MAST30001 Stochastic Modelling

EXAM SOLUTIONS, SEMESTER 2, 2012

1. (a) A set of subsets of Ω is a σ-algebra if

   1 $\Omega \in \mathcal{F}$.

   2 If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

   3 If $A_1, A_2, \ldots \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

   (i) $\Omega \in \mathcal{F}_1$
   The complement of each of the sets in $\mathcal{F}_1$ is also in $\mathcal{F}_1$.
   The union of any sequence of sets in $\mathcal{F}_1$ is also in $\mathcal{F}_1$.
   So $\mathcal{F}_1$ is a σ-algebra.

   (ii) $\{(0,0),(0,1),(1,0)\} \in \mathcal{F}_2$, but $\{(0,0),(0,1),(1,0)\}^c = \{(1,1)\} \notin \mathcal{F}_2$.
   So $\mathcal{F}_2$ is not a σ-algebra.

   (b) The probability axioms are

   P1 For all events A, $\mathbb{P}(A) \geq 0$.

   P2 $\mathbb{P}(\Omega) = 1$.

   P3 For disjoint events A and B, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

   So now,

   By P2, $\mathbb{P}(\Omega) = 1$ and, by P3, $\mathbb{P}(\Omega) + \mathbb{P}(\emptyset) = 1$, so $\mathbb{P}(\emptyset) = 0$.
   By P3, $\mathbb{P}(\{(0,0),(1,0),(1,1)\}) + \mathbb{P}(\{(0,1)\}) = 1$, so $\mathbb{P}(\{(0,1)\}) = 1/3$. Similarly $\mathbb{P}(\{(0,0)\}) = 1/3$.
   By P3, $\mathbb{P}(\{(0,0)\}) + \mathbb{P}(\{(0,1)\}) = \mathbb{P}(\{(0,0),(0,1)\})$, so $\mathbb{P}(\{(0,0),(0,1)\}) = 2/3$.
   By P3, $\mathbb{P}(\{(0,0)\}) + \mathbb{P}(\{(1,0),(1,1)\}) = \mathbb{P}(\{(0,0),(1,0),(1,1)\}) = 2/3$, so $\mathbb{P}(\{(1,0),(1,1)\}) = 1/3$.

   (c) (i) $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B) = \mathbb{P}(\{(0,0)\})/\mathbb{P}(B) = 1/2$.

   (ii) $\mathbb{P}(B \cap C) = \mathbb{P}(\{(1,0),(1,1)\}) = 1/3$. $\mathbb{P}(B)\mathbb{P}(C) = (2/3)(2/3) = 4/9$. So B and C are not independent.

   (d) (i) X can take on only the values 0 and 1.

$$\{\omega : X(\omega) \leq 0\} = \{(1,0),(1,1)\} \in \mathcal{F}_1$$
$$\{\omega : X(\omega) \leq 1\} = \Omega \in \mathcal{F}_1$$

   Therefore X is measurable with respect to $\mathcal{F}_1$.

   (ii)

$$\mathbb{P}\{\omega : X(\omega) \leq 0\} = \mathbb{P}(\{(1,0),(1,1)\}) = 1/3$$
$$\mathbb{P}\{\omega : X(\omega) \leq 1\} = \mathbb{P}(\Omega_) = 1.$$

Hence, the distribution function for X is given by

$$F_X(t) = \begin{cases} 0 & t < 0, \\ 1/3 & 0 \le t < 1, \\ 1 & t \ge 1. \end{cases}$$

The probability mass function is given by $p_X(0) = 1/3$, $p_X(1) = 2/3$.

16 marks

2. (a) (i) The communicating classes are $\mathcal{C}_1 = \{1,2\}$, $\mathcal{C}_2 = \{3\}$. $\mathcal{C}_1$ is transient with period 2, while $\mathcal{C}_2$ is recurrent and aperiodic.

(ii) There is a single communicating class $\{1,2,3,4\}$, which is recurrent and aperiodic.

