# *Week 2: FNCE10002 Principles of Finance*

THE UNIVERSITY OF MELBOURNE

## *Introduction to Financial Mathematics II*

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# *Introduction to Financial Mathematics II*

1. Calculate unknown time horizons and interest rates
2. Calculate the present value of ordinary and deferred perpetuities
3. Calculate the present and future values of ordinary annuities
4. Calculate present and future values of annuities due
5. Calculate the present value of growing ordinary perpetuities
6. Calculate the present and future values of growing ordinary annuities

# *Required Readings: Weeks 2 – 3*

- *Week 2*
  - GRAH, Ch. 3 (Sec. 3.5b – 3.6)
  - Lamba, A. S., 2019, Teaching Note 1: Introduction to Financial Mathematics (focus on sections 3.2, 3.6 and 3.7 of the teaching note)
- *Week 3*
  - GRAH, Ch. 3 (Sec 3.7) and Ch. 4 (Sec. 4.1 – 4.4)

# *2.1 Unknown Interest Rate and Time Period*

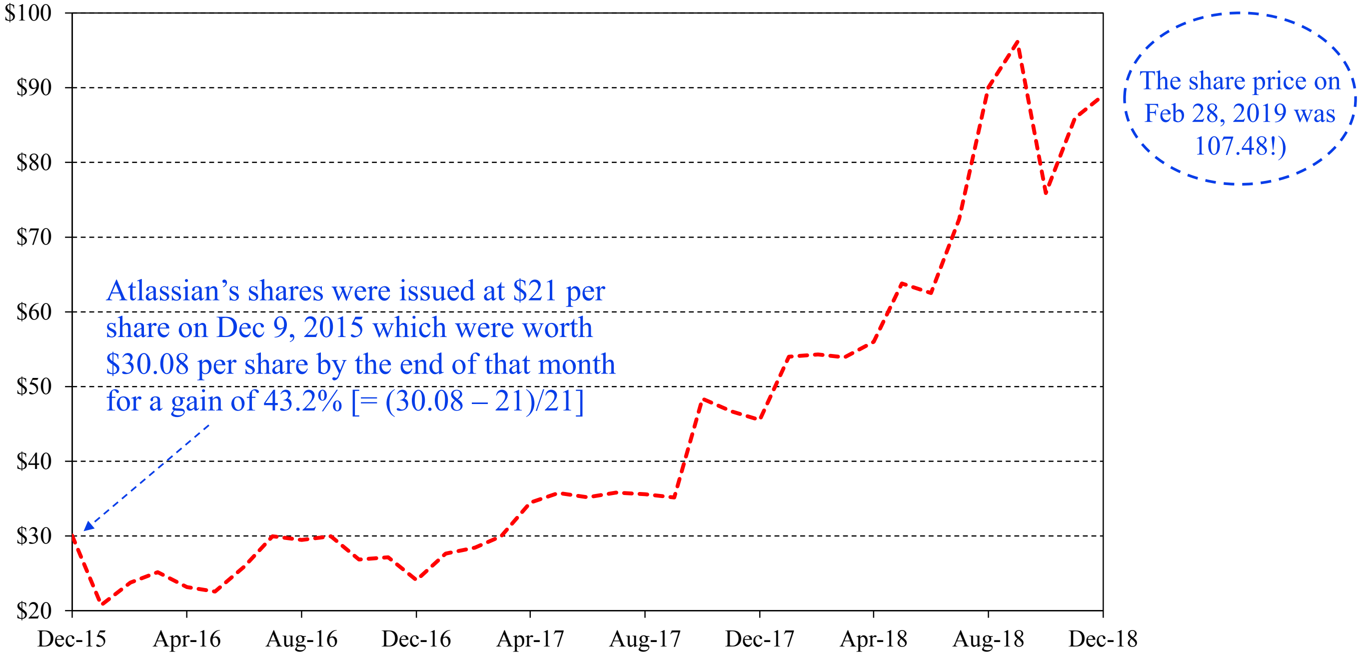
- So far we've assumed that we know the cash flows for which we need to calculate present or future values. We may also have situations where…
- We know the present (or future) value of a cash flow but the cash flow may be unknown. The *key rule* we use to calculate such cash flows is…
  - *PV*(Benefits or cash inflows) = *PV*(Costs or cash outflows) *or*
  - *FV*(Benefits or cash inflows) = *FV*(Costs or cash outflows)
  - *Examples:* An unknown credit card repayment amount (see tutorial 2 question); an unknown home loan or car loan repayment amount (examined in week 3)
- We know the present (or future) value of known cash flows but the time horizon or interest rate may be unknown

# *Case Study 1: It's a TEAM Effort*

- *Dec 9, 2015:* Atlassian Corp. (*Nasdaq code:* TEAM), the maker of cloud-based workplace collaboration software, on Wednesday evening priced its initial public offering (of its shares) at $21 per share. That was above Atlassian's proposed $19 to $20 per share offering range, which had been increased from an initial range of $16.50 to $18.50 per share. *At $21 per share, Atlassian raised $462 million at an initial market capitalization of nearly $4.38 billion*. (Here's a link to a story about the origin of this company.)
- By December 2018, Atlassian's market value was $23.19 billion. What has been the total growth rate in the market value of the company over this three-year period? What is the (compounded) annual growth rate in the market value? At this annual growth rate how long would it be before Atlassian becomes a trillion dollar company?

