Classical Guitar Intonation and Compensation: The Well-Tempered Guitar

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Abstract

Inspired by the pioneering work of luthier Greg Byers in 1996, we build an intuitive model of classical guitar intonation that includes the effects of the resonant length of the fretted string, linear mass density, tension, and bending stiffness. We begin by deriving an expression for the vibration frequencies of a stiff string using boundary conditions that are pinned at the saddle but clamped at the fret. Adopting logarithmic frequency differences based on "cents" that decouple these physical effects, we introduce Taylor series expansions that exhibit clearly the origins of frequency shifts of fretted notes from the corresponding Twelve-Tone Equal Temperament (12-TET) values. We demonstrate a simple *in situ* technique for measurement of the changes in frequency of open strings arising from small adjustments in length, and we propose a simple procedure that any interested guitarist can use to estimate the corresponding shifts in frequency due to tension and bending stiffness for their own guitars and favorite string sets. Based on these results, we employ an RMS frequency error method to select values of saddle and nut setbacks that map fretted frequencies — for a particular string set on a particular guitar — almost perfectly onto their 12-TET targets. This exercise allow us to discuss a general approach to tempering an "off-the-shelf" guitar to further reduce the tonal errors inherent in any fretted musical instrument.

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1 Introduction and Background

Any musician who has wrestled with the temperament of a fretted stringed instrument is well aware of the challenges presented by tuning and pitch. In addition to the mathematical physics of musical scales [1], the mechanical specifications of the instrument and the strings themselves [2, 3] require accommodation during both manufacturing [4, 5] and tuning to achieve harmonious results. We can gain an appreciation for this problem by analyzing the expression for the allowed vibration frequencies of an ideal string, given by [6, 7]

$$f_q = \frac{q}{2\,L_0}\sqrt{\frac{T_0}{\mu_0}}\,, \tag{1}$$

where $q \in \mathbb{N} = \{1, 2, \dots\}$ identifies the "harmonic" of the fundamental frequency f_1, L_0 is the length of the free (unfretted) string from the saddle to the nut, T_0 is the tension in the free string, and $\mu_0 \equiv M/L_0$ is the linear mass density of a free string of mass M. The act of fretting the string changes its length, and therefore its frequency. For example, modern classical guitars are manufactured with frets placed along the fretboard using the Twelve-Tone Equal Temperament (12-TET) system, whereby the resonant length of a string pressed behind fret n ideally should be $2^{-n/12}\,L_0$, thereby producing a note with frequency $2^{n/12}\,f_1$. But this result can never be achieved perfectly in reality.

First, the string is elevated above the frets by the saddle and nut, so the fretted string is slightly elongated relative to the free string, and the resulting frequency is flattened in pitch. In principle, this effect could be accommodated by minute changes in the positions of the frets, but there are additional practical complications. For example, the string's tension is increased by the change in length, causing the frequency to sharpen by an amount that significantly exceeds the reduction caused by the increase in the resonant string length. In addition, the string is by no means ideal, and its intrinsic stiffness results in an additional increase in pitch that depends on its mechanical characteristics. These guitar intonation difficulties seem to preclude successful temperament, but remarkably the instrument can be *compensated* by moving the positions of the saddle and the nut by small distances during the manufacturing process [4, 5]. This compensation process helps temper the guitar so that it is *playable*. It must also be *tunable*, so that the guitar strings can be brought into compliance with 12-TET quickly and accurately. This requirement places significant constraints on the physical properties of manufactured strings. Our goal in this work is to build an intuitive understanding of these effects to aid in the compensation and subsequent tuning of the classical guitar.

Throughout this work, we will use *cents* to describe small differences in pitch [5, 1]. One cent is one one-hundredth of a 12-TET half-step, so that there are 1200 cents per octave. An experienced guitar player can distinguish beat notes with a difference frequency of $\Delta f \approx 1$ Hz, which corresponds to 8 cents at A_3 ($f = 220$ Hz) or 5 cents at E_4 ($f = 329.63$ Hz). Using this approach, the difference in pitch between two frequencies f_1 and f_2 is defined as

$$\Delta\nu \equiv 1200 \log_2 \left(\frac{f_2}{f_1} \right) . \tag{2}$$

Let's choose the average frequency $f \equiv (f_1 + f_2)/2$ and the frequency difference $\Delta f \equiv f_2 - f_1$. Then

$$\Delta\nu = 1200 \log_2 \left(\frac{f + \Delta f/2}{f - \Delta f/2} \right) \approx \frac{1200}{\ln 2} \frac{\Delta f}{f} , \tag{3}$$

where the last approximation applies when $\Delta f \ll f$. As shown in Fig. 1, if the average frequency of the interval is used to compute $\Delta\nu$ — rather than the initial frequency f_1 — then the accuracy of Eq. (3) holds for almost an entire octave. In this plot, we chose $f_1 = A_3 = 220$ Hz, and allowed f_2 to vary from A_3 to $A_4 + 30$ Hz $= 450$ Hz. At the octave, the error in $\Delta\nu$ arising from Eq. (3) is only -46 cents, or -4%. The choice of a logarithmic frequency-difference scale *decouples* multiplicative factors that predict the frequency of a real fretted string, allowing us to build straightforward intuitive models of guitar intonation and compensation.

We present the basics of our model of classical guitar strings in Section 2, following the pioneering work of G. Byers [4, 8]. We offer an empirical reason to doubt the need for a complicated model of string fretting, and we explore this argument in greater detail in Appendix A. In Appendix B, we derive a new expression for the allowed vibration frequencies of a stiff string, derived under the assumption that the boundary conditions at the saddle and the nut are not symmetric. Based on this result, we then discuss in detail the four contributions to frequency shifts and errors of non-ideal strings pressed behind a fret: the change in the resonant length of the vibrating string; a decrease in the linear mass density of the entire string; an increase in the tension of the entire string; and an increase in the mechanical stiffness of the resonating string. Our goal is to simplify the decoupled equations describing these effects through Taylor series expansions to allow an intuitive picture of the string's behavior to emerge. Nylon strings behave very differently than the metal strings used on acoustic guitars [9]. They require time to "settle" and reach equilibrium after restringing, and it is unlikely that the uniform stiff rod approach used to develop equations for the resonant frequencies (including ours) apply to either the monofilament treble strings or the wound bass strings. Nevertheless, we are able to develop a phenomenological model of the mechanical characteristics of these strings that is consistent with measurements of frequency deviations on uncompensated classical guitars.

In Section 3, we suggest a simple experiment to estimate the response of the string's tension to the change in length caused by fretting, and we demonstrate the idea using a normal-tension string set on an Alhambra 8P guitar (as well as other string sets in Appendix D). Then, in Section 4, we use these estimates to demonstrate a straightforward analytic approach to compensating the errors in a guitar string, relying on a method — described in Appendix C

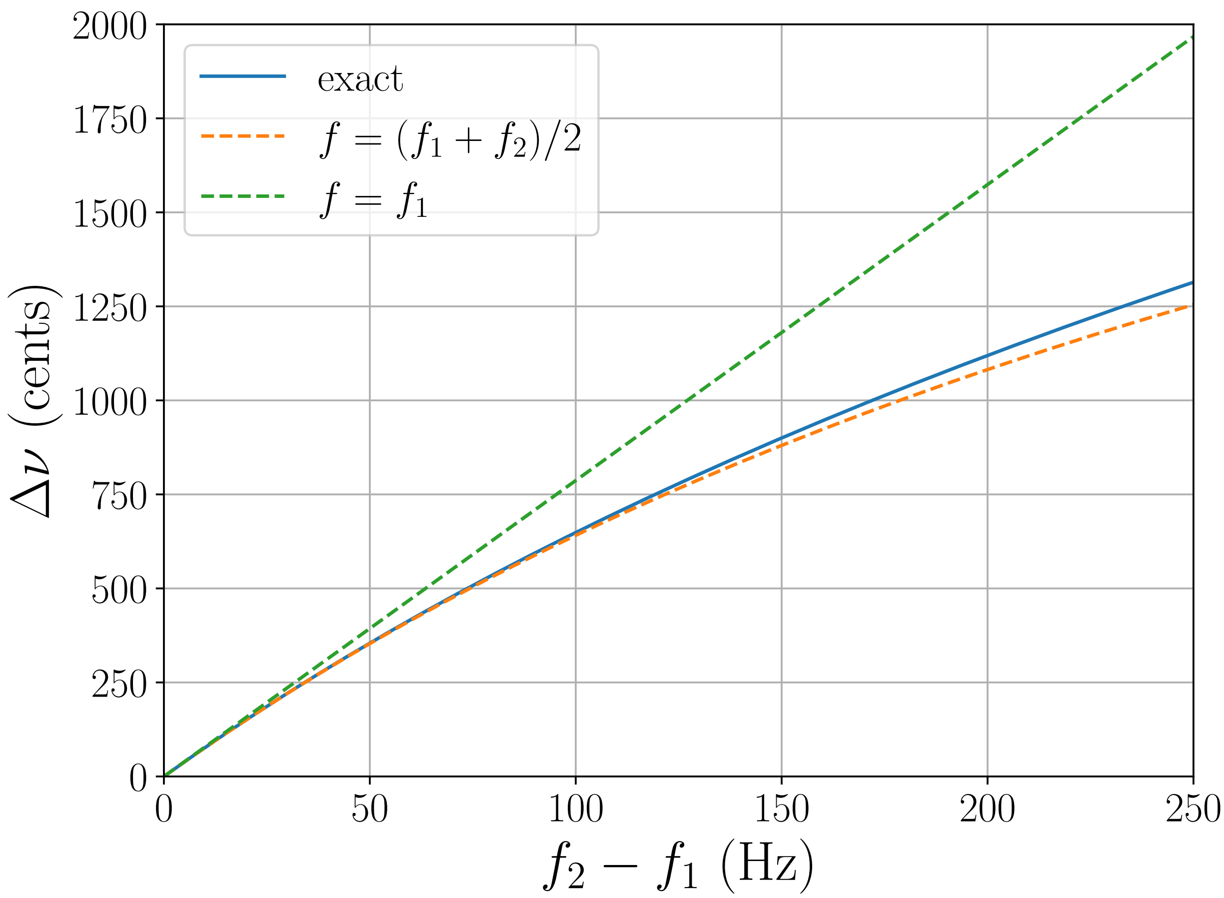


Figure 1: Plot of $\Delta\nu$ for $f_1 = A_3 = 220$ Hz and f_2 varying from A_3 to $A_4 + 30$ Hz $= 450$ Hz. We compare two different definitions of f in Eq. (3): the average of f_1 and f_2, and simply $f = f_1$. Using the average frequency leads to a significantly better approximation.

— to minimize the root-mean-squared (RMS) frequency deviation at each fret. Finally, in Section 5 we discuss a collaboration of guitar manufacturer and musician to temper the guitar using harmonic tuning and optimize it for a particular piece.

This document – as well as the Python computer code needed to reproduce the figures – is available at GitHub [10].

2 Simple Model of Guitar Intonation

The starting point for prior efforts to understand guitar intonation and compensation [4, 5] is a formula for f_q, the transverse vibration frequency harmonic q of a stiff string, originally published by Morse in 1936 [11, 12, 13]:

$$f_q = \frac{q}{2\,L}\sqrt{\frac{T}{\mu}}\left[1 + 2B + 4\left(1 + \frac{\pi^2\,q^2}{8}\right)B^2\right]. \tag{4}$$

Here L is the length of the string, T and μ are its tension and linear mass density, respectively, and B is a small "bending stiffness" coefficient to capture the relevant mechanical properties of the string. For a homogeneous string with a cylindrical cross-section, B is given by

$$B \equiv \sqrt{\frac{\pi\,\rho^4\,E}{4\,T\,L^2}}\,, \tag{5}$$

where ρ is the radius of the string and E is Young's modulus (or the modulus of elasticity). But it's unlikely that Eq. (4) accurately describes the resonant frequencies of a nylon string on a classical guitar, because it assumes that the string is "clamped" at both ends, so that a particular set of symmetric boundary conditions must be applied to the partial differential equation (PDE) describing transverse vibrations of the string. We believe that this assumption is correct for the end of the string held at either the nut or the fret, but that the string is "pinned" (and not clamped) at the saddle. In Appendix B, we solve the PDE using these non-symmetric boundary conditions, and find

$$f_q = \frac{q}{2\,L}\sqrt{\frac{T}{\mu}}\left[1 + B + \left(1 + \frac{1}{2}q^2\pi^2\right)B^2\right]. \tag{6}$$

Note that this expression is valid only when $B \ll 1$. For a typical nylon guitar string with $E \approx 5$ GPa, $T \approx 60$ N, $\rho \approx 0.5$ mm, and $L \approx 650$ mm, we have $B \approx 3 \times 10^{-3}$. (In this case, the quadratic B term in Eq. (6) is only 2% as large as the linear term, and can generally be neglected. We will include it in our analysis below only for completeness.) We should use Eq. (6) with some caution, because the chemistry, materials science, and physics of nylon strings (particularly the wound bass strings) are quite complicated [9].

Our model is based on the schematic of the guitar shown in Fig. 2. The scale length of the guitar is X_0, but we allow the edges of both the saddle and the nut to be set back an additional distance ΔS and ΔN, respectively. The location on the x-axis of the center of the n^{th} fret is X_n. In the y direction, $y = 0$ is taken as the surface of the fingerboard; the height of each fret is a, the height of the nut is $a + b$, and the height of the saddle is $a + b + c$. (For the time being, we are neglecting the art of *relief* practiced by expert luthiers that adjusts the value of b up the fretboard and strings.) L_n is the *resonant length* of the string from the saddle to the center of fret n, and L'_n is the length of the string from the fret to the nut. The total

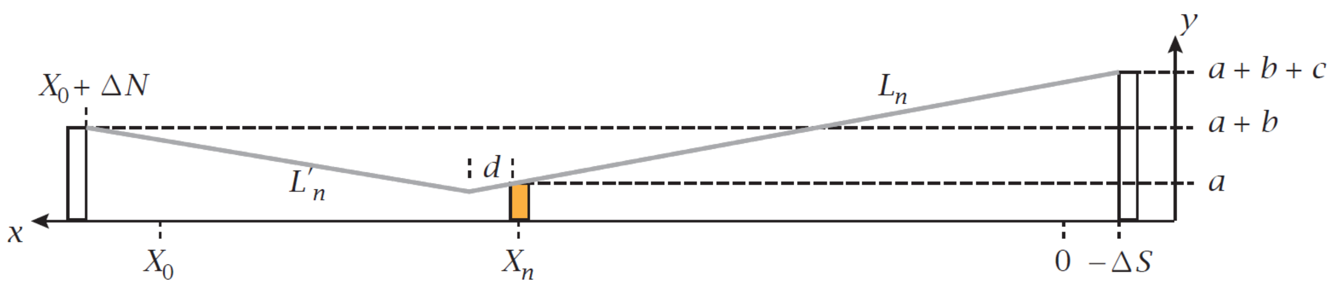


Figure 2: A simple (side-view) schematic of the classical guitar used in this model. The scale length of the guitar is X_0, but we allow the edges of both the saddle and the nut to be set back an additional distance ΔS and ΔN, respectively. The location on the x-axis of the center of the n^{th} fret is X_n. (Note that the x-axis is directed toward the left in this figure.) In the y direction, $y = 0$ is taken as the surface of the fingerboard; therefore the height of each fret above the fingerboard is a, the height of the nut is $a + b$, and the height of the saddle is $a + b + c$. L_n is the *resonant length* of the string from the saddle to the center of fret n, and L'_n is the length of the string from the fret to the nut. We have included a line-segment intersection at a distance d behind fret n to represent the slight increase in the distance L'_n caused by a finger.

length of the string is defined as $\mathcal{L}_n \equiv L_n + L'_n$. As discussed in more detail in Appendix A, we have chosen to include a line-segment intersection at a distance d behind fret n to represent the slight increase in the distance L'_n caused by a finger. This differs from previous studies of guitar intonation and compensation [4, 5], but our approach is consistent with empirical observations for nylon strings.

We start with the form of the fundamental frequency of a fretted string given by Eq. (6) with $q = 1$, and apply it to the frequency of a string pressed just behind the n^{th} fret:

$$f_n = \frac{1}{2\,L_n}\sqrt{\frac{T_n}{\mu_n}}\left[1 + B_n + \left(1 + \frac{\pi^2}{2}\right)B_n^2\right], \tag{7}$$

where T_n and μ_n are the modified tension and the linear mass density of the fretted string, and

$$B_n \equiv \sqrt{\frac{\pi\,\rho^4\,E}{4\,T_n\,L_n^2}}\,. \tag{8}$$

We note that T_n and μ_n depend on $\mathcal{L}_n$, the *total* length of the fretted string from the saddle to the nut. Ideally, in the 12-TET system [1],

$$f_n = \gamma_n\,f_0\,, \qquad \text{(12-TET ideal)} \tag{9}$$

where f_0 is the frequency of the open (unfretted) string, and

$$\gamma_n \equiv 2^{n/12}\,. \tag{10}$$

Therefore, the error interval — the difference between the fundamental frequency of the fretted string and the corresponding perfect 12-TET frequency — expressed in cents is given by

$$\begin{aligned}\Delta\nu_n &= 1200\,\log_2\left(\frac{f_n}{\gamma_n\,f_0}\right)\\ &= 1200\,\log_2\left(\frac{L_0}{\gamma_n\,L_n}\right) + 600\,\log_2\left(\frac{\mu_0}{\mu_n}\right) + 600\,\log_2\left(\frac{T_n}{T_0}\right)\\ &\quad + 1200\,\log_2\left[\frac{1+B_n+(1+\pi^2/2)\,B_n^2}{1+B_0+(1+\pi^2/2)\,B_0^2}\right],\end{aligned} \tag{11}$$

where $\log_2$ is the (binary) logarithm function calculated with base 2.

