MAST10008 Assignment 3 Solutions

1. (a) We can find the points of intersection (if any exist) of Π_1 and Π_2 by solving the linear system

$$\begin{aligned} x + b_1 y + c_1 z &= d_1 \\ x + b_2 y + c_2 z &= d_2 \end{aligned}$$

We do this via row operations:

$$\begin{bmatrix} 1 & b_1 & c_1 & d_1 \\ 1 & b_2 & c_2 & d_2 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_2} \begin{bmatrix} 1 & b_1 & c_1 & d_1 \\ 0 & b_2 - b_1 & c_2 - c_1 & d_2 - d_1 \end{bmatrix}$$

We see that Π_1 and Π_2 do not intersect if and only if the system is inconsistent if and only if the second row is $[0 \quad 0 \quad 0 \quad \text{nonzero}]$ if and only if ($b_2 - b_1 = 0$ and $c_2 - c_1 = 0$ and $d_2 - d_1 \neq 0$).

But the normals to the two planes are $(1, b_1, c_1)$ and $(1, b_2, c_2)$, so the above holds if and only if the normals are equal and $d_1 \neq d_2$.

(b) Let $P(x_0, y_0, z_0)$ be any point on Π_1, so $x_0 + b_1 y_0 + c_1 z_0 = d_1$. The distance between the point P and Π_2 is

$$D = \frac{|x_0 + b_2 y_0 + c_2 z_0 - d_2|}{\sqrt{1 + b_2^2 + c_2^2}} = \frac{|x_0 + b_1 y_0 + c_1 z_0 - d_2|}{\sqrt{1 + b_2^2 + c_2^2}} = \frac{|d_1 - d_2|}{\sqrt{1 + b_2^2 + c_2^2}}$$

2. Recall, from the lectures, the commutative diagram

$$\begin{tikzcd} M_{n\times n}(\mathbb{Z}) \arrow[r, "R"] \arrow[d, "\det"'] & M_{n\times n}(\mathbb{F}_2) \arrow[d, "\det"] \\ \mathbb{Z} \arrow[r, "r"'] & \mathbb{F}_2 \end{tikzcd}$$

where r is taking the remainder of the division of an integer by 2 and R is applying r to each entry in the matrix.

The commutativity of the diagram says that

$$\det \circ R = r \circ \det,$$

which applies in particular to our matrix A:

$$\det(R(A)) = r(\det(A)).$$

But

$$R(A) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is upper triangular, so its determinant is $\det(R(A)) = 1$.

Therefore $r(\det(A)) = 1$, which implies that $\det(A)$ is odd. In particular, $\det(A)$ must be nonzero, hence A is invertible.

(For those wanting independent verification, $\det(A) = -889231$ is indeed nonzero.)

3. (a) We prove this by contradiction (or contrapositive, if you insist).
   Suppose n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then

   $$n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1,$$

   which is manifestly odd. This is absurd since we are told that n^3 is even.

   (b) This is another proof by contradiction.
   Suppose $x = \frac{p}{q} \in \mathbb{Q}$, with p and q integers without common divisors, satisfies $x^3 = 2$. Then $p^3 = 2q^3$ is even, so by part (a) we must have that p is even. Write $p = 2r$, then we get $(2r)^3 = 2q^3$. This simplifies to $4r^3 = q^3$, so q^3 is even, hence by part (a) we must have that q is even.

   But then p and q are both divisible by 2, which contradicts the fact that they have no common divisors.

4. We prove this by induction on k.

   Base case: $k = 2$. The statement is $1 \cdot 2^0 = 2^2 - 2 - 1$, in other words $1 = 1$, true.

   For the induction step, fix $n \in \mathbb{N}$ with $n \geq 2$. We assume that the statement holds for n, that is
   $$1 \cdot 2^{n-2} + 2 \cdot 2^{n-3} + 3 \cdot 2^{n-4} + \cdots + (n-1) \cdot 2^0 = 2^n - n - 1.$$
   Multiply both sides by 2:
   $$1 \cdot 2^{n-1} + 2 \cdot 2^{n-2} + 3 \cdot 2^{n-3} + \cdots + (n-1) \cdot 2^1 = 2^{n+1} - 2n - 2.$$
   Now add n on both sides:
   $$1 \cdot 2^{n-1} + 2 \cdot 2^{n-2} + 3 \cdot 2^{n-3} + \cdots + (n-1) \cdot 2^1 + n \cdot 2^0 = 2^{n+1} - n - 2.$$
   Rewriting this as
   $$1 \cdot 2^{(n+1)-2} + 2 \cdot 2^{(n+1)-3} + 3 \cdot 2^{(n+1)-4} + \cdots + ((n+1)-1) \cdot 2^0 = 2^{n+1} - (n+1) - 1,$$
   we recognise the statement for $n + 1$.

   By the principle of mathematical induction, the statement is true for all $k \geq 2$.

5. (a) A short enumeration brings us to $4! = 24 > 16 = 2^4$, so we conjecture that the inequality $n! \geq 2^n$ holds for all $n \geq 4$.

   The base case $n = 4$ was dealt with already.

   For the induction step, fix $k \in \mathbb{N}$ with $k \geq 4$ and suppose the statement holds for k, that is $k! \geq 2^k$.

   Multiply both sides by $(k+1)$ to get $(k+1)! \geq (k+1)2^k$. Since $k \geq 4$ we have $k+1 \geq 2$ so $(k+1)2^k \geq 2^{k+1}$, so that $(k+1)! \geq 2^{k+1}$, which is the desired inequality for $k+1$.

   By the principle of mathematical induction, the inequality holds for all $n \geq 4$.

   (b) The claim is that $2^{2^k - 1}$ divides $(2^k)!$ for all $k \in \mathbb{N}$.

