Classical Guitar Intonation and Compensation: The Well-Tempered Guitar

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December 22, 2020

Abstract

TBD.

Contents

1 Introduction and Background

Discuss initial [1] and ongoing work by G. Byers [2], and recent studies of steel-string guitars [3]. General references on the physics of sound, music, and musical instruments are [4].

Fundamental frequency of a string [5]:

$$f_0 = \frac{1}{2\,L_0}\sqrt{\frac{T_0}{\mu_0}}\,, \tag{1}$$

where L_0 is the length of the free (unfretted) string from the saddle to the nut, T_0 is the tension in the free string, and $\mu_0 \equiv M/L_0$ is the linear mass density of a free string of mass M.

2 Simple Model of Guitar Intonation

The starting point for prior efforts to understand guitar intonation and compensation [2, 3] is a formula for f_m, the transverse vibration frequency harmonic m of a stiff string, originally published by Morse in 1936 [6]:

$$f_m = \frac{m}{2\,L}\sqrt{\frac{T}{\mu}}\left[1 + 2B + 4\left(1 + \frac{\pi^2\,m^2}{8}\right)B^2\right]. \tag{2}$$

Here L is the length of the string, T and μ are its tension and the linear mass density, respectively, and B is a small "bending stiffness" coefficient to capture the relevant mechanical properties of the string. For a homogeneous string with a cylindrical cross section, B is given by

$$B \equiv \sqrt{\frac{\pi\,\rho^4\,E}{4\,T\,L^2}}\,, \tag{3}$$

where ρ is the radius of the string and E is Young's modulus (or the modulus of elasticity). But it's unlikely that Eq. (2) accurately describes the resonant frequencies of a nylon string on a classical guitar, because it assumes that the string is "clamped" at both ends, so that a particular set of symmetric boundary conditions must be applied to the partial differential equation (PDE) describing transverse vibrations of the string. We believe that this assumption is correct for the end of the string held at either the nut or the fret, but that the string is "pinned" (and not clamped) at the saddle. In Appendix A, we solve the PDE using these non-symmetric boundary conditions, and find

$$f_m = \frac{m}{2\,L}\sqrt{\frac{T}{\mu}}\left[1 + B + \left(1 + \frac{1}{2}m^2\pi^2\right)B^2\right]. \tag{4}$$

Note that this expression is valid only when $B \ll 1$. For a typical nylon guitar string with $E \approx 4$ GPa, $T_0 \approx 60$ N, $\rho \approx 0.5$ mm, and $L_0 \approx 650$ mm, we have $B_0 \approx 6 \times 10^{-3}$. We'll use Eq. (4) with some caution, because the physics of nylon strings (particularly the wound base strings) are quite complicated [7].

Throughout this work, we will use *cents* to describe small differences in pitch [8]. One cent is one one-hundredth of a Twelve-Tone Equal Temperament (12-TET) half step, so that there are 1200 cents per octave. The difference in pitch between frequencies f_1 and f_2 is therefore defined as

$$\Delta\nu \equiv 1200\,\log_2\left(\frac{f_2}{f_1}\right). \tag{5}$$

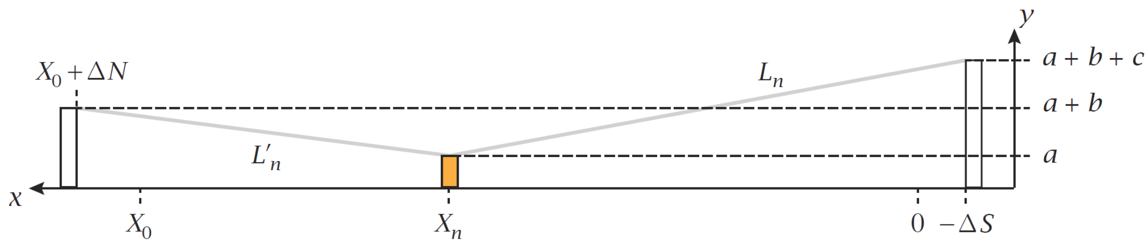


Figure 1: A simple (side-view) schematic of the classical guitar used in this model. The scale length of the guitar is X_0, but we allow the edges of both the saddle and the nut to be set back an additional distance ΔS and ΔN, respectively. The location on the x-axis of the center of the n^{th} fret is X_n. (Note that the x-axis is directed toward the left in this figure.) In the y direction, $y = 0$ is taken as the surface of the fingerboard; therefore the height of each fret above the fingerboard is a, the height of the nut is $a + b$, and the height of the saddle is $a + b + c$. L_n is the *resonant length* of the string from the saddle to the center of fret n, and L'_n is the length of the string from the fret to the nut.

We define $f \equiv (f_1 + f_2)/2$ and $\Delta f \equiv f_2 - f_1$. Then

$$\Delta\nu = 1200 \log_2 \left(\frac{f + \Delta f/2}{f - \Delta f/2} \right) \approx \frac{1200}{\ln 2} \frac{\Delta f}{f} \,, \tag{6}$$

where the last approximation applies when $\Delta f \ll f$. An experienced guitar player can distinguish beat notes with a difference frequency of $\Delta f \approx 1$ Hz, which corresponds to 8 cents at A_2 ($f = 220$ Hz) or 5 cents at E_4 ($f = 329.63$ Hz).

Our model begins with the schematic of the guitar shown in Fig. 1. The scale length of the guitar is X_0, but we allow the edges of both the saddle and the nut to be set back an additional distance ΔS and ΔN, respectively. The location on the x-axis of the center of the n^{th} fret is X_n. In the y direction, $y = 0$ is taken as the surface of the fingerboard; the height of each fret is a, the height of the nut is $a + b$, and the height of the saddle is $a + b + c$. L_n is the *resonant length* of the string from the saddle to the center of fret n, and L'_n is the length of the string from the fret to the nut. The total length of the string is defined as $\mathcal{L}_n \equiv L_n + L'_n$. For reasons discussed below, we have not (yet) adopted a more complicated fretting model [2, 3]. We start with the form of the fundamental frequency of a fretted string given by Eq. (4) with $m = 1$, and apply it to the frequency of a string pressed just behind the n^{th} fret:

$$f_n = \frac{1}{2\,L_n} \sqrt{\frac{T_n}{\mu_n}} \, (1 + B_n) \,, \tag{7}$$

where T_n and μ_n are the modified tension and the linear mass density of the fretted string,

$$B_n \equiv \sqrt{\frac{\pi\, \rho^4\, E}{4\, T_n\, L_n^2}} \,, \tag{8}$$

and we have neglected the smallest stiffness correction term proportional to B_n^2. We note that T_n and μ_n depend on $\mathcal{L}_n$, the *total* length of the fretted string from the saddle to the nut. Ideally, in the 12-TET system [8],

$$f_n = \gamma_n\, f_0 \,, \qquad \text{(12-TET ideal)} \tag{9}$$

where

$$\gamma_n \equiv 2^{n/12}\,. \tag{10}$$

Therefore, the error interval — the difference between the fundamental frequency of the fretted string and the corresponding perfect 12-TET frequency — expressed in cents is given by

$$\begin{aligned}\Delta\nu_n &= 1200\,\log_2\left(\frac{f_n}{\gamma_n\,f_0}\right)\\ &= 1200\,\log_2\left(\frac{L_0}{\gamma_n\,L_n}\sqrt{\frac{\mu_0}{\mu_n}\,\frac{T_n}{T_0}}\,\frac{1+B_n}{1+B_0}\right)\\ &= 1200\,\log_2\left(\frac{L_0}{\gamma_n\,L_n}\right)+600\,\log_2\left(\frac{\mu_0}{\mu_n}\right)+600\,\log_2\left(\frac{T_n}{T_0}\right)+1200\,\log_2\left(\frac{1+B_n}{1+B_0}\right)\,.\end{aligned} \tag{11}$$

The final form of Eq. (11) makes it clear that — for nylon guitar strings — there are four contributions to intonation:

1. *Resonant Length*: The first term represents the error caused by the increase in the length of the fretted string L_n compared to the ideal length X_n, which would be obtained if $b = c = 0$ and $\Delta S = \Delta N = 0$.

