

# Classical Guitar Intonation and Compensation: The Well-Tempered Guitar

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## Abstract

TBD.

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## 1 Introduction and Background

Discuss initial [1] and ongoing work by G. Byers [2], and recent studies of steel-string guitars [3]. General references on the physics of sound, music, and musical instruments are [4].

Fundamental frequency of a string [5]:

$$f_0 = \frac{1}{2L_0} \sqrt{\frac{T_0}{\mu_0}}, \quad (1)$$

where  $L_0$  is the length of the free (unfretted) string from the saddle to the nut,  $T_0$  is the tension in the free string, and  $\mu_0 \equiv M/L_0$  is the linear mass density of a free string of mass  $M$ .

## 2 Simple Model of Guitar Intonation

Here's the old *wrong* equation:

$$f_{mn} = \frac{m}{2L_n} \sqrt{\frac{T_n}{\mu_n}} \left[ 1 + 2B_n + 4 \left( 1 + \frac{\pi^2 m^2}{8} \right) B_n^2 \right], \quad (2)$$

Here's the new equation, from Appendix A, applied to fret  $n$ :

$$f_{mn} = \frac{m}{2L_n} \sqrt{\frac{T_n}{\mu_n}} (1 + B_n), \quad (3)$$

where  $L_n$  is the length of the vibrating string from the saddle to fret  $n$ , and  $T_n$  and  $\mu_n$  are the tension and the linear mass density of the fretted string, respectively. Here, we are using  $B_n$  to represent a small string “bending stiffness” coefficient to capture the relevant mechanical properties of the transversely vibrating string. For a homogeneous string with a cylindrical cross section,  $B_n$  is given by

$$B_n \equiv \sqrt{\frac{\pi \rho^4 E}{4 T_n L_n^2}}, \quad (4)$$

where  $\rho$  is the radius of the string and  $E$  is Young's modulus (or the modulus of elasticity). Note that Eq. (3) is valid only when  $B_n \ll 1$ . For a typical nylon guitar string with  $E \approx 4$  GPa,  $T_0 \approx 60$  N,  $\rho \approx 0.5$  mm, and  $L_0 \approx 650$  mm, we have  $B_0 \approx 6 \times 10^{-3}$ . We'll use Eq. (3) and Eq. (4) with considerable caution, because it's unlikely that they describe the physics of nylon strings (particularly the wound base strings) with a high degree of accuracy [6].

Throughout this work, we will use *cents* to describe small differences in pitch [7]. One cent is one one-hundredth of a Twelve-Tone Equal Temperament (12-TET) half step, so that there are 1200 cents per octave. The difference in pitch between frequencies  $f_1$  and  $f_2$  is therefore defined as

$$\Delta v \equiv 1200 \log_2 \left( \frac{f_2}{f_1} \right). \quad (5)$$

We define  $f \equiv (f_1 + f_2)/2$  and  $\Delta f \equiv f_2 - f_1$ . Then

$$\Delta v = 1200 \log_2 \left( \frac{f + \Delta f/2}{f - \Delta f/2} \right) \approx \frac{1200}{\ln 2} \frac{\Delta f}{f}, \quad (6)$$

where the last approximation applies when  $\Delta f \ll f$ . An experienced guitar player can distinguish beat notes with a difference frequency of  $\Delta f \approx 1$  Hz, which corresponds to 8 cents at  $A_2$  ( $f = 220$  Hz) or 5 cents at  $E_4$  ( $f = 329.63$  Hz).

Our model begins with the schematic of the guitar shown in Fig. 1. The scale length of the guitar is  $X_0$ , but we allow the edges of both the saddle and the nut to be set back an additional distance  $\Delta S$  and  $\Delta N$ , respectively. The location on the  $x$ -axis of the center of the  $n^{\text{th}}$  fret is  $X_n$ . In the  $z$  direction,  $z = 0$  is taken as the surface of the fingerboard; the height of each fret is  $a$ , the height of the nut is  $a + b$ , and the height of the saddle is  $a + b + c$ .  $L_n$  is the *resonant length* of the string from the saddle to the center of fret  $n$ , and  $L'_n$  is the length of the string

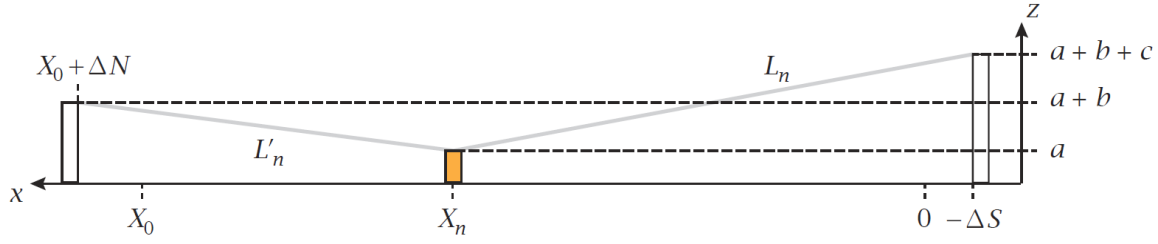


Figure 1: A simple (side-view) schematic of the classical guitar used in this model. The scale length of the guitar is  $X_0$ , but we allow the edges of both the saddle and the nut to be set back an additional distance  $\Delta S$  and  $\Delta N$ , respectively. The location on the  $x$ -axis of the center of the  $n^{\text{th}}$  fret is  $X_n$ . (Note that the  $x$ -axis is directed to the left in this figure.) In the  $z$  direction,  $z = 0$  is taken as the surface of the fingerboard; therefore the height of each fret above the fingerboard is  $a$ , the height of the nut is  $a + b$ , and the height of the saddle is  $a + b + c$ .  $L_n$  is the *resonant length* of the string from the saddle to the center of fret  $n$ , and  $L'_n$  is the length of the string from the fret to the nut.

