MAST30001 Stochastic Modelling

Tutorial Sheet 2

1. Show that the Markov property does not in general imply that for any event B, and states i, j,
$$P(X_{n+1} = j | X_n \in B, X_{n-1} = i) = P(X_{n+1} = j | X_n \in B).$$
(Define a Markov chain, event B, and states i, j, where the equality doesn't hold.)

2. Let $(Y_n)_{n\geq 1}$ be i.i.d. random variables with $P(Y_i = 1) = P(Y_i = -1) = 1/2$ and let $X_n = (Y_{n+1} + Y_n)/2$.
   (a) Find the transition probabilities $P(X_{n+m} = k | X_n = j)$ for $m = 1, 2, \ldots$ and $j, k = 0, \pm 1$.
   (b) Show that $(X_n)_{n\geq 1}$ is *not* a Markov chain.

3. Let (X_n) be a Markov chain with state space $\{1, 2, 3\}$ and transition matrix
$$\begin{pmatrix} 0 & 1/3 & 2/3 \\ 1/4 & 3/4 & 0 \\ 2/5 & 0 & 3/5 \end{pmatrix}$$
   (a) Compute $P(X_3 = 1, X_2 = 2, X_1 = 2 | X_0 = 1)$.
   (b) If X_0 is uniformly distributed on $\{1, 2, 3\}$, compute $P(X_3 = 1, X_2 = 2, X_1 = 2)$.
   (c) Now assuming $P(X_0 = 1) = P(X_0 = 3) = 2/5$, compute $P(X_1 = 2, X_4 = 2, X_6 = 2)$.

4. A simplified model for the spread of a contagion in a small population of size 4 is as follows. At each discrete time unit, two individuals in the population are chosen uniformly at random to meet. If one of these persons is healthy and the other has the contagion, then with probability 1/4 the healthy person becomes sick. Otherwise the system stays the same.
   (a) If X_n is the number of healthy people at step n, then explain why $X_0, X_1 \ldots$ is a Markov chain.
   (b) Specify the transition probabilities of X_n.
   (c) If initially the chance that a given person in the population has the disease equals 1/2, determined independently, then what is the chance everyone has the disease after two steps in the process?
   (d) Assume now that the process begins with exactly one person infected. Given that not everyone is infected after three steps of the process, what is the chance exactly one person is infected?

5. A Markov chain on $\{1, 2, 3, 4\}$ with transition matrix
$$P = \begin{pmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.4 & 0.3 \\ 0.1 & 0.4 & 0.3 & 0.2 \end{pmatrix}$$
starts with initial distribution uniform on the states 1, 2, 3, 4. For each $i = 1, 2, 3, 4$, and $n \geq 0$, compute the chance the chain is in state i at step n.

1. Show that the Markov property does not in general imply that for any event B and states i, j,

$$P(X_{n+1} = j | X_n \in B, X_{n-1} = i) = P(X_{n+1} = j | X_n \in B).$$

(Define a Markov chain, event B, and states i, j, where the equality doesn't hold.)

Ans. You can see there's a problem since if S denotes the state space and we set $B = S$, then the relation above becomes

$$P(X_{n+1} = j | X_{n-1} = i) = P(X_{n+1} = j);$$

that is, that X_{n+1} and X_{n-1} are independent, which isn't true in general. To be concrete, take simple random walk on $\mathbb{Z}$ with $P(X_0 = 1) = P(X_0 = 0) = 1/2$, and set $i = 0$, $B = \mathbb{Z}$ and $j = 1$. Then for $n \geq 1$, the left hand side of the equality is 0 but the right hand side is not.

2. Let $(Y_n)_{n\geq 1}$ be i.i.d. random variables with $P(Y_i = 1) = P(Y_i = -1) = 1/2$ and let $X_n = (Y_{n+1} + Y_n)/2$.

moving average

(a) Find the transition probabilities $P(X_{n+m} = k | X_n = j)$ for $m = 1, 2, \ldots$ and $j, k = 0, \pm 1$.

(b) Show that $(X_n)_{n\geq 1}$ is *not* a Markov chain.

