Semester 2, 2013 MAST20009 Vector Calculus Exam Answers

1. (a) Approach the origin along paths of the form $y = kx$ to show the limit does not exist.

   (b) Use the Sandwich theorem to prove that the limit is 0. Then, H is continuous at $(0, 0)$.

2. Find minimum of $f(x, y) = \sqrt{x^2 + y^2}$ subject to constraint $g(x, y) = x^2y - 16 = 0$. There is a minimum distance of $\sqrt{12}$ at the points $(\pm 2\sqrt{2}, 2)$.

3. (a) Sketch required.

   (b) $\mathbf{c}(t)$ is not a flowline of $\mathbf{G}$.

   (c) Differentiate $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ with respect to arclength s. Use $\dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$ and $\dfrac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$ to simplify expression.

4. (a) Calculate directly using definitions of divergence and curl.

   (b) (i) Show $\boldsymbol{\nabla} \cdot \mathbf{G} = 0$.

   (ii) If $\mathbf{G}$ is the velocity field of a fluid, then the rate at which fluid flows into a point is the same as the rate at which fluid flows out of the point.

   (iii) Solve coupled pdes $\dfrac{\partial F_2}{\partial z} = -3yz^2$, $\dfrac{\partial F_1}{\partial z} = -2xz$, $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 4xy^3$. One vector field is $\mathbf{F}(x, y, z) = (-xz^2, 2x^2y^3 - yz^3, 0)$.

5. (a) Sketch required.

   (b) Change the order of integration to get $\displaystyle\int_0^9 \int_0^{\sqrt{x}} y\cos(x^2)\, dy\, dx = \frac{1}{4}\sin(81)$.

6. (a) Sketch required.

   (b) Use cylindrical coordinates. Volume is $\displaystyle\int_0^2 \int_0^{2\pi} \int_{2\rho^2-2}^{10-\rho^2} \rho\, dz\, d\phi\, d\rho = 24\pi$ units^3

7. (a) Let $x = 5\sin\theta\cos\phi,\ y = 5\sin\theta\sin\phi,\ z = 5\cos\theta,\ 0 \le \theta \le \pi,\ 0 \le \phi \le \pi$.

   (b) Outward normal is $\big(25\sin^2\theta\cos\phi, 25\sin^2\theta\sin\phi, 25\sin\theta\cos\theta\big)$.

   (c) Tangent plane at $(\theta, \phi) = (\frac{\pi}{4}, 0)$ is $x + z = \frac{10}{\sqrt{2}}$.

8. Using vertical strips, mass of plate is $\displaystyle\iint_P x\, dS = \int_0^2 \int_0^{3-\frac{3x}{2}} \sqrt{14}x\, dy\, dx = 2\sqrt{14}$ grams.

9. (a) Sketch required.

   (b) Use divergence theorem in the plane and polar coordinates to get $\displaystyle\int_{\partial D} \mathbf{F} \cdot \hat{\mathbf{n}}\ ds = \int_0^1 \int_{\frac{\pi}{2}}^{\pi} 3r^3\ d\theta\ dr = \frac{3\pi}{8}.$

10. (a) State Stokes' theorem and all conditions.

(b) (i) Boundary curve is $x^2 + y^2 = 16,\ z = 5$, oriented clockwise.

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} 24 - 24\cos 2t - 20\sin t + 8\sin 2t\ dt = 48\pi.$$

(ii) Simplest surface is $D : x^2 + y^2 \leq 16,\ z = 5$ with $\hat{\mathbf{n}} = -\mathbf{k}$.

$$\iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot d\mathbf{S} = 3 \times \text{area } D = 48\pi.$$

11. (a) Use Gauss' theorem with $\mathbf{F} = f\boldsymbol{\nabla} g$ and vector identity 7.

(b) Let $f(r) = r$ and $g(r) = r^2$ in part (a). Then $\boldsymbol{\nabla} f = \frac{\mathbf{r}}{r},\ \boldsymbol{\nabla} g = 2\mathbf{r},\ \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g = 2r,\ \nabla^2 g = 6.$

12. (a) $h_u = h_v = \sqrt{a^2\sinh^2 u\cos^2 v + a^2\cosh^2 u\sin^2 v},\ h_z = 1.$

(b) $|\text{Jacobian}| = h_u h_v h_z$.

(c) Use the formulae sheet for curvilinear coordinates.

$$\frac{2u\hat{\mathbf{u}}}{\sqrt{a^2\sinh^2 u\cos^2 v + a^2\cosh^2 u\sin^2 v}} + \frac{3v^2 z\hat{\mathbf{v}}}{\sqrt{a^2\sinh^2 u\cos^2 v + a^2\cosh^2 u\sin^2 v}} + v^3\hat{\mathbf{z}}.$$