from the fret to the nut. The total length of the string is defined as  $\mathcal{L}_n \equiv L_n + L'_n$ . For reasons discussed below, we have not adopted a more complicated fretting model [2, 3]. We start with form of the fundamental frequency of a fretted string given by Eq. (3) with  $m = 1$ , and apply it to the frequency of a string pressed just behind the  $n^{\text{th}}$  fret:

$$f_n = \frac{1}{2L_n} \sqrt{\frac{T_n}{\mu_n}} (1 + B_n), \quad (7)$$

where we have neglected the smallest stiffness correction term proportional to  $B_n^2$ . We note that  $T_n$  and  $\mu_n$  depend on  $\mathcal{L}_n$ , the *total* length of the fretted string from the saddle to the nut. Ideally, in the 12-TET system [7],

$$f_n = \gamma_n f_0, \quad (12\text{-TET ideal}) \quad (8)$$

where

$$\gamma_n \equiv 2^{n/12}. \quad (9)$$

Therefore, the error interval expressed in cents is given by

$$\begin{aligned} \Delta v_n &= 1200 \log_2 \left( \frac{f_n}{\gamma_n f_0} \right) \\ &= 1200 \log_2 \left( \frac{L_0}{\gamma_n L_n} \sqrt{\frac{\mu_0}{\mu_n} \frac{T_n}{T_0} \frac{1 + B_n}{1 + B_0}} \right) \\ &= 1200 \log_2 \left( \frac{L_0}{\gamma_n L_n} \right) + 600 \log_2 \left( \frac{\mu_0}{\mu_n} \right) + 600 \log_2 \left( \frac{T_n}{T_0} \right) + 1200 \log_2 \left( \frac{1 + B_n}{1 + B_0} \right). \end{aligned} \quad (10)$$

The final form of Eq. (10) makes it clear that — for nylon guitar strings — there are four contributions to intonation:

1. *Resonant Length*: The first term represents the error caused by the increase in the length of the fretted string  $L_n$  compared to the ideal length  $X_n$ , which would be obtained if  $b = c = 0$  and  $\Delta S = \Delta N = 0$ .

2. *Linear Mass Density*: The second term is the error caused by the reduction of the linear mass density of the fretted string. This effect will depend on the *total* length of the string, given by  $\mathcal{L}_n = L_n + L'_n$ .
3. *Tension*: The third term is the error caused by the *increase* of the tension in the string arising from the stress and strain applied to the string by fretting. This effect will also depend on the total length of the string  $\mathcal{L}_n$ .
4. *Bending Stiffness*: The fourth and final term is the error caused by the change in the bending stiffness coefficient caused by changing the length of the string from  $L_0$  to  $L_n$ .

We will discuss each of these three sources of error in turn below.

## 2.1 Resonant Length

We can estimate the first term in the last line of Eq. (10) by referring to Fig. 1 and computing the resonant length  $L_n$ . We find:

$$L_n = \begin{cases} \sqrt{(X_0 + \Delta S + \Delta N)^2 + c^2}, & n = 0 \\ \sqrt{(X_n + \Delta S)^2 + (b + c)^2}. & n \geq 1 \end{cases} \quad (11)$$

When  $b + c \ll X_0$ , we can approximate  $L_n$  by

$$L_n \approx \begin{cases} X_0 + \Delta S + \Delta N + c^2/2X_0 & n = 0, \\ X_n + \Delta S + (b + c)^2/2X_n & n \geq 1. \end{cases} \quad (12)$$

Then — when the guitar has been manufactured such that  $X_n = X_0/y_n$  — the resonant length error is approximately

$$1200 \log_2 \left( \frac{L_0}{y_n L_n} \right) \approx \frac{1200}{\ln(2)} \left[ \frac{\Delta N - (y_n - 1) \Delta S}{X_0} - \frac{y_n^2 (b + c)^2 - c^2}{2 X_0^2} \right] \quad (13)$$

If the guitar is uncompensated, so that  $\Delta S = \Delta N = 0$ , this error is typically less than 0.25 cents. But, with  $\Delta S > 0$  and  $\Delta N < 0$ , we can significantly *increase* the magnitude of this “error” and cause the frequency to shift lower. We’ll see that this is our primary method of compensation.

TBD: Add a figure here?

## 2.2 Linear Mass Density

As discussed above, the linear mass density  $\mu_0$  of an open (unfretted) string is simply the total mass  $M$  of the string clamped between the saddle and the nut divided by the length  $L_0$ . Similarly, the mass density  $\mu_n$  of a string held onto fret  $N$  is  $M/\mathcal{L}_n$ . Therefore

$$\frac{\mu_0}{\mu_n} = \frac{\mathcal{L}_n}{L_0} \equiv 1 + Q_n, \quad (14)$$

where we have followed Byers and defined [2, 3]

$$Q_n \equiv \frac{\mathcal{L}_n - L_0}{L_0}. \quad (15)$$

Since we expect that  $Q_n \ll 1$ , we can approximate the second term in the final line of Eq. (10) as

$$600 \log_2 \left( \frac{\mu_0}{\mu_n} \right) \approx \frac{600}{\ln(2)} Q_n. \quad (16)$$

Previous studies of guitar intonation and compensation have chosen to include the apparent increase in length of the string caused by both the fretting depth and the shape of the fretted string under the finger [2, 3]. As the string is initially pressed to the fret, the total length  $\mathcal{L}_n$  increases and causes the tension in the string — which is clamped at the saddle and the nut — to increase. As the string is pressed further, does the additional deformation of the string increase its tension (throughout the resonant length  $L_n$ )? There are at least two purely empirical reasons to doubt this hypothesis. First, we can mark a string (with a fine-point felt pen) above a particular fret and then observe the mark with a magnifying glass. As the string is pressed all the way to the finger board, the mark does not move perceptibly — it has become effectively *clamped* on the fret. Second, we can use either our ears or a simple tool to measure frequencies [8] to listen for a shift as we use different fingers and vary the fretted depth of a string. The apparent modulation is far less than would be obtained by classical vibrato ( $\pm 15$  cents), so we assume that once the string is minimally fretted the length(s) can be regarded as fixed. (If this were not the case, then fretting by different people or with different fingers, at a single string or with a barre, would cause additional, varying frequency shifts that would be audible and difficult to compensate.)

