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Subject : Probability Assignment 1.
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Tutorial time : Monday 10 - 12 am

1. (a) from property 6 $P(A^c) = 1 - P(A)$
so $P((A \cup C)^c) = 1 - P(A \cup C) = 1 - 0.4 = \underline{\underline{0.6}}$

(b) from property 9 (addition theorem) $P(A \cup C) = P(A) + P(C) - P(A \cap C)$
so $P(A) = P(A \cup C) + P(A \cap C) - P(C) = 0.4 + 0.1 - 0.2 = \underline{\underline{0.3}}$

(c) from the definition of conditional probability $P(C | A \cup B \cup C) = \dfrac{P(C \cap (A \cup B \cup C))}{P(A \cup B \cup C)}$
from De Morgan's law $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$
so from property 6 $P(A \cup B \cup C) = 1 - P((A \cup B \cup C)^c) = 1 - P(A^c \cap B^c \cap C^c) = 1 - 0.4 = 0.6.$
$C \subset (A \cup B \cup C)$
so $C \cap (A \cup B \cup C) = C$ $P(C \cap (A \cup B \cup C)) = P(C)$
so $P(C | A \cup B \cup C) = \dfrac{P(C)}{P(A \cup B \cup C)} = \dfrac{0.2}{0.6} = \dfrac{1}{3} = \underline{\underline{0.333}}$

(d). $P(B | C^c) = \dfrac{P(B \cap C^c)}{P(C^c)} \underset{\text{property 6}}{=} \dfrac{P(B \cap C^c)}{1 - P(C)}.$

for $(B \cap C^c) = B \setminus C$.
since $(B \cap C^c) \cup (B \cap C) = B$, $(B \cap C^c) \cap (B \cap C) = \phi \Rightarrow (B \cap C^c)$ and $(B \cap C)$ are mutually exclusive events

from axiom 3 $P((B \cap C^c) \cup (B \cap C)) = P(B \cap C^c) + P(B \cap C) = P(B).$
so $P(B \cap C^c) = P(B) - P(B \cap C) = 0.4 - 0.2 = 0.2.$
(since $P(B \cap C) = P(B) + P(C) - P(B \cup C) = 0.4 + 0.2 - 0.4 = 0.2$).

then $P(B | C^c) = \dfrac{0.2}{1 - 0.2} = \underline{\underline{0.25}}.$

2. (a). ① prove if $P(A) = P(B)$, then $P(A \cap B^c) = P(A^c \cap B)$.
since $\begin{cases} (A \cap B) \cup (A \cap B^c) = A \\ (A \cap B) \cap (A \cap B^c) = \phi \end{cases} \Rightarrow$ $A \cap B$ and $A \cap B^c$ form a partition of A.
similarly $\begin{cases} (B \cap A) \cup (B \cap A^c) = B \\ (B \cap A) \cup (B \cap A^c) = \phi \end{cases} \Rightarrow B \cap A$ and $B \cap A^c$ form a partion of B.

from axiom 3. $P(A) = P((A \cap B) \cup (A \cap B^c)) = P(A \cap B) + P(A \cap B^c).$
$P(B) = P((B \cap A) \cup (B \cap A^c)) = P(B \cap A) + P(B \cap A^c).$

since $P(A \cap B) = P(B \cap A)$ and $P(A) = P(B)$
so $\cancel{P(A \cap B)} + P(A \cap B^c) = \cancel{P(B \cap A)} + P(B \cap A^c)$, $\underline{P(A \cap B^c) = P(B \cap A^c)}.$

(a) ② prove if $P(A\cap B^c) = P(A^c\cap B)$, then $P(A)=P(B)$.

$$\underbrace{P(A\cap B^c) + P(A\cap B)}_{\text{disjoint}} = \underbrace{P(A^c\cap B) + P(A\cap B)}_{\text{disjoint}}.$$

from axiom 3

$$P((A\cap B^c)\cup(A\cap B)) = P((A^c\cap B)\cup(A\cap B))$$

distributive law.

$$P(A\cap(B^c\cup B)) = P(B\cap(A^c\cup A)).$$

complement law $B^c\cup B = \Omega = A^c\cup A$

$$P(A\cap\Omega) = P(B\cap\Omega)$$

so $P(A) = P(B)$.

