Workshop 7

COMP20008 Elements of Data Processing

Learning outcomes

By the end of this class, you should be able to:

- explain the meaning of *correlation* and how it is measured
- compute the *Pearson correlation coefficient* by hand and in Python
- compute *mutual information* by hand and in Python

Review: What is correlation?

- A statistical relationship between two variables
- Doesn't have to be a *linear* relationship
- Can be measured in different ways, e.g.
  - mutual information
  - Pearson correlation coefficient
- E.g. income and education are correlated

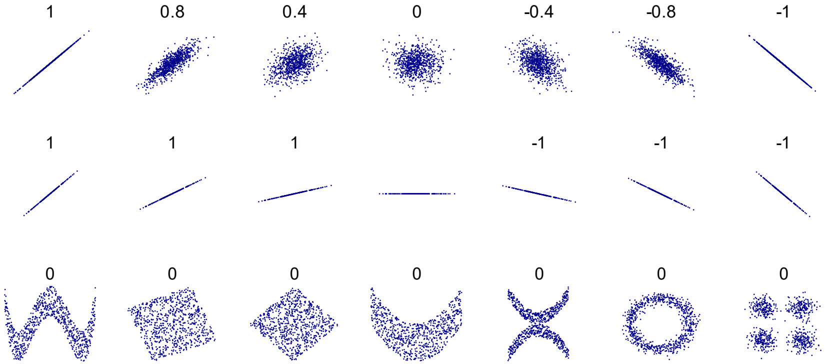


Image source: Wikimedia Commons

Pearson correlation coefficient (PCC)

Statistical definition (not examinable)

The PCC for a pair of random variables X and Y is

$$\rho_{XY} = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)}\,\sqrt{\mathrm{var}(Y)}}$$

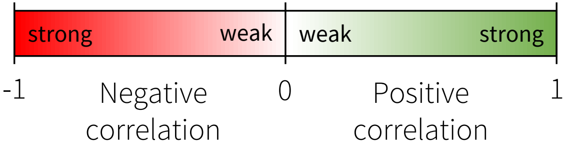
Sample definition

Given observations $\{(x_i, y_i)\}_{i=1\ldots n}$ the sample PCC is

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\,\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

- Measures *linear* correlation
- Many interpretations
  - standardised covariance
  - standardised slope of regression line



strong weak weak strong

-1 Negative correlation 0 Positive correlation 1

Q1: PCC

Compute the PCC between *average steps per day* ($\boldsymbol{X}$) and *average resting heart rate* ($\boldsymbol{Y}$)

Person ID	Avg. Steps per day (x_i)	Avg. resting heart rate (y_i)
1	1000	100
2	2500	105
3	3000	80
4	5000	77
5	6000	74
6	9000	70
7	11000	65
8	14000	63
9	18000	62
10	19000	61
11	19500	60.5
12	22000	55

Q1: PCC

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

Compute the PCC between *average steps per day* (***X***) and *average resting heart rate* (***Y***)

Person ID	Avg. Steps per day (x_i)	Avg. resting heart rate (y_i)
1	1000	100
2	2500	105
3	3000	80
4	5000	77
5	6000	74
6	9000	70
7	11000	65
8	14000	63
9	18000	62
10	19000	61
11	19500	60.5
12	22000	55

Step 1:

Compute the means

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{130000}{12} \approx 10833.3$$

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{872.5}{12} \approx 72.7083$$

Q1: PCC

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

Step 2:

Compute the sums

Person ID	Avg. Steps per day (x_i)	Avg. resting heart rate (y_i)	$x_i - \bar{x}$	$y_i - \bar{y}$	$(x_i - \bar{x})^2$	$(y_i - \bar{y})^2$	$(x_i - \bar{x}) \times (y_i - \bar{y})$
1	1000	100	-9833.33	27.2917	9.66944e7	744.835	-2.68368e5
2	2500	105	-8333.33	32.2917	6.94444e7	1042.75	-2.69097e5
3	3000	80	-7833.33	7.29167	6.13611e7	53.1684	-5.71181e4
4	5000	77	-5833.33	4.29167	3.40278e7	18.4184	-2.50347e4
5	6000	74	-4833.33	1.29167	2.33611e7	1.66840	-6.24306e3
6	9000	70	-1833.33	-2.70833	3.36111e6	7.33507	4.96528e3
7	11000	65	166.667	-7.70833	2.77778e4	59.4184	-1.28472e3
8	14000	63	3166.67	-9.70833	1.00277e7	94.2517	-3.07431e4
9	18000	62	7166.67	-10.7083	5.13611e7	114.668	-7.67431e4
10	19000	61	8166.67	-11.7083	6.66944e7	137.085	-9.56181e4
11	19500	60.5	8666.67	-12.2083	7.51111e7	149.043	-1.05806e5
12	22000	55	11166.7	-17.7083	1.24694e8	313.585	-1.97743e5
Sum	130000	872.5	0	0	6.16167e8	2736.23	-1.12883e6

Q1: PCC

Step 3:

Substitute intermediate results into formula

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}} = \frac{-1.12883 \times 10^6}{\sqrt{6.16167 \times 10^8}\sqrt{2736.23}}$$

$$= -0.8694$$

Q2: Interpretation of PCC

Does a sample PCC of −0.8694 imply doing *more* steps per day will cause one's resting heart rate to *decrease*?

