

# Classical Guitar Intonation and Compensation: The Well-Tempered Guitar

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## Abstract

TBD.

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## 1 Introduction and Background

Discuss initial [1] and ongoing work by G. Byers [2], and recent studies of steel-string guitars [3].

## 2 Simple Model of Guitar Intonation

Fundamental frequency of a string [4, 5]:

$$f_0 = \frac{1}{2L_0} \sqrt{\frac{T_0}{\mu_0}}, \quad (1)$$

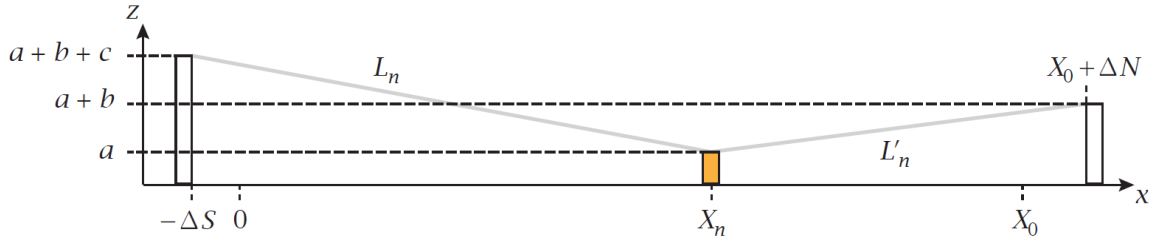


Figure 1: A simple (side-view) schematic of the classical guitar used in this model. The scale length of the guitar is  $X_0$ , but we allow the edges of both the saddle and the nut to be set back an additional distance  $\Delta S$  and  $\Delta N$ , respectively. The location on the  $x$ -axis of the center of the  $n^{\text{th}}$  fret is  $X_n$ . In the  $z$  direction,  $z = 0$  is taken as the surface of the fingerboard; therefore the height of each fret above the fingerboard is  $a$ , the height of the nut is  $a + b$ , and the height of the saddle is  $a + b + c$ .  $L_n$  is the *resonant length* of the string from the saddle to the center of fret  $n$ , and  $L'_n$  is the length of the string from the fret to the nut.

where  $L_0$  is the length of the free (unfretted) string from the saddle to the nut,  $T_0$  is the tension in the free string, and  $\mu_0 \equiv M/L_0$  is the linear mass density of a free string of mass  $M$ .

$$f_0 = \frac{1}{2L_0} \sqrt{\frac{T_0}{\mu_0}} \left[ 1 + B_0 + \left( 1 + \frac{\pi^2}{8} \right) B_0^2 \right], \quad (2)$$

where  $B_0$  is a “string stiffness parameter.” For a uniform string with a cylindrical cross section,  $B_0$  given by [6]

$$B_0 \equiv \sqrt{\frac{\pi R^4 E}{T_0 L_0^2}}, \quad (3)$$

where  $R$  is the radius of the string and  $E$  is Young’s modulus (or the modulus of elasticity). For a typical nylon guitar string with  $E \approx 2 - 4$  GPa,  $T_0 \approx 50 - 70$  N,  $R \approx 0.35 - 0.51$  mm, and  $L_0 \approx 650$  mm, we have  $B_0 \approx 0.007 - 0.026$ , indicating that the corrections in Eq. (2) are not significant.

Throughout this work, we will use *cents* to describe small differences in pitch [7]. One cent is one one-hundredth of a 12-TET half step, so that there are 1200 cents per octave. The difference in pitch between frequencies  $f_1$  and  $f_2$  is therefore defined as

$$\Delta v \equiv 1200 \log_2 \left( \frac{f_2}{f_1} \right). \quad (4)$$

We define  $f \equiv (f_1 + f_2)/2$  and  $\Delta f \equiv f_2 - f_1$ . Then

$$\Delta v = 1200 \log_2 \left( \frac{f + \Delta f/2}{f - \Delta f/2} \right) \approx \frac{1200}{\ln 2} \frac{\Delta f}{f}, \quad (5)$$

where the last approximation applies when  $\Delta f \ll f$ . An experienced guitar player can distinguish beat notes with a difference frequency of  $\Delta f \approx 1$  Hz, which corresponds to 8 cents at  $A_2$  ( $f = 220$  Hz) or 5 cents at  $E_4$  ( $f = 329.63$  Hz).

