Stochastic modelling exam 2010.

1. Given $X_n=i,\ X_{n-1}=x_{n-1}, \cdots, X_0=x_0$, the (distribution of the) number of working machines on day $n+1$ is determined by the number of break-downs of the i machines, so it's Markovian. Clearly, $p_{01}=1$, $p_{i0}=0$ for $i=1,2,3$.

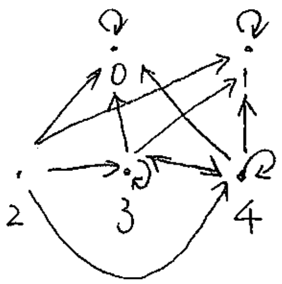
$p_{11}=P(\text{it breaks down} \mid X_n=1)=0.2$. Similarly, $p_{12}=0.8$;

$p_{21}=0.2\times0.2=0.4$, $p_{22}=\binom{2}{1}0.2\times0.8=0.32$, $p_{23}=0.8\times0.8=0.64$;

$p_{31}=\binom{3}{3}0.2^3=0.008$, $p_{32}=\binom{3}{2}0.2^2\times0.8=0.096$, $p_{33}=\binom{3}{1}0.2\times0.8^2$
$+\binom{3}{0}0.2^0\times0.8^3=0.896$.

In summary,

$$P=\begin{array}{c|c} & \begin{array}{cccc}0 & 1 & 2 & 3\end{array} \\ \begin{array}{c}0\\1\\2\\3\end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 \\ 0 & 0.04 & 0.32 & 0.64 \\ 0 & 0.008 & 0.096 & 0.896 \end{bmatrix}\end{array}$$

2. (a) 

2

(b) 0 & 1 are absorbing states
2, 3, 4 are all transient states. 3
The MC is reducible.

(c) $P(X_2=3, X_3=1 \mid X_1=4)=p_{43}p_{31}=0.35\times0.1=0.035$; 2

$P(X_2=3, X_4=1 \mid X_1=4)=p_{43}p_{31}^{(2)}=0.35\times(0.15\times0+0.1\times1+0\times0.3+0.25\times0.1+0.5\times0.05)$
$=0.0525$; 2

$$P(X_2=3 \mid X_1=4, X_3=4)=\frac{P(X_3=4, X_2=3 \mid X_1=4)}{P(X_3=4 \mid X_1=4)}=\frac{p_{43}p_{34}}{p_{44}^{(2)}}$$

$$=\frac{0.35\times0.5}{0.2\times0+0.05\times0+0\times0.5+0.35\times0.5+0.4\times0.4 \leftarrow 0.335}=0.522$$ 2

(d) There is only one closed class of essential states and the states are aperiodic.
The MC is not ergodic because it has two closed classes of (absorbing) states. 3

(e) Let $\tau = \min\{i: X_i \neq 4\}$, then

$$p(\tau = m \mid X_0 = 4) = p(X_m \neq 4, X_{m-1} = 4, \dots, X_1 = 4 \mid X_0 = 4)$$
$$= p_{44}^{m-1}(1 - p_{44}) = 0.6 \times 0.4^{m-1}, \quad m \geq 1. \qquad ③$$

3. (a) $N_{0.5} \sim Pn(1.5)$, so $p(N_{0.5} = 0) = e^{-1.5} = 0.223$ ②

(b) $p(N_2 = 5 \mid N_1 = 3) = p(N_2 - N_1 = 2 \mid N_1 = 3) \underset{\text{indept}}{=} p(\underset{\sim Pn(3)}{N_2 - N_1} = 2) = e^{-3} \cdot \frac{9}{2} = 0.224$ ②

$p(N_1 = 3 \mid N_2 = 5) = \binom{5}{3}\left(\frac{1}{2}\right)^5 = 0.3125$ ②

(c) The inter-arrival times $T_1, T_2, \dots$ are iid exp(3),

$E(T_1 + T_2 + T_3) = 3 E T_1 = 1$ hour. ②

(d) Let M_t = # high priority customers by t

L_t = # low " " " ",

by Poisson thinning property, $\{M_t\}$ & $\{L_t\}$ are indept Poisson processes with rates 0.6 & 2.4 resp. ②

