MAST30001 Stochastic Modelling

Tutorial Sheet 8

1. Consider a population consisting of particles arriving from outside according to a Poisson process with rate λ. The lifetime of each particle (after it arrives) is exponential with rate α and the lifetimes are all independent.

   (a) Model the system as a birth-death process and find the birth and death rates.

   (b) Show that the process is ergodic and find its stationary distribution.

   (c) What is the expected number of living particles in the population in stationary?

2. A system has N particles each of which at any given time are in one of the two energy states α or β. The particles switch between states α and β according to the following rules. When a particle enters state α, it switches to state β after an exponentially distributed with rate $\mu > 0$ amount of time, independent of the other particles' behaviour and the time the particle entered state α. Similarly, when a particle enters state β, it switches to state α after an exponentially distributed with rate $\lambda > 0$ amount of time, independent of the other particles' behaviour and the time the particle entered state β.

   (a) Model the number of particles in the energy state α as a continuous time Markov chain and define its generator.

   (b) Describe the long run behaviour of the chain.

   (c) If the chain starts with N particles in the α energy state and X_t is the number of α particles at time t, find the mean and variance of X_t as $t \to \infty$. Your answer should be a tidy formula.

3. The following continuous time Markov chain is used to model population growth without death. The basic assumption of the model is that every member of the population gives birth to a new member with rate λ (that is, at times with distribution exponential with rate λ), independently of the other members of the population. Let X_t be the size of the population at time t.

   (a) What is $\mathbb{P}(X_t = n | X_0 = 1)$?

   (b) If U is uniform on the interval $(0, 1)$, independent of X_t, find the distribution of $X_U | X_0 = 1$.

4. Show that in in $M/M/1$ queue with arrival rate λ and service rate $\mu > \lambda$, the expected lengths of the idle and busy periods are $1/\lambda$ and $1/(\mu - \lambda)$, respectively. *[Hint: the proportion of time the server is idle is equal to the stationary chance the system is empty.]*

1. Consider a population consisting of particles arriving from outside according to a Poisson process with rate λ. The lifetime of each particle (after it arrives) is exponential with rate α and the lifetimes are all independent.

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(a) birth rate λ $\quad\quad \lambda_i = \lambda$

death rate α. $\quad\quad \mu_i = i\alpha$ $\quad\quad$ i 个 exponential

(b). $\pi_k = \pi_0 \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}$

$\sum_{k=0}^{\infty} \prod_{i=1}^{k} \pi_0 \frac{\lambda_{i-1}}{\mu_i} < \infty$

$\pi_0 \prod_{i=0}^{k} \frac{\lambda}{i\alpha}$ $\quad \Rightarrow \quad$ $\sum_{k=0}^{\infty} \pi_0 \cdot \frac{\lambda^k}{k!\,\alpha^k} < \infty.$

$\pi_0 \cdot \frac{\lambda^k}{k!\,\alpha^k}$

$\underbrace{\pi_0 \sum_{k=0}^{\infty} \frac{(\frac{\lambda}{\alpha})^k}{k!}}_{\pi_0 e^{\frac{\lambda}{\alpha}}}$

$\pi_0 \sum_{k=0}^{\infty} \frac{(\frac{\lambda}{\alpha})^k}{k!} = \pi_0 e^{\frac{\lambda}{\alpha}} = 1$

$\pi_0 = e^{-\frac{\lambda}{\alpha}}$

$\therefore \pi_k = \pi_0 \cdot \frac{(\frac{\lambda}{\alpha})^k}{k!} = \frac{e^{-\frac{\lambda}{\alpha}} \cdot (\frac{\lambda}{\alpha})^k}{k!}$ $\sim Poi(\frac{\lambda}{\alpha})$ ②

stationary distribution is Poisson with $(\frac{\lambda}{\alpha})$.

so it is a Poisson process with

疑问: ① stationary distribution 指的是 $(\pi_1, \cdots \pi_k)$. 吗?

② Poisson process 和 Poisson distribution 的关联

(c). $E(X) = \sum k\pi_k = E(Poi(\frac{\lambda}{\alpha})) = \frac{\lambda}{\alpha}$.

$k \geq 0$

2. A system has N particles each of which at any given time are in one of the two energy states α or β. The particles switch between states α and β according to the following rules. When a particle enters state α, it switches to state β after an exponentially distributed with rate $\mu > 0$ amount of time, independent of the other particles' behaviour and the time the particle entered state α. Similarly, when a particle enters state β, it switches to state α after an exponentially distributed with rate $\lambda > 0$ amount of time, independent of the other particles' behaviour and the time the particle entered state β.

