: Frequency shift (in cents) for a classical guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). Four different strategies of saddle and nut compensation are illustrated.

D.2 Hard Tension – Nylon

Table 10: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). The corresponding scale length is 650 mm.

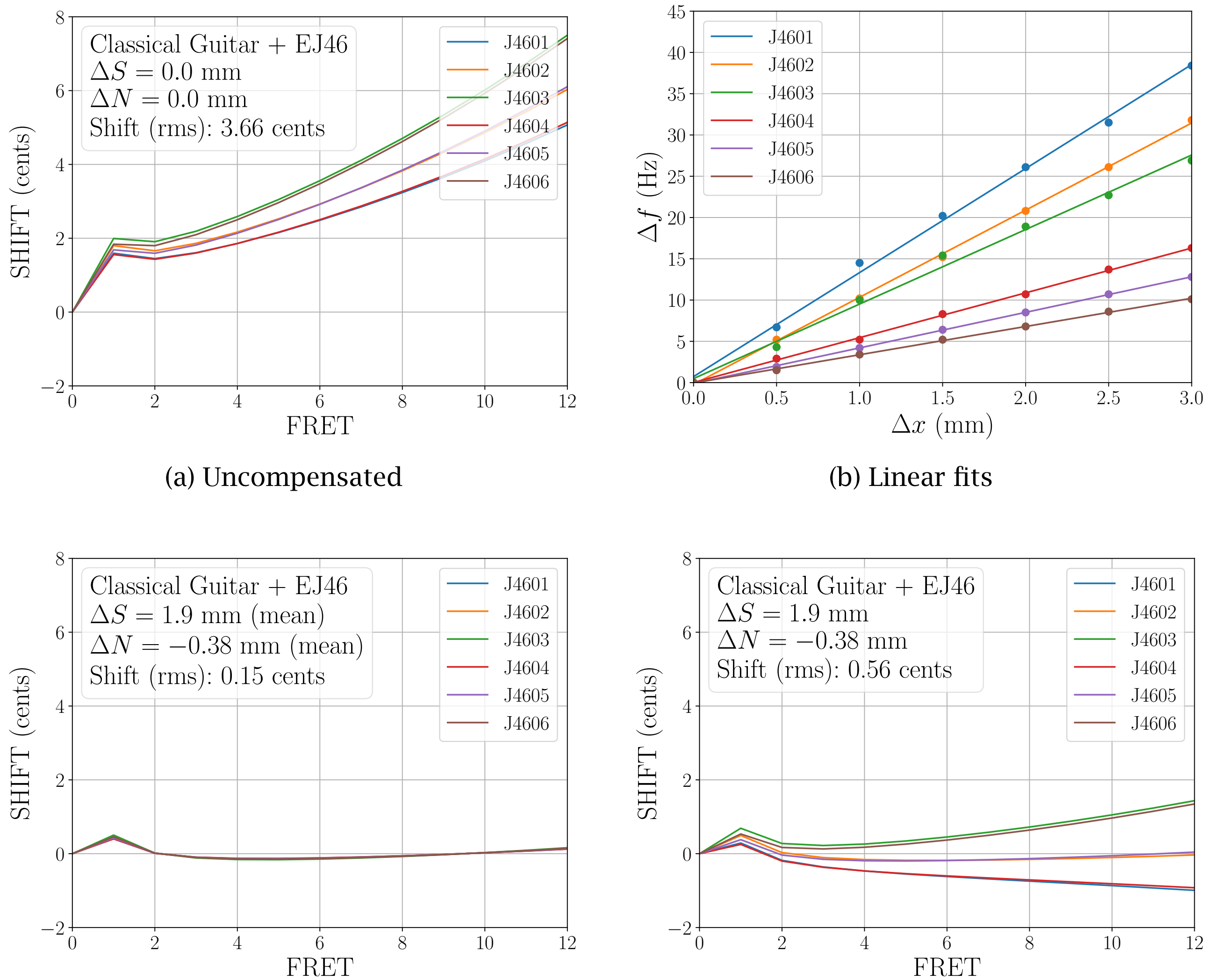
String	Note	ρ (mm)	μ (mg/mm)	T_0 (N)
J4601	E_4	0.362	0.386	70.9
J4602	B_3	0.415	0.522	53.8
J4603	G_3	0.521	0.856	55.6
J4604	D_3	0.381	2.007	73.1
J4605	A_2	0.457	3.486	71.3
J4606	E_2	0.559	5.666	65.0

Table 11: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46).

