THE UNIVERSITY OF MELBOURNE

Student Number	

Semester 1 Assessment, 2019

School of Mathematics and Statistics

MAST20004 Probability

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 8 pages (including this page)

Authorised Materials

- Mobile phones, smart watches, and internet or communication devices are forbidden.
- Students may bring one double-sided A4 sheet of handwritten notes into the exam room.
- Approved hand-held electronic scientific (but not graphing) calculators may be used.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the exam.
- This paper has 10 questions. Attempt as many questions, or parts of questions, as you can. Marks for individual questions are shown.
- Working and/or reasoning must be given to obtain full credit. Clarity, neatness, and style count.
- Statistical tables are not provided but you may use the MATLAB output at the end of the examination paper **FOR ANY QUESTION**.
- The total number of marks available for this exam is 110.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the exam.
- Initially students are to receive a 14 page script booklet.

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1. Consider a random experiment with sample space Ω.

   (a) Write down the axioms which must be satisfied by a probability mapping $\mathbb{P}$ defined on the events of the experiment.

   (b) Using the axioms show that for any event $A \subset \Omega$, $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

   (c) Using the axioms show that for any events $A, B \subset \Omega$ such that $A \subset B$, $\mathbb{P}(B \backslash A) = \mathbb{P}(B) - \mathbb{P}(A)$. (Recall that $B \backslash A$ is the event that B and not A will occur, that is, $B \cap A^c$.)

   (d) Let $C, D \subset \Omega$ be events. Using (c) and the axioms, show that the probability of exactly one of these events occurring is $\mathbb{P}(C) + \mathbb{P}(D) - 2\mathbb{P}(C \cap D)$.

[10 marks]

2. A box contains ten coins, of which three coins are of type one, four coins are of type two, and three coins are of type three. Type one coins are fair, type two coins are weighted so that when tossed show a head 70% of the time, and type three coins show a tail 60% of the time. A coin is selected randomly from the box and then tossed.

   (a) What is the probability that a tail is showing?

   (b) If a tail is showing, what is the probability that the coin is of type three?

   (c) Are the events that a tail is showing and a type three coin was selected positively related, negatively related, or independent? Justify your answer.

   (d) Given a tail is showing, and you toss the same coin again, what is the probability that a head will be showing?

[10 marks]

3. Let the probability mass function of X be given by

$$p_X(k) \;=\; c \cdot \frac{2^k}{k!}, \quad k = 1, 2, \ldots,$$

   for some constant c.

   (a) What is the value of the constant c?

   (b) Which is (are) the most probable value(s) of X? Justify your answer.

   (c) Derive an expression without a summation for the probability generating function P_X of X.

   For which values $z \in \mathbb{R}$ is P_X defined? Justify your answer.

   (d) Using (c), or otherwise, calculate $\mathbb{E}[X]$ and $V(X)$.

   (e) Is X more likely to be even or odd? Justify your answer.

[13 marks]

4. A factory has ten machines that are all operational at the beginning of an eight hour shift. Each machine relies on a critical component which carries a backup spare in the machine. The lifetimes of the components and their backup spares are independent and distributed according to an exponential distribution with a mean of four hours. Once a component fails, its backup spare immediately replaces it, but if the backup spare also fails, the machine is inoperative. Calculate the probability that at the end of a shift, at least two machines are operational.

[8 marks]

5. Let $X \stackrel{d}{=} \exp(1)$, $Y = \min(X, 2)$, and $Z = e^{X/3}$.

   (a) Find the cumulative distribution function F_Y of Y.

   (b) Is Y a discrete random variable, or a continuous random variable, or neither? Justify your answer.

   (c) Calculate $\mathbb{E}[Y]$.

   (d) Find the cumulative distribution function F_Z, and the probability density function f_Z, of Z.

   Identify the distribution of Z.

[10 marks]

6. Let $X \stackrel{d}{=} \exp(3)$ with pdf

$$f_X(x) = \begin{cases} 3e^{-3x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

and $Y = \psi(X) = e^X$.

   (a) Find the cdf of X and using the formula for computing the moments through the tail probabilities derive $\mu := \mathbb{E}[X]$ and $\mathbb{E}[X^2]$. Evaluate $V(X)$.

   (b) Calculate $\mathbb{E}[Y]$ and $V(Y)$.

   (c) Compute the approximate values of $\mathbb{E}[Y]$ and $V(Y)$ using

$$\mathbb{E}[\psi(X)] \approx \psi(\mu) + \frac{1}{2}\psi''(\mu)V(X) \quad \text{and} \quad V(\psi(X)) \approx \psi'(\mu)^2 V(X).$$

   Do you expect good approximations? Justify your answer.

