Analysis of variance
(Module 8)

Statistics (MAST20005) &
Elements of Statistics
(MAST90058)

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Semester 2, 2020

Aims of this module

- Introduce the **analysis of variance** technique, which builds upon the variance decomposition ideas in previous modules.
- Revisit linear regression and apply the ideas of hypothesis testing and analysis of variance.
- Discuss ways to derive optimal hypothesis tests.

Overview

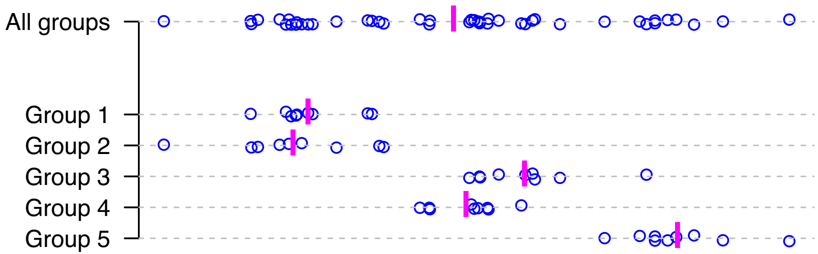
- **Analysis of variance (ANOVA).**
  Comparisons of more than two groups
- **Regression.**
  Hypothesis testing for simple linear regression
- **Likelihood ratio tests.**
  A method for deriving the best test for a given problem

Outline

Analysis of variance: introduction

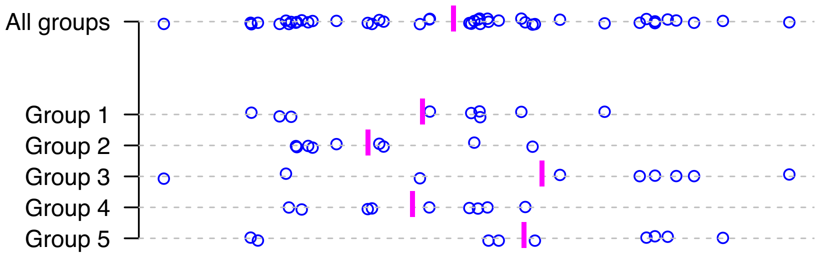
- Initial aim: compare the means of **more than two** populations
- Broader and more advanced aims:
  - Explore components of variation
  - Evaluate the fit a (general) linear model
- Formulated as hypothesis tests
- Referred to as analysis of variance, or ANOVA for short
- Involves comparing different summaries of variation
- Related to the 'analysis of variance' and 'variance decomposition' formulae we derived previously

Example: large variation between groups



compare variance between the group and within the group (inherited variation)

Example: smaller variation between groups



hard to say for sure that difference in the sample mean reflects difference in the true population mean or just the chance variation due to sampling

ANOVA: setting it up

- We have random samples from k populations, each having a normal distribution
- We sample n_i iid observations from the ith population, which has mean μ_i
- All populations assumed have the **same** variance, σ^2
- Question of interest: do the populations all have the same mean?
- Hypotheses:

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_k = \mu \quad \text{versus} \quad H_1: \bar{H}_0$$

($\bar{H}_0$ means 'not H_0')

- This model is known as a one-way ANOVA, or single-factor ANOVA

everything is signed to one of k groups

Notation

Population	Sample	Statistics	
$\mathrm{N}(\mu_1, \sigma^2)$	$X_{11}, X_{12}, \ldots, X_{1n_1}$	$\bar{X}_{1\cdot}$	S_1^2
$\mathrm{N}(\mu_2, \sigma^2)$	$X_{21}, X_{22}, \ldots, X_{2n_2}$	$\bar{X}_{2\cdot}$	S_2^2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{N}(\mu_k, \sigma^2)$	$X_{k1}, X_{k2}, \ldots, X_{kn_k}$	$\bar{X}_{k\cdot}$	S_k^2
Overall		$\bar{X}_{\cdot\cdot}$	

$$n = n_1 + \cdots + n_k \quad \text{(total sample size)}$$

$$\bar{X}_{i\cdot} = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij} \quad \text{(group means)}$$

$$\bar{X}_{\cdot\cdot} = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=1}^{n_i} X_{ij} = \frac{1}{n} \sum_{i=1}^{k} n_i \bar{X}_{i\cdot} \quad \text{(grand mean)}$$

Sum of squares (definitions)

- We now define statistics each called a sum of squares (SS)
- The total SS is:

$$SS(TO) = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \boxed{\bar{X}_{..}})^2$$

total mean (grand mean)

- The treatment SS, or between groups SS, is:

group mean

overall mean

$$SS(T) = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(\bar{X}_{i.} - \bar{X}_{..})^2 = \sum_{i=1}^{k} n_i(\bar{X}_{i.} - \bar{X}_{..})^2$$

- The error SS, or within groups SS, is: only be large when there is a lot of variation within group

$$SS(E) = \sum_{i=1}^{k}\boxed{\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_{i.})^2} = \sum_{i=1}^{k}(n_i - 1)S_i^2$$

$$s_i = \frac{1}{n_i - 1}\sum_{j=1}^{n_i}(x_{ij} - \bar{x}_{i.})^2$$

Analysis of variance decomposition

- It turns out that:

$$SS(TO) = SS(T) + SS(E)$$

- This is similar to the analysis of variance formulae we derived earlier, in simpler scenarios (iid model, regression model)
- We will use this relationship as a basis to derive a hypothesis test
- Let's first prove the relationship...

- Start with the ‘add and subtract’ trick:

$$
\begin{aligned}
SS(TO) &= \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{..})^2 \\
&= \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i.} + \bar{X}_{i.} - \bar{X}_{..})^2 \\
&= \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i.})^2 + \sum_{i=1}^{k} \sum_{j=1}^{n_i} (\bar{X}_{i.} - \bar{X}_{..})^2 \\
&\quad + 2 \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_{i.})(\bar{X}_{i.} - \bar{X}_{..}) \\
&= SS(E) + SS(T) + CP
\end{aligned}
$$

- The cross-product term is:

$$\begin{aligned} CP &= 2\sum_{i=1}^{k}\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_{i\cdot})(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot}) \\ &= 2\sum_{i=1}^{k}(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})\underline{\sum_{j=1}^{n_i}(X_{ij} - \bar{X}_{i\cdot})} \rightarrow n_i \overline{x_{i\cdot}} - n_i \overline{x_i} \\ &= 2\sum_{i=1}^{k}(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})(\underbrace{n_i\bar{X}_{i\cdot} - n_i\bar{X}_{i\cdot}}_{=0}) \\ &= 0 \end{aligned}$$

- Thus, we have:

$$SS(TO) = SS(T) + SS(E)$$

Sampling distribution of $SS(E)$

- The sample variance from the ith group, S_i^2, is an unbiased estimator of σ^2 and we know that $(n_i - 1)S_i^2/\sigma^2 \sim \chi^2_{n_i-1}$
- The samples from each group are independent so we can usefully combine them,

sample variance is independent

error ss, within the group ss.

$$\sum_{i=1}^{k} \frac{(n_i - 1)S_i^2}{\sigma^2} = \boxed{\frac{SS(E)}{\sigma^2} \sim \chi^2_{n-k}}$$

- Note that: $(n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) = n - k$
- This also gives us an unbiased pooled estimator of σ^2,

$$\boxed{\hat{\sigma}^2 = \frac{SS(E)}{n - k}}$$

?