2. *Linear Mass Density*: The second term is the error caused by the reduction of the linear mass density of the fretted string. This effect will depend on the *total* length of the string, given by $\mathcal{L}_n = L_n + L'_n$.

3. *Tension*: The third term is the error caused by the *increase* of the tension in the string arising from the stress and strain applied to the string by fretting. This effect will also depend on the total length of the string $\mathcal{L}_n$.

4. *Bending Stiffness*: The fourth and final term is the error caused by the change in the bending stiffness coefficient caused by changing the vibrating length of the string from L_0 to L_n.

We will discuss each of these three sources of error in turn below.

2.1 Resonant Length

We can estimate the first term in the last line of Eq. (11) by referring to Fig. 1 and computing the resonant length L_n. We find:

$$L_n = \begin{cases}\sqrt{(X_0+\Delta S+\Delta N)^2+c^2}\,, & n=0\\ \sqrt{(X_n+\Delta S)^2+(b+c)^2}\,. & n\geq 1\end{cases} \tag{12}$$

When $b + c \ll X_0$, we can approximate L_n by

$$L_n \approx \begin{cases}X_0+\Delta S+\Delta N+c^2/2\,X_0 & n=0\,,\\ X_n+\Delta S+(b+c)^2/2\,X_n & n\geq 1\,.\end{cases} \tag{13}$$

Then — when the guitar has been manufactured such that $X_n = X_0/\gamma_n$ — the resonant length error is approximately

$$1200 \log_2\left(\frac{L_0}{\gamma_n L_n}\right) \approx \frac{1200}{\ln(2)}\left[\frac{\Delta N - (\gamma_n - 1)\,\Delta S}{X_0} - \frac{\gamma_n^2 (b+c)^2 - c^2}{2\,X_0^2}\right] \tag{14}$$

If the guitar is uncompensated, so that $\Delta S = \Delta N = 0$, this error is typically less than 0.25 cents. But, with $\Delta S > 0$ and $\Delta N < 0$, we can significantly *increase* the magnitude of this "error" and cause the frequency to shift lower. We'll see that this is our primary method of compensation.

2.2 Linear Mass Density

As discussed above, the linear mass density μ_0 of an open (unfretted) string is simply the total mass M of the string clamped between the saddle and the nut divided by the length L_0. Similarly, the mass density μ_n of a string held onto fret N is $M/\mathcal{L}_n$. Therefore

$$\frac{\mu_0}{\mu_n} = \frac{\mathcal{L}_n}{L_0} \equiv 1 + Q_n\,, \tag{15}$$

where we have followed Byers and defined [2, 3]

$$Q_n \equiv \frac{\mathcal{L}_n - L_0}{L_0}\,. \tag{16}$$

Since we expect that $Q_n \ll 1$, we can approximate the second term in the final line of Eq. (11) as

$$600 \log_2\left(\frac{\mu_0}{\mu_n}\right) \approx \frac{600}{\ln(2)}\,Q_n\,. \tag{17}$$

Previous studies of guitar intonation and compensation have chosen to include the apparent increase in length of the string caused by both the fretting depth and the shape of the fretted string under the finger [2, 3]. As the string is initially pressed to the fret, the total length $\mathcal{L}_n$ increases and causes the tension in the string — which is clamped at the saddle and the nut — to increase. As the string is pressed further, does the additional deformation of the string increase its tension (throughout the resonant length L_n)? There are at least two purely empirical reasons to doubt this hypothesis. First, we can mark a string (with a fine-point felt pen) above a particular fret and then observe the mark with a magnifying glass. As the string is pressed all the way to the finger board, the mark does not move perceptibly — it has become effectively *clamped* on the fret. Second, we can use either our ears or a simple tool to measure frequencies [9] to listen for a shift as we use different fingers and vary the fretted depth of a string. The apparent modulation is far less than would be obtained by classical vibrato (± 15 cents), so we assume that once the string is minimally fretted the length(s) can be regarded as fixed. (If this were not the case, then fretting by different people or with different fingers, at a single string or with a barre, would cause additional, varying frequency shifts that would be audible and difficult to compensate.)

Referring to Fig. 1, we see that $\mathcal{L}_n = L_n + L'_n$, and we calculate L'_n as

$$L'_n = \begin{cases} 0\,, & n = 0 \\ \sqrt{(X_0 - X_n + \Delta N)^2 + b^2}\,. & n \geq 1 \end{cases} \tag{18}$$