*Sources:* http://www.fortune.com/2015/12/09/atlassian-ipo-price and https://finance.yahoo.com/quote/TEAM?p=TEAM

# *Case Study 1: It's a TEAM Effort*




*Source:* https://finance.yahoo.com/quote/TEAM?p=TEAM. Atlassian's share price over Dec 2015 – Dec 2018.

# *Case Study 1: It's a TEAM Effort*

- The *total* growth rate ($g_{\text{Total}}$) over the three-year period is…
- $PV_0$ = \$4.38 billion, $FV_3$ = \$23.19 billion, $n$ = 3 years, $g_{Total} = ?$
- $PV_0(1 + g_{Total}) = FV_n$
- $4.38(1 + g_{Total}) = 23.19$
- So, $(1 + g_{Total}) = 23.19/4.38 = 5.295$
- $g_{Total} = 5.295 – 1 = 4.295$ or 429.5%
- Why is the annual growth rate *not* 143.2% (= 429.5/3)?

# *Case Study 1: It's a TEAM Effort*

- The (compounded) *annual* growth rate ($g$) over the three-year period is…
- $PV_0$ = \$4.38 billion, $FV_3$ = \$23.19 billion, $n$ = 3 years, $g$ = ?
- $PV_0(1 + g)^n = FV_n$
- $4.38(1 + g)^3 = 23.19$
- So, $(1 + g)^3 = 23.19/4.38 = 5.2945$
- $g = (5.2945)^{1/3} - 1 = 0.7429$ or 74.3%
- *What is another interpretation for g?*

use the formula. !!

a compounded rate of return ?

# *Case Study 1: It's a TEAM Effort*

- How long would it take for Atlassian to become a trillion dollar company at this growth rate?
- $PV_0$ = \$23.19 billion, $FV_n$ = \$1,000 billion, $g$ = 74.3%, $n$ = ?
- $PV_0(1 + g)^n = FV_n$
- $23.19(1 + 0.743)^n = 1000$
- So, $(1 + 0.743)^n = 1000/23.19 = 43.122$
- Taking natural logs we have…
- $n \times ln(1.743) = ln(43.122)$
- So, $n = 3.764/0.5556 =$ 6.77 years (see next slide for the year-by-year market values)
- *What is the main caveat in the above analyses?*

警告.

not likely to grow in that growth rate

# *Case Study 1: It's a TEAM Effort*

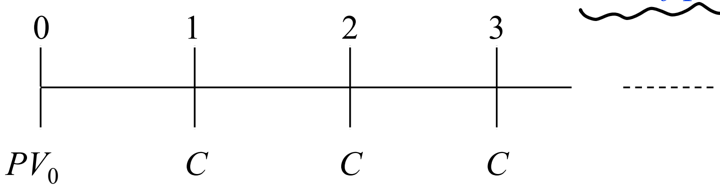
| *End of Year* | *Growth Rate* | *Market Value (b)* |
|---|---|---|
| 0 | | $23.19 |
| 1 | 74.3% | $40.42 |
| 2 | 74.3% | $70.45 |
| 3 | 74.3% | $122.80 |
| 4 | 74.3% | $214.04 |
| 5 | 74.3% | $373.08 |
| 6 | 74.3% | $650.29 |
| 7 | 74.3% | $1,133.46 |

*Note:* The table shows the end of the year market value of Atlassian over the next seven years growing at the expected growth rate of 74.3% per annum

# 2.1 Valuing Perpetuities and Annuities

- A *perpetuity* is an equal, periodic cash flow that goes on forever
  - *Note:* The first cash flow occurs at the *end of period 1*

$\lim_{n\to+\infty} \sum_{i=1}^{n} \left(\frac{1}{1+r}\right)^i$




*Note:* The text uses *PMT* or *CF* for *C*

*Cash flows occur at the end of each period*

$PV = PV_0 + \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \dots + \frac{C}{(1+r)^n} + \frac{C}{(1+r)^{n+1}} + \dots$

- Using value additivity, the present value of a perpetuity is…

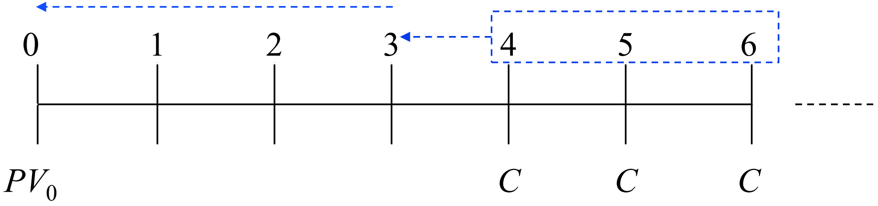
$= PV_0 + C\left[\frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots + \frac{1}{(1+r)^{n+1}} + \dots\right]$

  - $PV_0 = C/(1+r) + C/(1+r)^2 + \dots + C/(1+r)^n + C/(1+r)^{n+1} + \dots$ $=$
  - $PV_0 = C[1/(1+r) + 1/(1+r)^2 + \dots + 1/(1+r)^n + 1/(1+r)^{n+1} + \dots]$
- As $n$ approaches $\infty$, $[1/(1+r) + 1/(1+r)^2 + \dots + 1/(1+r)^{n+1} + \dots]$ approaches $1/r$
- So, the present value of a perpetuity, $PV_0 = \left(\frac{C}{r}\right)$

# *Valuing Perpetuities and Annuities*

- A *deferred perpetuity* is an equal, periodic cash flow that starts at some future date and then goes on forever

$PV_0 = [C/r]/(1 + r)^3$ $\quad$ $PV_3 = C/r$

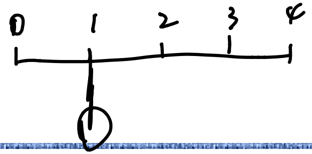



*Cash flows occur at the end of each period*

- In general, the present value of a perpetuity deferred to the end of time $n+1$ is…

$$PV_0 = \left(\frac{C}{r}\right)\left(\frac{1}{(1+r)^n}\right)$$

*Note: n*, not $n$+1!