   (d) The "rand" command in Matlab can be used to generate a realisation (an observation) of a random variable $U \stackrel{d}{=} R(0, 1)$. Explain how to generate a realisation of the random variable X.
   Explain how to calculate an estimate of $\mathbb{E}(\sin(X))$ from 100 realisations of the random variable $U \stackrel{d}{=} R(0, 1)$.

[15 marks]

7. Consider the bivariate random variable (X, Y) which has joint probability density function

$$f_{(X,Y)}(x, y) = \begin{cases} \frac{1}{2}, & \text{for } 0 < x, y < 1, \\ \frac{1}{2}, & \text{for } -1 < x, y \leq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) Derive the marginal probability density functions for X and Y.

   (b) Evaluate the following probabilities:

      (i) $\mathbb{P}(X > 0, Y > 0)$;

      (ii) $\mathbb{P}(X > 1/2, Y < 1/2)$;

      (iii) $\mathbb{P}(X + Y \leq 1)$;

      (iv) $\mathbb{P}(X^2 + Y^2 \leq 1)$.

   (c) Are X and Y independent? Justify your answer.

[9 marks]

8. Let X be a discrete random variable having the pmf $\mathbb{P}(X=-1)=\mathbb{P}(X=1)=0.5$.

   (a) Find the mean μ and variance σ^2 of X.

   (b) Derive the moment generating function of X and state the values for which it is defined.

   (c) Find the skewness and kurtosis of X. Comment on your findings.

   (d) Find the cumulant generating function of X.

   (e) Let $X_1, X_2, \ldots$ be a sequence of independent random variables where each one has the same distribution as X. Derive the moment generating function of

   $$S_n = \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}.$$

   (f) Using the moment generating function of S_n, prove that S_n converges to $N(0, \sigma^2)$ in distribution as $n \to \infty$, where σ^2 is the same as that in (a).
   Hint: you may wish to use $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ for small x.

   (g) Let

   $$T = X_1 + X_2 + \ldots + X_{100}.$$

   Give an approximate value of $\mathbb{P}(T < 12)$.

[18 marks]

9. Electricity usage during a summer day in Melbourne can be classified as 1=normal, 2=high, or 3=low. Weather conditions often make this level of usage change according to a Markov chain with the following transition matrix:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} \frac{3}{4} & \frac{1}{6} & \frac{1}{12} \\ \frac{2}{5} & \frac{1}{3} & \frac{4}{15} \\ \frac{1}{2} & \frac{2}{5} & \frac{1}{10} \end{array}\right] \end{array}.$$

   (a) Which underlying assumption has been made so that this situation can be modelled as a Markov chain?

   (b) Suppose on 1 January, the electricity usage is normal, what is the probability that the electricity usage is

   (i) normal on 3 January (in the same year)?

   (ii) normal on 4 January (in the same year)?

   (iii) normal on all days from 2 January to 4 January (in the same year)?

   Justify your answers where necessary.

   Hint: you may wish to use the Matlab output in the Appendix to reduce your calculations.

   (c) Without doing any calculations, explain two methods to obtain the stationary distribution of the Markov chain, one of which uses the Matlab output.
   Interpret your answer.

[9 marks]

10. Toss a biased coin with $\mathbb{P}(\text{Head}) = 1/4$ and $\mathbb{P}(\text{Tail}) = 3/4$ repeatedly. Let X be the number of tosses until two consecutive heads and Y be the number of heads until the first tail.

    (a) Find the distribution of Y.

    (b) For $i = 0, 1$, explain why $\mathbb{P}(X = n|Y = i) = \mathbb{P}(X = n - i - 1)$ and show that
    $$\mathbb{E}(X|Y = i) = \mathbb{E}(X) + i + 1.$$

    (c) For $i \geq 2$, explain why $\mathbb{P}(X = 2|Y = i) = 1$ and find $\mathbb{E}(X|Y = i)$.

    (d) Using $\mathbb{E}(X) = \mathbb{E}[\mathbb{E}(X|Y)]$, derive $\mathbb{E}(X)$.