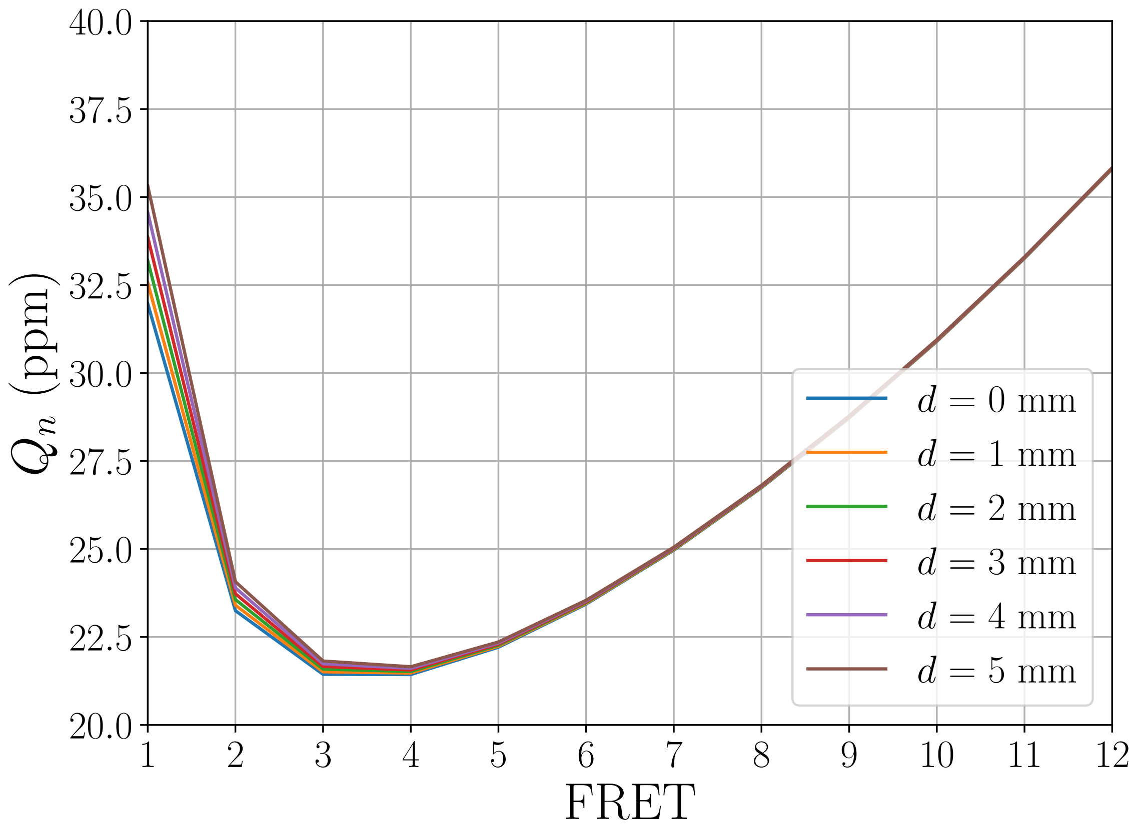


Figure 2: Plot of the normalized displacement Q_n as a function of the fret number for three different values of the parameter d. Here the guitar has $b = 1.0$ mm, $c = 3.5$ mm, no setbacks, and a scale length of 650 mm.

Assuming that $b^2 \ll X_0 - X_n$, we expand the $n \neq 1$ expression to obtain

$$L'_n \approx X_0 - X_n + \Delta N + \frac{b^2}{2\,(X_0 - X_n)} \tag{19}$$

Therefore, using Eq. (13), we have for $n \neq 1$

$$\mathcal{L}_n = L_n + L'_n \approx X_0 + \Delta S + \Delta N + \frac{(b+c)^2}{2\,X_n} + \frac{b^2}{2\,(X_0 - X_n)}\,, \tag{20}$$

and

$$\begin{aligned} Q_n &\approx \frac{1}{2\,X_0}\left[\frac{(b+c)^2}{X_n} + \frac{b^2}{X_0 - X_n} - \frac{c^2}{X_0}\right] \\ &= \frac{\gamma_n}{2\,X_0^2}\left[(b+c)^2 + \frac{b^2}{\gamma_n - 1} - \frac{c^2}{\gamma_n}\right]. \end{aligned} \tag{21}$$

When we substitute this expression into Eq. (17), the resulting error is generally quite small. Suppose that $b = 1.0$ mm, $c = 3.5$ mm, $X_0 = 650$ mm, $n = 12$, and $X_{12} = X_0/2 = 325$ mm. Then the error is about 0.03 cents, and will be even smaller for $n < 12$. If we add this shift due to the linear mass density to the residual quadratic resonant length shift given by Eq. (14), then we find the total error

$$\Delta\nu_n = \frac{300}{\ln(2)}\,\frac{\gamma_n}{X_0^2}\left[\frac{b^2}{\gamma_n - 1} + \frac{c^2}{\gamma_n} - (2\gamma_n - 1)(b+c)^2\right]. \tag{22}$$

For the same parameters, $\Delta\nu_{12} = -0.11$ cents, and $|\Delta\nu_n| < |\Delta\nu_{12}|$ for $n < 12$.

2.3 Tension

Elasticity properties [10]

$$\Delta T_n = E\,A\,\frac{\mathcal{L}_n - L_0}{L_0} = A\,E\,Q_n\,, \tag{23}$$

where we have used Eq. (16). Therefore, the tension of the fretted string is

$$T_n = T_0 + T_n = T_0\,(1 + \kappa\,Q_n)\,, \tag{24}$$

where we have defined the dimensionless "string constant"

$$\kappa \equiv \frac{A\,E}{T_0} = \frac{\pi\rho^2 E}{T_0}\,. \tag{25}$$

In this case, we assume that $\kappa Q_n \ll 1$, so that we can approximate the third term in the final line of Eq. (11) as

$$600\,\log_2\left(\frac{T_n}{T_0}\right) \approx \frac{600}{\ln(2)}\,\kappa\,Q_n\,. \tag{26}$$

This frequency shift is larger than that caused by the linear mass density error by a factor of κ.

2.4 Bending Stiffness

Since B_n is already relatively small, we only need to consider the largest contribution arising from the shortened length of the fretted string compared to that of the open string. We see from Eq. (12) that $L_n \approx L_0/\gamma_n$, so from Eq. (8) we have

$$B_n = \sqrt{\frac{\pi\,\rho^4\,E}{4\,T_n\,L_n^2}} \approx \frac{L_0}{L_n}\sqrt{\frac{\pi\,\rho^4\,E}{4\,T_0\,L_0^2}} = \gamma_n\,B_0\,. \tag{27}$$

Therefore, the fourth term in the final line of Eq. (11) can be approximated as

$$1200\,\log_2\left(\frac{1+B_n}{1+B_0}\right) \approx \frac{1200}{\ln(2)}\,(\gamma_n - 1)\,B_0\,. \tag{28}$$

When $n = 12$ and $B_0 = 6\times10^{-3}$, the corresponding shift is approximately 10 cents. Note that (to zero order in Q_n) this bending stiffness error does not depend on the tiny changes to the linear mass density or the tension that arises due to string fretting. Instead, it is an intrinsic property of the string.

Incorporating all of these effects, the total frequency shift is given by

$$\Delta\nu_n \approx \frac{1200}{\ln(2)}\left[(\gamma_n - 1)\left(B_0 - \frac{\Delta S}{X_0}\right) + \frac{\Delta N}{X_0} + \frac{1}{2}\,\kappa\,Q_n\right]. \tag{29}$$

3 Experimental Estimate of the String Constant

This section needs to be rewritten. How do we estimate the spring constant κ given by Eq. (25)? Recall how we introduced this concept in Section 2.3: increasing the tension by a quantity ΔT causes a shift in the frequency of a guitar string by $\Delta\nu$ in cents. Although we were considering fretted strings when we derived Eq. (11), the terms in that equation can be generalized to describe the case of an open string that has been stretched longitudinally. Suppose that we continue to clamp the string at the saddle and the nut, but that we tighten the tuning gear to stretch that string's length by an amount ΔL. The change in the string's frequency due to the change in the open resonant length is zero, because $L_0/\gamma_0 L_0 = 1$. The linear mass density of the string is smaller now because there is less material between the saddle and the nut, causing the frequency shift (in cents)

$$600\,\log_2\left(\frac{\mu}{\mu+\Delta\mu}\right) \approx \frac{600}{\ln(2)}\,\frac{\Delta L}{L}\,, \tag{30}$$

where L is the initial length of the string. Finally, the tension in the string increases by ΔT due to the elastic properties of the string. Following the discussion in Section 2.3, the corresponding frequency shift is

$$600\,\log_2\left(\frac{T+\Delta T}{T}\right) \approx \frac{600}{\ln(2)}\,\frac{\Delta L}{L}\,\kappa\,, \tag{31}$$

where T is the initial tension of the string. Therefore, the total frequency shift of the open string caused by a change ΔL in the string's length is

$$\Delta\nu \approx \frac{600}{\ln(2)}\,\frac{\Delta L}{L}\,(\kappa + 1)\,. \tag{32}$$

Solving this expression for the string constant, we find

$$\kappa = \frac{\ln(2)}{600} R - 1\,, \tag{33}$$

where

$$R \equiv \frac{L}{\Delta L}\,\Delta\nu \tag{34}$$

is a parameter originally defined by Byers[1] [2, 3].