$P(X_{n+1} = j \mid X_n, X_{n-1}, \cdots)$
$= P(X_{n+1} = j \mid X_n)$

give example

$P(X_{n+1} = 1 \mid X_n = 0, X_{n-1} = 1) = 0$

$P(X_{n+1} = 1 \mid X_n = 0) = \frac{1}{4}$

2. (a).

$$Y_i = \begin{cases} 1, & p = \frac{1}{2} \\ -1, & p = \frac{1}{2} \end{cases} \qquad X_n = \begin{cases} 1 & p = \frac{1}{4} \\ 0 & p = \frac{1}{2} \\ -1 & p = \frac{1}{4} \end{cases}$$

$P(X_{n+m} = k \mid X_n = j)$

n starting point

$m \in \mathbb{N}$

① $m = 1$

$P(Y_{n+2} + Y_{n+1} = 2k \mid Y_{n+1} + Y_n = 2j)$

(1a) $j = -1$

$P(Y_{n+2} + Y_{n+1} = 2k \mid Y_{n+1} + Y_n = -2)$

$= P(Y_{n+2} + Y_{n+1} = 2k \mid Y_{n+1} = Y_n = -1)$

$= P(Y_{n+2} + (-1) = 2k)$

$= P(Y_{n+2} = 2k + 1)$

$= \begin{cases} \frac{1}{2} & k = -1 \\ \frac{1}{2} & k = 0 \end{cases}$

0

$P(Y_{n+2} + Y_{n+1} = 2k \mid Y_{n+1} + Y_n = 0)$

$= \dfrac{P(Y_{n+2} + Y_{n+1} = 2k, Y_{n+1} + Y_n = 0)}{P(Y_{n+1} + Y_n = 0)}$

$= \dfrac{P(Y_{n+2} + Y_{n+1} = 2k, Y_{n+1} = 1, Y_n = 0) + P(Y_{n+2} + Y_{n+1} = 2k, Y_n = 1, Y_{n+1} = 1)}{\frac{1}{2}}$

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By symmetry

② $m \geq 2$ (eg $m = 3$)

$P(X_{n+3} = k \mid X_n = j)$.

$= P\left(\frac{Y_{n+4} + Y_{n+3}}{2} = k \;\middle|\; \frac{Y_{n+1} + Y_n}{2} = j\right)$

$= P\left(\frac{Y_{n+4} + Y_{n+3}}{2} = k\right)$

$= P[Y_{n+4} + Y_{n+3} = 2k]$.

$= \begin{cases} \frac{1}{4} & k = -1 \\ \frac{1}{2} & k = 0 \\ \frac{1}{4} & k = 1 \end{cases}$

transition prob $P(X_2 = 1 \mid X_1 = 1) = P_{11}$ $\neq$ $P(X_2 = 1, X_1 = 1)$

3. Let (X_n) be a Markov chain with state space $\{1, 2, 3\}$ and transition matrix

$$\begin{array}{c|c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1/3 & 2/3 \\ 1/4 & 3/4 & 0 \\ 2/5 & 0 & 3/5 \end{pmatrix} \end{array}$$

(a) Compute $P(X_3 = 1, X_2 = 2, X_1 = 2 | X_0 = 1)$.

(b) If X_0 is uniformly distributed on $\{1, 2, 3\}$, compute $P(X_3 = 1, X_2 = 2, X_1 = 2)$.

(c) Now assuming $P(X_0 = 1) = P(X_0 = 3) = 2/5$, compute $P(X_1 = 2, X_4 = 2, X_6 = 2)$.

$$P^2 = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{2}{5} & 0 & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{2}{5} & 0 & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{7}{20} & \frac{1}{4} & \frac{2}{5} \\ \frac{3}{16} & \frac{31}{48} & \frac{1}{6} \\ \frac{6}{25} & \frac{2}{15} & \frac{47}{75} \end{bmatrix}$$

$$P^3 = \begin{bmatrix} \frac{7}{20} & \frac{1}{4} & \frac{2}{5} \\ \frac{3}{16} & \frac{31}{48} & \frac{1}{6} \\ \frac{6}{25} & \frac{2}{15} & \frac{47}{75} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{2}{5} & 0 & \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{89}{400} & \frac{73}{240} & \frac{71}{150} \\ \frac{73}{320} & \frac{35}{64} & \frac{9}{40} \\ \frac{71}{250} & \frac{9}{50} & \frac{67}{125} \end{bmatrix}$$

(a). $P(X_3=1, X_2=2, X_1=2 | X_0=1)$

$= (P)_{12}(P)_{22}(P)_{21}$

$= \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{1}{16}$

(b) $P(X_0=1) = P(X_0=2) = P(X_0=3) = \frac{1}{3}$

$P(X_3=1, X_2=2, X_1=2) = \sum_{x_0 \in \{1,2,3\}} P(X_3=1, X_2=2, X_1=2 | X_0=x_0)\, P(X_0=x_0)$

$= \frac{1}{16} \cdot \frac{1}{3} + \frac{9}{64} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}$

$= \left(\frac{4}{64} + \frac{9}{64}\right) \cdot \frac{1}{3} = \frac{13}{64} \cdot \frac{1}{3} = \frac{13}{192}$