[8 marks]

Total marks = 110

End of Questions

Appendix: Some MATLAB output

```
>> x1=0.01:.01:1.00; x2=1.01:.01:2.00; x3=2.01:.01:3.00;
y1=cdf('norm',x1,0,1);  y2=cdf('norm',x2,0,1); y3=cdf('norm',x3,0,1);
[x1' y1' x2' y2' x3' y3']

ans =

    0.0100    0.5040    1.0100    0.8438    2.0100    0.9778
    0.0200    0.5080    1.0200    0.8461    2.0200    0.9783
    0.0300    0.5120    1.0300    0.8485    2.0300    0.9788
    0.0400    0.5160    1.0400    0.8508    2.0400    0.9793
    0.0500    0.5199    1.0500    0.8531    2.0500    0.9798
    0.0600    0.5239    1.0600    0.8554    2.0600    0.9803
    0.0700    0.5279    1.0700    0.8577    2.0700    0.9808
    0.0800    0.5319    1.0800    0.8599    2.0800    0.9812
    0.0900    0.5359    1.0900    0.8621    2.0900    0.9817
    0.1000    0.5398    1.1000    0.8643    2.1000    0.9821
    0.1100    0.5438    1.1100    0.8665    2.1100    0.9826
    0.1200    0.5478    1.1200    0.8686    2.1200    0.9830
    0.1300    0.5517    1.1300    0.8708    2.1300    0.9834
    0.1400    0.5557    1.1400    0.8729    2.1400    0.9838
    0.1500    0.5596    1.1500    0.8749    2.1500    0.9842
    0.1600    0.5636    1.1600    0.8770    2.1600    0.9846
    0.1700    0.5675    1.1700    0.8790    2.1700    0.9850
    0.1800    0.5714    1.1800    0.8810    2.1800    0.9854
    0.1900    0.5753    1.1900    0.8830    2.1900    0.9857
    0.2000    0.5793    1.2000    0.8849    2.2000    0.9861
    0.2100    0.5832    1.2100    0.8869    2.2100    0.9864
    0.2200    0.5871    1.2200    0.8888    2.2200    0.9868
    0.2300    0.5910    1.2300    0.8907    2.2300    0.9871
    0.2400    0.5948    1.2400    0.8925    2.2400    0.9875
    0.2500    0.5987    1.2500    0.8944    2.2500    0.9878
    0.2600    0.6026    1.2600    0.8962    2.2600    0.9881
    0.2700    0.6064    1.2700    0.8980    2.2700    0.9884
    0.2800    0.6103    1.2800    0.8997    2.2800    0.9887
    0.2900    0.6141    1.2900    0.9015    2.2900    0.9890
    0.3000    0.6179    1.3000    0.9032    2.3000    0.9893
    0.3100    0.6217    1.3100    0.9049    2.3100    0.9896
    0.3200    0.6255    1.3200    0.9066    2.3200    0.9898
    0.3300    0.6293    1.3300    0.9082    2.3300    0.9901
    0.3400    0.6331    1.3400    0.9099    2.3400    0.9904
    0.3500    0.6368    1.3500    0.9115    2.3500    0.9906
    0.3600    0.6406    1.3600    0.9131    2.3600    0.9909
    0.3700    0.6443    1.3700    0.9147    2.3700    0.9911
    0.3800    0.6480    1.3800    0.9162    2.3800    0.9913
    0.3900    0.6517    1.3900    0.9177    2.3900    0.9916
    0.4000    0.6554    1.4000    0.9192    2.4000    0.9918
    0.4100    0.6591    1.4100    0.9207    2.4100    0.9920
    0.4200    0.6628    1.4200    0.9222    2.4200    0.9922
    0.4300    0.6664    1.4300    0.9236    2.4300    0.9925
    0.4400    0.6700    1.4400    0.9251    2.4400    0.9927
    0.4500    0.6736    1.4500    0.9265    2.4500    0.9929
    0.4600    0.6772    1.4600    0.9279    2.4600    0.9931
    0.4700    0.6808    1.4700    0.9292    2.4700    0.9932
    0.4800    0.6844    1.4800    0.9306    2.4800    0.9934
    0.4900    0.6879    1.4900    0.9319    2.4900    0.9936
```

