THE UNIVERSITY OF MELBOURNE

Student Number

Semester 1 Assessment, 2018

School of Mathematics and Statistics

MAST20004 Probability

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 7 pages (including this page)

Authorised Materials

- Mobile phones, smart watches, and internet or communication devices are forbidden.
- Students may bring one double-sided A4 sheet of handwritten notes into the exam room.
- Approved hand-held electronic scientific (but not graphing) calculators may be used.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the exam.
- This paper has 9 questions. Attempt as many questions, or parts of questions, as you can. Marks for individual questions are shown.
- Working and/or reasoning must be given to obtain full credit. Clarity, neatness, and style count.
- Statistical tables are not provided but you may use the MATLAB output at the end of the examination paper **FOR ANY QUESTION**.
- The total number of marks available for this exam is 100.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the exam.
- Initially students are to receive a 14 page script booklet.

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1. Consider a random experiment with sample space Ω.

   (a) Write down the axioms which must be satisfied by a probability mapping $\mathbb{P}$ defined on the events of the experiment.

   (b) Using the axioms show that $\mathbb{P}(\emptyset) = 0$.

   (c) Using the axioms and part (b), prove that for disjoint sets A_i, $1 \le i \le n$, with finite n,
$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i).$$

   (d) Using part (c) show that for all events A and B, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

[10 marks]

2. Three jars each contain 10 balls, some are red, the rest green. A jar is selected according to some probability, and a ball is chosen at random. Suppose Jar A is selected with probability 0.2 and contains 7 red balls; Jar B is selected with probability 0.5 and contains 6 red balls; Jar C is selected with probability 0.3 and contains 9 red balls.

   (a) A jar is selected and a red ball is chosen. Find the probability that Jar A was selected.

   (b) Are the events that Jar A is selected and a red ball chosen positively related, negatively related, or independent? Justify your answer.

   (c) Now suppose Jar A and Jar B are taken aside, and two balls are randomly chosen independently, one from each jar. Only one of the two balls is green. Calculate the probability that the green ball came from Jar B.

[8 marks]

3. Grades on an exam of a certain subject at a certain university were not very good. Graphed, their distribution had a shape similar to the probability density function
$$f_X(x) = \begin{cases} ce^{-x/100}, & 0 \le x \le 100 \\ 0, & \text{otherwise.} \end{cases}$$

   As a way of rescaling the results, the lecturer announces that he will replace each person's grade, X, with a new grade, $Y = 10\sqrt{X}$.

   (a) Determine the value of c and calculate the cumulative distribution function of X.

   (b) Calculate the probability that a randomly chosen student achieved a high distinction by his/her own efforts, that is, achieved a score of 80 or more before scaling.

   (c) Compute $\mathbb{E}[X]$.

   (d) Find the cumulative distribution function of Y.

   (e) Find the probability density function of Y.

   (f) Calculate the probability that a randomly chosen student achieved a high distinction after scaling.

[11 marks]

4. Suppose that N is a random variable having a conditional Poisson distribution with probability mass function

$$p_N(i) \;=\; \frac{1}{2}\cdot\frac{(\log 3)^i}{i!}, \qquad i = 1, 2, 3, \ldots$$

   (a) Show that the mean of N is

$$\mu \;=\; \frac{3\log 3}{2} \;=\; 1.6479,$$

   and the variance of N is

$$\sigma^2 \;=\; \frac{3\log 3}{2} - \frac{3(\log 3)^2}{4} \;=\; 0.7427.$$

   (b) Calculate the probability $\mathbb{P}(|N-\mu| \le 2\sigma)$.

   (c) Use the Bienaymé-Chebyshev inequality to give a lower bound for the probability that N takes values within 2 standard deviations of its mean and compare it with the actual probability in part (b). Comment on your findings.

