MAST10008 Assignment 3

Due Thursday 4 April at 12pm in your tutor's assignment box

1. Consider two planes given by Cartesian equations

$$\begin{aligned} \Pi_1\colon \qquad & x + b_1 y + c_1 z = d_1 \\ \Pi_2\colon \qquad & x + b_2 y + c_2 z = d_2 \end{aligned}$$

   (a) Prove that Π_1 and Π_2 do not intersect if and only if their normal vectors are equal and $d_1 \neq d_2$.

   (b) Suppose Π_1 and Π_2 do not intersect. Determine the distance between Π_1 and Π_2 in terms of $b_1, c_1, d_1, b_2, c_2, d_2$.

2. Without any messy calculations, show that the matrix

$$A = \begin{bmatrix} 2019 & 2022 & 2027 & 2013 \\ 2020 & 2021 & 2026 & 2011 \\ 2018 & 2016 & 2023 & 2029 \\ 2014 & 2012 & 2010 & 2025 \end{bmatrix}$$

   is invertible.

   (Hint: the only "hard" arithmetic you need to do is to take remainders of integer division by 2.)

3. (a) Prove that if $n \in \mathbb{Z}$ and n^3 is even then n is even.

   (b) Prove that there does not exist $x \in \mathbb{Q}$ such that $x^3 = 2$.

4. Prove by induction that

$$1 \cdot 2^{k-2} + 2 \cdot 2^{k-3} + 3 \cdot 2^{k-4} + \cdots + (k-1) \cdot 2^0 = 2^k - k - 1$$

   for all integers $k \geq 2$.

5. (a) Find the smallest $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $n! \geq 2^n$.

   (b) Show that 2^{n-1} divides $n!$ whenever n is of the form $n = 2^k$ for some $k \in \mathbb{N}$. (If you need to, feel free to use the result of Question 4.)

   (c) (Not for credit.) Are there any $n \in \mathbb{N}$ such that 2^n divides $(n!)$?

6. Let A and B be sets and consider their Cartesian product $A \times B$.

   (a) Give surjective functions $\pi_A\colon A \times B \to A$ and $\pi_B\colon A \times B \to B$.

   (b) Suppose we are given a set X and functions $f_A\colon X \to A$ and $f_B\colon X \to B$. Prove that there is a unique function $g\colon X \to A \times B$ such that

$$f_A = \pi_A \circ g \qquad \text{and} \qquad f_B = \pi_B \circ g$$

   (c) Let Y be another set satisfying the same properties that we proved in (a) and (b) for $A \times B$. Prove that Y is in bijection with $A \times B$.