Author/s:
Mathematics and Statistics

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Vector Calculus, 2016 Semester 2, MAST20009

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THE UNIVERSITY OF MELBOURNE

Student Number	

Semester 2 Assessment, 2016

School of Mathematics and Statistics

MAST20009 Vector Calculus

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 7 pages (including this page)

Authorised Materials

- Mobile phones, smart watches, and internet or communication devices are forbidden.
- No materials are authorised.
- No calculators, tablet devices, or computers are authorised.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- There are 12 questions on this exam paper.
- All questions may be attempted.
- Marks for each question are indicated on the exam paper.
- Start each question on a new page.
- Clearly label each page with the number of the question that you are attempting.
- There is a separate 3 page formula sheet accompanying the examination paper, that you may use in this examination.
- The total number of marks available is 130.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.
- Initially students are to receive the exam paper, the 3 page formula sheet, and a 14 page script book.

This paper may be held in the Baillieu Library

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Question 1 (9 marks)

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \begin{cases} \dfrac{3x^2y}{x^4 + 2y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

(a) Is f continuous at $(0, 0)$?

Justify your answer.

(b) For $(x, y) \neq (0, 0)$, find $\dfrac{\partial f}{\partial y}$.

(c) Does $\dfrac{\partial f}{\partial y}$ exist at $(0, 0)$?

Justify your answer.

Question 2 (7 marks)

Consider the two functions $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$f(x, y, z) = (x - y^2 - z, 5x + z^2) \qquad \text{and} \qquad g(u, v) = (2v, u^2, u^3).$$

Evaluate the derivative $\boldsymbol{D}(g \circ f \circ g)(1, -1)$ using the matrix version of the chain rule.

Question 3 (17 marks)

Using the method of Lagrange multipliers, determine the maximum and the minimum of

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraints

$$x^2 + y^2 = 1 \qquad \text{and} \qquad x + y + z = 1.$$

Justify that the points you have found give the maximum and minimum of f.

Question 4 (11 marks)

Let

$$\boldsymbol{V}(x,y,z) = \left(\ x+2z, \quad -y+2z, \quad 2x+2y\ \right).$$

(a) Show that $\boldsymbol{V}$ is an incompressible vector field.

(b) Find a vector potential $\boldsymbol{F}$ for $\boldsymbol{V}$.

(c) Show that if

$$\phi(x,y,z) \ = \ \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 + 2xz + 2yz + 7,$$

then $\boldsymbol{V} = \boldsymbol{\nabla}\phi$.

(d) If the path C is the directed line segment starting at $(3,2,1)$ and finishing at $(1,2,3)$, evaluate the line integral

$$\int_C \boldsymbol{V} \cdot d\boldsymbol{s}.$$

Question 5 (10 marks)

Let $\boldsymbol{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 irrotational vector field and $g : \mathbb{R}^3 \to \mathbb{R}$ a C^3 scalar function.

(a) Use the vector identities to simplify

$$\boldsymbol{\nabla} \cdot (g\, \boldsymbol{\nabla} \times (g\boldsymbol{G}))\,.$$

At all times state which identity is being used.

(b) If $\boldsymbol{G} = \boldsymbol{\nabla} g$ find the vector field $\boldsymbol{H}$ such that $g\, \boldsymbol{\nabla} \times (g\boldsymbol{G}) = \boldsymbol{\nabla} \times \boldsymbol{H}$.

Question 6 (6 marks)

Evaluate the triple integral

$$\int_0^1 \int_0^z \int_{z^2}^1 \left(\exp\left(z^2\right) + z^2 \sin\left(x^3\right)\right) dx\, dy\, dz.$$

Question 7 (15 marks)

Determine the coordinates of the centre of mass of the solid region between the two spheres

$$x^2 + y^2 + z^2 \ = \ 1 \qquad \text{and} \qquad x^2 + y^2 + z^2 \ = \ 4$$

above the xy-plane, if the mass density is $\mu(x, y, z) = z$.

