Tutorial Sheet 1

Q1. Decide whether or not the following systems are linear:

(a) A Singaporean refinery processes oil from different sources, containing different amounts of natural gas (N), aromatics (A), petroleum (P), and heavy hydrocarbons (H); this is represented by the system:

$$\begin{array}{lllllll} \text{Bass Strait:} & 0.9\,G & +\,0.80\,A & +\,0.19\,P & +\,0.01\,H & = & 25,000 \text{ barrels /day} \\ \text{East Timor:} & 0.5\,G & +\,0.05\,A & +\,0.40\,P & +\,0.05\,H & = & 4,000 \text{ barrels /day} \\ & & & \vdots & & & \end{array}$$

(b) Ocean currents in the mid-depths of the Antarctic Circle locally undergo *circulation*, where the new position (y_1, y_2, y_3) of a particle, which was at (x_1, x_2, x_3), a fraction of a second later is approximately given by:

$$\begin{aligned} y_1 &= x_1 + \sigma(x_2 - x_1) \\ y_2 &= x_2 + r x_1 - x_2 - x_1 x_3 \\ y_3 &= x_1 x_2 - b x_3 \end{aligned}$$

where σ, r and b are constants.

(c) A (fictional) study on the effect of dosage of a particular medication on blood pressure gathered:

- the dosage applied D (mg), and
- the average change in systolic blood pressure ΔP (mmHg), for those patients with very high (> 160) blood pressure (< 140 is considered satisfactory, 100–120 normal).

It gave the results:

D	10	20	30	40	50	60	$\cdots$
ΔP	3	11	20	32	38	40	$\cdots$

which are being used to fit the full cubic model: $\Delta P = \alpha + \beta D + \gamma D^2 + \delta D^3$.

Q2. Consider the linear system:

$$\begin{cases} -2x - \ \ y = 44 \\ \ \ 5x + 8y = -22 \end{cases}$$

(a) Write this in the form of an augmented matrix.

(b) Reduce the matrix to row-echelon form, indicating the elementary row operations used.

(c) Hence solve the linear system.

Q3. I was making chocolates over Christmas, but already had the raw ingredients (dark, plain and white chocolate). I had 650g, 560g, and 420g of Dark, Plain and White Chocolate, respectively, and wanted to use all of it. Truffles use 75g of each kind; Torrone uses 50g Dark, 170g Plain and 240g White; and Fudge uses 150g Dark, 80g Plain and 10g White. Use matrices to find the number of each kind of chocolate I made.

Q4. Systems of linear equations have been written as augmented matrices and already row-reduced, as shown below. For each matrix:

(i) Is the system consistent?
(ii) How many solutions does it have?
(iii) Solve the system where possible.

(a) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 1 & -2 & 2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{array}\right]$

(c) $\left[\begin{array}{cccc|c} 1 & 2 & 2 & -1 & 7 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ (d) $\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Q5. If k is a real constant, then we can define a system by

$$\begin{cases} 2x - \ \ 4y + 4z = 12 \\ 3x + \ \ \ \ y - 8z = 4 \\ -5x + 11y + kz = -32 \end{cases}$$

(a) Use elementary row operations to reduce the system to row-echelon form.

(b) For what value(s) of k does the system have a unique solution? Infinitely many solutions?

(c) Solve the system when there is an infinite number of solutions.

Q6. (a) Find the inverses of the following matrices, where they exist.

(i) $\begin{bmatrix} -3 & 5 \\ 6 & -10 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 2 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} A & O \\ O & B \end{bmatrix}$

where A and B are 2×2 matrices and O is the 2×2 zero matrix.

(b) Use your answer to question (a) to help solve the system of equations:

$$\begin{cases} x + \ \ y + 2z = 4 \\ x + 3y \qquad\ \ = -1 \\ 2x + 2y + 5z = 5 \end{cases}$$

Q7. Let A and B be $n \times n$ matrices.

(a) If A and B are invertible, prove that their product AB is invertible and write its inverse in terms of A^{-1} and B^{-1}.

(b) **(*)** Suppose AB is invertible. Does it follow that both A and B must be invertible? If yes, give a proof; if no, give a counterexample.