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(a) suppose k particle in state α. $N-k$ in state β.

$$a_{k,k+1} = (N-k)\lambda$$

$$a_{k,k-1} = k\mu$$

$$a_{k,k} = -k\mu + (k-N)\lambda.$$

(b). birth-death process with birth rate $a_{k,k+1}$

deate. rate $a_{k,k-1}$.

long run $\pi_k = \pi_0 \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}$

$$\lambda_k = a_{k,k+1} = (N-k)\lambda$$

$$\mu_k = a_{k,k-1} = k\mu$$

$\to N \cdots N-k+1$

$\binom{N}{k}$.

$$\pi_0 \sum_{k\geq 0} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} = \pi_0 \sum_{k\geq 0} \prod_{i=1}^{k} \frac{(N-i+1)\lambda}{i\mu}$$

$$= \pi_0 \sum_{k\geq 0} \frac{N!\,\lambda^k}{(N-k)!\,k!\,\mu^k}$$

$$= \pi_0 \sum_{k\geq 0} \binom{N}{k} \left(\frac{\lambda}{\mu}\right)^k$$

$$= \pi_0$$

$$\pi_k = \pi_0 \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} = \pi_0 \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} = \pi_0 \binom{N}{k} \left(\frac{\lambda}{\mu}\right)^k$$

$\underline{\pi} A = 0$ $\quad [\pi_0\ \pi_1\ , \cdots]$

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & N-1 & N \\ \hline 0 & -N\lambda & N\lambda & 0 & 0 & & \\ 1 & \mu & -[\mu+(N-1)\lambda] & (N-1)\lambda & 0 & & \\ 2 & 0 & 2\mu & -[2\mu+(N-2)\lambda] & (N-2)\lambda & & \\ 3 & 0 & 0 & 3\mu & -[3\mu+(N-3)\lambda] & & \\ & & & & 4\mu & & \\ & & & & & & \lambda \\ N & & & & & N\mu & -N\mu \end{array}$$

$$\begin{cases} -N\lambda\pi_0 + \mu\pi_1 = 0 & ① \\ N\lambda\pi_0 - [\mu+(N-1)\lambda]\pi_1 + 2\mu\pi_2 = 0 & ② \\ (N-1)\lambda\pi_1 - [2\mu+(N-2)\lambda]\pi_2 + 3\mu\pi_3 = 0 \\ (N-2)\lambda\pi_2 - [3\mu+(N-3)\lambda]\pi_3 + 4\mu\pi_4 = 0 \\ \vdots \\ (N-k+1)\pi_{k-1} - [k\mu+(N-k)\lambda]\pi_k + (k+1)\mu\pi_{k+1} = 0. \\ \lambda\pi_{N-1} - N\mu\pi_N = 0. \end{cases}$$

from ①. $\mu\pi_1 = N\lambda\pi_0$.

$$\pi_1 = \frac{N\lambda}{\mu}\pi_0.$$

②. $N\lambda\pi_0 + 2\mu\pi_2 = [\mu + (N-1)\lambda] \cdot \frac{N\lambda}{\mu}\pi_0$

$$\mu \cancel{N\lambda\pi_0} + 2\mu^2\pi_2 = \mu\cancel{N\lambda\pi_0} + N(N-1)\lambda^2\pi_0.$$

$$\pi_2 = \frac{N(N-1)\lambda^2}{2\mu^2}\pi_0.$$

guess. $\pi_k = \binom{N}{k}\left(\frac{\lambda}{\mu}\right)^k \pi_0.$

prove. use $(N-k+1)\lambda\pi_{k-1} + (k+1)\mu\pi_{k+1} = [k\mu + (N-k)\lambda]\pi_k$

LHS. $(N-k+1)\lambda\binom{N}{k-1}\left(\frac{\lambda}{\mu}\right)^{k-1}\pi_0 + (k+1)\mu\binom{N}{k+1}\left(\frac{\lambda}{\mu}\right)^{k+1}\pi_0.$

$$\pi_0\left(\frac{\lambda}{\mu}\right)^k\left[\cancel{(N-k+1)}\frac{N!\ \lambda\left(\frac{\mu}{\lambda}\right)}{\underbrace{\cancel{(N-k+1)!}}_{(N-k)!}(k-1)!} + \cancel{(k+1)}\mu\frac{N!}{\underbrace{\cancel{(k+1)!}}_{k!}(N-k-1)!}\left(\frac{\lambda}{\mu}\right)\right]$$

$$= \pi_0\left(\frac{\lambda}{\mu}\right)^k\left[k\mu\binom{N}{k} + (N-k)\lambda\binom{N}{k}\right] = \text{RHS}.$$

for $k=N$. $\lambda\pi_{N-1} = \lambda\binom{N}{1}\left(\frac{\lambda}{\mu}\right)^{N-1}\pi_0$

$$= \lambda N\left(\frac{\lambda}{\mu}\right)^{N-1}\pi_0$$

$$N\mu\pi_N = \mu\left(\frac{\lambda}{\mu}\right)^N N\pi_0.$$

$$\pi_N = \left(\frac{\lambda}{\mu}\right)^N\pi_0$$

so $\underline{\pi_k = \binom{N}{k}\left(\frac{\lambda}{\mu}\right)^k\pi_0.}$ for $k = 0 \cdots N$.

