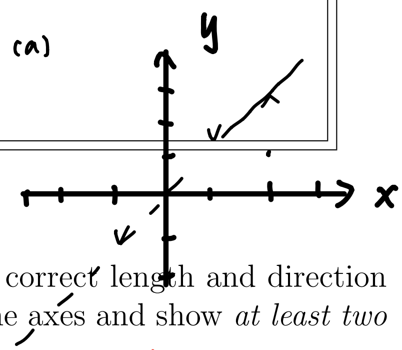
MAST20009 Vector Calculus

Practice Class 4 Questions

A path $\boldsymbol{c}(t)$ is a *flowline* of a vector field $\boldsymbol{F}$ if

$$\frac{d\boldsymbol{c}}{dt} = \boldsymbol{F}[\boldsymbol{c}(t)].$$

(a)



1. (a) Let

substitute a few point

$F(1,1) = (1,1)$ $F(-1,1) = (-1,-1)$

$F(1,-1) = (1,1)$

$$\boldsymbol{F}(x, y) = (x, x).$$

Sketch the vector field in the x-y plane, using the correct length and direction of the vectors. Make sure you include a scale on the axes and show *at least two vectors in each quadrant.*

(b) Let

$$\boldsymbol{F}(x, y, z) = y\boldsymbol{i} - x\boldsymbol{j} + \boldsymbol{k}.$$

Determine the equation of the flowlines for $\boldsymbol{F}$. Describe or sketch the flowlines.

(b) $c(t) = (x(t), y(t), z(t))$

$x'(t) = y$ $\quad z'(t) = 1$

$y'(t) = -x$

$\left\{ \begin{array}{l} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x \\ \frac{dz}{dt} = 1 \Rightarrow z = t + C \end{array} \right.$

$\underbrace{\frac{dy}{dx}} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy/dt}{dx/dt}$

$= \underbrace{\frac{-x}{y}}_{\text{seperable}}$

$\int y \frac{dy}{dx} \cdot dx = \int -x \, dx$

$\Rightarrow \frac{1}{2} y^2 = -\frac{1}{2} x^2 + C,$

$y^2 + x^2 = C$

If $f(x, y, z)$ is a C^1 scalar function, the *gradient* of f is:

$$\boldsymbol{\nabla} f = \boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y} + \boldsymbol{k}\frac{\partial f}{\partial z}$$

If $\boldsymbol{F}(x, y, z) = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$ is a C^1 vector field, the *divergence* of $\boldsymbol{F}$ is:

$$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

If $\boldsymbol{F}(x, y, z) = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$ is a C^1 vector field, the *curl* of $\boldsymbol{F}$ is:

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}$$

If $f(x, y, z)$ is a C^2 scalar function, the *Laplacian* of f is:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

If $\boldsymbol{F}(x, y, z) = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$ is a C^2 vector field, the *Laplacian* of $\boldsymbol{F}$ is:

$$\nabla^2 \boldsymbol{F} = \nabla^2 u\boldsymbol{i} + \nabla^2 v\boldsymbol{j} + \nabla^2 w\boldsymbol{k}$$

(a) $\nabla f = (2ye^{xy} - \frac{z}{x})\vec{i} + (2xe^{xy} - \frac{z}{y})\vec{j} + (-\log xy)\vec{k}$

$\nabla^2 f = 2y^2 e^{xy} + \frac{z}{x^2} + 2x^2 e^{xy} + \frac{z}{y^2} + 0$
$= 2(y^2+x^2)e^{xy} + z(\frac{1}{x^2} + \frac{1}{y^2})$

2. (a) Let $f(x, y, z) = 2e^{xy} - z\log(xy)$.
Calculate $\boldsymbol{\nabla} f$ and $\nabla^2 f$.

(b) Let $\boldsymbol{F}(x, y, z) = (xz\sin y, y\cos y, y^2 z^2)$.
Calculate $\boldsymbol{\nabla} \cdot \boldsymbol{F}$, $\boldsymbol{\nabla} \times \boldsymbol{F}$ and $\nabla^2 \boldsymbol{F}$.

> Let $\boldsymbol{V}(x, y, z)$ be a vector field. If $\boldsymbol{\nabla} \times \boldsymbol{V} = \boldsymbol{0}$ ($\boldsymbol{V}$ is *irrotational*), then $\boldsymbol{V}$ can be represented by
> $$\boldsymbol{V} = \boldsymbol{\nabla}\phi.$$
> $V_1 = \frac{\partial \phi}{\partial x}$.

3. Let $\boldsymbol{V}(x, y, z) = (2xyz + x^2)\boldsymbol{i} + (x^2 z + 1)\boldsymbol{j} + (x^2 y + z)\boldsymbol{k}$.

(a) Show that $\boldsymbol{V}$ is irrotational.

(b) Find the scalar potential function ϕ such that $\boldsymbol{V} = \boldsymbol{\nabla}\phi$.

