MAST30001 Stochastic Modelling

Tutorial Sheet 3

1. Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with probability mass function given by the following table.

k	0	1	2	3
$P(Y = k)$	0.2	0.2	0.3	0.3

Set $X_0 = 0$ and let $X_n = \max\{Y_1, \ldots, Y_n\}$ be the largest Y_i observed to time n.

   (a) Argue that (X_n) is a Markov chain and determine its transition matrix P.

   (b) Analyze the state space of the chain: communication classes, essentiality, reducibility.

   (c) Describe the long run behavior of the chain. In particular can you determine the matrix $\lim_{n\to\infty} P^n$?

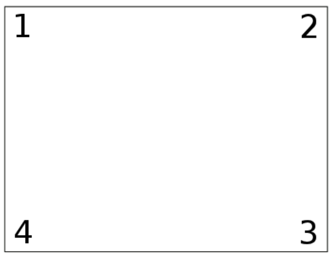
2. Two containers labelled α and ω have $2k$ balls distributed between them. At discrete time steps a ball is uniformly chosen from all $2k$ balls and it is moved from the container it is in to the other container. Let X_n be the number of balls in container α after the nth time step. This is a simple model for molecules diffusing through a membrane.

   (a) Is X_n a Markov chain? What are the transition probabilities?

   (b) Analyze the state space of the chain: communication classes, essentiality, reducibility.

   (c) If X_0 has a binomial distribution with parameters $(2k, 1/2)$ (meaning that X_0 balls are initially put into container α and $2k - X_0$ are put into container ω), what is the distribution of X_{10}? *[Hint: First compute the distribution of X_1.]*

3. A spider lives in a rectangular box with side lengths 3 and 4 centimeters. The spider sits in one of the four corners, marked with numbers 1, 2, 3 and 4 as illustrated below.



From time to time the spider runs from the corner it's in to another corner, chosen at random with probabilities inversely proportional to the distance from the current corner that the spider occupies. Denote by X_n the number of the corner the spider occupies after the nth change of corner.

   (a) Find the transition matrix P for the Markov chain X_n.

(b) Analyze the state space of the chain: communication classes, essentiality, reducibility.

(c) Assume that initially the spider is dropped into the center of the box and chooses a corner uniformly at random. What is $P(X_1 = 1, X_2 = 2, X_4 = 2)$?

4. A time *inhomogeneous* Markov chain (X_n) has one step transition matrix at the nth step given by $P(n)$:

$$(P(n))_{i,j} = P(X_n = j | X_{n-1} = i).$$

Show that

$$P(X_{n+m} = j | X_n = i) = (P(n+1)P(n+2) \cdots P(n+m))_{i,j}.$$

5. Let $(X_n)_{n \geq 1}$ be a Markov chain with state space $\{1, \ldots, k\}$ for some $k \geq 1$. Show that if i and j communicate, then the probability that the chain started in state i reaches state j in k steps or fewer is greater than 0.

1. Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with probability mass function given by the following table.

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$$P(X_{n+1}=k \mid X_n=j, \ldots X_1=x_1, X_0=0) = \begin{cases} P(Y_n=k) & k>j \\ P(Y_n\le j) & k=j \end{cases}$$

(a). $P[X_n = x_n \mid x_0=0, \cdots, X_{n-1} = x_{n-1})$

$= P(\max\{Y_1, \cdots Y_n\} = x_n \mid x_0=0,\ X_{n-1} = \max\{Y_1, \cdots Y_{n-1}\})$.

$= P[\max\{\max\{Y_1 \cdots Y_{n-1}\}, Y_n\} = x_n \mid \cancel{x_0=0},\ X_{n-1} = \max\{Y_1, \cdots Y_{n-1}\})$.

$= P[\max\{Y_n, X_{n-1}\} = x_n \mid X_{n-1} = x_{n-1}\}$

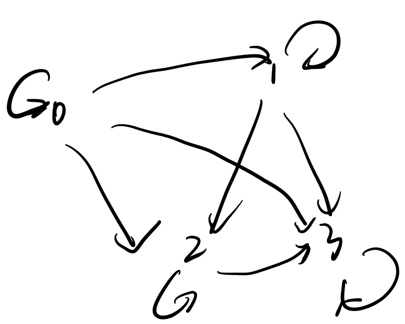
$= P(X_n = x_n \mid X_{n-1} = x_{n-1}\}$.

is a Markov chain.

$$P = \begin{bmatrix} 0.2 & 0.2 & 0.3 & 0.3 \\ 0 & 0.4 & 0.3 & 0.3 \\ 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

"$X_{n+1} \geq X_n$"

(b).



$S_0^n = \{0\}$

$S_1^n = \{1\}$

$S_2^n = \{2\}$

$S_3^e = \{3\}$ absorbing and essential

reducible DTMC

(c) 3 is absorbed state

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_0 = (p_0 \ p_1 \ p_2 \ p_3)$ $\qquad$ $x_0 P^n = [0 \ 0 \ 0 \ 1]$

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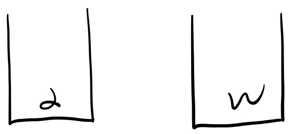
$X_0 \sim Bi(2k, \frac{1}{2})$.