String	R	σ	κ	B_0	E_{eff} (GPa)
J4601	23.5	0.5	47.9	0.00193	8.25
J4602	26.2	0.3	53.5	0.00234	5.31
J4603	28.3	1.0	57.5	0.00304	3.75
J4604	22.7	0.3	46.4	0.00200	7.43
J4605	24.0	0.2	49.0	0.00246	5.32
J4606	25.5	0.3	51.9	0.00310	3.44

Table 12: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46) on the Classical Guitar.

String	ΔS (mm)	ΔN (mm)	$\overline{\Delta\nu}_{\text{rms}}$ (cents)
J4601	1.50	-0.36	0.15
J4602	1.82	-0.40	0.17
J4603	2.34	-0.42	0.18
J4604	1.54	-0.35	0.15
J4605	1.88	-0.36	0.15
J4606	2.35	-0.38	0.17



(a) Uncompensated

(b) Linear fits

(c) Full compensation

(d) Mean compensation

Figure 17: Frequency shift (in cents) for a classical guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). Four different strategies of saddle and nut compensation are illustrated.

D.3 Extra Hard Tension – Nylon

Table 13: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). The corresponding scale length is 650 mm.

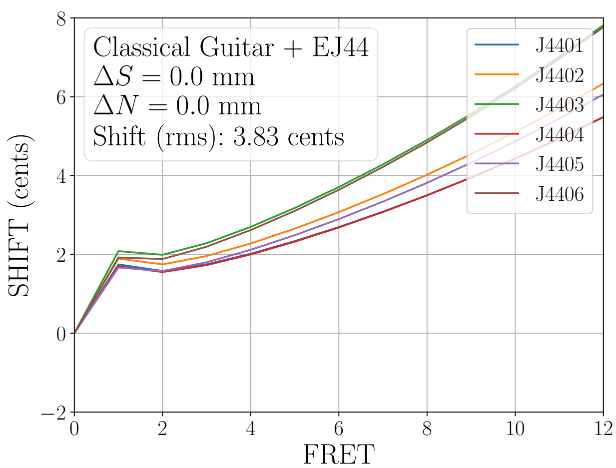
String	Note	ρ (mm)	μ (mg/mm)	T_0 (N)
J4401	E_4	0.368	0.401	73.6
J4402	B_3	0.423	0.544	56.1
J4403	G_3	0.528	0.891	57.8
J4404	D_3	0.381	2.007	73.1
J4405	A_2	0.457	3.486	71.3
J4406	E_2	0.571	6.134	70.4

Table 14: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). The corresponding scale length is 650 mm.

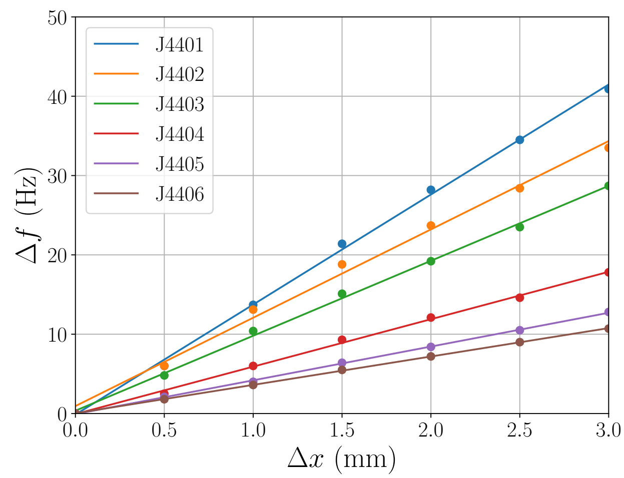
String	R	σ	κ	B_0	E_{eff} (GPa)
J4401	25.8	0.4	52.6	0.00206	9.09
J4402	27.7	0.9	56.3	0.00244	5.62
J4403	29.6	0.6	60.2	0.00315	3.97
J4404	25.0	0.5	51.0	0.00209	8.17
J4405	23.7	0.2	48.5	0.00245	5.26
J4406	26.6	0.2	54.3	0.00324	3.72

Table 15: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44) on the Classical Guitar.