The final form of Eq. (11) makes it clear that — for nylon guitar strings — there are four contributions to intonation:

1. *Resonant Length*: The first term represents the error caused by the increase in the length of the fretted string L_n compared to the ideal length X_n, which would be obtained if $b = c = 0$ and $\Delta S = \Delta N = 0$.

2. *Linear Mass Density*: The second term is the error caused by the reduction of the linear mass density of the fretted string. This effect will depend on the *total* length of the string $\mathcal{L}_n = L_n + L'_n$.

3. *Tension*: The third term is the error caused by the *increase* of the tension in the string arising from the stress and strain applied to the string by fretting. This effect will also depend on the total length of the string $\mathcal{L}_n$.

4. *Bending Stiffness*: The fourth and final term is the error caused by the change in the bending stiffness coefficient arising from the decrease in the vibrating length of the string from L_0 to L_n.

Note that the properties of the logarithm function have *decoupled* these physical effects by converting multiplication into addition. We will discuss each of these sources of error in turn below.

2.1 Resonant Length

The length L_0 of the open (unfretted) guitar string can be calculated quickly by referring to Fig. 2. We find:

$$L_0 = \sqrt{(X_0+\Delta S+\Delta N)^2 + c^2} \approx X_0 + \Delta S + \Delta N + \frac{c^2}{2\,X_0}, \tag{12}$$

where the approximation arises from the Taylor series that applies since $c^2 \ll X_0^2$. Similarly, since $(b+c)^2 \ll X_0^2$, the resonant length L_n is given by

$$L_n = \sqrt{(X_n+\Delta S)^2 + (b+c)^2} \approx X_n + \Delta S + \frac{(b+c)^2}{2\,X_n}. \tag{13}$$

Then — if the guitar has been manufactured such that $X_n = X_0/\gamma_n$ — the resonant length error determined by the first term in the last line of Eq. (11) is approximately

$$1200\,\log_2\left(\frac{L_0}{\gamma_n\,L_n}\right) \approx \frac{1200}{\ln(2)}\left[\frac{\Delta N - (\gamma_n - 1)\,\Delta S}{X_0} - \frac{\gamma_n^2(b+c)^2 - c^2}{2\,X_0^2}\right], \tag{14}$$

where ln is the natural logarithm function. If the guitar is uncompensated, so that $\Delta S = \Delta N = 0$, the magnitude of this error is typically less than 0.25 cents, and can be neglected. However, with $\Delta S > 0$ and $\Delta N < 0$, we can significantly flatten the frequency shift — for example, with $\Delta S = 1.75$ mm and $\Delta N = -0.35$ mm, the contribution of this term to the shift at the 12th fret is -6 cents. We'll see that this is our primary method of compensation.

2.2 Linear Mass Density

As discussed above, the linear mass density μ_0 of an open (unfretted) string is simply the total mass M of the string clamped between the saddle and the nut divided by the length L_0. Similarly, the mass density μ_n of a string held onto fret N is $M/\mathcal{L}_n$. Therefore

$$\frac{\mu_0}{\mu_n} = \frac{\mathcal{L}_n}{L_0} \equiv 1 + Q_n \, , \tag{15}$$

where we have followed Byers and defined [4, 5]

$$Q_n \equiv \frac{\mathcal{L}_n - L_0}{L_0} \, . \tag{16}$$

Since we expect that $Q_n \ll 1$, we can approximate the second term in the final line of Eq. (11) as

$$600 \log_2 \left(\frac{\mu_0}{\mu_n} \right) \approx \frac{600}{\ln(2)} \, Q_n \, . \tag{17}$$

Referring to Fig. 2, we see that $\mathcal{L}_n = L_n + L'_n$, and after judicious use of similar triangles and the Pythagorean Theorem we calculate L'_n for $n \geq 1$ as

$$\begin{aligned} L'_n &= \frac{L_n}{X_n + \Delta S} \, d + \sqrt{(X_0 - X_n + \Delta N - d)^2 + \left(b + \frac{b + c}{X_n + \Delta S} \, d \right)^2} \\ &\approx X_0 - X_n + \Delta N + \frac{b^2}{2\,(X_0 - X_n)} + \frac{[(b + c)\, X_0 - c\, X_n]^2}{2(X_0 - X_n)^2 X_n^2} \, d \end{aligned} \tag{18}$$

to first order in d/X_0. Therefore

$$\begin{aligned} Q_n &= \frac{L_n + L'_n - L_0}{L_0} \\ &\approx \frac{[\gamma_n \, b + (\gamma_n - 1)\, c]^2}{2\,(\gamma_n - 1)\, X_0^2} \left[1 + \gamma_n \sum_{k=1}^{\infty} \left(\frac{\gamma_n}{\gamma_n - 1} \, \frac{d}{X_0} \right)^k \right] , \end{aligned} \tag{19}$$

where we have retained b and c to second order, and (in practice) we'll need to keep d to second order as well so that we retain accuracy at the first fret. In Fig. 3, we plot a comparison between the exact expression for the normalized displacement Q_n given by Eq. (16) with the approximate expression given by Eq. (19). Here the guitar has $b = 1.0$ mm, $c = 4.0$ mm, $\Delta S = 1.75$ mm, $\Delta N = -0.35$ mm, and $X_0 = 650$ mm.

Although it is arguable whether the approximation given by Eq. (19) is simpler than the exact expression given by Eq. (16), it is quite clear from both Eq. (19) and Fig. 3 that Q_n *does not depend significantly on the setbacks* ΔS or ΔN. For the same parameters, when $d = 10$ mm, $\Delta \nu_1 \approx 0.04$ cents, and is smaller at all other frets. In general, the shift due to linear mass density can be neglected without significant loss of accuracy.

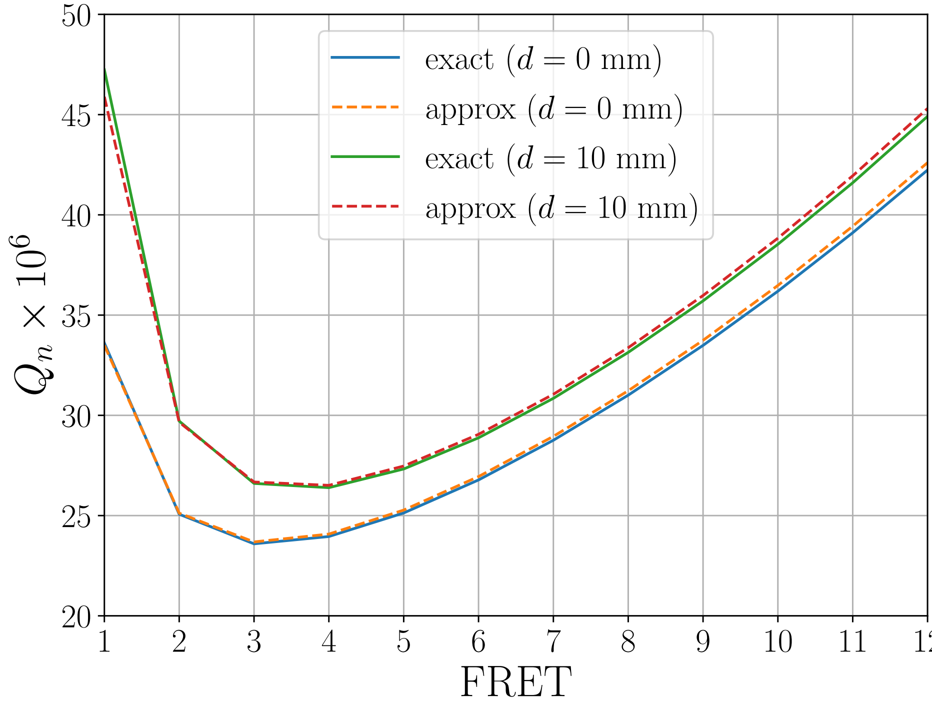


Figure 3: Comparison of the exact expression for the normalized displacement Q_n as a function of the fret number given by Eq. (16) with the approximate expression given by Eq. (19). For example, at the first fret with $d = 10$ mm, $Q_n \approx 45 \times 10^{-6}$. Here the guitar has $b = 1.0$ mm, $c = 4.0$ mm, $\Delta S = 1.75$ mm, $\Delta N = -0.25$ mm, and a scale length of 650 mm.

2.3 Tension

We've seen above that, as a guitar string is fretted its length increases, thereby increasing its tension through linear elastic deformation [14]. We will focus on the response of the string to a longitudinal strain, and neglect the transverse stress that causes negligible changes in the diameter of the string [9]. In this case, we can write the change in tension of a string experiencing an infinitesimal change in length from L to $\mathcal{L}$ as

$$\Delta T = \pi\,\rho^2\,E\,\frac{\mathcal{L} - L}{L}\,. \tag{20}$$

Therefore, the tension in a string clamped to fret n is

$$T_n = T_0 + \Delta T_n = T_0\,(1 + \kappa\,Q_n)\ , \tag{21}$$

where we have used Eq. (16) and defined the dimensionless "string constant"

$$\kappa \equiv \frac{\pi\rho^2 E}{T_0}\,. \tag{22}$$

If we assume that $\kappa\,Q_n \ll 1$, then we can approximate the third term in the final line of Eq. (11) as

$$600\,\log_2\left(\frac{T_n}{T_0}\right) \approx \frac{600}{\ln(2)}\,\kappa\,Q_n\,. \tag{23}$$

This frequency shift is larger than that caused by the linear mass density error by a factor of κ.

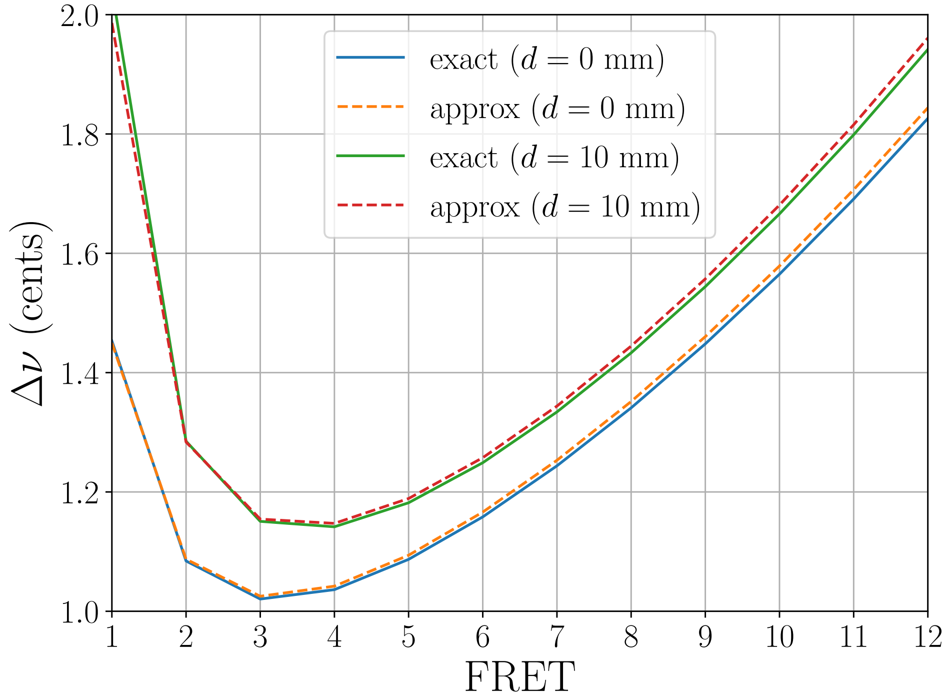


Figure 4: Comparison of the exact expression for the frequency shift due to tension increases as a function of the fret number given by the left-hand side of Eq. (23) with the approximate expression given by the right-hand side. Here the guitar has $b = 1.0$ mm, $c = 4.0$ mm, $\Delta S = 1.75$ mm, $\Delta N = -0.35$ mm, and a scale length of 650 mm.

In Fig. 4, we compare the exact expression for the frequency shift due to tension increases as a function of the fret number given by the left-hand side of Eq. (23) with the approximate expression given by the right-hand side . The exact curves for both $d = 0$ mm and $d = 10$ mm used exact expressions for $\mathcal{L}_n$ and L_0, while the approximate expressions relied only on Eq. (19). Here the guitar has the same parameters as in Fig. 3.

2.4 Bending Stiffness

The bending stiffness of a string clamped at the n^{th} fret is given by Eq. (8), Eq. (13), and Eq. (21) as

$$B_n = \sqrt{\frac{\pi\,\rho^4\,E}{4\,T_n\,L_n^2}} = \sqrt{1 + \kappa\,Q_n}\,\frac{L_0}{L_n}\,\sqrt{\frac{\pi\,\rho^4\,E}{4\,T_0\,L_0^2}} \approx \gamma_n\,B_0\,, \tag{24}$$

where the approximation applies when $B_0 \ll 1$ and the largest contribution arises from the shortened length of the fretted string compared to that of the open string. Therefore, the fourth term in the final line of Eq. (11) can be approximated as

$$1200\,\log_2\left[\frac{1 + B_n + (1 + \pi^2/2)\,B_n^2}{1 + B_0 + (1 + \pi^2/2)\,B_0^2}\right] \approx \frac{1200}{\ln(2)}\left[(\gamma_n - 1)\,B_0 + \frac{1}{2}\,(\gamma_n^2 - 1)\,(1 + \pi^2)\,B_0^2\right]. \tag{25}$$

In Fig. 5, we use Eq. (24) and Eq. (25) to compare the exact and approximate expressions for bending stiffness and the corresponding frequency shift. Note that we compare the approximate frequencies with and without the quadratic terms, and we see that the 2^{nd}-order contribution is about 0.2 cents at the 12^{th} fret. Once again, it is clear that B_n does not depend significantly on either ΔS or ΔN even when $\kappa = 100$. In other words, the bending stiffness error does not depend on the tiny changes to the linear mass density or the tension that arises due to string fretting. Instead, it is an intrinsic mechanical property of the string.

2.5 Total Frequency Shift

Incorporating all of these effects — and neglecting the term proportional to B_0^2 in Eq. (25) — we find that the total frequency shift is given approximately by

$$\Delta\nu_n \approx \frac{1200}{\ln(2)} \left[(\gamma_n - 1) \left(B_0 - \frac{\Delta S}{X_0} \right) + \frac{\Delta N}{X_0} + \frac{1}{2} \kappa\, Q_n \right] . \tag{26}$$

How do we determine the bending stiffness B_0 given by Eq. (5) and the spring constant κ given by Eq. (22)? It's impractical to assume that the modulus of elasticity of a particular string can be derived from published values of bulk nylon (particularly in the case of a wound string). Instead, let's assume that we know the value of κ, and then estimate the bending stiffness coefficient by comparing Eq. (22) and Eq. (5), and writing B_0 as

$$B_0 = \sqrt{\kappa}\, \frac{\rho}{2\, L_0} . \tag{27}$$

To measure κ, in Section 3 we will conduct an experiment that measures the change in the frequency of an open string — pinned at one end, and clamped at the other — as we make slight changes to its length [4, 5]. From Eq. (6), the derivative of the fundamental frequency of an open string is

$$\begin{aligned} \frac{d\,f}{d\,L} &= \frac{f}{L} \left(-1 + \frac{L}{2\,T} \frac{d\,T}{d\,L} - \frac{L}{2\,\mu} \frac{d\,\mu}{d\,L} + \frac{L}{1+B} \frac{d\,B}{d\,L} \right) \\ &= \frac{f}{L} \left(-1 + \frac{1}{2} \kappa + \frac{1}{2} - \frac{B}{1+B} \right) \\ &\approx \frac{f}{L} \times \frac{1}{2} (\kappa - 1) , \end{aligned} \tag{28}$$

where we have used the analyses above to determine that

$$\frac{d\,T}{d\,L} = \frac{T}{L} \kappa , \tag{29a}$$

$$\frac{d\,\mu}{d\,L} = -\frac{\mu}{L} , \text{ and} \tag{29b}$$

$$\frac{d\,B}{d\,L} = -\frac{B}{L} , \tag{29c}$$

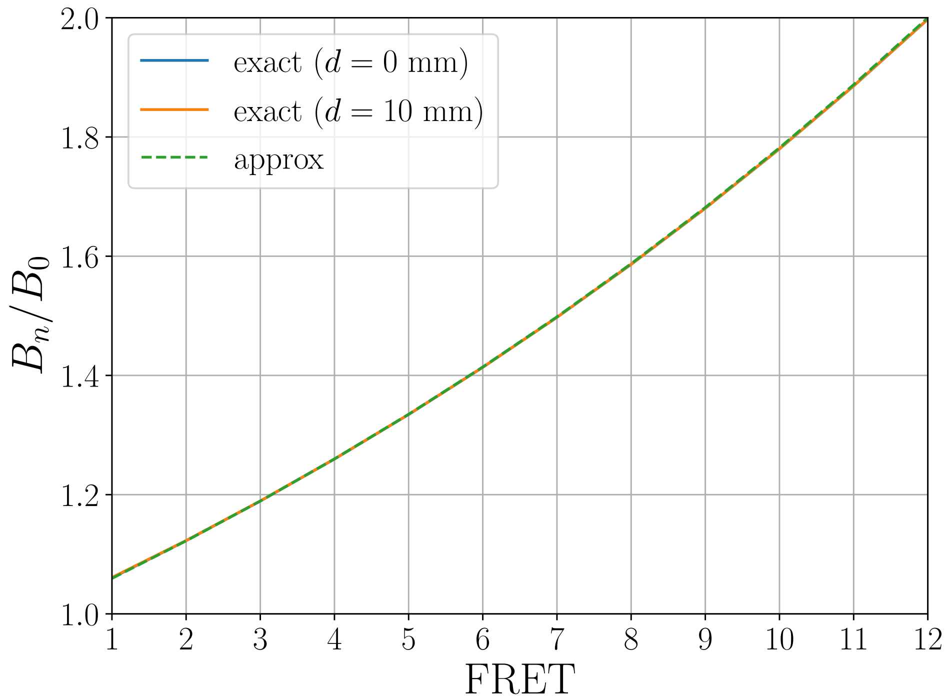
$$\tag{29d}$$

and we have again assumed that $B_0 \ll 1$. Therefore, following Byers [4, 5], we define the parameter R to be