- These results are true irrespective of whether H_0 is true or not

$x_{i1}, \ldots x_{in_i} \sim N(\mu_i, \sigma^2)$

$\frac{(n_i-1)S_i^2}{\sigma^2} \sim \chi^2_{n_i-1}$

$S_i^2 = \frac{1}{n_i-1}\sum_{j=1}^{n_i}(x_{ij} - \overline{x_{i\cdot}})^2$

since $\sum_{j=1}^{n_i}(x_{ij}-\mu_i)^2 = \sum_{j=1}^{n_i}(x_{ij} - \overline{x_{i\cdot}} + \overline{x_{i\cdot}} - \mu_i)^2$

$= \sum_{j=1}^{n_i}(x_{ij} - \overline{x_{i\cdot}})^2 + n_i(\overline{x_{i\cdot}} - \mu_i)^2$

take expectation

LHS.

$E[\sum_{j=1}^{n_i}(x_{ij}-\mu_i)^2] = n_i E[(x_{ij}-\mu_i)^2]$

$= n_i \sigma^2.$

RHS.

$(n_i-1)E(S_i^2). + \underbrace{E(n_i(\overline{x_i}-\mu_i)^2)}$

$n_i E((\overline{x_{i\cdot}}-\mu_i)^2)$

$= \sigma^2$

$\overline{x_{i\cdot}} \sim N(\mu_i, \frac{\sigma^2}{n_i}).$

$\therefore E(S_i^2) = \sigma^2$

Null sampling distribution of $SS(TO)$

$\Rightarrow \mu_1 = \cdots = \mu_k = \mu$ $\Rightarrow$ "sample from k different groups"

- If we assume H_0, we can derive simple expressions for the sampling distributions of the other sums of squares
- The combined data would be a sample of size n from $\mathrm{N}(\mu, \sigma^2)$. Hence $SS(TO)/(n-1)$ is an unbiased estimator of σ^2 and

"sample from a big group with size n"

$$\frac{SS(TO)}{\sigma^2} \sim \chi^2_{n-1}$$

$\bar{X}_1, \cdots \bar{X}_k$ iid $\sim N(\mu, \frac{\sigma^2}{n_i})$

$s^2 = \frac{1}{k-1}\sum_{i=1}^{k}(\bar{X}_i - \mu)^2$

$\frac{(k-1)s^2}{\sigma^2 / n_i} \sim \chi^2_{k-1}$

$\frac{n_i \sum_{i=1}^{k}(\bar{X}_i - \mu)^2}{\sigma^2} \sim \chi^2_{k-1}$

$\frac{SS(T)}{\sigma^2} \sim \chi^2_{k-1}$

Null sampling distribution of $SS(T)$

- Under H_0, we have $\bar{X}_{i\cdot} \sim \mathrm{N}(\mu, \frac{\sigma^2}{n_i})$
- $\bar{X}_{1\cdot}, \bar{X}_{2\cdot}, \ldots, \bar{X}_{k\cdot}$ are independent → $(k-1)$
- (Can think of this as a sample of sample means, and then think about what its variance estimator is)
- It is possible to show that (proof not shown):

$$\sum_{i=1}^{k} \frac{n_i(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2}{\sigma^2} = \frac{SS(T)}{\sigma^2} \sim \chi^2_{k-1}$$

and that this is independent of $SS(E)$

Null sampling distributions

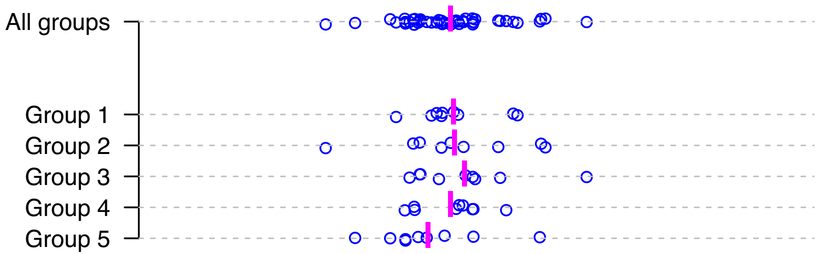
In summary, under H_0:

$$\frac{SS(TO)}{\sigma^2} = \frac{SS(E)}{\sigma^2} + \frac{SS(T)}{\sigma^2}$$

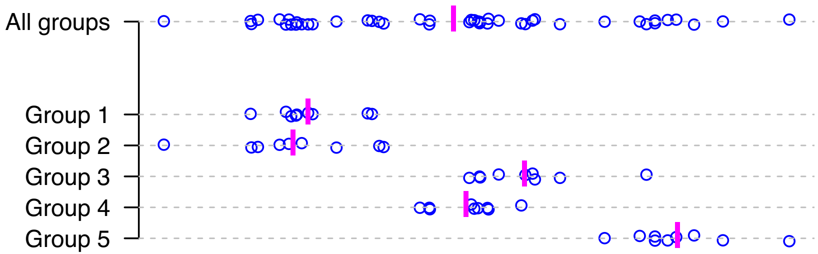
$$\frac{SS(TO)}{\sigma^2} \sim \chi^2_{n-1}, \quad \frac{SS(E)}{\sigma^2} \sim \chi^2_{n-k}, \quad \frac{SS(T)}{\sigma^2} \sim \chi^2_{k-1},$$

$SS(E)$ and $SS(T)$ are independent

H_0 is true → group mean is similar



H_1 is true large difference between group mean



$SS(T)$ under H_1

- What happens if H_1 is true?
- The population means differ, which will make $SS(T)$ larger
- Let's make this precise. . .

- Let $\bar{\mu} = n^{-1} \sum_{i=1}^{k} n_i \mu_i$, and then,

$$E(SS(T)) = E\left[\sum_{i=1}^{k} n_i (\bar{X}_{i.} - \bar{X}_{..})^2\right]$$

$$= E\left[\sum_{i=1}^{k} n_i \bar{X}_{i.}^2 + n_i \bar{X}_{..}^2 - 2 n_i (\bar{X}_{i.}) \bar{X}_{..}\right] \quad \sum_{i=1}^{k} n_i = n$$

$$= E\left[\sum_{i=1}^{k} n_i \bar{X}_{i.}^2 + n \bar{X}_{..}^2\right] - 2E\left(\sum_{i=1}^{k} n_i (\bar{X}_{i.})(\bar{X}_{..})\right)$$

$$= \sum_{i=1}^{k} n_i E(\bar{X}_{i.}^2) + n \bar{X}_{..}^2 - 2\bar{X}_{..} \underbrace{E\left(\sum_{i=1}^{k} n_i \bar{X}_{i.}\right)}_{E(n \bar{X}_{..})}$$

$$= \sum_{i=1}^{k} n_i E(\bar{X}_{i.}^2) - n E(\bar{X}_{..}^2)$$

$$= \sum_{i=1}^{k} n_i \left[var(\bar{X}_{i.}) + E(\bar{X}_{i.})^2\right] - n\left[var(\bar{X}_{..}) + E(\bar{X}_{..})^2\right]$$

