MAST30001 Stochastic Modelling

Tutorial Sheet 9

1. A rent-a-car washing facility can wash one car at a time. Cars arrive to be washed according to a Poisson process with rate 3 per day and the service time to wash a car is exponential with mean 7/24 days. It costs the company \$150 per day to operate the facility and the company loses \$10 per day for each car tied up in the washing facility. The company can increase the rate of washing to get down to a mean service time of 1/4 days at the additional cost of \$$C$ per day. What's the largest C can be for this upgrade to make economic sense?

2. Customers arrive at a bank according to a Poisson process rate λ. The bank's service policy is that

   - if there are fewer than 4 customers in the bank, then there is 1 teller,
   - if there are $4-9$ customers, there are 2 tellers,
   - if there are more than 9 customers, there are 3 tellers.

   Tellers' service times are independent and exponentially distributed with rate μ. Model the number of customers in the bank as a birth and death chain and determine for what values of λ and μ there is stable long run behavior and for these parameters compute the steady state distribution. *[Hint: This is similar to the analysis of the M/M/a queue done in lecture.]*

3. (M/M/∞ queue) Assume that in a queuing system customers arrive according to a rate λ Poisson process, customers are always served immediately (for example, customers making purchases on the internet), and the service time of a customer is exponential with rate μ, independent of arrival times and other service times.

   (a) Model this queue as a birth-death chain and write down its generator.

   (b) Describe the long run behaviour of the chain.

   (c) When the queue is in stationary (i.e., after its been running a long time), what is the expected number of customers in the system, number of customers in the queue, number of busy servers, and service time for an arriving customer?

   (d) Let X_t be the number of customers in the system (including those being served) at time t and set $X_0 = 0$. What is $E[X_t]$? *[Hint: if $m(t) = E[X_t]$, consider $m'(t)$.]* You should check your formula makes sense as t tends to infinity.

4. (M/G/∞ queue) In a certain communications system, information packets arrive according to a Poisson process with rate λ per second and each packet is processed in one second with probability p and in two seconds with probability $1-p$, independent of the arrival times and other service times. Let N_t be the number of packets that have entered the system up to time t and X_t be the number of packets in the system (including those being served) at time t.

   (a) Is $(X_t)_{t\geq 0}$ a Markov chain? (No detailed argument is necessary here, just think about it heuristically.)

   (b) If $X_0 = 0$, what is the distribution of X_2?

(c) If $X_0 = 0$, is there a “stationary” limiting distribution $\pi_k = \lim_{t\to\infty} P(X_t = k)$? If so, what is it?

(d) If $X_0 = N_0 = 0$, what is the joint distribution of X_t and N_t?

M/M/1

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c_1

c_2.

λ. ?.

$$\text{rate} = \frac{1}{\text{mean}} = \frac{24}{7}$$

$\lambda_i = 3$.

$\mu_i = \frac{24}{7}$

$$\rho = \frac{\lambda}{\mu} = \frac{3}{\frac{24}{7}} = \frac{7}{8}.$$

$\pi_k \sim \text{Geo}(\frac{1}{8})$.

$$E[\pi_k] = \frac{1-\frac{1}{8}}{\frac{1}{8}} = 7.$$

$$L = \sum_{k \geq 1} k\pi_k$$

$$= E[\text{Geo}(\tfrac{1}{8})] = 7.$$

$$\pi_0 = (1 - \tfrac{1}{8}) = \tfrac{7}{8}.$$

old cost $150 + 10L = 150 + 10 \times 7 = 220$ per day.

after $\lambda_i = 3$.

$$\mu_i = \frac{1}{\frac{1}{4}} = 4$$

$\rho = \frac{\lambda}{\mu} = \frac{3}{4}$ $\quad \pi_k \sim \text{Geo}(\frac{1}{4})$ $\quad E(\pi_k) = \frac{\frac{3}{4}}{\frac{1}{4}} = 3.$

new cost $150 + 10 \times 3 + C = 180 + C < 220$

$C_{max} = 40$

2. Customers arrive at a bank according to a Poisson process rate λ. The bank's service policy is that

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$$
\begin{array}{c|cccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
0 & -\lambda & \lambda & & & & & & & & \\
1 & \mu & -(\mu+\lambda) & \lambda & & & & & & & \\
2 & & \mu & -(\mu+\lambda) & \lambda & & & & & & \\
3 & & & \mu & -(\mu+\lambda) & \lambda & & & & & \\
4 & & & & 2\mu & -(2\mu+\lambda) & \lambda & & & & \\
5 & & & & & & \ddots & & & & \\
6 & & & & & & & & & & \\
7 & & & & & & & & & & \\
8 & & & & & & & & & & \\
9 & & & & & & & & & 3\mu & -(3\mu+\lambda) \quad \lambda \\
 & & & & & & & & & & 3\mu \qquad \lambda
\end{array}
$$

use $\underline{\pi}A=0 \Rightarrow$.

