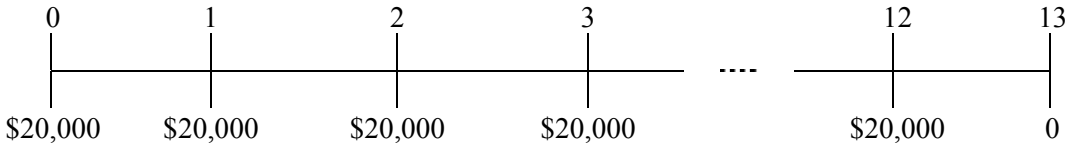
Note on Present and Future Values of Growing Ordinary Annuities

Example

Your second cousin is about to start kindergarten at Melbourne's third most prestigious private school. The tuition is $20,000 per year, payable at the *beginning* of the school year. You expect your second cousin to remain in private school through high school. What is the present value of the tuition payments if the interest rate is 5 percent per year? How does your answer change if you expect the tuition fees to increase at a rate of 5 percent per year over the 13 years of her schooling?

Analysis

In the first case, we have an annuity due and the timeline (assuming end-of-the-year cash flows) is as follows:



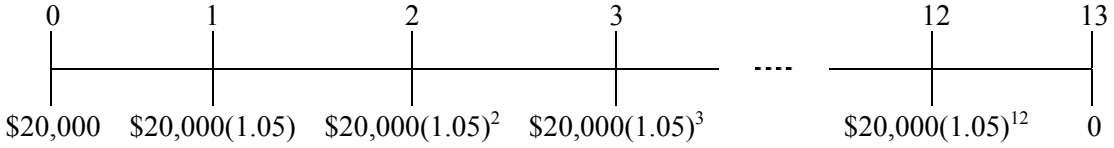
The present value of an annuity due is:

$$PV_0 = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^n}\right)(1+r).$$

In this case, we get:

$$PV_0 = \left(\frac{20000}{0.05}\right)\left(1 - \frac{1}{(1+0.05)^{13}}\right)(1+0.05) = \$197,265.$$

In the second case, the timeline (assuming end-of-the-year cash flows) is as follows:



The first payment is already at time 0. The remaining 12 payments represent a growing annuity cash flow which we would normally value using the following expression:

$$PV_0(GA) = \left(\frac{C_1}{r-g}\right)\left(1 - \left(\frac{1+g}{1+r}\right)^n\right).$$

Note, however, that in this case we **cannot** use the present value of a growing annuity expression above because $r = g$! So, we need to calculate the present value of each of the remaining 12 payments and add them up to get the present value today, as follows:

$$PV_0 = \frac{20000(1.05)}{(1.05)} + \frac{20000(1.05)^2}{(1.05)^2} + \frac{20000(1.05)^3}{(1.05)^3} + \cdots + \frac{20000(1.05)^{12}}{(1.05)^{12}}$$

Because the growth rate in the tuition fees equals the interest rate used to evaluate the cash flows the above set of calculations essentially is just adding up the cash flows:

$PV_0 = 20000 + 20000 + 20000 + \ldots + 20000 = 20000 \times 12 = \$240{,}000$

Adding the initial tuition payment today gives a total present value of 240000 + 20000 = $260,000.

If we needed the future value of the above cash flows we would need to compound them individually at the interest rate of 5%. We could, of course, do this directly by compounding the present value at the interest rate of 5%.