   We need to count the number of twos appearing in the factorial

$$(2^k)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (2^k),$$

   taking care not to overcount.

   I will do this by counting odd multiples of 2^j less than 2^k for j between 1 and k.

   There is only one odd multiple of 2^k that is less than 2^k, namely 2^k itself, and it contributes k twos.

   There is only 1 odd multiple of 2^{k-1} that is less than 2^k, namely $1 \cdot 2^{k-1}$, and it contributes $1 \cdot (k-1)$ twos.

   There are $2 = 2^{2-1}$ odd multiples of 2^{k-2} that are less than 2^k, namely $1 \cdot 2^{k-2}$ and $3 \cdot 2^{k-2}$, and they contribute $2^1 \cdot (k-2)$ twos.

   There are $4 = 2^{3-1}$ odd multiples of 2^{k-3} that are less than 2^k, namely $1 \cdot 2^{k-3}$, $3 \cdot 2^{k-3}$, $5 \cdot 2^{k-3}$, and $7 \cdot 2^{k-3}$, and they contribute $2^2 \cdot (k-3)$ twos.

   We continue like this until we reach the 2^{k-2} odd multiples of $2^1 = 2^{k-(k-1)}$ that are less than 2^k, and they contribute $2^{k-2} \cdot 1$ twos.

   In total, we have counted

$$k + 2^0 \cdot (k-1) + 2^1 \cdot (k-2) + \cdots + 2^{k-2} \cdot 1 = k + 2^k - k - 1 = 2^k - 1$$

   twos, where we used Question 4. We conclude that $(2^k)!$ is divisible by $2^{2^k - 1}$.

   (c) No, there aren't any.

6. (a) Let $\pi_A\colon A \times B \to A$ be given by $\pi_A(a,b) = a$.
I claim that this is surjective.
Let $a \in A$ be arbitrary. Take any $b \in B$ and consider $(a,b) \in A \times B$. Then $\pi_A(a,b) = a$.
A similar argument tells us that $\pi_B\colon A \times B \to B$ given by $\pi_B(a,b) = b$ is surjective.

(b) Given X and functions $f_A\colon X \to A$, $f_B\colon X \to B$, define $g\colon X \to A \times B$ by
$$g(x) = (f_A(x), f_B(x)).$$
Then for any $x \in X$ we have
$$\begin{aligned}\pi_A(g(x)) &= \pi_A(f_A(x), f_B(x)) = f_A(x)\\ \pi_B(g(x)) &= \pi_B(f_A(x), f_B(x)) = f_B(x),\end{aligned}$$
which means that $\pi_A \circ g = f_A$ and $\pi_B \circ g = f_B$.
To show that g is unique, we let $h\colon X \to A \times B$ satisfy the same properties. Let $x \in X$ and write $h(x) = (a_x, b_x)$ with $a_x \in A$, $b_x \in B$. Since $\pi_A \circ h = f_A$, we have
$$a_x = \pi_A(a_x, b_x) = \pi_A(h(x)) = f_A(x).$$
Since $\pi_B \circ h = f_B$, we have
$$b_x = \pi_B(a_x, b_x) = \pi_B(h(x)) = f_B(x).$$
Hence
$$h(x) = (a_x, b_x) = (f_A(x), f_B(x)) = g(x).$$
Since this holds for every $x \in X$, we conclude that $h = g$.

(c) We assume that there is a set Y together with (a) functions $\sigma_A\colon Y \to A$, $\sigma_B\colon Y \to B$, such that (b) given any set X and any functions $f_A\colon X \to A$, $f_B\colon X \to B$, there exists a unique function $g\colon X \to Y$ such that $\sigma_A \circ h = f_A$ and $\sigma_B \circ h = f_B$.
Letting $X = Y$, $f_A = \sigma_A$, and $f_B = \sigma_B$ in property (b) for $A \times B$, we get a unique map $g\colon Y \to A \times B$ such that $\pi_A \circ g = \sigma_A$, $\pi_B \circ g = \sigma_B$.
Letting $X = A \times B$, $f_A = \pi_A$, and $f_B = \pi_B$ in property (b) for Y, we get a unique map $h\colon A \times B$ such that $\sigma_A \circ h = \pi_A$, $\sigma_B \circ h = \pi_B$.
I claim that $h \circ g = \mathrm{id}_Y$ and $g \circ h = \mathrm{id}_{A\times B}$ (so that Y is in bijection with $A \times B$.
We do this by magic.[1]
We let $X = A \times B$, $f_A = \pi_A$, and $f_B = \pi_B$ in property (b) for $A \times B$, and observe that both functions $\mathrm{id}_{A\times B}$ and $g \circ h$ satisfy the desired property:
$$\begin{aligned}\pi_A \circ \mathrm{id}_{A\times B} &= \pi_A\\ \pi_A \circ (g \circ h) = (\pi_A \circ g) \circ h = \sigma_A \circ h &= \pi_A\\ \pi_B \circ \mathrm{id}_{A\times B} &= \pi_B\\ \pi_B \circ (g \circ h) = (\pi_B \circ g) \circ h = \sigma_B \circ h &= \pi_B\end{aligned}$$
But such a map is unique, so we must have $\mathrm{id}_{A\times B} = g \circ h$.
The same argument using $X = Y$, $f_A = \sigma_A$, and $f_B = \sigma_B$ shows that $\mathrm{id}_Y = h \circ g$.

Note: Property (b) that we proved for $A \times B$ above is known as *the universal mapping property of the Cartesian product.* Universal mapping properties are great tools in pure mathematics, as most algebraic constructions can be characterised and studied via their universal mapping properties.

[1] No, not really, but it will seem that way.