621 HW 5

Problem 1

a. What is an explosive AR TS (order 1)

The process to find a stationary AR(1) model is called explosive.

$$AR(1): \quad x_{t+1} = \phi x_t + w_{t+1}$$

$$x_t = \phi^{-1} x_{t+1} - \phi^{-1} w_{t+1} = \phi^{-1}(\phi^{-1} x_{t+2} - \phi^{-1} w_{t+2}) - \phi^{-1} w_{t+1}$$

$$\vdots$$

$$= \phi^{-k} x_{t+k} - \sum_{j=1}^{k-1} \phi^{-j} w_{t+j} = -\sum_{j=1}^{\infty} \phi^{-j} w_{t+j}$$

(using future's data to get. ∴ it depends on future)

b. Discuss what is meant by the fact there is a causal counterpart.

If there is a causal counterpart, the time series is a stationary process. ∴ even not considering explosive process, there will not be problems.

eg $\quad x_t = \phi x_{t-1} + \sigma_w^2 \equiv y_t = \phi^{-1} y_{t-1} + v_t$

$v_t \sim \text{iid } N(0, \sigma_w^2 \phi^{-2})$

Problem 2

AR(1) model with non-zero intercept.

$$x_t = \phi_0 + \phi_1 x_{t-1} + \varepsilon_t. \qquad E(\varepsilon_t) = 0$$

$$E(x_t) = E(\phi_0 + \phi_1 x_{t-1} + \varepsilon_t) = \phi_0 + \phi_1 E(x_{t-1})$$

$$E(x_t) = E(x_{t-1}) \Rightarrow \quad E(x_t) = \frac{\phi_0}{1-\phi_1}$$

$$\operatorname{var}(x_t) = \operatorname{var}(\phi_0 + \phi_1 x_{t-1} + \varepsilon_t) = \phi_1^2 \operatorname{var}(x_{t-1}) + \operatorname{var}(\varepsilon_t).$$

if $\operatorname{var}(\varepsilon_t) = \sigma^2$. $\quad \operatorname{var}(x_t) = \dfrac{\sigma^2}{1-\phi_1^2}$

considering h. $h = s-t$

$$\gamma_0 = \operatorname{cov}(x_t, x_t) = \operatorname{cov}(\phi_0 + \phi_1 x_{t-1} + \varepsilon_t,\ \phi_0 + \phi_1 x_{t-1} + \varepsilon_t)$$

$$= \phi_1^2 \operatorname{var}(x_{t-1}) + \operatorname{var}(\varepsilon_t)$$

$$\gamma_0 = \frac{\sigma^2}{1-\phi_1^2}$$

$$\gamma_h = \operatorname{cov}(x_t, x_{t-h}) = E[(x_t - \phi_0)(x_{t-h} - \phi_0)]$$

$$= E[(\phi_1(x_{t-1} - \phi_0) + \varepsilon_t)(x_{t-h} - \phi_0)]$$

$$= E[\phi_1(x_{t-1} - \phi_0)(x_{t-h} - \phi_0) + \varepsilon_t(x_{t-h} - \phi_0)]$$

$$= \phi_1 E[(x_{t-1} - \phi_0)(x_{t-h} - \phi_0)] + E[\varepsilon_t(x_{t-h} - \phi_0)]$$

$$= \phi_1 \gamma_{h-1} + E[\varepsilon_t(x_{t-h} - \phi_0)] = \phi_1 \gamma_{h-1}$$

$$\gamma_1 = \phi_1 \gamma_0$$

$$\gamma_2 = \phi_1 \gamma_1 = \phi_1^2 \gamma_0.$$

$$\vdots$$

$$\gamma_h = \phi_1^h \gamma_0$$

Problem 3

$$x_t = \delta + x_{t-1} + w_t \qquad w_t \sim N(0, \sigma^2)$$

$$x_t - x_{t-1} = \delta + (x_{t-1} - x_{t-1}) + w_t$$

$$x_t - x_{t-1} = \delta + w_t$$

$$E(x_t - x_{t-1}) = \delta.$$

$$\operatorname{var}(x_t - x_{t-1}) = \sigma^2.$$

$$\gamma_k = \int_{-\pi}^{\pi} \underset{\text{谱密度}}{\underbrace{f(\lambda)}} e^{ik\lambda}\, d\lambda.$$

$$\gamma_0 = \operatorname{cov}(\delta + w_t,\ \delta + w_t) = \operatorname{var}(w_t) = \sigma^2.$$

$$\gamma_1 = \operatorname{cov}(\delta + w_{t+1},\ \delta + w_t) = 0.$$

$$f(\omega) = \frac{1}{2\pi}\left[\gamma(0) + 2\sum_{h=1} \gamma(h)\cos(\omega h) \right]$$
$$= \frac{1}{2\pi}\left[\sigma^2 \right]$$
$$= \frac{\sigma^2}{2\pi}$$

Problem 4.

(a). $x_t = 0.8x_{t-1} - 0.15x_{t-2} + w_t - 0.3w_{t-1}$.

root of MA >1. invertible

root of AR >1 causal.

ϕ — $\cancel{2}$

3ϕ — $\cancel{10}$

$(\phi-2)(3\phi-10)=0$

AR polynomial: $\quad x_t = 0.8x_{t-1} - 0.15x_{t-2}$

$$1 = 0.8\phi - 0.15\phi^2.$$
$$\Rightarrow 0.15\phi^2 - 0.8\phi + 1 = 0.$$
$$\Rightarrow 15\phi^2 - 80\phi + 100 = 0.$$
$$\Rightarrow 3\phi^2 - 16\phi + 20 = 0$$
$$\Rightarrow (\phi-2)(3\phi-10) = 0.$$
$$\Rightarrow \phi_1 = 2 \quad \phi_2 = \frac{10}{3}$$

MA polynomial. $\quad 1 - 0.3\phi = 0. \Rightarrow \phi = 0.3.$

$\because$ roots of AR polynomial >1. $\therefore$ the model is not causal

$\because$ roots of MA polynomial <1. $\therefore$ the model is not invertible

(b) $\quad x_t = x_{t-1} - 0.5x_{t-2} + w_t - w_{t-1}$.

AR polynomial: $\quad -0.5\phi^2 + \phi - 1 = 0.$

$$\Rightarrow 0.5\phi^2 - \phi + 1 = 0$$

$\Rightarrow \quad \phi^2 - 2\phi + 2 = 0$

$\Rightarrow$ no roots.

$\therefore$ the model is casual

MA polynomial $\quad 1 - \phi = 0. \Rightarrow \phi = 1.$

the model is not invertible