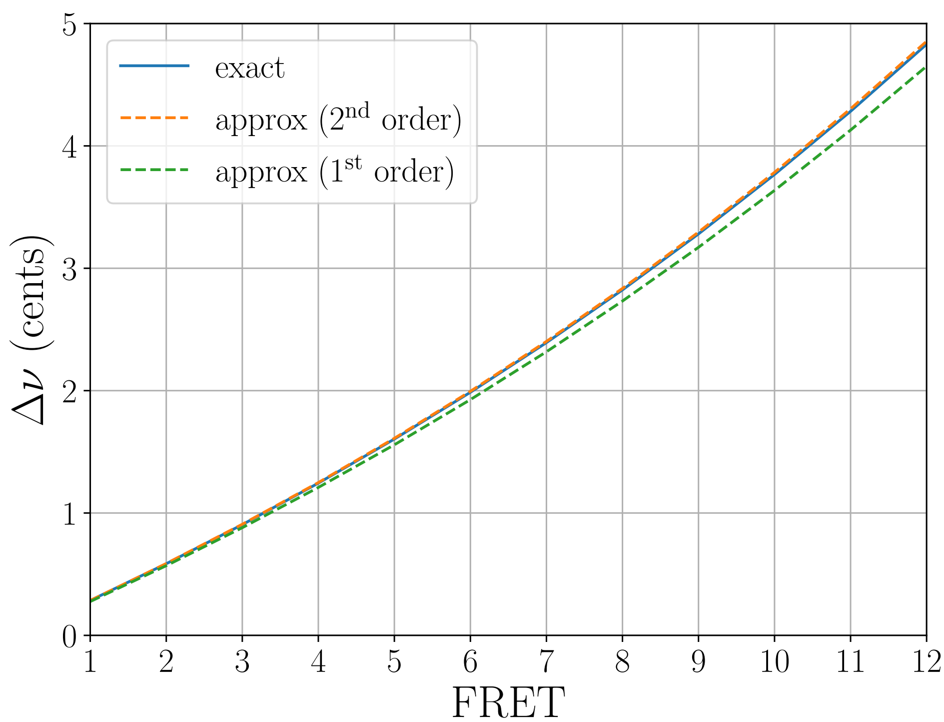
$$R \equiv \frac{d\,\ln(f)}{d\,\ln(L)} = \frac{L}{f} \frac{d\,f}{d\,L} = \frac{1}{2} (\kappa - 1) , \tag{30}$$

which gives

$$\kappa = 2\,R + 1 . \tag{31}$$



(a) Bending stiffness



(b) Frequency shifts due to stiffness

Figure 5: A comparison of exact and approximate expressions for (a) bending stiffness, given by Eq. (24), and (b) the corresponding frequency shift, given by Eq. (25). Here the guitar has $b = 1.0$ mm, $c = 4.0$ mm, $\Delta S = 1.75$ mm, $\Delta N = -0.35$ mm, $\kappa = 100$, and a scale length of 650 mm.

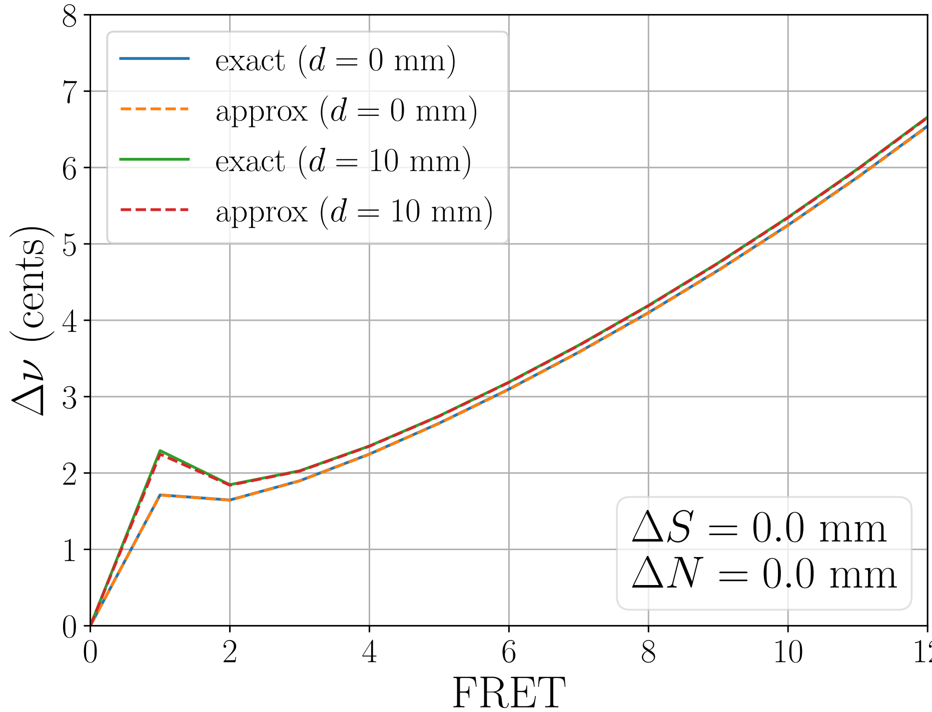


Figure 6: The total frequency shift given by Eq. (11) for a classical guitar with a scale length of 650 mm, $b = 1.0$ mm, $c = 4.0$ mm, $\Delta S = \Delta N = 0$ mm, and a string with $R = 24$ and a radius $\rho = 0.5$ mm.

We can estimate the typical value of R for nylon classical guitar strings through a simple observation. On a classical guitar with a scale length of 650 mm, we can usually tune an open string a full step by winding the tuner/machine head three half turns. As we shall see below, this increases or decreases the string length above the first fret by about 3 mm, where the corresponding length of the open string from the saddle to the first fret is about 614 mm. Since a full step is (by definition) 200 cents, Eq. (3) tells us that

$$\frac{\Delta f}{f} \approx \frac{\ln(2)}{1200}\,\Delta\nu = \frac{200}{1731} = 0.116\,. \tag{32}$$

In this case, we estimate R to be

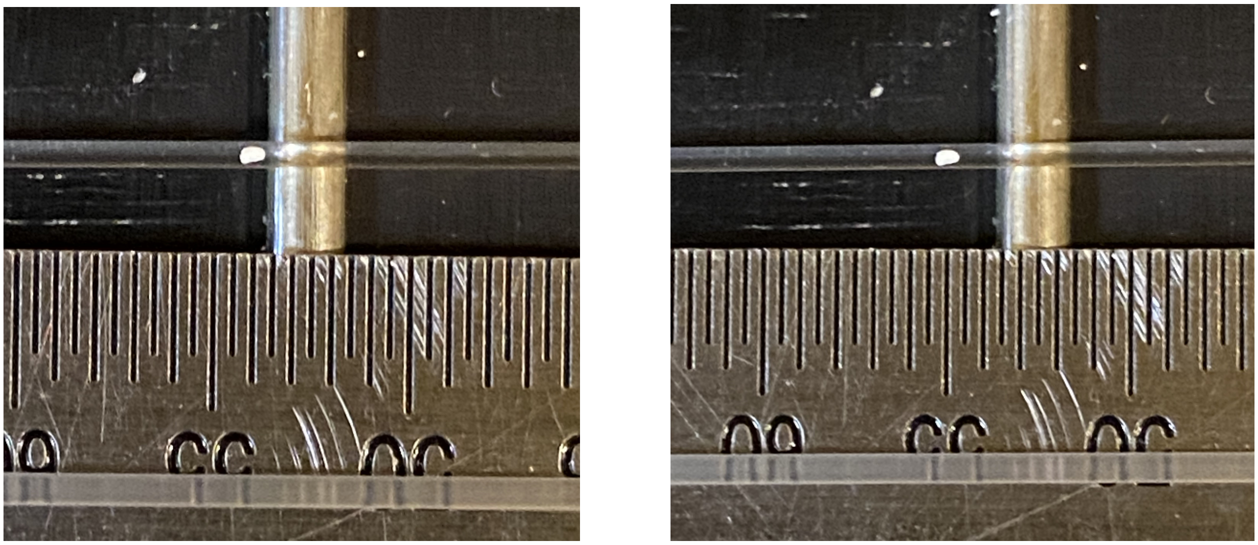
$$R \approx \frac{614}{3}\,0.116 = 24\,, \tag{33}$$

giving $\kappa \approx 49$ and $\sqrt{\kappa} \approx 7$. In Fig. 6, we plot the total frequency shift given by Eq. (11) for a classical guitar with a scale length of 650 mm, $b = 1.0$ mm, $c = 4.0$ mm, $\Delta S = \Delta N = 0$ mm, and a string with $R = 24$ and a radius $\rho = 0.5$ mm. The corresponding linear string constant and bending stiffness coefficient are $\kappa = 49$ and $B_0 = 0.00268$, respectively. Note that the string is sharp at every fret, but a nonzero value of d is only important at the first fret.

Equation (26) provides us with a clue on how to modify this guitar to improve the tone of this string. We see that the bending stiffness and the increase in string tension due to fretting sharpen the pitch, but that we can reduce these effects with a positive saddle setback and negative nut setback. In fact, a simple compensation strategy would be to choose

$$\Delta S = B_0\,X_0\,,\ \text{and} \tag{34a}$$

$$\Delta N = -\kappa\,X_0\,\overline{Q}/2\,, \tag{34b}$$



(a) Δx_1 (b) Δx_2

Figure 7: Two examples of displacement measurements of a small deposit of white correction fluid relative to a D'Addario string-depth gauge marked in half-millimeter increments.

Table 1: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 650 mm.

String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4501	E_4	0.356	3.74	68.6
J4502	B_3	0.409	5.05	52.0
J4503	G_3	0.512	8.36	54.3
J4504	D_3	0.368	19.21	70.0
J4505	A_2	0.445	32.90	67.3
J4506	E_2	0.546	54.72	62.8

where $\overline{Q}$ is the relative displacement averaged over a particular set of frets, to offset string stiffness and tension, respectively. If we select the first twelve frets, then for $d = 0$ mm

$$\overline{Q} \approx \frac{77.9\, b^2 + 35.6\, b\, c + 5.82\, c^2}{24\, X_0^2}\,. \tag{35}$$

In the case of the parameters used in Fig. 6, we use this simple approximate approach to estimate $\Delta S = 1.75$ mm, and $\Delta N = -0.49$ mm. We'll rely on a more accurate method for compensation in Section 4, but we'll obtain results that are similar to those found using this basic approach.

3 Experimental Estimate of the String Constant

It is relatively easy to measure *in situ* the value of R (and therefore κ) for any guitar string with the aid of a device that can measure frequency [15], a simple ruler with fine markings (e.g., a string depth gauge), a camera with a macro mode, and white correction fluid. For example, in

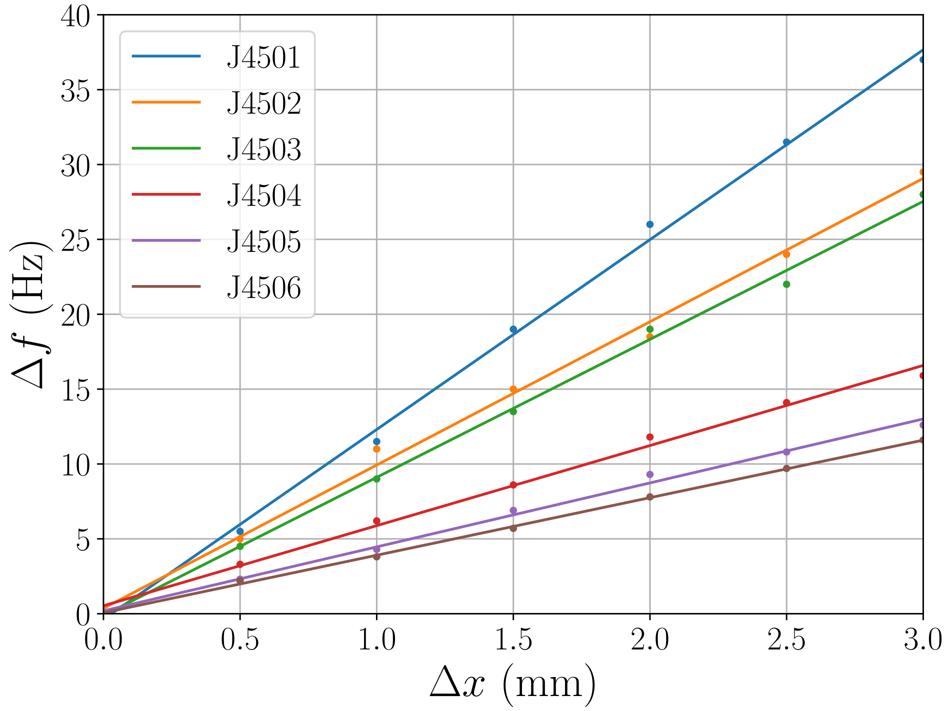


Figure 8: Results of experiments to measure R for each string in the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45) set. The points represent the measurement data, while the lines are the results of linear least-squares fits to that data.

Fig. 7 we show photographs of the nylon normal-tension first string on an Alhambra 8P classical guitar. By depositing a small sample of white correction fluid on the string, we can measure small displacements within 0.25 mm against a string-depth gauge marked in half-millimeter increments. Then we can pluck the open string and measure its vibration frequency. We found that significantly stretching a string that had settled into equilibrium could result in a nonlinear reduction in $\Delta f/\Delta L$, so prior to our measurements we tuned each string down one whole step. The string stretches uniformly along its length, so at any position x the relative displacement $\Delta x/x$ should be invariant. We chose to work near the first fret, which is located about 614 mm from the saddle, and we typically made seven measurements of displacement over a 3 mm range, as well as the corresponding frequencies.

For example, we began with a normal-tension nylon classical string set with the specifications listed in Table 1 using metric units.[1] In Fig. 8, we plot our measurements of Δf as a function of the displacement Δx relative to the frequency of the string when $\Delta x = 0$. We then performed a least-squares fit to a straight line [16] (also shown in Fig. 8), determined the derivative $\Delta f/\Delta L$, and then computed R using Eq. (30) with $L = 614$ mm and f defined as the average frequency over the range. The results are shown in Table 2. Here σ is the covariant (diagonal) uncertainty in R (so that, for example, the first string in the table has $R = 23.6 \pm 0.5$), and $\kappa = 2R + 1$. We also estimate an effective modulus of elasticity E from Eq. (22), expressed in units of gigapascals (1 GPa = 10^9 N/m^2.). Similar measurements and results for other string sets are provided in Appendix D. Note — as predicted in Section 2.5 — the expectation that the guitar will be *tunable* results in R values of manufactured strings that are in the range 20 – 30.

[1]Note that the correct unit of force in the metric system is Newtons (N), rather than kilograms, which is a unit of mass.

Table 2: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 650 mm.

String	R	σ	κ	B_0	E (GPa)
J4501	23.6	0.5	48.2	0.00190	8.33
J4502	23.8	0.7	48.6	0.00219	4.81
J4503	28.8	0.7	58.7	0.00302	3.87
J4504	22.4	0.8	45.7	0.00192	7.51
J4505	23.8	0.8	48.6	0.00238	5.27
J4506	28.6	0.4	58.2	0.00321	3.90

Table 3: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45) on the Classical Guitar.