X.

high relation

no causality

Q2: Interpretation of PCC

Does a sample PCC of -0.8694 imply doing *more* steps per day will cause one's resting heart rate to *decrease*?

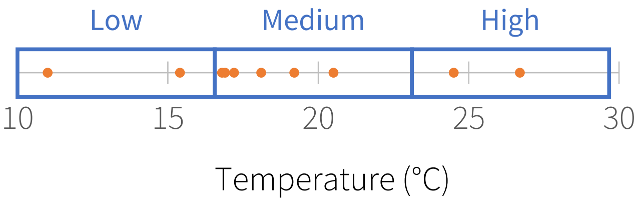
- There's a strong negative correlation, but correlation *does not* imply causation
- There might be a *confounding variable* responsible for the correlation

  e.g. high blood pressure might cause high heart rate and less physical activity
- Also need to be careful about the data. Is there sufficient data? Is it unbiased?

Discretisation

- Convert continuous variables to discrete variables
- Useful for density estimation (we'll use it to estimate mutual information)
- Various methods:
  - Equal frequency binning
  - Equal width binning
  - Custom method based on domain knowledge

ID	Temp	Temp (Disc)
1	15.4	Low
2	16.9	Medium
3	20.5	Medium
4	24.5	High
5	18.1	Medium
6	17.2	Medium
7	16.8	Medium
8	19.2	Medium
9	11	Low
10	26.7	High



Q3: Equal-frequency discretisation

Apply 3-bin equal-frequency discretisation to *average steps per day* ($\boldsymbol{X}$) and 4-bin equal-frequency discretisation to *average resting heart rate* ($\boldsymbol{Y}$).

Show the values of the discretised features.

Q3: Equal-frequency discretisation

Person ID	Avg. Steps per day (X)
1	1000
2	2500
3	3000
4	5000
5	6000
6	9000
7	11000
8	14000
9	18000
10	19000
11	19500
12	22000

Step 1:
Sort column

Person ID	Avg. Steps per day (X)
1	1000
2	2500
3	3000
4	5000
5	6000
6	9000
7	11000
8	14000
9	18000
10	19000
11	19500
12	22000

Step 2:
Split rows
into 3 blocks

Person ID	Avg. Steps per day (X)	X (discrete)
1	1000	1
2	2500	1
3	3000	1
4	5000	1
5	6000	2
6	9000	2
7	11000	2
8	14000	2
9	18000	3
10	19000	3
11	19500	3
12	22000	3

Q3: Equal-frequency discretisation

Person ID	Avg. resting heart rate (Y)
1	100
2	105
3	80
4	77
5	74
6	70
7	65
8	63
9	62
10	61
11	60.5
12	55

Step 1:
Sort column

Person ID	Avg. resting heart rate (Y)
12	55
11	60.5
10	61
9	62
8	63
7	65
6	70
5	74
4	77
3	80
1	100
2	105

Step 2:
Split rows
into 4 blocks

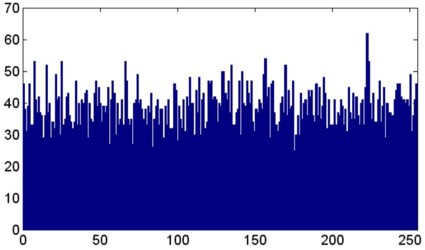
Person ID	Avg. resting heart rate (Y)	Y (discrete)
12	55	1
11	60.5	1
10	61	1
9	62	2
8	63	2
7	65	2
6	70	3
5	74	3
4	77	3
3	80	4
1	100	4
2	105	4

Q3: Equal-frequency discretisation

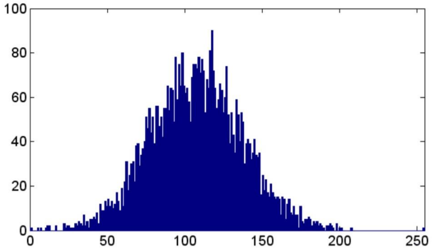
Person ID	Avg. Steps per day (X)	Avg. resting heart rate (Y)	X (discrete)	Y (discrete)
1	1000	100	1	4
2	2500	105	1	4
3	3000	80	1	4
4	5000	77	1	3
5	6000	74	2	3
6	9000	70	2	3
7	11000	65	2	2
8	14000	63	2	2
9	18000	62	3	2
10	19000	61	3	1
11	19500	60.5	3	1
12	22000	55	3	1