$x^2yz + \frac{1}{3}x^3 + C_1(y,z)$
$(x^2yz + y) + C_2(x,z)$
$(x^2yz + \frac{1}{2}z^2) + C_3(x,y)$ (b)

$V_1 = \frac{\partial \phi}{\partial x}$ $V_2 = \frac{\partial \phi}{\partial y}$ $V_3 = \frac{\partial \phi}{\partial z}$

$\phi = x^2yz + \frac{1}{3}x^3 + y + \frac{1}{2}z^2 + C$

3(a). $\underset{\sim}{\nabla} \times \underset{\sim}{V} = \underset{\sim}{0}$

$$\det \begin{array}{c} \begin{matrix} \vec{i} & \vec{j} & \vec{k} \end{matrix} \\ \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz+x^2 & x^2z+1 & x^2y+z \end{bmatrix} \end{array}$$

$= \underset{\sim}{i}\left[\frac{\partial}{\partial y}(x^2y+z) - \frac{\partial}{\partial z}(x^2z+1)\right]$
sequence.
$- \underset{\sim}{j}\left[\frac{\partial}{\partial x}(x^2y+z) - \frac{\partial}{\partial z}(2xyz+x^2)\right]$
$+ \underset{\sim}{k}\left[\frac{\partial}{\partial x}(x^2z+1) - \frac{\partial}{\partial y}(2xyz+x^2)\right]$

$= \underset{\sim}{i}(x^2 - x^2) - \underset{\sim}{j}(2xy - 2xy)$
$+ \underset{\sim}{k}(2xz - 2xz)$
$= \underset{\sim}{0}$

> Let $\boldsymbol{V}(x, y, z)$ be a vector field. If $\boldsymbol{\nabla} \cdot \boldsymbol{V} = 0$ ($\boldsymbol{V}$ is *incompressible*), then $\boldsymbol{V}$ can be represented by
> $$\boldsymbol{V} = \boldsymbol{\nabla} \times \boldsymbol{F}$$
> where $\boldsymbol{F} = (F_1, F_2, F_3)$ and
> $$F_1 = \int_0^z V_2(x, y, t)dt - \int_0^y V_3(x, t, 0)dt$$
> $$F_2 = -\int_0^z V_1(x, y, t)dt$$
> $$F_3 = 0.$$

4. Let $\boldsymbol{V}(x, y, z) = xe^{2z}\boldsymbol{i} + ye^{2z}\boldsymbol{j} - (e^{2z} + 2xe^{-y^2})\boldsymbol{k}$.

(a) Show that $\boldsymbol{V}$ is incompressible.

(b) Find a vector potential $\boldsymbol{F}$ such that $\boldsymbol{V} = \boldsymbol{\nabla} \times \boldsymbol{F}$.

Let f, g be scalar functions of (x, y, z). Let $\boldsymbol{F}$, $\boldsymbol{G}$ be vector fields in R^3.

1. $\boldsymbol{\nabla}(f+g) = \boldsymbol{\nabla} f + \boldsymbol{\nabla} g$
2. $\boldsymbol{\nabla}(\beta f) = \beta \boldsymbol{\nabla} f \qquad (\beta \text{ constant})$
3. $\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f$
4. $\boldsymbol{\nabla}\left(\dfrac{f}{g}\right) = \dfrac{g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g}{g^2} \qquad \text{provided } g \neq 0$
5. $\boldsymbol{\nabla} \cdot (\boldsymbol{F} + \boldsymbol{G}) = \boldsymbol{\nabla} \cdot \boldsymbol{F} + \boldsymbol{\nabla} \cdot \boldsymbol{G}$
6. $\boldsymbol{\nabla} \times (\boldsymbol{F} + \boldsymbol{G}) = \boldsymbol{\nabla} \times \boldsymbol{F} + \boldsymbol{\nabla} \times \boldsymbol{G}$
7. $\boldsymbol{\nabla} \cdot (f\boldsymbol{F}) = f\boldsymbol{\nabla} \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \boldsymbol{\nabla} f$
8. $\boldsymbol{\nabla} \cdot (\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\boldsymbol{\nabla} \times \boldsymbol{G})$
9. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) = 0$
10. $\boldsymbol{\nabla} \times (f\boldsymbol{F}) = f\boldsymbol{\nabla} \times \boldsymbol{F} + \boldsymbol{\nabla} f \times \boldsymbol{F}$
11. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \boldsymbol{0}$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g$
13. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g) = 0$
14. $\boldsymbol{\nabla} \cdot (f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f) = f\nabla^2 g - g\nabla^2 f$
15. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F}$

5. (a) Let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $r = \sqrt{x^2+y^2+z^2}$. Using the vector identities, simplify
$$\boldsymbol{\nabla} \times (e^{2r}\,\boldsymbol{r}).$$
   (b) Assuming that $f(x, y, z)$ and $g(x, y, z)$ are C^2 scalar functions, prove identity 12 by direct calculation.

When you have finished the above questions, continue working on the questions in the Vector Calculus Problem Sheet Booklet.