It is relatively easy to estimate the value of R for any guitar string with the aid of a simple device that can measure frequency shifts in cents [9] and either calipers or a ruler with finely marked graduations. With a fine-point felt pen, make a small mark on the string at some convenient point (say, directly above the first fret). Then tighten that string's tuning gear to increase the frequency by an amount measured using the electronic tuner. (A particularly convenient shift is four half-steps, corresponding to $\Delta\nu = 400$ cents, which will require a stretch of approximate 5 – 6 mm.)

Table 1: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 650 mm.

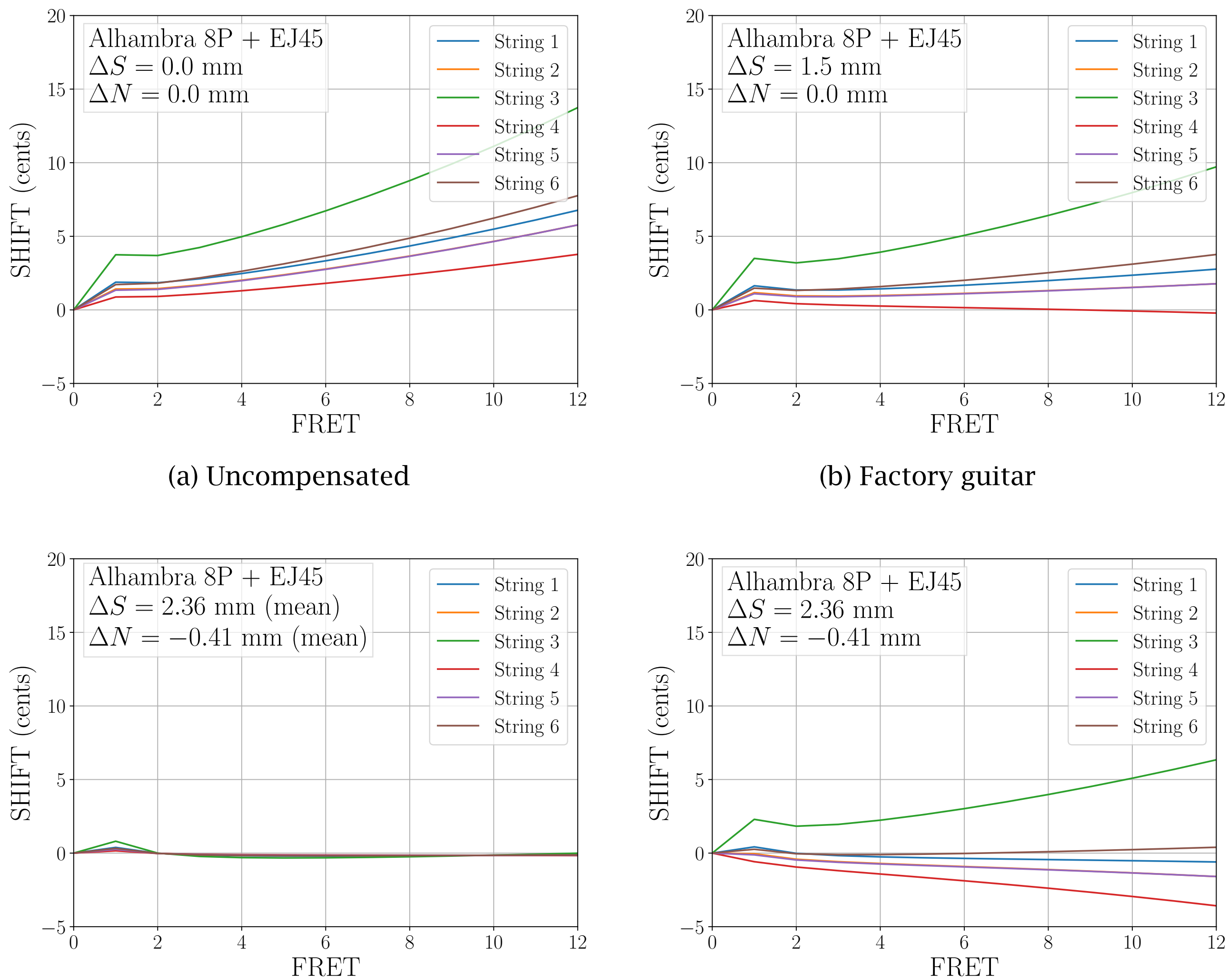
String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4501	E_4	0.36	3.74	68.62
J4502	B_3	0.41	5.05	52.04
J4503	G_3	0.51	8.36	54.26
J4504	D_3	0.37	19.21	69.99
J4505	A_2	0.44	32.90	67.27
J4506	E_2	0.55	54.72	62.80

Table 2: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 650 mm.

String	$\Delta\nu_{12}$ (cents)	R ($\times 10^4$)	κ	E (GPa)	B_0 ($\times 10^{-3}$)
J4501	3	5.2	58.6	10.1	2.1
J4502	2	3.8	42.5	4.2	2.1
J4503	10	10.1	115.9	7.6	4.2
J4504	0	2.3	26.0	4.3	1.4
J4505	2	3.5	39.6	4.3	2.2
J4506	4	4.4	49.8	3.3	3.0

Table 3: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45) on the Alhambra 8P classical guitar.

String	ΔS (mm)	ΔN (mm)
J4501	2.15	-0.44
J4502	1.90	-0.32
J4503	4.30	-0.86
J4504	1.29	-0.19
J4505	1.93	-0.30
J4506	2.59	-0.37



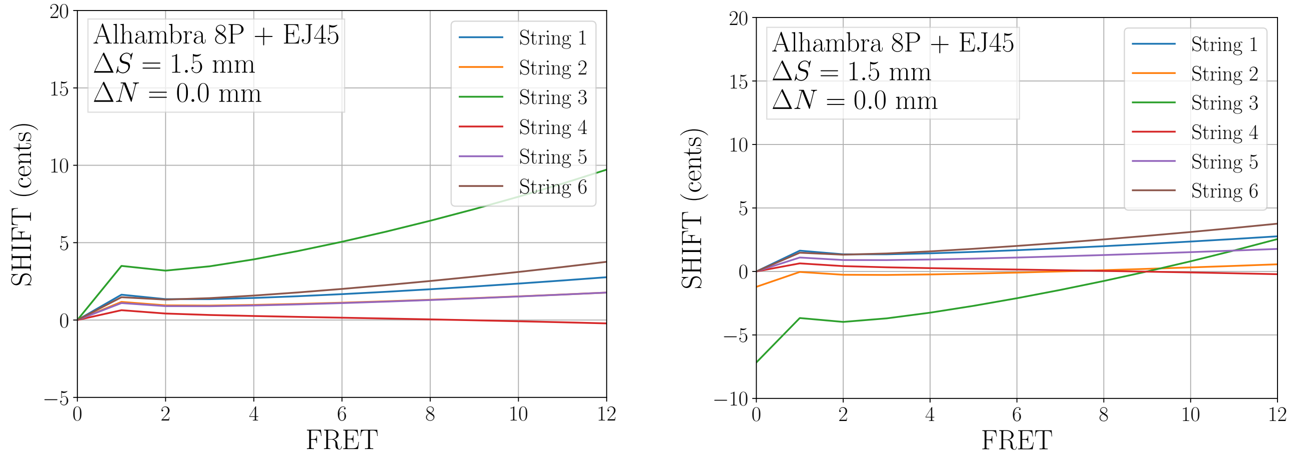
(a) Uncompensated

(b) Factory guitar

(c) Full compensation

(d) Mean compensation

Figure 3: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). Four different strategies of saddle and nut compensation are illustrated.



