MAST20009 Vector Calculus

Practice Class 7 Questions

<u>Path Integrals</u>

Let $f(x, y, z)$ be a continuous scalar function and $\boldsymbol{c}(t) = (x(t), y(t), z(t))$ be a C^1 path.

The *path integral* of f along $\boldsymbol{c}$ from $t = a$ to $t = b$ is:

$$\int_{\boldsymbol{c}} f\, ds = \int_a^b f\,[\boldsymbol{c}(t)]\,|\boldsymbol{c}'(t)|\,dt.$$

If f is the mass per unit length of a cable $\boldsymbol{c}$, then $\int_{\boldsymbol{c}} f ds$ is the total mass of the cable.

1. Let $\boldsymbol{c}$ be a thin straight cable joining $(2, 1, 6)$ to $(3, 0, 1)$.

(a)

$\vec{c} = (\vec{u_0}) + t(\vec{u} - \vec{u_0}) = (2,1,6) + t\big((3,0,1) - (2,1,6)\big)$

$= (2,1,6) + t(1,-1,-5)$

$= (2+t, 1-t, 6-5t)$

$0 \le t \le 1$

   (a) Write down a parametrisation for $\boldsymbol{c}$ in terms of an increasing parameter t.

   (b) If the mass per unit length of $\boldsymbol{c}$ is $\mu(x, y, z) = x + yz$ grams, determine the total mass of the cable.

$|\vec{c}'(t)| = |(1,-1,-5)|$

$= \sqrt{27}$

$\int_0^1 (2+t) + (1-t)(6-5t) \cdot \sqrt{30}\, dt = \sqrt{27}\int_0^1 (2+t) + (5t^2 - 11t + 6)\, dt$

$= \sqrt{27}\int_0^1 (5t^2 - 10t + 8)\, dt$

$= \sqrt{27}\left[\frac{5}{3}t^3 - 5t^2 + 8t\right]_{t=0}^{t=1} = \sqrt{27} \cdot (\frac{5}{3} - 5 + 8)$

$= \cancel{\sqrt{30}} \cdot (\frac{5}{3} + 3)$

$= \cancel{\frac{14\sqrt{30}}{3}}\ \sqrt{27}$

<u>Line Integrals</u>

Let $\boldsymbol{F}(x, y, z)$ be a continuous vector field and $\boldsymbol{c}(t) = (x(t), y(t), z(t))$ be a C^1 path.

The *line integral* of $\boldsymbol{F}$ along $\boldsymbol{c}$ from $t = a$ to $t = b$ is:

$$\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_a^b \boldsymbol{F}\,[\boldsymbol{c}(t)] \cdot \boldsymbol{c}'(t)\,dt.$$

If $\boldsymbol{F}$ is a force, then $\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s}$ is the work done by $\boldsymbol{F}$ to move a particle along $\boldsymbol{c}$.

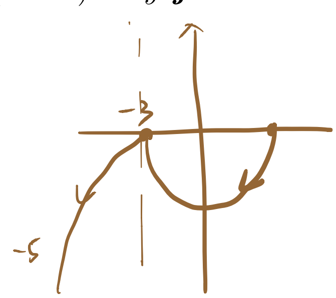
2. Let $\boldsymbol{c}$ be the path consisting of the circle $x^2+y^2 = 9$ traversed in a clockwise direction, starting at $(3, 0)$ and finishing at $(-3, 0)$, followed by the parabola $y = -(x + 3)^2$ starting at $(-3, 0)$ and finishing at $(-5, -4)$.

   Calculate the work done by the force $\boldsymbol{F}(x, y) = (x - 2)\boldsymbol{i} - y^2\boldsymbol{j}$ to move a particle along $\boldsymbol{c}$.

$\int_0^{\pi} (-9\cos t \sin t)\, dt$

$= \int_0^{\pi} (-9 \sin t)\, d(\sin t)$

$= \int_0^{\pi} (-9u)$



Tangents and Normals to Surfaces

Let S be a differentiable surface. Consider curves on S given by:

$\boldsymbol{c}_1 = \Phi(u_0, v)$ — u is constant

$\boldsymbol{c}_2 = \Phi(u, v_0)$ — v is constant

- $\boldsymbol{T}_v$ is the *tangent* vector to $\boldsymbol{c}_1$ at $\Phi(u_0, v_0)$

$$\boldsymbol{T}_v = \left.\frac{d\boldsymbol{c}_1}{dv}\right|_{v=v_0} = \left.\left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)\right|_{(u_0,v_0)}$$

- $\boldsymbol{T}_u$ is the *tangent* vector to $\boldsymbol{c}_2$ at $\Phi(u_0, v_0)$

$$\boldsymbol{T}_u = \left.\frac{d\boldsymbol{c}_2}{du}\right|_{u=u_0} = \left.\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)\right|_{(u_0,v_0)}$$

- There are 2 *normal* vectors to surface at (x_0, y_0, z_0).

$$\boldsymbol{n} = \boldsymbol{T}_u \times \boldsymbol{T}_v \qquad \text{OR} \qquad \boldsymbol{n}' = -\boldsymbol{n} = \boldsymbol{T}_v \times \boldsymbol{T}_u$$

- If $\boldsymbol{n} \neq \boldsymbol{0}$, the surface is *smooth*.

If S is a smooth surface, then the *tangent plane* to S at (x_0, y_0, z_0) is

$$(x - x_0, y - y_0, z - z_0) \cdot \left.\boldsymbol{n}\right|_{(u_0,v_0)} = 0.$$

3. Let S be the surface of the paraboloid $z = 3x^2 + 3y^2 - 4$ for $z \leq 5$.

   (a) Write down a parametrisation for S in terms of u and v, based on cylindrical coordinates.

   (b) Find the point (u, v) that corresponds to the point $(x, y, z) = \left(\frac{1}{4}, \frac{\sqrt{3}}{4}, -\frac{13}{4}\right)$.

   (c) Find the tangent vectors $\boldsymbol{T}_u$ and $\boldsymbol{T}_v$ to the surface.

   (d) Find the outward normal vector to the surface in terms of u and v.

   (e) Find the equation of the tangent plane to S at $\left(\frac{1}{4}, \frac{\sqrt{3}}{4}, -\frac{13}{4}\right)$.

When you have finished the above questions, continue working on the questions in the Vector Calculus Problem Sheet Booklet.