combining ① and ②, we get $P(A)=P(B)$ if and only if $P(A\cap B^c)=P(A^c\cap B)$

(b). De Morgan's Law $A^c\cup B^c = (A\cap B)^c$

from property 6 $P((A\cap B)^c) = 1-P(A\cap B) = 1$

so $P(A\cap B)=0$ which means that event A and B have no intersecti (event A and event B are disjoint)

so from question $P(A)+P(B)=1$ so we can set $B=A^c$ which satis the conditi

so it is possible to have $P(A)=0.7$, $P(B)=0.3$ and $P(A^c\cup B^c)=1$

(c). since $(A\cap B^c)\cap(A\cap B)=\phi$

$(A\cap B^c)\cup(A\cap B) = A\cap(B^c\cup B) = A\cap\Omega = A$ > $A\cap B^c$, $A\cap B$ are disjoint

so. $P(A) = P(A\cap B^c)+P(A\cap B)$. ①

from addition law $P(A\cup B^c) = P(A)+P(B^c) - P(A\cap B^c)$. ②

① can write $-P(A\cap B^c) = P(A\cap B) - P(A)$

then substitute to ②

$$P(A\cup B^c) = \cancel{P(A)} + P(B^c) + P(A\cap B) - \cancel{P(A)} \quad ③.$$

since $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = 0.7$.

so $P(A\cap B) = P(A|B)\cdot P(B) = 0.7\times 0.35 = 0.245$.

substitute to ③ LHS $= P(A\cup B^c) = 1$

RHS $= P(B^c)+P(A\cap B) = 1-P(B)+P(A\cap B) = 1-0.35+0.245$
$= 0.895 < 1$

so LHS $\neq$ RHS

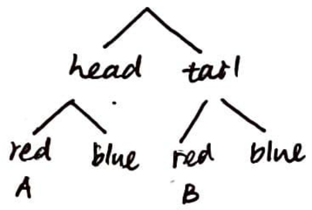
not possible..

3.

(a) Let A be the event a red ball is chosen from Jar A

Let B be the event a red ball is chosen from Jar B

Let R be the event that a red ball is chosen, "head" be tossing a coin when "head up"
"tail" be tossing a coin where "tail up"



$$P(A) = \frac{\#red_A}{\#red_A + \#blue_A} = \frac{2}{2+5} = \frac{2}{7} = P(R \mid head)$$

$$P(B) = \frac{\#red_B}{\#red_B + \#blue_B} = \frac{4}{4+6} = \frac{2}{5} = P(R \mid tail).$$

$$P(head) = \frac{1}{3} \qquad P(tail) = 1 - P(head) = \frac{2}{3}$$

$$P(R) = P(R \mid head) \cdot P(head) + P(R \mid tail) \cdot P(tail)$$

$$= \frac{2}{7} \times \frac{1}{3} + \frac{2}{5} \times \frac{2}{3}$$

$$= \frac{38}{105}$$

(b). $$P(tail \mid R) = \frac{P(tail \cap R)}{P(R)} = \frac{P(R \mid tail) \cdot P(tail)}{P(R \mid head) \cdot P(head) + P(R \mid tail) \cdot P(tail)} = \frac{\frac{2}{5} \cdot \frac{2}{3}}{\frac{38}{105}} = \frac{14}{19}$$

↑ law of total probability and Bayer's formula
event "head" and "tail" is partition of Ω.

(c). tail = {a tail is showing}

R = {a red ball is chosen}

$p(tail) = \frac{2}{3}$ $\qquad P(tail \mid R) = \frac{14}{19}$ ......

since $\frac{14}{19} > \frac{2}{3}$, so $P(tail \mid R) > P(tail)$, which means "a red ball is chosen" increase the probability of "a tail is showing", so they are positively related.

4. (a) Let E: be the event that roll a die when it's an even number.

O be the event that roll a die when it's an odd number.

$P(E) = P(O) = \frac{1}{2} P(\Omega) = \frac{1}{2}$.

$B_1^c = \{(OOO)\}$ $\qquad P(B_1) = 1 - P(B_1^c) = 1 - (\frac{1}{2})^3 = \frac{7}{8}$

$B_2 = \{(OEE), (EOE), (EEO), (EEE)\}$

$P(B_2) = \binom{3}{1}(\frac{1}{2})^3 + (\frac{1}{2})^3 = \frac{1}{2}$

$B_3 = \{(EEE), (OOO)\}$

$P(B_3) = (\frac{1}{2})^3 + (\frac{1}{2})^3 = \frac{1}{4}$

$B_1 \cap B_2 = \{(OEE), (EOE), (EEO), (EEE)\} = B_2$

$P(B_1 \cap B_2) = \frac{1}{2}$ while $P(B_1) \cdot P(B_2) = \frac{7}{16}$ so B_1 and B_2 are dependent.

$B_1 \cap B_3 = \{(EEE)\}$.

