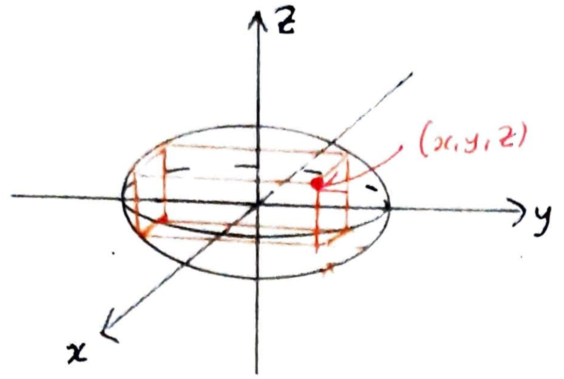
MAST20009 Assignment 2 Semester 1 2020

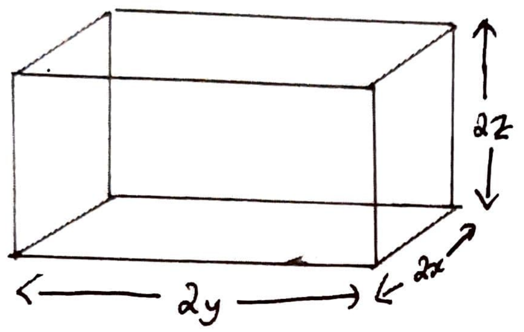
Q1



Rectangular box is inscribed inside ellipsoid. Largest box will touch ellipsoid at some points.

(a) Let (x,y,z) be a point in $\mathbb{R}^3$ at the corner of the box, so that

$\Rightarrow$ Rectangular box has dimensions $2x \times 2y \times 2z$, so $x,y,z \geq 0$.



Volume of box $= (2x)(2y)(2z) = 8xyz$

To find dimensions of largest box we let $x,y,z \geq 0$ and let

$$f(x,y,z) = 8xyz$$

and $g(x,y,z) = x^2 + 2y^2 + 4z^2 - 12 = 0 \quad \ldots(1)$

(b) Use Lagrange multipliers.

There exists a $\lambda \in \mathbb{R}$ such that $\underline{\nabla} f = \lambda \underline{\nabla} g$

$$\Rightarrow (8yz, 8xz, 8xy) = \lambda(2x, 4y, 8z)$$

Equating components gives

$8yz = 2x\lambda \Rightarrow 4yz = x\lambda \quad \ldots(2)$

$8xz = 4y\lambda \Rightarrow 2xz = y\lambda \quad \ldots(3)$

$8xy = 8z\lambda \Rightarrow xy = z\lambda \quad \ldots(4)$

Equations (2), (3) give

$xy\lambda = 4y^2 z = 2x^2 z$

$\Rightarrow 2z(x^2 - 2y^2) = 0$

$\Rightarrow z = 0$ or $x^2 = 2y^2 \quad \ldots (5)$

Equations (3), (4) give

$yz\lambda = 2xz^2 = xy^2$

$\Rightarrow x(y^2 - 2z^2) = 0$

$\Rightarrow x = 0$ or $y^2 = 2z^2 \quad \ldots (6)$

Equations (1), (5), (6) give the following 4 cases for critical points if $x, y, z \geq 0$.

* $z = 0, \ x = 0$

$\Rightarrow 2y^2 = 12 \quad \Rightarrow y^2 = 6 \quad \Rightarrow y = \sqrt{6}.$

$\Rightarrow (0, \sqrt{6}, 0)$

* $z = 0, \ y^2 = 2z^2$

$\Rightarrow y = 0$ and $x^2 = 12 \quad \Rightarrow x = \sqrt{12} = 2\sqrt{3}$

$\Rightarrow (2\sqrt{3}, 0, 0)$

* $x^2 = 2y^2, \ x = 0$

$\Rightarrow y = 0$ and $4z^2 = 12 \Rightarrow z^2 = 3 \quad \Rightarrow z = \sqrt{3}$

