Exponential families

Poisson Regression

① with log link

$y_i \sim Poisson(\lambda_i)$ $y = 0, 1, 2, 3.$

$\lambda_i > 0$

$\lambda = f(t_i)$

why $\beta_0 + \beta_1 t_i$ not work here ? $\Rightarrow$ range $(-\infty, +\infty)$

but $\lambda_i > 0$.

$\Rightarrow$ so log. link $\log \lambda_i = \beta_0 + \beta_1 t_i$

glm(y~t, family=poisson, data).

Learning goals

- Be able to show whether a given distribution is an exponential family.
- Be able to compute mean and variance of random variables belonging to exponential families.
- Understand the interpretation of the variance function and be able to compute the variance function for exponential families.

Exponential families

Y comes from an exponential family if it has density/mass function of the form

linear function of y

$$f(y;\theta,\phi) = \exp\left[\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right]$$

no θ for second term

- θ is the *canonical parameter* (captures location)
- ϕ is the *dispersion parameter* (captures scale)

Example: normal

$Y \sim N(\mu, \sigma^2)$

var(Y) = σ^2

$= v(\mu)\sigma^2$

1

or $v(\mu) = b''((b')^{-1}(\mu))$

$b''(\text{anything}) = 1$

$$
\begin{aligned}
f(y) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}} = \exp\left[\frac{-1}{2}\cdot\frac{(y^2-2\mu y+\mu^2)}{\sigma^2} - \frac{1}{2}\log(2\pi\sigma^2)\right] \\
&= \exp\left[\frac{y\mu - \mu^2/2}{\sigma^2} - \frac{1}{2}\left(\frac{y^2}{\sigma^2} + \log(2\pi\sigma^2)\right)\right] = \exp\left[\frac{y\mu - \frac{1}{2}\mu^2}{\sigma^2} - \frac{1}{2}\left[\frac{y^2}{\sigma^2} + \log(2\pi\sigma^2)\right]\right] \\
&= \exp\left[\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right]
\end{aligned}
$$

$a(\phi) = \sigma^2$

where $\theta = \mu$, $\phi = \sigma^2$, and

$E[Y] = b'(\theta) = \theta$ $\quad b(\theta) = \frac{1}{2}\theta^2 \quad (\theta = \mu)$

$\text{var}[Y] = b''(\theta)\, a(\phi) = \phi$

$= \sigma^2$

☆ NOT unique.

$$
\begin{aligned}
b(\theta) &= \theta^2/2 \\
a(\phi) &= \phi \\
c(y,\phi) &= -\frac{1}{2}\left(\frac{y^2}{\phi} + \log(2\pi\phi)\right)
\end{aligned}
$$

Example: Poisson

$Y \sim \text{pois}(\lambda)$

$= \exp[y \log \lambda - \lambda - \log y!]$

$\text{var}(Y) = \lambda$

$= V(\mu)\, a(\phi)$

$= V(\lambda) \times 1$

$= V(\lambda)$

$$
\begin{aligned}
f(y) &= e^{-\lambda}\lambda^y/y! \text{ for } y = 0, 1, 2, \ldots \\
&= \exp\left[y \log \lambda - \lambda - \log y!\right] \\
&= \exp\left[\frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi)\right]
\end{aligned}
$$

where $\theta = \log \lambda$, $\phi = 1$, and

$\underline{\theta = \log \lambda.}$

$V(\lambda) = \lambda$

identity function

$E[Y] = b'(\theta) = e^{\log \lambda} = \lambda$

$$
\begin{aligned}
b(\theta) &= e^{\theta} \\
a(\phi) &= \phi \\
c(y, \phi) &= -\log y!
\end{aligned}
$$

$b(\log \lambda) = \lambda$

$b(\theta) = e^{\theta}$

$c(y, \phi) = -\log y!$

$\text{var}(Y) = b''(\theta)\, a(\phi)$

$= \lambda$

$\underline{\phi = 1}$

$a(\phi) = \phi$

or

$V(\mu) = b''\left((b')^{-1}(\mu)\right)$

$= b''(\log \mu) = e^{\log \mu} = \mu$

Example: binomial

$Y \sim \text{bin}(m, p)$ for known m (not a parameter)

$$f(y) = \binom{m}{y} p^y (1-p)^{m-y} \text{ for } y = 0, 1, \ldots, m$$

Lab problem in the week 3.

Other examples of exponential families are the gamma and the inverse Gaussian.

Exponential family: mean and variance

Lemma If Y is from an exponential family then

$$\mathbb{E}\,Y = b'(\theta)$$
$$\mathrm{Var}\,Y = b''(\theta)a(\phi)$$

[Proof] Exercise.

$L = L(\theta) = f_\theta(x)$

property ① $E\left[\frac{\partial \log L}{\partial \theta}\right] = 0$

$\Downarrow$

$= \int \cancel{f_\theta(x)} \cdot \frac{f'_\theta(x)}{\cancel{f_\theta(x)}} \Rightarrow \frac{\partial}{\partial\theta}\int f_\theta(x)\,dx = 0.$

② $E\left[\frac{\partial^2 \log L}{\partial \theta^2}\right] = -E\left[\left(\frac{\partial \log L}{\partial\theta}\right)^2\right]$

STEP1 first derivative

$\frac{\partial}{\partial\theta}[\log f_\theta(x)] = \frac{f'_\theta(x)}{f_\theta(x)}$

STEP2 second derivative

$\left(\frac{f'_\theta(x)}{f_\theta(x)}\right)' = \frac{f''_\theta(x) f_\theta(x) - [f'_\theta(x)]^2}{[f_\theta(x)]^2}$

$= \frac{f''_\theta(x)}{f_\theta(x)} - \left[\frac{f'_\theta(x)}{f_\theta(x)}\right]^2$

$\Downarrow$

$\frac{\partial \log L}{\partial\theta}$

STEP3. take expection

$\int \cancel{f_\theta(x)} \cdot \frac{f''_\theta(x)}{\cancel{f_\theta(x)}}\,dx - E\left[\left(\frac{\partial \log L}{\partial\theta}\right)^2\right]$

$= 0 - E\left[\left(\frac{\partial \log L}{\partial\theta}\right)^2\right] = -E\left[\left(\frac{\partial \log L}{\partial\theta}\right)^2\right]$

Exponential family: variance function

$\mu = E[Y] = b'(\theta)$

$\theta = (b')^{-1}(\mu)$

$var(Y) = b''(\theta)\, a(\phi)$

$= b''((b')^{-1}(\mu))\, a(\phi)$

$= v(\mu)\, a(\phi)$

Let $\mu = \mathbb{E}\, Y$ and write

$$\mathrm{Var}\ Y = v(\mu) a(\phi)$$

(so $v = b'' \circ (b')^{-1}$). v is called the *variance function*

relationship between mean & var

Examples:

normal $v(\mu) = 1$ → mean & var independent

mean affect on var through $v(\mu)$

Poisson $v(\mu) = \mu$

binomial $v(\mu) = \mu(1 - \mu/m)$

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