MAST20009 Vector Calculus

Practice Class 1 Questions

We say $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has the limit L as (x, y) approaches (x_0, y_0):

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) \;=\; L$$

if when (x, y) approaches (x_0, y_0) along ANY path in the domain D, $f(x, y)$ gets close to L.

The limit can exist if $f(x, y)$ is undefined at (x_0, y_0).

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1. Evaluate the following limits if they exist. If they do not exist, explain why.

   (a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy}{x+y+1}$ $= 0$

   (b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ $y=kx$ $\quad \frac{x^2-k^2x^2}{x^2+k^2x^2} = \frac{1-k^2}{1+k^2}$ depend on k

   (c) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{xy}{x^3-y^3}$ $y=kx$ Along $x=0$ $\quad \lim_{y\to 0} \frac{y}{-y^3} = \lim_{y\to 0} -\frac{1}{y^2} = \infty$ not exist.

   (d) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2y^4}{3x^4+y^4}$ Along $x=0$ $\quad \lim_{(x,y)\to(0,0)} \frac{0}{y^4} = 0$

   $y=kx$ $\quad \lim_{(x,y)\to(0,0)} \frac{k^4x^6}{3x^4+k^4x^4} = \frac{k^4x^2}{3+k^4} = 0$

   Sandwich theorem

   $0 \le \frac{x^2y^4}{3x^4+y^4} \le \frac{x^2(y^4+\cancel{3x^4})}{\cancel{3x^4+y^4}} \to 0.$

$f(x, y)$ is continuous at $(x, y) = (x_0, y_0)$ if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0).$$

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2. Show that the following function f is *not* continuous at $(x, y) = (1, 0)$.

$$f(x,y) \;=\; \begin{cases} \dfrac{x^2-2x+1+y^2}{x^3-2x^2+x+xy^2}, & (x,y) \neq (1,0) \\[2ex] 2, & (x,y) = (1,0) \end{cases}$$

Can $f(x, y)$ be defined differently at $(1, 0)$ so that the function is continuous there?

$\lim_{(x,y)\to(1,0)} \frac{x^2-2x+1+y^2}{x^3-2x^2+x+xy^2} = \lim_{(x,y)\to(1,0)} \frac{1}{x} = 1.$

$f(x, y)$ is differentiable at (x_0, y_0) if (i) $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ exist at (x_0, y_0) (ii) a tangent plane exists at (x_0, y_0) and is a good approximation to $f(x, y)$ at (x_0, y_0).

If $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ exist and are continuous at (x_0, y_0) then $f(x, y)$ is differentiable at (x_0, y_0).

3. Consider the function

$$f(x, y) = \frac{y^3}{x^2 + y^2}.$$

(a) $f_x = \dfrac{-2xy^3}{(x^2+y^2)^2}$

$f_y = \dfrac{3y^2(x^2+y^2) - y^3 \cdot 2y}{(x^2+y^2)^2} = \dfrac{y^4 + 3x^2y^2}{(x^2+y^2)^2}$. ✓

(a) What is the largest possible domain for f? Is f a C^1 function on this domain?

$(x,y) \in \mathbb{R} \setminus \{0,0\})$.

(b) Define $f(0, 0) = 0$.

(b). $\lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \dfrac{\frac{0}{h^2} - 0}{h} = \cancel{\lim_{h\to 0} \frac{1}{h^2}}\ 0$ $\cancel{= \infty}$.

$\lim_{h\to 0} \dfrac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \dfrac{\frac{h^3}{h^2} - 0}{h} = 1$.

$y = kx$

$f_x = \lim_{x\to 0} \dfrac{-2k^3x^4}{(x^2+k^2x^2)^2} = \dfrac{-2k^3}{(1+k^2)^2}$

depend on k

(i) Calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at the origin.

(ii) Are $\dfrac{\partial f}{\partial x}$ ✗ and $\dfrac{\partial f}{\partial y}$ continuous at the origin? ✗

(iii) What conclusion can we make about the differentiability of f at the origin?

When you have finished the above questions, continue working on the questions in the Vector Calculus Problem Sheet Booklet.

$y = kx$

$f_y = \lim_{x\to 0} \dfrac{k^4x^4 + 3k^2\cancel{x^4}}{(1+k^2)}$ de