   (d) Assume that N is the number of claims received by an insurance company in a day and the amount of each claim can be modelled by a $R(100, 1300)$ random variable independent of other claims. Let T be the total value of claims during the day. Make reasonable assumptions and compute $\mathbb{E}[T|N]$, $V(T|N)$, $\mathbb{E}[T]$, $V(T)$, and the correlation coefficient $\rho(T, N)$.

[18 marks]

5. Let X and Y be two independent random variables with $X \stackrel{d}{=} R(0,2)$ and $Y \stackrel{d}{=} \exp(1)$.

   (a) Use the convolution formula to calculate the probability density function of $W = X + Y$.

   (b) Derive the probability density function of $U = XY$.

[8 marks]

6. Let $X \stackrel{d}{=} N(0,1)$ and $Y = \psi(X) = e^X$.

   (a) Find $\mathbb{E}[Y]$ and $V(Y)$.

   (b) Compute the approximate values of $\mathbb{E}[Y]$ and $V(Y)$ using $\mathbb{E}[\psi(X)] \approx \psi(\mu) + \frac{1}{2}\psi''(\mu)V(X)$ and $V(\psi(X)) \approx \psi'(\mu)^2 V(X)$. Do you expect good approximations? Justify your answer.

[8 marks]

7. Let X a be random variable with probability density function given by

$$f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the mean μ and variance σ^2 of X.

(b) Derive the moment generating function of X and state the values for which it is defined.

(c) For the value(s) at which the moment generating function found in part (b) is (are) *not* defined, what should the moment generating function be defined as? Justify your answer.

(d) Let $X_1, X_2, \ldots$ be a sequence of independent random variables where each one has the same distribution as X. Derive the moment generating function of

$$S_n = \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}.$$

(e) Use the moment generating function of S_n to prove that S_n converges to a normal distribution in distribution as $n \to \infty$, and specify the parameters of the limiting distribution.

(f) Let

$$T = X_1 + X_2 + \ldots + X_{100}.$$

Give an approximate value of $\mathbb{P}(T < 13)$.

[15 marks]

8. Consider the bivariate random variable (R, Θ) which has joint probability density function

$$f_{(R,\Theta)}(r,\theta) = \begin{cases} \dfrac{2r^3}{\pi}, & \text{for } 0 < r < 1,\ 0 < \theta < 2\pi \\ 0, & \text{otherwise.} \end{cases}$$

(a) Write down the marginal probability density functions for R and Θ.

(b) Write down the marginal distribution functions for R and Θ.

(c) Are R and Θ independent? Justify your answer.

(d) The "rand" command in Matlab can be used to generate a realisation of a random variable $U \stackrel{d}{=} R(0,1)$. Explain how to generate a realisation of the random variable

(i) R;

(ii) Θ.

(e) Using your results from part (d), explain how to calculate an estimate of $\mathbb{E}(R\cos\Theta)$ from 2000 realisations of the random variable $U \stackrel{d}{=} R(0,1)$.

[11 marks]

9. A gambler with only \$1 in his pocket plays a game in which he can only bet \$1 for each game. He wins the game with probability 0.4, in which case he receives \$2 (he gets his \$1 back and receives another \$1) and he loses his \$1 with probability 0.6. He plays the game repeatedly until either he loses all his money, or he has \$3.

   (a) Model this situation as a Markov chain. Specify the state space S, and give the state transition probability matrix, $\boldsymbol{P}$.

   (b) Which underlying assumption has been made so that this situation can be modelled as a Markov chain?

   (c) Assuming that he has played three games, what is the probability that he has

      (i) \$1?

      (ii) lost all his money?

   (d) Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \pi_3)$. Solve the system of equations

$$\begin{aligned} \boldsymbol{\pi P} &= \boldsymbol{\pi}, \\ \boldsymbol{\pi 1} &= 1, \end{aligned}$$

   where $\mathbf{1}$ is the 4×1 column vector of 1s.
   Interpret your answer.

