Semester 2 Assessment, 2015

School of Mathematics and Statistics

MAST20009 Vector Calculus

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 5 pages (including this page)

Authorised materials:

- No materials are authorised.

Instructions to Students

- You may remove this question paper at the conclusion of the examination
- There are 11 questions on this exam paper.
- All questions may be attempted.
- Marks for each question are indicated on the exam paper.
- Start each question on a new page.
- Clearly label each page with the number of the question that you are attempting.
- There is a separate 3 page formula sheet accompanying the examination paper, that you may use in this examination.
- The total number of marks available is 125.

Instructions to Invigilators

- Students may remove this question paper at the conclusion of the examination
- Initially students are to receive the exam paper, the 3 page formula sheet, and a 14 page script book.

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Question 1 (11 marks)

(a) Evaluate the following limits, if they exist. If the limit does not exist, explain why it does not exist.

(i) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x-y}{x^2+xy-2y^2}$;

(ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^3-2xy+5y^2}{x^3+2y^2}$.

(b) Consider the function

$$f(x,y) = e^{3x}\sin(\pi y).$$

(i) Determine the second order Taylor polynomial for f about $(0,1)$.

(ii) Using part (b)(i), approximate $f\left(\frac{1}{10}, \frac{3}{4}\right)$.

Question 2 (10 marks)

Using Lagrange Multipliers, determine the maximum and minimum of the function

$$f(x,y,z) = 2x + y$$

subject to the constraints

$$x+y+z = 3 \quad \text{and} \quad x^2+z^2 = 4.$$

Justify that the points you have found give the maximum and minimum of f.

Question 3 (9 marks)

(a) Determine the curvature κ of the path

$$\mathbf{c}(t) = (2+\sqrt{2}\cos t, 1-\sin t, 3+\sin t),\ \ t>0.$$

(b) Let $\mathbf{T}$ be the unit tangent vector and $\mathbf{B}$ be the unit binormal vector to an arbitrary C^3 path. Prove that

$$\frac{d\mathbf{B}}{dt}\cdot\mathbf{T} = 0.$$

Question 4 (8 marks)

Consider the vector identity

$$\nabla \cdot (f\nabla g - g\nabla f) \;=\; f\nabla^2 g - g\nabla^2 f$$

where $f(x, y, z)$ and $g(x, y, z)$ are C^2 scalar functions.

(a) Prove the identity by direct calculation.

(b) Prove the identity using the basic identities of vector calculus.

Question 5 (15 marks)

Consider the vector field

$$\mathbf{F}(x, y, z) = 2y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}$$

and the path

$$\mathbf{c}(t) = \left(t^2, \frac{2}{3}t^3, \frac{1}{4}t^4\right),\; 0 \le t \le 1.$$

(a) Is $\mathbf{c}$ a flow line of $\mathbf{F}$? Justify your answer.

(b) Show that $\mathbf{F}$ is a conservative vector field in $\mathbb{R}^3$.

(c) Determine a scalar potential ϕ such that $\mathbf{F} = \nabla\phi$.

(d) Determine the work done by $\mathbf{F}$ in moving a particle along $\mathbf{c}$.

(e) Consider an electrical cable in the shape of the path $\mathbf{c}$. Determine the total charge in the cable, if the charge per unit length in the cable is $\mu = 2t + 1$.

Question 6 (10 marks)

Let V be the solid region within the cylinder $x^2 + y^2 = 1$ that is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and bounded above by the plane $z = 6$.

(a) Sketch the region V.

(b) Determine the total mass of V if the mass per unit volume is $\mu = z(x^2 + y^2)$.

Question 7 (12 marks)

Let S be the hemisphere $z = -\sqrt{9 - x^2 - y^2}$.

(a) Write down a parametrization for S based on spherical coordinates.

(b) Using part (a), find an outward normal vector to S.

(c) Determine the Cartesian equation of the tangent plane to S at $\left(0, \frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$.

Question 8 (10 marks)

Let C be the boundary of the triangle with vertices at $(0,0), (1,3), (4,0)$ traversed in the clockwise direction.

(a) Sketch C.

(b) Evaluate the line integral

$$\int_C [\log(1+y) + xy]\; dx \;+\; \frac{x}{1+y}\, dy.$$

Question 9 (15 marks)

Let S be the capped surface given by the union of two surfaces S_1 and S_2 where

$$S_1: \quad z = x^2 + y^2, \quad 2 \leq z \leq 3,$$

and

$$S_2: \quad z = 6 - x^2 - y^2, \quad 3 \leq z \leq 6.$$

Let S be oriented using the outward unit normal.

(a) Sketch S.

(b) If $\mathbf{F}(x,y,z) = -yz^3\mathbf{i} + 3x\mathbf{j} + x^5\mathbf{k}$, evaluate

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S}$$

using

(i) a line integral,

(ii) a surface integral.

Question 10 (15 marks)

Let S be the sphere $x^2 + y^2 + z^2 = 4$, oriented using the outward unit normal. Let

$$\mathbf{F}(x,y,z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$$

(a) State Gauss' Divergence theorem. Explain all symbols used and any required conditions.

(b) Using Gauss' theorem, evaluate the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Question 11 (10 marks)

Define *parabolic cylindrical* coordinates (u, v, z) by

$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv, \quad z = z$$

where $u \in \mathbb{R}$, $v \geq 0$, $z \in \mathbb{R}$ and $u^2 + v^2 > 0$.

(a) Compute the scale factors h_u, h_v and h_z.

(b) Show that the coordinate system is orthogonal.

(c) Write down an expression for the |Jacobian|.

(d) Let $f(u,v,z) = u^4 v^2 + z^3$ and $\mathbf{F}(u,v,z) = v^2 \mathbf{e}_v$. Find an expression for

(i) $\boldsymbol{\nabla} f$ (ii) $\boldsymbol{\nabla} \times \mathbf{F}$

in terms of u, v and z.

End of Exam—Total Available Marks = 125.