Our model begins with the schematic of the guitar shown in Fig. 1. The scale length of the guitar is  $X_0$ , but we allow the edges of both the saddle and the nut to be set back an additional distance  $\Delta S$  and  $\Delta N$ , respectively. The location on the  $x$ -axis of the center of the  $n^{\text{th}}$  fret is  $X_n$ . In the  $z$  direction,  $z = 0$  is taken as the surface of the fingerboard; the height of each fret

is  $a$ , the height of the nut is  $a + b$ , and the height of the saddle is  $a + b + c$ .  $L_n$  is the *resonant length* of the string from the saddle to the center of fret  $n$ , and  $L'_n$  is the length of the string from the fret to the nut. The total length of the string is defined as  $\mathcal{L}_n \equiv L_n + L'_n$ . For reasons discussed below, we have not adopted a more complicated fretting model [2, 3]. We start with the simple form of the fundamental frequency of a string given by Eq. (1), and apply it to the frequency of a string pressed just behind the  $n^{\text{th}}$  fret:

$$f_n = \frac{1}{2L_n} \sqrt{\frac{T_n}{\mu_n}}, \quad (6)$$

where  $T_n$  and  $\mu_n$  are respectively the tension and linear mass density of the fretted string. We note that  $T_n$  and  $\mu_n$  depend on  $\mathcal{L}_n$ , the *total* length of the fretted string from the saddle to the nut. Ideally, in the 12-TET equal-temperament system [7],

$$f_n = \gamma_n f_0, \quad (12\text{-TET ideal}) \quad (7)$$

where

$$\gamma_n \equiv 2^{n/12}. \quad (8)$$

Therefore, the error interval expressed in cents is given by

$$\begin{aligned} \Delta v_n &= 1200 \log_2 \left( \frac{f_n}{\gamma_n f_0} \right) \\ &= 1200 \log_2 \left( \frac{L_0}{\gamma_n L_n} \sqrt{\frac{\mu_0}{\mu_n} \frac{T_n}{T_0}} \right) \\ &= 1200 \log_2 \left( \frac{L_0}{\gamma_n L_n} \right) + 600 \log_2 \left( \frac{\mu_0}{\mu_n} \right) + 600 \log_2 \left( \frac{T_n}{T_0} \right). \end{aligned} \quad (9)$$

The final form of Eq. (9) makes it clear that — for nylon guitar strings — there are three contributions to intonation:

1. *Resonant Length*: The first term represents the error caused by the increase in the length of the fretted string  $L_n$  compared to the ideal length  $X_n$ , which would be obtained if  $b = c = 0$  and  $\Delta S = \Delta N = 0$ .
2. *Linear Mass Density*: The second term is the error caused by the reduction of the linear mass density of the fretted string. This effect will depend on the *total* length of the string, given by  $\mathcal{L}_n = L_n + L'_n$ .
3. *Tension*: The third (and most complex) term is the error caused by the *increase* of the tension in the string caused by the stress and strain applied to the string by fretting. This effect will also depend on the total length of the string  $\mathcal{L}_n$ .

We will discuss each of these three sources of error in turn below.

## 2.1 Resonant Length Error

We can estimate the first term in the last line of Eq. (9) by referring to Fig. 1 and computing the resonant length  $L_n$ . We find:

$$L_n = \begin{cases} \sqrt{(X_0 + \Delta S + \Delta N)^2 + c^2}, & n = 0 \\ \sqrt{(X_n + \Delta S)^2 + (b + c)^2}. & n \geq 1 \end{cases} \quad (10)$$

When  $b + c \ll X_0$ , we can approximate  $L_n$  by

$$L_n \approx \begin{cases} X_0 + \Delta S + \Delta N + c^2/2 X_0 & n = 0, \\ X_n + \Delta S + (b + c)^2/2 X_n & n \geq 1. \end{cases} \quad (11)$$

Then — when the guitar has been manufactured such that  $X_n = X_0/\gamma_n$  — the resonant length error is approximately

$$1200 \log_2 \left( \frac{L_0}{\gamma_n L_n} \right) \approx -\frac{1200}{\ln(2)} \left[ \frac{(\gamma_n - 1) \Delta S - \Delta N}{X_0} + \frac{\gamma_n^2 (b + c)^2 - c^2}{2 X_0^2} \right] \quad (12)$$

If the guitar is uncompensated, so that  $\Delta S = \Delta N = 0$ , this error is typically less than 0.25 cents. But, with  $\Delta S > 0$  and  $\Delta N < 0$ , we can significantly *increase* the magnitude of this “error” and cause the frequency to shift lower. We’ll see that this is our primary method of compensation.