[11 marks]

End of Questions

Appendix: Some MATLAB output

```
>> x1=0.01:.01:1.00; x2=1.01:.01:2.00; x3=2.01:.01:3.00;
y1=cdf('norm',x1,0,1);  y2=cdf('norm',x2,0,1); y3=cdf('norm',x3,0,1);
[x1' y1' x2' y2' x3' y3']

ans =

    0.0100    0.5040    1.0100    0.8438    2.0100    0.9778
    0.0200    0.5080    1.0200    0.8461    2.0200    0.9783
    0.0300    0.5120    1.0300    0.8485    2.0300    0.9788
    0.0400    0.5160    1.0400    0.8508    2.0400    0.9793
    0.0500    0.5199    1.0500    0.8531    2.0500    0.9798
    0.0600    0.5239    1.0600    0.8554    2.0600    0.9803
    0.0700    0.5279    1.0700    0.8577    2.0700    0.9808
    0.0800    0.5319    1.0800    0.8599    2.0800    0.9812
    0.0900    0.5359    1.0900    0.8621    2.0900    0.9817
    0.1000    0.5398    1.1000    0.8643    2.1000    0.9821
    0.1100    0.5438    1.1100    0.8665    2.1100    0.9826
    0.1200    0.5478    1.1200    0.8686    2.1200    0.9830
    0.1300    0.5517    1.1300    0.8708    2.1300    0.9834
    0.1400    0.5557    1.1400    0.8729    2.1400    0.9838
    0.1500    0.5596    1.1500    0.8749    2.1500    0.9842
    0.1600    0.5636    1.1600    0.8770    2.1600    0.9846
    0.1700    0.5675    1.1700    0.8790    2.1700    0.9850
    0.1800    0.5714    1.1800    0.8810    2.1800    0.9854
    0.1900    0.5753    1.1900    0.8830    2.1900    0.9857
    0.2000    0.5793    1.2000    0.8849    2.2000    0.9861
    0.2100    0.5832    1.2100    0.8869    2.2100    0.9864
    0.2200    0.5871    1.2200    0.8888    2.2200    0.9868
    0.2300    0.5910    1.2300    0.8907    2.2300    0.9871
    0.2400    0.5948    1.2400    0.8925    2.2400    0.9875
    0.2500    0.5987    1.2500    0.8944    2.2500    0.9878
    0.2600    0.6026    1.2600    0.8962    2.2600    0.9881
    0.2700    0.6064    1.2700    0.8980    2.2700    0.9884
    0.2800    0.6103    1.2800    0.8997    2.2800    0.9887
    0.2900    0.6141    1.2900    0.9015    2.2900    0.9890
    0.3000    0.6179    1.3000    0.9032    2.3000    0.9893
    0.3100    0.6217    1.3100    0.9049    2.3100    0.9896
    0.3200    0.6255    1.3200    0.9066    2.3200    0.9898
    0.3300    0.6293    1.3300    0.9082    2.3300    0.9901
    0.3400    0.6331    1.3400    0.9099    2.3400    0.9904
    0.3500    0.6368    1.3500    0.9115    2.3500    0.9906
    0.3600    0.6406    1.3600    0.9131    2.3600    0.9909
    0.3700    0.6443    1.3700    0.9147    2.3700    0.9911
    0.3800    0.6480    1.3800    0.9162    2.3800    0.9913
    0.3900    0.6517    1.3900    0.9177    2.3900    0.9916
    0.4000    0.6554    1.4000    0.9192    2.4000    0.9918
    0.4100    0.6591    1.4100    0.9207    2.4100    0.9920
    0.4200    0.6628    1.4200    0.9222    2.4200    0.9922
    0.4300    0.6664    1.4300    0.9236    2.4300    0.9925
    0.4400    0.6700    1.4400    0.9251    2.4400    0.9927
    0.4500    0.6736    1.4500    0.9265    2.4500    0.9929
    0.4600    0.6772    1.4600    0.9279    2.4600    0.9931
    0.4700    0.6808    1.4700    0.9292    2.4700    0.9932
    0.4800    0.6844    1.4800    0.9306    2.4800    0.9934
    0.4900    0.6879    1.4900    0.9319    2.4900    0.9936
```

