MAST20004 Probability

Tutorial Set 6

1. (a) If $X \stackrel{d}{=} \mathrm{R}(0,1)$ and $Y = -\frac{1}{\alpha}\log X$, where $\alpha > 0$, show that $Y \stackrel{d}{=} \exp(\alpha)$.
   (b) Imagine that you have a computer that can generate random numbers between 0 and 1. Explain what you would do to simulate observations from an exponential random variable with mean 5.

   Solution:

   (a)

$$\begin{aligned}\mathbb{P}(Y \le y) &= \mathbb{P}\left(-\frac{1}{\alpha}\log X \le y\right)\\ &= \mathbb{P}(X \ge e^{-\alpha y})\\ &= 1 - e^{-\alpha y}.\end{aligned}$$

   So Y is exponentially distributed with parameter α.

   (b) Use the random number generator to give numbers X_i that are uniformly distributed on the interval $[0,1]$, and then put $Y_i = -5\log X_i$. By part (a), the Y_i will be exponentially distributed with mean $1/\alpha = 5$.

2. If $Z \stackrel{d}{=} \gamma(r,1)$ and $X = \frac{1}{\alpha}Z$ where $\alpha > 0$, show that $X \stackrel{d}{=} \gamma(r,\alpha)$.
   Hence show that if $X \stackrel{d}{=} \gamma(r,\alpha)$ and $Y \stackrel{d}{=} \gamma(r,\beta)$ where $\alpha > 0$ and $\beta > 0$, then $\alpha X \stackrel{d}{=} \beta Y$.

   Solution:

$$\begin{aligned}\mathbb{P}(X \le x) &= \mathbb{P}\left(\frac{1}{\alpha}Z \le x\right)\\ &= \mathbb{P}(Z \le \alpha x)\\ &= \frac{1}{\Gamma(r)}\int_0^{\alpha x} z^{r-1}e^{-z}dz\\ &= \frac{1}{\Gamma(r)}\int_0^{x} \alpha^r u^{r-1}e^{-\alpha u}du.\end{aligned}$$

   So $X \stackrel{d}{=} \gamma(r,\alpha)$. By this result $\alpha X \stackrel{d}{=} \gamma(r,1)$ and $\beta Y \stackrel{d}{=} \gamma(r,1)$, so $\alpha X \stackrel{d}{=} \beta Y$.

3. Let X be a discrete random variable with possible values $\{0,1,2,\ldots\}$.
   (a) Show that

$$\mathbb{E}(X) = \sum_{x=0}^{\infty}\mathbb{P}(X > x).$$

   (b) Use this result to evaluate the mean of a geometric random variable.

   Solution:

   (a) Method 1: one can use the formula for computing moments through tail probabilities to obtain

$$\mathbb{E}(X) = \int_0^{\infty}(1 - F_X(x))dx.$$

However, since $1 - F_X(x) = 1 - F_X(i)$ for $i \leq x < i+1$, we have

$$\begin{aligned}
\mathbb{E}(X) &= \int_0^\infty (1 - F_X(x))dx \\
&= \sum_{i=0}^\infty \int_i^{i+1} (1 - F_X(x))dx \\
&= \sum_{i=0}^\infty \int_i^{i+1} (1 - F_X(i))dx \\
&= \sum_{i=0}^\infty (1 - F_X(i)) = \sum_{i=0}^\infty \mathbb{P}(X > i).
\end{aligned}$$

Method 2: let the probability mass function of X be $p_X(k)$, then

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=1}^\infty k p_X(k) \\
&= \sum_{k=1}^\infty \left(\sum_{x=1}^k 1 \right) p_X(k) \\
&= \sum_{x=1}^\infty \sum_{k=x}^\infty p_X(k) \\
&= \sum_{x=1}^\infty \mathbb{P}(X \geq x) \\
&= \sum_{x=0}^\infty \mathbb{P}(X > x).
\end{aligned}$$

(b) For a geometric random variable X with parameter p, $\mathbb{P}(X > x) = (1-p)^{x+1}$. So

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{x=0}^\infty \mathbb{P}(X > x) \\
&= \sum_{x=0}^\infty (1-p)^{x+1} \\
&= \sum_{x=1}^\infty (1-p)^x \\
&= \frac{1-p}{1-(1-p)} \\
&= \frac{1-p}{p}.
\end{aligned}$$

4. Suppose X has pdf given by $f_X(x) = \dfrac{1}{(1+x)^2}$, $x > 0$. Let $Y = \dfrac{1}{X}$. Show that $Y \stackrel{d}{=} X$.

