Tutorial 7: Bases of subspaces

Q1. You are given that

$$A = \begin{bmatrix} 2 & 0 & -2 & 3 & 0 & 4 \\ -11 & 8 & 43 & 9 & 12 & 17 \\ -3 & -1 & -1 & 0 & 0 & 3 \\ 2 & -1 & -6 & 1 & -1 & 1 \\ 1 & 2 & 7 & 0 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

(a) Write down a basis for the row space of A.

(b) Write down a basis for the column space of A.

(c) Does the set

$$\{(2, -11, -3, 2, 1), (0, 8, -1, -1, 2), (3, 9, 0, 1, 0), (0, 12, 0, -1, 2), (4, 17, 3, 1, -3)\}$$

span $\mathbb{R}^5$? Explain your answer.

(d) What is the dimension of the solution space of A?

(e) Find a basis for the solution space of A.

(f) Write the vectors $(-2, 43, -1, -6, 7)$ and $(4, 17, 3, 1, -3)$ as linear combinations of the other columns of A.

Q2. Let

$$S = \{2 + x - x^3, 3 - x^2 + 2x^3, 1 + x + x^2 + x^3, 1 - x^2 - 2x^3\}.$$

Find a subset of S that is a basis for the subspace of $\mathcal{P}_3$ spanned by S.

[Hint: Use coordinate vectors to reduce to a problem in $\mathbb{R}^4$.]

Q3. Find a basis for the (i) row space, (ii) column space and (iii) solution space of the following matrix where the entries are in $\mathbb{F}_2$. State the dimension of each space. How many elements are in each space?

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Q4. Consider the linear transformation $T : \mathcal{P}_3 \to \mathcal{P}_3$ defined by

$$T(p(x)) = (1 - x^2)p''(x) - 2xp'(x) + 12p(x),$$

and the ordered basis $\mathcal{B} = (1, x, x^2, x^3)$ for $\mathcal{P}_3$.

(a) (i). Find the image of each basis vector, $T(1)$, $T(x)$, $T(x^2)$ and $T(x^3)$, and express each as a linear combination of the basis vectors in $\mathcal{B}$.

(ii). Find the coordinate vectors: $[T(1)]_{\mathcal{B}}$, $[T(x)]_{\mathcal{B}}$, $[T(x^2)]_{\mathcal{B}}$ and $[T(x^3)]_{\mathcal{B}}$

(iii). Write down the matrix of the transformation T with respect to the ordered basis $\mathcal{B}$.

(b) Find bases for the kernel and image of T.

(c) Using these bases, answer the following questions:

(i). What are the degree ≤ 3 polynomial solutions to the differential equation

$$(1 - x^2)p''(x) - 2xp'(x) + 12p(x) = 0\,?$$

(ii). Do the following differential equations have a solution in $\mathcal{P}_3$?

$$(1 - x^2)p''(x) - 2xp'(x) + 12p(x) = x^3$$

$$(1 - x^2)p''(x) - 2xp'(x) + 12p(x) = 10$$

Q5. Let $T : \mathbb{R} \to \mathbb{R}$ be the linear transformation defined by $T(x, y) = (5x - 3y, 6x - 4y)$. Let $\mathcal{B} = ((2, 2), (1, 2))$ and $\mathcal{S} = ((1, 0), (0, 1))$ be ordered bases of $\mathbb{R}^2$.

(a) Write down the matrix of T with respect to the standard basis of $\mathbb{R}^2$. Call this matrix A.

(b) Find $T(\mathbf{b}_1)$ and $T(\mathbf{b}_2)$ and hence determine λ_1 and λ_2 such that $T(\mathbf{b}_1) = \lambda_1\mathbf{b}_1$ and $T(\mathbf{b}_2) = \lambda_2\mathbf{b}_2$.

(c) Use your answer to (b) to find $T(2, 0)$. Note that $(2, 0) = (4, 4) + (-2, -4)$.

(d) On the same diagram mark the points $(2, 2)$, $(1, 2)$, $T(2, 2)$, $T(1, 2)$, $(2, 0) = (4, 4) + (-2, -4)$ and $T(2, 0)$ clearly showing how $T(2, 0)$ is obtained from $T(2, 2)$ and $T(1, 2)$.

(e) Write down $P_{\mathcal{S}\leftarrow\mathcal{B}}$ and hence find $P_{\mathcal{B}\leftarrow\mathcal{S}}$.

(f) Use your answer to (e) to find $[T]_{\mathcal{B}}$, the matrix of T with respect to the ordered basis $\mathcal{B}$.

(g) What do you notice about your answer to (f) and your answer to (b)?