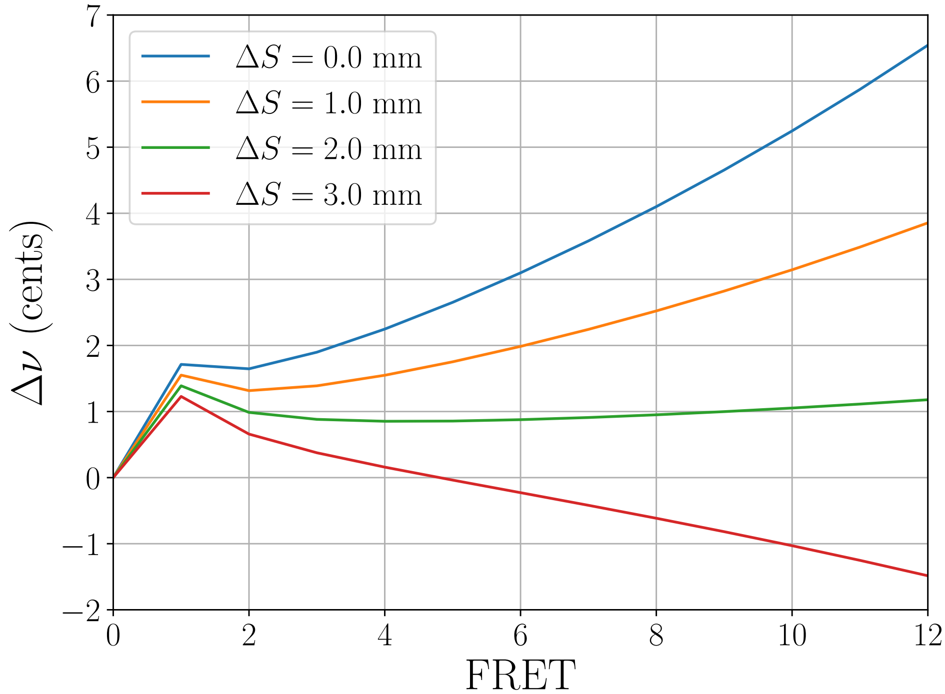
String	ΔS (mm)	ΔN (mm)	$\overline{\Delta\nu}_{\mathrm{rms}}$ (cents)
J4501	1.54	-0.35	0.16
J4502	1.74	-0.36	0.17
J4503	2.38	-0.43	0.20
J4504	1.53	-0.34	0.16
J4505	1.88	-0.36	0.17
J4506	2.51	-0.42	0.19

The still more important requirement that the guitar be *playable* leads us to the discussion of compensation in the next section.

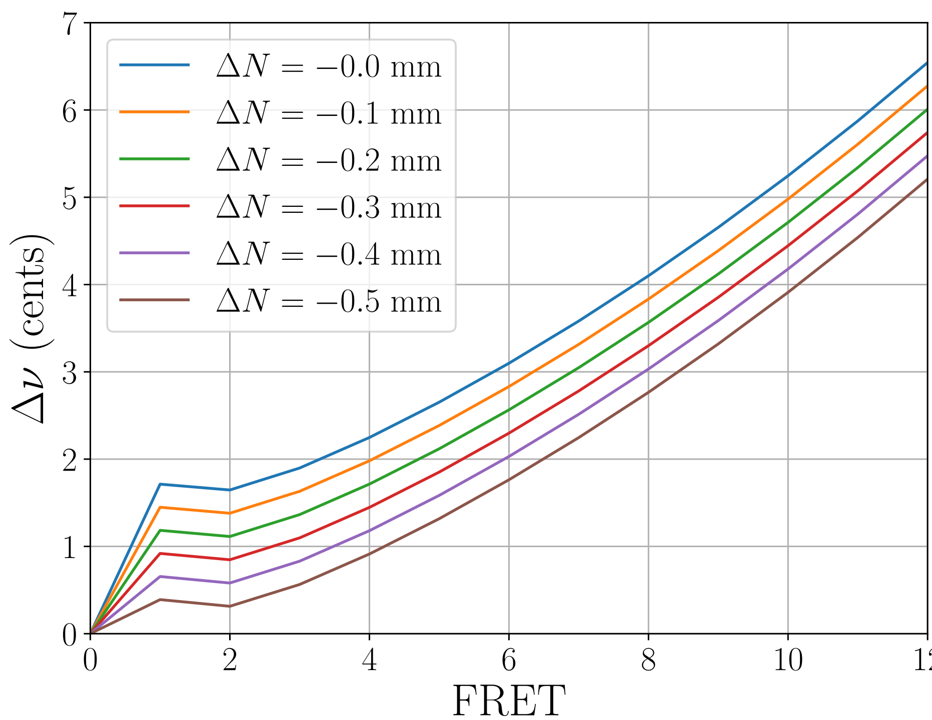
4 Classical Guitar Compensation

As we discussed in Section 2.5, Eq. (26) provides a guide to how to modify our prototypical classical guitar to improve the tone of the string shown in Fig. 6. We noted that the bending stiffness and the increase in string tension due to fretting sharpen the pitch, but that we can flatten it with a positive saddle setback and negative nut setback. As in Fig. 6, let's choose $b = 1$ mm, $c = 4$ mm, $R = 24$ and $\rho = 0.5$ mm as in Fig. 6. Then the corresponding string constant and bending stiffness are $\kappa = 49$ and $B_0 = 0.00268$ respectively. In this case, Figure 9 shows that increasing the saddle setback tends to rotate the pitch curve downward, and increasing the magnitude of the negative nut setback displaces the pitch curve almost uniformly downward. In this context — referring to Eq. (26) — we follow Eq. (34) and choose $\Delta S = B_0 X_0$ to compensate for stiffness, and then select $\Delta N = -\kappa X_0 \overline{Q}/2$, where $\overline{Q}$ is the relative displacement averaged over a particular set of frets, to compensate for tension. If we compute this mean over the first 12 frets, then we estimate $\Delta S = 1.75$ mm, $\Delta N = -0.49$ mm for $d = 0$, and $\Delta N = -0.55$ mm for $d = 10$. The pitch curve shown in Fig. 10 illustrates the dramatic improvement in tone that can be obtained via classical guitar compensation.

As an alternative to this simple approach, we adopt the method described in Appendix C and adjust the setbacks to minimize the root-mean-squared (RMS) average of the frequency deviations for each string. This mean (over the first 12 frets) can be computed by squaring the frequency deviations shown in Fig. 11, averaging those values, and then taking the square root



(a) Frequency shifts ($\Delta N = 0$)



(b) Frequency shifts ($\Delta S = 0$)

Figure 9: In (a), we plot the frequency shifts for our classical guitar for several saddle setbacks with $\Delta N = 0$. Here $R = 24$ and the string radius is 0.5 mm. In (b), we set $\Delta S = 0$ and plot the frequency shifts for several nut "setbacks."

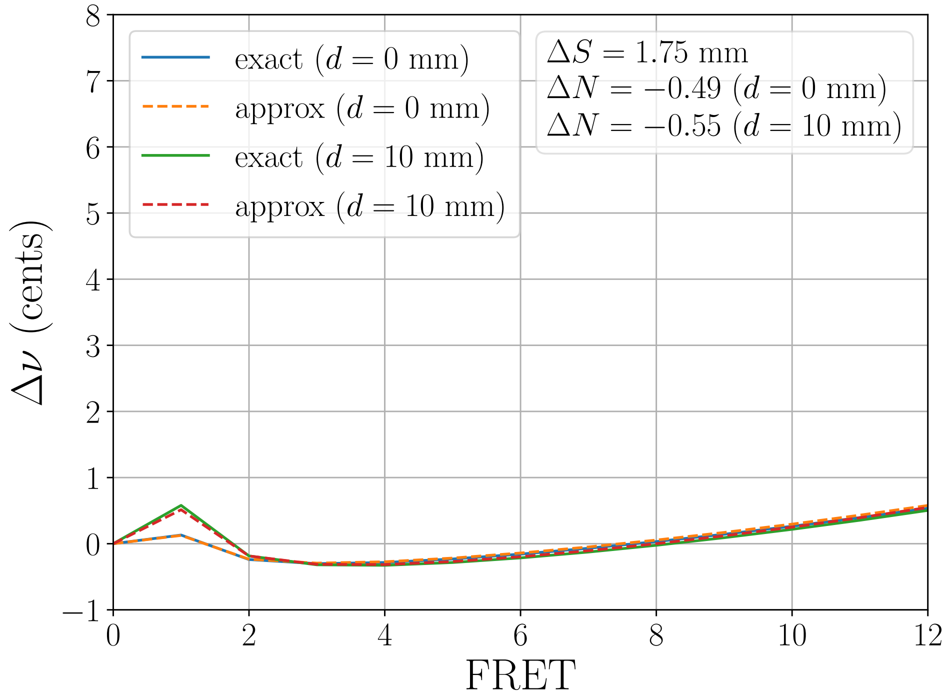


Figure 10: The total frequency shift given by Eq. (11) for a classical guitar with a scale length of 650 mm, $b = 1.0$ mm, $c = 4.0$ mm, and a string with $R = 24$ and a radius $\rho = 0.5$ mm.

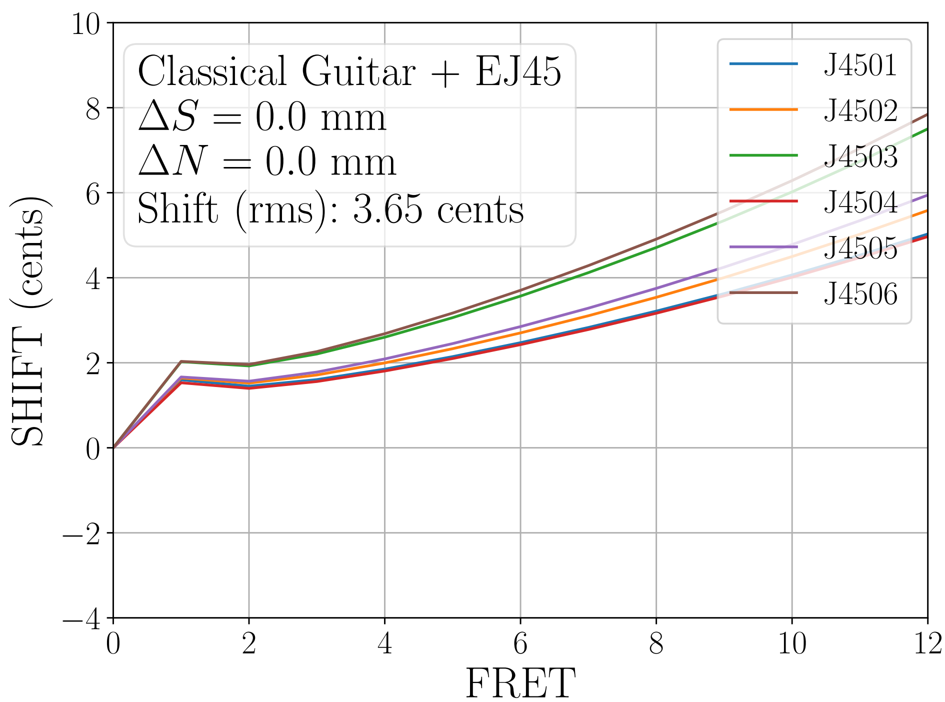


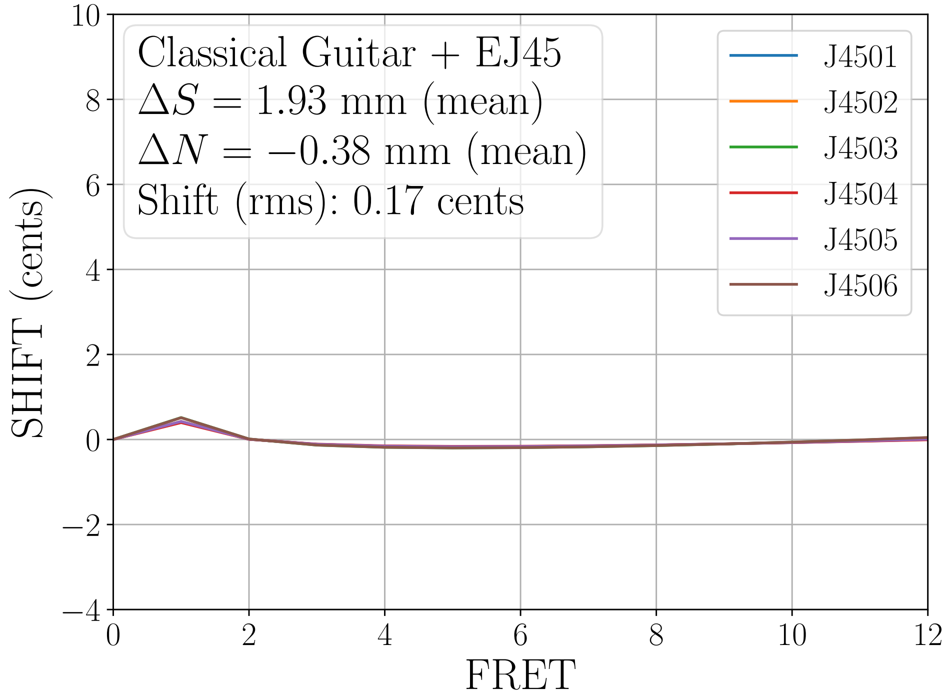
Figure 11: Frequency errors for an uncompensated classical guitar with a scale length of 650 mm, $b = 1.0$ mm, $c = 4.0$ mm, $d = 0$ mm, and normal tension nylon strings (D'Addario EJ45).

of the result. Using Eq. (66), the setbacks we obtain are listed in Table 3, and the corresponding frequency deviations — obtained with the listed setback for each string — are shown in Fig. 12a (assuming that all other aspects of the guitar remain unchanged). Of course, manufacturing a guitar with unique saddle and nut setbacks for each string (of a particular tension) can be challenging, so we also plot in Fig. 12b the shifts obtained by setting single values of ΔS and ΔN to the mean of the corresponding column in Table 3. Note that the saddle setbacks tend to be larger — and the nut setbacks smaller — than the simple estimates that we made above. This is easily understood by examining Eq. (26): the portion of ΔS that exceeds $B_0 X_0$ scales with $\gamma_n - 1$, and helps to compensate for tension errors as n increases.

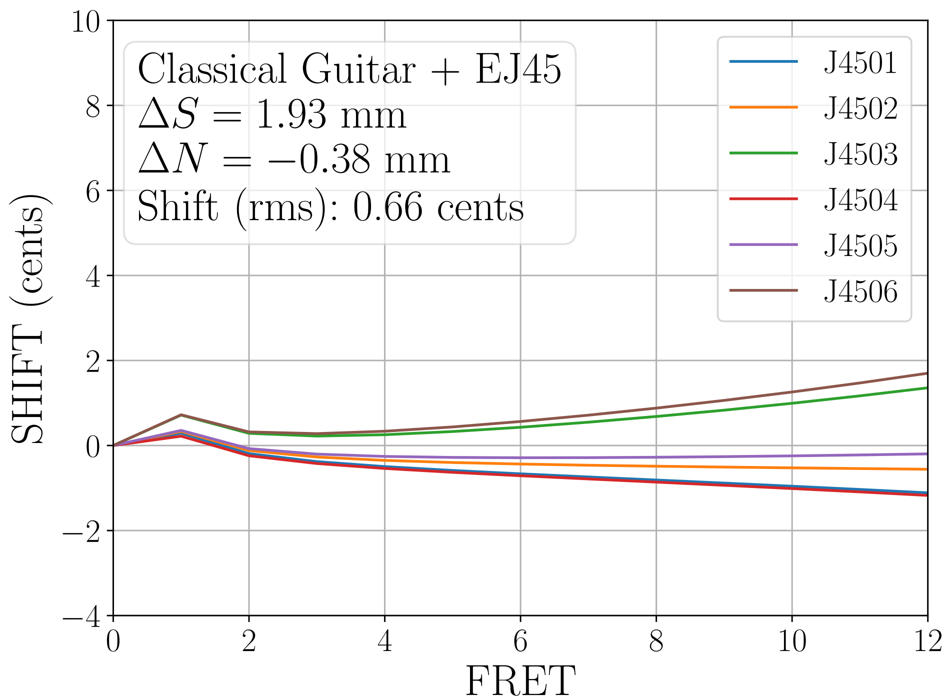
We close this section with several comments. First, as mentioned above, it is nontrivial to manufacture a guitar with different setbacks for each string [4], and it is unlikely that the exact values listed in Table 3 are applicable to other string sets. We have measured the values of R for five other string sets (with $d = 0$), and in Appendix D we have reproduced the compensation procedure for them. Although each set exhibits variation between strings (and with respect to other sets) in individual setbacks for each string, they are similar enough that we suspect that there is the potential for great simplification in guitar design. For example, following the analysis of Appendix C, it is possible to determine a single setback pair $\{\Delta S, \Delta N\}$ that minimizes the RMS frequency errors of an ensemble of strings over a collection of frets simply by computing the mean of the setbacks over all strings, and then using these mean values when manufacturing the guitar. If we consider five of the six string sets we have measured here — neglecting the light tension strings because of their pathologically high values of R — we obtain the global mean setbacks $\Delta S = 1.90$ mm and $\Delta N = -0.38$ mm for the saddle and nut, respectively. Note that these results are remarkably similar to the values used in Fig. 12b; if we plot the frequency deviations of those five string sets with these particular setback values, then we find that the maximum error always occurs at the twelfth fret, and it is always less than 2 cents. In the next section, we discuss a method to temper the guitar to reduce these errors further.

Second, after paying close attention to the impact of the act of fretting on our approximations in Section 2, we have simply set $d = 0$ in this section. This is because nonzero values of d have an impact on the frequency deviation of a string at the first fret, and is otherwise negligible. This tends to increase the required mean value of the nut setback, but does not require a significant change to the saddle setback. As an example, in Fig. 13, we plot the mean optimum saddle and nut setbacks for the classical guitar parameters used in Fig. 12b with the D'Addario Nylon Normal Tension EJ45 string set as a function of the fretting distance d. Paying careful attention to the y-axis of these plots, we see that the value of the mean saddle setback *decreases* by less than 5% as d increases to 10 mm, and the magnitude of the nut setback increases by almost 30% at the same fretting offset. This behavior is essentially the same for all string sets considered in this work, and can be used by the luthier to determine the optimum setback values for their designs. Note that — when the RMS approach is used to compute the mean setbacks — the difference $\Delta S - \Delta N$ (the sum of the saddle setback and the magnitude of the nut setback) is virtually independent of d, as shown in Fig. 14.

