M_t $(\lambda_m = p\lambda)$

$\uparrow p$

N_t $(\lambda = 2)$

$\downarrow 1-p$

R_t $((1-p)\lambda)$

MAST30001 Stochastic Modelling

Tutorial Sheet 6

注意单位 rate 2 per minute → ask 1 hour ××

$\lambda = 2$

1. Yeast microbes from the air outside of a culture float by according to a Poisson process with rate 2 per minute. Each microbe that floats by joins the population of the culture with probability p and with probability $1-p$ the microbe doesn't join the culture, and this choice is made independent from the times of arrival and choice to join of all other microbes.

   $P(N_3 = 4) = P(Poi(3\lambda) = 4) = \frac{e^{-3\lambda}(3\lambda)^4}{4!}$

   (a) What is the chance that exactly four microbes float by in the first 3 minutes?

   (b) What is the chance that exactly four microbes join the culture in the first 3 minutes? $P(M_3 = 4) = P(Poi(3\lambda p) = 4)$.

   (c) Given that 7 microbes have floated by the culture in first 3 minutes, what is the chance that at least two of the seven join the culture? $P(M_3 \geq 2 \mid N_3 = 7) = P(Bi(n=7, p) \geq 2)$ $= 1 - \binom{7}{0} p^0 (1-p)^7 - \binom{7}{1} p^1 (1-p)^6$

   (d) Given that 7 microbes have floated by the culture in first 3 minutes, what is the chance that exactly 3 floated by in the first 1 minute?

   (e) What is the chance that in the first 3 minutes, exactly four microbes join the culture and 3 float by that don't join the culture? $P(M_t = 4, R_t = 3) \Rightarrow$ M_t, R_t independent $P(M_t = 4) \cdot P(R_t = 3)$ $= P(Po(\lambda p t) = 4) \times P(Po(\lambda(1-p)t) = 3)$

   $= P(N_t = 7)\, P(M_t = 4, R_t = 3 \mid N_t = 7)$ — $P(Po(3\lambda) = 7)$, $P(Bi(7, p) = 3)$

   (d) $P(N_1 = 3 \mid N_3 = 7) = \frac{P(N_1 = 3, N_3 = 7)}{P(N_3 = 7)} = \frac{P(Po(\lambda) = 3)\, P(Po(2\lambda) = 4)}{P(Po(3\lambda) = 7)} = \frac{7!\, 2^4}{3!\, 4!\, 3^7} = \binom{7}{3}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^3 = P(Bi(7, \tfrac{1}{3}) = 3)$

   $X_i \mid N_3 = 7 \sim U(0, 3)$

   $P(N_1 = 3)$ → 3 three out of $X_1, \dots, X_7$ so that $X_i \in [0, 1]$ $= P(Bi(7, \tfrac{1}{3}) = 3)$

Assume now that a second strain of yeast microbes independently float by the culture according to a Poisson process with rate 1, and each microbe joins the culture with probability q, analogous to the previous process.

   (f) What is the chance that exactly four yeast microbes from either strain float by in the first 3 minutes? $P(N_3^{(1)} + N_3^{(2)} = 4)$.

   $\lambda = \lambda_1 p$, $\lambda = \lambda_2 q$; $M_t = M_{N_t^{(1)}} + M_{N_t^{(2)}}$; $N_t^{(1)}$ $(\lambda_1 = 2)$, $N_t^{(2)}$ $(\lambda_2 = 1)$; R_t.

   (g) What is the chance that exactly four yeast microbes from either strain join the culture in the first 3 minutes? add two poisson $\Rightarrow P(Po(\lambda_1 + \lambda_2)t) = P(Poi(\lambda_1 + \lambda_2)t) = 4$. $P(M_3 = 4) = P(Po(\lambda_1 p + \lambda_2 q)t = 4)$, $t = 3$

2. In a Poisson process with rate 1, what is the joint density of the times of the first and second jumps? What is the joint density of the times of the ith and jth jump for $i < j$? Can you interpret these formulas similar to our discussion in lecture deriving the joint densities of order statistics?

