recall from $f: D \subset \mathbb{R} \to \mathbb{R}$ such that $f(x) \le f(x_0)$ for all $x \in I$

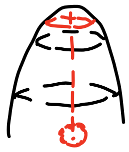
$\updownarrow$

$f(x) \ge f(x_0)$ for all $x \in I$

5.4 Optimisation in two variables

Consider $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ and let (x_0, y_0) be a point in the interior of D.

We say that *f has a local maximum at (x_0, y_0)* if there exists an open neighborhood $I \subset D$ containing (x_0, y_0) such that



$f(x,y) \le f(x_0, y_0)$ for all $(x_0, y_0) \in I$

We say that *f has a local minimum at (x_0, y_0)* if

$f(x,y) \ge f(x_0, y_0)$ for all $(x,y) \in I$

$f: D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$

$f_x \quad f_y$

nabla

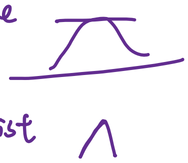
$\nabla f = (f_x, f_y)$

$\downarrow$

gradient of f

recall

If f has a local maximum or local minim at x_0, then either



- $f'(x_0) = 0$
- or $f'(x_0)$ does not exist

Theorem 5.18. If f has a local maximum or local minimum at (x_0, y_0), then either

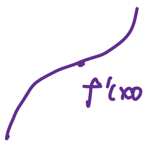
- $(\nabla f)(x_0, y_0) = (0, 0)$
- or at least one of f_x, f_y does not exist at (x_0, y_0).

The point (x_0, y_0) is called a *critical point of f* if $(\nabla f)(x_0, y_0) = (0, 0)$.

Be careful! Not all critical points are local maxima or minima.

not only if

recall



$f'(x_0) = 0$ not critical point

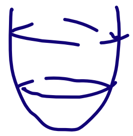
Example 5.19. Find the critical points of $f(x, y) = x^2 + y^2$.

$$f_x = \frac{\partial f}{\partial x} = 2x$$

$$f_y = \frac{\partial f}{\partial y} = 2y$$

$$\nabla f = (2x, 2y) = (0, 0)$$

critical point $(0, 0)$



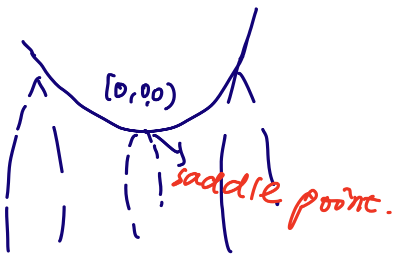
Example 5.20. Find the critical points of $f(x, y) = y^2 - x^2$.

$$f_y = \frac{\partial f}{\partial y} = 2y$$

$$f_x = \frac{\partial f}{\partial x} = -2x$$

$$\nabla f = (2y, -2x)$$

critical point $(x_0, y_0) = (0, 0)$



Example 5.21. Find the critical points of $f(x, y) = x^2 + 6xy + 4y^2 + 2x - 4y$.

$f_x = 2x + 6y + 2$

$f_y = 8y - 4 + 6x$

$\nabla f = 0$

$2x + 6y = -2$

$6x + 8y = 4$

$$\begin{bmatrix} 2 & 6 & -2 \\ 6 & 8 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -1 \\ 3 & 4 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -1 \\ 0 & -5 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$(x_0, y_0) = (2, -1)$

Suppose f has continuous second order partial derivatives in a neighbourhood of (x_0, y_0).

The symmetric 2×2 matrix

$$\mathbf{H} = \mathbf{H}_f(x_0, y_0) = \begin{bmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{bmatrix}$$

is called *the Hessian of f at (x_0, y_0)*.

Note that we know that $\mathbf{H}$ has real eigenvalues.

Theorem 5.22 (Second Derivative Test)**.** Let (x_0, y_0) be a critical point of f. Then (x_0, y_0) is a

(a) local minimum if $\mathbf{H}$ has positive eigenvalues

(b) local maximum if $\mathbf{H}$ has negative eigenvalues

(c) saddle point if $\mathbf{H}$ has one positive eigenvalue and one negative eigenvalue.

The test is inconclusive if $\mathbf{H}$ is not invertible.

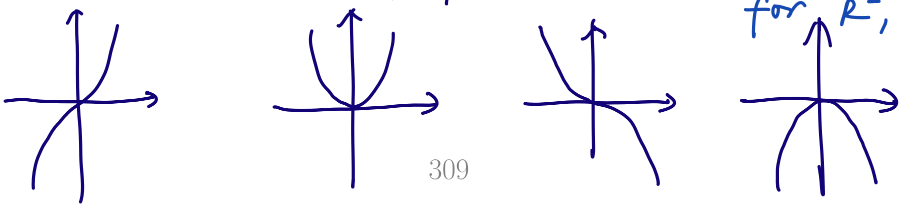
one variable second derivative test $f: D \subset \mathbb{R} \to \mathbb{R}$

let x_0 be a critical point of f, then x_0 is a

(a) local min if $f''(x_0) > 0$

(b) local max if $f''(x_0) < 0$

Test is inconclusive if $f''(x_0) = 0$ → for $\mathbb{R}$, we need one value
for $\mathbb{R}^2$, we need one matrix



Corollary 5.23 (Second Derivative Test, Alternative Version)**.** Let (x_0, y_0) be a critical point of f. Then (x_0, y_0) is a

(a) local minimum if $\det \mathbf{H} > 0$ and $f_{xx}(x_0, y_0) > 0$

(b) local maximum if $\det \mathbf{H} > 0$ and $f_{xx}(x_0, y_0) < 0$

(c) saddle point if $\det \mathbf{H} < 0$.

The test is inconclusive if $\det \mathbf{H} = 0$.

Example 5.24. Classify the critical points of $f(x, y) = x^2 + 6xy + 4y^2 + 2x - 4y$.

$f_{xx} = 2$

$f_{xy} = 6$

$f_{yy} = 8$

$$H = \begin{bmatrix} 2 & 6 \\ 6 & 8 \end{bmatrix}$$

$\det H = -20 < 0$

so $(-2, 1)$ is a saddle point

Example 5.25. Classify the critical points of $f(x, y) = x^2y + x^4 - y^3/3$.

$f_x = 2xy + 4x^3$

$f_y = x^2 - y^2$

$2xy + 4x^3 = 0$

$x^2 - y^2 = 0$

$2x(y + 2x^2) = 0$

$(x+y)(x-y) = 0$

so $y = \frac{1}{2}$ or 0

$\downarrow \qquad \downarrow$

$x = \frac{1}{2} \qquad x = 0$

$(0,0)$ $(\frac{1}{2}, -\frac{1}{2})$

① if $x + y = 0$

$x = -y$

$2(-y)(y + 2y^2) = 0$

$-2y \cdot y(1 + 2y) = 0$

$y = 0$ or $y = -\frac{1}{2}$

② if $x - y = 0$

$x = y$

$2y(y + 2y^2) = 0$

$2y^2(1 + 2y) = 0$

$y = 0 \to x = 0$

$(0,0)$

$y = -\frac{1}{2} \rightarrow x = -\frac{1}{2}$ $\left(-\frac{1}{2}, -\frac{1}{2}\right)$