$$= \sum_{i=1}^{k} n_i \left[\frac{\sigma^2}{n_i} + \mu_i^2\right] - n\left[\frac{\sigma^2}{n} + \bar{\mu}^2\right]$$

$$= \left[\sum_{i=1}^{k} \sigma^2 + n_i \mu_i^2\right] - \sigma^2 - n\bar{\mu}^2$$

$$\mathbb{E}[SS(T)] = \mathbb{E}\left[\sum_{i=1}^{k} n_i (\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2\right] = \mathbb{E}\left[\sum_{i=1}^{k} n_i \bar{X}_{i\cdot}^2 - n\bar{X}_{\cdot\cdot}^2\right]$$

$$= \sum_{i=1}^{k} n_i \, \mathbb{E}(\bar{X}_{i\cdot}^2) - n \, \mathbb{E}(\bar{X}_{\cdot\cdot}^2)$$

$$= \sum_{i=1}^{k} n_i \left[\text{var}(\bar{X}_{i\cdot}) + \mathbb{E}(\bar{X}_{i\cdot})^2\right] - n\left[\text{var}(\bar{X}_{\cdot\cdot}) + \mathbb{E}(\bar{X}_{\cdot\cdot})^2\right]$$

$$= \sum_{i=1}^{k} n_i \left[\frac{\sigma^2}{n_i} + \mu_i^2\right] - n\left[\frac{\sigma^2}{n} + \bar{\mu}^2\right]$$

$$= \underbrace{(k-1)\sigma^2}_{\text{natural variation in the data}} + \underbrace{\sum_{i=1}^{k} n_i (\mu_i - \bar{\mu})^2}_{\text{depend on mean (how groups relates to each other}}$$

$\bar{\mu}$: weighted average

$= k\sigma^2 - \sigma^2 + \sum_{i=1}^{k} n_i \mu_i^2 - n\bar{\mu}^2$

$= (k-1)\sigma^2 + \sum_{i=1}^{k} n_i(\mu_i - \bar{\mu})^2$

$$\mathbb{E}[SS(T)] = (k-1)\sigma^2 + \sum_{i=1}^{k} n_i(\mu_i - \bar{\mu})^2$$

- Under H_0 the second term is zero and we have,

μ_i equal $\bar{\mu}$

$$\frac{\mathbb{E}(SS(T))}{k-1} = \sigma^2$$

- Otherwise (under H_1), the second term is positive and gives,

$$\frac{\mathbb{E}(SS(T))}{k-1} > \sigma^2$$

hypothesis test

$\Rightarrow$ reject if $SS(T)$ to big

$SS(T) > c$

- In contrast, we always have,

$$\frac{E(SS(E))}{n-k} = \sigma^2$$

always hold no matter H_0 or H_1

$\frac{SS(T)}{\sigma^2} \sim \chi^2_{k-1}$ (H_0)

since we don't know σ (parameter)

F-test statistic

- This movitates using the following as our test statistic:

$$F = \frac{SS(T)/(k-1)}{SS(E)/(n-k)}$$

- Under H_0, we have $F \sim \mathrm{F}_{k-1,n-k}$, since it is the ratio of independent χ^2 random variables
- Under H_1, the numerator will tend to be larger
- Therefore, reject H_0 if $F > c$
- This is known as an F-test

substitude σ^2 with s^2 get a t distribution

$\frac{SS(T)}{\sigma^2}$

since $\frac{E(SS(E))}{n-k} = \sigma^2$

get an estimator $\hat{\sigma}^2 = \frac{SS(E)}{n-k}$

$\frac{SS(T)}{\frac{SS(E)}{n-k}}$

we don't know the distribution

$$\frac{\frac{SS(T)}{\sigma^2} / k-1}{\frac{SSE}{\sigma^2} / n-k} = F$$

difference b/t groups

inheritance variation

kind of a reference to judge SS(T)

ANOVA table

intuition
let $k=2$
degree of freedom $n-1$
treatment df $= 1$
need one extra parameter to explain having 2 groups rather than only one group

The test quantities are often summarised using an ANOVA table:

think about # of parameters

Source	df	SS	MS	F
Treatment	$k-1$	$SS(T)$	$MS(T) = \frac{SS(T)}{k-1}$	$\frac{MS(T)}{MS(E)}$
Error	$n-k$	$SS(E)$	$MS(E) = \frac{SS(E)}{n-k}$	
Total	$n-1$	$SS(TO)$		

$= SS(T) + SS(E)$

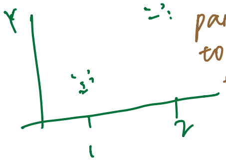
k groups

$k-1$ can move around

$k-1$ things to set to explain the data

Notes:

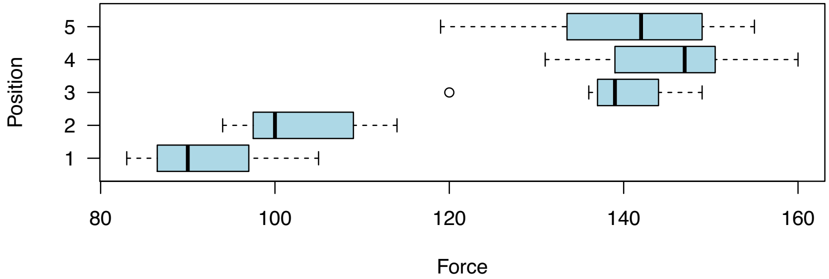
- MS = 'Mean square'
- $\hat{\sigma}^2 = MS(E)$ is an unbiased estimator



need one parameter to explain the difference between groups

Example (one-way ANOVA)

Force required to pull out window studs in 5 positions on a car window.



```
> head(data1)
  Position Force
1        1    92
2        1    90
3        1    87
4        1   105
5        1    86
6        1    83

> table(data1$Position)

1 2 3 4 5
7 7 7 7 7
```