Solution:

$$\begin{aligned} F_X(x) &= \int_0^x \frac{1}{(1+u)^2} du \\ &= 1 - \frac{1}{1+x}. \end{aligned}$$

Now

$$\begin{aligned} \mathbb{P}(Y \le y) &= \mathbb{P}\left(\frac{1}{X} \le y\right) \\ &= \mathbb{P}\left(X \ge \frac{1}{y}\right) \\ &= 1 - \mathbb{P}\left(X < \frac{1}{y}\right) \\ &= 1 - \left(1 - \frac{1}{1+1/y}\right) \\ &= 1 - \frac{1}{1+y}. \end{aligned}$$

X and Y have the same distribution function, so $Y \stackrel{d}{=} X$.

5. Let X be a random variable with pdf given by $f_X(x) = xe^{-x^2/2}$, $x > 0$. Use the formula for computing higher moments through tail probabilities to show that

$$\mathbb{E}(X^i) = i \cdot 2^{i/2-1}\Gamma(i/2),\ i > 0.$$

Solution: The cdf of X is $F(x) = \begin{cases} 1 - e^{-x^2/2}, & \text{if } x \ge 0, \\ 0, & \text{if } x < 0. \end{cases}$

Hence

$$\begin{aligned} \mathbb{E}(X^i) &= i\int_0^\infty x^{i-1}(1-F(x))dx \\ &= i\int_0^\infty x^{i-1}e^{-x^2/2}dx \\ &\underbrace{=}_{y=\frac{x^2}{2}} i \cdot 2^{\frac{i}{2}-1}\int_0^\infty y^{\frac{i}{2}-1}e^{-y}dy \\ &= i \cdot 2^{\frac{i}{2}-1}\Gamma(i/2). \end{aligned}$$

6. Let X be a random variable with distribution function F_X and $Y = \Phi(X)$ where Φ is a strictly decreasing function mapping S_X to the set S_Y.

   (a) Derive an expression for $F_Y(y)$ in terms of $F_X(x)$ that is valid for continuous X, and another that is valid for discrete X.

   (b) Assuming that X is a continuous random variable, and with F_X and Φ both differentiable functions on S_X, derive a formula for the density function of Y in terms of the derivative of Φ^{-1}.

   (c) Give a version of the formula in (b) that doesn't contain the derivative of Φ^{-1}.

   (d) For the random variable $Y = e^{-X}$ where $X \stackrel{d}{=} R(0,1)$, verify that your formulae for (b) and (c) give the same result.

   (e) Use your result from (c) and the corresponding formula for increasing Φ derived on lecture slide 295 to write down a formula valid for either increasing or decreasing Φ.

Solution: First observe that, because Φ is strictly decreasing, it must be one-to-one and so its inverse Φ^{-1}, which maps S_Y to S_X, must exist. Then for $y \in S_Y$.

(a)

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(\Phi(X) \leq y) \\ &= P(X \geq \Phi^{-1}(y)) \\ &= 1 - P(X < \Phi^{-1}(y)) \\ &= \begin{cases} 1 - F_X(\Phi^{-1}(y)) & \text{if } X \text{ is continuous} \\ 1 - F_X(z) & \text{if } X \text{ is discrete.} \end{cases} \end{aligned}$$

In the last line z is the largest number less than $\Phi^{-1}(y)$ that X can take with positive probability. A neat way to incorporate both the cases is to write

$$F_Y(y) = \lim_{u \to \Phi^{-1}(y)^-} (1 - F_X(u)).$$

(b) For $y \in S_Y$,

$$f_Y(y) = -f_X(\Phi^{-1}(y)) \frac{d}{dy} \left(\Phi^{-1}(y)\right).$$

(c) For $y \in S_Y$,

$$f_Y(y) = -f_X(\Phi^{-1}(y)) \frac{1}{\Phi'(\Phi^{-1}(y))}.$$

(d) For $X \stackrel{d}{=} R(0,1)$,

$$f_X(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Here, $S_X = [0,1]$ and $S_Y = [e^{-1}, 1]$. So, for $y \in S_Y$, $\Phi^{-1}(y) = -\log y$. So the formula from (b) gives

$$f_Y(y) = \begin{cases} \frac{1}{y} & \text{if } y \in [e^{-1}, 1] \\ 0 & \text{otherwise.} \end{cases}$$

The formula from (c) gives the same thing.