Referring to Fig. 1, we see that  $\mathcal{L}_n = L_n + L'_n$ , and we calculate  $L'_n$  as

$$L'_n = \begin{cases} 0, & n = 0 \\ \sqrt{(X_0 - X_n + \Delta N)^2 + b^2}, & n \geq 1 \end{cases} \quad (17)$$

Assuming that  $b^2 \ll X_0 - X_n$ , we expand the  $n \neq 1$  expression to obtain

$$L'_n \approx X_0 - X_n + \Delta N + \frac{b^2}{2(X_0 - X_n)} \quad (18)$$

Therefore, using Eq. (12), we have for  $n \neq 1$

$$\mathcal{L}_n = L_n + L'_n \approx X_0 + \Delta S + \Delta N + \frac{(b+c)^2}{2X_n} + \frac{b^2}{2(X_0 - X_n)}, \quad (19)$$

and

$$\begin{aligned} Q_n &\approx \frac{1}{2X_0} \left[ \frac{(b+c)^2}{X_n} + \frac{b^2}{X_0 - X_n} - \frac{c^2}{X_0} \right] \\ &= \frac{\gamma_n}{2X_0^2} \left[ (b+c)^2 + \frac{b^2}{\gamma_n - 1} - \frac{c^2}{\gamma_n} \right]. \end{aligned} \quad (20)$$

When we substitute this expression into Eq. (16), the resulting error is generally quite small. Suppose that  $b = 1.0$  mm,  $c = 3.5$  mm,  $X_0 = 650$  mm,  $n = 12$ , and  $X_{12} = X_0/2 = 325$  mm. Then the error is about 0.03 cents, and will be even smaller for  $n < 12$ . If we add this shift due to the linear mass density to the residual quadratic resonant length shift given by Eq. (13), then we find the total error

$$\Delta v_n = \frac{300}{\ln(2)} \frac{\gamma_n}{X_0^2} \left[ \frac{b^2}{\gamma_n - 1} + \frac{c^2}{\gamma_n} - (2\gamma_n - 1)(b+c)^2 \right]. \quad (21)$$

For the same parameters,  $\Delta v_{12} = -0.11$  cents, and  $|\Delta v_n| < |\Delta v_{12}|$  for  $n < 12$ .

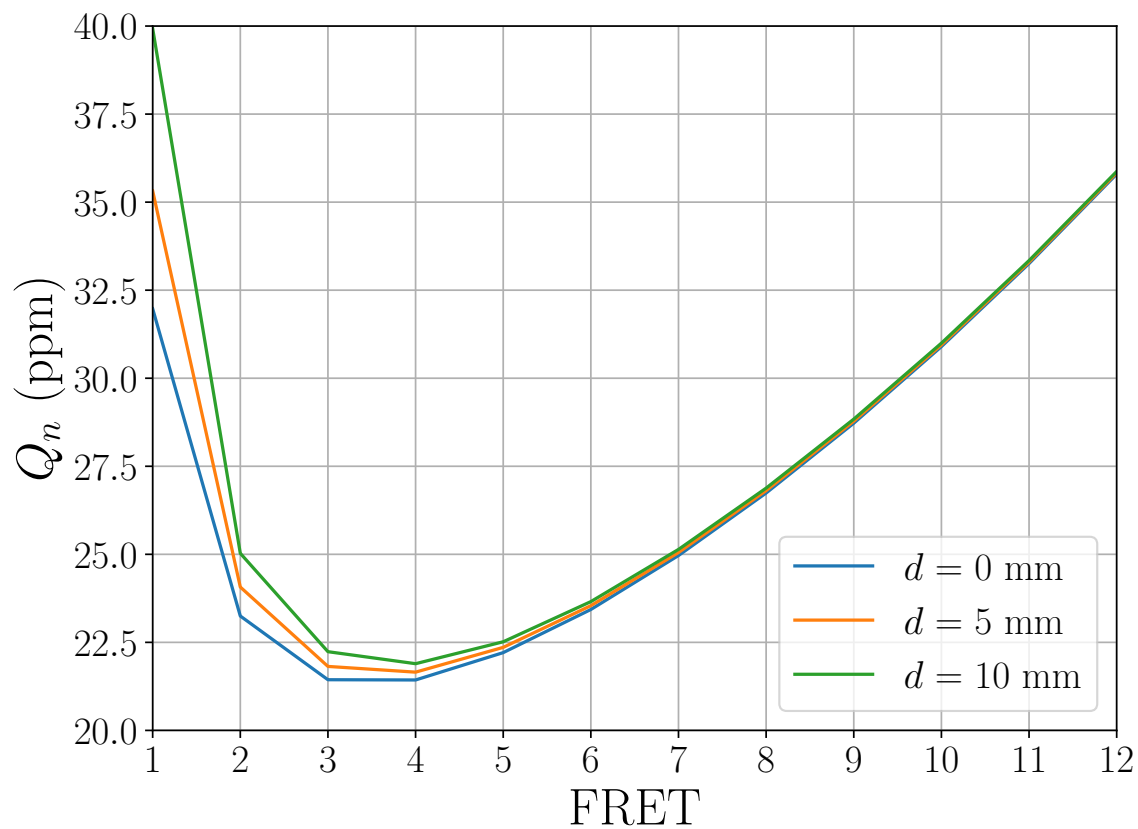


Figure 2: Plot of the normalized displacement  $Q_n$  as a function of the fret number for three different values of the parameter  $d$ . Here the guitar has  $b = 1.0$  mm,  $c = 3.5$  mm, no setbacks, and a scale length of 650 mm.

## 2.3 Tension

Elasticity properties [9]

$$\Delta T_n = E A \frac{\mathcal{L}_n - L_0}{L_0} = A E Q_n, \quad (22)$$

where we have used Eq. (15). Therefore, the tension of the fretted string is

$$T_n = T_0 + \Delta T_n = T_0 (1 + \kappa Q_n), \quad (23)$$

where we have defined the dimensionless “string constant”

$$\kappa \equiv \frac{A E}{T_0} = \frac{\pi \rho^2 E}{T_0}. \quad (24)$$

In this case, we assume that  $\kappa Q_n \ll 1$ , so that we can approximate the third term in the final line of Eq. (10) as

$$600 \log_2 \left( \frac{T_n}{T_0} \right) \approx \frac{600}{\ln(2)} \kappa Q_n. \quad (25)$$

This frequency shift is larger than that caused by the linear mass density error by a factor of  $\kappa$ .