(a) Factory guitar — 12-TET tuned (b) Factory guitar — harmonically tuned

Figure 4: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). Here we compare the factory guitar tuned to 12-TET with the same guitar harmonically tuned.

4 Classical Guitar Compensation

5 Tempering the Classical Guitar

6 Conclusion

A Vibration Frequencies of a Stiff String

Here we outline the calculation of the normal mode frequencies of a vibrating stiff string with non-symmetric boundary conditions. We begin with the wave equation [11]

$$\mu \frac{\partial^2}{\partial t^2} y(x) = T \frac{\partial^2}{\partial x^2} y(x) - E\, S\, \mathcal{K}^2 \frac{\partial^4}{\partial x^4} y(x)\,, \tag{35}$$

where μ and T are respectively the linear mass density and the tension of the string, E is its Young's modulus (or the modulus of elasticity), S is the cross-sectional area, and $\mathcal{K}$ is the radius of gyration of the string. (For a uniform cylindrical string/wire with radius ρ, $S = \pi\rho^2$ and $\mathcal{K} = \rho/2$.) If we scale x by the length L of the string, and t by $1/\omega_0 \equiv (L/\pi)\sqrt{\mu/T}$, then we obtain the dimensionless wave equation

$$\pi^2 \frac{\partial^2}{\partial t^2} y(x) = \frac{\partial^2}{\partial x^2} y(x) - B^2 \frac{\partial^4}{\partial x^2} y(x)\,, \tag{36}$$

where B is the "bending stiffness parameter" given by

$$B \equiv \sqrt{\frac{E\, S\, \mathcal{K}^2}{L^2 T}}\,. \tag{37}$$

[1]Byers expressed this parameter in terms of the fractional frequency shift (in Hertz) as $R = (L/\Delta L)(\Delta f/f) \approx (\ln(2)/1200)(L/\Delta L)\Delta\nu$. Therefore, our dimensionless value of R is larger than Byers' by a factor of about 1730.

We assume that $y(x)$ is a sum of terms of the form

$$y(x) = C\,e^{k\,x - i\,\omega\,t}\,, \tag{38}$$

requiring that k and ω satisfy the expression

$$B^2 k^4 - k^2 - (\pi\,\omega)^2 = 0\,, \tag{39}$$

or

$$k^2 = \frac{1 \pm \sqrt{1 + (2\,\pi\,B\,\omega)^2}}{2\,B^2}\,. \tag{40}$$

Therefore, given ω, we have four possible choices for k: $\pm k_1$, or $\pm i k_2$, where

$$k_1^2 = \frac{\sqrt{1 + (2\,\pi\,B\,\omega)^2} + 1}{2\,B^2}\,, \text{ and} \tag{41a}$$

$$k_2^2 = \frac{\sqrt{1 + (2\,\pi\,B\,\omega)^2} - 1}{2\,B^2}\,. \tag{41b}$$

The corresponding general solution to Eq. (36) has the form

$$y(x) = e^{-i\omega t}\left(C_1^+ e^{k_1 x} + C_1^- e^{-k_1 x} + C_2^+ e^{ik_2 x} + C_2^- e^{-ik_2 x}\right)\,. \tag{42}$$

As discussed in Section 2, the boundary conditions for the case of a classical guitar string are not symmetric. At $x = 0$ (the saddle), the string is pinned (but not clamped), so that $y = 0$ and $\partial^2 y/\partial x^2 = 0$. However, at $x = 1$ (the fret) the string is clamped, so that $y = 0$ and $\partial y/\partial x = 0$. Applying these constraints to Eq. (42), we obtain

$$0 = C_1^+ + C_1^- + C_2^+ + C_2^-\,, \tag{43a}$$

$$0 = k_1^2\left(C_1^+ + C_1^-\right) - k_2^2\left(C_2^+ + C_2^-\right)\,, \tag{43b}$$

$$0 = C_1^+ e^{k_1} + C_1^- e^{-k_1} + C_2^+ e^{ik_2} + C_2^- e^{-ik_2}\,, \text{ and} \tag{43c}$$

$$0 = k_1\left(C_1^+ e^{k_1} - C_1^- e^{-k_1}\right) + ik_2\left(C_2^+ e^{ik_2} - C_2^- e^{-ik_2}\right)\,. \tag{43d}$$

Since $k_1^2 + k_2^2 \neq 0$, the first two of these equations tell us that $C_1^- = -C_1^+ \equiv -C_1$, and $C_2^- = -C_2^+ \equiv -C_2$. Therefore, the second two equations become

$$C_1 \sinh(k_1) = -i\,C_2 \sin(k_2)\,, \text{ and} \tag{44a}$$

$$k_1\,C_1 \cosh(k_1) = -i\,k_2\,C_2 \cos(k_2)\,. \tag{44b}$$

Dividing the first of these equations by the second, we find

$$\tan(k_2) = \frac{k_2}{k_1}\tanh k_1\,. \tag{45}$$

From Eq. (41), we see that $k_1^2 - k_2^2 = 1/B^2$, so that

$$k_1 = \frac{1}{B}\sqrt{1 + (B\,k_2)^2}\,. \tag{46}$$

In the case of a classical guitar, we expect that $B \ll 1$, so $k_1 \approx 1/B \gg 1$, and therefore $\tanh k_1 \longrightarrow 1$. Substituting Eq. (46) into Eq. (45), we obtain

$$\tan(k_2) = \frac{B\,k_2}{\sqrt{1 + (B\,k_2)^2}}\,. \tag{47}$$

We expect that $B\,k_2 \ll 1$, so we assume that $k_2 = m\pi(1+\epsilon)$, where m is an integer greater than or equal to 1, and $\epsilon \ll 1$. Therefore, to second order in ϵ, we have $\tan(k_2) \approx m\pi\epsilon$, and

$$\epsilon = \frac{B\,(1+\epsilon)}{\sqrt{1+[m\,\pi\,B\,(1+\epsilon)]^2}}\,. \tag{48}$$

The denominator of the right-hand side of this equation has a Taylor expansion given by $1-\frac{1}{2}\,[m\,\pi\,B\,(1+\epsilon)]^2$, indicating that it will not contribute to ϵ to second order in B. Therefore, to this order,

$$\epsilon \approx \frac{B}{1-B} \approx B + B^2\,. \tag{49}$$

We substitute $k = \pm i k_2$ into Eq. (39) with $k_2 = m\pi/(1-B)$ to obtain

$$\begin{aligned}
\omega &= \frac{k_2}{\pi}\sqrt{1+(B\,k_2)^2} \\
&= \frac{m}{1-B}\sqrt{1+m^2\pi^2\left(\frac{B}{1-B}\right)^2} \\
&\approx m\left[1+B+\left(1+\frac{1}{2}m^2\pi^2\right)B^2\right].
\end{aligned} \tag{50}$$

Restoring the time scaling by $1/\omega_0$, and defining the frequency (in cycles/second) $f = \omega/2\pi$, we finally have

$$f_m = \frac{m}{2\,L}\sqrt{\frac{T}{\mu}}\left[1+B+\left(1+\frac{1}{2}m^2\pi^2\right)B^2\right]. \tag{51}$$

We use this result to build our model in Section 2.