$$\pi_k = \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}\pi_0 = \begin{cases} \left(\frac{\lambda}{\mu}\right)^k \pi_0 & k<4 \\ \left(\frac{\lambda}{\mu}\right)^k \pi_0 \left(\frac{1}{2}\right)^{k-3} & 4\le k\le 9. \\ \left(\frac{\lambda}{\mu}\right)^k \pi_0 \left(\frac{1}{64}\right)\cdot\left(\frac{1}{3}\right)^{k-9} & 9<k \end{cases}$$

$$\sum \pi_k = 1 = \sum_{k=0}^{3}\left(\frac{\lambda}{\mu}\right)^k\pi_0 + \sum_{k=4}^{9}\left(\frac{\lambda}{\mu}\right)^k\pi_0\left(\frac{1}{2}\right)^{k-3} + \sum_{k=10}^{\infty}\left(\frac{\lambda}{\mu}\right)^k\cdot\pi_0\left(\frac{1}{64}\right)\underbrace{\left(\frac{1}{3}\right)^{k-9}}_{\left(\frac{\lambda}{3\mu}\right)^k\cdot\left(\frac{1}{3}\right)^{-9}}.$$

converge $\frac{\lambda}{3\mu}<1$

$$\pi_0^{-1} = \sum_{k=1}^{3}\left(\frac{\lambda}{\mu}\right)^k + \sum_{k=4}^{9}\left(\frac{\lambda}{\mu}\right)^k\left(\frac{1}{2}\right)^{k-3} + \sum_{k=10}^{\infty}\left(\frac{\lambda}{\mu}\right)^k\frac{1}{64}\left(\frac{1}{3}\right)^{k-9}$$

$$= \sum_{k=0}^{3}\rho^k \qquad 8\sum_{k=4}^{9}\left(\tfrac{1}{2}\rho\right)^k \qquad 2^{-6}3^9\sum_{k=10}^{\infty}\left(\tfrac{1}{3}\rho\right)^k$$

$$\frac{1-\rho^4}{1-\rho} \qquad 8\cdot\frac{\left(\frac{1}{2}\rho\right)^4\cdot\left(1-\left(\frac{1}{2}\rho\right)^6\right)}{1-\frac{1}{2}\rho} \qquad 2^{-6}3^9\cdot\frac{\left(\frac{1}{3}\rho\right)^{10}}{1-\frac{1}{3}\rho}$$

$$= \frac{\rho^4[1-(\frac{1}{2}\rho)^6]}{2-\rho} = \frac{\rho^{10}}{2^6(3-\rho)}$$

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   (a) Model this queue as a birth-death chain and write down its generator.

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   (c) When the queue is in stationary (i.e., after its been running a long time), what is the expected number of customers in the system, number of customers in the queue, number of busy servers, and service time for an arriving customer?

   (d) Let X_t be the number of customers in the system (including those being served) at time t and set $X_0 = 0$. What is $E[X_t]$? *[Hint: if $m(t) = E[X_t]$, consider $m'(t)$.]* You should check your formula makes sense as t tends to infinity.

(a). suppose now i customers.

$a_{i,i+1} = \lambda$. $\quad a_{i,i} = -(\lambda + \mu i)$

$\boxed{a_{i,i-1} = i\mu}$. $\quad a_{00} = -\lambda$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ 0 & -\lambda & \lambda & & \\ 1 & \mu & -(\mu+\lambda) & \lambda & \\ 2 & & 2\mu & -(2\mu+\lambda) & \lambda \\ 3 & & & 3\mu & -(3\mu+\lambda) \quad \lambda. \end{array}$$

(b). $\pi A = 0$

$$\pi_k = \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} \pi_0 = \frac{\lambda^k}{k!\mu^k}\pi_0 = \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \pi_0$$

$$\sum_{k\geq 0} \pi_k = 1 \Rightarrow \pi_0 \sum_{k\geq 0} \frac{(\frac{\lambda}{\mu})^k}{k!} = 1 \qquad \pi_0 e^{\frac{\lambda}{\mu}} = 1$$

$$\pi_0 = e^{-\frac{\lambda}{\mu}}.$$

$$\therefore \pi_k = e^{-\frac{\lambda}{\mu}} \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k$$

(c) When the queue is in stationary (i.e., after its been running a long time), what is the expected number of customers in the system, number of customers in the queue, number of busy servers, and service time for an arriving customer?