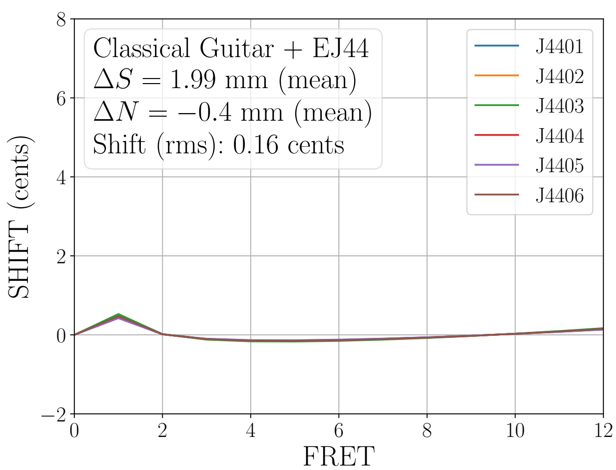
String	ΔS (mm)	ΔN (mm)	$\overline{\Delta\nu}_{\text{rms}}$ (cents)
J4401	1.62	-0.39	0.17
J4402	1.91	-0.42	0.18
J4403	2.43	-0.44	0.19
J4404	1.63	-0.38	0.16
J4405	1.87	-0.36	0.15
J4406	2.47	-0.40	0.17



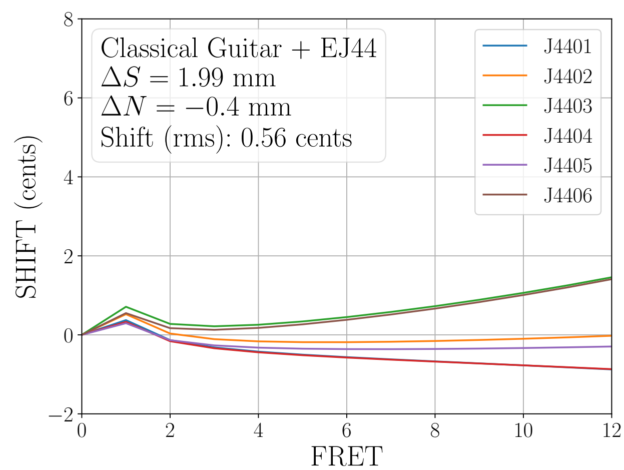
(a) Uncompensated



(b) Linear fits



(c) Full compensation



(d) Mean compensation

Figure 18: Frequency shift (in cents) for a classical guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). Four different strategies of saddle and nut compensation are illustrated.

D.4 Normal Tension – Carbon

Table 16: String specifications for the D'Addario Pro-Arte Carbon Classical Guitar Strings – Normal Tension (EJ45FF). The corresponding scale length is 650 mm.

String	Note	ρ (mm)	μ (mg/mm)	T_0 (N)
J4501FF	E_4	0.305	0.464	85.3
J4502FF	B_3	0.345	0.607	62.6
J4503FF	G_3	0.420	0.893	58.0
J4504FF	D_3	0.356	1.643	59.9
J4505FF	A_2	0.445	3.089	63.2
J4506FF	E_2	0.559	5.715	65.6

Table 17: Derived physical properties of the D'Addario Pro-Arte Carbon Classical Guitar Strings – Normal Tension (EJ45FF). The corresponding scale length is 650 mm.

String	R	σ	κ	B_0	E_{eff} (GPa)
J4501FF	21.8	0.4	44.6	0.00157	13.04
J4502FF	26.0	0.6	53.1	0.00194	8.86
J4503FF	26.9	0.8	54.7	0.00239	5.71
J4504FF	24.3	0.4	49.6	0.00193	7.47
J4505FF	26.9	0.4	54.7	0.00253	5.57
J4506FF	23.5	0.9	47.9	0.00298	3.20

Table 18: Predicted setbacks for the D'Addario Pro-Arte Carbon Classical Guitar Strings – Normal Tension (EJ45FF) on the Classical Guitar.

String	ΔS (mm)	ΔN (mm)	$\overline{\Delta\nu}_{\mathrm{rms}}$ (cents)
J4501FF	1.23	-0.33	0.14
J4502FF	1.53	-0.40	0.17
J4503FF	1.86	-0.41	0.17
J4504FF	1.51	-0.37	0.16
J4505FF	1.96	-0.41	0.17
J4506FF	2.24	-0.35	0.15