## 2.4 Bending Stiffness

Since  $B_n$  is already relatively small, we only need to consider the largest contribution arising from the shortened length of the fretted string compared to that of the open string. We see from Eq. (11) that  $L_n \approx L_0 / \gamma_n$ , so from Eq. (4) we have

$$B_n = \sqrt{\frac{\pi \rho^4 E}{T_n L_n^2}} \approx \frac{L_0}{L_n} \sqrt{\frac{\pi \rho^4 E}{T_0 L_0^2}} = \gamma_n B_0. \quad (26)$$

Therefore, the fourth term in the final line of Eq. (10) can be approximated as

$$1200 \log_2 \left( \frac{1 + B_n}{1 + B_0} \right) \approx \frac{1200}{\ln(2)} (\gamma_n - 1) B_0. \quad (27)$$

When  $n = 12$  and  $B_0 = 6 \times 10^{-3}$ , the corresponding shift is approximately 10 cents. Note that (to zero order in  $Q_n$ ) this bending stiffness error does not depend on the tiny changes to the linear mass density or the tension that arises due to string fretting. Instead, it is an intrinsic property of the string.

## 3 Experimental Estimate of the String Constant

**This section needs to be rewritten.** How do we estimate the spring constant  $\kappa$  given by Eq. (24)? Recall how we introduced this concept in Section 2.3: increasing the tension by a quantity  $\Delta T$  causes a shift in the frequency of a guitar string by  $\Delta \nu$  in cents. Although we were considering fretted strings when we derived Eq. (10), the terms in that equation can be generalized to describe the case of an open string that has been stretched longitudinally. Suppose that we continue to clamp the string at the saddle and the nut, but that we tighten the tuning gear to stretch that string’s length by an amount  $\Delta L$ . The change in the string’s frequency due to

Table 1: Derived physical properties of the D’Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

String	$R$	$\kappa$	Modulus (GPa)	Stiffness
J4301	$2.67 \times 10^4$	29.8	5.17	$2.94 \times 10^{-3}$
J4302	$1.09 \times 10^4$	11.6	1.14	$2.11 \times 10^{-3}$
J4303	$6.63 \times 10^4$	75.6	4.97	$6.75 \times 10^{-3}$
J4304	$2.60 \times 10^4$	29.1	4.86	$2.95 \times 10^{-3}$
J4305	$2.66 \times 10^4$	29.7	3.02	$3.52 \times 10^{-3}$
J4306	$1.40 \times 10^4$	15.2	1.01	$3.20 \times 10^{-3}$

the change in the open resonant length is zero, because  $L_0/\gamma_0 L_0 = 1$ . The linear mass density of the string is smaller now because there is less material between the saddle and the nut, causing the frequency shift (in cents)

$$600 \log_2 \left( \frac{\mu}{\mu + \Delta\mu} \right) \approx \frac{600}{\ln(2)} \frac{\Delta L}{L}, \quad (28)$$

where  $L$  is the initial length of the string. Finally, the tension in the string increases by  $\Delta T$  due to the elastic properties of the string. Following the discussion in Section 2.3, the corresponding frequency shift is

$$600 \log_2 \left( \frac{T + \Delta T}{T} \right) \approx \frac{600}{\ln(2)} \frac{\Delta L}{L} \kappa, \quad (29)$$

where  $T$  is the initial tension of the string. Therefore, the total frequency shift of the open string caused by a change  $\Delta L$  in the string’s length is

$$\Delta\nu \approx \frac{600}{\ln(2)} \frac{\Delta L}{L} (\kappa + 1). \quad (30)$$

Solving this expression for the string constant, we find

$$\kappa = \frac{\ln(2)}{600} R - 1, \quad (31)$$

where

$$R \equiv \frac{L}{\Delta L} \Delta\nu \quad (32)$$

is a parameter originally defined by Byers<sup>1</sup> [2, 3].

It is relatively easy to estimate the value of  $R$  for any guitar string with the aid of a simple device that can measure frequency shifts in cents [8] and either calipers or a ruler with finely marked graduations. With a fine-point felt pen, make a small mark on the string at some convenient point (say, directly above the first fret). Then tighten that string’s tuning gear to increase the frequency by an amount measured using the electronic tuner. (A particularly convenient shift is four half-steps, corresponding to  $\Delta\nu = 400$  cents, which will require a stretch of approximate 5 – 6 mm.)

<sup>1</sup>Byers expressed this parameter in terms of the fractional frequency shift (in Hertz) as  $R = (L/\Delta L)(\Delta f/f) \approx (\ln(2)/1200)(L/\Delta L)\Delta\nu$ . Therefore, our dimensionless value of  $R$  is larger than Byers’ by a factor of about 1730.



Table 2: Derived physical properties of the D’Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). The corresponding scale length is 650 mm.

String	$R$	$\kappa$	Modulus (GPa)	Stiffness
J4601	$10.2 \times 10^4$	116.5	20.1	$6.01 \times 10^{-3}$
J4602	$4.13 \times 10^4$	46.7	4.65	$4.37 \times 10^{-3}$
J4603	$2.96 \times 10^4$	33.2	2.16	$4.61 \times 10^{-3}$
J4604	$4.66 \times 10^4$	52.8	8.47	$4.26 \times 10^{-3}$
J4605	$1.79 \times 10^4$	19.7	2.13	$3.12 \times 10^{-3}$
J4606	$2.65 \times 10^4$	29.6	1.96	$4.68 \times 10^{-3}$

Table 3: Derived physical properties of the D’Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). The corresponding scale length is 650 mm.

String	$R$	$\kappa$	Modulus (GPa)	Stiffness
J4401	$2.47 \times 10^4$	27.6	20.1	$2.98 \times 10^{-3}$
J4402	$5.64 \times 10^4$	64.2	4.65	$5.21 \times 10^{-3}$
J4403	$6.21 \times 10^4$	70.7	2.16	$6.83 \times 10^{-3}$
J4404	$4.66 \times 10^4$	52.8	8.47	$4.26 \times 10^{-3}$
J4405	$1.79 \times 10^4$	19.7	2.13	$3.12 \times 10^{-3}$
J4406	$2.56 \times 10^4$	28.5	1.96	$4.70 \times 10^{-3}$

## 4 Classical Guitar Compensation

$$\Delta v_n \approx \frac{1200}{\ln(2)} \left[ (\gamma_n - 1) \left( B_0 - \frac{\Delta S}{X_0} \right) + \frac{\Delta N}{X_0} + \frac{1}{2} \kappa Q_n \right]. \quad (33)$$

Strategy: Use  $\Delta S$  to compensate for bending stiffness, and  $\Delta N$  to compensate for tension increases arising from fretting. Doing so predicts that  $\Delta S \approx 3$  mm for the Alhambra 8P with light-tension strings, or about twice the saddle setback that is actually manufactured.

