PHYC10003 Physics 1

Lecture 18: Equilibrium

Static equilibrium and elasticity

Last lecture

- Conservation of angular momentum
- Analogies between linear and rotational motion

Equilibrium and stability

- We often want objects to be stable despite forces acting on them
- Consider a book resting on a table, an ice puck sliding with constant velocity, a rotating ceiling fan, a rolling bicycle wheel with constant velocity
- These objects have the characteristics that:
  1. The linear momentum of the center of mass is constant
  2. The angular momentum about the center of mass, or any other point, is constant

Equilibrium

- Such objects are in **equilibrium**

$$\vec{P} = \text{a constant} \quad \text{and} \quad \vec{L} = \text{a constant}.$$ **Eq. (12-1)**

- Here, we are largely concerned with objects that are not moving at all; $P = L = 0$
- These objects are in **static equilibrium**
- The only one of the examples from the previous page in static equilibrium is the book at rest on the table

Stable and unstable equilibrium

- As discussed in 8-3, if a body returns to static equilibrium after a slight displacement, it is in *stable* static equilibrium
- If a small displacement ends equilibrium, it is *unstable*
- Despite appearances, this rock is in stable static equilibrium, otherwise it would topple at the slightest gust of wind

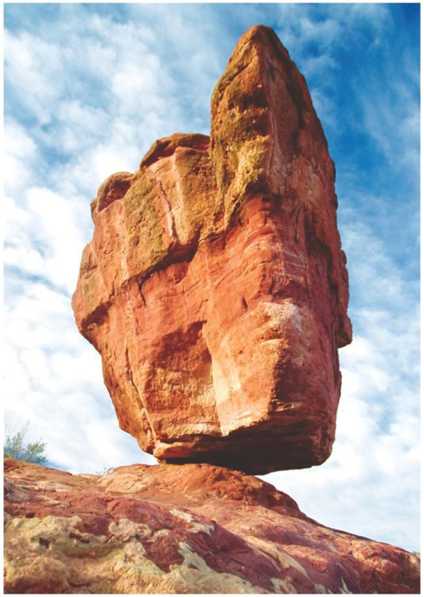

Figure 12-1

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Stable and unstable equilibrium

- In part (a) of the figure, we have unstable equilibrium
- A small force to the right results in (b)
- In (c) equilibrium is stable, but push the domino so it passes the position shown in (a) and it falls
- The block in (d) is even more stable

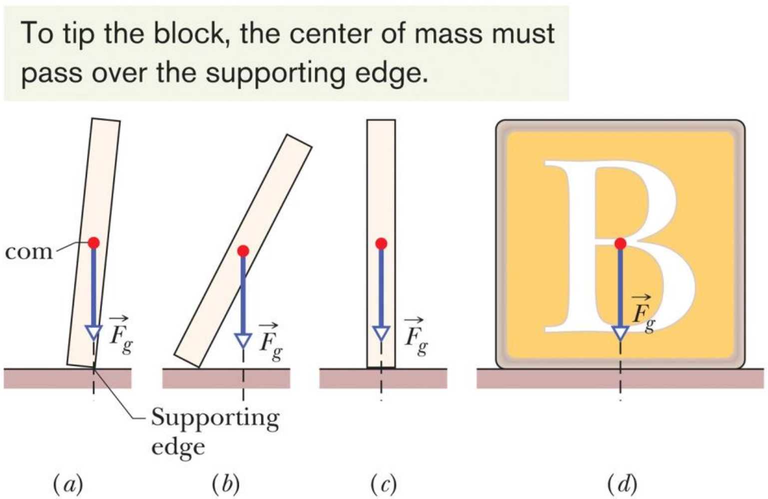


Figure 12-2

Equilibrium - requirements

- Requirements for equilibrium are given by Newton's second law, in linear and rotational form

$$\vec{F}_{\text{net}} = 0 \qquad \text{(balance of forces).}$$ **Eq. (12-3)**

$$\vec{\tau}_{\text{net}} = 0 \qquad \text{(balance of torques).}$$ **Eq. (12-5)**

- Therefore we have for equilibrium:

1. The vector sum of the external forces that act on the body is zero.
2. The vector sum of all torques that act on the body is zero, measured around *any* possible fixed point.