Third, recall that we recalculated the expected frequency shift of a classical guitar string with asymmetric boundary conditions in Appendix B, and found an expression for f_q given by Eq. (55) that reduces — by about a factor of 2 — the impact of the bending stiffness relative to the symmetric boundary condition case given by the symmetric (clamped) case in Eq. (4). Furthermore, we have not used more sophisticated techniques to calculate the bending stiffness for either monofilament nylon strings or wound nylon strings, opting instead for the phe-

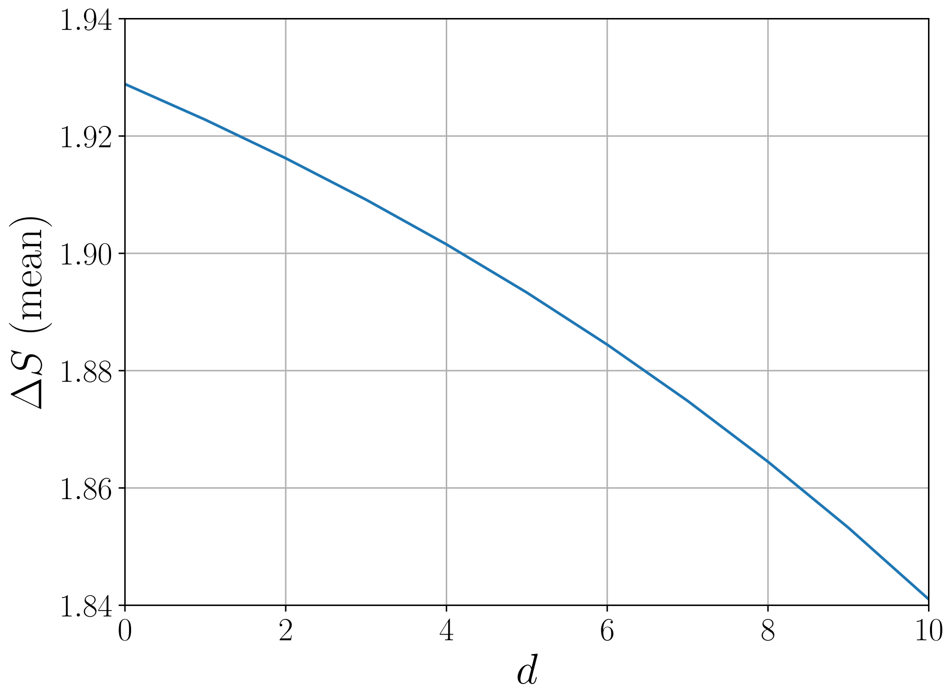


(a) Full compensation

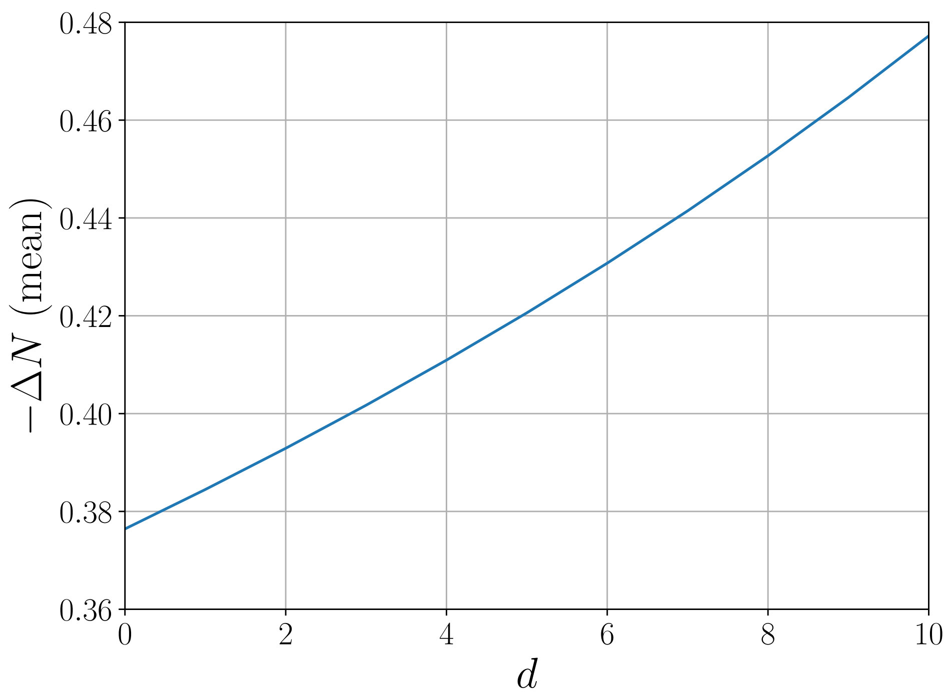


(b) Mean compensation

Figure 12: Frequency shifts (in cents) for a classical guitar with normal tension nylon strings (D'Addario EJ45). In (a) we use the individual values for each string that are listed in Table 3. In (b), we set ΔS and ΔN to the mean of the corresponding column in that table.



(a) Mean saddle setback



(b) Mean nut setback

Figure 13: The mean optimum saddle and nut setbacks for the D'Addario Nylon Normal Tension EJ45 string set as a function of the fretting distance d.

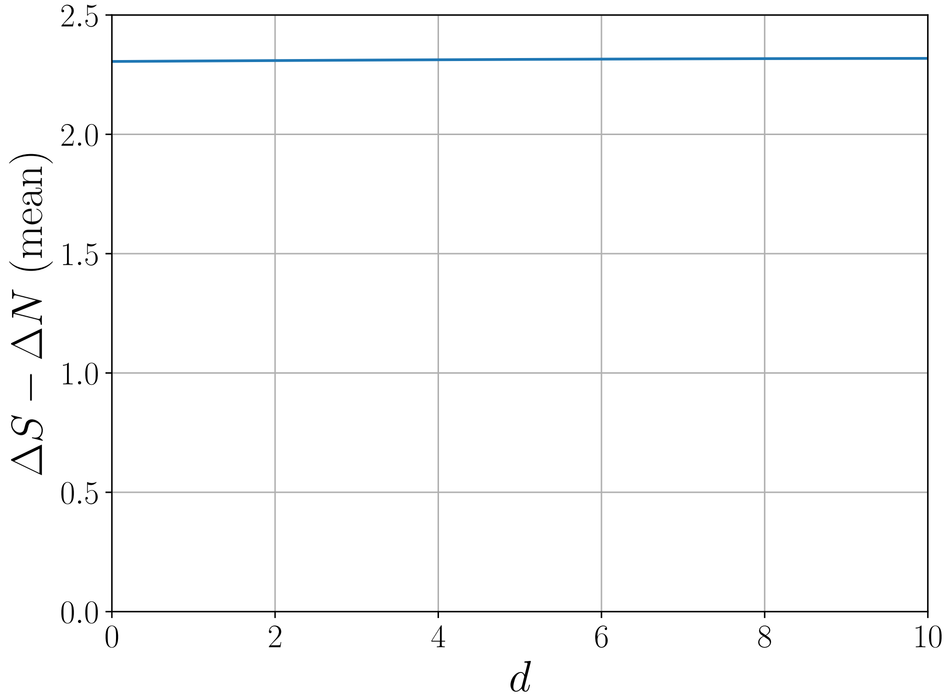


Figure 14: The mean value of $\Delta S - \Delta N$ for the D'Addario Nylon Normal Tension EJ45 string set as a function of the fretting distance d.

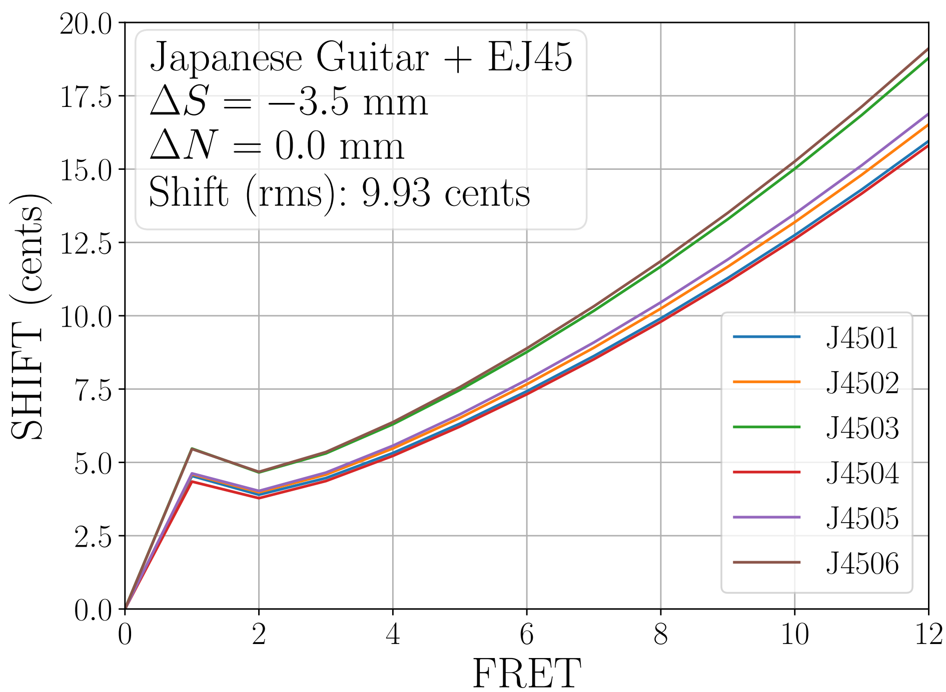


Figure 15: Expected frequency shifts for a classical guitar using normal-tension nylon strings with $b = 1.4$ mm, $c = 5.5$ mm, $d = 10$ mm, $X_0 = 652$ mm, $\Delta S = -3.5$ mm, and $\Delta N = 0$ mm.

nomenological model given by Eq. (27). But is this approach valid? As a test, we compared our frequency shift estimates based on Section 2 with experimental measurements made using a guitar manufactured in Japan based on a few unique design choices: $b = 1.4$ mm, $c = 5.5$ mm, $d = 10$ mm, $X_0 = 652$ mm, $\Delta S = -3.5$ mm, and $\Delta N = 0$ mm. The larger value of b and the large *negative* saddle setback result in very large frequency deviations at all frets. We measured the shifts at the first fret and found that they fell into the range of 4.5 – 5.5 cents, consistent with our predictions in Fig. 15. At the 12th fret, we measured $\Delta f = 18$ cents for the third and sixth strings, and $\Delta f = 14.5 - 16$ cents for the other strings, in reasonably close agreement with our predictions. By contrast, the corresponding equation arising from symmetric (clamped) boundary conditions — Eq. (4) — predicts shifts that are 30 – 45% higher, and setbacks that are more than a factor of two larger. We conclude that Eq. (55) and Eq. (27) can be used to reliably predict setbacks for classical guitars.

Finally, many luthiers provide "relief" to enlarge the effective height of the string (particularly for the wound bass strings) as the fret number grows to provide clearance for vibration amplitude at higher volume. In practice, this is accomplished by pivoting the fret board shown in Fig. 2 clockwise about $x = X_0$, increasing the height of the string above fret n by an amount

$$\begin{aligned} \Delta y_n &= m\,(X_0 - X_n) \\ &= \frac{\gamma_n - 1}{\gamma_n}\, m\, X_0 \\ &= \frac{\gamma_n - 1}{\gamma_n}\, 2\,\Delta y_{12}\,, \end{aligned} \tag{36}$$

where Δy_{12} is the relief at the twelfth fret and $m = 2\,\Delta y_{12}/X_0 \geq 0$ is the downward slope of the fret board. If we update Eq. (13) and Eq. (18) (with $d = 0$), then we obtain

$$L_n = \sqrt{(X_n + \Delta S)^2 + (b + \Delta y_n + c)^2}\,, \text{ and} \tag{37a}$$

$$L'_n = \sqrt{(X_0 - X_n + \Delta N)^2 + (b + \Delta y_n)^2}\,. \tag{37b}$$

These equations indicate that we could update the approximation for Q_n given by Eq. (19) by replacing $b \longrightarrow b + \Delta y_n$, which results in the numerator

$$\gamma_n\, b + (\gamma_n - 1)\, c \longrightarrow \gamma_n\, b + (\gamma_n - 1)\,\left(c + 2\,\Delta y_{12}\right)\,, \tag{38}$$

indicating that the intuitive substitution $c \longrightarrow c + 2\,\Delta y_{12}$ in Eq. (19) captures the effect of relief. (Note that this should *not* be done when computing the length L_0 of the open string!)

5 Tempering the Classical Guitar

> Temperament: A compromise between the acoustic purity of theoretically exact intervals, and the harmonic discrepancies arising from their practical employment.
> — Dr. Theo. Baker [17]

In Fig. 12b, a uniformly compensated classical guitar with normal tension strings tuned to 12-TET shows (of the treble strings) the third string has the greatest error in tuning across the fretboard. Tuning this guitar to 12-TET exacts a perfect-fifth in the third string while playing a C major chord in first position. This results in the third string being too sharp for the other common chords of E major (G#), A major, and D major (A), particularly when the guitar is played at a higher fret position. One way to reduce this error is by lowering the pitch of the third string below 12-TET with an electronic tuner. Another more comprehensive system is to tune all the strings harmonically to the fifth string, which lowers the third string by 7 cents as well as tempering the remaining strings.

In this particular case, the "Harmonic Tuning Method" can be followed using these steps:

1. Begin by tuning the fifth string to A_2 = 110 Hz, resulting in a fifth-fret harmonic of A_4 = 440 Hz. (This can also be tuned by ear using an A_4 tuning fork).

2. Tune that harmonic to the seventh fret harmonic of the fourth string, which is also A_4 = 440 Hz.

3. Tune the seventh fret harmonic on the fifth string (330 Hz, or 0.37 Hz sharper than 12-TET E_4) to the fifth-fret harmonic of the sixth string.

4. The seventh fret harmonic on the fifth string can tune the remaining fretted strings: the ninth fret on the third string, the fifth fret on the second string, and the open first string.

Table 4: Harmonic tuning methodology based on A_4 and E_4. The asterisk indicates a harmonic with a null at the designated fret.

Reference String/Fret	Target String/Fret
A*/5 (A_4)	D*/7
A*/7 (E_4)	E*/5
A*/7 (E_4)	G/9
A*/7 (E_4)	B/5
A*/7 (E_4)	E/0

We have summarized these steps in Table 4, and in Fig. 16 we show the same guitar tuned in this fashion. Although the RMS shift over all strings is similar to that obtained by 12-TET tuning, the reduction in errors by strings 2 and 3 on the second and higher frets is significant. Note that other tuning choices can be made depending on the piece being played. For example, the third string could also be tuned at the second fret to A_3 = 220 Hz using the fifth-string harmonic at the 12^{th} fret, and/or the first string could be tuned at the fifth fret to A_4 using the fifth-fret harmonic of the fifth string. The flexibility of the harmonic tuning method — and its reliance on only an A_4 tuning fork — is a great asset for the classical guitarist.

6 Conclusion: The Recipe

In this work, we have constructed a model of classical guitar intonation that includes the effects of the resonant length of the fretted string, linear mass density, tension, and bending stiffness. We have described a simple experimental approach to estimating the increase in string tension arising from an increase in its length, and then the corresponding mechanical stiffness. This allows us to determine the saddle and nut positions needed to compensate the guitar for a particular string, and we propose a simple approach to find averages of these positions to accommodate a variety of strings. This "mean" method benefits further from

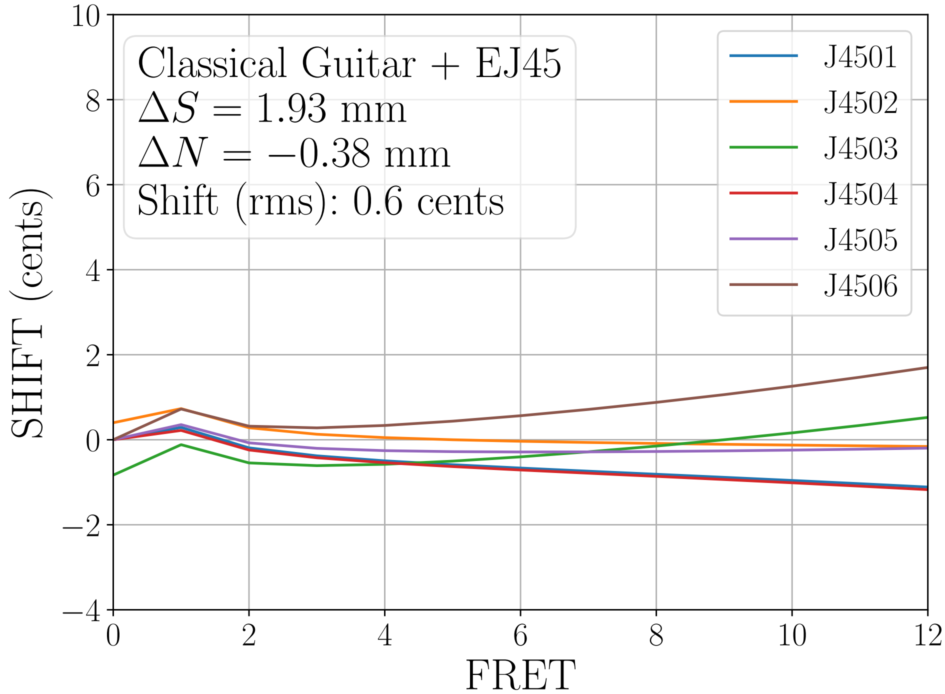


Figure 16: Frequency shift (in cents) for the mean compensated classical guitar with normal tension nylon strings (D'Addario EJ45) shown with 12-TET tuning in Fig. 12b. Here the same guitar has been harmonically tuned using the approach outlined in Table 4.

temperament techniques — such as harmonic tuning — that can enhance the intonation of the classical guitar for particular musical pieces.

