Student Number: ____________

The University of Melbourne
Semester 1 Assessment 2014

Department of Mathematics and Statistics
MAST20009 VECTOR CALCULUS

Reading Time: 15 minutes

Writing Time: 3 hours

Open Book Status: Closed book

This paper has 6 pages (including this page)

Authorized Materials:

- No materials are authorised.
- No calculators, computers or mobile phones are permitted.

Paper to be held by Baillieu Library: Yes

Instructions to Invigilators:

- Script books only are required.

Instructions to Students:

- This examination consists of ten questions.
- All questions may be attempted.
- Marks for each question are indicated on the paper.
- The total number of marks is 100.
- There are formulas on pages 5 and 6 that you may use in this examination.

Extra Materials Required: None

1. Find the following limits of functions of two variables or show that the limits do not exist. State clearly any theorems that you use.

   (a) $\lim_{(x,y)\to(1,0)} \sqrt{9 - x^2 - e^y}$

   (b) $\lim_{(x,y)\to(0,0)} \dfrac{x^3y^2}{(x^3 + y^2)^2}$

   (c) $\lim_{(x,y)\to(0,0)} \dfrac{e^{3(x+y)} - 1}{2(x + y)}$

10 marks

2. Consider the function of two variables

$$f(x, y) = \frac{1}{1 + x^2 + y^2}$$

   (a) Obtain the partial derivatives $f_x(x, y)$, $f_y(x, y)$, $f_{xx}(x, y)$, $f_{xy}(x, y)$ and $f_{yy}(x, y)$ simplifying your answers as much as possible.

   (b) Hence obtain the second order (quadratic) Taylor polynomial for $f(x, y)$ about $(x, y) = (0, 0)$.

   (c) Use one-variable Taylor series to obtain the Taylor series for $f(x, y)$ about the origin $(x, y) = (0, 0)$. State clearly the domain on which this Taylor series converges.

10 marks

3. (a) The curve C is given by intersecting the ellipsoid

$$x^2 + y^2 + 2z^2 = 1$$

   with the plane

$$x + y + 2z = 0$$

   Use Lagrange multipliers to find the point on C which lies closest to the point P with coordinates $(0, 0, 4)$.

   (b) Consider the C^1 transformation

$$u(x, y) = \tfrac{1}{2}(x + y^2), \qquad v(x, y) = \tfrac{1}{2}(x - y^2), \qquad y > 0$$

   (i) Find the inverse transformation.

   (ii) Verify the Jacobian relation

$$\frac{\partial(u, v)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(u, v)} = 1$$

10 marks

4. A suspended cable takes the shape of a catenary curve given by

$$y = \cosh x = \tfrac{1}{2}(e^x + e^{-x}), \qquad -1 \le x \le 1$$

   (a) Sketch a graph of this curve.

   (b) Using x as a parameter, find the unit tangent vector $\hat{\boldsymbol{t}}$ and hence the curvature $\kappa = \kappa(x)$ of the cable.

   (c) If the linear mass density of the cable is given by $\rho = \dfrac{dM}{ds} = 3 + x$ where s is the arc length, find the total mass M of the cable.

10 marks

5. (a) If $f = f(x, y, z)$ and $g = g(x, y, z)$ are scalar fields in $\mathbb{R}^3$ and ∇ is the vector differential operator nabla, prove directly from first principles the vector identity

$$\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$$

(b) If $\boldsymbol{A}(x, y, z) = (x^2 y, -2xz, 2yz)$, find

(i) $\nabla \cdot \boldsymbol{A}$ (ii) $\nabla \cdot \nabla \times \boldsymbol{A}$ (iii) $\nabla \times (\nabla \times \boldsymbol{A})$

(c) If $\boldsymbol{r} = (x, y, z) = r\,\hat{\boldsymbol{r}}$ where $|\boldsymbol{r}| = r = \sqrt{x^2 + y^2 + z^2}$ and $|\hat{\boldsymbol{r}}| = 1$, show

$$\nabla^2 \left(\frac{1}{r}\right) = 0, \qquad r \neq 0$$

10 marks

6. (a) Sketch the region R enclosed by the curves $x = -1 - y$ and $x = 1 - y^2$ and use a double integral to find its area.