→ summarise

y ~ x

```
> model1 <- lm(Force ~ factor(Position), data = data1)
> anova(model1)
Analysis of Variance Table

Response: Force
                  Df  Sum Sq Mean Sq F value    Pr(>F)
factor(Position)   4 16672.1  4168.0  44.202 3.664e-12 ***
Residuals         30  2828.9    94.3
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

p-value

→ rejection of null

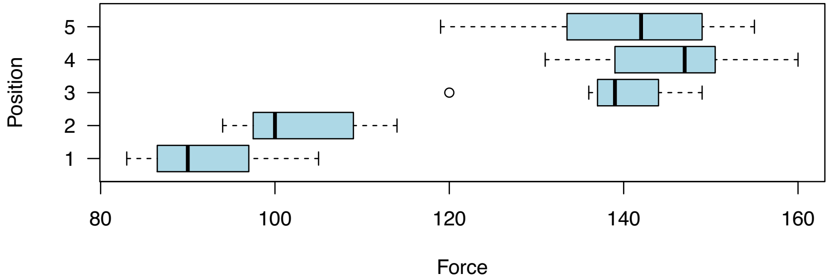
Notes:

- Need to use `factor()` to denote categorical variables
- R doesn't provide a 'Total' row, but we don't need it
- Residuals is the 'Error' row
- `Pr(>F)` is the p-value for the F-test

not a numerical variable.

We conclude that the mean force required to pull out the window studs varies between the 5 positions on the car window (e.g. p-value < 0.01)

This was obvious from the boxplots: positions 1 & 2 are quite different from 3, 4 & 5



Two factors

- In one-way ANOVA, the observations were partitioned into k groups
- In other words, they were defined by a single categorical variable (‘factor’)
- What if we had two such variables?
- We can extend the procedure to give two-way ANOVA, or two-factor ANOVA
- For example, the fuel consumption of a car may depend on type of petrol and the brand of tyres

Two-way ANOVA: setting it up

- Factor 1 has a levels, Factor 2 has b levels
- Suppose we have exactly one observation per factor combination
- Observe X_{ij} with factor 1 at level i and factor 2 at level j
- Gives a total of $n = ab$ observations
- Assume $X_{ij} \sim \mathrm{N}(\mu_{ij}, \sigma^2)$, $i = 1, \ldots, a$, $j = 1, \ldots, b$, and that these are independent
- Consider the model:

deviation from the average

overall mean

$$\mu_{ij} = \mu + \boxed{\alpha_i + \beta_j}$$

contribution of factor 2

contribution of factor 1

$$\text{with } \sum_{i=1}^{a} \alpha_i = 0, \ \sum_{j=1}^{b} \beta_j = 0$$

- μ is an overall effect, α_i is the effect of the ith row and β_j the effect of the jth column.
- For example, $a = 4$ and $b = 4$,

	1	2	3	4
1	$\mu + \alpha_1 + \beta_1$	$\mu + \alpha_1 + \beta_2$	$\mu + \alpha_1 + \beta_3$	$\mu + \alpha_1 + \beta_4$
2	$\mu + \alpha_2 + \beta_1$	$\mu + \alpha_2 + \beta_2$	$\mu + \alpha_2 + \beta_3$	$\mu + \alpha_2 + \beta_4$
3	$\mu + \alpha_3 + \beta_1$	$\mu + \alpha_3 + \beta_2$	$\mu + \alpha_3 + \beta_3$	$\mu + \alpha_3 + \beta_4$
4	$\mu + \alpha_4 + \beta_1$	$\mu + \alpha_4 + \beta_2$	$\mu + \alpha_4 + \beta_3$	$\mu + \alpha_4 + \beta_4$

- We are usually interested in $\boxed{H_{0A}}: \alpha_1 = \alpha_2 = \cdots = \alpha_a = 0$ or $H_{0B}: \beta_1 = \beta_2 = \cdots = \beta_b = 0$

hyphothesis for A

- Let

$$\bar{X}_{..} = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} X_{ij}, \quad \bar{X}_{i.} = \frac{1}{b}\sum_{j=1}^{b} X_{ij}, \quad \bar{X}_{.j} = \frac{1}{a}\sum_{i=1}^{a} X_{ij}$$

$$\bar{X}_{..} = \sum_{i=1}^{a} \frac{1}{a\cancel{b}} \cancel{b}\, \bar{X}_{i.}$$

$$= \frac{1}{a}\sum_{i=1}^{a} \bar{X}_{i.}$$

- Arguing as before,

$$\overbrace{SS(TO)}^{} = \sum_{i=1}^{a} \sum_{j=1}^{b} (X_{ij} - \bar{X}_{..})^2$$

row col

$$= \sum_{i=1}^{a} \sum_{j=1}^{b} [(\boxed{\bar{X}_{i.}} - \bar{X}_{..}) + (\boxed{\bar{X}_{.j}} - \bar{X}_{..}) + (X_{ij} - \boxed{\bar{X}_{i.}} - \boxed{\bar{X}_{.j}} + \bar{X}_{..})]^2$$

$$= b\sum_{i=1}^{a} (\bar{X}_{i.} - \bar{X}_{..})^2 + a\sum_{j=1}^{b} (\bar{X}_{.j} - \bar{X}_{..})^2$$

$$+ \sum_{i=1}^{a} \sum_{j=1}^{b} (X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})^2$$

$$= SS(A) + SS(B) + SS(E)$$

$\bar{x}_{..} \sum_{i=1}^{a}\sum_{j=1}^{b} (\bar{x}_{i.} + \bar{x}_{.j})$

$= \bar{x}_{..} 2 (ab \bar{x}_{..})$

$\sum_{i=1}^{a}\sum_{j=1}^{b} (x_{ij} - \bar{x}_{..})^2$

$= \sum_{i=1}^{a}\sum_{j=1}^{b} [(\bar{x}_{i.} - \bar{x}_{..}) + (\bar{x}_{.j} - \bar{x}_{..}) + (x_{ij} - \bar{x}_{i.} - \bar{x}_{.j} + \bar{x}_{..})]^2$

$= \sum_{i=1}^{a}\sum_{j=1}^{b} [(\bar{x}_{i.} - \bar{x}_{..})^2 + (\bar{x}_{.j} - \bar{x}_{..})^2 + (x_{ij} - \bar{x}_{i.} - \bar{x}_{.j} + \bar{x}_{..})^2]$

cross term

① $\sum_{i=1}^{a}\sum_{j=1}^{b} (\bar{x}_{i.} - \bar{x}_{..})(\bar{x}_{.j} - \bar{x}_{..})$

$= \sum_{i=1}^{a}\sum_{j=1}^{b} (\bar{x}_{i.})(\bar{x}_{.j}) - (\bar{x}_{i.} + \bar{x}_{.j})(\bar{x}_{..}) + (\bar{x}_{..})^2$

$= \cancel{ab(\bar{x}_{..})^2} - 2ab(\bar{x}_{..})^2 + \cancel{(ab)(\bar{x}_{..})^2} = 0.$

9.4-4. Show that the cross-product terms formed from $(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot})$, $(\overline{X}_{\cdot j} - \overline{X}_{\cdot\cdot})$, and $(X_{ij} - \overline{X}_{i\cdot} - \overline{X}_{\cdot j} + \overline{X}_{\cdot\cdot})$ sum to zero, $i = 1, 2, \ldots a$ and $j = 1, 2, \ldots, b$. Hint: For example, write

$$\sum_{i=1}^{a}\sum_{j=1}^{b} (\overline{X}_{\cdot j} - \overline{X}_{\cdot\cdot})(X_{ij} - \overline{X}_{i\cdot} - \overline{X}_{\cdot j} + \overline{X}_{\cdot\cdot})$$

$$= \sum_{j=1}^{b} (\overline{X}_{\cdot j} - \overline{X}_{\cdot\cdot}) \sum_{i=1}^{a} [(X_{ij} - \overline{X}_{\cdot j}) - (\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot})]$$

and sum each term in the inner summation, as grouped here, to get zero.

② $\sum_{i=1}^{a}\sum_{j=1}^{b} (\bar{x}_{i.} - \bar{x}_{..})(x_{ij} - \bar{x}_{i.} - \bar{x}_{.j} + \bar{x}_{..})$

$= \sum_{i=1}^{a} (\bar{x}_{i.} - \bar{x}_{..}) \underbrace{\sum_{j=1}^{b} [(x_{ij} - \bar{x}_{i.}) - (\bar{x}_{.j} - \bar{x}_{..})]}$

$b(\bar{x}_{i.} - \bar{x}_{i.}) - b(\bar{x}_{..} - \bar{x}_{..})$

$= 0.$

similar for ③

- If both $\alpha_1 = \cdots = \alpha_a = 0$ and $\beta_1 = \cdots = \beta_b = 0$, then we have $SS(A)/\sigma^2 \sim \chi^2_{a-1}$, $SS(B)/\sigma^2 \sim \chi^2_{b-1}$ and $SS(E)/\sigma^2 \sim \chi^2_{(a-1)(b-1)}$ and these variables are independent (proof not shown)
- Reject H_{0A}: $\alpha_1 = \cdots = \alpha_a = 0$ at significance level α if:

$$F_A = \frac{SS(A)/(a-1)}{SS(E)/((a-1)(b-1))} > c$$

  where c is the $1-\alpha$ quantile of $\mathrm{F}_{a-1,(a-1)(b-1)}$
- Reject H_{0B}: $\beta_1 = \cdots = \beta_b = 0$ at significance level α if:

$$F_B = \frac{SS(B)/(b-1)}{SS(E)/((a-1)(b-1))} > c$$

  where c is the $1-\alpha$ quantile of $\mathrm{F}_{b-1,(a-1)(b-1)}$

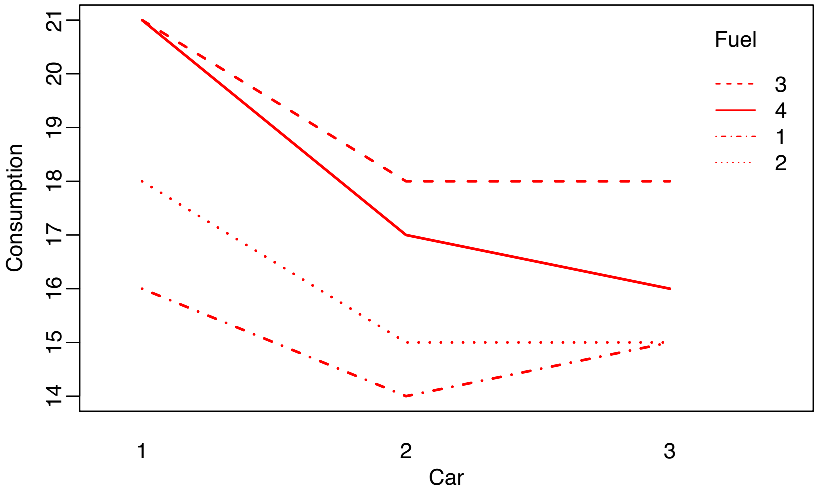
ANOVA table

Source	df	SS	MS	F
Factor A	$a-1$	$SS(A)$	$MS(A)=\frac{SS(A)}{a-1}$	$\frac{MS(A)}{MS(E)}$
Factor B	$b-1$	$SS(B)$	$MS(B)=\frac{SS(B)}{b-1}$	$\frac{MS(B)}{MS(E)}$
Error	$(a-1)(b-1)$	$SS(E)$	$MS(E)=\frac{SS(E)}{(a-1)(b-1)}$	
Total	$ab-1$	$SS(TO)$		

Example (two-way ANOVA)

Data on fuel consumption for three types of car (A) and four types of fuel (B).

