Introductory Macroeconomics

Lecture 15: Solow-Swan model, part one

Bruce Preston & Daeha Cho

1st Semester 2021

This Lecture

- Firm's demand for factors
  - interpretation of Cobb-Douglas production function
- Growth accounting
- Outline of Solow-Swan model
- BOFAH chapter 14.4 and 15

Decisions Facing Competitive Firm

- What is a competitive firm?

Assumption

  - very small player in the market in which it trades
  - takes the prices of its output and its inputs as given by market conditions

prices are given

- A firm seeks to maximise profits by hiring workers and using capital by taking prices as given

$$\Pi_t = \overbrace{pY_t}^{\text{revenue}} - \overbrace{\underbrace{WL_t}_{\text{labor cost}} - \overbrace{(r+\delta)K_t}^{\text{capital cost}}}^{\text{total cost}}$$

  - p is the output price fixed
  - W is the nominal wage
  - r is the real interest rate, δ is the depreciation rate

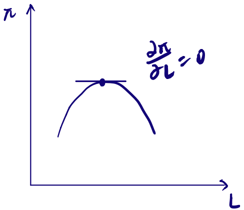
$$\frac{\partial \pi_t}{\partial L_t} = \frac{\partial (pY_t)}{\partial L_t} - W = 0$$

$$\Rightarrow \frac{\partial (pY_t)}{\partial L_t} = W$$

marginal revenue product of labour (MRPL).

marginal cost of labor

Demand for Labour



- Holding all factors apart from L_t fixed, profit max requires

$$\frac{\partial(pY_t)}{\partial L_t} = W$$

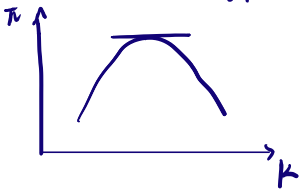
  - LHS is the *marginal revenue product of labour* (MRPL), defined as the extra revenue received by a firm from selling the output produced from an extra unit of labour
  - MRPL is the product of the output price and the marginal product of labour (MPL).

$$\frac{\partial(pY_t)}{\partial L_t} = p \underbrace{\frac{\partial Y_t}{\partial L_t}}_{MPL}$$

MRPL = p × MPL.

Demand for Capital

$\frac{\partial \pi}{\partial K} = 0$



- Holding all factors apart from K_t fixed, profit max requires

$$\frac{\partial (pY_t)}{\partial K_t} = r + \delta$$

$\frac{\partial (pY_t)}{\partial K_t}$

MRPK　　MC of capital

  - LHS is the *marginal revenue product of capital* (MRPK), defined as the extra revenue received by a firm from selling the output produced from an extra unit of capital

  - MRPK is the product of the output price and the marginal product of capital (MPK)

$$\frac{\partial (pY_t)}{\partial K_t} = p \underbrace{\frac{\partial Y_t}{\partial K_t}}_{MPK}$$

MRPK = p · MPK.

Capital and Labour Share

- Capital (labour) share is the share of national income allocated to capital (labour)

- Assuming Cobb-Douglas production function,

  - capital share is

$$\frac{(r+\delta)K_t}{pY_t} = \frac{\frac{\partial(pY_t)}{\partial K_t}K_t}{pY_t} = \frac{\cancel{p}\alpha \cancel{A_t}K_t^{\alpha-1}L_t^{1-\alpha}K_t}{pY_t} = \frac{\cancel{p}\alpha A_t K_t^{\alpha}L_t^{1-\alpha}}{\cancel{p}\cancel{A_t}K_t^{\alpha}L_t^{1-\alpha}} = \alpha$$

firm pays to capital → capital income (for capital owner)

revenue of firm ↓ assume 1 firm ↓ national income

capital income / national income = capital share.

derived.
$r+\delta = \frac{\partial(pY_t)}{\partial K_t}$

Cobb-Douglas function $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$

$\frac{\partial(pY_t)}{\partial K_t} = \alpha p A_t K_t^{\alpha-1} L_t^{1-\alpha}$

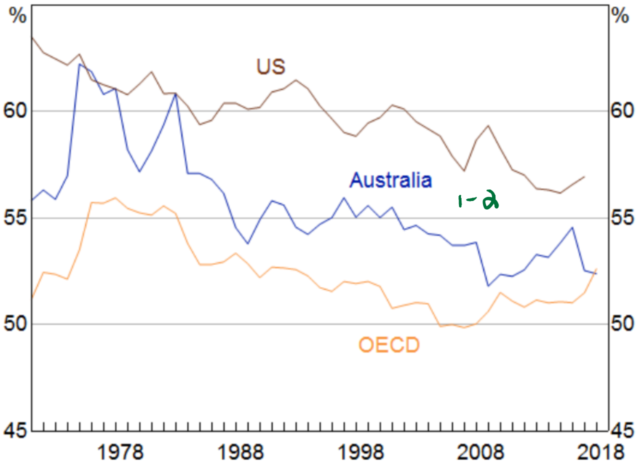
  - by a similar derivation, labour share $\frac{WL_t}{pY_t}$ is $1-\alpha$