$p(M_{0.5} = 1, L_{0.5} = 1) = p(M_{0.5} = 1)\, p(L_{0.5} = 1) = e^{-0.3} \times 0.3 \times e^{-1.2} \times 1.2 = 0.36 e^{-1.5} = 0.08$ ②

(e) 0.2 since each has prob. 0.2 of being high priority, indept of others. ①

(f) $E(N_1 \mid M_1 = 1) = E(L_1 + M_1 \mid M_1 = 1) = E L_1 + 1 = 3.4$ ②

(g) Yes, by Poisson thinning property. ②

(h) No, because the inter-arrival times are not exponentially dist. ②

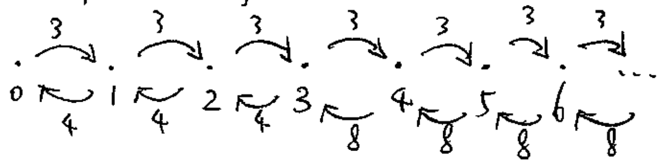
(i) Let $\xi_1, \xi_2, \dots$ be the profit from high priority customer,

$\eta_1, \eta_2, \dots$ " low ",

then hourly total profit $T = \sum_{i=1}^{M_1} \xi_i + \sum_{j=1}^{L_1} \eta_j \Rightarrow ET = EM_1 \cdot E\xi_1 + EL_1 \cdot E\eta_1$

$= 0.6 \times 200 + 2.4 \times 120 = 408.$ ③

4. (a) $S = \{0, 1, 2, \dots\}$ ①



②

Birth & death?

(b) Arrival rate $\lambda = 3$, maximum service rate $= 8 > 3$, so ergodic. ①

$k_0 = 1$, $k_1 = \frac{3}{4}$, $k_2 = \frac{9}{16}$, $k_3 = \frac{27}{64}$, $k_j = \frac{27}{64} \cdot \left(\frac{3}{8}\right)^{j-3}$, $j \geq 4$.

$\sum_{i=0}^{\infty} k_i = 2\frac{79}{80} = 2.9875$,

$\pi_0 = \frac{80}{239}$, $\pi_1 = \frac{60}{239}$, $\pi_2 = \frac{45}{239}$, $\pi_i = \frac{135}{956}\left(\frac{3}{8}\right)^{i-3}$, $i \geq 3$. ②

(c) $$E_{st} X_t = 1 \times \frac{60}{239} + 2 \times \frac{45}{239} + \sum_{i=3}^{\infty} i \cdot \frac{135}{956}\left(\frac{3}{8}\right)^{i-3}$$

$$= \frac{60}{239} + \frac{90}{239} + \sum_{i=3}^{\infty}(i-2)\frac{135}{956}\left(\frac{3}{8}\right)^{i-3} + \sum_{i=3}^{\infty} 2 \cdot \frac{135}{956} \cdot \left(\frac{3}{8}\right)^{i-3}$$

$$= 1\frac{527}{1195} = 1.441.$$ ③

(d) Direct computation of $1 \times \pi_2 + 2 \times \pi_3 + \sum_{i=4}^{\infty}(i-2)\pi_i$ or by Little's law,

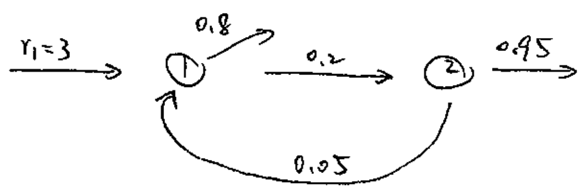
$L_q = \frac{3303}{4780} = 0.691$ ②

(e) $P(\text{server B idle}) = P_{st}(X_t \leq 3) = \pi_0 + \pi_1 + \pi_2 + \pi_3 = \frac{875}{956} = 0.915.$ ②

(f) By Little's law, $W = \frac{1101}{4780} = 0.2303$ (see p.7 for direct computation) ②

(g) " , $D = \frac{574}{1195} = 0.4803$ Or: $D = W + E(\text{service time})$
$= W + 0.25$ ②

5. (a)