Equilibrium - conditions

Simple case: consider forces only in the *xy* plane,:

$$F_{\text{net},x} = 0 \quad \text{(balance of forces),}$$ **Eq. (12-7)**

$$F_{\text{net},y} = 0 \quad \text{(balance of forces),}$$ **Eq. (12-8)**

$$\tau_{\text{net},z} = 0 \quad \text{(balance of torques).}$$ **Eq. (12-9)**

For static equilibrium additional requirements are:

3. The linear momentum, $\vec{P}$, of the body must be zero
4. The angular momentum of the body, $\vec{L}$, must be zero.

Centre of gravity

- The gravitational force on a body is the sum of gravitational forces acting on individual elements (atoms) of the body
- The gravitational force, $\boldsymbol{F}_g$: on a body effectively acts at a single point called the centre of gravity (cog).
- Until now we have assumed that the gravitational force acts at the centre of mass (com)
- This is approximately true for the everyday case: if $\boldsymbol{g}$ is the same everywhere in the body then the centre of mass coincides with the centre of gravity.

Torque and gravitational force

Consider a sum of torques on each element vs. the torque caused by the gravitational force at the cog

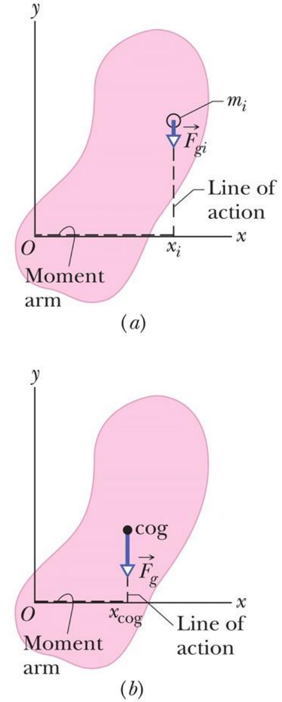
$$x_{\text{cog}} \sum F_{gi} = \sum x_i F_{gi}.$$

Substitute $m_i g_i$ for F_{gi}:

$$x_{cog} \sum m_i g_i = \sum x_i m_i g_i$$

$$x_{cog}\, g \sum m_i = g \sum x_i m_i$$

$$x_{cog} = \frac{\sum x_i m_i}{\sum m_i} \qquad x_{cog} = \frac{1}{M} \sum x_i m_i$$



Equilibrium - example

Balancing a horizontal beam

- M=2.7 kg, m=1.8 kg
- Set rotation axis at x=0

Sum the torques

$\frac{1}{4} Mg\, L + \frac{1}{2}\, mgL = F_r L$

F_r: force at RHS of beam

Hence, F_r = 15 N.