Table 4: Predicted setbacks for the D’Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43) on the Alhambra 8P classical guitar.

String	$\Delta S$ (mm)	$\Delta N$ (mm)
J4301	1.91	-0.31
J4302	1.37	-0.12
J4303	4.38	-0.79
J4304	1.92	-0.30
J4305	2.29	-0.31
J4306	2.08	-0.16
Mean	2.32	-0.33

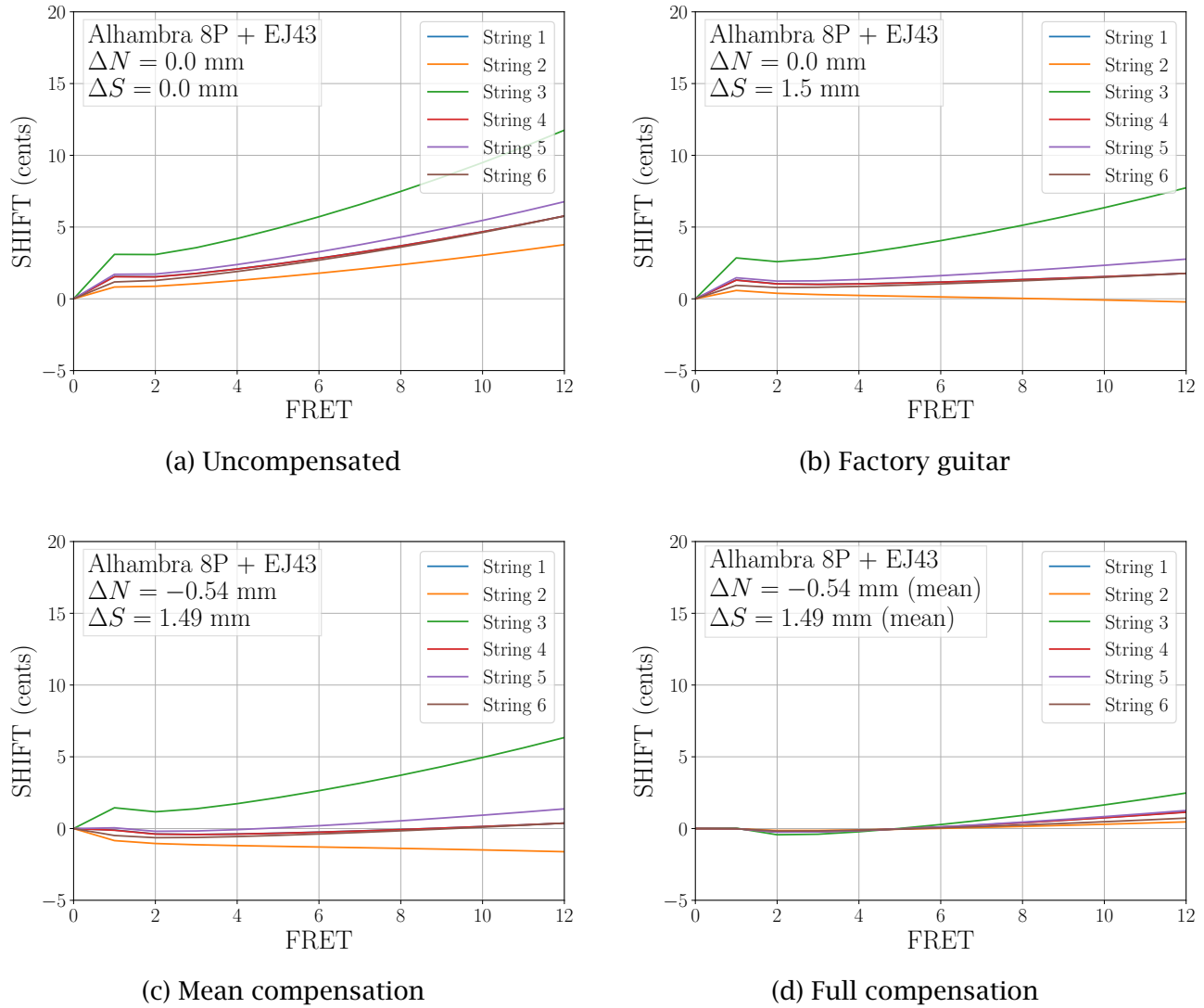


Figure 3: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings - Light Tension (EJ43). Four different strategies of saddle and nut compensation are illustrated.

Table 5: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings - Hard Tension (EJ46) on the Alhambra 8P classical guitar.

String	$\Delta S$ (mm)	$\Delta N$ (mm)
J4601	3.91	-1.20
J4602	2.84	-0.49
J4603	3.00	-0.34
J4604	2.77	-0.55
J4605	2.03	-0.20
J4606	3.04	-0.31
Mean	2.93	-0.52