①

Define 0 = "outside", then the routing matrix is $P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 0.8 & 0 & 0.2 \\ 2 & 0.95 & 0.05 & 0 \end{array}$ ①

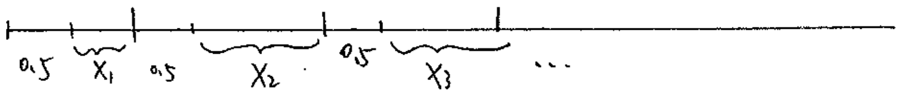
(b) $\left.\begin{array}{l} \lambda_1 = r_1 + 0.05\lambda_2 \\ \lambda_2 = r_2 + \lambda_1 \times 0.2 \end{array}\right\} \Rightarrow \lambda_1 = \frac{100}{33} = 3.03$, $\lambda_2 = \frac{20}{33}$.

Since $\lambda_1 < \mu_1$, $\lambda_2 < \mu_2$, the network is ergodic. 2

(c) (i) $L = \dfrac{\lambda_1}{\mu_1 - \lambda_1} + \dfrac{\lambda_2}{\mu_2 - \lambda_2} = 1\dfrac{753}{1157} = 1.651$. 2

(ii) $L = \lambda D \Rightarrow D = \dfrac{1910}{3471} = 0.550$. 2

6. (a)



1

$X_1, X_2, \cdots$ are iid $\exp(1)$ random variables.

Since the times between any two consecutive recorded particles are the sum of recuperating time + exp waiting time, indept of others, $\{M_t\}$ is a renewal process.

Let $T_i = 0.5 + X_i$, then $F_{T_i}(x) = \begin{cases} 0, & x < 0.5 \\ 1 - e^{-(x-0.5)}, & x \geq 0.5. \end{cases}$ 2

(b) $\mu = E T_i = E X_i + 0.5 = 1.5$ seconds

$\sigma^2 = Var(X_i) = 1$ 2

(c) $M_t \approx \dfrac{t}{\mu} = \dfrac{2t}{3}$, t large. $\dfrac{M_t - \frac{t}{\mu}}{\sqrt{\frac{t}{\mu} \cdot \frac{\sigma^2}{\mu^2}}} \sim N(0,1)$, so

$P\left(-1.645 < \dfrac{M_t - \frac{t}{\mu}}{\sqrt{\frac{t}{\mu} \cdot \frac{\sigma^2}{\mu^2}}} < 1.645\right) \approx 0.90$

$\Rightarrow$ 90% CI: $\dfrac{t}{\mu} \pm 1.645\sqrt{\dfrac{t\sigma^2}{\mu^3}} \overset{t=9000}{=} \cdots = (5915.05, 6084.95)$ 3

(d) $P(Y_t \geq y) \approx \dfrac{1}{\mu}\displaystyle\int_y^\infty (1 - F_{T_1}(z))\,dz = \begin{cases} \dfrac{1}{1.5}\displaystyle\int_y^\infty e^{-(s-0.5)}\,ds, & y \geq 0.5 \\ \dfrac{1}{1.5}\left(1 + \displaystyle\int_y^{0.5} 1\,ds\right), & 0 \leq y < 0.5 \end{cases}$

$= \begin{cases} \dfrac{1}{1.5} e^{-(y-0.5)}, & y \geq 0.5 \\ \dfrac{1}{1.5}(1.5 - y), & 0 \leq y < 0.5. \end{cases}$ 3

(e) Take n renewal periods, the proportion of recuperating time is

$$\frac{0.5\,n}{(X_1+0.5)+\cdots+(X_n+0.5)} \xrightarrow{LLN} \frac{0.5}{EX_1+0.5} = \frac{1}{3}.$$

So $p_t = \frac{1}{3}$ for t large. ②

(f) Renewal function is $H(t) = EN_t$, $t \geq 0$. ①

By renewal equation,

$$H(t) = F(t) + \int_0^t H(t-s)\,dF(s).$$ ①

Since $F(t) = 0$ for $0 \leq t \leq 0.5$, we have $H(t) = 0$ for $0 \leq t \leq 0.5$. ①