B Compensation by Minimizing RMS Error

The root-mean-square (RMS) frequency error (in cents) averaged over the frets $n \in \{1, n_{\max}\}$ *of a particular string* is given by

$$\overline{\Delta\nu} \equiv \sqrt{\frac{\sum_{n=1}^{n_{\max}}\Delta\nu_n^2}{n_{\max}}}\,, \tag{52}$$

where $\Delta\nu_n$ is given by Eq. (29). Here we will vary both ΔS and ΔN to minimize $\overline{\Delta\nu}$. In this case, it is sufficient to minimize the quantity

$$\chi^2 = \sum_{n=1}^{n_{\max}}\left[\frac{\ln(2)}{1200}\,\Delta\nu_n\right]^2 \tag{53}$$

such that the gradient of χ^2 with respect to ΔS and ΔN vanishes. The components of this gradient are

$$\frac{\partial}{\partial\Delta S}\chi^2 = -\frac{2}{X_0}\sum_n(\gamma_n-1)\left[(\gamma_n-1)\left(B_0-\frac{\Delta S}{X_0}\right)+\frac{\Delta N}{X_0}+\frac{\kappa}{2}\,Q_n\right], \text{ and} \tag{54a}$$

$$\frac{\partial}{\partial\Delta N}\chi^2 = \frac{2}{X_0}\sum_n\left[(\gamma_n-1)\left(B_0-\frac{\Delta S}{X_0}\right)+\frac{\Delta N}{X_0}+\frac{\kappa}{2}\,Q_n\right]. \tag{54b}$$

Setting both of these components to zero, we can rewrite them as the matrix equation

$$\begin{bmatrix} \sigma_2 & -\sigma_1 \\ \sigma_1 & -\sigma_0 \end{bmatrix} \begin{bmatrix} \Delta S \\ \Delta N \end{bmatrix} = X_0 \begin{bmatrix} \sigma_2\, B_0 + \frac{1}{2}\,\kappa\, \sum_n (\gamma_n - 1) Q_n \\ \sigma_1\, B_0 + \frac{1}{2}\,\kappa\, \sum_n Q_n \end{bmatrix}, \tag{55}$$

where

$$\sigma_k \equiv \sum_{n=1}^{n_{\rm max}} (\gamma_n - 1)^k\,. \tag{56}$$

Equation (55) has the straightforward analytic solution

$$\begin{bmatrix} \Delta S \\ \Delta N \end{bmatrix} = \frac{X_0}{\sigma_1^2 - \sigma_0\,\sigma_2} \begin{bmatrix} -\sigma_0 & \sigma_1 \\ -\sigma_1 & \sigma_2 \end{bmatrix} \begin{bmatrix} \sigma_2\, B_0 + \frac{1}{2}\,\kappa\, \sum_n (\gamma_n - 1) Q_n \\ \sigma_1\, B_0 + \frac{1}{2}\,\kappa\, \sum_n Q_n \end{bmatrix}. \tag{57}$$

The corresponding Hessian matrix for this problem is

$$H = \begin{bmatrix} \frac{\partial^2 \chi^2}{\partial \Delta S^2} & \frac{\partial^2 \chi^2}{\partial \Delta N\, \partial \Delta S} \\ \frac{\partial^2 \chi^2}{\partial \Delta S\, \partial \Delta N} & \frac{\partial^2 \chi^2}{\partial \Delta N^2} \end{bmatrix} = \frac{2}{X_0^2} \begin{bmatrix} \sigma_2 & -\sigma_1 \\ -\sigma_1 & \sigma_0 \end{bmatrix}. \tag{58}$$

The Hessian is positive definite if and only if all of its eigenvalues are positive, and in the case of a 2×2 real matrix, this holds when the determinant is greater than zero. It is easy to verify numerically (and with some effort algebraically) that $\mathrm{Det}(H) > 0$ for $n_{\rm max} > 1$. Therefore, the solution for ΔS and ΔN given by Eq. (57) minimizes the RMS frequency error.

C Other Classical Guitar String Sets

C.1 Light Tension

Table 4: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

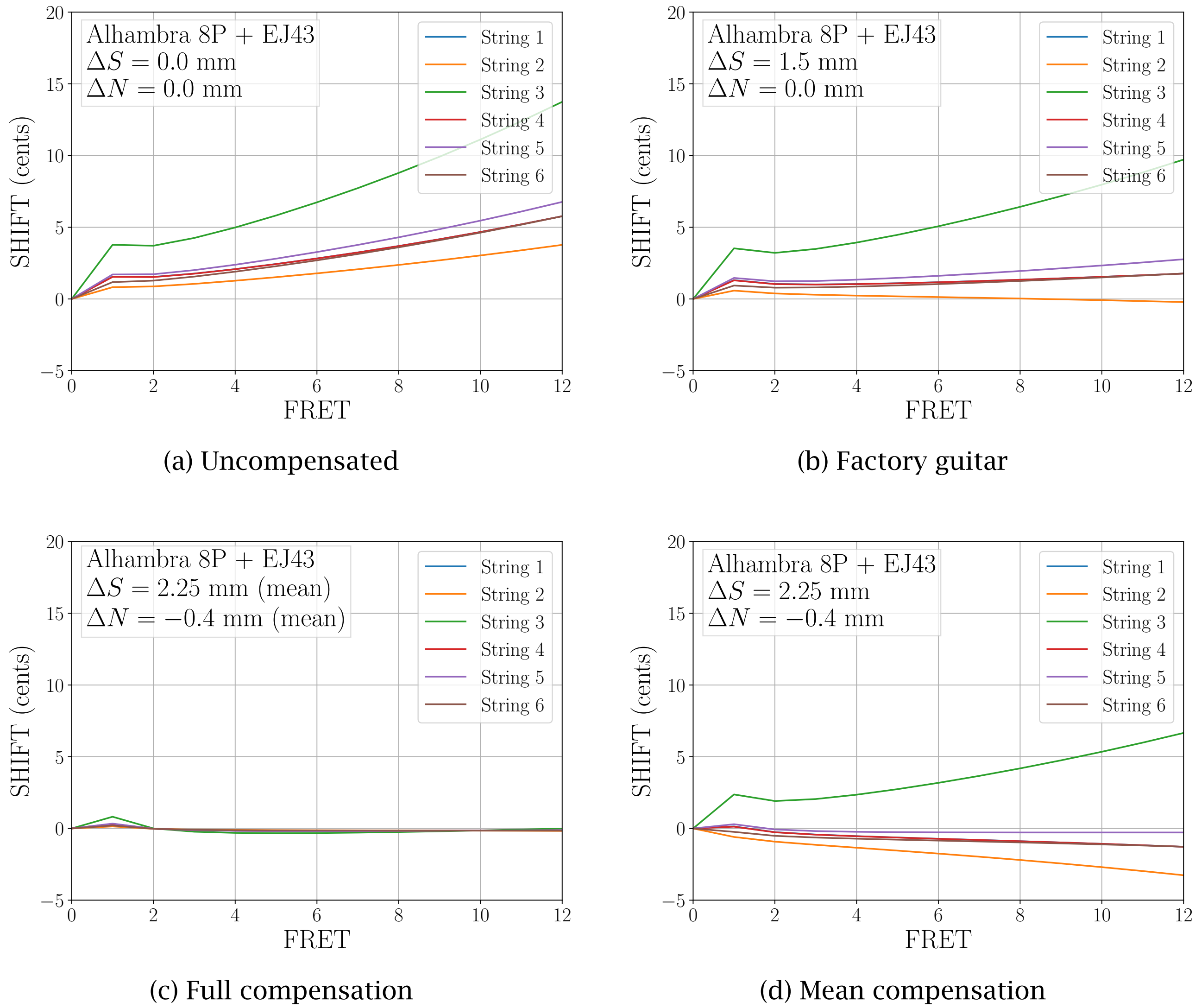
String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4301	E_4	0.35	3.62	66.39
J4302	B_3	0.40	4.87	50.24
J4303	G_3	0.50	8.08	52.48
J4304	D_3	0.36	18.23	66.41
J4305	A_2	0.42	27.41	56.06
J4306	E_2	0.53	51.59	59.21