Recall that

$$x_{\text{cm}} \ = \ \frac{\displaystyle\iiint_D x\,\mu(x, y, z)\,dx\,dy\,dz}{\text{mass}}$$

$$y_{\text{cm}} \ = \ \frac{\displaystyle\iiint_D y\,\mu(x, y, z)\,dx\,dy\,dz}{\text{mass}}$$

$$z_{\text{cm}} \ = \ \frac{\displaystyle\iiint_D z\,\mu(x, y, z)\,dx\,dy\,dz}{\text{mass}},$$

where

$$\text{mass} \ = \ \iiint_D \mu(x, y, z)\,dx\,dy\,dz.$$

Question 8 (11 marks)

Consider the surface S parameterised by

$$\Phi(\rho, \phi) \ = \ \big(\ 2\rho \sin\phi, \quad 2\rho\cos\phi, \quad \phi \ \big),$$

where $0 \le \rho \le 4$ and $0 \le \phi \le \pi$.

(a) Determine a normal vector to S.

(b) Find the equation of the tangent plane to the surface at the point $(0, 2, 0)$.

(c) Let

$$f(x, y, z) \ = \ z\sqrt{1 + x^2 + y^2}.$$

Calculate the surface integral

$$\iint_S f\,dS.$$

Question 9 (12 marks)

(a) State Green's theorem in the plane. Explain all symbols used and any required conditions.

(b) Using Green's theorem in the plane, prove that the area of a region D in the xy-plane is given by the line integral

$$\frac{1}{2}\int_C x\,dy - y\,dx,$$

where C is a simple closed curve that bounds D.

(c) Using part (b), calculate the total area enclosed by both loops of the lemniscate (see Figure 1) parameterised by

$$x = \frac{\cos t}{1+\sin^2 t}, \qquad y = \frac{\cos t \sin t}{1+\sin^2 t},$$

where $0 \le t < 2\pi$.

Hint: Recognise that $y = x\sin t$ and use $\displaystyle\int \frac{\cos^3 t}{\left(1+\sin^2 t\right)^2}\,dt = \frac{\sin t}{1+\sin^2 t} + c$, where c is an arbitrary constant.

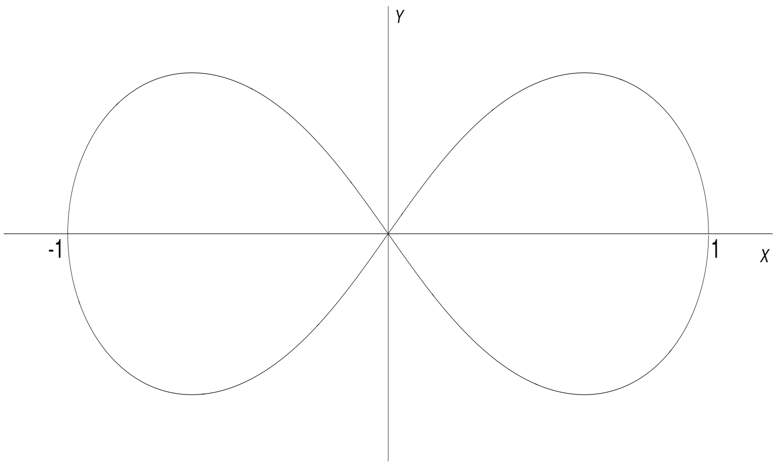

Figure 1: Plot of the lemniscate.

Question 10 (11 marks)

Use Stokes' theorem to evaluate the line integral

$$\int_C -y^3 dx + x^3 dy - z^3 dz,$$

where the curve C is the intersection of the cylinder

$$x^2 + y^2 = 1$$

and the plane

$$x + y + z = 2,$$

and the orientation on C corresponds to anticlockwise motion in the xy-plane.