For $0.5 < t \leq 1$,

$$H(t) = 1 - e^{-(t-0.5)} + \int_{0.5}^{t} \underbrace{H(t-s)}_{0 \leq t-s < 0.5,\ \text{so } H(t-s)=0} e^{-(s-0.5)}\,ds$$

$$= 1 - e^{-(t-0.5)}.$$ ②

7. (a) set $Y_t = \begin{cases} \text{its price at } t & \text{if not sold yet} \\ 0, & \text{if sold.} \end{cases}$ ①

Actions: $a=0$, do nothing; $a=1$, sell. ①

The transition probabilities are $p_{x0}(1) = 1$, $p_{00}(a) = 1$, and if $x \neq 0$,

$p_{xy}(0)$ = prob. of price y on the resp. day. ①

Reward function: $R(x,1) = x$, $R(x,0) = 0$, total reward

$$\sum_t R(Y_t, a_t) = \text{price at which it is sold.}$$ ①

(b) $V_1(0) = 0$, $V_1(x) = x$ for $x \neq 0$: if not sold on days 1 & 2, sell now.

$E_0 V_1(X_3) = 8050$. ①

$V_2(x) = \max_a \{R(x,a) + E_a V_1(X_3)\}$, $V_3(x) = \max_a \{R(x,a) + E_a V_2(X_2)\}$ ②

Now, $V_2(x) = \max\{\underbrace{x}_{a=1}, \underbrace{8050}_{a=0}\}$, sell if 2^{nd} offer > 8050, hold on otherwise.

② $$E_0 V_2(X_2) = E \max(X_2, 8050) = 8430.$$

$V_3(x) = \max\{\underbrace{x}_{a=1}, \underbrace{8430}_{a=0}\}$, sell if 1^{st} offer > 8430, hold on otherwise.

$$E_0 V_3(X_1) = 8658.$$ ②

The optimum policy: If first offer is 9000, sell, otherwise, hold on;
2nd " " , " , " ";
for the third offer, sell regardless of the offer

The expected price is \$8658. ①

#

Q 4 (f):

Direct computation for waiting time: when consider the waiting time, we should include the "extra" person, hence

# customers in QS	# in the queue	probability (PASTA)	waiting time
0	0	π_0	0
1	0	π_1	$\frac{1}{\mu}$
2	1	π_2	$\frac{2}{\mu}$
3 (4 in total due to the extra person)	1	π_3	$\frac{2}{2\mu}$
4	2	π_4	$\frac{3}{2\mu}$
⋮	⋮	⋮	⋮

So
$$W = \frac{1}{\mu}\pi_1 + \frac{2}{\mu}\pi_2 + \frac{2}{2\mu}\pi_3 + \frac{3}{2\mu}\pi_4 + \cdots$$
$$= \frac{1}{\mu}\left(\pi_1 + 2\pi_2 + \frac{1}{2}\left(2\pi_3 + 3\pi_4 + \cdots\right)\right)$$
$$= \frac{1}{\mu}\left(\frac{60}{239} + 2\times\frac{45}{239} + \frac{1}{2}\sum_{i=3}^{\infty}(i-1)\cdot\frac{135}{956}\cdot\left(\frac{3}{8}\right)^{i-3}\right)$$
$$= \frac{1}{\mu}\left(\frac{150}{239} + \frac{1}{2}\sum_{i=3}^{\infty}(i-2)\frac{135}{956}\left(\frac{3}{8}\right)^{i-3} + \frac{1}{2}\sum_{i=3}^{\infty}\frac{135}{956}\left(\frac{3}{8}\right)^{i-3}\right)$$
$$= \frac{1}{4}\left(\frac{150}{239} + \frac{1}{2}\cdot\frac{135}{956}\left(\frac{1}{(1-\frac{3}{8})^2} + \frac{1}{1-\frac{3}{8}}\right)\right)$$
$$= \frac{1}{4}\left(\frac{150}{239} + \frac{1}{2}\cdot\frac{135}{956}\cdot\left(\frac{64}{25} + \frac{40}{25}\right)\right) = \frac{1}{4}\left(\frac{150}{239} + \frac{1}{2}\cdot\frac{135}{956}\cdot\frac{104}{25}\right) = \frac{1101}{4780}.$$