$\Rightarrow (0, 0, \sqrt{3})$

* $x^2 = 2y^2, \ y^2 = 2z^2$

$\Rightarrow x^2 = 4z^2$

and $4z^2 + 2(2z^2) + 4z^2 = 12$

$\Rightarrow 12z^2 = 12$

$\Rightarrow z^2 = 1 \quad \Rightarrow z = 1$

Hence $x^2 = 4, \ y^2 = 2 \quad \Rightarrow x = 2, y = \sqrt{2}.$

$\Rightarrow (2, \sqrt{2}, 1)$

This gives 4 critical points
$(0,\sqrt{6},0)$, $(2\sqrt{3},0,0)$, $(0,0,\sqrt{3})$, $(2,\sqrt{2},1)$

The volume is only non zero if $(x,y,z) = (2,\sqrt{2},1)$

Here $V = 8(2)(\sqrt{2})(1) = 16\sqrt{2}$.

The dimension of the largest box inscribed in ellipsoid are
$4 \times 2\sqrt{2} \times 2$ units
as $(x,y,z) = (2,\sqrt{2},1)$ and Volume $= 16\sqrt{2}$ $(\text{units})^3$.

(c) The constraint is an ellipsoid which is closed and bounded so maxima and minima must exist

* $f = 0$ at $(0,\sqrt{6},0)$, $(2\sqrt{3},0,0)$, $(0,0,\sqrt{3})$

* $f = 16\sqrt{2}$ at $(2,\sqrt{2},1)$

Hence $(x,y,z) = (2,\sqrt{2},1)$ gives maximum volume of $16\sqrt{2}$ (units)

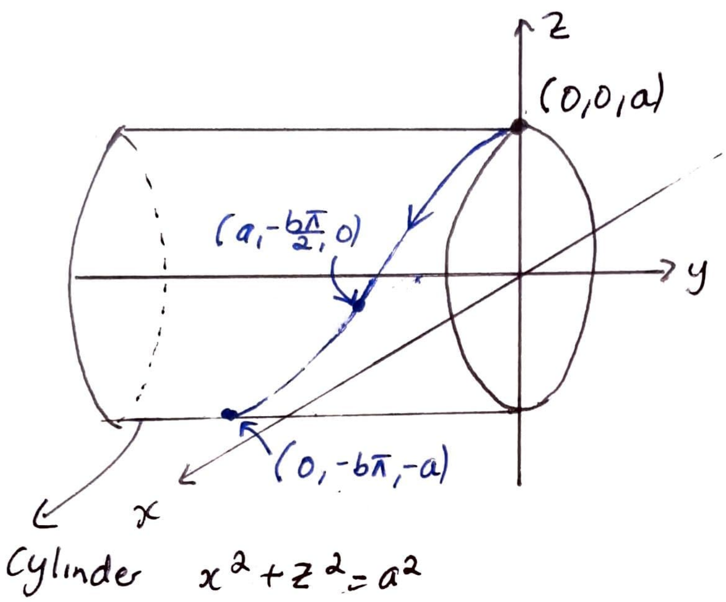
and $(x,y,z) = (0,\sqrt{6},0)$, $(2\sqrt{3},0,0)$, $(0,0,\sqrt{3})$ give the minimum volume of 0 $(\text{units})^3$.

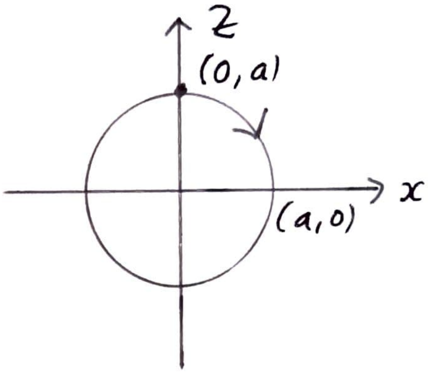
$\boxed{Q2}$ $\underline{c}(t) = (a\sin t, -bt, a\cos t)$, $t \geq 0$, $a>0$, $b>0$

(a) $x(t) = a\sin t$, $y(t) = -bt$, $z(t) = a\cos t$

$\Rightarrow x^2 + z^2 = a^2$ which is a circle in xz plane traversed clockwise.