**TBD: Add a figure here?**

Previous studies of guitar intonation and compensation have chosen to include the apparent increase in length of the string caused by both the fretting depth (**Matt: How does this sound?**) and the shape of the fretted string under the finger [2, 3]. As the string is initially pressed to the fret, the total length  $\mathcal{L}_n$  increases and causes the tension in the string — which is clamped at the saddle and the nut — to increase. As the string is pressed further, does the additional deformation of the string increase its tension (throughout the resonant length  $L_n$ )? There are at least two purely empirical reasons to doubt this hypothesis. First, we can mark a string (with a fine-point felt pen) above a particular fret and then observe the mark with a magnifying glass. As the string is pressed all the way to the finger board, the mark does not move perceptibly — it has become effectively *clamped* on the fret. Second, we can use either our ears or a simple tool to measure frequencies [8] to listen for a shift as we use different fingers and vary the fretted depth of a string. The apparent modulation is far less than would be obtained by classical vibrato ( $\pm 15$  cents), so we assume that once the string is minimally fretted the length(s) can be regarded as fixed. (If this were not the case, then fretting by different people or with different fingers, at a single string or with a barre, would cause additional, varying frequency shifts that would be audible and difficult to compensate.)

## 2.2 Linear Mass Density Error

As discussed above, the linear mass density  $\mu_0$  of an unfretted string is simply the total mass  $M$  of the string clamped between the saddle and the nut divided by the length  $L_0$ . Similarly, the mass density  $\mu_n$  of a string held onto fret  $N$  is  $M/\mathcal{L}_n$ . Therefore

$$\frac{\mu_0}{\mu_n} = \frac{\mathcal{L}_n}{L_0} \equiv 1 + \lambda_n, \quad (13)$$

where

$$\lambda_n \equiv \frac{\mathcal{L}_n - L_0}{L_0}. \quad (14)$$

Since we expect that  $\lambda_n \ll 1$ , we can approximate the second term in the final line of Eq. (9) as

$$600 \log_2 \left( \frac{\mu_0}{\mu_n} \right) \approx \frac{600}{\ln(2)} \lambda_n. \quad (15)$$

Referring to Fig. 1, we see that  $\mathcal{L}_n = L_n + L'_n$ , and we calculate  $L'_n$  as

$$L'_n = \begin{cases} 0, & n = 0 \\ \sqrt{(X_0 - X_n + \Delta N)^2 + b^2}. & n \geq 1 \end{cases} \quad (16)$$

Assuming that  $b^2 \ll X_0 - X_n$ , we expand the  $n \neq 1$  expression to obtain

$$L'_n \approx X_0 - X_n + \Delta N + \frac{b^2}{2(X_0 - X_n)} \quad (17)$$

Therefore, using Eq. (11), we have for  $n \neq 1$

$$\mathcal{L}_n = L_n + L'_n \approx X_0 + \Delta S + \Delta N + \frac{(b + c)^2}{2X_n} + \frac{b^2}{2(X_0 - X_n)}, \quad (18)$$

and

$$\begin{aligned} \lambda_n &\approx \frac{1}{2X_0} \left[ \frac{(b + c)^2}{2X_n} + \frac{b^2}{2(X_0 - X_n)} - \frac{c^2}{2X_0} \right] \\ &= \frac{\gamma_n}{2X_0^2} \left[ (b + c)^2 + \frac{b^2}{\gamma_n - 1} - \frac{c^2}{\gamma_n} \right]. \end{aligned} \quad (19)$$

This error is generally quite small. Suppose that  $b = 1.0$  mm,  $c = 3.5$  mm,  $X_0 = 650$  mm,  $n = 12$ , and  $X_{12} = X_0/2 = 325$  mm. Then the error is about 0.03 cents, and will be even smaller for  $n < 12$ . If we add this shift due to the linear mass density to the residual quadratic resonant length shift given by Eq. (12), then we find the total error