Entropy

- Scalar quantity $\mathrm{H}(X)$ associated with a random variable X
- Interpretation: measures the average level of “information” / “surprise” / “uncertainty” in the outcomes of X



High entropy



Low entropy

Image source: Vazquez-Fernandez et al. (2010). Entropy of Gabor Filtering for Image Quality Assessment. Lecture Notes in Computer Science. 52-61.

Entropy of a discrete random variable

Entropy:

$$\mathrm{H}(X) = -\sum_{x \in \mathcal{X}} p_x \log p_x$$

Conditional entropy:

$$\mathrm{H}(Y|X) = \sum_{x \in \mathcal{X}} p_x \mathrm{H}(Y|X = x)$$

Interpretation: average information given we know the outcome of X

where $\boldsymbol{p_x}$ is the relative frequency of category $\boldsymbol{x}$

Q4: Entropy

Compute $\mathrm{H}(X)$, $\mathrm{H}(Y)$, $\mathrm{H}(Y|X)$ and $\mathrm{H}(X|Y)$ where X is the discretised avg. steps per day and Y is the discretised avg. resting heart rate

Person ID	X (discrete)	Y (discrete)
1	1	4
2	1	4
3	1	4
4	1	3
5	2	3
6	2	3
7	2	2
8	2	2
9	3	2
10	3	1
11	3	1
12	3	1

Q4: Entropy

Compute $\mathrm{H}(X)$, $\mathrm{H}(Y)$, $\mathrm{H}(Y|X)$ and $\mathrm{H}(X|Y)$ where X is the discretised avg. steps per day and Y is the discretised avg. resting heart rate

$$\begin{aligned}
\mathrm{H}(X) &= -\sum_{x=1}^{3} p_x \log p_x \\
&= -\frac{4}{12}\log\frac{4}{12} - \frac{4}{12}\log\frac{4}{12} - \frac{4}{12}\log\frac{4}{12} \\
&= -3\left(\frac{4}{12}\log\frac{4}{12}\right) \\
&= 1.585
\end{aligned}$$

x	p_x
1	4/12
2	4/12
3	4/12

Person ID	X (discrete)	Y (discrete)
1	1	4
2	1	4
3	1	4
4	1	3
5	2	3
6	2	3
7	2	2
8	2	2
9	3	2
10	3	1
11	3	1
12	3	1

Q4: Entropy

$$
\begin{aligned}
\mathrm{H}(Y) &= -\sum\nolimits_{y=1}^{4} p_y \log p_y \\
&= -\frac{3}{12}\log\frac{3}{12} - \frac{3}{12}\log\frac{3}{12} - \frac{3}{12}\log\frac{3}{12} - \frac{3}{12}\log\frac{3}{12} \\
&= -4\left(\frac{3}{12}\log\frac{3}{12}\right) \\
&= 2
\end{aligned}
$$

y	p_y
1	3/12
2	3/12
3	3/12
4	3/12

Person ID	X (discrete)	Y (discrete)
1	1	4
2	1	4
3	1	4
4	1	3
5	2	3
6	2	3
7	2	2
8	2	2
9	3	2
10	3	1
11	3	1
12	3	1

Q4: Entropy

x	p_x
1	4/12
2	4/12
3	4/12

$$\mathrm{H}(Y|X) = \sum_{x=1}^{3} p_x \mathrm{H}(Y|X=x)$$
$$= \frac{4}{12} \times 0.811 + \frac{4}{12} \times 1 + \frac{4}{12} \times 0.811$$
$$= 0.874$$

Person ID	X (discrete)	Y (discrete)
1	1	4
2	1	4
3	1	4
4	1	3
5	2	3
6	2	3
7	2	2
8	2	2
9	3	2
10	3	1
11	3	1
12	3	1

x	y	p_y
1	1	0/4
1	2	0/4
1	3	1/4
1	4	3/4

$$\mathrm{H}(Y|X=1) = -\frac{1}{4}\log\frac{1}{4} - \frac{3}{4}\log\frac{3}{4} = 0.811$$

$$\mathrm{H}(Y|X=2) = -\frac{2}{4}\log\frac{2}{4} - \frac{2}{4}\log\frac{2}{4} = 1$$

x	y	p_y
2	1	0/4
2	2	2/4
2	3	2/4
2	4	0/4

x	y	p_y
3	1	3/4
3	2	1/4
3	3	0/4
3	4	0/4

$$\mathrm{H}(Y|X=1) = -\frac{1}{4}\log\frac{1}{4} - \frac{3}{4}\log\frac{3}{4} = 0.811$$

