**MAST30001 Stochastic Modelling – 2020**

**Assignment 1**

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Please complete the Plagiarism Declaration Form on the LMS before submitting this assignment.

**Submission Instructions:**

- Write your answers on blank paper. Write on one side of the paper only. Start each question on a new page. Write the question number at the top of each page.
- Scan your exam submission to a single PDF file with a mobile phone or a scanner. Scan from directly above to avoid any excessive keystone effect. Check that all pages are clearly readable and cropped to the A4 borders of the original page. Poorly scanned submissions may be impossible to mark.
- Upload the PDF file to Gradescope via the LMS. Gradescope will ask you to identify on which of the uploaded pages your answers to each question are located.
- The submission deadline is **5:00pm on Thursday, 17 September, 2020**.

There are 3 questions, all of which will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness, and style count.

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1. A discrete time Markov chain with state space $S = \{1, 2, 3, 4, 5, 6, 7\}$ has the following transition matrix.

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3/7 & 4/7 & 0 \\ 0 & 0 & 0 & 0 & 3/4 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

1 (a) Write down the communication classes of the chain. $S_1 = \{1,2,3,4\}$ $S_2 = \{5,6,7\}$

2 (b) Find the period of each communicating class. for $S_1$ $d = \cancel{4}\ 2$ for $S_2$ aperiodic (loop 1.

3 (c) Determine which classes are essential. $S_1$, $S_2$ both essential.

4 (d) Classify each essential communicating class as transient or positive recurrent or null recurrent.

5 (e) Describe the long run behaviour of the chain (including deriving long run probabilities where appropriate).

6 (f) Given $X_0 = 4$, find the long run proportion of time the chain spends in state $j$ for each $j \in S$.

7 (g) Find the expected number of steps taken for the chain to first reach state 3, given the chain starts at state 4. expected hitting time

2. An irreducible Markov chain on $\{0, 1, \ldots\}$ has transition probabilities

$$p_{ij} = \begin{cases} \frac{1}{(i+1)(i+2)}, & 0 \le j \le i \\ \frac{i+1}{i+2}, & j = i+1. \end{cases}$$

Determine whether the chain is positive recurrent, null recurrent, or transient. *[Hint: It is possible to find an exact simple expression for $p_{ij}^{(n)}$.]*

3. (a) Fix $p \in (0,1)$ and define the Markov chain $(X_n)_{n\ge0}$ on $\{0,1,\ldots,\}$ with transition probabilities, for $i \ge 1$,

$$p_{i,i+1} = 1 - p_{i,i-1} = p,$$

and $p_{0,1} = 1 - p_{0,0} = p$. Setting $T = \inf\{n \ge 1 : X_n = 0\}$, show that if $(X_n)_{n\ge0}$ is positive recurrent, then for all $i$ in the state space,

first hitting time to 0 when start from 0

$$\mathbb{E}[T|X_0 = i] < \infty,$$

and if the chain is not positive recurrent, then

$$\mathbb{E}[T|X_0 = i] = \infty.$$

(b) Fix $p \in (0,1)$ and define the Markov chain $(Y_n)_{n\ge0}$ on $\{0,1,\ldots,\}$ with transition probabilities, for $i \ge 1$,

$$p_{i,i+1} = 1 - p_{i,i-1} = p,$$

and $p_{0,j} = q_j$ for $j \ge 1$ where $\sum_{j=1}^{K} q_j = 1$. Determine the values of $p$ such that the chain is positive recurrent.

$E(T|X_0=i)$

$= \sum_{n=1}^{\infty} n f_{i0}^{(n)}$

positive recurrent

$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$

$\lim_{n\to\infty} p_{ii}^{(n)} \neq 0$

$\sum_{n=1}^{n} n f_{ii}^{(n)} < \infty$

$\Rightarrow \sum_{n=1}^{n} n f_{00}^{(n)} < \infty$

$m_{i,0} = \begin{cases} 0 & \text{if } i = 0 \\ 1 + \sum_{k\neq 0} p_{ik} m_{k,0} & \end{cases}$

$1 + p_{i,i}$