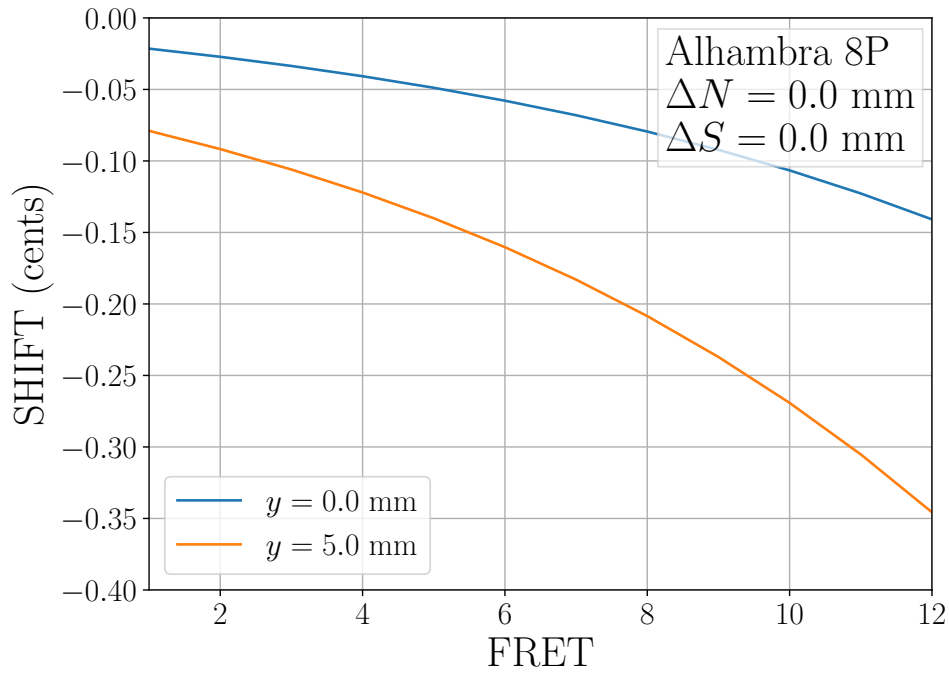
$$\Delta v_n = \frac{300}{\ln(2)} \frac{\gamma_n}{X_0^2} \left[ \frac{b^2}{\gamma_n - 1} + \frac{c^2}{\gamma_n} - (2\gamma_n - 1)(b + c)^2 \right]. \quad (20)$$

For the same parameters,  $\Delta v_{12} = -0.11$  cents, and  $|\Delta v_n|$  is smaller for  $n < 12$ .

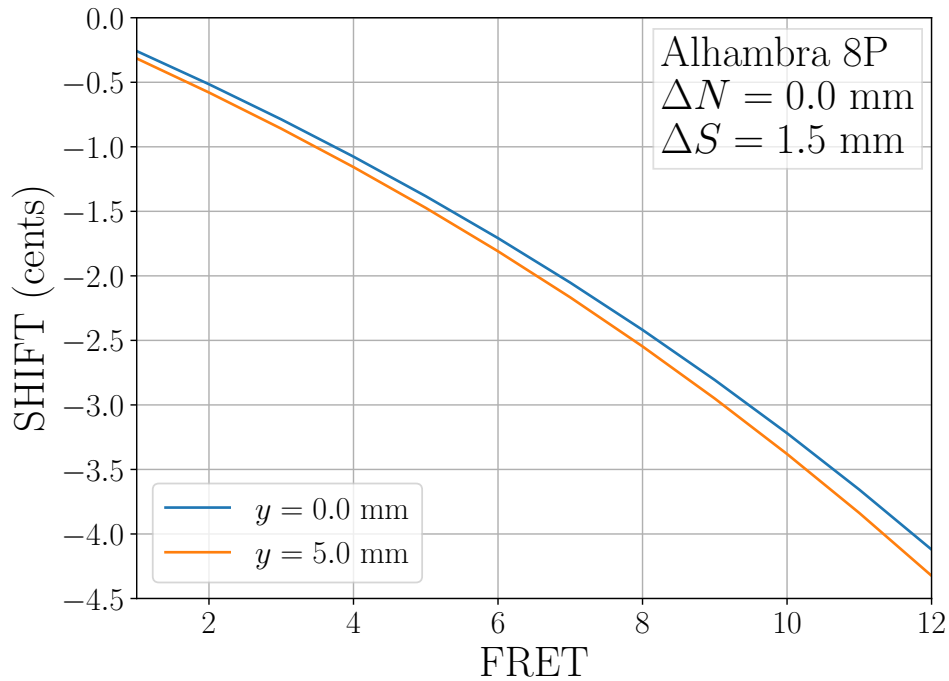
### 3 Experimental Estimate of the String Constant