Question 11 (11 marks)

Let $\boldsymbol{F}(x, y, z) = (z + e^y, z - \sin x, 2z - x + \cosh y)$ and S be the closed surface that bounds the region above the paraboloid

$$z = 3x^2 + 3y^2 - 1,$$

and below the cone

$$z = \sqrt{x^2 + y^2} + 1.$$

Evaluate the flux integral

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}.$$

Question 12 (10 marks)

Define paraboloidal coordinates (u, v, ϕ) by

$$\begin{aligned} x &= uv\cos\phi \\ y &= uv\sin\phi \\ z &= \frac{u^2 - v^2}{2}, \end{aligned}$$

where $u \geq 0$, $v \geq 0$, and $0 \leq \phi < 2\pi$.

(a) Let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Write down the expressions for

$$\frac{\partial \boldsymbol{r}}{\partial u}, \qquad \frac{\partial \boldsymbol{r}}{\partial v}, \qquad \text{and} \qquad \frac{\partial \boldsymbol{r}}{\partial \phi}.$$

(b) Find the scale factors h_u, h_v, and h_ϕ.

(c) For $u, v \neq 0$ and $0 \leq \phi < 2\pi$, let

$$f(u, v, \phi) = uv^2\phi^3.$$

Write down a simplified expression for $\boldsymbol{\nabla} f(u, v, \phi)$ in the form

$$g_1(u, v, \phi)\boldsymbol{e}_u + g_2(u, v, \phi)\boldsymbol{e}_v + g_3(u, v, \phi)\boldsymbol{e}_\phi,$$

for some functions g_1, g_2, and g_3.

(d) For $u, v \neq 0$ and $0 \leq \phi < 2\pi$, let

$$\boldsymbol{F}(u, v, \phi) = \frac{1}{\sqrt{u^2 + v^2}}\left(\boldsymbol{e}_u + \boldsymbol{e}_v + \boldsymbol{e}_\phi\right).$$

Write down a simplified expression for $\boldsymbol{\nabla} \cdot \boldsymbol{F}(u, v, \phi)$.

End of Exam—Total Available Marks = 130

MAST20009 Vector Calculus Formulae Sheet

INTEGRATION FORMULAE AND IDENTITIES

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec x\,dx = \log|\sec x + \tan x| + C \qquad \int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \operatorname{cosec}^2 x\,dx = -\cot x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + C \qquad \int \operatorname{cosech}^2 x\,dx = -\coth x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \arccos\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \operatorname{arccosh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\operatorname{arctanh}\left(\frac{x}{a}\right) + C$$

where $a > 0$ is constant and C is an arbitrary constant of integration.

$\cos^2 x + \sin^2 x = 1$ $\qquad$ $\cosh^2 x - \sinh^2 x = 1$
$1 + \tan^2 x = \sec^2 x$ $\qquad$ $1 - \tanh^2 x = \operatorname{sech}^2 x$
$\cot^2 x + 1 = \operatorname{cosec}^2 x$ $\qquad$ $\coth^2 x - 1 = \operatorname{cosech}^2 x$

$\cos 2x = \cos^2 x - \sin^2 x$ $\qquad$ $\cosh 2x = \cosh^2 x + \sinh^2 x$
$\cos 2x = 2\cos^2 x - 1$ $\qquad$ $\cosh 2x = 2\cosh^2 x - 1$
$\cos 2x = 1 - 2\sin^2 x$ $\qquad$ $\cosh 2x = 1 + 2\sinh^2 x$
$\sin 2x = 2\sin x\cos x$ $\qquad$ $\sinh 2x = 2\sinh x\cosh x$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$ $\qquad$ $\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$
$\sin(x+y) = \sin x\cos y + \cos x\sin y$ $\qquad$ $\sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$

$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$ $\qquad$ $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$

$e^{ix} = \cos x + i\sin x$
$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$ $\qquad$ $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$

$\operatorname{arcsinh} x = \log(x + \sqrt{x^2+1})$ $\qquad$ $\operatorname{arccosh} x = \log(x + \sqrt{x^2-1})$

$$\operatorname{arctanh} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

VECTOR IDENTITIES

Let f and $g : \mathbb{R}^3 \to \mathbb{R}$ be scalar functions, $\boldsymbol{F}$ and $\boldsymbol{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be vector fields, and $\beta \in \mathbb{R}$ be any constant.