$$\sum_{k\geq 0}\binom{N}{k}\left(\frac{\lambda}{\mu}\right)^k\pi_0 = 1.$$

$$\pi_0\sum_{k\geq 0}\binom{N}{k}\left(\frac{\lambda}{\mu}\right)^k 1^{N-k}$$

$$= \pi_0\left(1+\frac{\lambda}{\mu}\right)^N = 1$$

$$\pi_0 = \left(1+\frac{\lambda}{\mu}\right)^{-N}$$

$$= \left(\frac{\mu}{\lambda+\mu}\right)^N.$$

$$\therefore \pi_k = \binom{N}{k}\left(\frac{\lambda}{\mu}\right)^k \cdot \left(\frac{\mu}{\lambda+\mu}\right)^N$$

$$\left(\frac{\lambda}{\mu}\right)^k \cdot \left(\frac{\mu}{\lambda+\mu}\right)^k \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{N-k}$$

$$= \binom{N}{k}\left(\frac{\lambda}{\lambda+\mu}\right)^k \cdot \left(\frac{\mu}{\lambda+\mu}\right)^{N-k}$$

$\sim$ Binomial $\left(\frac{\lambda}{\lambda+\mu}\right)$.

Mean $= \frac{N\lambda}{\lambda+\mu}$ var $= \frac{N\lambda\mu}{(\lambda+\mu)^2}$.

$\lim_{t\to\infty} EX_t = \frac{N\lambda}{\lambda+\mu}$ $\lim_{t\to\infty} Var(X_t) = \frac{N\lambda\mu}{(\lambda+\mu)^2}$

3. The following continuous time Markov chain is used to model population growth without death. The basic assumption of the model is that every member of the population gives birth to a new member with rate λ (that is, at times with distribution exponential with rate λ), independently of the other members of the population. Let X_t be the size of the population at time t.

(a) What is $\mathbb{P}(X_t = n | X_0 = 1)$?

(b) If U is uniform on the interval $(0, 1)$, independent of X_t, find the distribution of $X_U | X_0 = 1$.

$[\, p_{11} \ p_{12} \ p_{13} \cdots]$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & -\lambda & \lambda & & \\ 2 & & -2\lambda & 2\lambda & \\ & & & -3\lambda & 3\lambda \\ & & & & -4\lambda \end{array}$$

(a) generator $a_{i,i+1} = i\lambda$

$a_{i,i} = -i\lambda$.

$$P(X_t = n | X_0 = 1) = (P^{(t)})_{1n}$$

$$\frac{d}{dt} P^{(t)} = P^{(t)} A$$

$$\left(\frac{d}{dt} P^{(t)}\right)_{1n} = \sum_j (P^{(t)})_{1j} (A)_{jn}.$$

$$= P^{(t)}_{1,n-1} \cdot a_{n-1,n} + P^{(t)}_{1,n} \cdot a_{n,n}$$

$$= P^{(t)}_{1,n-1} \cdot (n-1)\lambda + P^{(t)}_{1,n} (-n\lambda).$$

$$\left(\frac{d}{dt} P^{(t)}\right)_{11} = (P^{(t)})_{11} \cdot (-\lambda).$$

$$(P^{(t)})_{11} = e^{-\lambda t}$$

$$\left(\frac{d}{dt} P^{(t)}\right)_{12} = (P^{(t)})_{11} \cdot (\lambda) - 2\lambda (P^{(t)})_{12}$$

$$= \lambda e^{-\lambda t} - 2\lambda (P^{(t)})_{12}$$

$$\left[(P^{(t)})_{12}\right]' + 2\lambda (P^{(t)})_{12} = \lambda e^{-\lambda t}.$$

$$I(x) = \exp\left(\int 2\lambda \, dt\right) = \exp(2\lambda t).$$

$$e^{2\lambda t} \cdot \frac{d}{dt}(P^{(t)})_{12} + 2\lambda e^{2\lambda t} (P^{(t)})_{12} = \lambda e^{\lambda t}$$

$$\frac{d}{dt}\left(e^{2\lambda t} p^{(t)} \right) = \frac{d}{dt} (e^{\lambda t})$$

