MAST20009 Vector Calculus

Practice Class 6 Questions

Change of variables for double integrals

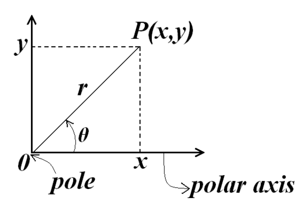
Let D, D^* be elementary regions in $\mathbb{R}^2$ and $T : D^* \to D$ be C^1. If T is one-to-one and $D = T\left(D^*\right)$

$$\iint_D f(x,y)\,dx\,dy \;=\; \iint_{D^*} f\left[x(u,v), y(u,v)\right]\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

where the |Jacobian| is:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \;=\; \left|\det\begin{bmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{bmatrix}\right|$$

Polar Coordinates (r, θ)



- $r = \text{length } \overrightarrow{OP} = \sqrt{x^2+y^2}$, $0 \le r < \infty$.
- $\theta =$ angle measured anticlockwise from positive x axis to $\overrightarrow{OP}$, $0 \le \theta \le 2\pi$.
- $x = r\cos\theta$, $y = r\sin\theta$
- Jacobian $= r$

1. Let D be the parallelogram with vertices $(0,0)$, $(1,2)$, $(2,2)$, and $(3,4)$. Evaluate

$$\iint_D \left(x^2+y^2\right) dx\,dy$$

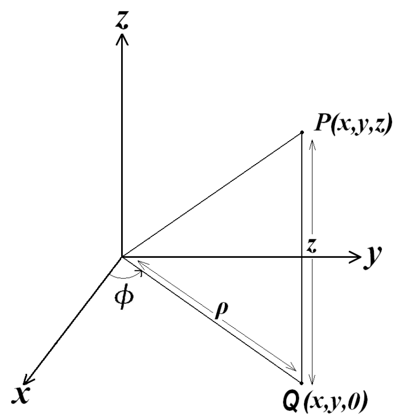
by making the change of variables $u = x - \dfrac{y}{2}, \quad v = -x + y.$

2. Evaluate

$$\iint_R \left(x^2+y^2\right)^{\frac{3}{2}} dx\,dy$$

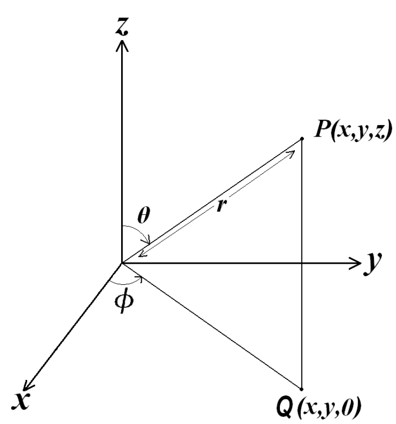
where R is the semi-circular disk $x^2 + y^2 \le 4, \quad y \ge 0.$

Cylindrical Coordinates (ρ, ϕ, z)



- $\rho =$ length $\overrightarrow{OQ} = \sqrt{x^2 + y^2}$, $0 \leq \rho < \infty$.
- $\phi =$ angle measured anticlockwise from positive x axis to $\overrightarrow{OQ}$, $0 \leq \phi < 2\pi$.
- $x = \rho \cos\phi,\ \ y = \rho \sin\phi,\ \ z = z \qquad (z \in \mathbb{R})$.
- Jacobian $= \rho$

Spherical Coordinates (r, θ, ϕ)



- $r =$ length $\overrightarrow{OP} = \sqrt{x^2 + y^2 + z^2}$, $0 \leq r < \infty$.
- $\theta =$ angle measured from positive z axis to $\overrightarrow{OP}$, $0 \leq \theta < \pi$.
- $\phi =$ angle measured anticlockwise from positive x axis to $\overrightarrow{OQ}$, $0 \leq \phi < 2\pi$.
- $x = r \sin\theta \cos\phi,\ \ y = r \sin\theta \sin\phi,\ \ z = r \cos\theta$.
- Jacobian $= r^2 \sin\theta$

Change of variables for triple integrals

Let D, D^* be elementary regions in $\mathbb{R}^3$ and $T : D^* \to D$ be C^1. If T is one-to-one and $D = T(D^*)$

$$\iiint\limits_D f(x,y,z)\,dx\,dy\,dz \;=\; \iiint\limits_{D^*} f\left[x(u,v,w), y(u,v,w), z(u,v,w)\right]\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

where

$$|\text{Jacobian}| = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$$

3. Let B be the region inside the cone $z = \sqrt{x^2+y^2}$ and inside the hemisphere $x^2+y^2+z^2 = 2,\ z \geq 0$.

   Write down a triple integral to calculate the volume of B using

   (a) spherical coordinates,

   (b) cylindrical coordinates.

4. Let D be the region between the two paraboloids $z = x^2+y^2-7$ and $z = 9-3x^2-3y^2$. Using an appropriate change of variables, evaluate

$$\iiint\limits_D x\,dV.$$

   (*Note*: You evaluated this integral using cartesian coordinates in practice class 5.)

Average Values

The *average* of $f(x,y,z)$ in $D \subset \mathbb{R}^3$ is

$$\bar{f} \;=\; \frac{\iiint\limits_D f(x,y,z)\,dx\,dy\,dz}{\iiint\limits_D dx\,dy\,dz}$$

5. The temperature at all points in the hemisphere $z = -\sqrt{4-x^2-y^2}$ is given by

$$T(x,y,z) = \left(x^2+y^2+z^2\right)^3.$$

   Find the average temperature in the hemisphere.

When you have finished the above questions, continue working on the questions in the Vector Calculus Problem Sheet Booklet.