Q4: Entropy

y	p_y
1	3/12
2	3/12
3	3/12
4	3/12

Person ID	X (discrete)	Y (discrete)
1	1	4
2	1	4
3	1	4
4	1	3
5	2	3
6	2	3
7	2	2
8	2	2
9	3	2
10	3	1
11	3	1
12	3	1

$$\begin{aligned}\mathrm{H}(X|Y) &= \sum_{y=1}^{4} p_y \mathrm{H}(X|Y=y)\\ &= \frac{3}{12}\times 0 + \frac{3}{12}\times 0.918 + \frac{3}{12}\times 0.918 + \frac{3}{12}\times 0\\ &= 2\left(\frac{3}{12}\times 0.918\right)\\ &= 0.459\end{aligned}$$

x	y	p_x
1	4	3/3
2	4	0/3
3	4	0/3

$\mathrm{H}(X|Y=4) = 0$

$\mathrm{H}(X|Y=3) = -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3} = 0.918$

x	y	p_x
1	3	1/3
2	3	2/3
3	3	0/3

x	y	p_x
1	2	0/3
2	2	2/3
3	2	1/3

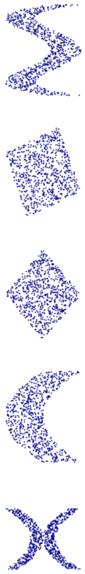
$\mathrm{H}(X|Y=2) = -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3} = 0.918$

$\mathrm{H}(X|Y=1) = 0$

x	y	p_x
1	1	0/3
2	1	0/3
3	1	3/3

Mutual information

- Measures *non-linear* correlation
- $\mathrm{MI}(X, Y)$ denotes the mutual information of X and Y
- Interpretations:
    - the amount of information obtained about X from observing Y
    - the price paid for encoding X, Y as independent variables when they're not
- $\mathrm{MI}(X, Y) \geq 0$
    - A value of 0 means X and Y are independent
    - Larger values indicate stronger dependence



Mutual information

Mutual information can be expressed in terms of entropy $\mathrm{H}(X)$ and conditional entropy $\mathrm{H}(X|Y)$:

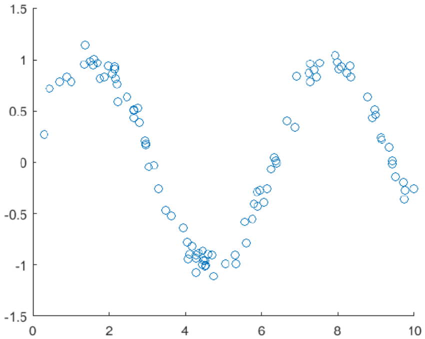
$$\begin{aligned}\mathrm{MI}(X,Y) &= \mathrm{H}(X) - \mathrm{H}(X|Y)\\ &= \mathrm{H}(Y) - \mathrm{H}(Y|X)\end{aligned}$$

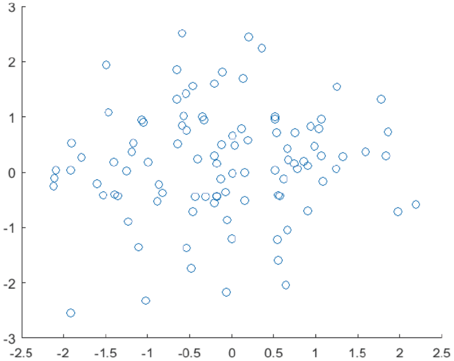
There’s also a normalized version:

Varies between 0 and 1

$$\mathrm{NMI}(X,Y) = \frac{\mathrm{MI}(X,Y)}{\min\{\mathrm{H}(X),\mathrm{H}(Y)\}}$$

Mutual information vs. Pearson correlation





PCC:	-0.086	PCC:	0.08
NMI:	0.43	NMI:	0.009

Q5: Mutual information

Compute the mutual information between discretised *average steps per day* ($\boldsymbol{X}$) and discretised *average resting heart rate* ($\boldsymbol{Y}$)

Q5: Mutual information

Compute the mutual information between discretised *average steps per day* (***X***) and discretised *average resting heart rate* (***Y***)

Substitute earlier results:

$$\mathrm{MI}(X,Y) = \mathrm{H}(X) - \mathrm{H}(X|Y) = 2\ - 0.874 = 1.126$$
$$\mathrm{MI}(X,Y) = \mathrm{H}(Y) - \mathrm{H}(Y|X) = 1.585 - 0.459 = 1.126$$