THE UNIVERSITY OF MELBOURNE

Semester 2 Assessment, 2015

School of Mathematics and Statistics

MAST30001 Stochastic Modelling

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam.

This paper consists of 4 pages (including this page)

Authorised materials:

- Students may bring one double-sided A4 sheet of handwritten notes into the exam room.
- Hand-held electronic scientific (but not graphing) calculators may be used.

Instructions to Students

- You may remove this question paper at the conclusion of the examination.
- This paper has **7 questions**. Attempt as many questions, or parts of questions, as you can. The number of marks allocated to each question is shown in the brackets after the question statement. There are **80 total marks** available for this examination. A table of **normal distribution probabilities** can be found at the end of the exam. Working and/or reasoning must be given to obtain full credit. Clarity, neatness and style count.

Instructions to Invigilators

- Students may remove this question paper at the conclusion of the examination.

1. (a) Analyse the state space $S = \{1, 2, 3, 4\}$ for each of the three Markov chains given by the following transition matrices. That is, write down the communication classes and their periods, label each class as essential or not, and as transient or positive recurrent or null recurrent.

   i.
$$\begin{pmatrix} 1/6 & 5/6 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   ii.
$$\begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 3/5 & 2/5 & 0 \end{pmatrix}.$$

   iii.
$$\begin{pmatrix} 1/6 & 1/3 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{pmatrix}.$$

   (b) For the Markov chain given by the transition matrix in part iii above, discuss the long run behaviour of the chain including deriving long run probabilities.

   (c) For the Markov chain given by the transition matrix in part iii above, find the expected number of steps taken for the chain to first reach state 3 given the chain starts at state 1.

   (d) For the Markov chain given by the transition matrix in part iii above, find the expected number of steps taken for the chain to first return to state 2 given the chain starts at state 2.

[15 marks]

2. A continuous time Markov chain $(X_t)_{t\geq 0}$ has generator

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

The states are labelled $\{1, 2, 3\}$. Use the Kolmogorov forward equation to find $p_{1,j}(t) = P(X_t = j | X_0 = 1)$ for $j = 1, 2, 3$.

[6 marks]

3. A Markov chain $(X_n)_{n\geq 0}$ on $\{0, 1, 2, \ldots\}$ has transition probabilities for $i = 0, 1, 2, \ldots$,

$$p_{i,i+1} = 1 - p_{i,0} = e^{-(i+1)^{-\alpha}},$$

where we only consider $\alpha \geq 1$. This chain is irreducible. For which values of α is the chain transient? Null recurrent? Positive recurrent?

[6 marks]

4. A petrol station in the country is at the intersection of two roads, one running north-south and the other east-west. Cars drive from the north-south road according to a Poisson process with rate 3 per hour and from the east-west road according to an independent Poisson process with rate 5 per hour. Cars that drive by stop at the petrol station independently, with probability 1/10 if they are coming from the north-south road and with probability 1/20 if they are coming from the east-west road.

   (a) What is the chance that exactly five cars from the north-south road drive by the petrol station between 9am and 11am?

   (b) What is the chance that exactly five cars from the north-south road stop at the petrol station between 9am and 11am?

   (c) What is the expected amount of time between when the petrol station opens at 7am and the first car from either road drives by?

   (d) What is the expected amount of time between when the petrol station opens at 7am and the first car from either road stops?

   (e) Given exactly five cars from the north-south road have stopped at the petrol station between 9am and 11am, what is the chance that exactly two cars from the north-south road have driven by and not stopped in that time period?

   (f) Given exactly five cars from the north-south road have stopped at the petrol station between 9am and 11am, what is the chance that exactly two cars in total (that is, from either road) have driven by and not stopped in that time period?

   (g) Given that exactly five cars from the north-south road have stopped at the petrol station between 9am and 11am, what is the chance that exactly two of those five arrived between 9am and 9:30am?

   (h) Given that exactly five cars from the north-south road have stopped at the petrol station between 9am and 11am, what is the chance that exactly two cars in total (that is, from either road) arrived between 9am and 9:30am?

[15 marks]

5. At a certain tram stop, trams run from 5am to 1am. We model the times of arrival of trams starting from 5am as a renewal process with inter-arrival times, in minutes, uniform on the interval $(0, 20)$.

   (a) Compute the mean and variance of the inter-arrival distribution.

   (b) On average, about how many trams arrive at the stop between 5am and 5pm?

   (c) Give an interval around your estimate from (b) that will have a 95% chance of covering the true number of trams that arrive at the stop over the course of the 12 hours.

   (d) If you arrive at the tram stop at 10pm, what would you estimate to be the mean and variance of the time until the next tram arrives?

[11 marks]

6. In a certain computer system, jobs arrive according to a Poisson process with rate λ. There are two servers that process jobs, Server A works at exponential rate μ_A and Server B at exponential rate $\mu_B < \mu_A$. Since Server A is faster than Server B, the system works as follows. When there is one job in the system, Server A processes it, and if there is more than one job, both servers process separate jobs. If there are exactly two jobs in the system and Server A finishes its job before the arrival of an additional job, then Server B instantly passes its job to Server A to process. When both servers are busy jobs queue in an infinite buffer.

   (a) Model the number of jobs in the system (including those being worked on) as a continuous time Markov chain $(X_t)_{t\geq 0}$, and write down its generator.

   (b) Determine for what parameters λ, μ_A, μ_B, the Markov chain is ergodic and for these values write down the steady state distribution.

   Assume for the rest of the problem that the parameters satisfy the constraints of part (b) so that a steady state exists.

   (c) What is the average number of jobs in the system?

   (d) What is the average number of jobs waiting in the queue (that is, in the system but not in service)?

   (e) What is the average amount of time an arriving job waits for service?

   (f) What is the average amount of time an arriving job is in the system?

   (g) What proportion of time is Server B idle?

[15 marks]

7. Let $(B_t)_{t\geq 0}$ and $(\widetilde{B}_t)_{t\geq 0}$ be two independent standard Brownian motions, let $-1 < \rho < 1$, and for $t \geq 0$ set $W_t = \rho B_t + \sqrt{1-\rho^2}\widetilde{B}_t$.

   (a) Use the axioms of Brownian motion to show that $(W_t)_{t\geq 0}$ is a standard Brownian motion.

   (b) Find $P(W_2 \leq 0 | W_1 = 1)$.

   (c) Find $P(W_1 \leq 0 | W_2 = 1)$.

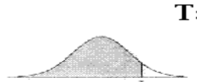
   For the remaining parts of the problem, set $\rho = \sqrt{2}/3$.

   (d) Find $P(W_2 \leq -1 | B_1 = 0)$.

   (e) Find $P(W_1 \leq 0 | B_2 = -1)$.

[12 marks]

Tables of the Normal Distribution



Probability Content from -oo to Z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015

End of Exam