Semester 1 Assessment, 2016

School of Mathematics and Statistics

MAST20009 Vector Calculus

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 5 pages (including this page)

Authorised materials:

- No materials are authorised.

Instructions to Students

- You may remove this question paper at the conclusion of the examination
- There are 11 questions on this exam paper.
- All questions may be attempted.
- Marks for each question are indicated on the exam paper.
- Start each question on a new page.
- Clearly label each page with the number of the question that you are attempting.
- There is a separate 3 page formula sheet accompanying the examination paper, that you may use in this examination.
- The total number of marks available is 125.

Instructions to Invigilators

- Students may remove this question paper at the conclusion of the examination
- Initially students are to receive the exam paper, the 3 page formula sheet, and a 14 page script book.

This paper may be held in the Baillieu Library

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Question 1 (12 marks)

Consider the function f defined for all (x, y) such that $y \neq 0$, with the rule:

$$f(x, y) = \frac{y}{\sqrt{x^2 + y^2} + x}$$

(a) Show that

$$f(r\cos\theta, r\sin\theta) = \tan\left(\frac{\theta}{2}\right)$$

for all $r > 0$ and all $\theta \in (0, 2\pi)$, with $\theta \neq \pi$.

(b) Evaluate the limits below, if they exist. If the limit does not exist, explain why it does not exist. You must clearly state if you use continuity, l'Hopital's rule or the sandwich theorem in your working. You do not need to justify using limit laws.

(i) $\displaystyle\lim_{(x,y)\to(0,-1)} f(x, y)$

(ii) $\displaystyle\lim_{(x,y)\to(1,0)} \log|f(x, y)|$

(iii) $\displaystyle\lim_{(x,y)\to(0,0)} f^2(x, y)$

(iv) $\displaystyle\lim_{(x,y)\to(-1,0)} \arctan(f(x, y))$

Question 2 (11 marks)

Using Lagrange Multipliers, find all the extremum points and the extremum values of the function

$$f(x, y, z) = x^2 - y^2 - z^2$$

subject to the constraints:

$$x^2 + y^2 = 4$$
$$y = \sqrt{3 + z^2}$$

Question 3 (7 marks)

Consider the function:

$$f(x, y) = \arctan\left(\frac{y}{x}\right)$$

(a) Determine the second order Taylor polynomal for f about $(1, 1)$.

(b) Using your result in part (a), approximate $f(\frac{4}{5}, \frac{5}{4}) - \frac{\pi}{4}$. You can leave your answer as a sum of fractions.

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Question 4 (9 marks)

Let f, g and h be scalar functions of order C^2 on $\mathbb{R}^3$.

(a) Show that
$$\boldsymbol{\nabla}\left(e^f\right) = e^f\ \boldsymbol{\nabla} f$$

(b) Show that
$$\nabla^2\left(e^f\right) = e^f\left(\nabla^2 f + \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} f\right)$$

(c) Assume that g and h are such that $\nabla^2 g = \nabla^2 h = 0$ and $\boldsymbol{\nabla} g \cdot \boldsymbol{\nabla} h = 0$. Obtain an expression for $\nabla^2\left(ge^h - he^g\right)$ in the simplest possible form.

Question 5 (15 marks)

Consider the constant vector field $\mathbf{A}(x, y, z) = (\alpha, \beta, \gamma)$, where α, β and γ are three real numbers, not all zero. Define two more vector fields on $\mathbb{R}^3$ as $\mathbf{B} = \mathbf{r} \times \mathbf{A}$ and $\mathbf{C} = \mathbf{r} \times \mathbf{B}$, where $\mathbf{r} = (x, y, z)$.

(a) Show that $\mathbf{A}$ and $\mathbf{r}$ are gradient vector fields and that $\mathbf{B}$ is incompressible.

(b) Using the identity $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$, valid for any three vectors in $\mathbb{R}^3$, prove that $\mathbf{C}$ is *not* an irrotational vector field.

(c) Assume that $\alpha = 1$, $\beta = 0$ and $\gamma = 0$. Find a vector potential field $\mathbf{V}$ for $\mathbf{B}$.

(d) Assuming same values for α, β and γ as at part (c), find all the flow lines of the vector field $\mathbf{B}$.

Question 6 (8 marks)

Let V be the region given by $x^2 + y^2 + z^2 \leq 100$ and $z \geq 0$. Calculate the average value of $g(x, y, z) = x^2 + y^2$ on V, given by:

$$\bar{g} = \frac{\iiint_V g(x, y, z)\,dxdydz}{\iiint_V dxdydz}$$

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Question 7 (11 marks)

Let S be the surface given by $z = \sqrt{x^2 + y^2 - 1}$, $z > 0$.

(a) Write down a parametrisation of S based on cylindrical coordinates.

(b) Find the downward normal vector to S at a point (x, y, z).

(c) Determine the Cartesian equation of the tangent plane to S at the point $(3, -4, 2\sqrt{6})$.

Question 8 (11 marks)

Let C be the closed curve consisting of the straight line segment from the origin to $(\sqrt{2}, 0)$, followed by the anticlockwise arc of the circle of radius $\sqrt{2}$ and with center in the origin, joining $(\sqrt{2}, 0)$ and $(1, 1)$ and followed by the line segment from $(1, 1)$ to the origin. Let D be the region bounded by C.

(a) Sketch C.

(b) Verify Green's Theorem for $P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = y^3\mathbf{i} + x^3\mathbf{j}$ on the region D.

Question 9 (12 marks)

(a) State Stokes' Theorem and explain all the symbols used and all the required conditions.

(b) Let S be the surface given by $(2 - z)^2 = x^2 + y^2$, where $0 \leq z \leq 2$. Let S be oriented using the outward unit normal. Consider the vector field:

$$\mathbf{F}(x, y, z) = -y^3\mathbf{i} + x^3\mathbf{j} + e^{-z}\mathbf{k}$$

(i) Sketch S, clearly labelling its boundary and indicating all the orientations considered.

(ii) Using Stokes' Theorem, evaluate

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S}$$

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Question 10 (16 marks)

Consider the vector field $\mathbf{F}(x,y,z) = (x^3-y^2)\mathbf{i}+(y^3-z^2)\mathbf{j}+(z^3-x^2)\mathbf{k}$. Let D be the region bounded above by the disk S_1 given by $x^2+y^2 \leq 25$ and $z=0$, and bounded below by the hemisphere S_2 given by $z = -\sqrt{25-x^2-y^2}$. The boundary of D is oriented using the outward normal vector.

(a) Draw the region D indicating the chosen normal vector on each face of the boundary of D.

(b) Calculate the flux of $\mathbf{F}$ through the surface S_1.

(c) Calculate the flux of $\mathbf{F}$ through the surface S_2.

Question 11 (13 marks)

Define *prolate spheroidal coordinates* (u,v,w) by

$$x = \sinh u \sin v \cos w$$

$$y = \sinh u \sin v \sin w$$

$$z = \cosh u \cos v$$

where $u > 0$, $v \in (0,\pi)$ and $w \in (0,2\pi)$.

(a) Show that the scalar factors verify:

$$h_1 = h_2 = \sqrt{\sinh^2 u + \sin^2 v} \text{ and } h_3 = \sinh u \sin v$$

(b) Show that the coordinate system is orthogonal.

(c) Write down an expression for the absolute value of the Jacobian.

(d) Calculate $\nabla^2 f$, where $f(u,v,w) = \cosh u \cos v$.

End of Exam—Total Available Marks = 125.

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