(iii) The communicating classes are $\mathcal{C}_1 = \{1,2\}$, which is recurrent and aperiodic, $\mathcal{C}_2 = \{4\}$, which is transient and aperiodic and $\mathcal{C}_3 = \{3,5,6\}$, which is recurrent and has period 3.

(b) Let $f_{i,0}$ be the probability that the DTMC ever reaches state 0 given that it starts in state i. For $i \ge 1$, the $f_{i,0}$ must satisfy

$$\begin{aligned} f_{1,0} &= pf_{2,0} + (1-p), \\ f_{i,0} &= (p/i)f_{i+1,0} + (1-1/i)f_{i,0} + ((1-p)/i)f_{i-1,0} \quad \text{for } i \ge 2. \end{aligned} \tag{1}$$

The *minimal non-negative* solution of these equations gives $f_{i,0}$. The second equation above reduces to

$$(p/i)f_{i+1,0} - (1/i)f_{i,0} + ((1-p)/i)f_{i-1,0} = 0$$

which is the same as

$$pf_{i+1,0} - f_{i,0} + (1-p)f_{i-1,0} = 0$$

These equations make up a system of second-order linear difference equations with constant coefficients. The characteristic equation is

$$pm^2 - m + (1-p) = 0 \Longrightarrow (pm - (1-p))(m-1) = 0.$$

The roots of the characteristic equation are $(1-p)/p$ and 1. The general solution to (1) is of the form

$$f_{i,0} = \begin{cases} A + B((1-p)/p)^i & p \ne 1/2, \\ A + Bi & p = 1/2. \end{cases}$$

In the case $p = 1/2$, substituting the solution into the boundary equation yields $A = 1$, and, in order for $f_{i,0}$ to be bounded, we need $B = 0$. In the case $p \ne 1/2$, the boundary equation gives $B = (1 - A)$ and so $f_{i,0} = A + (1 - A)((1 - p)/p)^i$. Now, if $((1 - p)/p) > 1$, the minimal non-negative solution is $f_{i,0} = 1$. If $((1 - p)/p) < 1$, the minimal non-negative solution is $f_{i,0} = ((1 - p)/p)^i$. Hence,

$$f_{i,0} = \min\left\{\left(\frac{(1-p)}{p}\right)^i, 1\right\}.$$

The DTMC is transient if $f_{i,0} < 1$, which occurs when $p > 1/2$. The DTMC is recurrent if $f_{i,0} = 1$, which occurs when $p \leq 1/2$.
To distinguish between the null-recurrent and the positiv-recurrent cases, we examine the equations $\pi = \pi P$. These reduce to

$$\pi_0 = (1-p)\pi_0 + (1-p)\pi_1 \tag{2}$$
$$\pi_1 = p\pi_0 + ((1-p)/2)\pi_2 \tag{3}$$
$$\pi_i = (p/(i-1))\pi_{i-1} + (1 - 1/i)\pi_i + ((1-p)/(i+1))\pi_{i+1} \quad i \geq 2. \tag{4}$$

Equation (4) reduces to

$$(p/(i-1))\pi_{i-1} - (1/i)\pi_i + ((1-p)/(i+1))\pi_{i+1} = 0 \quad i \geq 2.$$

Putting $x_i = \pi_i/i$, for $i \geq 1$, this becomes

$$px_{i-1} - x_i + (1-p)x_{i+1} = 0 \quad i \geq 2.$$

This is also a system of linear difference equations with constant coefficients. Its characteristic equation is $(1 - p)m^2 - m + p = 0$. Analogous to the working above, the general solution is

$$x_i = \begin{cases} A + B(p/(1-p))^i & p \neq 1/2, \\ A + Bi & p = 1/2, \end{cases}$$

and so, for $i \geq 1$,

$$\pi_i = \begin{cases} Ai + Bi(p/(1-p))^i & p \neq 1/2, \\ Ai + Bi^2 & p = 1/2, \end{cases}$$