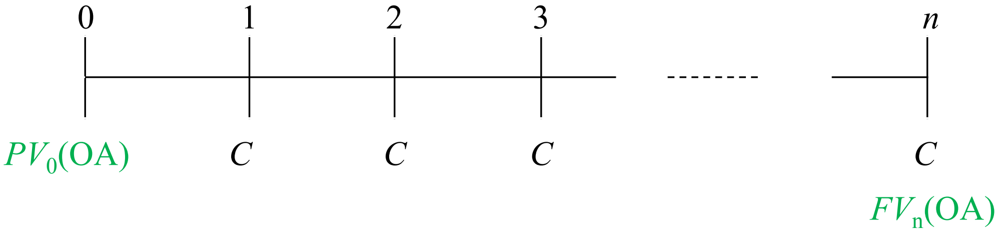



# *Valuing Perpetuities and Annuities*

- *Example 1:* A prize guarantees you $10,000 per year forever with the first payment to be made at the end of year 1. How much would you sell the prize for today if the interest rate is 10% p.a.? What would the prize's value be if it were *deferred* to the end of year 4? That is, the first cash flow occurs at the end of year 4 (see the timeline in the previous slide)
- Present value of perpetuity, $PV_0$ = 10000/0.10 = $100,000 $\frac{C}{0.1}$
- Present value of *deferred* perpetuity…
  - $PV_3$ = 10000/0.10 = $100,000
  - $PV_0$ = 100000/(1.10)$^3$ = $75,131.48

# 2.2 Valuing Ordinary Annuities

- An *ordinary annuity* is a series of equal, periodic cash flows occurring at the *end of each period* and lasting for *n* periods
  - *Note:* The *first* cash flow occurs at the *end of period 1* and the *last* cash flow occurs at the *end of period n*

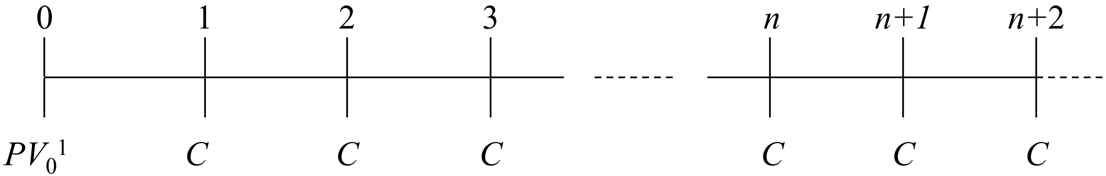



*Cash flows occur at the end of each period*

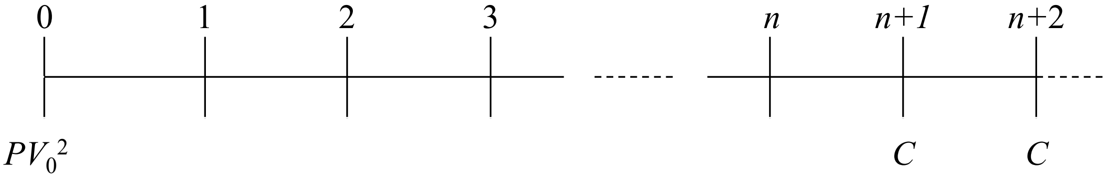
- The present value of an ordinary annuity can be obtained as the *difference* between an ordinary perpetuity and a deferred perpetuity

# Valuing Ordinary Annuities

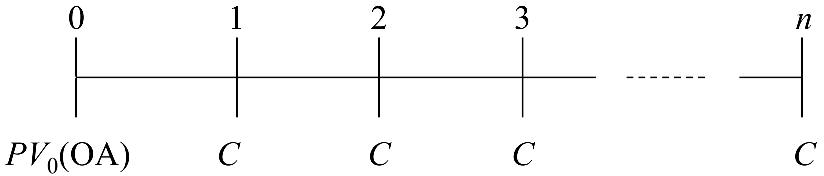
Ordinary perpetuity, $PV_0^1$




Minus: Deferred perpetuity, $PV_0^2$




Equals: Ordinary annuity, $PV_0(\mathrm{OA}) = PV_0^1 - PV_0^2$



# *Valuing Ordinary Annuities*

- The *present value* of an ordinary annuity over *n* time periods is the difference between an *ordinary* perpetuity (starting at the end of period 1) and a *deferred* perpetuity (starting at the end of period *n + 1*), $PV_0(\text{OA}) = PV_0^{\ 1} - PV_0^{\ 2}$

$$PV_0(OA) = \left(\frac{C}{r}\right) - \left(\frac{C}{r}\right)\left(\frac{1}{(1+r)^n}\right)$$

- Simplifying the above expression, we get…

$$PV_0(OA) = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^n}\right)$$

# *Valuing Ordinary Annuities*

- The *future value* of an ordinary annuity can be obtained by compounding the present value above to the end of time period $n$ using $(1 + r)^n$

$$FV_n(OA) = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^n}\right)(1+r)^n$$