$P(B_1 \cap B_3) = (\frac{1}{2})^3 = \frac{1}{8}$

P $P(B_1) \cdot P(B_3) = \frac{7}{8} \cdot \frac{1}{4} = \frac{7}{32} \neq P(B_1 \cap B_3)$ so B_1 and B_3 are dependent

$B_2 \cap B_3 = \{(EEE)\}$.

$P(B_2 \cap B_3) = (\frac{1}{2})^3 = \frac{1}{8}$

$P(B_2) \cdot P(B_3) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} = P(B_2 \cap B_3)$ so B_2 and B_3 are independent.

(b). the die is rolled four times.

$B_2 = \{(OEEE), (EOEE), (EEOE), (EEEO), (EEEE)\}$.

$B_3 = \{(EEEE), (OOOO)\}$.

$B_2 \cap B_3 = \{(EEEE)\}$

$P(B_2 \cap B_3) = (\frac{1}{2})^4 = \frac{1}{16}$

$P(B_2) = \binom{4}{1}(\frac{1}{2})^4 + (\frac{1}{2})^4 = \frac{5}{16}$

$P(B_3) = (\frac{1}{2})^4 \cdot 2 = \frac{1}{8}$

$P(B_2) \cdot P(B_3) = \frac{5}{16} \cdot \frac{1}{8} = \frac{5}{128} \neq P(B_2 \cap B_3)$.

so B_2 and B_3 are dependent.

$B_1 \cap B_2 \cap B_3 = \{(EEEE)\}$.

$P(B_1 \cap B_2 \cap B_3) = (\frac{1}{2})^4 = \frac{1}{16}$

$B_1 = \{\Omega \setminus (OOOO)\}$ $\quad P(B_1) = 1 - (\frac{1}{2})^4 = \frac{15}{16}$

$P(B_1) \cdot P(B_2) \cdot P(B_3) = \frac{15}{16} \cdot \frac{5}{16} \cdot \frac{1}{8} = \frac{75}{2048} \neq P(B_1 \cap B_2 \cap B_3)$.

so B_1, B_2 and B_3 are not mutually independent.

5. (a) Let X be the sum of the two numbers.

All possible combination to select two numbers from $\{1,2,3,4\}$

$= \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$

$\Omega = \{2,3,4,5,6,7,8\}$

$p_X(2) = P(X=2) = P(\{w: X(w)=2\}) = P(\{(1,1)\}) = \frac{1}{16}$

$p_X(3) = P(X=3) = P(\{w: X(w)=3\}) = P(\{(1,2),(2,1)\}) = \frac{1}{8}$

$p_X(4) = P(X=4) = P(\{w: X(w)=4\}) = P(\{(1,3),(3,1),(2,2)\}) = \frac{3}{16}$

$p_X(5) = P(X=5) = P(\{w: X(w)=5\}) = P(\{(1,4),(4,1),(2,3),(3,2)\}) = \frac{1}{4}$

$p_X(6) = P(X=6) = P(\{w: X(w)=6\}) = P(\{(2,4),(4,2),(3,3)\}) = \frac{3}{16}$

$p_X(7) = P(X=7) = P(\{w: X(w)=7\}) = P(\{(3,4),(4,3)\}) = \frac{1}{8}$

$p_X(8) = P(X=8) = P(\{w: X(w)=8\}) = P(\{(4,4)\}) = \frac{1}{16}$

probability mass function

cummulative distribution function

x	2	3	4	5	6	7	8
$P_X(x)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{16}$
$F_X(x)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{13}{16}$	$\frac{15}{16}$	1

$$F_X(x) = \begin{cases} \frac{1}{16} & x < 2 \\ \frac{3}{16} & 2 \le x < 3 \end{cases}$$

(b). let x be the absolute value of the difference between the two numbers.

x can be $0, 1, 2, 3$.

$P_X(0) = P(X=0) = P(\{(1,1), (2,2), (3,3), (4,4)\}) = \frac{1}{4}$

$P_X(1) = P(X=1) = P(\{(1,2), (2,3), (3,4), (4,3), (3,2), (2,1)\}) = \frac{6}{16} = \frac{3}{8}$

$P_X(2) = P(X=2) = P(\{(1,3), (2,4), (4,2), (3,1)\}) = \frac{4}{16} = \frac{1}{4}$

$P_X(3) = P(X=3) = P(\{(1,4), (4,1)\}) = \frac{2}{16} = \frac{1}{8}$

x	0	1	2	3
$P_X(x)$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
$F_X(x)$	$\frac{1}{4}$	$\frac{5}{8}$	$\frac{7}{8}$	1

$F_X(x) = \{$