Clearly this is not summable when $p = 1/2$, which tells us that the DTMC is null-recurrent when $p = 1/2$. When $p < 1/2$, we must have $A = 0$ for π_i to be summable. To solve for B, substitute our expression for π_i into equation (3).

$$B\left(\frac{p}{1-p}\right) = \pi_0 p + B\left(\frac{p^2}{1-p}\right)$$

which yields $\pi_0 = B$. Our solution now becomes

$$\pi_i = \pi_0 i \left(\frac{p}{1-p}\right)^i, i \geq 1. \tag{5}$$

20 marks

3. We set this up as a Markov reward process. The possible states are $\{0, 1, \ldots, 5\}$ and the entire payoff occurs on the last step. The decisions to be made are the amounts b to bet at each step. When we have k dollars, these must lie in the set $\{0, 1, \ldots, \min(k, 5-k)\}$.

   Let $V_n(k)$ be the reward associated with being in state k with n time periods to go, assuming that we adopt an optimal betting strategy. Then

$$V_n(k) = \max_{b \in \{0,\ldots,\min(k,5-k)\}} \left[0.6V_{n-1}(k-b) + 0.4V_{n-1}(k+b)\right], \tag{6}$$

   with

$$V_1(k) = \begin{cases} 0 & \text{if } k = 0, 1, 2 \\ 0.4 & \text{if } k = 3, 4 \\ 1 & \text{if } k = 5 \end{cases}$$

   In the second case, the optimal decision is to bet $5 - k$, and in the final case to bet nothing. Using (6), we derive

$$V_2(k) = \begin{cases} 0 & \text{if } k = 0, 1 \\ 0.16 & \text{if } k = 2, \text{ betting 1 or 2} \\ 0.4 & \text{if } k = 3, \text{ betting 0 or 2} \\ 0.64 & \text{if } k = 4, \text{ betting 1} \\ 1 & \text{if } k = 5 \end{cases}$$

$$V_3(k) = \begin{cases} 0 & \text{if } k = 0 \\ 0.064 & \text{if } k = 1, \text{ betting 1} \\ 0.256 & \text{if } k = 2, \text{ betting 2} \\ 0.4 & \text{if } k = 3, \text{ betting 0 or 2} \\ 0.64 & \text{if } k = 4, \text{ betting 0 or 1} \\ 1 & \text{if } k = 5 \end{cases}$$

   So, in the first instance, the gambler should bet 2. If she wins, she then bets 1. If she wins that bet, her final bet is 0. If she loses, she still has a chance to get to five dollars by betting 2. The probability of her winning is 0.256. **8 marks**

4. (a) The time until the first earthquake, in years, is exponentially-distributed with mean 25.

   (b) The probability that there is no earthquake in 100 years is $e^{-(0.04\times100)} = e^{-4}$.

   (c) Tsunamis occur according to a Poisson process with parameter 0.03, so the probability that there is no tsunami in 100 years is $e^{-(0.03\times100)} = e^{-3}$.

   (d) The expected number of tsunamis in 100 years is $0.03 \times 100 = 3$.

(e) This is distributed according to a binomial distribution with $n = 6$ and $p = 0.75$. So the probability that there are less than or equal to two tsunamis is

$$\sum_{i=0}^{2} \binom{6}{i} (0.75)^i (0.25)^{6-i} = (0.25)^6 + 6(0.75)(0.25)^5 + 15(0.75)^2(0.25)^4.$$

(f) The number of earthquakes that generate tsunamis is independent of the number of earthquakes that don't, and the latter is distributed as Poisson with rate $.01 \times 100$ with an expectation of 1. So the expected number of earthquakes is $2 + 1 = 3$.

12 marks

5. (a) The generator for this CTMC is

$$A = \begin{bmatrix} -1 & 1 \\ 12 & -12 \end{bmatrix}$$

The equations $\pi A = 0$ lead to

$$-\pi_1 + 12\pi_2 = 0$$

from which we get $\pi_2 = \pi_1/12$. Normalisation gives us $\pi_1 = 12/13$ and then $\pi_2 = 1/13$.