Labour Share

$1-\alpha \in (0.5, 0.6)$

$\alpha \in (0.5, 0.4)$

Labour Share of Income*



* Compensation of employees divided by total factor income

Sources: OECD; RBA

Growth Accounting

- *Growth accounting* is a method of decomposing a country's historical growth in output per capita into factors of production

- The growth rate of X between $t-1$ and t is

growth rate

$$\boxed{\frac{X_t - X_{t-1}}{X_{t-1}}} \approx \log\left(\frac{X_t}{X_{t-1}}\right)$$

→ approximation : from Taylor expansion

  – example: $X_{t-1} = 100, X_t = 101$

$$\frac{X_t - X_{t-1}}{X_{t-1}} = 0.01$$

$$\log\left(\frac{X_t}{X_{t-1}}\right) = 0.00995033085316$$

Growth Accounting

- Let labour L_t denote the population size
- Rewriting Cobb-Douglas production function in per person terms,

output per person

$$\boxed{\frac{Y_t}{L_t}} = A_t \left(\frac{K_t}{L_t}\right)^{\alpha}$$

$$\rightarrow y_t = A_t k_t^{\alpha},$$

capital per person

tfp

total factor productivity

$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$

$\frac{Y_t}{L_t} = A_t K_t^{\alpha} L_t^{-\alpha}$

$= A_t \left(\frac{K_t}{L_t}\right)^{\alpha}$

capital person

  - where $y_t = \frac{Y_t}{L_t}$ and $k_t = \frac{K_t}{L_t}$

Growth Accounting

- Dividing the production function for period t by that for period $t-1$,

$y_t = A_t k_t^\alpha$

$$\frac{y_t}{y_{t-1}} = \frac{A_t}{A_{t-1}} \left(\frac{k_t}{k_{t-1}} \right)^\alpha$$

$\log(xY^\alpha) = \log(x) + \alpha \log(Y)$

- Taking logs, we have the growth accounting formula

from previous

$\frac{x_t - x_{t-1}}{x_{t-1}} \approx \log\left(\frac{x_t}{x_{t-1}}\right)$

$$\boxed{\log\left(\frac{y_t}{y_{t-1}}\right)} = \log\left(\frac{A_t}{A_{t-1}}\right) + \alpha \log\left(\frac{k_t}{k_{t-1}}\right)$$

$$\boxed{\frac{y_t - y_{t-1}}{y_{t-1}}} = \frac{A_t - A_{t-1}}{A_{t-1}} + \alpha \left(\frac{k_t - k_{t-1}}{k_{t-1}} \right)$$

TFP growth rate

→ capital per person growth rate

- Output per person growth is explained by TFP growth and capital per person growth

factor rather than kt. Lt

eg technology, political environment

- TFP growth is often called as *Solow residual*

Growth Accounting (1960 - 2017)

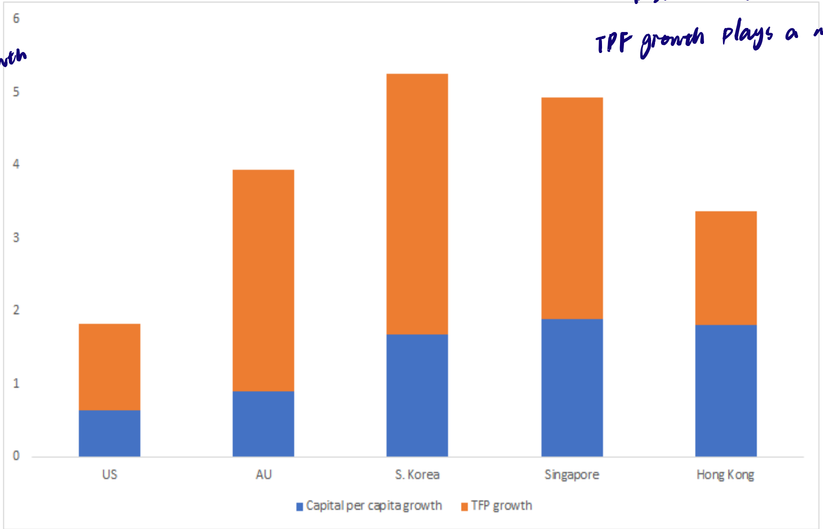
S & P 500

K Δ   K small, TFP dominates. (high tech)

TPF growth plays a major role.