```
> head(data2)
  Car Fuel Consumption
1   1    1          16
2   1    2          18
3   1    3          21
4   1    4          21
5   2    1          14
6   2    2          15
```


Fuel
3
4
1
2
Consumption
14
15
16
17
18
19
20
21
1
2
3
Car

2 factors

```
> model2 <- lm(Consumption ~ factor(Car) + factor(Fuel),
+                   data = data2)
> anova(model2)
Analysis of Variance Table

Response: Consumption
             Df Sum Sq Mean Sq F value   Pr(>F)
factor(Car)   2     24 12.0000      18 0.002915 **
factor(Fuel)  3     30 10.0000      15 0.003401 **
Residuals     6      4  0.6667
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

From this we conclude there is a clear difference in fuel consumption between cars (we reject $H_{0A}: \alpha_1 = \alpha_2 = \alpha_3$) and also between fuels (we reject $H_{0B}: \beta_1 = \beta_2 = \beta_3 = \beta_4$).

Interaction terms

- In the previous example we assumed an additive model:

$$\mu_{ij} = \mu + \alpha_i + \beta_j$$

- This assumes, for example, that the relative effect of petrol 1 is the same for all cars.
- If it is not true, then there is a statistical interaction (or simply an interaction) between the factors

- A more general model, which includes interactions, is:

overall mean

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$$

impact of particular level of factor a

  where γ_{ij} is the interaction term associated with combination (i, j).
- In addition to our previous assumptions, we also impose:

$$\sum_{i=1}^{a} \gamma_{ij} = 0, \quad \text{and} \quad \sum_{j=1}^{b} \gamma_{ij} = 0$$

- The terms α_i and β_j are called main effects
- When written out as a table they are also often referred to as the row effects and column effects respectively

- Writing this out as a table:

	1	2	$\cdots$
1	$\mu+\alpha_1+\beta_1+\gamma_{11}$	$\mu+\alpha_1+\beta_2+\gamma_{12}$	$\cdots$
2	$\mu+\alpha_2+\beta_1+\gamma_{21}$	$\mu+\alpha_2+\beta_2+\gamma_{22}$	$\cdots$
3	$\mu+\alpha_3+\beta_1+\gamma_{31}$	$\mu+\alpha_3+\beta_2+\gamma_{32}$	$\cdots$
4	$\mu+\alpha_4+\beta_1+\gamma_{41}$	$\mu+\alpha_4+\beta_2+\gamma_{42}$	$\cdots$

- We are now interested in testing whether:
  - the row effects are zero
  - the column effects are zero
  - the interactions are zero (do this first!)
- To make inferences about the interactions we need more than one observation per cell
- Let X_{ijk}, $i=1,\ldots,a$, $j=1,\ldots,b$, $k=1,\ldots,c$ be the kth observation for combination (i,j)

$X_{ijk} \sim N(\mu_{ij}, \sigma^2)$

$\mu_{ij} \to \mu + a_i + b_j + \gamma_{ij}$

if we only have one observation and have deviation from what we expected, cannot tell from γ_{ij} or σ^2

the deviation is γ_{ij}

σ^2: natural noise

- Let

$$\bar{X}_{ij\cdot} = \frac{1}{c}\sum_{k=1}^{c} X_{ijk}$$

$$\bar{X}_{i\cdot\cdot} = \frac{1}{bc}\sum_{j=1}^{b}\sum_{k=1}^{c} X_{ijk}$$

$$\bar{X}_{\cdot j\cdot} = \frac{1}{ac}\sum_{i=1}^{a}\sum_{k=1}^{c} X_{ijk}$$

$$\bar{X}_{\cdots} = \frac{1}{abc}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} X_{ijk}$$

- and as before

$$
\begin{aligned}
SS(TO) &= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(X_{ijk} - \bar{X}_{...}\right)^2 \\
&= bc\sum_{i=1}^{a}(\bar{X}_{i..} - \bar{X}_{...})^2 + ac\sum_{j=1}^{b}(\bar{X}_{.j.} - \bar{X}_{...})^2 \\
&\quad + c\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{X}_{ij.} - \bar{X}_{i..} - \bar{X}_{.j.} + \bar{X}_{...})^2 \\
&\quad + \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(X_{ijk} - \bar{X}_{ij.})^2 \\
&= SS(A) + SS(B) + SS(AB) + SS(E)
\end{aligned}
$$

Test statistics

- Familiar arguments show that to test

all interaction term is 0

$$H_{0AB}: \gamma_{ij} = 0, \quad i = 1, \ldots, a, \quad j = 1, \ldots, b$$

we may use the statistic

$$F = \frac{SS(AB)/[(a-1)(b-1)]}{SS(E)/[ab(c-1)]}$$

which has a F distribution with $(a-1)(b-1)$ and $ab(c-1)$ degrees of freedom.