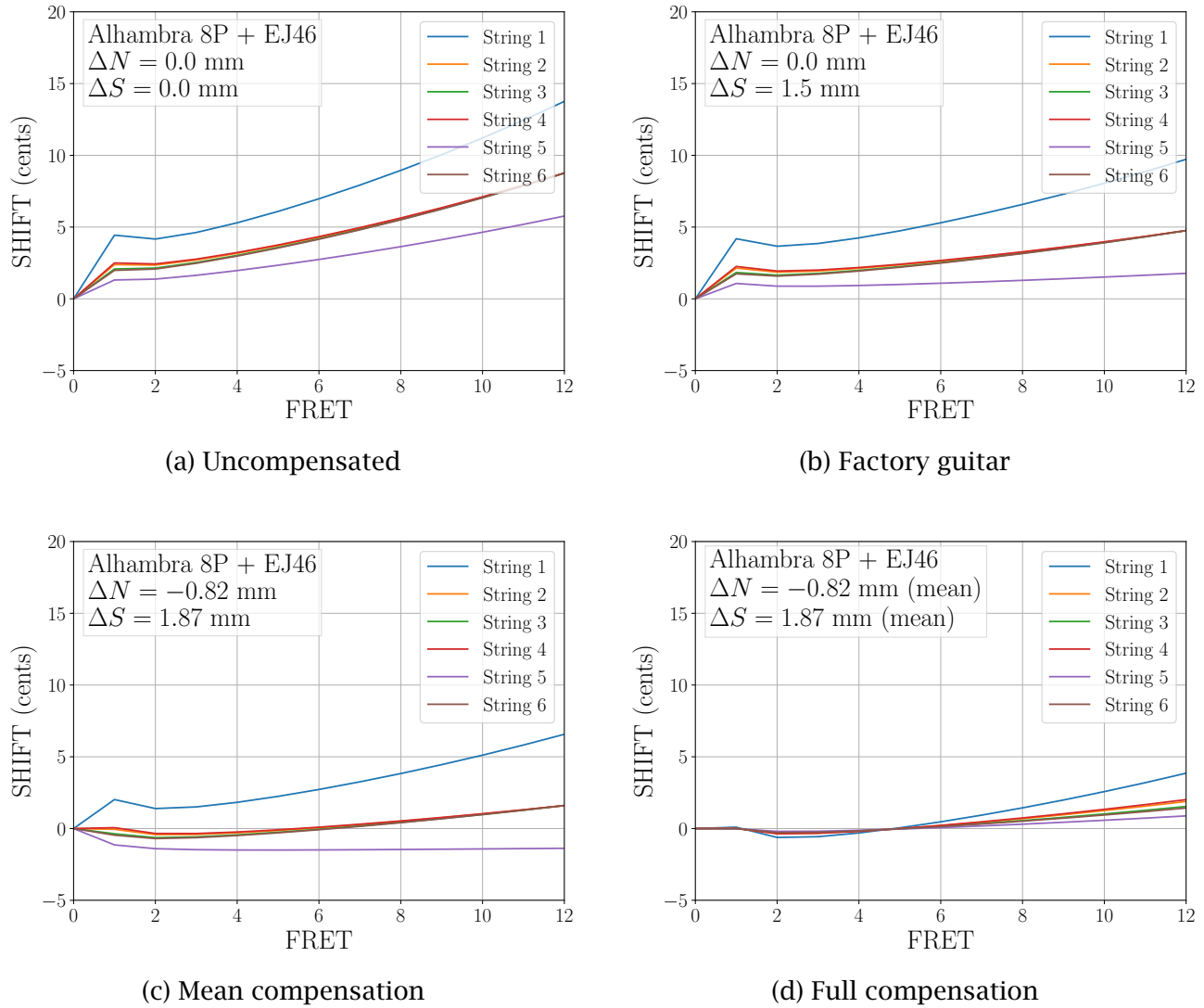
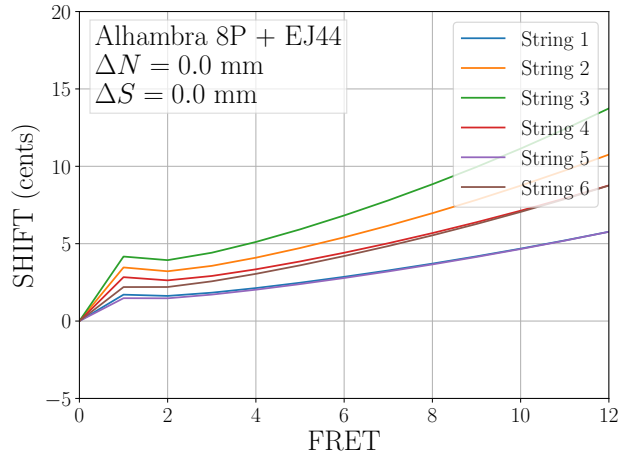


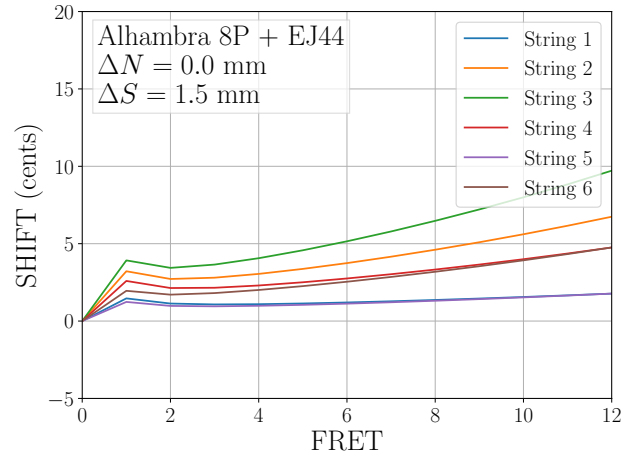
Figure 4: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings - Hard Tension (EJ46). Four different strategies of saddle and nut compensation are illustrated.

Table 6: Predicted setbacks for the D'Addario Pro-Arte Nylon Classical Guitar Strings - Extra Hard Tension (EJ44) on the Alhambra 8P classical guitar.

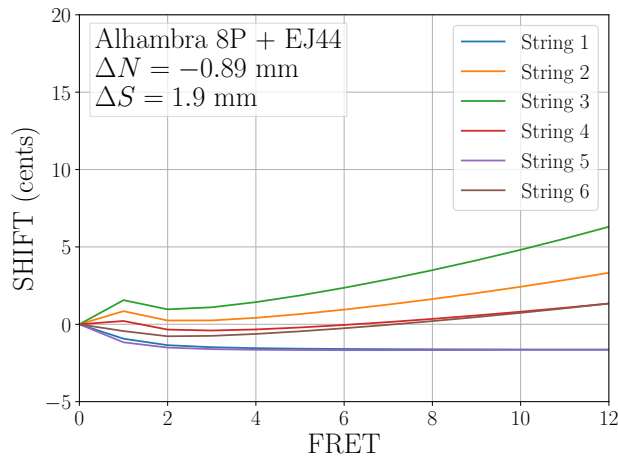
String	$\Delta S$ (mm)	$\Delta N$ (mm)
J4401	1.93	-0.29
J4402	3.39	-0.67
J4403	4.44	-0.73
J4404	2.77	-0.55
J4405	2.03	-0.20
J4406	3.05	-0.30
Mean	2.94	-0.46