$$e^{2\lambda t}(p^{(t)})_{12} = e^{\lambda t} + C.$$

$$p^{(t)} = e^{-\lambda t} + e^{-2\lambda t} \cdot C.$$

$$(p^{(0)})_{12} = 0$$

$$1 + C = 0$$

$$C = -1$$

$$(p^{(t)})_{12} = e^{-\lambda t} - e^{-2\lambda t}$$

$$= e^{-\lambda t}(1 - e^{-\lambda t})$$

since $(p^{(t)})_{11} = e^{-\lambda t}(1 - e^{-\lambda t})^0$

$(p^{(t)})_{12} = e^{-\lambda t}(1 - e^{-\lambda t})^1$

so guess $(p^{(t)})_{1n} = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}$

prove $\left(\frac{d}{dt} p^{(t)}\right)_{1,n} = (n-1)\lambda\, p^{(t)}_{1,n-1} - n\lambda\, p^{(t)}_{1,n}$

LHS. $\frac{d}{dt}\left[e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}\right]$

$$= -\lambda e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}$$

$$+ e^{-\lambda t}(n-1)(1 - e^{-\lambda t})^{n-2} \cdot \lambda e^{-\lambda t}$$

$$= -\lambda e^{-\lambda t}(1 - e^{-\lambda t})^{n-1} + \lambda e^{-2\lambda t}(n-1)(1 - e^{-\lambda t})^{n-2}$$

(b) If U is uniform on the interval $(0, 1)$, independent of X_t, find the distribution of $X_U | X_0 = 1$.

(b). $U \sim$ uinform $(0,1)$

$X_U | X_0 = 1$

$U \sim$ uniform $(0,1)$.

$\rightarrow$ function of U

$Y = X_U$ mean $(Y | U = u) \overset{d}{=} X_u$

$X_u = n$.

$$P(X_U=n) = \sum_u P(X_U=n \mid U=u) \cdot \underbrace{P(U=u)}_{=1 \text{ since uniform}}.$$

X_u

$Y = X_U$ means $(Y \mid U=u) \overset{d}{=} X_u$

$$P(Y=n) = \int_0^1 P(Y=n \mid U=u)\, \underset{f_U(u) \Rightarrow 1.}{du}$$

$$= \int_0^1 P(X_u=n)\, du$$

$$= \int_0^1 P(X_U=n \mid U=u) f_U(u)\, du$$

$$= \int_0^1 P(X_U=n \mid U=u)\, du$$

independent

$$= \int_0^1 P(X_U=n)\, du$$

$$= \int_0^1 P(X_U=n \mid X_0=1) \cdot P(X_0=1)\, du$$

$$P(X_U=n \mid X_0=1)$$

$$= \sum_{\text{all } t} P\big((X_U=n \mid X_0=1) \mid U=t\big)\, P(U=t).$$

$$= \sum_{\text{all } t} P(X_t=n \mid X_0=1)\, P(U=t).$$

$$= \int_0^1 P(X_t=n \mid X_0=1)\, \underbrace{f_U(t)}_{=1.}\, du. = \int_0^1 e^{-\lambda t}(1-e^{-\lambda t})^{n-1}\, dt$$

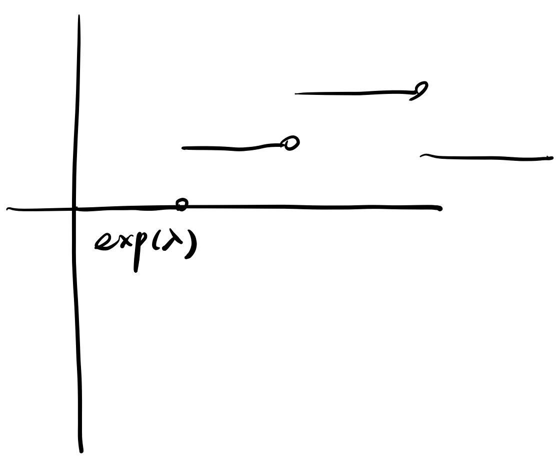
4. Show that in in $M/M/1$ queue with arrival rate λ and service rate $\mu > \lambda$, the expected lengths of the idle and busy periods are $1/\lambda$ and $1/(\mu-\lambda)$, respectively. *Hint: the proportion of time the server is idle is equal to the stationary chance the system is empty.*

π_0

length $= \frac{1}{\lambda}$.

$\pi_0 = 1 - \frac{\lambda}{\mu}$.

proportion of busy



$\pi_0 = 1 - \frac{\lambda}{\mu}$ is the long run proportion of time the system is empty.

$$\frac{\frac{1}{\lambda}}{\frac{1}{\lambda} + L} = \pi_0$$

$$L = \frac{1}{\mu - \lambda}.$$