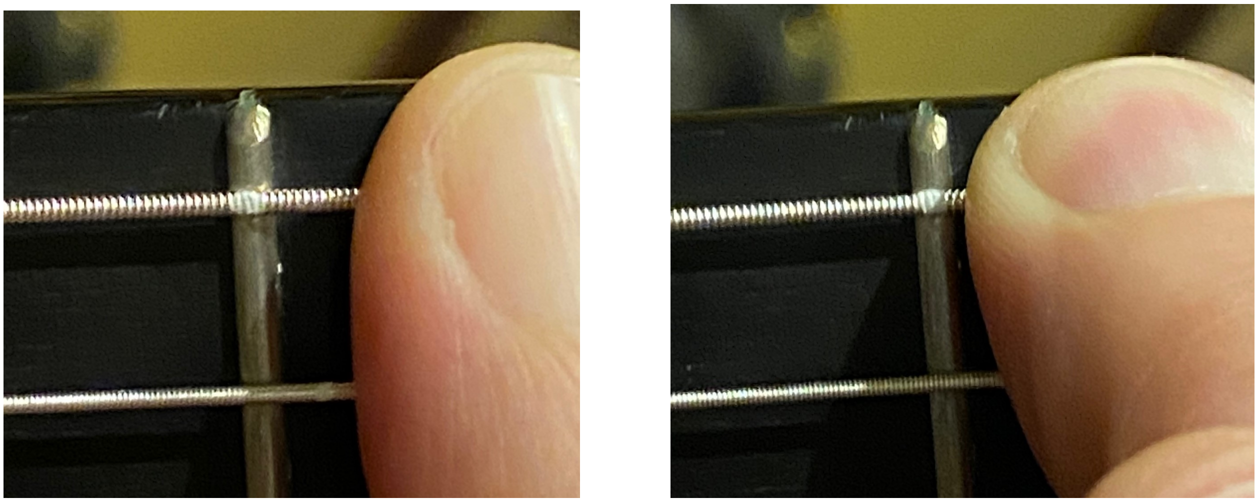
Our calculations have relied on Eq. (6), which was derived by noting that the string was pinned to the saddle rather than clamped. We then separated the contributions to the frequency deviations from ideal values caused by fretting by measuring these differences using the logarithmic "cents" approach defined by Eq. (2). This approach has allowed us to determine Taylor series approximations to each of these contributions that are valid when the height of a string above each fret is small compared to the scale length. From these approximations, we have been able to create a simple arithmetic "recipe" that predicts saddle and nut setbacks that enable the guitar to compensate for the frequency effects of fretting:

1. Determine the scale length of the guitar by doubling the distance between the inside edge of the nut and the center of the 12$^{\text{th}}$ fret.

2. Using Fig. 2 as a guide, carefully measure the values of b and c. It is possible that the luthier has selected a saddle with vertical curvature, resulting in different values of c for each string.

3. Estimate the relief Δy_{12} at the 12$^{\text{th}}$ fret. Measure the action (height) y_{12} of the string above fret 12; then $\Delta y_{12} = y_{12} - b - c/2$.

4. Select a string set with values of κ and B_0 listed in one of the derived physical properties tables in this paper, or follow the procedure developed in Section 3 to determine these quantities for a different string set.



(a) Before fretting (b) After fretting

Figure 17: Location of a small marker of white correction fluid before and after fretting.

5. Compute the mean value of Q using Eq. (35):

$$\overline{Q} \approx \frac{77.9\, b^2 + 35.6\, b\ (c + 2\,\Delta y_{12}) + 5.82\ (c + 2\,\Delta y_{12})^2}{24\, X_0^2}\,.$$

6. Compute the setbacks using Eq. (34):

$$\Delta S = B_0\, X_0\,, \text{ and}$$
$$\Delta N = -\kappa\, X_0\, \overline{Q}/2\,,$$

These setback estimates can be averaged across the string set to design compensated nuts and saddles that should be relatively easy to fabricate. Note that the resulting mean values are reasonably close to the results we'd obtain using the more exact approach presented in Section 4 and Appendix C for a fretting distance $d = 10$ mm. Nevertheless, we understand that high-end (concert) guitars that are likely to rely on one or two string sets (and the appropriate value of d for one guitar player) will benefit from the full, more accurate treatment of individual string setbacks.

We have placed the text of this manuscript (as well as the computational tools needed to reproduce our numerical results and all of the graphs presented here) online [10] so as to invite comment and contributions from and collaboration with interested luthiers and musicians.

A Fretting Classical Guitar Strings

Previous studies of guitar intonation and compensation [4, 5] included a contribution to the incremental change in the length of the fretted string caused by both the depth and the shape of the string under the finger. As the string is initially pressed to the fret, the total length $\mathcal{L}_n$ increases and causes the tension in the string — which is pinned at the saddle and clamped at the nut — to increase. As the string is pressed further, does the additional deformation of the string increase its tension (throughout the resonant length L_n)? There are at least two

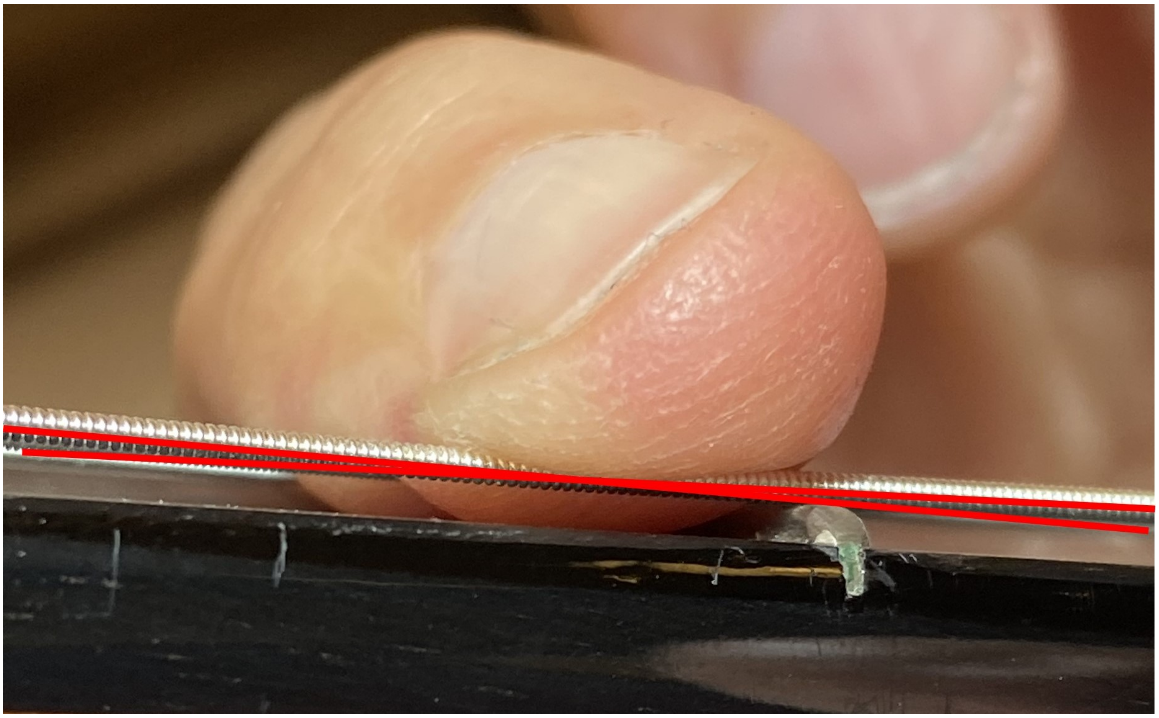


Figure 18: Photo of a wound nylon E_2 string clamped at the first fret of a classical guitar. The shape of the fretted string can be well approximated by two line segments intersecting about 5–6 mm behind the fret.

purely empirical reasons to doubt this hypothesis. First, as shown in Fig. 17, we can mark a string (with a small deposit of white correction fluid) above a particular fret and then observe the mark with a magnifying glass. As the string is pressed flat on the finger board with two fingers, the mark does not move perceptibly — it has become *clamped* on the fret. Second, we can use either our ears or a simple tool to measure frequencies [15] to listen for a shift as we use different fingers and vary the fretted depth of a string. The apparent modulation is far less than would be obtained by classical vibrato (± 15 cents), so we assume that once the string is minimally fretted the length(s) can be regarded as fixed. (If this were not the case, then fretting by different people or with different fingers, at a single string or with a barre, would cause additional, varying frequency shifts that would be audible and difficult to compensate.)

In Section 2, we have included this concept in a simple way to determine the effect it will have on the frequency shift due to increased string tension. First, as shown in Fig. 18, as the string is pressed onto the fret, its shape is described quite well by two line segments intersecting behind the fret. Here it is clear that the finger is shaped by the string more than the string is shaped by the finger. We have taken this observation into consideration in Fig. 2 by introducing such an intersection point at a distance d behind fret n to represent the slight increase in the distance L'_n caused by a finger. The consequences of this choice are discussed in Section 2.2, and the impact it has on (for example) the relative displacement Q_n is shown in Fig. 3.

B Vibration Frequencies of a Stiff String

Here we outline the calculation of the normal mode frequencies of a vibrating stiff string with non-symmetric boundary conditions. We begin with the wave equation [12]

$$\mu \frac{\partial^2}{\partial t^2} y(x) = T \frac{\partial^2}{\partial x^2} y(x) - E\, S\, \mathcal{K}^2 \frac{\partial^4}{\partial x^4} y(x)\,, \tag{39}$$

where μ and T are respectively the linear mass density and the tension of the string, E is its Young's modulus (or the modulus of elasticity), S is the cross-sectional area, and $\mathcal{K}$ is the radius of gyration of the string. (For a uniform cylindrical string/wire with radius ρ, $S = \pi\rho^2$ and $\mathcal{K} = \rho/2$.) If we scale x by the length L of the string, and t by $1/\omega_0 \equiv (L/\pi)\sqrt{\mu/T}$, then we obtain the dimensionless wave equation

$$\pi^2 \frac{\partial^2}{\partial t^2} y(x) = \frac{\partial^2}{\partial x^2} y(x) - B^2 \frac{\partial^4}{\partial x^2} y(x)\,, \tag{40}$$

where B is the "bending stiffness parameter" given by

$$B \equiv \sqrt{\frac{E\, S\, \mathcal{K}^2}{L^2 T}}\,. \tag{41}$$

We assume that $y(x)$ is a sum of terms of the form

$$y(x) = C\, e^{k x - i \omega t}\,, \tag{42}$$

requiring that k and ω satisfy the expression

$$B^2 k^4 - k^2 - (\pi\, \omega)^2 = 0\,, \tag{43}$$

or

$$k^2 = \frac{1 \pm \sqrt{1 + (2\, \pi\, B\, \omega)^2}}{2\, B^2}\,. \tag{44}$$

Therefore, given ω, we have four possible choices for k: $\pm k_1$, or $\pm i k_2$, where

$$k_1^2 = \frac{\sqrt{1 + (2\, \pi\, B\, \omega)^2} + 1}{2\, B^2}\,, \text{ and} \tag{45a}$$

$$k_2^2 = \frac{\sqrt{1 + (2\, \pi\, B\, \omega)^2} - 1}{2\, B^2}\,. \tag{45b}$$

The corresponding general solution to Eq. (40) has the form

$$y(x) = e^{-i\omega t} \left(C_1^+ e^{k_1 x} + C_1^- e^{-k_1 x} + C_2^+ e^{i k_2 x} + C_2^- e^{-i k_2 x} \right)\,. \tag{46}$$

As discussed in Section 2, the boundary conditions for the case of a classical guitar string are not symmetric. At $x = 0$ (the saddle), the string is pinned (but not clamped), so that $y = 0$ and $\partial^2 y/\partial x^2 = 0$. However, at $x = 1$ (the fret) the string is clamped, so that $y = 0$ and $\partial y/\partial x = 0$. Applying these constraints to Eq. (46), we obtain

$$0 = C_1^+ + C_1^- + C_2^+ + C_2^-\,, \tag{47a}$$

$$0 = k_1^2 \left(C_1^+ + C_1^-\right) - k_2^2 \left(C_2^+ + C_2^-\right)\,, \tag{47b}$$

$$0 = C_1^+ e^{k_1} + C_1^- e^{-k_1} + C_2^+ e^{i k_2} + C_2^- e^{-i k_2}\,, \text{ and} \tag{47c}$$

$$0 = k_1 \left(C_1^+ e^{k_1} - C_1^- e^{-k_1}\right) + i k_2 \left(C_2^+ e^{i k_2} - C_2^- e^{-i k_2}\right)\,. \tag{47d}$$

Since $k_1^2 + k_2^2 \neq 0$, the first two of these equations tell us that $C_1^- = -C_1^+ \equiv -C_1$, and $C_2^- = -C_2^+ \equiv -C_2$. Therefore, the second two equations become

$$C_1 \sinh(k_1) = -i\, C_2 \sin(k_2)\,, \text{ and} \tag{48a}$$
$$k_1\, C_1 \cosh(k_1) = -i\, k_2\, C_2 \cos(k_2)\,. \tag{48b}$$

Dividing the first of these equations by the second, we find

$$\tan(k_2) = \frac{k_2}{k_1} \tanh k_1\,. \tag{49}$$

From Eq. (45), we see that $k_1^2 - k_2^2 = 1/B^2$, so that

$$k_1 = \frac{1}{B}\sqrt{1 + (B\, k_2)^2}\,. \tag{50}$$

In the case of a classical guitar, we expect that $B \ll 1$, so $k_1 \approx 1/B \gg 1$, and therefore $\tanh k_1 \longrightarrow 1$. Substituting Eq. (50) into Eq. (49), we obtain

$$\tan(k_2) = \frac{B\, k_2}{\sqrt{1 + (B\, k_2)^2}}\,. \tag{51}$$

We expect that $B\, k_2 \ll 1$, so we assume that $k_2 = q\pi(1+\epsilon)$, where $q \in \mathbb{N}$ is an integer greater than or equal to 1, and $\epsilon \ll 1$. Therefore, to second order in ϵ, we have $\tan(k_2) \approx q\pi\epsilon$, and

$$\epsilon = \frac{B\,(1+\epsilon)}{\sqrt{1 + [q\,\pi\, B\,(1+\epsilon)]^2}}\,. \tag{52}$$

The denominator of the right-hand side of this equation has a Taylor expansion given by $1 - \frac{1}{2}\,[q\,\pi\, B\,(1+\epsilon)]^2$, indicating that it will not contribute to ϵ to second order in B. Therefore, to this order,

$$\epsilon \approx \frac{B}{1-B} \approx B + B^2\,. \tag{53}$$

We substitute $k = \pm i k_2$ into Eq. (43) with $k_2 = q\pi/(1-B)$ to obtain

$$\begin{aligned}\omega &= \frac{k_2}{\pi}\sqrt{1 + (B\, k_2)^2} \\ &= \frac{q}{1-B}\sqrt{1 + q^2\pi^2\left(\frac{B}{1-B}\right)^2} \\ &\approx q\left[1 + B + \left(1 + \frac{1}{2}q^2\pi^2\right)B^2\right].\end{aligned} \tag{54}$$

Restoring the time scaling by $1/\omega_0$, and defining the frequency (in cycles/second) $f = \omega/2\pi$, we finally have

$$f_q = \frac{q}{2\,L}\sqrt{\frac{T}{\mu}}\left[1 + B + \left(1 + \frac{1}{2}q^2\pi^2\right)B^2\right]. \tag{55}$$

We use this result to build our model in Section 2.