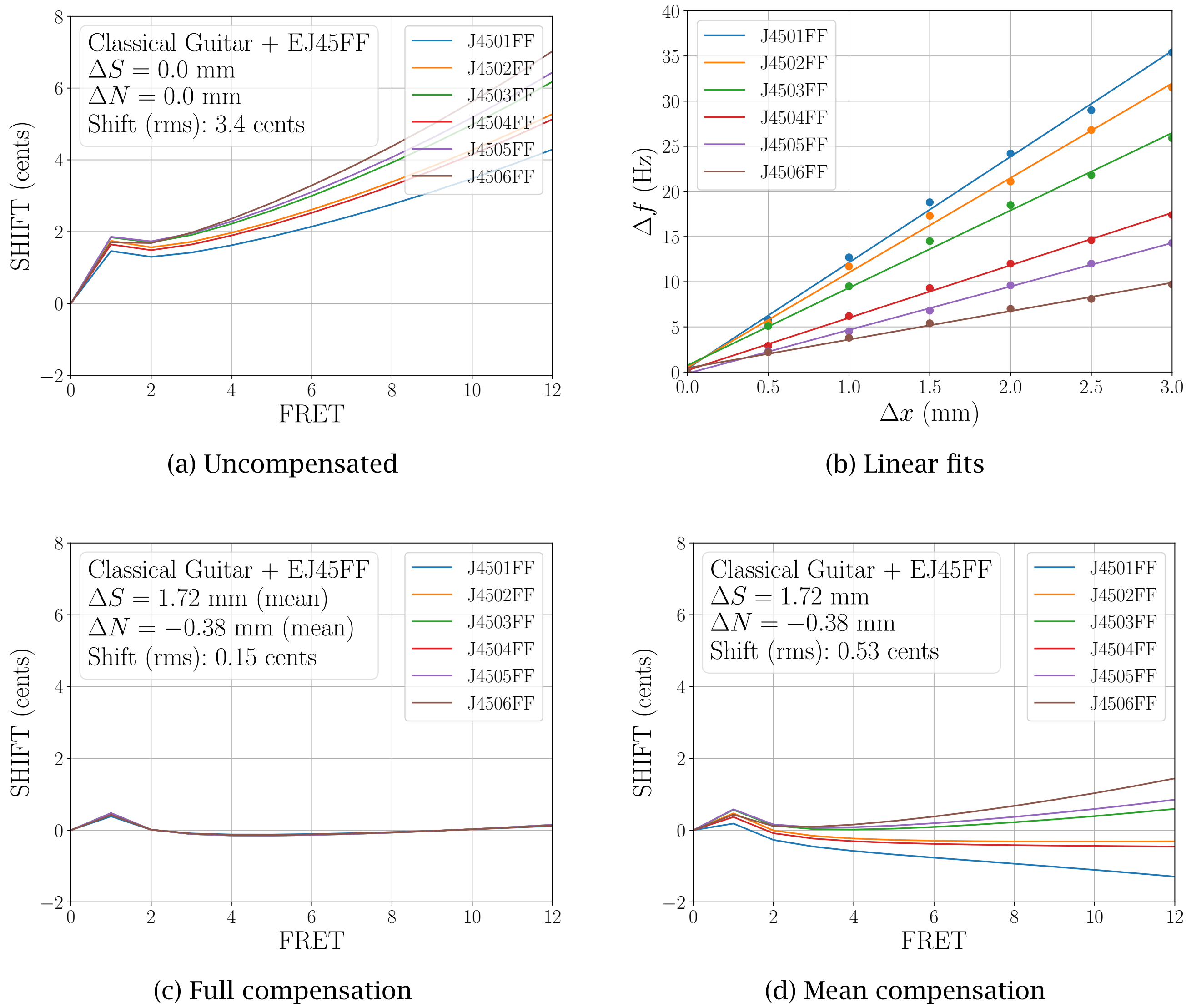


Figure 19: Frequency shift (in cents) for a classical guitar with D'Addario Pro-Arte Carbon Classical Guitar Strings – Normal Tension (EJ45FF). Four different strategies of saddle and nut compensation are illustrated.

D.5 Hard Tension – Carbon

Table 19: String specifications for the D'Addario Pro-Arte Carbon Classical Guitar Strings – Hard Tension (EJ46FF). The corresponding scale length is 650 mm.

String	Note	ρ (mm)	μ (mg/mm)	T_0 (N)
J4601FF	E_4	0.315	0.500	91.8
J4602FF	B_3	0.356	0.643	66.3
J4603FF	G_3	0.431	0.946	61.4
J4604FF	D_3	0.368	1.839	67.0
J4605FF	A_2	0.457	3.554	72.7
J4606FF	E_2	0.584	6.125	70.3

Table 20: Derived physical properties of the D'Addario Pro-Arte Carbon Classical Guitar Strings – Hard Tension (EJ46FF). The corresponding scale length is 650 mm.

String	R	σ	κ	B_0	E_{eff} (GPa)
J4601FF	22.2	0.3	45.4	0.00163	13.38
J4602FF	23.1	0.4	47.1	0.00188	7.86
J4603FF	25.8	0.6	52.6	0.00240	5.55
J4604FF	23.1	0.4	47.2	0.00195	7.42
J4605FF	23.8	0.2	48.6	0.00245	5.37
J4606FF	23.1	0.4	47.2	0.00309	3.09

Table 21: Predicted setbacks for the D'Addario Pro-Arte Carbon Classical Guitar Strings – Hard Tension (EJ46FF) on the Classical Guitar.