(b) We write down and solve the Kolmogorov Forwrad equations for $P_{11}(t)$ and $P_{12}(t)$, subject to $P_{11}(0) = 1$.

$$\frac{dP_{11}(t)}{dt} = -P_{11}(t) + 12P_{12}(t)$$
$$\frac{dP_{12}(t)}{dt} = -12P_{12}(t) + P_{11}(t).$$

We know that $P_{12}(t) = 1 - P_{11}(t)$, and so the first equation becomes

$$\frac{dP_{11}(t)}{dt} = -P_{11}(t) + 12(1 - P_{11}(t)),$$

which we can rewrite as

$$\frac{dP_{11}(t)}{dt} + 13P_{11}(t) = 12.$$

Any solution to this has the form

$$P_{11}(t) = \frac{12}{13} + A\exp(-13t)$$

The condition $P_{11}(0) = 1$ gives us $A = 1/13$, and so

$$P_{11}(t) = \frac{12 + \exp(-13t)}{13}.$$

It follows that

$$P_{12}(t) = \frac{1 - \exp(-13t)}{13}.$$

(c)

$$\lim_{t\to\infty} P_{11}(t) = \frac{12}{13}$$
$$\lim_{t\to\infty} P_{12}(t) = \frac{1}{13}$$

which agrees with our solution for the stationary distribution.

(d) If the CTMC starts in its stationary distribution, then it stays in its stationary distribution for all time. Thus $P(X(t) = 1) = 12/13$ and $P(X(t) = 2) = 1/13$ for all t.

(e) We need to interpret management's decree in terms of the stationary distribution of the CTMC. With the repair rate given by γ, we can repeat the reasoning in (a) above to derive the fact that the stationary probability of being in state 1 is $\gamma/(1+\gamma)$, and we want this to be greater than or equal to 0.95. This is achieved if $\gamma = 19$. Thus the minimal rate of repair that will achieve management's objective is 19.

12 marks

6. (a) (i) In lectures we saw that a birth and death process is positive recurrent if

$$\sum_{k=0}^{\infty}\left[\prod_{l=1}^{k}\frac{\lambda_{l-1}}{\mu_l}\right] < \infty,$$

where the birth and death rates are λ_l and μ_l, respectively. In this $M/M/1$ queue, $\lambda_l = 3$ and $\mu_l = 5$ for $l = 1, 2, \ldots$, so

$$\prod_{l=1}^{k}\frac{\lambda_{l-1}}{\mu_l} = \left(\frac{3}{5}\right)^k$$

and

$$\sum_{k=0}^{\infty}\left[\prod_{l=1}^{k}\frac{\lambda_{l-1}}{\mu_l}\right] = \frac{1}{1-3/5} = 5/2 < \infty.$$

The stationary distribution is thus

$$\pi_k = (2/5)(3/5)^k$$

(ii) PASTA tells us that customers who arrive according to a Poisson process "see" the stationary distribution so an arriving customer finds k customers already present with probability $\pi_k = (2/5)(3/5)^k$.

(iii) Because of the memoryless property of the exponential distribution, the residual service time is still exponential with parameter $\mu = 5$. Thus the LST of the remaining service time is $5/(s+5)$.

(iv) The remaining $k-1$ service times are all exponential with parameter $\mu = 5$ and the time taken for all of them is the convolution of these $k-1$ service times. By the Convolution Theorem, its LST is $(5/(s+5))^{k-1}$.