$P(X_3=1, X_2=2, X_1=2 | X_0=2)$

$= (P)_{22}(P)_{22}(P)_{21}$

$= \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} = \frac{9}{64}$

$P(X_3=1, X_2=2, X_1=2 | X_0=3)$

$= (P)_{32}(P)_{22}(P)_{21}$

$= 0$

(c) $P(X_0=1) = P(X_0=3) = \frac{2}{5}$ $\quad P(X_0=2) = \frac{1}{5}$

$P(X_1=2, X_4=2, X_6=2) = \sum_{x_0 \in \{1,2,3\}} P(X_1=2, X_4=2, X_6=2 | X_0=x_0) \cdot P(X_0)$

$= \frac{2}{5} \cdot \overset{\frac{1}{3}}{(P)_{12}} (P^3)_{22} (P^2)_{22}$

$+ \frac{1}{5} \cdot \overset{\frac{3}{4}}{(P)_{22}} (P^2)_{22} (P^3)_{22}$

$+ \frac{2}{5} \underset{0}{(P)_{32}} (P^3)_{22} (P^2)_{22}$

$= \left(\frac{2}{15} + \frac{3}{20}\right) \cdot (P^3)_{22} \cdot (P^2)_{22}$

$= 0.1$

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   (a) If X_n is the number of healthy people at step n, then explain why $X_0, X_1 \ldots$ is a Markov chain.

   (b) Specify the transition probabilities of X_n.

   (c) If initially the chance that a given person in the population has the disease equals 1/2, determined independently, then what is the chance everyone has the disease after two steps in the process?

   (d) Assume now that the process begins with exactly one person infected. Given that not everyone is infected after three steps of the process, what is the chance exactly one person is infected?

initial 2, 3, 4.

initial 4. $P: (\frac{1}{2})^4 = \frac{1}{16}$. $\xrightarrow{2\text{ steps}}$ $\frac{1}{16}$

initial 3. $P.\ C_4^3 \cdot (\frac{1}{2})^4 = 4 \cdot \frac{1}{16} = \frac{1}{4}$

$\frac{1}{4} \cdot \frac{7}{8} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{1}{8} \cdot 1 =$ $\frac{15}{256}$

initial 2 $P: C_4^2 \cdot (\frac{1}{2})^4 = \frac{6}{16} = \frac{3}{8}$

$\frac{3}{8} \cdot \frac{1}{6} \cdot \frac{1}{8} = \frac{1}{128}$

$\frac{33}{256}$

to prove it's a Markov chain

(a) $P(X_{n+1} = x_n - 1 \mid x_{n-1}, \cdots x_0) = \frac{x_n(4-x_n)}{C_4^2} \times \frac{1}{4}$ depend only on x_n

(b).

$$\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{7}{8} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{5}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{7}{8} & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) $\sum_{k=0}^{4} (P^2)_{k0} (\frac{1}{2})^4$.

(d).

$P(X_3 = 3 \mid X_3 > 0, X_0 = 3)$

$= \frac{P(X_3 = 3, X_0 = 3)}{P(X_3 > 0, X_0 = 3)}$ $\rightarrow = P(X_3 > 0 \mid X_0 = 3) \cdot P(X_0 = 3)$ $= P(X_3 > 0 \mid X_0 = 3)$. $= 1$.

$= \frac{(P^3)_{33}}{1 - P(X_3 = 0, X_0 = 3)}$

$= \frac{P^3}{1 - P_{3,2} P_{2,1} P_{1,0}}$

$p_{i,i-1} = 1 - p_{i,i} = i(4-i)/24, i = 0, \ldots, 4,$

$P(X_{n+1} = X_n - 1) = \frac{X_n(4 - X_n)}{C_3^2} \times \frac{1}{4}$

$\Rightarrow p_{i,i-1} = \frac{i(4-i)}{24}$. $i = 0, 1, 2, 3, 4$.

5. A Markov chain on $\{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.4 & 0.3 \\ 0.1 & 0.4 & 0.3 & 0.2 \end{pmatrix}$$

starts with initial distribution uniform on the states 1, 2, 3, 4. For each $i = 1, 2, 3, 4$, and $n \geq 0$, compute the chance the chain is in state i at step n.

$\vec{x} = [\frac{1}{4}\ \ \frac{1}{4}\ \ \frac{1}{4}\ \ \frac{1}{4}]$

$\vec{x} \cdot \vec{P} = [\frac{1}{4}\ \ \frac{1}{4}\ \ \frac{1}{4}\ \ \frac{1}{4}] \begin{bmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.4 & 0.3 \\ 0.1 & 0.4 & 0.3 & 0.2 \end{bmatrix}$

since $\vec{X}\vec{P} = \vec{X}$

$\vec{x} \cdot P^n = \vec{x}$

the chance of the chain is state i at step n is $\frac{1}{4}$.