- Simplifying the above expression, we get…

$$FV_n(OA) = \left(\frac{C}{r}\right)\left[(1+r)^n - 1\right]$$

# *Valuing Perpetuities and Annuities*

- *Example 2:* Suppose a prize guarantees you $10,000 per year for the next three years with the first payment to be made at the end of year 1. How much would you sell the prize for today if the interest rate is 10% p.a.? How is the value of this prize related to what we did in the previous example?
- This is a 3-year annuity whose present value can be calculated using…

$$PV_0(OA) = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^n}\right)$$

$$PV_0(OA) = \left(\frac{10000}{0.10}\right)\left[1 - \frac{1}{(1+0.10)^3}\right] = \$24,868.52$$

# *Valuing Perpetuities and Annuities*

- Note that the value of the prize is equal to the *difference* in the present value of the $10,000 perpetuity and the present value of the $10,000 perpetuity deferred until the end of year 4
- From the previous example, we had…
  - Present value of perpetuity, $PV_0 = 10000/0.10 = \$100,000$
  - Present value of deferred perpetuity…
    - $PV_3 = 10000/0.10 = \$100,000$
    - $PV_0 = 100000/(1.10)^3 = \$75,131.48$
  - Difference = 100000 – 75131.48 = $24,868.52

# *Case Study 2: No Latte for You!*

- Assume that you drink a \$4 latte a day. Over the month the cost is \$80 (= \$4 × 5 × 4). Assume that you choose to forgo your daily latte and instead invest this amount at the end of each month in an investment fund that earns an interest rate of 12% p.a. What would the value of your investment be at the end of: (a) 10 years and (b) 50 years? What would these values be if the interest rate were 8% p.a.? What are the present values of your investments? What is the general relation between the future and present values calculated?
- The cash flows here are *monthly* so we first need to calculate the *monthly interest rate*, which is 0.12/12 or 1% per month
  - In the second scenario, the monthly interest rate is 0.08/12 or 0.667%

$r\%$ p.a.

$n \to$ month

- The time horizons in months are…

  a) 10 × 12 = 120 months in the first scenario

  b) 50 × 12 = 600 months in the second scenario

# *Case Study 2: No Latte for You!*

- The future values at the two interest rates are…
- Interest rate earned is 1% per month

$$FV_{120} = \left(\frac{80}{0.01}\right)\left[(1+0.01)^{10\times 12} - 1\right] = \$18,403.10$$

$$FV_{600} = \left(\frac{80}{0.01}\right)\left[(1+0.01)^{50\times 12} - 1\right] = \$3,124,667.18$$

- Interest rate earned is 0.667% per month

$$FV_{120} = \left(\frac{80}{0.00667}\right)\left[(1+0.00667)^{10\times 12} - 1\right] = \$14,635.68$$

$$FV_{600} = \left(\frac{80}{0.00667}\right)\left[(1+0.00667)^{50\times 12} - 1\right] = \$643,538.20$$

# *Case Study 2: No Latte for You!*

- The present values at the two interest rates are…
- Interest rate earned is 1% per month

$$PV_0 = \left(\frac{80}{0.01}\right)\left[1 - \frac{1}{(1+0.01)^{10\times 12}}\right] = \$5,576.04$$

$$PV_0 = \left(\frac{80}{0.01}\right)\left[1 - \frac{1}{(1+0.01)^{50\times 12}}\right] = \$7,979.57$$

- Interest rate earned is 0.667% per month

$$PV_0 = \left(\frac{80}{0.00667}\right)\left[1 - \frac{1}{(1+0.00667)^{10\times 12}}\right] = \$6,593.72$$

$$PV_0 = \left(\frac{80}{0.00667}\right)\left[1 - \frac{1}{(1+0.00667)^{50\times 12}}\right] = \$11,777.28$$

# *Case Study 2: No Latte for You!*

- What is the general relation between the future and present values?
- Consider the case where the monthly interest rate is 1% and the time horizon is 600 months (or 50 years). If you invested \$7,979.57 *today* then in 600 months' time your investment would be worth…

$$FV_{600} = 7979.57\left(1+0.01\right)^{50\times 12} = \$3,124,667.18$$

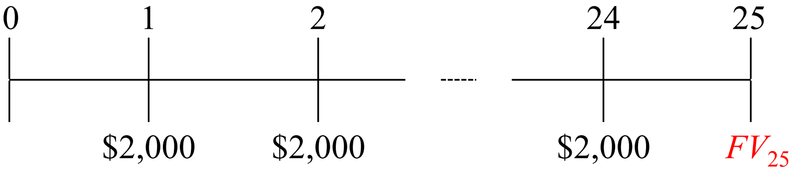
- Here's a link to an article on an app (Acorns) that lets you invest your spare change in the Australian sharemarket
  - *http://www.businessinsider.com.au/review-i-tried-acorns-the-app-that-turns-your-spare-change-into-investments-2016-3*
- Here's a link to an article reviewing Acorns (US) and its competitors
  - *https://www.listenmoneymatters.com/acorns-vs-betterment-vs-wealthfront/*

# *Case Study 3: To Charge or Not to Charge*

- Your friend has just started a Masters program and has decided to pay for her living expenses using a credit card that has no minimum monthly payment. She intends to charge $2,000 every month on the credit card for the next 24 months. The card carries an annual interest rate of 18% p.a. which translates to a monthly interest rate of 1.5% (= 0.18/12). How much money will she owe on the credit card 25 months from now when she receives her first credit card statement after graduation? (*Round your calculations to the nearest dollar.*)
- *We need to be careful here as we cannot just use the future value of an ordinary annuity formula to get the answer!*
  - Using the formula will give us the future value at the end of month **24**
  - We need the future value at the end of month **25**
  - *Huge Hint:* Use a timeline to visualize the cash flows!