Since $x^2+z^2 = a^2$ and $y = -bt$, $(b>0)$, the curve winds around the cylinder $x^2+z^2=a^2$ starting at $(0,0,a)$ and moving towards the negative direction on the y axis as t increases.





(b) Now $\underline{c}'(t) = (a\cos t, -b, -a\sin t)$

$\Rightarrow |\underline{c}'(t)| = \sqrt{a^2\cos^2 t + b^2 + a^2\sin^2 t} = \sqrt{a^2+b^2}$

So tangent vector is

$$\underline{T}(t) = \frac{\underline{c}'(t)}{|\underline{c}'(t)|} = \frac{1}{\sqrt{a^2+b^2}}(a\cos t, -b, -a\sin t)$$

$$\Rightarrow \underline{T}'(t) = \frac{1}{\sqrt{a^2+b^2}}(-a\sin t, 0, -a\cos t)$$

$$\Rightarrow |\underline{T}'(t)| = \frac{1}{\sqrt{a^2+b^2}}\sqrt{a^2\sin^2 t + a^2\cos^2 t} = \frac{a}{\sqrt{a^2+b^2}}$$

So curvature is

$$\kappa(t) = \frac{|\underline{T}'(t)|}{|\underline{c}'(t)|} = \frac{a}{\sqrt{a^2+b^2}} \cdot \frac{1}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2}$$

(c) The arclength parameter is

$$s(t) = \int_0^t |\underline{c}'(\tau)| d\tau$$

$$= \int_0^t \sqrt{a^2+b^2}\, d\tau$$

$$= \sqrt{a^2+b^2}\, [\tau]_{\tau=0}^{\tau=t}$$

$$= t\sqrt{a^2+b^2}$$

(d) Since $s = t\sqrt{a^2+b^2} \Rightarrow t = \dfrac{s}{\sqrt{a^2+b^2}}$

Hence path can be parametrised in terms of arclength by

$$\underline{c}(s) = \left(a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{-bs}{\sqrt{a^2+b^2}}, a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right)\right), \quad s \geq 0$$

(e) Now $\underline{c}'(s) = \left(\dfrac{a}{\sqrt{a^2+b^2}}\cos\left(\dfrac{s}{\sqrt{a^2+b^2}}\right), \dfrac{-b}{\sqrt{a^2+b^2}}, \dfrac{-a}{\sqrt{a^2+b^2}}\sin\left(\dfrac{s}{\sqrt{a^2+b^2}}\right)\right)$

$$\Rightarrow |\underline{c}'(s)| = \left(\frac{a^2}{a^2+b^2}\cos^2\left(\frac{s}{\sqrt{a^2+b^2}}\right) + \frac{b^2}{a^2+b^2} + \frac{a^2}{a^2+b^2}\sin^2\left(\frac{s}{\sqrt{a^2+b^2}}\right)\right)^{1/2}$$

$$= \left(\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}\right)^{1/2}$$

$$= 1$$

$$\Rightarrow \underline{T}(s) = \frac{\underline{c}'(s)}{|c'(s)|} = \frac{1}{\sqrt{a^2+b^2}}\left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), -b, -a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right)\right)$$

$$\Rightarrow \frac{d\underline{T}}{ds} = \frac{1}{\sqrt{a^2+b^2}}\left(\frac{-a}{\sqrt{a^2+b^2}}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0, \frac{-a}{\sqrt{a^2+b^2}}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right)\right)$$

$$= \frac{-a}{a^2+b^2}\left(\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0, \cos\left(\frac{s}{\sqrt{a^2+b^2}}\right)\right)$$