String	ΔS (mm)	ΔN (mm)	$\overline{\Delta\nu}_{\mathrm{rms}}$ (cents)
J4601FF	1.28	-0.34	0.14
J4602FF	1.46	-0.35	0.15
J4603FF	1.86	-0.39	0.17
J4604FF	1.51	-0.35	0.15
J4605FF	1.87	-0.36	0.15
J4606FF	2.32	-0.35	0.15

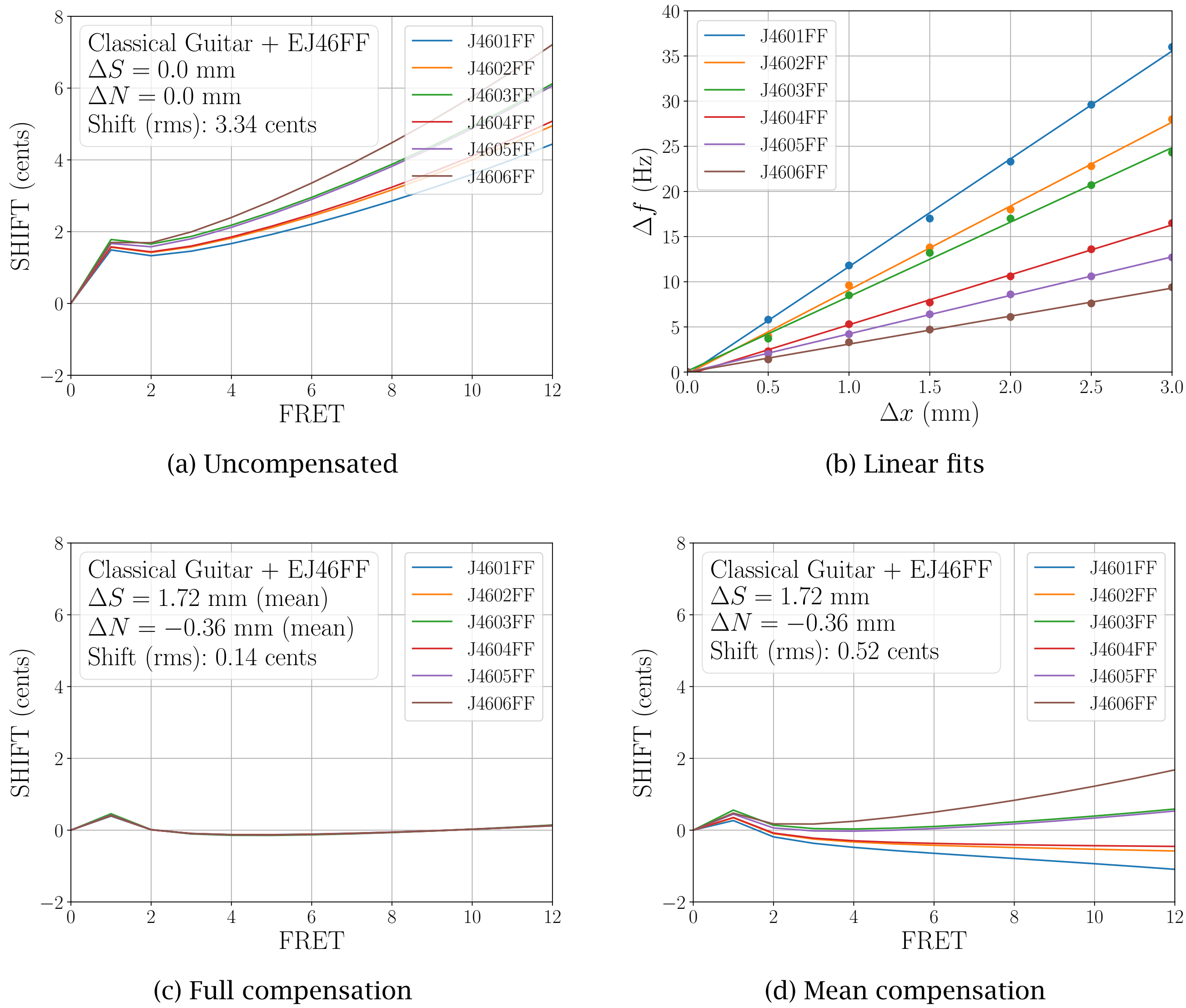


Figure 20: Frequency shift (in cents) for a classical guitar with D'Addario Pro-Arte Carbon Classical Guitar Strings – Hard Tension (EJ46FF). Four different strategies of saddle and nut compensation are illustrated.

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