Vector Calculus : Lecture 1

Let $f : \mathbb{R}^m \to \mathbb{R}^n$ and $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ and

$L = (L_1, \ldots, L_n) \in \mathbb{R}^n$.

The limit of f as $x \to a$ is L, $\lim_{x \to a} f(x) = L$,

if f satisfies:

if $k \in \mathbb{Z}_{>0}$ is a tolerance

then there exists a calibration accuracy $l \in \mathbb{Z}_{>0}$

such that if $b \in \mathbb{R}^m$ and $d(b,a) < 10^{-l}$

then $d(f(b), L) < 10^{-k}$

The function f is continuous at a if f satisfies:

$$\lim_{x \to a} f(x) = f(a).$$

The function f is C^r at a if f satisfies

$$\left.\frac{\partial^r f}{\partial x_{i_1} \cdots \partial x_{i_r}}\right|_a$$ exist and are continuous.

<u>Theorem</u>

$$\cdots \Rightarrow \underset{\text{at } a}{f \text{ is } C^2} \Rightarrow \underset{\text{at } a}{f \text{ is } C^1} \Rightarrow \underset{\text{at } a}{f \text{ is differentiable}} \Rightarrow \left.\frac{\partial f_i}{\partial x_j}\right|_a \text{ exist}$$

Conceptually,

$f: \mathbb{R}^2 \to \mathbb{R}^1$ is differentiable at (a_1, a_2)

if the graph of f has a tangent plane at (a_1, a_2)

The <u>derivative matrix of f</u> is

$$D(f) = \left(\frac{\partial f_i}{\partial x_j} \right)$$

The <u>Jacobian</u> of f is $\det(D(f))$.

<u>Theorem</u> (Chain rule) Let $a = (a_1, \dots, a_m) \in \mathbb{R}^m$

If $f: \mathbb{R}^m \to \mathbb{R}^n$ is C^1 at a and

$g: \mathbb{R}^n \to \mathbb{R}^p$ is C^1 at $f(a)$ then

$$D(g \circ f)\big|_a = \left(D(g)\big|_{f(a)} \right) \left(D(f)\big|_a \right)$$

Limit theorems

Theorem Let $a=(a_1,\dots,a_m)\in\mathbb{R}^m$ and let

$f:\mathbb{R}^m\to\mathbb{R}^n$ and $g:\mathbb{R}^m\to\mathbb{R}^n$ and $c\in\mathbb{R}$.

and assume

$\lim_{x\to a} f$ exists and $\lim_{x\to a} g$ exists.

Then

(a) $\lim_{x\to a}(f+g)=\left(\lim_{x\to a} f\right)+\left(\lim_{x\to a} g\right)$

(b) $\left(\lim_{x\to a} fg\right)=\left(\lim_{x\to a} f\right)\left(\lim_{x\to a} g\right)$

(c) $\left(\lim_{x\to a} cf\right)=c\left(\lim_{x\to a} f\right)$

(d) If $\left(\lim_{x\to a} g\right)\neq 0$ then $\lim_{x\to a}\frac{f}{g}=\dfrac{\lim_{x\to a} f}{\lim_{x\to a} g}$

Theorem Let $a=(a_1,\dots,a_m)\in\mathbb{R}^m$ and $f:\mathbb{R}^m\to\mathbb{R}$ and $g:\mathbb{R}^m\to\mathbb{R}$. Assume $\lim_{x\to a} f$ exists and $\lim_{x\to a} g$ exists.

If $f(x)\le g(x)$ then $\left(\lim_{x\to a} f\right)\le\left(\lim_{x\to a} g\right)$

§1.1 Example 1:

Evaluate $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x+3}{5xy-y^3}$.

Solution

Using the limit laws:

$$\lim_{(x,y)\to(0,1)} \frac{x+3}{5xy-y^3} = \frac{\displaystyle\lim_{(x,y)\to(0,1)} (x+3)}{\displaystyle\lim_{(x,y)\to(0,1)} (5xy-y^3)}$$

$$= \frac{3}{\left(\displaystyle\lim_{(x,y)\to(0,1)} 5xy\right) - \left(\displaystyle\lim_{(x,y)\to(0,1)} y^3\right)} = \frac{3}{5\cdot 0\cdot 1 - 1^3}$$

$$= \frac{3}{-1} = -3.$$

Note that $\displaystyle\lim_{(x,y)\to(0,1)} (5xy-y^2)$ is not 0, so the limit law applies and we are not dividing by 0.

§1.1 Example 2

Evaluate $\displaystyle\lim_{(x,y)\to(2,1)} \frac{x^2-3xy+2y^2}{x-2y}$.

Solution:

$$\lim_{(x,y)\to(2,1)} \frac{x^3-3xy+2y^2}{x-2y} = \lim_{(x,y)\to(2,1)} \frac{(x-2y)(x-y)}{(x-2y)}$$

$$= \lim_{(x,y)\to(2,1)} (x-y) = 2-1=1.$$

§1.1 Example 3

Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$

Solution:

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{x^2}{x^2+y^2} = \lim_{y\to 0} \frac{0^2}{0^2+y^2} = \lim_{y\to 0} 0 = 0$$

and

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+0^2} = \lim_{x\to 0} 1 = 1.$$

If the limit approaches more than one number from different directions then the limit is not well determined. So

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} \text{ does not exist.}$$

§1.1 Example 4

Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Solution:

$$\lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0\cdot y}{0^2+y^2} = \lim_{y\to 0} 0 = 0$$

and

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y}} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x\cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

If the limit approaches more than one number (from different directions) then the limit is not determined in any exact way.

So

$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

§1.1 Example 5:

Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{\sqrt{x^2+y^2}}$

Solution:

$$0 \leq \frac{x^2}{\sqrt{x^2+y^2}} \leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}.$$

So

$$\lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} \leq \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}$$

So

$$0 \leq \lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} \leq \sqrt{\lim_{(x,y)\to(0,0)} (x^2+y^2)}$$

So

$$0 \leq \lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} \leq 0.$$

So

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0.$$