(b) Evaluate the following triple iterated integral by changing the order of integration as appropriate

$$\int_0^4 \int_0^1 \int_{2y}^2 \frac{2\cos x^2}{\sqrt{z}}\, dx\, dy\, dz$$

10 marks

7. Consider the vector field

$$\boldsymbol{F}(x, y, z) = (6xy^2 + z, 6x^2 y + 3y^2, x)$$

(a) Show that $\boldsymbol{F}$ is irrotational. In what domain is $\boldsymbol{F}$ conservative and why?

(b) Find a scalar potential φ such that $\boldsymbol{F} = \nabla\varphi$.

(c) Hence evaluate the path integral

$$\int_{(0,0,0)}^{(1,1,2)} \boldsymbol{F} \cdot d\boldsymbol{r}$$

10 marks

8. Verify Stokes' theorem

$$\iint_S \nabla \times \boldsymbol{A} \cdot d\boldsymbol{S} = \oint_C \boldsymbol{A} \cdot d\boldsymbol{r}$$

for the vector field

$$\boldsymbol{A}(x, y, z) = (2x, 3xy^2, 5z)$$

where S is the curved surface of the paraboloid of revolution

$$S: \quad z = 4 - x^2 - y^2, \quad z \geq 0$$

and $C = \partial S$ is its boundary.

10 marks

9. Consider the vector field

$$\boldsymbol{A}(x, y, z) = (3x + yz, y - e^z, 3z + 2)$$

(a) By direct calculation, evaluate the surface integral

$$\iint_R \boldsymbol{A} \cdot d\boldsymbol{S}$$

of the vector field $\boldsymbol{A}$ over the disk in the xy-plane

$$R: \quad x^2 + y^2 \leq 1, \quad z = 0$$

Assume that the normal to R is in the downward direction.

(b) Obtain the upward flux

$$\iint_S \boldsymbol{A} \cdot d\boldsymbol{S}$$

through the curved surface of the hemisphere

$$S: \quad x^2 + y^2 + z^2 = 1, \quad z \geq 0.$$

10 marks

10. Consider paraboloidal coordinates defined by

$$x = uv\cos\theta, \quad y = uv\sin\theta, \quad z = \tfrac{1}{2}(u^2 - v^2), \qquad u \geq 0, \quad v \geq 0, \quad \theta \in [0, 2\pi)$$

and the vector fields

$$\boldsymbol{A} = A_1(u, v, \theta)\, \boldsymbol{e}_u + A_2(u, v, \theta)\, \boldsymbol{e}_v, \qquad \boldsymbol{B} = B_3(u, v, \theta)\, \boldsymbol{e}_\theta$$

where $\boldsymbol{e}_u$, $\boldsymbol{e}_v$ and $\boldsymbol{e}_\theta$ are unit vectors along the coordinate curves and A_1, A_2 and B_3 are C^2 functions of u, v and θ.

(a) Show that paraboloidal coordinates form an orthogonal curvilinear coordinate system.

(b) Using paraboloidal coordinates, obtain expressions for

(i) $\nabla \cdot \boldsymbol{A}$ (ii) $\nabla \times \boldsymbol{B}$

(c) Verify that the vector field $\boldsymbol{B}$ satisfies

$$\nabla \cdot \nabla \times \boldsymbol{B} = 0$$

10 marks

Nabla Identities

Let $f(x, y, z)$ and $g(x, y, z)$ be scalar functions, $\boldsymbol{F}$ and $\boldsymbol{G}$ be vector fields in R^3 and β be any constant.

1. $\nabla(f + g) = \nabla f + \nabla g$
2. $\nabla(\beta f) = \beta \nabla f$
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$ provided $g \neq 0$.
5. $\nabla \cdot (\boldsymbol{F} + \boldsymbol{G}) = \nabla \cdot \boldsymbol{F} + \nabla \cdot \boldsymbol{G}$
6. $\nabla \times (\boldsymbol{F} + \boldsymbol{G}) = \nabla \times \boldsymbol{F} + \nabla \times \boldsymbol{G}$
7. $\nabla \cdot (f\boldsymbol{F}) = f\nabla \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \nabla f$
8. $\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot (\nabla \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\nabla \times \boldsymbol{G})$
9. $\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$
10. $\nabla \times (f\boldsymbol{F}) = f\nabla \times \boldsymbol{F} + \nabla f \times \boldsymbol{F}$
11. $\nabla \times (\nabla f) = \boldsymbol{0}$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\nabla f \cdot \nabla g$
13. $\nabla \cdot (\nabla f \times \nabla g) = 0$
14. $\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$

