Markov Processes

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Markov processes

A discrete-time Markov process on $\mathbb{R}^d$ is a sequence of r.v.s $X_0, X_1, X_2, \ldots$ such that

$$X_{n+1} | X_n, X_{n-1}, \ldots \stackrel{\mathrm{d}}{=} X_{n+1} | X_n.$$

That is, we only need to know the current state to determine the distribution for the next state. We only care about where we are, not how we got here.

We describe a Markov process using its transition density $p : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$:

$$\mathbb{P}(X_{n+1} \in A | X_n = x) = \int_{y \in A} p(x, y) dy.$$

We will assume that p does not depend on n (we say the process is time homogeneous).

Stationarity and limiting behaviour

A probability density π is stationary for $\{X_n\}$ if for all y

$$\pi(y) = \int \pi(x)p(x, y)\,dx$$

That is, if your state is distributed according to π, then after a single step it still is.

A distribution π is limiting if for all x

$$\mathbb{P}(X_n \in A \mid X_0 = x) \to \int_A \pi(y)\,dy =: \pi(A).$$

Limiting distributions are always unique and stationary. Under certain conditions stationary distributions (if they exist) are unique and limiting.

Conditions for stationary to be limiting

Definition The process $\{X_n\}$ is *aperiodic* if there do not exist sets $A_0, A_1, \ldots, A_{d-1}$ such that $X_n \in A_i$ implies $X_{n+1} \in A_{(i+1) \bmod d}$. The process is *irreducible* if for any $A \in \mathbb{R}^d$ with $\int_A dx > 0$ and $x \in \mathbb{R}^d$ there exists an n such that $\mathbb{P}(X_n \in A \mid X_0 = x) > 0$. That is, you can get anywhere from anywhere else.

Theorem If a Markov process is aperiodic and irreducible then any stationary distribution is unique and limiting.

Ergodicity

Theorem For an aperiodic irreducible Markov process, if π is a limiting distgribution then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\text{a.s.}} \int x\pi(x)\,dx.$$

We say that the process is *ergodic*.

In fact, if the process is ergodic then for any function h

$$\frac{1}{n}\sum_{i=1}^{n} h(X_i) \xrightarrow{\text{a.s.}} \int h(x)\pi(x)\,dx.$$

For example, if $X \sim \pi$ then $\sum_{i=1}^{n} 1_A(X_i)/n \xrightarrow{\text{a.s.}} \mathbb{P}(X \in A)$.

Existence of a limiting distribution

Definition The process $\{X_n\}$ is *reversible* with respect to π, if

$$\pi(x)p(x,y) = \pi(y)p(y,x) \text{ for all } x, y.$$

Corollary If $\{X_n\}$ is reversible w.r.t. π then π is stationary, since

$$\begin{aligned} \int \pi(x)p(x,y)dx &= \int \pi(y)p(y,x)dx \\ &= \pi(y)\int p(y,x)dx \\ &= \pi(y) \end{aligned}$$