MAST20009 Vector Calculus

Practice Class 8 Questions

<u>Integrals of scalar functions over surfaces</u>

Let $f(x, y, z)$ be a continuous function defined on a smooth parametrised surface S. We define

$$\iint_S f dS = \iint_D f\left[\Phi(u, v)\right] \; |\boldsymbol{T}_u \times \boldsymbol{T}_v| \, du \, dv$$

where $\Phi : D \to S$ and $\Phi(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$.

1. Let S be the surface of the hemisphere $x^2 + y^2 + z^2 = 2, \; z \geq 0$.

   (a) Using spherical coordinates, determine a parametrisation for S.

   (b) Using part (a), evaluate

   $$\iint_S z \, dS.$$

<u>A special case</u>

If the surface S can be written as $z = f(x, y)$, then

$$\iint_S g \, dS = \iint_D \frac{g[x, y, f(x, y)]}{\left|\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}\right|} \, dx \, dy$$

where $\hat{\boldsymbol{n}}$ is the unit normal to S.

2. Using the special case formula for $z = f(x, y)$, evaluate

   $$\iint_S x + y + 2z \, dS$$

   where S is the triangle with vertices $(2, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 2)$.

<u>Integrals of vector fields over surfaces</u>

Let $\boldsymbol{F}(x, y, z)$ be a continuous vector field defined on a smooth, orientable parametrised surface S. We define

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot \hat{\boldsymbol{n}} \, dS = \iint_D \boldsymbol{F} \cdot (\boldsymbol{T}_u \times \boldsymbol{T}_v) \, du \, dv$$

where $\Phi : D \to S$ and $\Phi(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$

and $\hat{\boldsymbol{n}}$ is the unit outward normal to S.

3. Let S be the surface of the open cone $z = \sqrt{x^2 + y^2}$, $z \leq 1$, oriented with outward unit normal.

   (a) Using cylindrical coordinates, determine a parametrisation for S.

   (b) Using part (a), evaluate

   $$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

   where $\boldsymbol{F}(x, y, z) = 3x\boldsymbol{i} - 2\boldsymbol{j} + y^2\boldsymbol{k}$.

When you have finished the above questions, continue working on the questions in the Vector Calculus Problem Sheet Booklet.