### 4 Classical Guitar Compensation

### 5 Conclusion

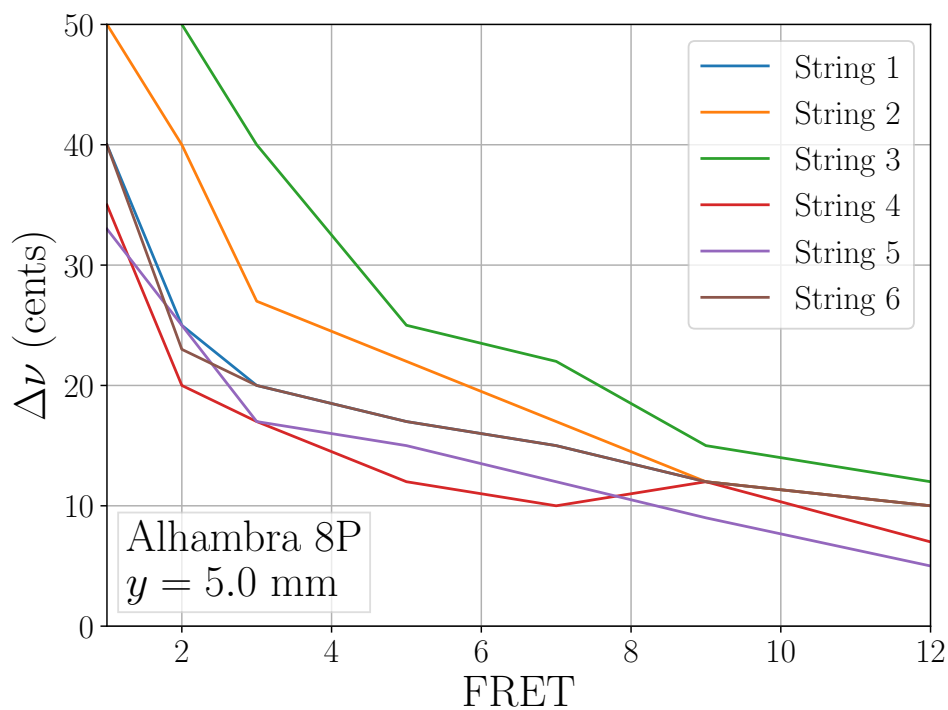


(a) Frequency shift for an uncompensated guitar



(b) Frequency shift for a factory guitar

Figure 2: Frequency shift (in cents) due to the fretted length  $L_n$  for an uncompensated (a) and factory (b) Alhambra 8P guitar, for both zero and nonzero lateral displacement  $y$ .