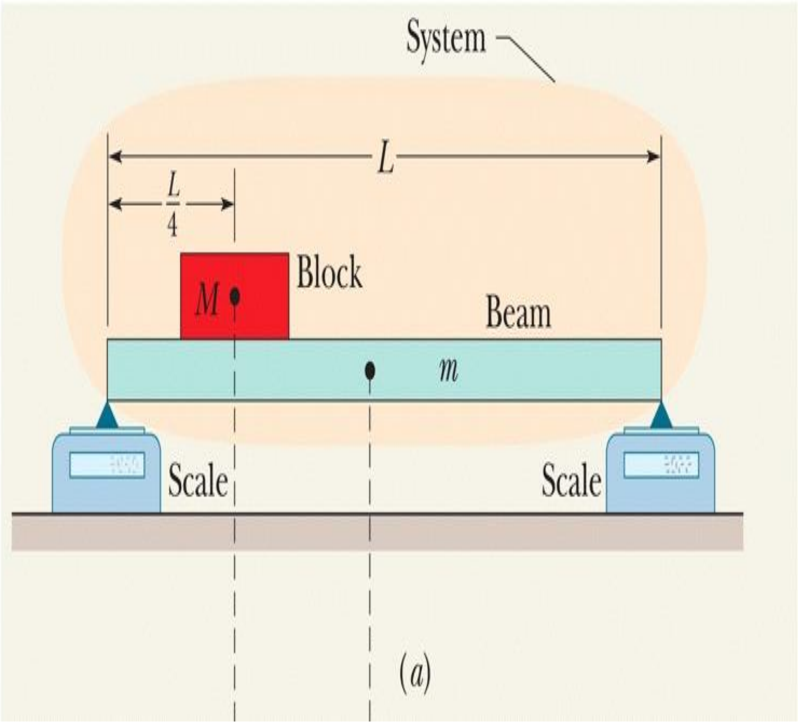
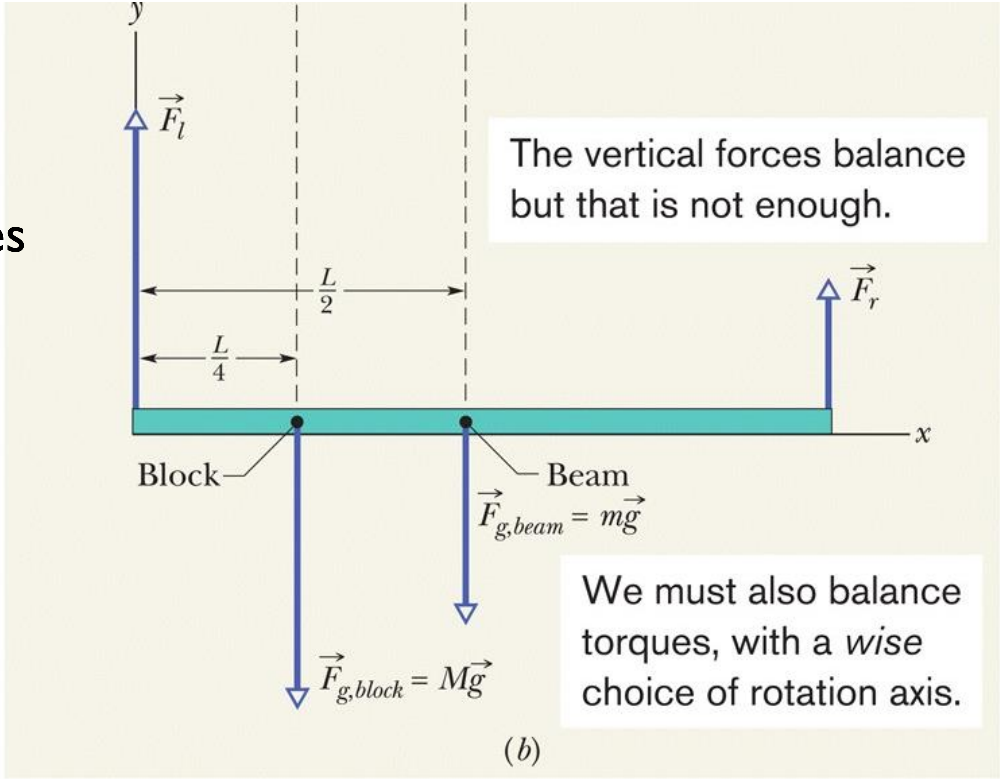