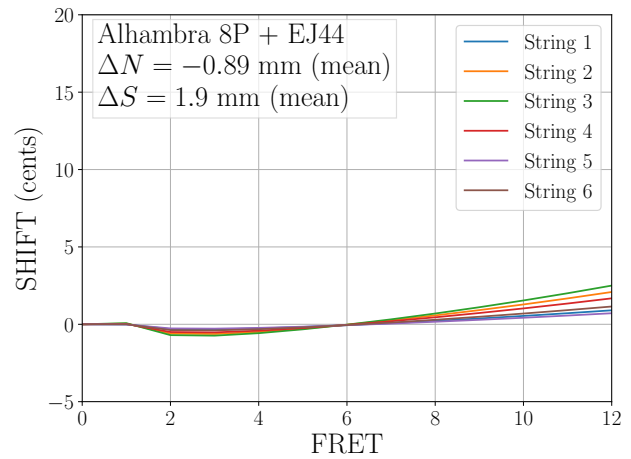
(a) Uncompensated



(b) Factory guitar



(c) Mean compensation



(d) Full compensation

Figure 5: Frequency shift (in cents) for an Alhambra 8P guitar with D'Addario Pro-Arte Nylon Classical Guitar Strings - Extra Hard Tension (EJ44). Four different strategies of saddle and nut compensation are illustrated.

## 5 Conclusion

### A Vibration Frequencies of a Stiff String

Here we outline the calculation of the normal mode frequencies of a vibrating stiff string with non-symmetric boundary conditions. We begin with the wave equation [10]

$$\mu \frac{\partial^2}{\partial t^2} y(x) = T \frac{\partial^2}{\partial x^2} y(x) - E S \mathcal{K}^2 \frac{\partial^4}{\partial x^2} y(x), \quad (34)$$

where  $\mu$  and  $T$  are respectively the linear mass density and the tension of the string,  $E$  is its Young's modulus (or the modulus of elasticity),  $S$  is the cross-sectional area, and  $\mathcal{K}$  is the radius of gyration of the string. If we simplify this equation by scaling  $x$  by the length  $L$  of the string, and  $t$  by  $1/\omega_0 \equiv (L/\pi)\sqrt{\mu/T}$ , then we obtain the dimensionless wave equation

$$\pi^2 \frac{\partial^2}{\partial t^2} y(x) = \frac{\partial^2}{\partial x^2} y(x) - B^2 \frac{\partial^4}{\partial x^2} y(x), \quad (35)$$

where  $B$  is the “bending stiffness parameter” given by

$$B \equiv \sqrt{\frac{E S \mathcal{K}^2}{L^2 T}}. \quad (36)$$

If we assume that  $y(x)$  is a sum of terms of the form

$$y(x) = C e^{kx - i\omega t}, \quad (37)$$

then  $k$  and  $\omega$  must satisfy the expression

$$B^2 k^4 - k^2 - (\pi \omega)^2 = 0, \quad (38)$$

or

$$k^2 = \frac{1 \pm \sqrt{1 + (2\pi B \omega)^2}}{2B^2}. \quad (39)$$

Therefore, given  $\omega$ , we have four possible choices for  $k$ :  $\pm k_1$ , or  $\pm i k_2$ , where

$$k_1^2 = \frac{\sqrt{1 + (2\pi B \omega)^2} + 1}{2B^2}, \text{ and} \quad (40a)$$

$$k_2^2 = \frac{\sqrt{1 + (2\pi B \omega)^2} - 1}{2B^2}. \quad (40b)$$

Therefore, the general solution to Eq. (35) has the form

$$y(x) = e^{-i\omega t} \left( C_1^+ e^{k_1 x} + C_1^- e^{-k_1 x} + C_2^+ e^{i k_2 x} + C_2^- e^{-i k_2 x} \right). \quad (41)$$

As discussed in Section 2, the boundary conditions for the case of a classical guitar string are not symmetric. At  $x = 0$  (the saddle), the string is pinned (but not clamped), so that  $y = 0$  and  $\partial^2 y / \partial x^2 = 0$ . However, at  $x = 1$  (the fret) the string is clamped, so that  $y = 0$  and  $\partial y / \partial x = 0$ . Applying these boundary conditions to Eq. (41), we obtain

$$0 = C_1^+ + C_1^- + C_2^+ + C_2^-, \quad (42a)$$

$$0 = k_1^2 (C_1^+ + C_1^-) - k_2^2 (C_2^+ + C_2^-), \quad (42b)$$

$$0 = C_1^+ e^{k_1} + C_1^- e^{-k_1} + C_2^+ e^{i k_2} + C_2^- e^{-i k_2}, \text{ and} \quad (42c)$$

$$0 = k_1 (C_1^+ e^{k_1} - C_1^- e^{-k_1}) + i k_2 (C_2^+ e^{i k_2} - C_2^- e^{-i k_2}). \quad (42d)$$

Since  $k_1^2 + k_2^2 \neq 0$ , the first two of these equations tell us that  $C_1^- = -C_1^+ \equiv -C_1$ , and  $C_2^- = -C_2^+ \equiv -C_2$ . Therefore, the second two equations become

$$C_1 \sinh(k_1) = -i C_2 \sin(k_2), \text{ and} \quad (43a)$$

$$k_1 C_1 \cosh(k_1) = -i k_2 C_2 \cos(k_2). \quad (43b)$$

Dividing the first of these equations by the second, we find

$$\tan(k_2) = \frac{k_2}{k_1} \tanh k_1. \quad (44)$$

In the case of a classical guitar, we expect that  $B \ll 1$ , so from Eq. (40a) we see that  $k_1 \approx 1/B \gg 1$ , and therefore  $\tanh k_1 \rightarrow 1$ . Writing  $k_2 = m\pi(1 + \epsilon)$ , where  $m$  is an integer greater than or equal to 1, and  $\epsilon \ll 1$ , we have  $\tan(k_2) \approx m\pi\epsilon$ , and

$$\frac{k_2}{k_1} \approx m\pi B(1 + \epsilon) = m\pi\epsilon, \quad (45)$$

indicating that  $\epsilon = B$  to lowest order in  $B$ . To lowest (zero) order in  $B$ , we see from Eq. (40b) that  $k_2 = \pi\omega = m\pi(1 + B)$ , or  $\omega = m(1 + B)$ . Restoring the time scaling by  $1/\omega_0$ , and defining the frequency  $f = \omega/2\pi$ , we finally have

$$f_m = \frac{m}{2L} \sqrt{\frac{T}{\mu}} (1 + B). \quad (46)$$

We use this result to build our model in Section 2.

