MAST30001 Stochastic Modelling – 2020

Assignment 1

Please complete the Plagiarism Declaration Form on the LMS before submitting this assignment.

Submission Instructions:

- Typed submissions (ideally using LaTeX) are preferred. For handwritten solutions:
  - Write your answers on blank paper. Write on one side of the paper only. Start each question on a new page. Write the question number at the top of each page.
  - Scan your solutions to a single PDF file with a mobile phone or a scanner. Scan from directly above to avoid any excessive keystone effect. Check that all pages are clearly readable and cropped to the A4 borders of the original page. Poorly scanned submissions may be impossible to mark.
- Upload the PDF file to Gradescope via the LMS. Gradescope will ask you to identify on which of the uploaded pages your answers to each question are located.
- The submission deadline is **5:00pm on Thursday, 17 September, 2020**.

There are 3 questions, all of which will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness, and style count.

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1. A discrete time Markov chain with state space $S = \{1, 2, 3, 4, 5, 6, 7\}$ has the following transition matrix.

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3/7 & 4/7 & 0 \\ 0 & 0 & 0 & 0 & 3/4 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

   (a) Write down the communication classes of the chain.

   (b) Find the period of each communicating class.

   (c) Determine which classes are essential.

   (d) Classify each essential communicating class as transient or positive recurrent or null recurrent.

   (e) Describe the long run behaviour of the chain (including deriving long run probabilities where appropriate).

   (f) Given $X_0 = 4$, find the long run proportion of time the chain spends in state j for each $j \in S$.

   (g) Find the expected number of steps taken for the chain to first reach state 3, given the chain starts at state 4.

Ans.

(a) The communicating classes of the chain are $S_1 = \{1, 2, 3, 4\}$, $S_2 = \{5, 6\}$ and $S_3 = \{7\}$.

(b) S_1 has period 2 and the other classes are aperiodic due to loops.

(c) S_1 and S_2 are essential.

(d) S_1 and S_2 are positive recurrent because they are finite and essential.

(e) The chain will eventually end up in S_1 or S_2. Because S_2 is positive recurrent and aperiodic, it is ergodic, with long run probabilities given by the unique stationary distribution $\pi^{(2)} = (\pi_5, \pi_6)$ satisfying

$$\pi^{(2)} \begin{pmatrix} 3/7 & 4/7 \\ 3/4 & 1/4 \end{pmatrix} = \pi^{(2)},$$

and $\pi_5 + \pi_6 = 1$. Solving these equations yields

$$\pi^{(2)} = \left(\frac{21}{37}, \frac{16}{37}\right).$$

For S_1, the periodic subclasses are $\{1, 3\}$ and $\{2, 4\}$. The square of the transition matrix restricted to S_3 is

$$P = \begin{pmatrix} 2/3 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix},$$

and thus the stationary probability vector for the two step (reducible chain) are given by

$$\tilde{\pi}^{(1)} = (\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3, \tilde{\pi}_4) = \left(\frac{3}{5}, \frac{3}{5}, \frac{2}{5}, \frac{2}{5}\right),$$

and

$$P(X_{2n+k} = j | X_0 \in \{1, 3\}) \overset{n\to\infty}{\longrightarrow} \begin{cases} 3/5, & k = 0, j = 1 \text{ or } k = 1, j = 2; \\ 2/5, & k = 0, j = 3 \text{ or } k = 1, j = 4; \\ 0, & \text{else}; \end{cases}$$

and

$$P(X_{2n+k} = j | X_0 \in \{2, 4\}) \overset{n\to\infty}{\longrightarrow} \begin{cases} 3/5, & k = 0, j = 2 \text{ or } k = 1, j = 1; \\ 2/5, & k = 0, j = 4 \text{ or } k = 1, j = 3; \\ 0, & \text{else}. \end{cases}$$

(f) If $X_0 = 4$, then the chain is in class S_1 and never visits classes S_2 or S_3, so the proportion of time spent in states $\{5, 6, 7\}$ is zero.
For state $j \in \{1, 2, 3, 4\}$, the proportion of time spent in each state is

$$\frac{1}{2}\tilde{\pi}_j.$$

(g) For $j \in S_1, j \neq 3$, let e_j be the expected amount of time to reach state 3 given the chain starts in state j. We want to find e_4. First step analysis implies

$$\begin{aligned} e_4 &= 1 + \frac{1}{2}e_1, \\ e_1 &= 1 + e_2, \\ e_2 &= 1 + \frac{2}{3}e_1, \end{aligned}$$

and solving gives $e_4 = 4$.