Table 5: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

String	$\Delta\nu_{12}$ (cents)	R ($\times 10^4$)	κ	E (GPa)	B_0 ($\times 10^{-3}$)
J4301	3	4.3	48.2	8.3	1.9
J4302	2	2.2	23.9	2.4	1.5
J4303	10	10.2	117.1	7.7	4.2
J4304	0	4.2	47.5	7.9	1.9
J4305	2	4.6	51.8	5.3	2.3
J4306	4	3.0	33.2	2.2	2.4

Table 6: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43) on the Alhambra 8P classical guitar.

String	ΔS (mm)	ΔN (mm)
J4301	1.86	-0.36
J4302	1.30	-0.18
J4303	4.29	-0.87
J4304	1.86	-0.35
J4305	2.20	-0.39
J4306	1.98	-0.25



(a) Uncompensated

(b) Factory guitar

(c) Full compensation

(d) Mean compensation

Figure 5: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). Four different strategies of saddle and nut compensation are illustrated.

C.2 Hard Tension

Table 7: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). The corresponding scale length is 650 mm.

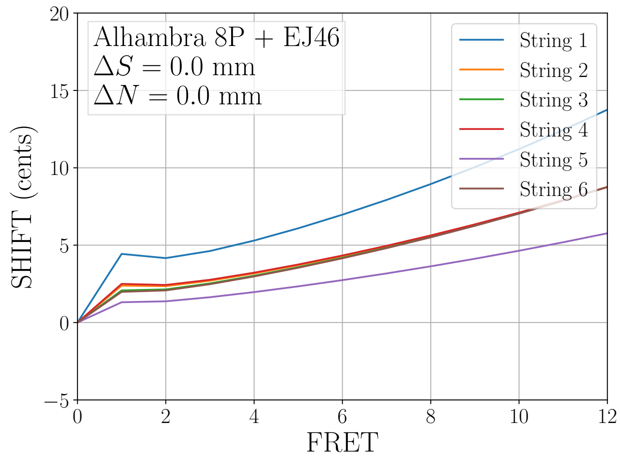
String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4601	E_4	0.36	3.86	70.88
J4602	B_3	0.42	5.22	53.83
J4603	G_3	0.52	8.57	55.61
J4604	D_3	0.38	20.07	73.14
J4605	A_2	0.46	34.87	71.31
J4606	E_2	0.56	56.67	65.04

Table 8: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). The corresponding scale length is 650 mm.

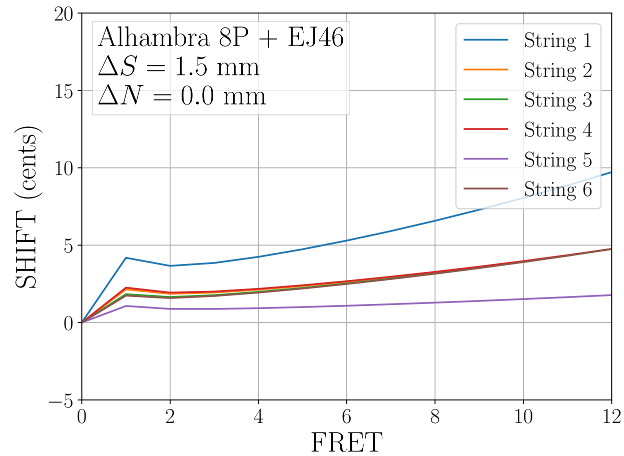
String	$\Delta\nu_{12}$ (cents)	R ($\times 10^4$)	κ	E (GPa)	B_0 ($\times 10^{-3}$)
J4601	3	12.5	143.1	24.6	3.3
J4602	2	6.5	73.8	7.3	2.7
J4603	10	5.4	61.8	4.0	3.1
J4604	0	6.9	78.3	12.6	2.6
J4605	2	3.4	38.6	4.2	2.2
J4606	4	5.1	58.0	3.8	3.3

Table 9: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46) on the Alhambra 8P classical guitar.

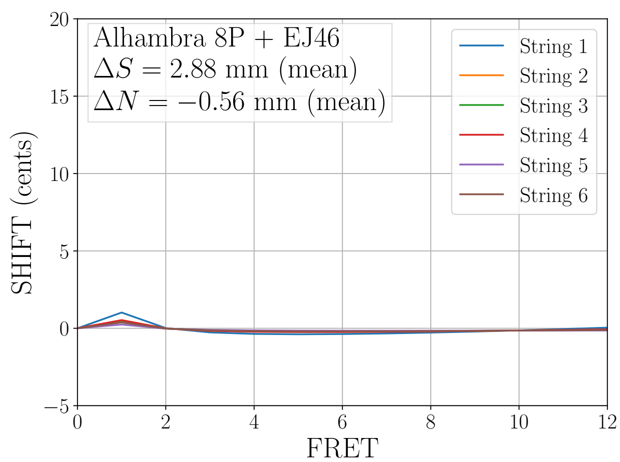
String	ΔS (mm)	ΔN (mm)
J4601	4.08	-1.07
J4602	2.77	-0.55
J4603	2.87	-0.46
J4604	2.73	-0.58
J4605	1.94	-0.29
J4606	2.90	-0.43



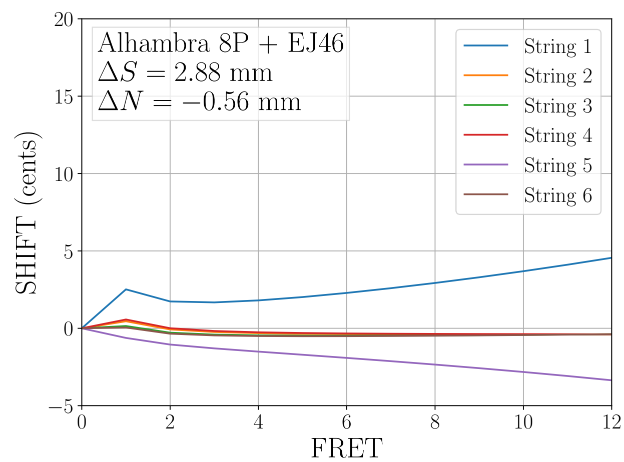
(a) Uncompensated



(b) Factory guitar



(c) Full compensation



(d) Mean compensation

Figure 6: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). Four different strategies of saddle and nut compensation are illustrated.