0.5000	0.6915	1.5000	0.9332	2.5000	0.9938
0.5100	0.6950	1.5100	0.9345	2.5100	0.9940
0.5200	0.6985	1.5200	0.9357	2.5200	0.9941
0.5300	0.7019	1.5300	0.9370	2.5300	0.9943
0.5400	0.7054	1.5400	0.9382	2.5400	0.9945
0.5500	0.7088	1.5500	0.9394	2.5500	0.9946
0.5600	0.7123	1.5600	0.9406	2.5600	0.9948
0.5700	0.7157	1.5700	0.9418	2.5700	0.9949
0.5800	0.7190	1.5800	0.9429	2.5800	0.9951
0.5900	0.7224	1.5900	0.9441	2.5900	0.9952
0.6000	0.7257	1.6000	0.9452	2.6000	0.9953
0.6100	0.7291	1.6100	0.9463	2.6100	0.9955
0.6200	0.7324	1.6200	0.9474	2.6200	0.9956
0.6300	0.7357	1.6300	0.9484	2.6300	0.9957
0.6400	0.7389	1.6400	0.9495	2.6400	0.9959
0.6500	0.7422	1.6500	0.9505	2.6500	0.9960
0.6600	0.7454	1.6600	0.9515	2.6600	0.9961
0.6700	0.7486	1.6700	0.9525	2.6700	0.9962
0.6800	0.7517	1.6800	0.9535	2.6800	0.9963
0.6900	0.7549	1.6900	0.9545	2.6900	0.9964
0.7000	0.7580	1.7000	0.9554	2.7000	0.9965
0.7100	0.7611	1.7100	0.9564	2.7100	0.9966
0.7200	0.7642	1.7200	0.9573	2.7200	0.9967
0.7300	0.7673	1.7300	0.9582	2.7300	0.9968
0.7400	0.7704	1.7400	0.9591	2.7400	0.9969
0.7500	0.7734	1.7500	0.9599	2.7500	0.9970
0.7600	0.7764	1.7600	0.9608	2.7600	0.9971
0.7700	0.7794	1.7700	0.9616	2.7700	0.9972
0.7800	0.7823	1.7800	0.9625	2.7800	0.9973
0.7900	0.7852	1.7900	0.9633	2.7900	0.9974
0.8000	0.7881	1.8000	0.9641	2.8000	0.9974
0.8100	0.7910	1.8100	0.9649	2.8100	0.9975
0.8200	0.7939	1.8200	0.9656	2.8200	0.9976
0.8300	0.7967	1.8300	0.9664	2.8300	0.9977
0.8400	0.7995	1.8400	0.9671	2.8400	0.9977
0.8500	0.8023	1.8500	0.9678	2.8500	0.9978
0.8600	0.8051	1.8600	0.9686	2.8600	0.9979
0.8700	0.8078	1.8700	0.9693	2.8700	0.9979
0.8800	0.8106	1.8800	0.9699	2.8800	0.9980
0.8900	0.8133	1.8900	0.9706	2.8900	0.9981
0.9000	0.8159	1.9000	0.9713	2.9000	0.9981
0.9100	0.8186	1.9100	0.9719	2.9100	0.9982
0.9200	0.8212	1.9200	0.9726	2.9200	0.9982
0.9300	0.8238	1.9300	0.9732	2.9300	0.9983
0.9400	0.8264	1.9400	0.9738	2.9400	0.9984
0.9500	0.8289	1.9500	0.9744	2.9500	0.9984
0.9600	0.8315	1.9600	0.9750	2.9600	0.9985
0.9700	0.8340	1.9700	0.9756	2.9700	0.9985
0.9800	0.8365	1.9800	0.9761	2.9800	0.9986
0.9900	0.8389	1.9900	0.9767	2.9900	0.9986
1.0000	0.8413	2.0000	0.9772	3.0000	0.9987

```
>> y=[3/4 1/6 1/12; 2/5 1/3 4/15; 1/2 2/5 1/10]; y2=y^2;
y3=y^3; y4=y^4; y5=y^5; y20=y^(20); y50=y^(50)

ans

y =

    0.7500    0.1667    0.0833
    0.4000    0.3333    0.2667
    0.5000    0.4000    0.1000


y2 =

    0.6708    0.2139    0.1153
    0.5667    0.2844    0.1489
    0.5850    0.2567    0.1583


y3 =

    0.6463    0.2292    0.1245
    0.6132    0.2488    0.1380
    0.6206    0.2464    0.1330


y4 =

    0.6387    0.2339    0.1274
    0.6284    0.2403    0.1312
    0.6305    0.2388    0.1307


y5 =

    0.6363    0.2354    0.1283
    0.6331    0.2373    0.1296
    0.6337    0.2370    0.1293



y20 =

    0.6352    0.2361    0.1288
    0.6352    0.2361    0.1288
    0.6352    0.2361    0.1288


y50 =

    0.6352    0.2361    0.1288
    0.6352    0.2361    0.1288
    0.6352    0.2361    0.1288
```