1. $\boldsymbol{\nabla}(f+g) = \boldsymbol{\nabla} f + \boldsymbol{\nabla} g$
2. $\boldsymbol{\nabla}(\beta f) = \beta \boldsymbol{\nabla} f$
3. $\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f$
4. $\boldsymbol{\nabla}\left(\frac{f}{g}\right) = \frac{g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g}{g^2}$ provided $g \neq 0$.
5. $\boldsymbol{\nabla} \cdot (\boldsymbol{F} + \boldsymbol{G}) = \boldsymbol{\nabla} \cdot \boldsymbol{F} + \boldsymbol{\nabla} \cdot \boldsymbol{G}$
6. $\boldsymbol{\nabla} \times (\boldsymbol{F} + \boldsymbol{G}) = \boldsymbol{\nabla} \times \boldsymbol{F} + \boldsymbol{\nabla} \times \boldsymbol{G}$
7. $\boldsymbol{\nabla} \cdot (f\boldsymbol{F}) = f\boldsymbol{\nabla} \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \boldsymbol{\nabla} f$
8. $\boldsymbol{\nabla} \cdot (\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\boldsymbol{\nabla} \times \boldsymbol{G})$
9. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) = 0$
10. $\boldsymbol{\nabla} \times (f\boldsymbol{F}) = f\boldsymbol{\nabla} \times \boldsymbol{F} + \boldsymbol{\nabla} f \times \boldsymbol{F}$
11. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \boldsymbol{0}$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g$
13. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g) = 0$
14. $\boldsymbol{\nabla} \cdot (f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f) = f\nabla^2 g - g\nabla^2 f$
15. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F}$

Note:
The identities require $f, g, \boldsymbol{F}$ and $\boldsymbol{G}$ to be suitably differentiable, either order C^1 or C^2.

IDENTITIES FOR ORTHOGONAL CURVILINEAR COORDINATES

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^2 scalar function and $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field where

$$\boldsymbol{F}(u_1, u_2, u_3) = F_1(u_1, u_2, u_3)\boldsymbol{e}_1 + F_2(u_1, u_2, u_3)\boldsymbol{e}_2 + F_3(u_1, u_2, u_3)\boldsymbol{e}_3.$$

Then

1. $\boldsymbol{\nabla} f = \dfrac{1}{h_1}\dfrac{\partial f}{\partial u_1}\,\boldsymbol{e}_1 + \dfrac{1}{h_2}\dfrac{\partial f}{\partial u_2}\,\boldsymbol{e}_2 + \dfrac{1}{h_3}\dfrac{\partial f}{\partial u_3}\,\mathbf{e_3}$

2. $\boldsymbol{\nabla} \cdot \boldsymbol{F} = \dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial\left(h_2 h_3 F_1\right)}{\partial u_1} + \dfrac{\partial\left(h_1 h_3 F_2\right)}{\partial u_2} + \dfrac{\partial\left(h_1 h_2 F_3\right)}{\partial u_3}\right]$

3. $\boldsymbol{\nabla} \times \boldsymbol{F} = \dfrac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$

4. $\nabla^2 f = \dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial}{\partial u_1}\left(\dfrac{h_2 h_3}{h_1}\dfrac{\partial f}{\partial u_1}\right) + \dfrac{\partial}{\partial u_2}\left(\dfrac{h_1 h_3}{h_2}\dfrac{\partial f}{\partial u_2}\right) + \dfrac{\partial}{\partial u_3}\left(\dfrac{h_1 h_2}{h_3}\dfrac{\partial f}{\partial u_3}\right)\right]$

Note:
Equations 1-4 reduce to the usual expressions for cartesian coordinates if

$$h_1 = h_2 = h_3 = 1; \qquad \boldsymbol{e}_1 = \boldsymbol{i}, \boldsymbol{e}_2 = \boldsymbol{j}, \boldsymbol{e}_3 = \boldsymbol{k}; \qquad (u_1, u_2, u_3) = (x, y, z).$$