## B Classical Guitar String Specifications

Table 7: String properties for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 25.5 inches.

String	Note	Diameter (in)	Density (lb/in)	Tension (lb)
J4501	E <sub>4</sub>	0.0280	$2.092 \times 10^{-5}$	15.3
J4502	B <sub>3</sub>	0.0322	$2.827 \times 10^{-5}$	11.6
J4503	G <sub>3</sub>	0.0403	$4.679 \times 10^{-5}$	12.1
J4504	D <sub>3</sub>	0.0290	$1.075 \times 10^{-4}$	15.6
J4505	A <sub>2</sub>	0.0350	$1.842 \times 10^{-4}$	15.0
J4506	E <sub>2</sub>	0.0430	$3.063 \times 10^{-4}$	14.0

Table 8: String properties for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Normal Tension (EJ45). The corresponding scale length is 650 mm.

String	Note	Radius (mm)	Density (kg/mm)	Tension (N)
J4501	E <sub>4</sub>	0.356	$3.737 \times 10^{-7}$	68.6
J4502	B <sub>3</sub>	0.409	$5.050 \times 10^{-7}$	52.0
J4503	G <sub>3</sub>	0.512	$8.358 \times 10^{-7}$	54.3
J4504	D <sub>3</sub>	0.368	$1.921 \times 10^{-6}$	70.0
J4505	A <sub>2</sub>	0.445	$3.290 \times 10^{-6}$	67.3
J4506	E <sub>2</sub>	0.546	$5.472 \times 10^{-6}$	62.8

Table 9: String properties for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 25.5 inches.

String	Note	Diameter (in)	Density (lb/in)	Tension (lb)
J4301	E <sub>4</sub>	0.0275	$2.024 \times 10^{-5}$	14.8
J4302	B <sub>3</sub>	0.0317	$2.729 \times 10^{-5}$	11.2
J4303	G <sub>3</sub>	0.0397	$4.525 \times 10^{-5}$	11.7
J4304	D <sub>3</sub>	0.0280	$1.020 \times 10^{-4}$	14.8
J4305	A <sub>2</sub>	0.0330	$1.535 \times 10^{-4}$	12.5
J4306	E <sub>2</sub>	0.0420	$2.888 \times 10^{-4}$	13.2

Table 10: String properties for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Light Tension (EJ43). The corresponding scale length is 650 mm.

String	Note	Radius (mm)	Density (kg/mm)	Tension (N)
J4301	E <sub>4</sub>	0.349	$3.615 \times 10^{-7}$	66.4
J4302	B <sub>3</sub>	0.403	$4.875 \times 10^{-7}$	50.2
J4303	G <sub>3</sub>	0.504	$8.083 \times 10^{-7}$	52.5
J4304	D <sub>3</sub>	0.356	$1.823 \times 10^{-6}$	66.4
J4305	A <sub>2</sub>	0.419	$2.741 \times 10^{-6}$	56.1
J4306	E <sub>2</sub>	0.533	$5.159 \times 10^{-6}$	59.2

Table 11: String properties for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). The corresponding scale length is 25.5 inches.

String	Note	Diameter (in)	Density (lb/in)	Tension (lb)
J4601	E <sub>4</sub>	0.0285	$2.161 \times 10^{-5}$	15.8
J4602	B <sub>3</sub>	0.0327	$2.924 \times 10^{-5}$	12.0
J4603	G <sub>3</sub>	0.0410	$4.795 \times 10^{-5}$	12.4
J4604	D <sub>3</sub>	0.0300	$1.124 \times 10^{-4}$	16.3
J4605	A <sub>2</sub>	0.0360	$1.952 \times 10^{-4}$	15.9
J4606	E <sub>2</sub>	0.0440	$3.173 \times 10^{-4}$	14.5

Table 12: String properties for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ46). The corresponding scale length is 650 mm.

String	Note	Radius (mm)	Density (kg/mm)	Tension (N)
J4601	E <sub>4</sub>	0.362	$3.860 \times 10^{-7}$	70.9
J4602	B <sub>3</sub>	0.415	$5.223 \times 10^{-7}$	53.8
J4603	G <sub>3</sub>	0.521	$8.565 \times 10^{-7}$	55.6
J4604	D <sub>3</sub>	0.381	$2.007 \times 10^{-6}$	73.1
J4605	A <sub>2</sub>	0.457	$3.487 \times 10^{-6}$	71.3
J4606	E <sub>2</sub>	0.559	$5.667 \times 10^{-6}$	65.0

Table 13: String properties for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Extra Hard Tension (EJ44). The corresponding scale length is 25.5 inches.

String	Note	Diameter (in)	Density (lb/in)	Tension (lb)
J4401	E <sub>4</sub>	0.0290	$2.243 \times 10^{-5}$	16.4
J4402	B <sub>3</sub>	0.0333	$3.046 \times 10^{-5}$	12.5
J4403	G <sub>3</sub>	0.0416	$4.989 \times 10^{-5}$	12.9
J4404	D <sub>3</sub>	0.0300	$1.124 \times 10^{-4}$	16.3
J4405	A <sub>2</sub>	0.0360	$1.952 \times 10^{-4}$	15.9
J4406	E <sub>2</sub>	0.0450	$3.435 \times 10^{-4}$	15.7

Table 14: String properties for the D'Addario Pro-Arte Nylon Classical Guitar Strings – Hard Tension (EJ44). The corresponding scale length is 650 mm.

String	Note	Radius (mm)	Density (kg/mm)	Tension (N)
J4401	E <sub>4</sub>	0.368	$4.007 \times 10^{-7}$	73.6
J4402	B <sub>3</sub>	0.423	$5.441 \times 10^{-7}$	56.1
J4403	G <sub>3</sub>	0.528	$8.912 \times 10^{-7}$	57.9
J4404	D <sub>3</sub>	0.381	$2.007 \times 10^{-6}$	73.1
J4405	A <sub>2</sub>	0.457	$3.487 \times 10^{-6}$	71.3
J4406	E <sub>2</sub>	0.571	$6.136 \times 10^{-6}$	70.4



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