# *Case Study 3: To Charge or Not to Charge*

- We want to calculate the future value of your friend's account balance. The timeline over the next 25 months is…




- The future value of this 24-month annuity is…

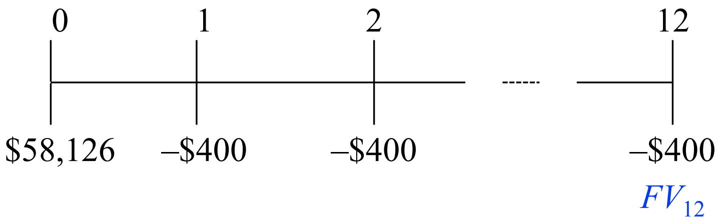
$$FV_{24} = \left(\frac{C}{r}\right)\left[(1+r)^n - 1\right] = \left(\frac{2000}{0.015}\right)\left[(1+0.015)^{24} - 1\right] = \$57,267$$

- So, the amount owed at the *end of month 25* is…

$$FV_{25} = 57267\left(1+0.015\right) = \$58,126$$

# *Case Study 3: To Charge or Not to Charge*

- *Case Study (continued):* Your friend, who now owes $58,126 on her credit card, has decided to start paying off the debt on a monthly basis, starting next month. If she can only afford to repay $400 every month how much would she still owe on her credit card by the end of one year (that is, 12 months from now)?
- The timeline of her repayment schedule looks as follows…



# *Case Study 3: To Charge or Not to Charge*

- *Here is how the process would work…*
- The \$58,126 at the end of what is now month 0 would accrue interest at 1.5% over month 1 to be worth…
  - $58126(1 + 0.015) =$ \$58,998 at the end of month 1
- She repays \$400, leaving a debt of: $58998 – 400 =$ \$58,598
- This amount would accrue interest at 1.5% over month 2 to be worth…
  - $58598(1 + 0.015) =$ \$59,477 at the end of month 2
- She repays \$400, leaving a debt of: $59477 – 400 =$ \$59,077

  … and so on…
- *Is there a simpler way to do these calculations?*

# *Case Study 3: To Charge or Not to Charge*

- *Step 1:* The future value of the current balance in 12 months is…

$$FV_{12} = 58126(1+0.015)^{12} = \$69,497$$

- *Step 2:* The future value of the \$400 repaid every month in 12 months is…

$$FV_{12} = \frac{-400}{0.015}\left[(1+0.015)^{12} - 1\right] = -\$5,216$$

- The *net* amount still owed at the end of the 12 month period is…
- Amount still owed = 69497 – 5216 = \$64,281
- *At this rate, how long do think it will take your friend to repay her debt?*

# *2.3 Valuing Annuities Due*

- Unlike an ordinary annuity, an *annuity due* is a series of equal, periodic cash flows occurring at the *beginning of each period*
- Note that the beginning of period $n$ is the same as the end of period $n - 1$. The overall effect is to move the annuity *back* one period on our standard (that is, end-of-period) timeline

*Example 1*

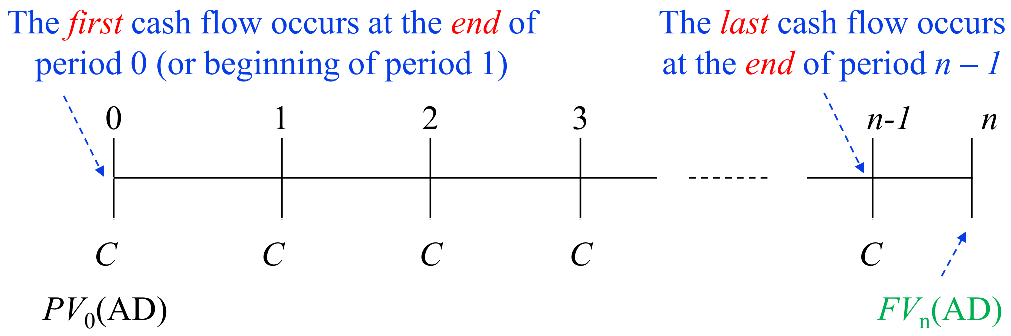
| *Beginning of year* | *End of previous year* |
|---|---|
| 1 Jan 2015 | 31 Dec 2014 |
| 1 Jan 2016 | 31 Dec 2015 |
| 1 Jan 2017 | 31 Dec 2016 |
| 1 Jan 2018 | 31 Dec 2017 |

*Example 2*

| *Beginning of month* | *End of previous month* |
|---|---|
| 1 Dec 2018 | 30 Nov 2018 |
| 1 Nov 2018 | 31 Oct 2018 |
| 1 Oct 2018 | 30 Sep 2018 |
| 1 Sep 2018 | 31 Aug 2018 |

# *Valuing Annuities Due*

- The beginning of period *n* is the same as the end of period *n – 1*. The overall effect is to move the annuity *back* one period on our standard (that is, end-of-period) time line




*Cash flows occur at the end of each period*

# *Valuing Annuities Due*

- The *present value* of an annuity due at *r*% p.a. is *equivalent* to the present value of an ordinary annuity compounded *one additional period*