Figure 12-5

Equilibrium - example

Balance vertical forces

$$F_l = (M+m)g - F_r$$
$$= 29N$$



Equilibrium - example

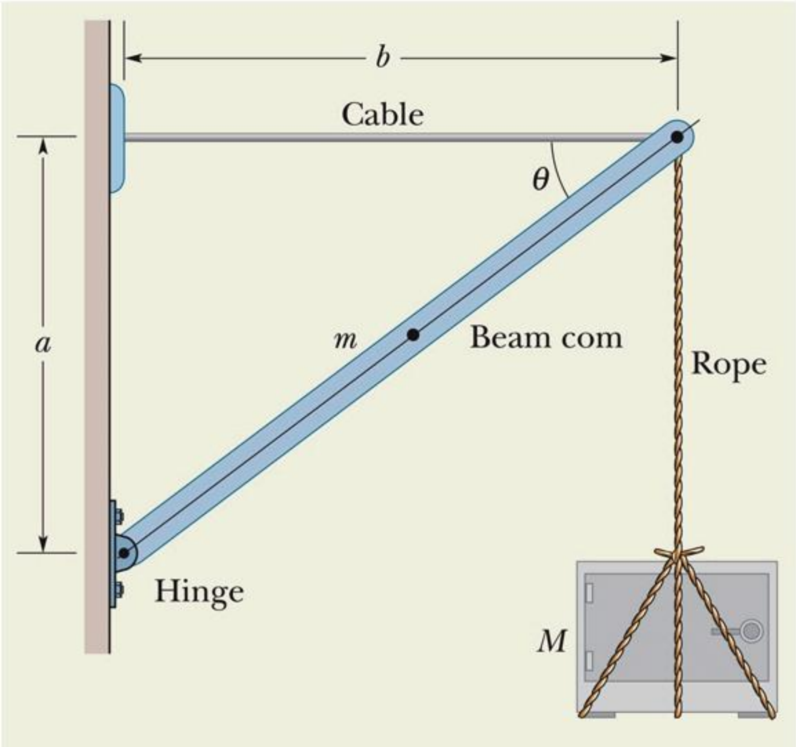
Balancing a leaning beam:

Find the tension in the cable, in the rope and the size of force *F* at the hinge.

M = 430 kg, m = 85 kg,

a = 1.9 m, b = 2.5 m

Set rotation axis at x=0, y=0



Equilibrium - example

1: Sum the torques, using $T_r = Mg$

$$aT_c - bT_r - \frac{1}{2}bmg = 0$$

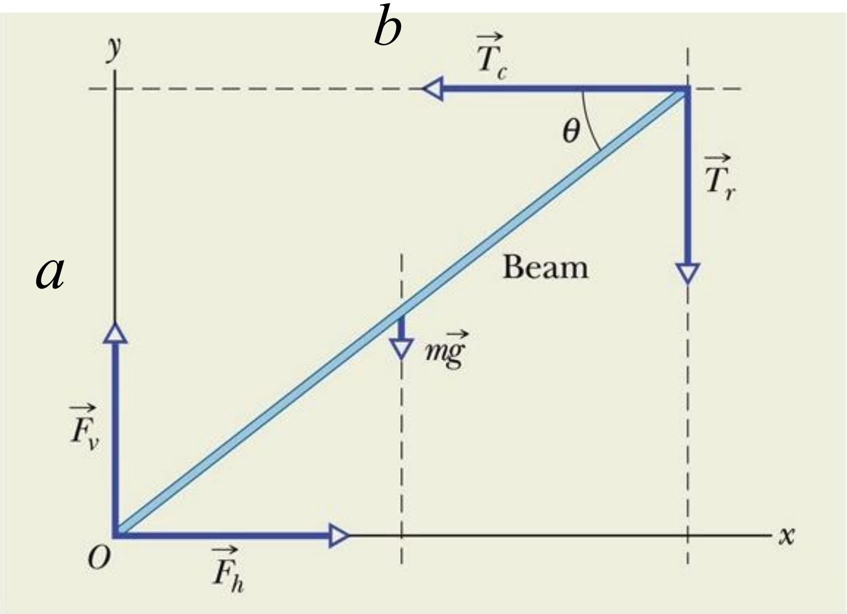
$$\therefore T_c = 6100N$$

2: Balance the forces

$$F_h = T_c = 6100 \text{ N}$$

$$F_v = (m + M)g = 5050 \text{ N}$$

$$F = 7900 \text{ N}$$



Elasticity

- For problems in the *xy* plane we have 3 independent equations
- Therefore we can solve for 3 unknowns
- If we have more unknown forces, we cannot solve for them and the situation is **indeterminate**
- This assumes that bodies are rigid and do not deform (there are no such bodies)
- With some knowledge of elasticity, we can solve more problems

Elasticity

- All rigid bodies are partially **elastic**, meaning we can change their dimensions by applying forces
- A **stress**, deforming force per unit area, produces a **strain**, or unit deformation
- There are 3 main types of stress:

  (a) Tensile, (b) Shearing, (c) Hydraulic

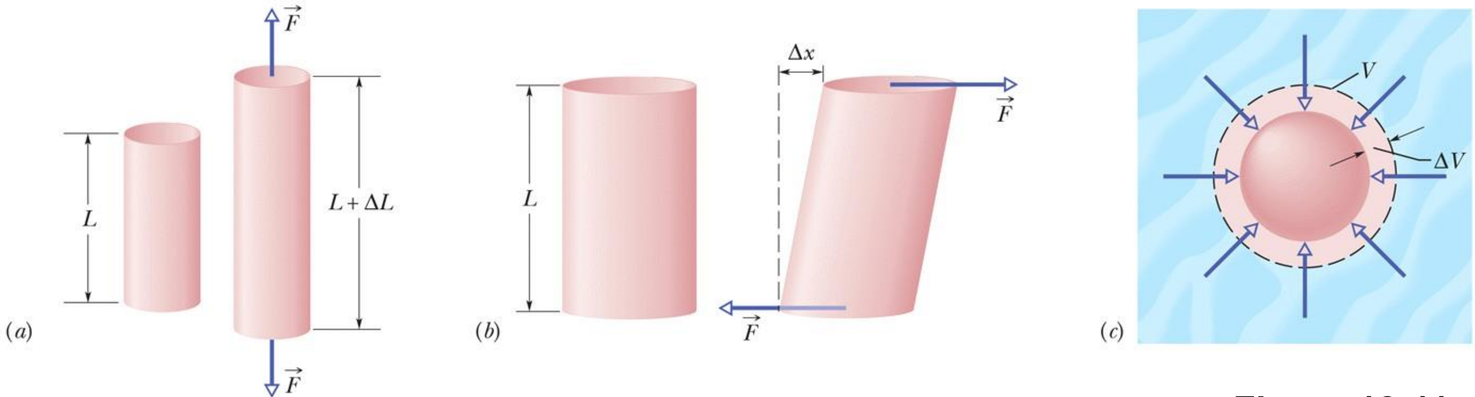


Figure 12-11

Elasticity – stress and strain

- Stress and strain are proportional in the elastic range
- Related by the **modulus of elasticity**:

$$\text{stress} = \text{modulus} \times \text{strain}. \qquad \textbf{Eq. (12-22)}$$

- As stress increases, eventually a **yield strength** is reached and the material deforms permanently
- At the **ultimate strength**, the material breaks

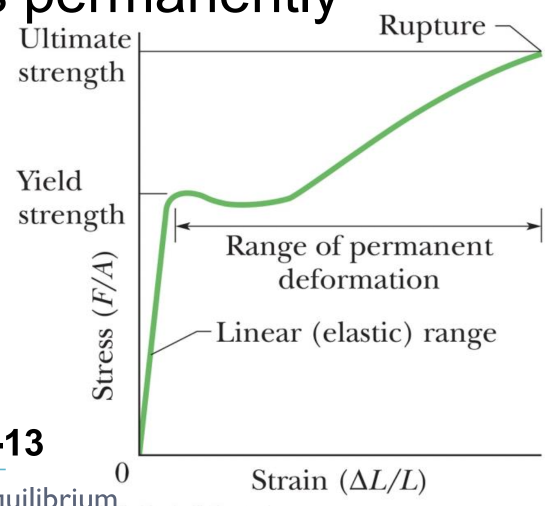


Figure 12-13

Elasticity- Young's modulus

- In simple tension/compression, stress is *F/A*
- The strain is the dimensionless quantity *ΔL/L*
- **Young's modulus**, *E*, used for tension/compression

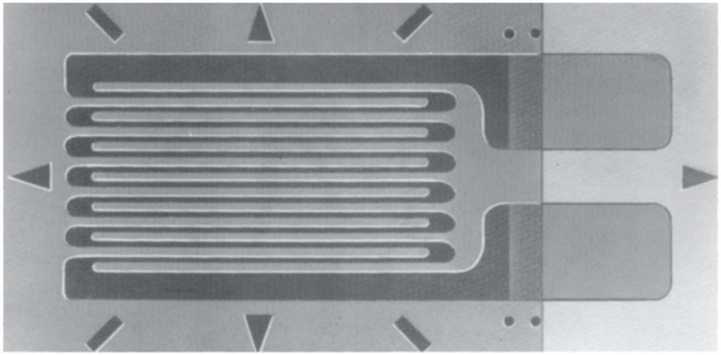
$$\frac{F}{A} = E\,\frac{\Delta L}{L}. \qquad \textbf{Eq. (12-23)}$$