(c) $E(X) = \sum_{k \geq 0} k \pi_k$

$$= \sum_{k \geq 0} e^{-\rho} \cdot \frac{1}{k!} \rho^k \cdot k$$

$$= e^{-\rho} \sum_{k \geq 1} \frac{\rho^{k-1} \rho}{(k-1)!}$$

$$= e^{-\rho} \cdot \rho e^{\rho} = \rho$$

since it is $M/M/\infty$ queue

so customer get serviced immediately

$L_q = 0$

number of busy servers = expected number of customer $= \rho$

number of customer in the system = mean(ρ).

service time $= \frac{1}{\mu}$

(d) Let X_t be the number of customers in the system (including those being served) at time t and set $X_0 = 0$. What is $E[X_t]$? *[Hint: if $m(t) = E[X_t]$, consider $m'(t)$.]* You should check your formula makes sense as t tends to infinity.

$m(t) = E[X_t]$

$P(X_t = n \mid X_0 = 0) = (P^{(n)})_{0t}$

using forward equation

$$\frac{d}{dt} p^{(t)}_{0,n} = \sum_j p^{(t)}_{0,j} a_{j,n}$$

$a_{i,i+1} = \lambda$

$a_{i,i-1} = i\mu$

$a_{i,i} = -(\lambda + i\mu)$.

$$\frac{d}{dt}P_{0,n}^{(t)} = (P^{(t)})_{0,n-1}\, a_{n-1,n} + (P^{(t)})_{0,n}\, a_{n,n} + (P^{(t)})_{0,n+1} \cdot a_{n+1,n}$$

① $\frac{d}{dt}P_{0,n}(t) = P_{0,n-1}(t)\lambda - P_{0,n}(t)[(\lambda + n\mu)] + P_{0,n+1}(t)\, n\mu$

② $\frac{d}{dt}P_{0,n-1}(t) = P_{0,n-2}(t)\lambda - P_{0,n-1}(t)[\lambda + (n-1)\mu] + P_{0,n}(t)(n-1)\mu$

③ $\boxed{m'(t) = \lambda - \mu m(t).}$

$m'(t) + \mu m(t) = \lambda$

$I(t) = \exp\left(\int \mu\, dt\right) = e^{\mu t}$

$e^{\mu t} m'(t) + \mu e^{\mu t} m(t) = \lambda e^{\mu t}$

$\frac{d}{dt}\left(m(t) \cdot e^{\mu t}\right) = \lambda e^{\mu t}.$

$m(t) \cdot e^{\mu t} = \frac{\lambda}{\mu} e^{\mu t} + C.$

$m(t) = \frac{\lambda}{\mu} + C \cdot e^{-\mu t}$

$m(0) = 1$

$\frac{\lambda}{\mu} + C = 0$

$C = -\frac{\lambda}{\mu}.$

$\therefore m(t) = \frac{\lambda}{\mu} - \frac{\lambda}{\mu} e^{-\mu t} = \frac{\lambda}{\mu}(1 - e^{-\mu t}).$

Po(λt).

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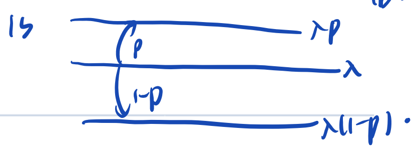
(b) If $X_0 = 0$, what is the distribution of X_2?



(b). by thining theorem, then $(A_t)_{t\geq 0}$ is package processed in 1s, and $(B_t)_{t\geq 0}$ is package processed in 2s.

since A_t and B_t are independent

so. the total process is Poisson with rate $\lambda(2-p)$

(c) If $X_0 = 0$, is there a "stationary" limiting distribution $\pi_k = \lim_{t\to\infty} P(X_t = k)$? If so, what is it?

(d) If $X_0 = N_0 = 0$, what is the joint distribution of X_t and N_t?

(c).