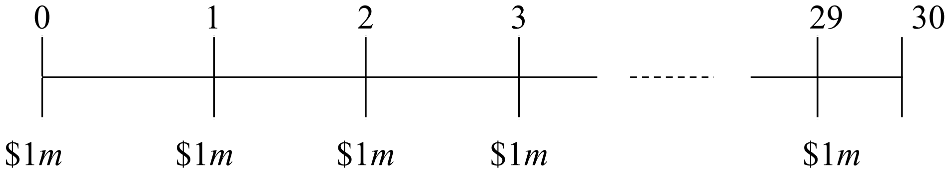
$$PV_0(AD) = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^n}\right)(1+r)$$

- The *future value* of an annuity due at *r*% p.a. is also *equivalent* to compounding by *one additional period* the future value of an ordinary annuity

$$FV_n(AD) = \left(\frac{C}{r}\right)\left[(1+r)^n - 1\right](1+r)$$

# *Valuing Annuities Due*

- *Example:* You're the lucky winner of the $30 million state lottery. You can take your prize money either as: (a) 30 payments of $1 million per year (starting today), or (b) $15 million paid today. If the interest rate is 8% p.a., which option should you take? (*Round your calculations to the nearest dollar*.)




$PV_0$(AD)

*Cash flows occur at the end of each period*

# *Valuing Annuities Due*

- We value the annuity due *as if* it were an ordinary annuity and then adjust it by a factor of $(1 + r)$…

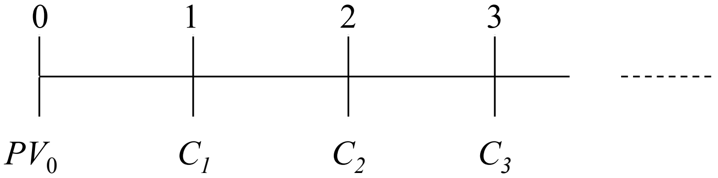
$$PV_0(AD) = \left(\frac{1000000}{0.08}\right)\left(1 - \frac{1}{(1+0.08)^{30}}\right)(1+0.08) = \$12,158,406$$

- *Alternative method:* Calculate the present value of the 29-year annuity and then add the cash flow in year 0…

$$PV_0(AD) = 1000000 + \left(\frac{1000000}{0.08}\right)\left(1 - \frac{1}{(1+0.08)^{29}}\right) = \$12,158,406$$

# *2.4 Valuing Growing Cash Flows*

- A *growing perpetuity* is a series of periodic cash flows occurring at the *end of each period* and growing at a constant rate forever
  - *Note:* The first cash flow occurs at the *end of period 1*




*Cash flows occur at the end of each period*

- A *constant* growth rate ($g$) in cash flows implies…

*Note:* $n$–1, not $n$!

$$C_2 = C_1(1+g),\ C_3 = C_1(1+g)^2, \ldots,\ C_n = C_1(1+g)^{n-1}$$

$T: \frac{1}{1+r} + \frac{1+g}{(1+r)^2} + \frac{(1+g)^2}{(1+r)^3} + \cdots + \frac{(1+g)^{n-1}}{(1+r)^n}$

# *Valuing Growing Cash Flows*

- The present value of a growing perpetuity is…
  - $PV_0 = C_1/(1+r) + C_1(1+g)/(1+r)^2 + \ldots + C_1(1+g)^{n-1}/(1+r)^n \ldots$
  - $PV_0 = C_1\{[1/(1+r)] + [(1+g)/(1+r)^2] + \ldots + [(1+g)^{n-1}/(1+r)^n] \ldots\}$
- As $n$ approaches $\infty$…
  $\{[1/(1+r)] + [(1+g)/(1+r)^2] + \ldots + [(1+g)^{n-1}/(1+r)^n] \ldots\}$ approaches $1/(r-g)$
- The present value of a growing perpetuity, $PV_0 = C_1/(r-g)$
  - *Note 1:* In the above expression it *must* be the case that $r > g$
  - *Note 2:* When $g = 0$ we get an *ordinary perpetuity* (as before) with a present value, $PV_0 = C/r$
  - *Note 3:* It is possible for $g < 0$ (negative growth in cash flows)

# *Valuing Growing Cash Flows*

- *Example:* A prize guarantees you $10,000 per year growing at 6% p.a. forever with the first payment to be made at the end of year 1. How much would the prize be worth today if the interest rate is 10% p.a.? What are the dollar values of the prize at the end of 10, 20, 50 and 100 years? What are the respective present values of these amounts today?

# *Valuing Growing Cash Flows*

- The present value of a growing perpetuity is…
  - $PV_0 = 10000/(0.10 - 0.06) = \$250{,}000$
- The dollar value of the prize at the end of various years can be calculated using:
  - Dollar value in year $n$, $C_n = 10000(1 + g)^{n-1}$
- The present value of these amounts can be calculated using:
  - Present value in year 0 = $\underbrace{C_n/(1 + r)^{n}}$ *Note: n* not *n – 1!*