(a) Experimental data

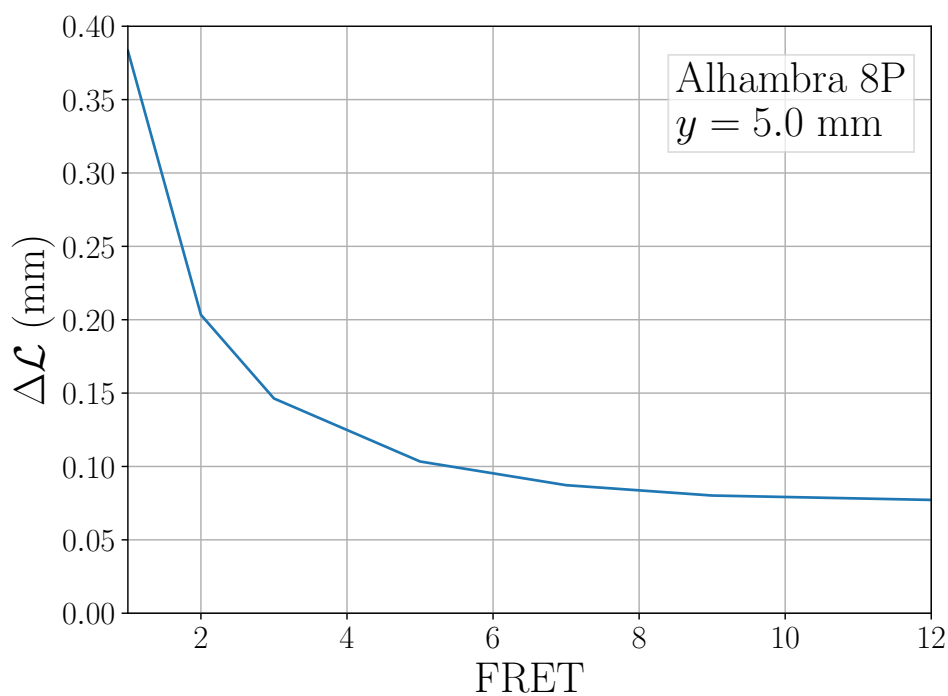
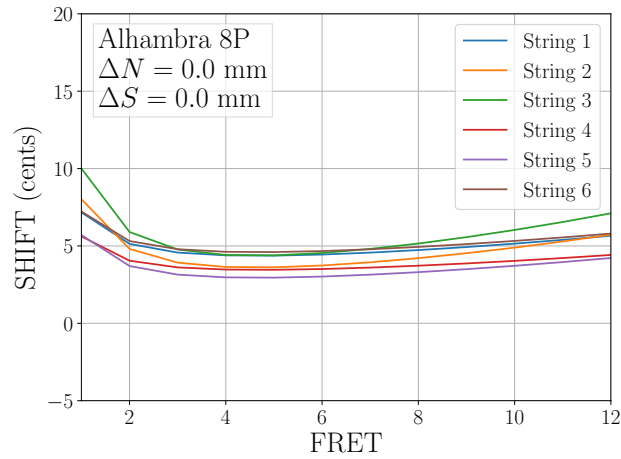
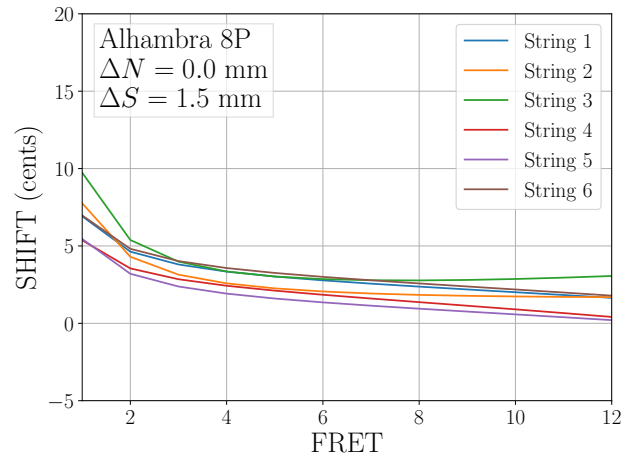
(b) Calculated change in total string length  $\mathcal{L}$ 

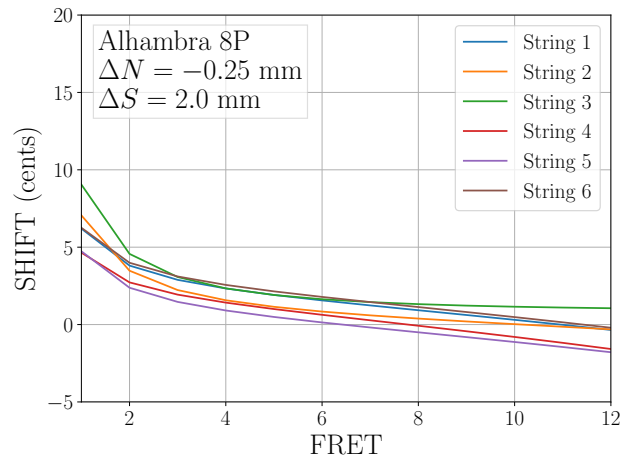
Figure 3: Frequency shift (in cents) (a) and change in total string length  $\mathcal{L}$  (b) due to lateral displacement  $y$ .



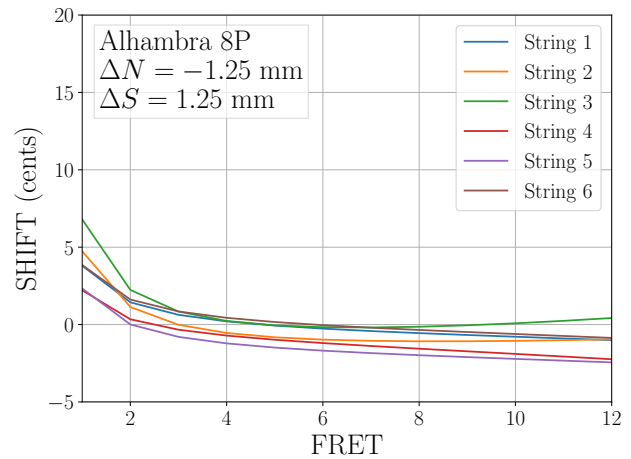
(a) Uncompensated



(b) Factory guitar



(c) Saddle and nut



(d) Preserved scale length

Figure 4: Frequency shift (in cents) for four different strategies of saddle and nut compensation.



## References

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