C Compensation by Minimizing RMS Error

The root-mean-square (RMS) frequency error (in cents) averaged over the frets $n \in \{1, n_{\max}\}$ (for $n_{\max} > 1$) *of a particular string* is given by

$$\overline{\Delta\nu}_{\text{rms}} \equiv \sqrt{\frac{\sum_{n=1}^{n_{\max}} \Delta\nu_n^2}{n_{\max}}}\,, \tag{56}$$

where $\Delta\nu_n$ is given by Eq. (26). Here we will vary both ΔS and ΔN to minimize $\overline{\Delta\nu}_{\text{rms}}$. In this case, it is sufficient to minimize the quantity

$$\chi^2 = \sum_{n=1}^{n_{\max}} \left[\frac{\ln(2)}{1200}\,\Delta\nu_n\right]^2 \tag{57}$$

such that the gradient of χ^2 with respect to ΔS and ΔN vanishes. The components of this gradient are

$$\frac{\partial}{\partial \Delta S}\,\chi^2 = -\frac{2}{X_0}\sum_n (\gamma_n - 1)\left[(\gamma_n - 1)\left(B_0 - \frac{\Delta S}{X_0}\right) + \frac{\Delta N}{X_0} + \frac{\kappa}{2}\,Q_n\right]\,, \text{ and} \tag{58a}$$

$$\frac{\partial}{\partial \Delta N}\,\chi^2 = \frac{2}{X_0}\sum_n \left[(\gamma_n - 1)\left(B_0 - \frac{\Delta S}{X_0}\right) + \frac{\Delta N}{X_0} + \frac{\kappa}{2}\,Q_n\right]\,. \tag{58b}$$

Setting both of these expressions to zero, we can rewrite them as the matrix equation

$$\begin{bmatrix} \sigma_2 & -\sigma_1 \\ \sigma_1 & -\sigma_0 \end{bmatrix} \begin{bmatrix} \Delta S \\ \Delta N \end{bmatrix} = X_0 \begin{bmatrix} \sigma_2\, B_0 + \frac{1}{2}\,\kappa\,\overline{Q}_1 \\ \sigma_1\, B_0 + \frac{1}{2}\,\kappa\,\overline{Q}_0 \end{bmatrix}\,, \tag{59}$$

where

$$\sigma_k \equiv \sum_{n=1}^{n_{\max}} (\gamma_n - 1)^k\,, \text{ and} \tag{60}$$

$$\overline{Q}_k \equiv \sum_{n=1}^{n_{\max}} (\gamma_n - 1)^k\, Q_n\,. \tag{61}$$

We note that

$$g_k \equiv \sum_{n=1}^{n_{\max}} \gamma_n^k = \frac{\gamma_k\,(\gamma_{k n_{\max}} - 1)}{\gamma_k - 1}\,, \tag{62}$$

and therefore

$$\sigma_0 = n_{\max}\,, \tag{63a}$$

$$\sigma_1 = g_1 - n_{\max}\,, \text{ and} \tag{63b}$$

$$\sigma_2 = g_2 - 2g_1 + n_{\max}\,. \tag{63c}$$

Equation (59) has the straightforward analytic solution

$$\begin{bmatrix} \Delta S \\ \Delta N \end{bmatrix} = \frac{X_0}{\sigma_0\,\sigma_2 - \sigma_1^2} \begin{bmatrix} \sigma_0 & -\sigma_1 \\ \sigma_1 & -\sigma_2 \end{bmatrix} \begin{bmatrix} \sigma_2\, B_0 + \frac{1}{2}\,\kappa\,\overline{Q}_1 \\ \sigma_1\, B_0 + \frac{1}{2}\,\kappa\,\overline{Q}_0 \end{bmatrix}\,, \tag{64}$$

or

$$\Delta S = \left(B_0 - \frac{\kappa}{2} \frac{\sigma_1 \overline{Q}_0 - \sigma_0 \overline{Q}_1}{\sigma_0 \sigma_2 - \sigma_1^2} \right) X_0 \,, \text{ and} \tag{65a}$$

$$\Delta N = -\frac{\kappa}{2} \frac{\sigma_2 \overline{Q}_0 - \sigma_1 \overline{Q}_1}{\sigma_0 \sigma_2 - \sigma_1^2} X_0 \,. \tag{65b}$$

If we approximate $\overline{Q}_k \approx \sigma_k \overline{Q}$, where $\overline{Q}$ is the relative displacement averaged over a particular set of frets, we obtain the estimates given by Eq. (34). The corresponding solution when the quadratic stiffness term is included is given by

$$\begin{bmatrix} \Delta S \\ \Delta N \end{bmatrix} = \frac{X_0}{\sigma_0 \sigma_2 - \sigma_1^2} \begin{bmatrix} \sigma_0 & -\sigma_1 \\ \sigma_1 & -\sigma_2 \end{bmatrix} \begin{bmatrix} \sigma_2 B_0 + \frac{1}{2}(1+\pi^2)(2\sigma_2 + \sigma_3) B_0^2 + \frac{1}{2}\kappa \overline{Q}_1 \\ \sigma_1 B_0 + \frac{1}{2}(1+\pi^2)(2\sigma_1 + \sigma_2) B_0^2 + \frac{1}{2}\kappa \overline{Q}_0 \end{bmatrix} . \tag{66}$$

The corresponding Hessian matrix for this problem is the symmetric matrix

$$H = \begin{bmatrix} \frac{\partial^2 \chi^2}{\partial \Delta S^2} & \frac{\partial^2 \chi^2}{\partial \Delta N \, \partial \Delta S} \\ \frac{\partial^2 \chi^2}{\partial \Delta S \, \partial \Delta N} & \frac{\partial^2 \chi^2}{\partial \Delta N^2} \end{bmatrix} = \frac{2}{X_0^2} \begin{bmatrix} \sigma_2 & -\sigma_1 \\ -\sigma_1 & \sigma_0 \end{bmatrix} . \tag{67}$$

We can apply the second partial derivative test to the Hessian to determine whether we've found an extremum of χ^2. If the determinant of the Hessian is positive, and (in the case of a 2×2 matrix) one of the diagonal elements is positive, then we have found a minimum. The second condition is satisfied by $\sigma_0 = n_{\max} > 0$ when $n_{\max} \geq 1$. The determinant is given by

$$\mathrm{Det}(H) = \frac{4}{X_0^4} \left(n_{\max} g_2 - g_1^2 \right) , \tag{68}$$

which is indeed greater than 0 for $n_{\max} > 1$ since the quantity

$$\frac{n_{\max} g_2}{g_1^2} = \frac{n_{\max}(\gamma_1 - 1)(\gamma_{n_{\max}} + 1)}{(\gamma_1 + 1)(\gamma_{n_{\max}} - 1)} \approx 1 + \frac{\ln^2(2)}{12^3} (n_{\max}^2 - 1) > 1 \,. \tag{69}$$

Therefore, we can be confident that the solution for ΔS and ΔN given by Eq. (65) minimizes the RMS frequency error provided that we are averaging over at least the first two frets.

The setback solution given by Eq. (65) is valid for a single string, and results like those shown in Table 3 and Fig. 12a assume that the guitar is built such that each string — from a particular set of strings — has a unique saddle and nut setback. Suppose that we'd prefer to engineer a guitar with single, uniform values of both ΔS and ΔN that provide reasonable compensation across an entire string set (or an ensemble of strings from a variety of manufacturers). In this case, Eq. (56) becomes

$$\overline{\Delta \nu}_{\mathrm{rms}} \equiv \sqrt{\frac{\sum_{m=1}^{m_{\max}} \sum_{n=1}^{n_{\max}} \left[\Delta \nu_n^{(m)} \right]^2}{m_{\max} \, n_{\max}}} \,, \tag{70}$$

where m labels the strings in the set, and Eq. (26) has been updated to become

$$\Delta \nu_n^{(m)} \approx \frac{1200}{\ln(2)} \left\{ (\gamma_n - 1) \left[B_0^{(m)} - \frac{\Delta S}{X_0} \right] + \frac{\Delta N}{X_0} + \frac{1}{2} \kappa^{(m)} Q_n \right\} . \tag{71}$$

If we rigorously follow the same approach that we used to arrive at Eq. (65), in the multi-string case we obtain

$$\begin{bmatrix} \Delta S \\ \Delta N \end{bmatrix} = \frac{1}{m_{\max}} \begin{bmatrix} \sum_{m=1}^{m_{\max}} \Delta S^{(m)} \\ \sum_{m=1}^{m_{\max}} \Delta N^{(m)} \end{bmatrix} , \tag{72}$$

where

$$\begin{bmatrix} \Delta S^{(m)} \\ \Delta N^{(m)} \end{bmatrix} = \frac{X_0}{\sigma_1^2 - \sigma_0\,\sigma_2} \begin{bmatrix} -\sigma_0 & \sigma_1 \\ -\sigma_1 & \sigma_2 \end{bmatrix} \begin{bmatrix} \sigma_2\,B_0^{(m)} + \frac{1}{2}\,\kappa^{(m)}\,\overline{Q}_1 \\ \sigma_1\,B_0^{(m)} + \frac{1}{2}\,\kappa^{(m)}\,\overline{Q}_0 \end{bmatrix}. \tag{73}$$

In other words, we can find the optimum values for both ΔS and ΔN simply by averaging the corresponding setbacks over a commercially interesting collection of string sets.

D Other Classical Guitar String Sets

D.1 Light Tension – Nylon

Table 5: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

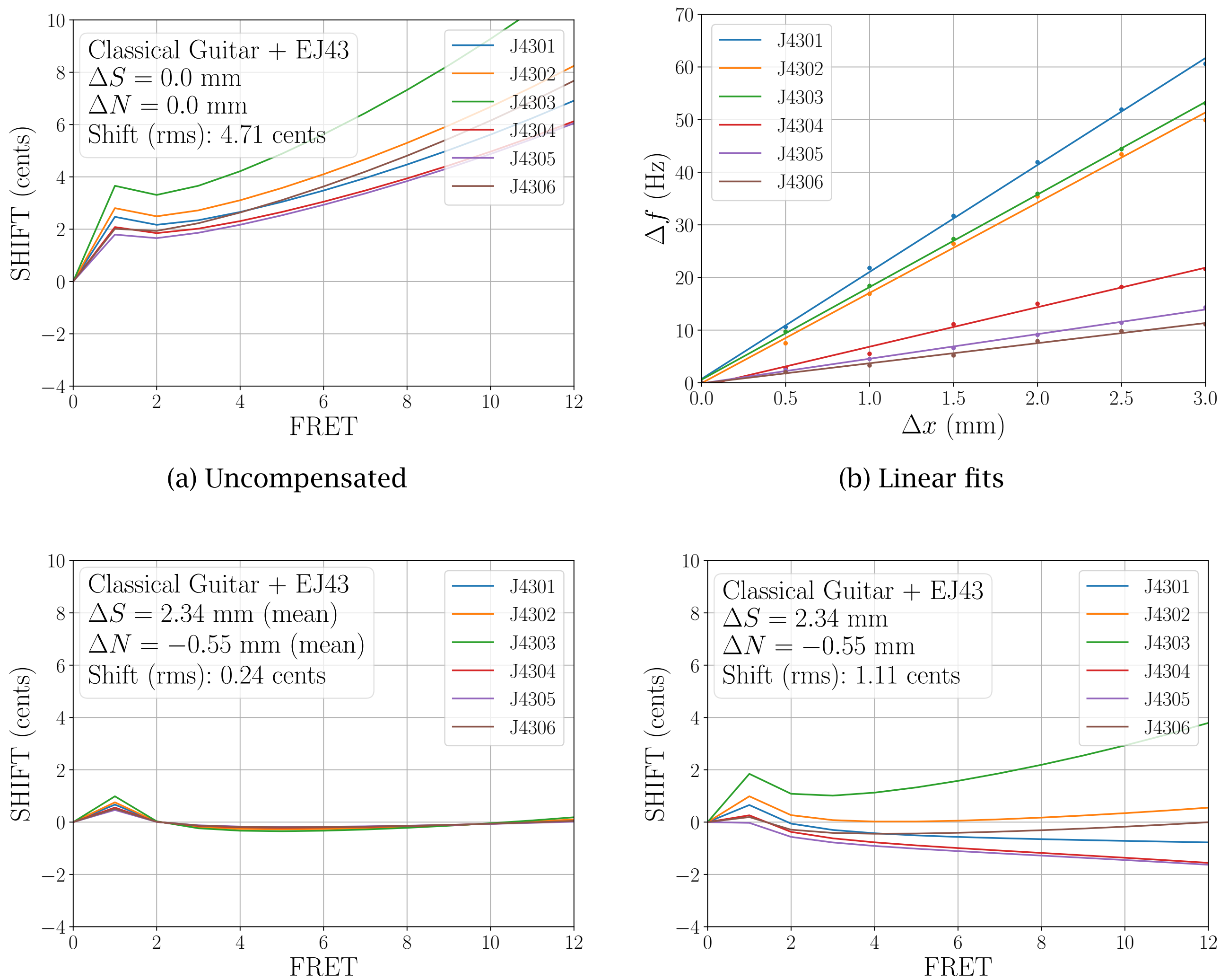
String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4301	E_4	0.349	3.62	66.4
J4302	B_3	0.403	4.87	50.2
J4303	G_3	0.504	8.08	52.5
J4304	D_3	0.356	18.23	66.4
J4305	A_2	0.419	27.41	56.1
J4306	E_2	0.533	51.59	59.2

Table 6: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

String	R	σ	κ	B_0	E (GPa)
J4301	37.8	0.5	76.6	0.00235	13.28
J4302	42.6	1.0	86.2	0.00287	8.50
J4303	55.0	0.4	111.1	0.00409	7.30
J4304	31.4	1.2	63.7	0.00218	10.65
J4305	26.1	0.5	53.2	0.00235	5.40
J4306	28.5	1.1	57.9	0.00312	3.84

Table 7: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43) on the Classical Guitar.

String	ΔS (mm)	ΔN (mm)	$\overline{\Delta\nu}_{\rm rms}$ (cents)
J4301	2.01	-0.56	0.25
J4302	2.43	-0.63	0.28
J4303	3.44	-0.81	0.36
J4304	1.82	-0.47	0.21
J4305	1.88	-0.39	0.18
J4306	2.45	-0.42	0.19



(a) Uncompensated

(b) Linear fits

(c) Full compensation

(d) Mean compensation

Figure 19: Frequency shift (in cents) for a classical guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). Four different strategies of saddle and nut compensation are illustrated.

D.2 Hard Tension – Nylon

Table 8: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). The corresponding scale length is 650 mm.

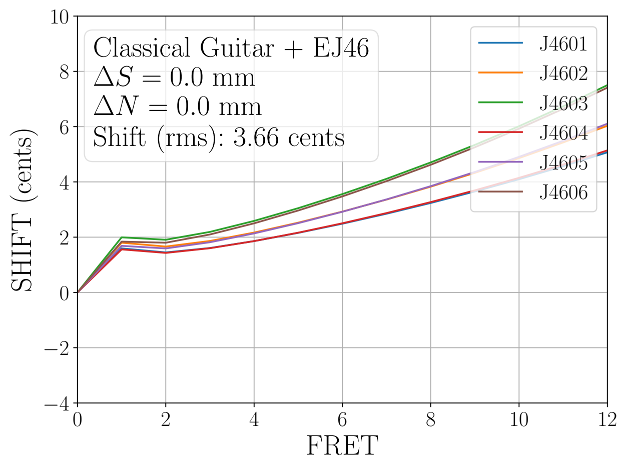
String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4601	E_4	0.362	3.86	70.9
J4602	B_3	0.415	5.22	53.8
J4603	G_3	0.521	8.57	55.6
J4604	D_3	0.381	20.07	73.1
J4605	A_2	0.457	34.87	71.3
J4606	E_2	0.559	56.67	65.0

Table 9: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46).

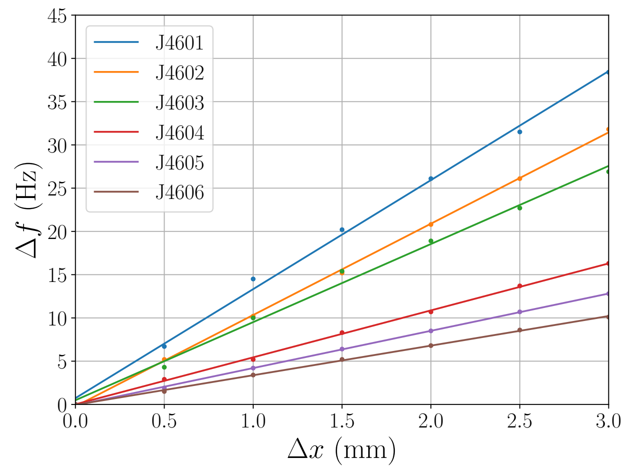
String	R	σ	κ	B_0	E (GPa)
J4601	23.5	0.5	47.9	0.00193	8.25
J4602	26.2	0.3	53.5	0.00234	5.31
J4603	28.3	1.0	57.5	0.00304	3.76
J4604	22.7	0.3	46.4	0.00200	7.44
J4605	24.0	0.2	49.0	0.00246	5.33
J4606	25.5	0.3	51.9	0.00310	3.44

Table 10: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46) on the Classical Guitar.

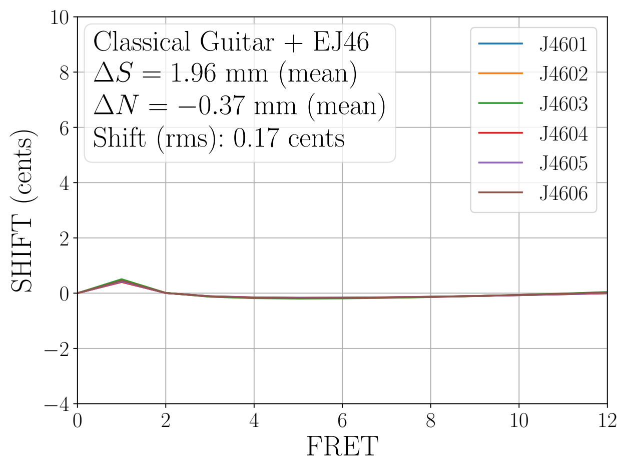
String	ΔS (mm)	ΔN (mm)	$\overline{\Delta \nu}_{\mathrm{rms}}$ (cents)
J4601	1.55	-0.35	0.16
J4602	1.87	-0.39	0.18
J4603	2.39	-0.42	0.19
J4604	1.59	-0.34	0.16
J4605	1.93	-0.36	0.17
J4606	2.40	-0.38	0.18



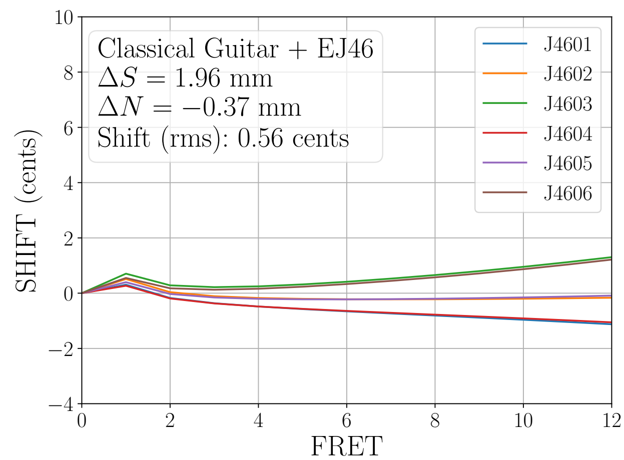
(a) Uncompensated



(b) Linear fits



(c) Full compensation



(d) Mean compensation

Figure 20: Frequency shift (in cents) for a classical guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). Four different strategies of saddle and nut compensation are illustrated.