- To test

$$H_{0A}\colon \alpha_i = 0, \quad i = 1, \ldots, a$$

we may use the statistic

$$F = \frac{SS(A)/[(a-1)]}{SS(E)/[ab(c-1)]}$$

which has a F distribution with $(a-1)$ and $ab(c-1)$ degrees of freedom.

- To test

$$H_{0B}\colon \beta_j = 0, \quad j = 1, \ldots, b$$

we may use the statistic

$$F = \frac{SS(B)/[(b-1)]}{SS(E)/[ab(c-1)]}$$

which has a F distribution with $(b-1)$ and $ab(c-1)$ degrees of freedom.

ANOVA table

Source	df	SS	MS	F
Factor A	$a-1$	$SS(A)$	$MS(A) = \frac{SS(A)}{a-1}$	$\frac{MS(A)}{MS(E)}$
Factor B	$b-1$	$SS(B)$	$MS(B) = \frac{SS(B)}{b-1}$	$\frac{MS(B)}{MS(E)}$
Factor AB	$(a-1)(b-1)$	$SS(AB)$	$MS(AB) = \frac{SS(AB)}{(a-1)(b-1)}$	$\frac{MS(AB)}{MS(E)}$
Error	$ab(c-1)$	$SS(E)$	$MS(E) = \frac{SS(E)}{ab(c-1)}$	
Total	$abc-1$	$SS(TO)$		

Example (two-way ANOVA with interaction)

- Six groups of 18 people
- Each person takes an arithmetic test: the task is to add three numbers together
- The numbers are presented either in a down array or an across array; this defines 2 levels of factor A
- The numbers have either one, two or three digits; this defines 3 levels of factor B
- The response variable, X, is the average number of problems completed correctly over two 90-second sessions

- Example of adding **one-digit** numbers in an **across** array:

$$2 + 5 + 1 = ?$$

- Example of adding **two-digit** numbers in an **down** array:

$$\begin{array}{rr} & 13 \\ & 87 \\ + & 51 \\ \hline & ? \end{array}$$

```
> head(data3)
     A B    X
1 down 1 19.5
2 down 1 18.5
3 down 1 32.0
4 down 1 21.5
5 down 1 28.5
6 down 1 33.0

> table(data3[, 1:2])
        B
A          1  2  3
  down    18 18 18
  across 18 18 18
```

→ consider two factors and interaction

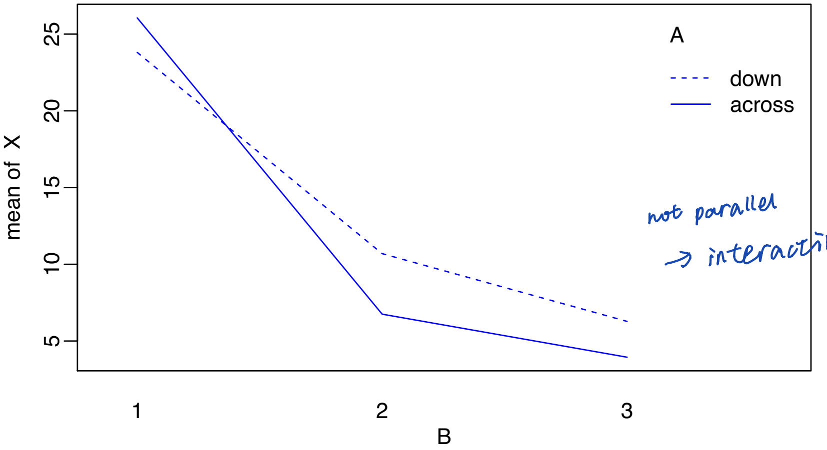
```
> model3 <- lm(X ~ factor(A) * factor(B), data = data3)
> anova(model3)
Analysis of Variance Table

Response: X
                     Df Sum Sq Mean Sq  F value  Pr(>F)
factor(A)             1   48.7    48.7   2.8849 0.09246 .
factor(B)             2 8022.7  4011.4 237.7776 < 2e-16 ***
factor(A):factor(B)   2  185.9    93.0   5.5103 0.00534 **
Residuals           102 1720.8    16.9
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

interaction →

Note the use of '∗' in the model formula.
The interaction is significant at a 5% level (or even at 1%).

Interaction plot

```
with(data3, interaction.plot(B, A, X, col = "blue"))
```



not parallel
→ interaction

Beyond the F-test

- We have rejected the null. . . now what?
- This is often only the beginning of a statistical analysis of this type of data
- Will be interested in more detailed inferences, e.g. CIs/tests about individual parameters
- You know enough to be able to work some of this out. . .
- . . . and later subjects will go into this in more detail (e.g. MAST30025)

Outline

Recap of simple linear regression

- Y a response variable, e.g. student's grade in first-year calculus
- x a predictor variable, e.g. student's high school mathematics mark
- Data: pairs $(x_1, y_1), \ldots, (x_n, y_n)$
- Linear regression model:

$$Y_i = \alpha + \beta(x_i - \bar{x}) + \epsilon_i$$

  where $\epsilon_i \sim \mathrm{N}(0, \sigma^2)$ is a random error
- **Note:** α here plays the same role as α_0 from Module 5. We have dropped the '0' subscript for convenience, and also to avoid confusion with its use to denote null hypotheses.

- The MLE (and OLS) estimators are:

$$\hat{\alpha} = \bar{Y}, \quad \hat{\beta} = \frac{\sum_{i=1}^{n} Y_i(x_i - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

- and

$$\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n}[Y_i - \hat{\alpha} - \hat{\beta}(x_i - \bar{x})]^2$$

- We also derived:

$$\hat{\alpha} \sim \mathrm{N}\left(\alpha, \frac{\sigma^2}{n}\right)$$

$$\hat{\beta} \sim \mathrm{N}\left(\beta, \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\right)$$

- and

$$\frac{(n-2)\hat{\sigma}^2}{\sigma^2} = \frac{\sum_{i=1}^{n}\left[Y_i - \hat{\alpha} - \hat{\beta}(x_i - \bar{x})\right]^2}{\sigma^2} \sim \chi^2_{n-2}$$

- From these we obtain,

$$T_\alpha = \frac{\hat{\alpha} - \alpha}{\hat{\sigma}/\sqrt{n}} \sim t_{n-2}$$

$$T_\beta = \frac{\hat{\beta} - \beta}{\hat{\sigma}/\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}} \sim t_{n-2}$$

- We used these previously to construct confidence intervals
- We can also use them to construct hypothesis tests
- For example, to test $H_0 : \beta = \beta_0$ versus $H_1 : \beta \neq \beta_0$ (or $\beta > \beta_0$ or $\beta < \beta_0$), we use T_β as the test statistic

Example: testing the slope parameter (β)

- Data: 10 pairs of scores on a preliminary test and a final exam
- Estimates: $\hat{\alpha} = 81.3$, $\hat{\beta} = 0.742$, $\hat{\sigma}^2 = 27.21$
- Test $H_0 : \beta = 0$ versus $H_1 : \beta \neq 0$ with a 1% significance level
- Reject H_0 if: two sides test

$$|T_\beta| \geqslant 3.36 \quad \text{(0.995 quantile of } t_8\text{)}$$

- For the observed data,

$$t_\beta = \frac{0.742 - 0}{\sqrt{27.21/756.1}} = 3.91$$

so we reject H_0, concluding there is sufficient evidence that the slope differs from zero.

Note regarding the intercept parameter (α)

- Software packages (such as R) will typically fit the model:

$$Y_i = \alpha + \beta x_i + \epsilon_i$$

- This is equivalent to

$$Y_i = \alpha^* + \beta(x_i - \bar{x}) + \epsilon_i$$

  where $\alpha = \alpha^* - \beta\bar{x}$

- The formulation $Y_i = \alpha^* + \beta(x - \bar{x}) + \epsilon$ is easier to examine theoretically.
- We saw that

$$\hat{\alpha}^* = \bar{Y}, \quad \text{and} \quad \hat{\alpha} = \bar{Y} - \hat{\beta}\bar{x}$$

- $\hat{\alpha}$ or $\hat{\alpha}^*$ are rarely of direct interest

Using R

Use R to fit the regression model for the slope example:

```
                        y  ~   x
> m1 <- lm(final_exam ~ prelim_test)
> summary(m1)

Call:
lm(formula = final_exam ~ prelim_test)

Residuals:
   Min      1Q  Median      3Q     Max
-6.883 -3.264 -0.530  3.438  8.470
```

two sides alternative
t statistic
p value for test relating to those parameter

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  30.6147    13.0622   2.344  0.04714 *
prelim_test   0.7421     0.1897   3.912  0.00447 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 5.217 on 8 degrees of freedom
Multiple R-Squared: 0.6567,  Adjusted R-squared: 0.6137
F-statistic:  15.3 on 1 and 8 DF,  p-value: 0.004471
```

test $\alpha=0$

?

test $\beta=0$

The t-value and the p-value are for testing $H_0: \alpha = 0$ and $H_0: \beta = 0$ respectively.

Interpreting the R output

- Usually most interested in testing $H_0: \beta = 0$ versus $H_1: \beta \neq 0$
- If we reject H_0 then we conclude there is sufficient evidence of (at least) a linear relationship between the mean response and x
- In the example,
$$t = \frac{0.7421}{0.1897} = 3.912$$
- This test statistic has a t-distribution with $10 - 2 = 8$ degrees of freedom, and the associated p-value is $0.00447 < 0.05$ so at the 5% level of significance we reject H_0
- It is also possible to represent this test using an ANOVA table

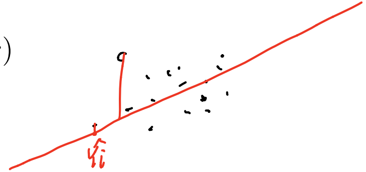
Deriving the variance decomposition formula

- Independent pairs $(x_1, Y_1), \ldots, (x_n, Y_n)$
- Parameter estimates,

$$\hat{\alpha} = \bar{Y}, \quad \hat{\beta} = \frac{\sum_{i=1}^{n} Y_i (x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

- Fitted value (estimated mean), for each y

$$\hat{Y}_i = \bar{Y} + \hat{\beta}(x_i - \bar{x})$$



- Do the ‘add and subtract’ trick again:

$$\begin{aligned}\sum_{i=1}^{n}(Y_i - \bar{Y})^2 &= \sum_{i=1}^{n}(Y_i \boxed{- \hat{Y}_i + \hat{Y}_i} - \bar{Y})^2 \\ &= \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 + \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2 \\ &\quad + 2\sum_{i=1}^{n}(Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y})\end{aligned}$$

- Deal with the cross-product term,

$$\sum_{i=1}^{n}(Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y}) = \sum_{i=1}^{n}\left[Y_i - \bar{Y} - \hat{\beta}(x_i - \bar{x})\right]\hat{\beta}(x_i - \bar{x})$$

$$= \hat{\beta}\sum_{i=1}^{n}\left[Y_i - \bar{Y} - \hat{\beta}(x_i - \bar{x})\right](x_i - \bar{x})$$

$$= \hat{\beta}\left[\sum_{i=1}^{n}(Y_i - \bar{Y})(x_i - \bar{x}) - \hat{\beta}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]$$

$$= \underbrace{\hat{\beta}\left[\sum_{i=1}^{n}Y_i(x_i - \bar{x}) - \hat{\beta}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]}_{\Downarrow\ 0}$$

$$= 0$$

$\hat{Y}_i = \bar{Y} + \hat{\beta}(x_i - \bar{x})$

$\sum_{i=1}^{n}(Y_i - \hat{Y}_i)(\hat{Y}_i - \bar{Y})$

$= \sum_{i=1}^{n}[Y_i - \bar{Y} - \hat{\beta}(x_i - \bar{x})]\hat{\beta}(x_i - \bar{x})$

$= \hat{\beta}\sum_{i=1}^{n}(Y_i - \underset{\text{constant}}{\bar{Y}})(x_i - \bar{x}) - \hat{\beta}^2\sum_{i=1}^{n}(x_i - \bar{x})^2$

$= \hat{\beta}\sum_{i=1}^{n}Y_i(x_i - \bar{x}) - \hat{\beta}^2\sum_{i=1}^{n}(x_i - \bar{x})^2$

$\hat{\beta} = \dfrac{\sum(x_i - \bar{x})Y_i}{\sum(x_i - \bar{x})^2}$

- That gives us,

$$\sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 + \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2$$

- We can write this as follows,

$$SS(TO) = SS(E) + SS(R)$$

"between group"

where $SS(R)$ is the regression SS or model SS

- The regression SS quantifies the variation **due to** the straight line
- The error SS quantifies the variation **around** the straight line

- To complete the specification,

MS(E) is a unbiased estimator for σ

$$MS(E) = \frac{SS(E)}{n-2} = \frac{1}{n-2}\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 = \hat{\sigma}^2$$

$$MS(R) = \frac{SS(R)}{1} = \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2$$

- Then we have the test statistic,

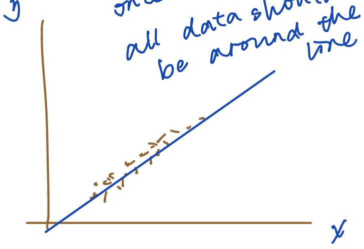
$$F = \frac{MS(R)}{MS(E)} \sim \mathrm{F}_{1,n-2}$$

ANOVA table

slope parameter
relationship between x & Y
all inside β

if the data do have
linear trend, high F
→ reject null → slope ⇒
model doesn't
fit very
well

once setting β,
all data should
be around the
line



Source	df	SS	MS	F
Model	1	$SS(R)$	$MS(R) = \frac{SS(R)}{1}$	$\frac{MS(R)}{MS(E)}$
Error	$n-2$	$SS(E)$	$MS(E) = \frac{SS(E)}{n-2}$	
Total	$n-1$	$SS(TO)$		

is there relationship
between x & Y

⇒ $\beta = 0$ test H_0

Using R

```
> anova(m1)
Analysis of Variance Table

Response: final_exam
            Df Sum Sq Mean Sq F value   Pr(>F)
prelim_test  1 416.39  416.39  15.301 0.004471 **
Residuals    8 217.71   27.21
---
Signif. codes:  0 ’***’ 0.001 ’**’ 0.01 ’*’ 0.05 ’.’ 0.1 ’ ’ 1
```

Notes:

- The F-statistic tests the ‘significance of the regression’
- That is, $H_0: \beta = 0$ versus $H_1: \beta \neq 0$

Outline

Is there a ‘best’ test?