0.5000	0.6915	1.5000	0.9332	2.5000	0.9938
0.5100	0.6950	1.5100	0.9345	2.5100	0.9940
0.5200	0.6985	1.5200	0.9357	2.5200	0.9941
0.5300	0.7019	1.5300	0.9370	2.5300	0.9943
0.5400	0.7054	1.5400	0.9382	2.5400	0.9945
0.5500	0.7088	1.5500	0.9394	2.5500	0.9946
0.5600	0.7123	1.5600	0.9406	2.5600	0.9948
0.5700	0.7157	1.5700	0.9418	2.5700	0.9949
0.5800	0.7190	1.5800	0.9429	2.5800	0.9951
0.5900	0.7224	1.5900	0.9441	2.5900	0.9952
0.6000	0.7257	1.6000	0.9452	2.6000	0.9953
0.6100	0.7291	1.6100	0.9463	2.6100	0.9955
0.6200	0.7324	1.6200	0.9474	2.6200	0.9956
0.6300	0.7357	1.6300	0.9484	2.6300	0.9957
0.6400	0.7389	1.6400	0.9495	2.6400	0.9959
0.6500	0.7422	1.6500	0.9505	2.6500	0.9960
0.6600	0.7454	1.6600	0.9515	2.6600	0.9961
0.6700	0.7486	1.6700	0.9525	2.6700	0.9962
0.6800	0.7517	1.6800	0.9535	2.6800	0.9963
0.6900	0.7549	1.6900	0.9545	2.6900	0.9964
0.7000	0.7580	1.7000	0.9554	2.7000	0.9965
0.7100	0.7611	1.7100	0.9564	2.7100	0.9966
0.7200	0.7642	1.7200	0.9573	2.7200	0.9967
0.7300	0.7673	1.7300	0.9582	2.7300	0.9968
0.7400	0.7704	1.7400	0.9591	2.7400	0.9969
0.7500	0.7734	1.7500	0.9599	2.7500	0.9970
0.7600	0.7764	1.7600	0.9608	2.7600	0.9971
0.7700	0.7794	1.7700	0.9616	2.7700	0.9972
0.7800	0.7823	1.7800	0.9625	2.7800	0.9973
0.7900	0.7852	1.7900	0.9633	2.7900	0.9974
0.8000	0.7881	1.8000	0.9641	2.8000	0.9974
0.8100	0.7910	1.8100	0.9649	2.8100	0.9975
0.8200	0.7939	1.8200	0.9656	2.8200	0.9976
0.8300	0.7967	1.8300	0.9664	2.8300	0.9977
0.8400	0.7995	1.8400	0.9671	2.8400	0.9977
0.8500	0.8023	1.8500	0.9678	2.8500	0.9978
0.8600	0.8051	1.8600	0.9686	2.8600	0.9979
0.8700	0.8078	1.8700	0.9693	2.8700	0.9979
0.8800	0.8106	1.8800	0.9699	2.8800	0.9980
0.8900	0.8133	1.8900	0.9706	2.8900	0.9981
0.9000	0.8159	1.9000	0.9713	2.9000	0.9981
0.9100	0.8186	1.9100	0.9719	2.9100	0.9982
0.9200	0.8212	1.9200	0.9726	2.9200	0.9982
0.9300	0.8238	1.9300	0.9732	2.9300	0.9983
0.9400	0.8264	1.9400	0.9738	2.9400	0.9984
0.9500	0.8289	1.9500	0.9744	2.9500	0.9984
0.9600	0.8315	1.9600	0.9750	2.9600	0.9985
0.9700	0.8340	1.9700	0.9756	2.9700	0.9985
0.9800	0.8365	1.9800	0.9761	2.9800	0.9986
0.9900	0.8389	1.9900	0.9767	2.9900	0.9986
1.0000	0.8413	2.0000	0.9772	3.0000	0.9987