So curvature is

$$\kappa(t) = \left|\frac{d\underline{T}}{ds}\right| = \frac{a}{a^2+b^2}\sqrt{\sin^2\left(\frac{s}{\sqrt{a^2+b^2}}\right) + \cos^2\left(\frac{s}{\sqrt{a^2+b^2}}\right)}$$

$$= \frac{a}{a^2+b^2}$$

Q3 Let $\underline{F}(x,y,z) = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$ be a C^2 vector field.

$$\underline{\nabla}\times(\underline{\nabla}\times\underline{F}) = \underline{\nabla}\times\left[\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\underline{i} - \left(\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right)\underline{j} + \left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\underline{k}\right]$$

$$= \det\begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y} \end{bmatrix}$$

$$= \underline{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right)\right]$$
$$-\underline{j}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\right]$$
$$+\underline{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)\right]$$

$$= \underline{i}\left[\frac{\partial^2 F_2}{\partial y\partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F_3}{\partial z\partial x}\right]$$
$$-\underline{j}\left[\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x\partial y} - \frac{\partial^2 F_3}{\partial z\partial y} + \frac{\partial^2 F_2}{\partial z^2}\right]$$
$$+\underline{k}\left[\frac{\partial^2 F_1}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial x^2} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y\partial z}\right]$$

$$= \underline{i}\left[\left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial y\partial x} + \frac{\partial^2 F_3}{\partial z\partial x}\right) - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2}\right)\right]$$
$$+\underline{j}\left[\left(\frac{\partial^2 F_1}{\partial x\partial y} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial z\partial y}\right) - \left(\frac{\partial^2 F_2}{\partial x^2} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_2}{\partial z^2}\right)\right]$$
$$+\underline{k}\left[\left(\frac{\partial^2 F_1}{\partial x\partial z} + \frac{\partial^2 F_2}{\partial y\partial z} + \frac{\partial^2 F_3}{\partial z^2}\right) - \left(\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_3}{\partial z^2}\right)\right]$$

$$= \underline{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \nabla^2 F_1\right]$$
$$+\underline{j}\left[\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \nabla^2 F_2\right]$$
$$+\underline{k}\left[\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \nabla^2 F_3\right]$$

Since $\underline{F}$ is C^2 the order of differentiation is not important in the mixed partial derivatives. Hence

$$\frac{\partial^2 F_1}{\partial x \partial y} = \frac{\partial^2 F_1}{\partial y \partial x} \quad , \quad \frac{\partial^2 F_1}{\partial x \partial z} = \frac{\partial^2 F_1}{\partial z \partial x}$$

$$\frac{\partial^2 F_2}{\partial x \partial y} = \frac{\partial^2 F_2}{\partial y \partial x} \quad , \quad \frac{\partial^2 F_2}{\partial y \partial z} = \frac{\partial^2 F_2}{\partial z \partial y}$$

$$\frac{\partial^2 F_3}{\partial x \partial z} = \frac{\partial^2 F_3}{\partial z \partial x} \quad , \quad \frac{\partial^2 F_3}{\partial y \partial z} = \frac{\partial^2 F_3}{\partial z \partial y}$$

Then

$\underline{\nabla} \times (\underline{\nabla} \times \underline{F})$

$$= \left[\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\right]\underline{i} + \left[\frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\right]\underline{j} + \left[\frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)\right]\underline{k}$$

$$- \left[\nabla^2 F_1 \underline{i} + \nabla^2 F_2 \underline{j} + \nabla^2 F_3 \underline{k}\right]$$

$$= \underline{\nabla}\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \nabla^2 \underline{F}$$

$$= \underline{\nabla}\,(\underline{\nabla} \cdot \underline{F}) - \nabla^2 \underline{F}$$