D.3 Extra Hard Tension – Nylon

Table 11: String specifications for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). The corresponding scale length is 650 mm.

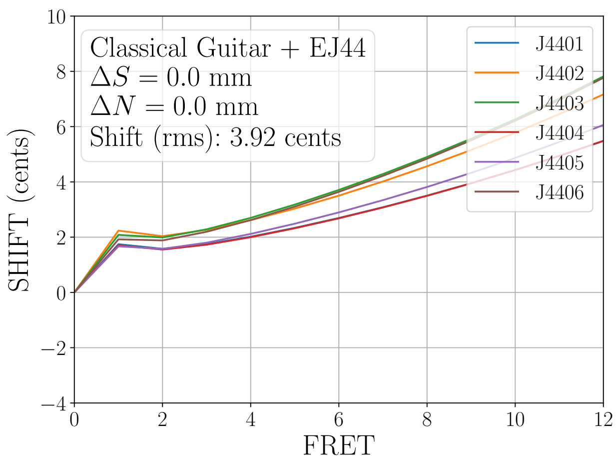
String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4401	E_4	0.368	4.01	73.6
J4402	B_3	0.423	5.44	56.1
J4403	G_3	0.528	8.91	57.9
J4404	D_3	0.381	20.07	73.1
J4405	A_2	0.457	34.87	71.3
J4406	E_2	0.571	61.36	70.4

Table 12: Derived physical properties of the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). The corresponding scale length is 650 mm.

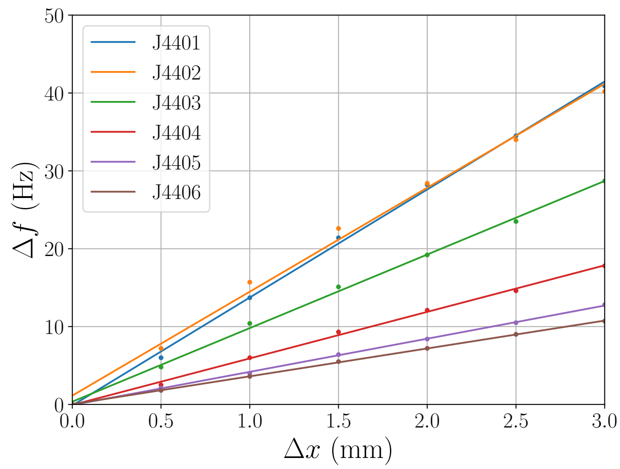
String	R	σ	κ	B_0	E (GPa)
J4401	25.8	0.4	52.6	0.00206	9.09
J4402	33.2	1.1	67.3	0.00267	6.72
J4403	29.6	0.6	60.2	0.00315	3.97
J4404	25.0	0.5	51.0	0.00209	8.17
J4405	23.7	0.2	48.5	0.00245	5.27
J4406	26.6	0.2	54.3	0.00324	3.73

Table 13: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44) on the Classical Guitar.

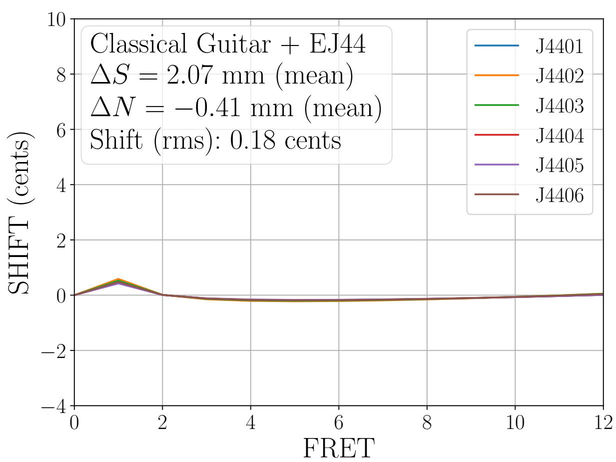
String	ΔS (mm)	ΔN (mm)	$\overline{\Delta\nu}_{\rm rms}$ (cents)
J4401	1.67	-0.39	0.18
J4402	2.18	-0.49	0.22
J4403	2.48	-0.44	0.20
J4404	1.68	-0.37	0.17
J4405	1.92	-0.36	0.17
J4406	2.51	-0.40	0.18



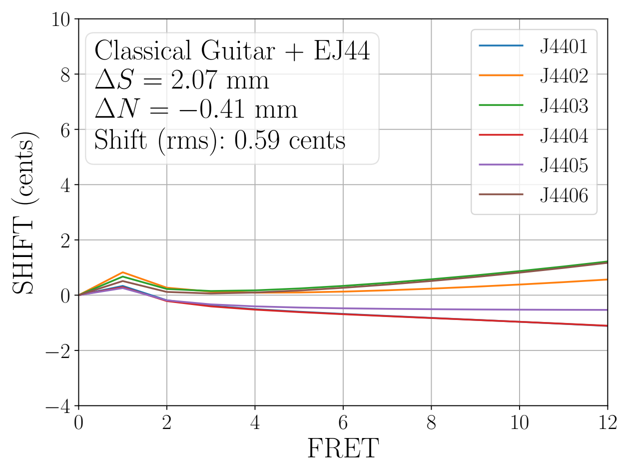
(a) Uncompensated



(b) Linear fits



(c) Full compensation



(d) Mean compensation

Figure 21: Frequency shift (in cents) for a classical guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). Four different strategies of saddle and nut compensation are illustrated.

D.4 Normal Tension – Carbon

Table 14: String specifications for the D'Addario Pro-Arte Carbon Classical Guitar Strings – Normal Tension (EJ45FF). The corresponding scale length is 650 mm.

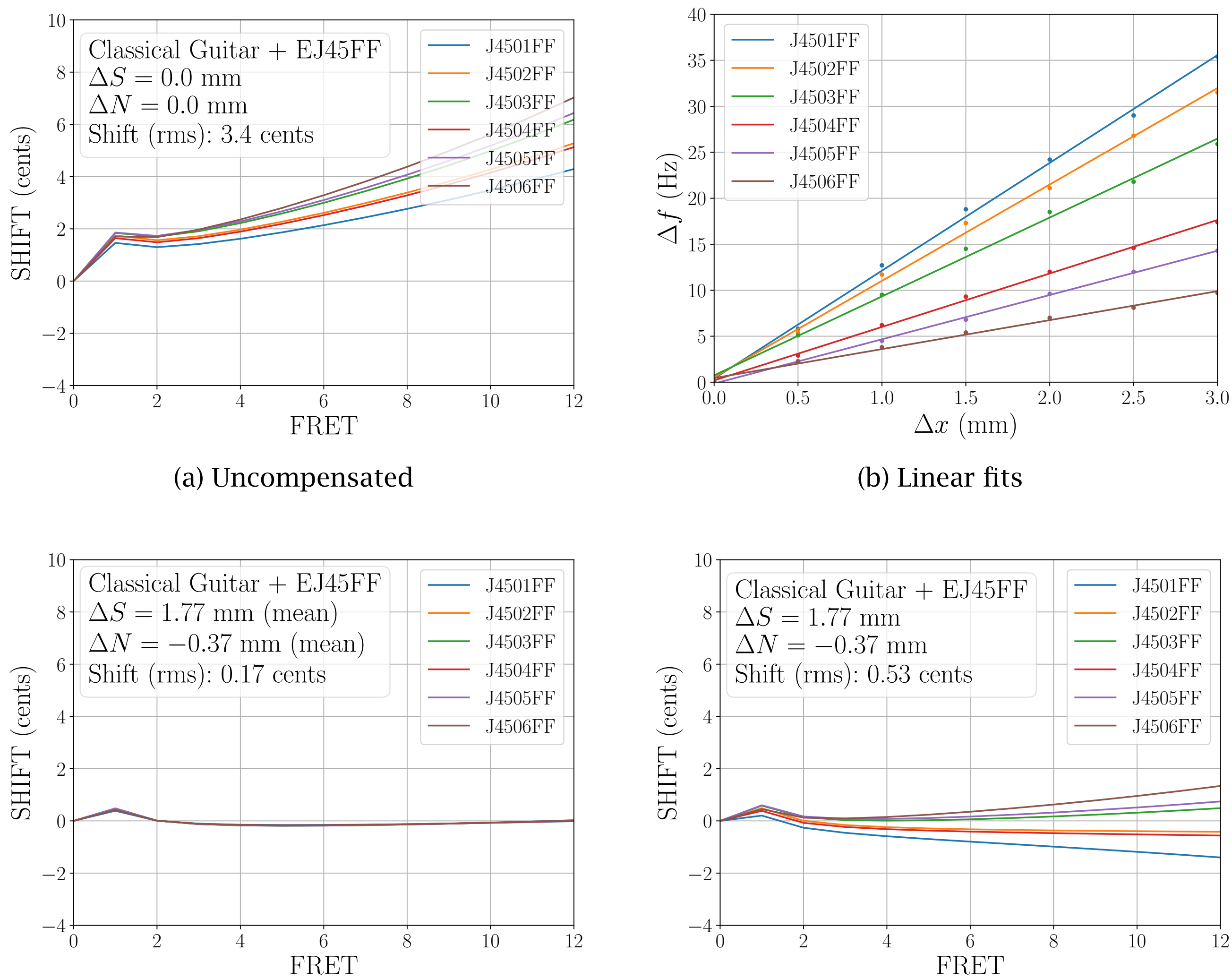
String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4501FF	E_4	0.305	4.64	85.3
J4502FF	B_3	0.345	6.07	62.6
J4503FF	G_3	0.420	8.93	58.0
J4504FF	D_3	0.356	16.43	59.9
J4505FF	A_2	0.445	30.90	63.2
J4506FF	E_2	0.559	57.16	65.6

Table 15: Derived physical properties of the D'Addario Pro-Arte Carbon Classical Guitar Strings – Normal Tension (EJ45FF). The corresponding scale length is 650 mm.

String	R	σ	κ	B_0	E (GPa)
J4501FF	21.8	0.4	44.6	0.00157	13.04
J4502FF	26.0	0.6	53.1	0.00194	8.86
J4503FF	26.9	0.8	54.7	0.00239	5.71
J4504FF	24.3	0.4	49.6	0.00193	7.47
J4505FF	26.9	0.4	54.7	0.00253	5.57
J4506FF	23.5	0.9	47.9	0.00298	3.20

Table 16: Predicted setbacks for the D'Addario Pro-Arte Carbon Classical Guitar Strings – Normal Tension (EJ45FF) on the Classical Guitar.

String	ΔS (mm)	ΔN (mm)	$\overline{\Delta\nu}_{\mathrm{rms}}$ (cents)
J4501FF	1.29	-0.33	0.15
J4502FF	1.59	-0.39	0.18
J4503FF	1.91	-0.40	0.18
J4504FF	1.56	-0.37	0.17
J4505FF	2.01	-0.40	0.18
J4506FF	2.29	-0.35	0.16



(a) Uncompensated

(b) Linear fits

(c) Full compensation

(d) Mean compensation

Figure 22: Frequency shift (in cents) for a classical guitar with D'Addario Pro-Arte Carbon Classical Guitar Strings – Normal Tension (EJ45FF). Four different strategies of saddle and nut compensation are illustrated.

D.5 Hard Tension – Carbon

Table 17: String specifications for the D'Addario Pro-Arte Carbon Classical Guitar Strings – Hard Tension (EJ46FF). The corresponding scale length is 650 mm.

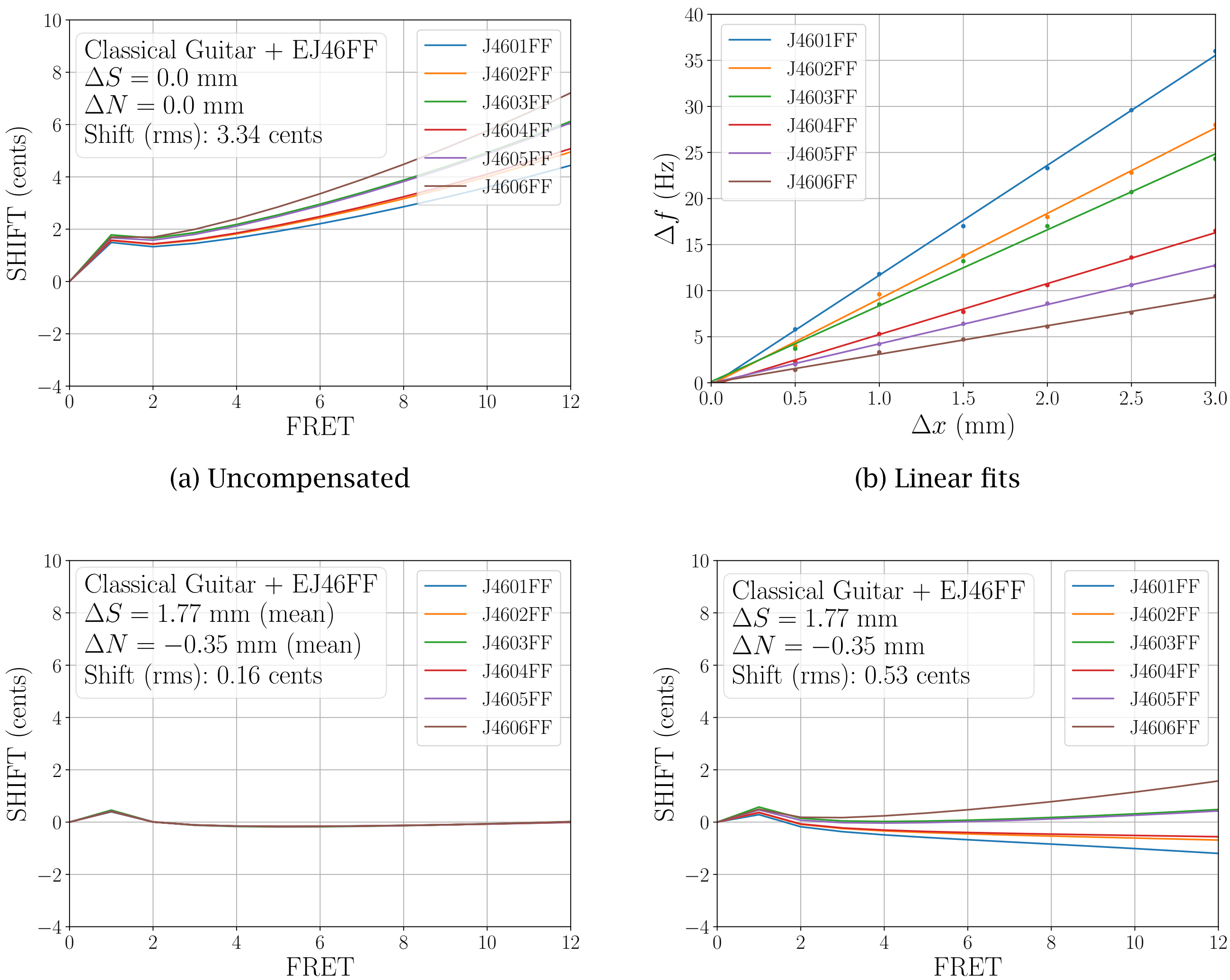
String	Note	Radius (mm)	Density ($\times 10^{-7}$ kg/mm)	Tension (N)
J4601FF	E_4	0.315	5.00	91.8
J4602FF	B_3	0.356	6.43	66.3
J4603FF	G_3	0.431	9.47	61.5
J4604FF	D_3	0.368	18.40	67.0
J4605FF	A_2	0.457	35.55	72.7
J4606FF	E_2	0.584	61.27	70.3

Table 18: Derived physical properties of the D'Addario Pro-Arte Carbon Classical Guitar Strings – Hard Tension (EJ46FF). The corresponding scale length is 650 mm.

String	R	σ	κ	B_0	E (GPa)
J4601FF	22.2	0.3	45.4	0.00163	13.39
J4602FF	23.1	0.4	47.1	0.00188	7.86
J4603FF	25.8	0.6	52.6	0.00240	5.55
J4604FF	23.1	0.4	47.2	0.00195	7.43
J4605FF	23.8	0.2	48.6	0.00245	5.38
J4606FF	23.1	0.4	47.2	0.00309	3.09

Table 19: Predicted setbacks for the D'Addario Pro-Arte Carbon Classical Guitar Strings – Hard Tension (EJ46FF) on the Classical Guitar.

String	ΔS (mm)	ΔN (mm)	$\overline{\Delta\nu}_{\mathrm{rms}}$ (cents)
J4601FF	1.34	-0.33	0.16
J4602FF	1.52	-0.35	0.16
J4603FF	1.91	-0.39	0.18
J4604FF	1.56	-0.35	0.16
J4605FF	1.92	-0.36	0.17
J4606FF	2.36	-0.34	0.16



(a) Uncompensated

(b) Linear fits

(c) Full compensation

(d) Mean compensation

Figure 23: Frequency shift (in cents) for a classical guitar with D'Addario Pro-Arte Carbon Classical Guitar Strings – Hard Tension (EJ46FF). Four different strategies of saddle and nut compensation are illustrated.

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