C.3 Extra Hard Tension

Table 10: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). The corresponding scale length is 650 mm.

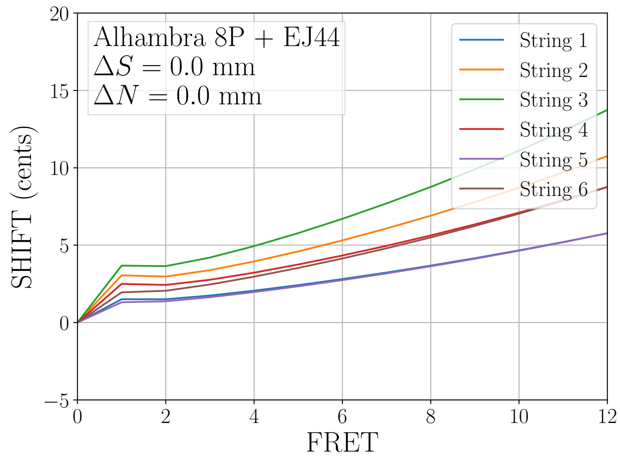
String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4401	E_4	0.37	4.01	73.57
J4402	B_3	0.42	5.44	56.07
J4403	G_3	0.53	8.91	57.86
J4404	D_3	0.38	20.07	73.14
J4405	A_2	0.46	34.87	71.31
J4406	E_2	0.57	61.36	70.42

Table 11: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). The corresponding scale length is 650 mm.

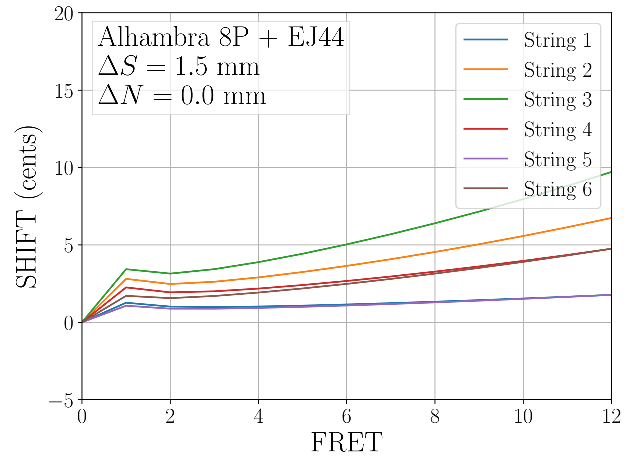
String	$\Delta\nu_{12}$ (cents)	R ($\times 10^4$)	κ	E (GPa)	B_0 ($\times 10^{-3}$)
J4401	3	4.1	46.3	8.0	1.9
J4402	2	8.4	95.5	9.5	3.2
J4403	10	9.9	113.2	7.5	4.3
J4404	0	6.9	78.3	12.6	2.6
J4405	2	3.4	38.6	4.2	2.2
J4406	4	5.0	56.8	3.9	3.3

Table 12: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44) on the Alhambra 8P classical guitar.

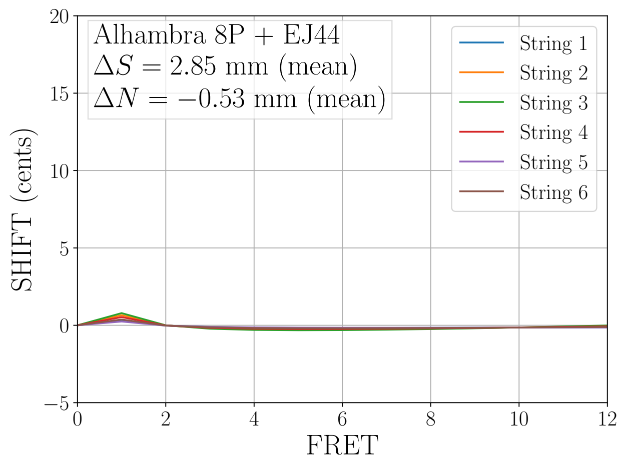
String	ΔS (mm)	ΔN (mm)
J4401	1.87	-0.35
J4402	3.34	-0.71
J4403	4.33	-0.84
J4404	2.73	-0.58
J4405	1.94	-0.29
J4406	2.91	-0.42



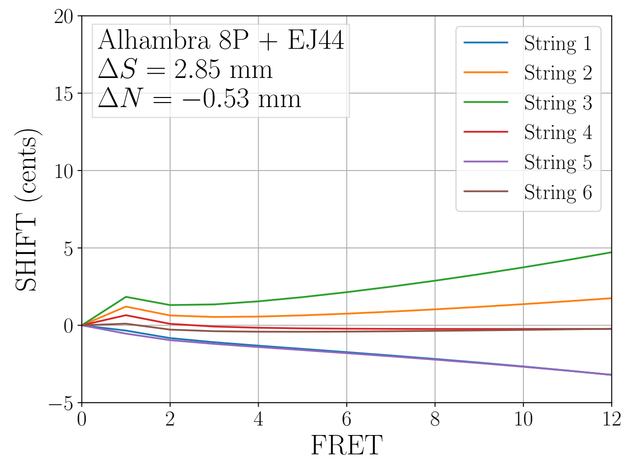
(a) Uncompensated



(b) Factory guitar



(c) Full compensation



(d) Mean compensation

Figure 7: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). Four different strategies of saddle and nut compensation are illustrated.

References

[1] G. Byers, Guitars of Gregory Byers: Intonation (2020). See http://byersguitars.com/intonation and http://www.byersguitars.com/Research/Research.html.

[2] G. Byers, "Classical Guitar Intonation," American Lutherie **47**, 368 (1996).

[3] G. U. Varieschi and C. M. Gower, "Intonation and Compensation of Fretted String Instruments," Amer. J. Phys. **78**, 47 (2010).

[4] P. M. Morse, *Vibration and Sound* (Acoustical Society of America, New York, 1981).

[5] P. M. Morse, *Vibration and Sound*, pp. 84–85, in [4] (1981).

[6] P. M. Morse, *Vibration and Sound*, pp. 166–170, in [4] (1981).

[7] N. Lynch-Aird and J. Woodhouse, "Mechanical Properties of Nylon Harp Strings," Materials **10**, 497 (2017).

[8] D. S. Durfee and J. S. Colton, "The physics of musical scales: Theory and experiment," Amer. J. Phys. **83**, 835 (2015).

[9] J. Larsson, ProGuitar Tuner (2020). See https://www.proguitar.com/guitar-tuner.

[10] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity*, *Course of Theoretical Physics*, vol. 7 (Butterworth Heinemann, Oxford, 1986), 3rd edn.

[11] H. Fletcher, "Normal Vibration Frequencies of a Stiff Piano String," J. Acoust. Soc. Am. **36**, 203 (1964).