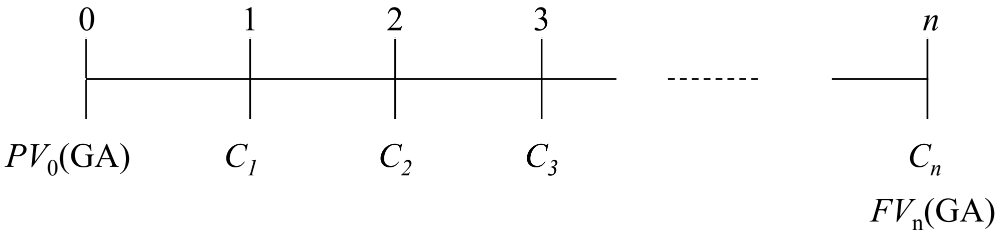
# *Valuing Growing Cash Flows*

- The dollar values (and their respective present values today) of the prize at the end of various years are as follows

| *End of Year* | *Prize Amount* | *Present Value Today* |
|---|---|---|
| 1 | $10,000 | $9,091 |
| 2 | $10,600 | $8,760 |
| 3 | $11,236 | $8,442 |
| 4 | $11,910 | $8,135 |
| 5 | $12,625 | $7,839 |
| 10 | $16,895 | $6,514 |
| 20 | $30,256 | $4,497 |
| 50 | $173,775 | $1,480 |
| 100 | $3,200,963 | $232! |

# 2.5 Valuing Growing Ordinary Annuities

- A *growing ordinary annuity* is a series of periodic cash flows occurring at the *end of each period* and lasting for $n$ periods where the cash flows grow at a constant rate



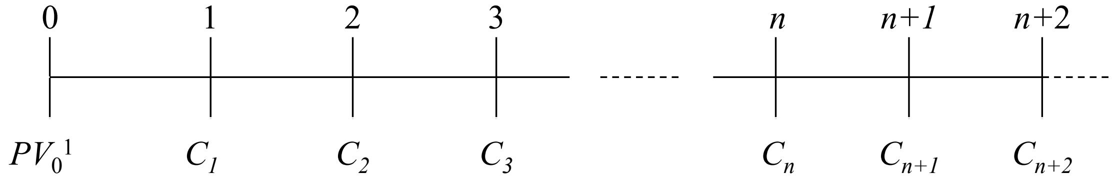

- A *constant* growth rate ($g$) in cash flows implies…

$$C_2 = C_1(1+g),\ C_3 = C_1(1+g)^2,\ \ldots,\ C_n = C_1(1+g)^{n-1}$$

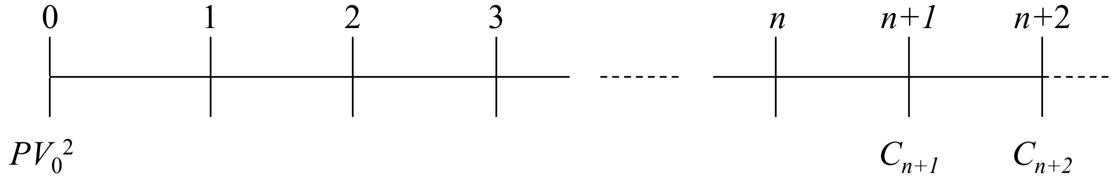
- Growing ordinary annuities can be valued as the *difference* between a growing ordinary perpetuity and a growing *deferred* perpetuity

# *Valuing Growing Ordinary Annuities*

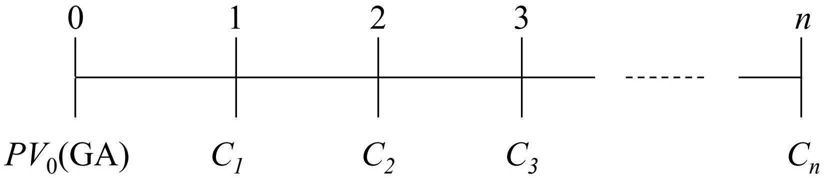
*Growing ordinary perpetuity, $PV_0^1$*




*Minus: Growing deferred perpetuity, $PV_0^2$*




*Equals: Growing ordinary annuity, $PV_0(\text{GA}) = PV_0^1 - PV_0^2$*



# *Present Value of a Growing Ordinary Annuity*

- The *present value* of a growing ordinary annuity over $n$ time periods can be obtained as the difference between a growing *ordinary* perpetuity (starting at the end of period 1) and a growing *deferred* perpetuity (starting at the end of period $n + 1$), $PV_0(\text{GA}) = PV_0^{\,1} - PV_0^{\,2}$
- Note that in the growing deferred perpetuity, the cash flow at the end of period $n + 1$, $C_{n+1}$ is…
  - $C_{n+1} = C_1(1 + g)^n$

# *Present Value of a Growing Ordinary Annuity*

- The *present value* of a growing ordinary annuity $\underbrace{\text{over } n \text{ time}}$ periods is…

$$PV_0(GA) = \frac{C_1}{r-g} - \left[\left(\frac{C_1(1+g)^n}{r-g}\right) \Big/ (1+r)^n\right]$$

- Simplifying the above equation, we get…

$$PV_0(GA) = \left(\frac{C_1}{r-g}\right)\left(1 - \left(\frac{1+g}{1+r}\right)^n\right)$$

# *Future Value of a Growing Ordinary Annuity*

- The *future value* of a growing ordinary annuity at the end of $n$ time periods is simply the present value compounded over that time horizon, $FV_n(GA) = PV_0(GA)(1 + r)^n$

$$\underbrace{FV_n(GA) = \left(\frac{C_1}{r-g}\right)\left(1-\left(\frac{1+g}{1+r}\right)^n\right)(1+r)^n}$$

- In these expressions, can $r < g$? when $r<g$ $1+r<1+g$ $1-\left(\frac{1+g}{1+r}\right)^n<0$ $+$ $\frac{1}{r-g}<0$ $\rightarrow$ can $r<g$
- Can $r = g$? ✓ cannot use the formula

# *Case Study 4: Thanks Aunty!*

- Your rich and generous aunty has decided that she (or her estate) will contribute \$1,000 at the end of next year (year 1) with the contribution growing at 5% per annum for: (i) 10 years and (ii) 50 years. If you're able to invest these funds and earn an annual interest rate of 10% what will the future values of your investment be at the point where your aunty stops making contributions? If you had to convince her to make a *single* contribution today (year 0) how much should you ask for in each case? What is the relation between the future and present values calculated?