3. Let $U_{(1)}, \dots, U_{(n)}$ be order statistics of independent variables, uniform on the interval $(0, 1)$. For $0 < x < y < 1$ what is

   (a) $\mathbb{P}(U_{(1)} > x, U_{(n)} < y)$, $= P(U_1, \dots U_n \in (x, y)) = (y-x)^n$

   (b) $\mathbb{P}(U_{(1)} < x, U_{(n)} < y)$,

   (c) $\mathbb{P}(U_{(k)} < x, U_{(k+1)} > y)$?

   deal with smallest, always use greater sign

   $\Rightarrow P(U_{(n)} < y) = P(U_{(1)} < x, U_{(n)} < y) + P(U_{(1)} \geq x, U_{(n)} < y)$; y^n, $y^n - (y-x)^n$, $(y-x)^n$

   $P(U_{(k)} < x) - P(U_{(k)} < x, U_{(k+1)} < y)$

   $1 - P(U_{(k)} > x) - (P(U_{(k+1)} < y) - P(U_{(k)} > x, U_{(k+1)} < y))$

   $= 1 - P(U_{(k)} > x) - P(U_{(k+1)} < y) + P(U_{(k)} > x, U_{(k+1)} < y)$

   $= 1 - (1-x)^{n-k+1} - y^{k+1} + (y-x)!$

4. From Tutorial 1: If N is geometric with parameter p ($\mathbb{P}(N = j) = p(1-p)^j$, $j = 0, 1, 2, \dots$) and given $N = n$, X is gamma with parameter $n+1$, what is the density of X? Another question: If S is exponential with rate λ and given $S = s$, M is Poisson with mean s, then what is the distribution of M? A third question: If K is Poisson with mean μ and given $K = k$, J is binomial with parameters k and p, then what is the distribution of J? Can you explain (or even derive) the answers to these three questions through superposition and thinning of Poisson processes?

   $K \sim Poi(\mu)$

   $J \mid K = k \sim Bi(k, p)$.

   $PPo(1)$, $\frac{1}{\lambda+1}$, $PPo(\lambda+1)$, $\frac{\lambda}{\lambda+1}$, $PPo(\lambda)$ $S \sim exp(\lambda)$

   $M \mid S = s \sim PPoi(s)$

   $M = \#$ point in 1st arrival in before.

   $M \sim Geo(\frac{\lambda}{\lambda+1})$

$N \sim Geo(p)$.

$X \mid N=n \sim Gamma(n+1)$

$PProc(p\mu)$.

$J = \#$ arrival in $[0,1]$ ~~#~~ for $PPro(p\mu)$

$J \sim Poisson(p\mu)$

N+1 arrival

$X = T_{N+1}$

$X \sim exp(p)$

N+1

p.

1-p

5. Customers enter a bank according to a Poisson process $(N_t)_{t\geq 0}$ with rate $\lambda = 10$ per hour and each customer makes a deposit or withdrawal. If X_j is the amount brought in by the jth customer, assume that the X_j are i.i.d. and independent of the arrivals of customers with distribution uniform on $\{-4, -3, \ldots, 4, 5\}$ (negative amounts correspond to withdrawals). Then the balance of the bank over t hours is given by a compound Poisson process

$$Y_t = \sum_{j=1}^{N_t} X_j.$$

(a) Draw a typical trajectory of the process Y_t.

(b) Calculate the mean and variance of the money brought into the bank over an eight hour business day.

(c) Use the central limit theorem to approximate the probability that the bank has a total balance greater than \$4500 over 100 business days.

6. For $r > 0$ and $0 < p < 1$, let N_t be a Poisson process with rate $\lambda = r\log(1/p)$ and $X_1, X_2, \ldots$ be i.i.d. with distribution

$$P(X_1 = k) = \frac{(1-p)^k}{k\log(1/p)}, \quad k = 1, 2, \ldots$$

Use moment generating functions to show that the compound Poisson variable

$$Y_t = \sum_{j=1}^{N_t} X_j$$

has the negative binomial distribution (started from zero) with parameters rt and p; that is, that

$$P(Y_t = k) = \binom{k+rt-1}{k}(1-p)^k p^{rt}, \quad k = 0, 1, 2, \ldots$$