(v) Conditional on a customer arriving to find $k \geq 1$ customers in the queue, its waiting time is the convolution of the residual service time of the customer in service, plus the total service time of the $k-1$ customers in front of it. This has LST $(5/(s+5))^k$. The stationary probability of the arriving customer finding k customers, given that it arrives to a busy queue, is $(2/5)(3/5)^{k-1}$. Therefore, conditional on a customer arriving to find a busy queue, the waiting time has LST $(5/(s+5))^k$ with probability $(2/5)(3/5)^{k-1}$. The waiting time LST, conditional on a customer arriving to a busy queue is thus

$$\frac{2}{s+5}\sum_{k=1}^{\infty}\left(\frac{3}{s+5}\right)^{k-1} = \frac{2}{s+2}.$$

We recognise this as the LST of an exponential distribution with parameter 2.

(vi) The stationary waiting time is zero with probability 2/5 and, with probability 3/5, is exponentially distributed with parameter 2.

(b) (i) 1. $a[(n_1,n_2),(n_1+1,n_2)] = 6$.
2. $a[(n_1,n_2),(n_1-1,n_2)] = 10I(n_1>0)$.
3. $a[(n_1,n_2),(n_1,n_2-1)] = 8I(n_2>0)$.

(ii)

$$\begin{aligned}\pi(n_1,n_2)[6+10I(n_1>0)+8I(n_2>0)]&\\ &= 6I(n_1>0)\cdot\pi(n_1-1,n_2)\\ &\quad+8\cdot\pi(n_1,n_2+1)\\ &\quad+10I(n_2>0)\cdot\pi(n_1+1,n_2-1).\end{aligned}$$

(iii) Substituting the proposed solution into the equation in (ii), we have

$$\begin{aligned}\text{RHS} &= K\big[6I(n_1>0)(3/5)^{n_1-1}(3/4)^{n_2}\\ &\qquad+8(3/5)^{n_1}(3/4)^{n_2+1}+10I(n_2>0)(3/5)^{n_1+1}(3/4)^{n_2-1}\big]\\ &= K(3/5)^{n_1}(3/4)^{n_2}\big[6(5/3)I(n_1>0)+8(3/4)+10(3/5)(4/3)I(n_2>0)\big]\\ &= K(3/5)^{n_1}(3/4)^{n_2}[10I(n_1>0)+6+8I(n_2>0)]\\ &= \text{LHS}.\end{aligned}$$

(iv) We require

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\pi(n_1,n_2)<\infty.$$

Here we have

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}K\left(\frac{3}{5}\right)^{n_1}\left(\frac{3}{4}\right)^{n_2}=\left[\left(1-\frac{3}{5}\right)\left(1-\frac{3}{4}\right)\right]^{-1}$$

so the stationary distribution exists and

$$\pi((0,0))=\left(1-\frac{3}{5}\right)\left(1-\frac{3}{4}\right)$$

(v) If 30% of customers go back to queue 1 from queue 2, then the traffic equations become

$$\nu_1 = 6+0.3\nu_2$$
$$\nu_2=\nu_1$$

which gives us $\nu_1=\nu_2=60/7$. Since this is greater than $\nu_2 = 8$, the second queue is no longer stable and so no stationary distribution exists.

12 marks

7. (a) The density of the τ_i is given by $f(x)=2x$ for $x\in[0,1]$. Therefore

$$E[\tau_i]=\int_0^1 2x^2dx=2/3$$
$$E[\tau_i^2]=\int_0^1 2x^3dx=1/2$$

which gives us that $V(\tau_i)=1/18$.

(b) For $y\in[0,1]$,

$$\lim_{t\to\infty}P(Y_t\le y)=3/2\int_0^y(1-x^2)dx=3y/2-y^3/2.$$

(c) We know that

$$\frac{N_{300}-900/2}{\sqrt{900/2\times 9/72}}=4(N_{300}-900/2)/30$$

is approximately distributed as a standard normal random variable. Thus

$$P(-1.96<4(N_{300}-900/2)/30\le 1.96)\approx 0.95$$

which tells us that

$$P(58.04 < 4N_{300}/30 \leq 61.96) \approx 0.95$$

and so

$$P(435.3 < N_{300} \leq 464.7) \approx 0.95.$$

10 marks

Total: 100 marks