Orthogonal Curvilinear Coordinates

Let $f(u_1, u_2, u_3)$ be a C^1 scalar function, $g(u_1, u_2, u_3)$ a C^2 scalar function and

$$\boldsymbol{F} = F_1(u_1, u_2, u_3)\boldsymbol{e}_1 + F_2(u_1, u_2, u_3)\boldsymbol{e}_2 + F_3(u_1, u_2, u_3)\boldsymbol{e}_3$$

be a C^1 vector field. Then

1. $\nabla f = \dfrac{1}{h_1}\dfrac{\partial f}{\partial u_1}\boldsymbol{e}_1 + \dfrac{1}{h_2}\dfrac{\partial f}{\partial u_2}\boldsymbol{e}_2 + \dfrac{1}{h_3}\dfrac{\partial f}{\partial u_3}\boldsymbol{e}_3$

2. $\nabla \cdot \boldsymbol{F} = \dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial\left(h_2 h_3 F_1\right)}{\partial u_1} + \dfrac{\partial\left(h_1 h_3 F_2\right)}{\partial u_2} + \dfrac{\partial\left(h_1 h_2 F_3\right)}{\partial u_3}\right]$

3. $\nabla \times \boldsymbol{F} = \dfrac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$

4. $\nabla^2 g = \dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial}{\partial u_1}\left(\dfrac{h_2 h_3}{h_1}\dfrac{\partial g}{\partial u_1}\right) + \dfrac{\partial}{\partial u_2}\left(\dfrac{h_1 h_3}{h_2}\dfrac{\partial g}{\partial u_2}\right) + \dfrac{\partial}{\partial u_3}\left(\dfrac{h_1 h_2}{h_3}\dfrac{\partial g}{\partial u_3}\right)\right]$

INTEGRATION FORMULAE AND IDENTITIES

$$\int \sin x \, dx = -\cos x + C \qquad \int \cos x \, dx = \sin x + C$$

$$\int \sec x \, dx = \log|\sec x + \tan x| + C \qquad \int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + C$$

$$\int \sec^2 x \, dx = \tan x + C \qquad \int \operatorname{cosec}^2 x \, dx = -\cot x + C$$

$$\int \sinh x \, dx = \cosh x + C \qquad \int \cosh x \, dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x \, dx = \tanh x + C \qquad \int \operatorname{cosech}^2 x \, dx = -\coth x + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}} \, dx = \arccos\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \operatorname{arccosh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{a^2 - x^2} \, dx = \frac{1}{a} \operatorname{arctanh}\left(\frac{x}{a}\right) + C$$

where $a > 0$ is constant and C is an arbitrary constant of integration.

$\cos^2 x + \sin^2 x = 1$ $\qquad$ $\cosh^2 x - \sinh^2 x = 1$

$1 + \tan^2 x = \sec^2 x$ $\qquad$ $1 - \tanh^2 x = \operatorname{sech}^2 x$

$\cot^2 x + 1 = \operatorname{cosec}^2 x$ $\qquad$ $\coth^2 x - 1 = \operatorname{cosech}^2 x$

$\cos 2x = \cos^2 x - \sin^2 x$ $\qquad$ $\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cos 2x = 2\cos^2 x - 1$ $\qquad$ $\cosh 2x = 2\cosh^2 x - 1$

$\cos 2x = 1 - 2\sin^2 x$ $\qquad$ $\cosh 2x = 1 + 2\sinh^2 x$

$\sin 2x = 2 \sin x \cos x$ $\qquad$ $\sinh 2x = 2 \sinh x \cosh x$

$\cos(x + y) = \cos x \cos y - \sin x \sin y$ $\qquad$ $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

$\sin(x + y) = \sin x \cos y + \cos x \sin y$ $\qquad$ $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$ $\qquad$ $\sinh x = \frac{1}{2}(e^x - e^{-x})$

$e^{ix} = \cos x + i \sin x$

$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ $\qquad$ $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

$\operatorname{arcsinh} x = \log(x + \sqrt{x^2 + 1})$ $\qquad$ $\operatorname{arccosh} x = \log(x + \sqrt{x^2 - 1})$

$$\operatorname{arctanh} x = \frac{1}{2} \log\left(\frac{1 + x}{1 - x}\right)$$

END OF EXAMINATION