THE UNIVERSITY OF MELBOURNE

Student Number

Semester 1 Assessment, 2018

School of Mathematics and Statistics

MAST20009 Vector Calculus

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 5 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- No written or printed materials may be brought into the examination.
- No calculators of any kind may be brought into the examination.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- There are 11 questions on this exam paper.
- All questions may be attempted.
- Marks for each question are indicated on the exam paper.
- Start each question on a new page.
- Clearly label each page with the number of the question that you are attempting.
- There is a separate 3 page formula sheet accompanying the examination paper, which you may use in this examination.
- The total number of marks available is 125.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.
- Initially students are to receive the exam paper, the 3 page formula sheet, and two 14 page script books.

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Question 1 (14 marks)

(a) Evaluate the following limits, if they exist. If the limit does not exist, explain why it does not exist.

(i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x-y}$;

(ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y-x+3\sin x}{x+y}$.

(b) Consider the functions $f:\mathbb{R}^2\to\mathbb{R}^3$, $g:\mathbb{R}^3\to\mathbb{R}^2$ and $h:\mathbb{R}^2\to\mathbb{R}^2$ given by

$$\begin{aligned} f(x,y) &= (2x^2, 3y, y^2-x), \\ g(x,y,z) &= (y+z^2, x^2+z), \\ h(x,y) &= (y^2-x, 2x+y). \end{aligned}$$

Evaluate the derivative $\mathbf{D}[h(g[f(x,y)])]$ at $(x,y)=(0,1)$ using the matrix version of the chain rule.

Question 2 (11 marks)

Using Lagrange Multipliers, determine the maximum and minimum of

$$f(x,y,z) = x^2+y^2+z^2$$

subject to the constraints

$$z^2 = x^2+y^2 \quad \text{and} \quad x-2z=3.$$

Justify that the points you have found give the maximum and minimum of f.

Question 3 (11 marks)

(a) Consider the vector field

$$\mathbf{F}(x,y) = y^3\mathbf{i} - x^3\mathbf{j}.$$

(i) Determine $\nabla^2\mathbf{F}$.

(ii) Determine the equation for the flow line of $\mathbf{F}$ passing through the point $(1,1)$ in terms of x and y.

(b) Let $\mathbf{u}:\mathbb{R}\to\mathbb{R}^3$ be a C^3 path parametrised in terms of t. Evaluate and simplify

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{u}'\cdot(\mathbf{u}\times\mathbf{u}'')\right].$$

Question 4 (11 marks)

Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field and $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^1 scalar function. Let $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $r = |\mathbf{r}|$, $r \neq 0$.

(a) If f has the form $f(x, y, z) = g(r)$ where $g : \mathbb{R} \to \mathbb{R}$ is a C^1 scalar function, prove that

$$\boldsymbol{\nabla} f = \frac{1}{r}\frac{\mathrm{d}g}{\mathrm{d}r}\,\mathbf{r}.$$

(b) Prove vector identity (7), that is:

$$\boldsymbol{\nabla} \cdot (f\mathbf{F}) = f\boldsymbol{\nabla} \cdot \mathbf{F} + \mathbf{F} \cdot \boldsymbol{\nabla} f.$$

(c) Using parts (a) and (b), evaluate

$$\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{r}}{r^7}\right).$$

Question 5 (10 marks)

Let D be the part of the elliptical region $2x^2 + 3y^2 \leq 1$ for $x \leq 0$.

(a) Sketch the region D.

(b) Evaluate the double integral

$$\iint_D e^{(4x^2+6y^2)}\, dxdy$$

by making the change of variables $x = \dfrac{r}{\sqrt{2}}\cos\theta$ and $y = \dfrac{r}{\sqrt{3}}\sin\theta$.

Question 6 (10 marks)

Let V be the solid region inside the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$.

(a) Sketch the region V.

(b) Determine the total mass of V if the mass per unit volume is $\mu = z(x^2 + y^2)$.

Question 7 (16 marks)

Let R be the hemisphere $z = -\sqrt{9 - x^2 - y^2}$.

(a) Write down a parametrisation for R based on spherical coordinates.

(b) Using part (a) and a suitable cross product, find a normal vector to R. Determine whether the normal is pointing inwards or outwards to R.

(c) Using part (b) and a suitable integral, determine the surface area of R.

(d) Determine the Cartesian equation of the tangent plane to R at $\left(\dfrac{3}{\sqrt{2}}, \dfrac{3}{\sqrt{2}}, 0\right)$.

Question 8 (11 marks)

Let S be the surface of the tetrahedron bounded by the plane $x+y+z=1$ and the coordinate planes $x=0$, $y=0$ and $z=0$.

(a) Sketch S.

(b) Determine the flux of the velocity field

$$\mathbf{F}(x,y,z) = \left(2x^3y + e^{2y} + 2xz, xy - 3x^2y^2 + e^{3z}, xe^y - z^2\right)$$

across S in the direction of the outward unit normal.

Question 9 (11 marks)

Let S be the capped surface given by the union of two surfaces S_1 and S_2 where

$$S_1: \quad z = 10 - x^2 - y^2, \quad z \geq 1,$$

and

$$S_2: \quad x^2 + y^2 = 9, \quad -2 \leq z \leq 1.$$

Let

$$\mathbf{F}(x,y,z) = (zx + z^2y + x)\mathbf{i} + (z^3yx + y)\mathbf{j} + z^4x^2\mathbf{k}.$$

(a) Sketch S.

(b) Determine the curl of $\mathbf{F}$.

(c) If S is oriented using the outward unit normal, evaluate the surface integral

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S}.$$

Question 10 (13 marks)

(a) State Green's theorem. Explain all symbols used and any required conditions.

(b) Let C be the ellipse $4x^2 + 9y^2 = 1$, oriented anticlockwise. Using Green's theorem, evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{s}$$

when

$$\mathbf{F}(x,y) = \frac{2y}{x^2+y^2}\mathbf{i} - \frac{2x}{x^2+y^2}\mathbf{j}.$$

Question 11 (7 marks)

Define *paraboloidal* coordinates (u, v, ϕ) by

$$x = uv\cos\phi, \quad y = uv\sin\phi, \quad z = \frac{1}{2}(u^2 - v^2)$$

where $u > 0$, $v > 0$ and $0 \leq \phi < 2\pi$.

(a) Compute the scale factors h_u, h_v and h_ϕ.

(b) Find an expression for the following in terms of u, v and ϕ.

(i) $\boldsymbol{\nabla}(u^2v^4 + \cos\phi)$

(ii) $\boldsymbol{\nabla} \cdot \left(\sqrt{u^2 + v^2}\,\mathbf{e}_u\right)$

End of Exam—Total Available Marks = 125