FAANG → google
facebook Apply netflix Amazon

output per person growth



Solow-Swan Model Overview

- Solow-Swan model is a theoretical model that can rationalize the growth accounting facts
  - given the initial capital per person K_0/L_0
  - describes of how capital per person evolves over time and contributes to the growth in output per person
  - TFP growth rates can be easily added in this model

Production and Saving Function

constant return to scale

- Production function $Y_t = Af(K_t, L_t)$ satisfying CRS and positve marginal products and diminishing marginal products

- TFP is constant over time but will be relaxed later on

  assume $\lambda = \frac{1}{L_t}$

  – CRS assumption implies

$$\frac{Y_t}{L_t} = Af(\frac{K_t}{L_t}, \frac{L_t}{L_t}) = Af(\frac{K_t}{L_t}, 1) = Af(\frac{K_t}{L_t})$$

output per person

capital per person

$$\frac{Y_t}{L_t} = A_t f(\frac{K_t}{L_t}, \frac{L_t}{L_t}) = A_t f(\frac{K_t}{L_t}, 1) = A_t f(\frac{K_t}{L_t})$$

national saving

$S_t = Y_t - T_t - C_t + (T_t - G_t) = Y_t - C_t - G_t$ ①

Accounting identity

$Y_t = C_t + I_t + G_t$ ②

- Fraction θ of output (income) per person is saved

  = GDP

  $\theta Y_t = S_t$

  – in a closed economy

① + ②

$S_t = I_t$

$$\frac{S_t}{L_t} = \theta\frac{Y_t}{L_t} = \frac{I_t}{L_t}$$

saving per person

Assumptions on Investment

- Capital depreciates at a rate d (γ)
- Population grows at a constant rate n
- Investment per person $\frac{I_t}{L_t}$ is the sum of *replacement investment* and *net investment*
  - *replacement investment* is defined as investment that maintains the capital per person
  - *net investment* is defined as investment that adds to the size of the capital per person

$\frac{k_t}{L_t}$ depreciate at rate d ⇒ if you want to maintain original level of capital ① $d\frac{k_t}{L_t}$

population growth at a rate n.

to maintain original capital $n\frac{k_t}{L_t}$ ②

①+② replacement investment

$$\frac{I_t}{L_t} = \underbrace{\left(\frac{K_{t+1}}{L_{t+1}} - \frac{K_t}{L_t}\right)}_{\textit{net investment}} + \underbrace{(n+d)\frac{K_t}{L_t}}_{\textit{replacement investment}}$$

Steady State

- Results

steady state

  – If net investment $= 0$, capital per person remains constant $\frac{K_{t+1}}{L_{t+1}} = \frac{K_t}{L_t}$

  – If net investment > 0, capital per person increases $\frac{K_{t+1}}{L_{t+1}} > \frac{K_t}{L_t}$

  – If net investment < 0, capital per person decreases

- *Steady state* is a state of the economy where capital per person is unchanged

  – steady state is reached when net investment $=0$

$$\underbrace{\theta \frac{Y_t}{L_t} = (n+d)\frac{K_t}{L_t}}$$

$\Rightarrow \frac{I_t}{L_t} = (n+d)\frac{K_t}{L_t}$

closed economy $\frac{S_t}{L_t} = \frac{\theta Y_t}{L_t}$

Steady State

- Steady state capital per person $\left(\frac{K}{L}\right)^*$ solves

$\theta \frac{Y_t}{L_t} = \theta A f\left(\frac{K}{L}\right)^*$

LHS

steady state

$\Rightarrow$ no need time subscript

$$\boxed{\theta A f\left(\left(\frac{K}{L}\right)^*\right)} = (n+d)\left(\frac{K}{L}\right)^*$$

RHS.

- Steady state output per person $\left(\frac{Y}{L}\right)^*$ is

$$\left(\frac{Y}{L}\right)^* = Af\left(\left(\frac{K}{L}\right)^*\right)$$

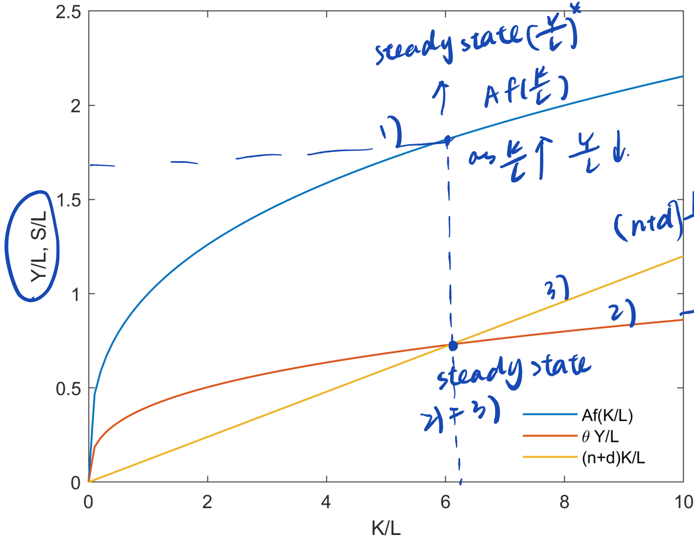
$y_t = A k_t^{\alpha}$

$\frac{Y_t}{L_t} = A\left(\frac{K_t}{L_t}\right)^{\alpha}$

$\uparrow$

$\left(\frac{K}{L}\right)^*$

Solow Diagram



1) production f

2) saving f

3) replacement inv

$(n+d)\frac{K}{L}$ slope n+d

saving

$\theta A f(\frac{K}{L})$

$\theta \in (0,1)$

Next Lecture

- More on Solow-Swan Model
  - transitional dynamics
  - empirical performance of the model
  - effect of a change in TFP