# *Case Study 4: Thanks Aunty!*

- This is a growing annuity where…
  - $C_1$ = \$1,000, $g$ = 5%, $r$ = 10% and $n$ = 10 years or 50 years
- Future value of the investment in 10 years…

$$FV_{10}(GA) = \left(\frac{1000}{0.10-0.05}\right)\left(1-\left(\frac{1+0.05}{1+0.10}\right)^{10}\right)(1+0.10)^{10} = \$19,296.96$$

- Future value of the investment in 50 years…

$$FV_{50}(GA) = \left(\frac{1000}{0.10-0.05}\right)\left(1-\left(\frac{1+0.05}{1+0.10}\right)^{50}\right)(1+0.10)^{50} = \$2,118,469.06$$

# *Case Study 4: Thanks Aunty!*

- Present value of the investments over 10 years…

$$PV_0(GA) = \left(\frac{1000}{0.10-0.05}\right)\left(1-\left(\frac{1+0.05}{1+0.10}\right)^{10}\right) = \$7,439.81 \quad or \quad PV_0(GA) = \frac{19296.96}{(1+0.10)^{10}}$$

- Present value of the investment over 50 years…

$$PV_0(GA) = \left(\frac{1000}{0.10-0.05}\right)\left(1-\left(\frac{1+0.05}{1+0.10}\right)^{50}\right) = \$18,046.29 \quad or \quad PV_0(GA) = \frac{2118469.06}{(1+0.10)^{50}}$$

- *What is the relation between the future and present values?*

# *A Review*

- Consider the *present value* of a growing ordinary annuity…

$$PV_0(GA) = \left(\frac{C_1}{r-g}\right)\left(1 - \left(\frac{1+g}{1+r}\right)^n\right)$$

- We get the *future value* of a growing ordinary annuity by compounding the above present value over $n$ periods using $(1 + r)^n$
- As the time horizon, $n$ approaches infinity we get the *present value* of a growing perpetuity…

$$PV_0 = \left(\frac{C_1}{r-g}\right)$$

- When the growth rate $g$ is 0, we get the *present value* of a perpetuity…

$$PV_0 = \frac{C_1}{r}$$

# A Review

- Consider the *present value* of a growing ordinary annuity again…

$$PV_0(GA) = \left(\frac{C_1}{r-g}\right)\left(1-\left(\frac{1+g}{1+r}\right)^n\right)$$

- When the growth rate $g$ is 0, we get the *present value* of an ordinary annuity…

$$PV_0(OA) = \left(\frac{C_1}{r}\right)\left(1-\left(\frac{1}{1+r}\right)^n\right)$$

- We get the *future value* of an ordinary annuity by compounding the above present value over $n$ periods using $(1 + r)^n$
- We get the *present and future values* of annuities *due* by adjusting the respective present and future of the above ordinary annuities by $(1 + r)$

# *Key Concepts*

- A perpetuity is an equal, periodic cash flow that goes on forever
- A deferred perpetuity is an equal, periodic cash flow that starts at some future date and then goes on forever
- Ordinary annuities are periodic, end-of-the-period cash flows
- Annuities due are periodic, beginning-of-the-period cash flows
- The present (future) value of an annuity due involves a simple adjustment to the present (future) value of an ordinary annuity
- The present value of a growing ordinary annuity can be calculated as the difference between the present value of a growing perpetuity and a growing deferred perpetuity
- The future value of a growing ordinary annuity can be calculated by compounding its present value to the end of the annuity's time horizon

# *Formula Sheet*

- Present value of a perpetuity

$$PV_0 = \frac{C}{r}$$

- Present value of a deferred perpetuity

$$PV_0 = \left(\frac{C}{r}\right)\left(\frac{1}{(1+r)^n}\right)$$

- Present value of ordinary annuity

$$PV_0(OA) = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^n}\right)$$

- Future value of ordinary annuity

$$FV_n(OA) = \left(\frac{C}{r}\right)\left[(1+r)^n - 1\right]$$

# *Formula Sheet*

- Present value of an annuity due

$$PV_0(AD) = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^n}\right)(1+r)$$

- Future value of an annuity due

$$FV_n(AD) = \left(\frac{C}{r}\right)\left[(1+r)^n - 1\right](1+r)$$

- The present value of a growing perpetuity

$$PV_0 = \frac{C_1}{r-g}$$

# *Formula Sheet*

- Present value of a growing ordinary annuity

$$PV_0(GA) = \left(\frac{C_1}{r-g}\right)\left(1-\left(\frac{1+g}{1+r}\right)^n\right)$$

- Future value of a growing ordinary annuity

$$FV_n(GA) = \left(\frac{C_1}{r-g}\right)\left(1-\left(\frac{1+g}{1+r}\right)^n\right)(1+r)^n$$

(*Note:* The formula sheets on the mid semester and final exams will contain all the formulas covered in lectures but *without* the descriptions)