$P(X_1 = x_0 + 1) = 1 - \frac{x_0}{2k}$

$P(X_1 = x_0 - 1) = \frac{x_0}{2k}$



(a) $P(X_n = x_n \mid X_{n-1} = x_{n-1}, X_{n-2} = x_{n-2}, \cdots X_1 = x_1, X_0 = x_0)$

$= P(X_n = X_{n-1} + Y_n \mid X_{n-1} = x_{n-1}, \cdots, \cancel{X_0 = x_0})$.

$= P(X_n = x_{n-1} + Y_n \mid X_{n-1} = x_{n-1})$

$Y_n = \begin{cases} 1 & p = 1 - \frac{X_{n-1}}{2k} \\ -1 & p = \frac{X_{n-1}}{2k} \end{cases}$ $\qquad$ depend on X_{n-1}.

so X_n is a Markov chain

$$P = \begin{cases} 1 - \frac{x_{n-1}}{2k} & \text{if } x_n = x_{n-1} + 1 \\ \frac{x_{n-1}}{2k} & \text{if } x_n = x_{n-1} - 1 \\ 0 & \text{else.} \end{cases}$$

(b). the state space $\{0, \cdots, 2k\}$

like random walk, only one communication class
essential

irreducible.

(c). c) If X_0 has a binomial distribution with parameters $(2k, 1/2)$ (meaning that X_0 balls are initially put into container α and $2k - X_0$ are put into container ω), what is the distribution of X_{10}? *[Hint: First compute the distribution of X_1.]*

$\boxed{X_0}$ $\boxed{2k\text{-}X_0}$
α ω

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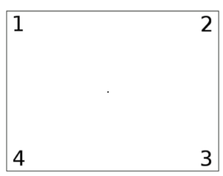

4

(a).

$$P = \begin{bmatrix} 0 & \frac{1}{3} & \frac{5}{12} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{1}{4} & \frac{5}{12} \\ \frac{5}{12} & \frac{1}{4} & 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{5}{12} & \frac{1}{3} & 0 \end{bmatrix}$$

(b). one communication class
essential
irreducible.

From time to time the spider runs from the corner it's in to another corner, chosen at random with probabilities inversely proportional to the distance from the current corner that the spider occupies. Denote by X_n the number of the corner the spider occupies after the nth change of corner.

μ

(a) Find the transition matrix P for the Markov chain X_n.

(c).

$P(X_1=1, X_2=2, X_4=2)$

$= P(X_4=2 \mid X_2=2) \cdot P(X_2=2 \mid X_1=1) \cdot P(X_1=1)$

$= (P^3)_{14} \cdot \frac{1}{4}$

(b) Analyze the state space of the chain: communication classes, essentiality, reducibility.

(c) Assume that initially the spider is dropped into the center of the box and chooses a corner uniformly at random. What is $P(X_1 = 1, X_2 = 2, X_4 = 2)$?

4. A time *inhomogeneous* Markov chain (X_n) has one step transition matrix at the nth step given by $P(n)$:

$$(P(n))_{i,j} = P(X_n = j | X_{n-1} = i).$$

Show that

$$P(X_{n+m} = j | X_n = i) = (P(n+1)P(n+2)\cdots P(n+m))_{i,j}\,.$$

if $m=1$

$P(X_{n+1}=j \mid X_n=i) = P(n+1)_{ij}$

$P(AB|C)$
$= P(A|BC)\,P(B|C)$

suppose $m=k$

$P(X_{n+k}=j \mid X_n=i) = (P(n+1)\cdots P(n+k))_{i,j}$

$m=k+1$

$P(X_{n+k+1}=j \mid X_n=i) = \sum_{l} P(X_{n+k+1}=j,\ X_{n+k}=l \mid X_n=i)$

$= \sum_{l} P(X_{n+k+1}=j \mid X_{n+k}=l) \cdot P(n+k=l \mid X_n=i)$

$= \sum_{j} P(n+k+1)_{lj} \, (P(n+1)P(n+2)\cdots P(n+k))_{i,l}$

$\underline{\underline{c\text{-}k}}$ $= (P(n+1)\cdots P(n+k)\, P(n+k+1))_{ij}$

5. Let $(X_n)_{n\geq 1}$ be a Markov chain with state space $\{1, \ldots, k\}$ for some $k \geq 1$. Show that if i and j communicate, then the probability that the chain started in state i reaches state j in k steps or fewer is greater than 0.

$i \leftrightarrow j \quad \Rightarrow \quad \sum_{n=1}^{k} P_{ij}^{(n)} > 0$

$i \leftrightarrow j \quad \exists\, n_1, n_2 \geq 1$ that $P_{ij}^{(n_1)} > 0$, $P_{ji}^{(n_2)} > 0$