2. An irreducible Markov chain on $\{0, 1, \dots\}$ has transition probabilities

$$p_{ij} = \begin{cases} \frac{1}{(i+1)(i+2)}, & 0 \le j \le i \\ \frac{i+1}{i+2}, & j = i+1. \end{cases}$$

Determine whether the chain is positive recurrent, null recurrent, or transient. *[Hint: It is possible to find an exact simple expression for $p_{ij}^{(n)}$.]*

Ans. Since the chain is irreducible, it's enough to check the recurrence at state 0. We find that for $j = 0, \dots, n$,

$$p_{0j}^{(n)} = \sum_{k=j-1}^{n-1} p_{0k}^{(n-1)} p_{kj} = p_{0,j-1}^{(n-1)} \frac{j}{j+1} + \sum_{k=j}^{n-1} p_{0k}^{(n-1)} \frac{1}{(k+1)(k+2)}.$$

Trying small cases, we have

$$\begin{aligned} p_{0k}^{(1)} &= 1/2, \quad k = 1, 2, \\ p_{0k}^{(2)} &= 1/3, \quad k = 1, 2, 3, \\ p_{0k}^{(3)} &= 1/4, \quad k = 1, 2, 3, 4. \end{aligned}$$

We can then guess $p_{0k}^{(n)} = 1/(n+1)$ for $0 \le k \le n$, which satisfies the recursion above. With this formula, the expected number of visits to 0 started from 0 is

$$\sum_{n \ge 1} p_{00}^{(n)} = \sum_{n \ge 1} \frac{1}{n+1} = \infty,$$

and so the chain is <u>recurrent</u>. Since the chain is aperiodic (loops) and irreducible, if it were positive recurrent then it would be ergodic, but

$$\lim_{n \to \infty} p_{00}^{(n)} = 0,$$

so it must be <u>null recurrent</u>.

3. (a) Fix $p \in (0, 1)$ and define the Markov chain $(X_n)_{n \ge 0}$ on $\{0, 1, \dots, \}$ with transition probabilities, for $i \ge 1$,

$$p_{i,i+1} = 1 - p_{i,i-1} = p,$$

and $p_{0,1} = 1 - p_{0,0} = p$. Setting $T = \inf\{n \ge 1 : X_n = 0\}$, show that if $(X_n)_{n \ge 0}$ is positive recurrent, then for all i in the state space,

$$\mathbb{E}[T | X_0 = i] < \infty,$$

and if the chain is not positive recurrent, then

$$\mathbb{E}[T | X_0 = i] = \infty.$$

(b) Fix $p \in (0, 1)$ and define the Markov chain $(Y_n)_{n \ge 0}$ on $\{0, 1, \dots, \}$ with transition probabilities, for $i \ge 1$,

$$p_{i,i+1} = 1 - p_{i,i-1} = p,$$

and $p_{0,j} = q_j$ for $j \ge 1$ where $\sum_{j=1}^{K} q_j = 1$ and K is a positive integer. Determine the values of p such that the chain is positive recurrent.

Ans.

(a) If the chain is positive recurrent, then $\mathbb{E}[T|X_0 = 0] < \infty$ and is otherwise infinite. The first step analysis equations for $e_j := \mathbb{E}[T|X_0 = j]$ are

$$\begin{aligned} e_0 &= 1 + pe_1, \\ e_1 &= 1 + pe_2, \\ e_j &= 1 + pe_{j+1} + (1-p)e_{j-1}, \;\; j \geq 2, \end{aligned}$$

and so inductively e_j is infinite or finite with e_0.

(b) Since positive recurrence is a class property and the chain is irreducible, we only need to check at one state, say 0. Then first step analysis implies that

$$\mathbb{E}[T|Y_0 = 0] = 1 + \sum_{j=1}^{K} q_j \mathbb{E}[T|Y_0 = j].$$

But the distribution of $(Y_n)_{0 \leq n \leq T}$ given $Y_0 = j \geq 1$ is the same as $(X_n)_{0 \leq n \leq T}$ given $X_0 = j \geq 1$; this is because the transition probabilities are the same away from state 0; and therefore $\mathbb{E}[T|Y_0 = j] = \mathbb{E}[T|X_0 = j]$ for $j \geq 1$, and so if $p < 1/2$, then

$$\mathbb{E}[T|Y_0 = 0] = 1 + \sum_{j=1}^{K} q_j e_j \leq 1 + \left(\max_{1 \leq i \leq k} e_i \right) < \infty,$$

and if $p \geq 1/2$,

$$\mathbb{E}[T|Y_0 = 0] = 1 + \sum_{j=1}^{K} q_j e_j = \infty.$$