- Note that many materials have very different tensile and compressive strengths, despite the same modulus being used for both
- Example - concrete: high compressive strength, very low tensile strength

Elasticity - measurement

- Strain can be measured by a strain gage
- Placed on the material, it becomes subject to the same strain
- Strain can be read out as a change in electrical resistance, for strains up to 3%

Figure 12-14



Courtesy Micro Measurements, a Division of Vishay Precision Group, Raleigh, NC

Elasticity – shear and bulk modulus

- **Shear modulus**, *G*, used for shearing

$$\frac{F}{A} = G\,\frac{\Delta x}{L}.$$

Eq. (12-24)

- *Δx* is along a *different* axis than *L*
- **Bulk modulus**, *B*, used for hydraulic compression

$$p = B\,\frac{\Delta V}{V}.$$

Eq. (12-25)

- Relates pressure to volume change

Elasticity – some examples

The table shows some elastic properties for common materials, for comparison purposes

Table 12-1 Some Elastic Properties of Selected Materials of Engineering Interest

Material	Density ρ (kg/m^3)	Young's Modulus E ($10^9\ N/m^2$)	Ultimate Strength S_u ($10^6\ N/m^2$)	Yield Strength S_y ($10^6\ N/m^2$)
Steel[a]	7860	200	400	250
Aluminum	2710	70	110	95
Glass	2190	65	50[b]	—
Concrete[c]	2320	30	40[b]	—
Wood[d]	525	13	50[b]	—
Bone	1900	9[b]	170[b]	—
Polystyrene	1050	3	48	—

[a]Structural steel (ASTM-A36). [b]In compression.
[c]High strength [d]Douglas fir.

Table 12-1

Elasticity - example

Balancing a wobbly table:

Three legs are 1.00m long but the fourth is longer by 0.50mm

- Compressed by M = 290 kg: legs are compressed but not buckled and the table does not wobble
- Legs are wooden cylinders with area A = 1.0 cm^2
- E = 1.3 x 10^{10} N/m^2 (value for Douglas fir)
- 3 shorter legs must compress the same amount, the longer leg compresses more
- Write length comparison, use the stress-strain equation, and approximate all legs to be length L

Elasticity - example

1: Use elasticity equation

$$\frac{F}{A} = E\frac{\Delta L}{L}.$$

$$\frac{F_3 L}{AE} = \Delta L, \quad \frac{F_4 L}{AE} = \Delta L + d, \qquad \therefore \quad \frac{F_4 L}{AE} = \frac{F_3 L}{AE} + d$$

2: Balance the forces $$3F_3 + F_4 - Mg = 0$$

3: Solve for F_3 and F_4: F_3=550 N F_4=1200 N

Each short leg is compressed by 0.42 mm, and the long leg is compressed by 0.92 mm

Summary

Static Equilibrium

$$\vec{F}_{\text{net}} = 0 \qquad \text{(balance of forces).}$$

Eq. (12-3)

$$\vec{\tau}_{\text{net}} = 0 \qquad \text{(balance of torques).}$$

Eq. (12-5)

Center of Gravity

- If the gravitational acceleration is the same for all elements of the body, the cog is at the com.

Elastic Moduli

- Three elastic moduli
- Strain: fractional length change
- Stress: force per unit area

$$\text{stress} = \text{modulus} \times \text{strain}.$$

Eq. (12-22)

Tension and Compression

- *E* is Young's modulus

$$\frac{F}{A} = E\,\frac{\Delta L}{L}.$$

Eq. (12-23)

Summary

Shearing

- *G* is the shear modulus

$$\frac{F}{A} = G\frac{\Delta x}{L}.$$ **Eq. (12-24)**

Hydraulic Stress

- *B* is the bulk modulus

$$p = B\frac{\Delta V}{V}.$$ **Eq. (12-25)**