(e)

$$f_Y(y) = \left| f_X((\Phi^{-1}(y)) \frac{1}{\Phi'(\Phi^{-1}(y))} \right|.$$

7. Let the joint probability mass function of random variables X and Y be given by

$$p(x, y) = \begin{cases} \frac{1}{7}x^2 y & \text{if} (x, y) = (1, 1), (1, 2), (2, 1) \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the probability $P(X \leq Y)$.

(b) Are X and Y independent? Why or why not?

Solution:

(a) $P(X \leq Y) = P(\{(1, 1), (1, 2)\}) = \frac{1}{7} + \frac{2}{7} = \frac{3}{7}$.

(b) They are not independent, because, for example, $P((1, 1)) = \frac{1}{7}$, $P(X = 1) = \frac{3}{7}$, $P(Y = 1) = \frac{5}{7}$, and $P((1, 1)) \neq P(X = 1)P(Y = 1)$.

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Computer Lab 6

Matlab GUI - Lab6ExA

This lab makes use of a MATLAB GUI (graphical user interface) called 'Lab6ExA'. Copy the Lab 6 files from the LMS and then run the GUI by typing 'Lab6ExA' in the Command Window. The GUI contains three main windows displaying the functional relationship ($Y = \psi(X)$) between two random variables, as well as the probability density functions for both X and its transformation Y.
You can choose various probability density functions for X using the drop down list, and also change the parameters 'a' and 'b' by typing in the appropriate box and hitting return. Similarly you can choose various transformations and set relevant constants.

If you click on the 'Step' button the GUI generates a small random sample from X and plots the transformed values to give you an idea how the transformation is working.

If you click on the 'Run' button the GUI generates a large random sample and displays the empirical pdf for the transform Y and in some cases also the theoretical answer (making use of the same general formulae you have been given in lectures). Please be patient when you click on the 'Run' button - it may take 10 seconds or more for Matlab to complete all the necessary calculations.

You can reset the 'pdf of Y' window by clicking on the 'Reset' button.

You can rescale the axes by clicking on the appropriate box to change the minimum or maximum values.

Thanks are due to Alan Motyer who undertook the GUI programming.

Exercise A - taken from Ghahramani Section 6.2 Problem 1

Let X be a continuous random variable with density function

$$f(x) = \begin{cases} 1/4 & \text{if } x \in (-2, 2) \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density functions of $Y = X^3$ and $Z = X^4$. Then use the GUI to simulate the distributions for both $Y = X^3$ and $Z = X^4$. Check that the shapes of the empirical pdfs are broadly consistent with your theoretical answers (a rough sketch of the theoretical pdf may assist). The main point of this exercise is to see how the transformation works - use the 'Step' button. A precise comparison is not required.

Exercise B - Tutorial Question 2

Use the GUI to simulate this transformation and your check your theoretical result. This is an example of an important general technique using the inverse of the distribution function to generate an observation on any random variable by transforming a uniform random variable on $(0, 1)$.

Exercise C - Linear transforms

Find values of the constants 'a' and 'b' in a linear transform so it transforms $X \stackrel{d}{=} R(0,1)$ to $Y \stackrel{d}{=} R(3,5)$ and check using the GUI. Now do the reverse - find 'a' and 'b' so that $X \stackrel{d}{=} R(3,5)$ transforms to $Y \stackrel{d}{=} R(0,1)$. Express this second result in terms of the distribution function for X.

Exercise D - Quadratic transforms

1. Use theory to obtain the distribution of $Y = X^2$ for $X \stackrel{d}{=} R(-1,1)$. Check your result using the GUI.
2. Try to obtain the theoretical distribution of $Y = X^2$ for $X \stackrel{d}{=} R(-1,2)$ (the GUI graphics should help you to write down the distribution function for Y for values in $(0,1)$ and also in $(1,4)$. If you use the lecture slide formula be very careful with limits - remember the distribution function of X is zero below -1. Check your result by running the GUI. Make sure you understand the discontinuity in the pdf at 1.

Exercise E - Optional exploration

The variety of transformations you can explore with the GUI are endless. Play around with different pdfs and transforms and see what you can discover.