MAST30025: Linear Statistical Models

Week 2 Lab

1. Show that $X^T X$ is a symmetric matrix.

2. (a) Let
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
be a nonsingular 2×2 matrix. Show by direct multiplication that
$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$
   (b) Find the inverse of
$$\begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}.$$

3. Is
$$X = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$
orthogonal? If not, what value of c makes the matrix cX orthogonal?

4. (a) Find the eigenvalues, and an associated eigenvector for each eigenvalue, of the matrix
$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$
   (b) Find an orthogonal matrix P such that $P^T AP$ is diagonal.
   (c) Write down $P^T AP$ for the P given in part (b).

5. Let
$$A = \begin{bmatrix} 1 & 4 & 3 \\ -2 & 0 & 2 \\ 4 & 4 & 0 \end{bmatrix}.$$
   (a) Write down the trace of A.
   (b) Are the columns of A linearly independent? Justify your answer.
   (c) Find the rank of A.

6. Show that if X is of full rank, then
$$I - X(X^T X)^{-1} X^T$$
is an idempotent matrix.

7. Prove that a (real) symmetric matrix A is positive semidefinite if and only if all of its eigenvalues are non-negative, and positive definite if and only if all of its eigenvalues are strictly positive.

8. (Not examinable) Prove that for any matrix A
$$r(A) = r(A^T) = r(A^T A).$$
You may use the fact that pre- or post-multiplying by a non-singular matrix does not change the rank.

R exercises

The following are taken from Chapter 2 of spuRs (Introduction to Scientific Programming and Simulation Using R).

1. Give R assignment statements that set the variable z to
   (a) x^{a^b}
   (b) $(x^a)^b$
   (c) $3x^3 + 2x^2 + 6x + 1$ (try to minimise the number of operations required)
   (d) the second-to-last digit of x before the decimal point (hint: use `floor(x)` and/or `%%`)
   (e) $z + 1$

2. Give R expressions that return the following matrices and vectors
   (a) $(1, 2, 3, 4, 5, 6, 7, 8, 7, 6, 5, 4, 3, 2, 1)$
   (b) $(1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5)$
   (c) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$
   (d) $\begin{pmatrix} 0 & 2 & 3 \\ 0 & 5 & 0 \\ 7 & 0 & 0 \end{pmatrix}$

3. Suppose `vec` is a strictly positive vector of length 2. Interpreting `vec` as the coordinates of a point in $\mathbb{R}^2$, use R to express it in polar coordinates. You will need (at least one of) the inverse trigonometric functions: `acos(x)`, `asin(x)`, and `atan(x)`.

4. Use R to produce a vector containing all integers from 1 to 100 that are not divisible by 2, 3, or 7.

5. Suppose that `queue <- c("Steve", "Russell", "Alison", "Liam")` and that `queue` represents a supermarket queue with Steve first in line. Using R expressions update the supermarket queue as successively:
   (a) Barry arrives;
   (b) Steve is served;
   (c) Pam talks her way to the front with one item;
   (d) Barry gets impatient and leaves;
   (e) Alison gets impatient and leaves.

   For the last case you should not assume that you know where in the queue Alison is standing.

   Finally, using the function `which(x)`, find the position of Russell in the queue.

   Note that when assigning a text string to a variable, it needs to be in quotes.

6. Which of the following assignments will be successful? What will the vectors `x`, `y`, and `z` look like at each stage?

```
rm(list = ls())
x <- 1
x[3] <- 3
y <- c()
y[2] <- 2
y[3] <- y[1]
y[2] <- y[4]
z[1] <- 0
```

7. Build a 10×10 identity matrix. Then make all the non-zero elements 5. Do this latter step in at least two different ways.