- We have examined a variety of commonly used tests
- We used test statistics that:
  - Seemed useful
  - We were familiar with
- Did we use the ‘best’ one?
- Is there a general procedure for finding a good/best test statistic?
- We will introduce a general procedure now, and discuss why it is optimal later in the semester

Likelihood ratio test

- The likelihood ratio test (LRT) is a general procedure that can find the best test for a given problem
- Suppose we have H_0 and H_1 and both are composite and of the form:

$$H_0: \theta \in A_0 \quad \text{versus} \quad H_1: \theta \in A_1$$

  where A_0 and A_1 are sets of possible parameter values consistent with each of the hypotheses.
- Note: we have mostly dealt with A_0 that has only one element (simple null hypothesis)
- The likelihood ratio is:

$$\boxed{\lambda = \frac{L_0}{L_1} = \frac{\max_{\theta \in A_0} L(\theta)}{\max_{\theta \in A_1} L(\theta)}}$$

how well is the data explained by one hypothesis than the other

$$\lambda = \frac{L_0}{L_1} = \frac{\max_{\theta \in A_0} L(\theta)}{\max_{\theta \in A_1} L(\theta)}$$

- L is the likelihood function
- Clearly $\lambda \geqslant 0$
- Large $\lambda \Rightarrow$ more support for H_0 over H_1
- λ near zero $\Rightarrow$ more support for H_1 over H_0
- Therefore, we want a critical region of the form,

$$\lambda \leqslant k$$

once λ is too small
(let k as threshold)
the reject H0

- Choose k to give the desired significance level

Example 1 (likelihood ratio test)

- $X_i \sim \mathrm{N}(\mu, \sigma^2 = 5)$, i.e. σ is known
- $H_0 : \mu = 162$ versus $H_1 : \mu \neq 162$
- When H_0 is true, $\mu = 162$ so $L_0 = L(162)$
- When H_1 is true, need to maximise the likelihood, $L_1 = L(\hat{\theta}) = L(\bar{x})$ $\longrightarrow$ maximum likelihood estimator
- The likelihood ratio is,

$$\lambda = \frac{L_0}{L_1} = \frac{L(162)}{L(\bar{x})} = \frac{(10\pi)^{-n/2} \exp\left[-\frac{1}{10}\sum_{i=1}^{n}(x_i - 162)^2\right]}{(10\pi)^{-n/2} \exp\left[-\frac{1}{10}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]}$$
$$= \exp\left[-\frac{n}{10}(\bar{x} - 162)^2\right]$$

$$\lambda = \boxed{\exp\left[-\frac{n}{10}(\bar{x} - 162)^2\right] \le k}$$

- $\lambda \leqslant k$ same as

z-test

follow standard normal.

$$\frac{|\bar{x} - 162|}{\sigma/\sqrt{n}} \geqslant c$$

$$-\frac{n}{10}(\bar{x} - 162)^2 \le \log k$$

$$n(\bar{x} - 162)^2 \ge -10\log k$$

- A critical region for a size α test is

$$\frac{|\bar{x} - 162|}{\sigma/\sqrt{n}} \geqslant \Phi^{-1}(1 - \alpha/2)$$

- Note: this required knowledge of the distribution of $\bar{X}$!

Example 2 (likelihood ratio test)

- $X_i \sim \mathrm{N}(\mu, \sigma^2)$, i.e. σ is unknown
- $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$
- Under H_0 we have $\mu = \mu_0$, and under H_1 we need to use its MLE
- Under either hypothesis, σ^2 is unspecified, so in both cases we need its MLE (conditional on the specified value of μ).
- So, under H_0 we use:

$$\hat{\mu} = \mu_0, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_0)^2$$

- And under H_1 we use:

$$\hat{\mu} = \bar{x}, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Sampling (iid) from: $X \sim \mathrm{N}(\theta_1, \theta_2)$

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} \exp\left[-\frac{(x_i - \theta_1)^2}{2\theta_2}\right]$$

$$\ln L(\theta_1, \theta_2) = -\frac{n}{2}\ln(2\pi\theta_2) - \frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i - \theta_1)^2$$

σ^{-2}

$-2\sigma^{-3}$

$$\ln L(\mu, \sigma^2) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2$$

$$\frac{\partial \ln L(\mu, \sigma^2)}{\partial \sigma} = -\frac{n}{2}\cdot\frac{\cancel{2\sigma 2\pi}}{\cancel{2\pi\sigma^2}} + \frac{1}{2}\frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i - \mu)^2$$

$$\sigma^{-2}\cancel{\frac{1}{2\sigma^3}}\sum_{i=1}^{n}(x_i - \mu)^2 = \cancel{\frac{n}{2}}\cdot\cancel{\frac{1}{\sigma}}$$

$$\sigma^2 = \sum_{i=1}^{n}(x_i - \mu)^2 \cdot \frac{1}{n}$$

$$\therefore \hat{\sigma}^2 = \sum_{i=1}^{n}(x_i - \mu_0)^2 \cdot \frac{1}{n} \text{ under } H_0$$

- Some simplification yields

$$\lambda = \left[\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{\sum_{i=1}^{n}(x_i - \mu_0)^2} \right]^{n/2}$$

$$L(\widehat{\Omega}) = \left[\frac{1}{2\pi \left(\frac{1}{n}\right) \sum_{i=1}^{n} (x_i - \bar{x})^2} \right]^{n/2} \exp\left[-\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{\left(\frac{2}{n}\right) \sum_{i=1}^{n} (x_i - \bar{x})^2} \right]$$

$$= \left[\frac{ne^{-1}}{2\pi \sum_{i=1}^{n} (x_i - \bar{x})^2} \right]^{n/2}.$$

$$L(\hat{\mu}, \hat{\sigma}^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \sigma^2}} \exp\left[-\frac{(x_i - \hat{\mu})^2}{2\hat{\sigma}^2} \right]$$

under H_0

$$= \left[2\pi \sum_{i=1}^{n} (x_i - \mu_0)^2 \right]^{-\frac{n}{2}} \exp\left[- \cancel{\frac{\sum_{i=1}^{n}(x_i - \mu_0)^2}{2 \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu_0)^2}} \right]$$

$$= \left[2\pi \frac{\sum_{i=1}^{n} (x_i - \mu_0)^2}{n} \right]^{-\frac{n}{2}} \cdot \exp\left(-\frac{n}{2}\right).$$

under H_1

$$= \left[2\pi \sum_{i=1}^{n} (x_i - \bar{x}) \right]^{-\frac{n}{2}} \exp\left(-\frac{n}{2}\right).$$

$$\lambda = \left[\frac{\sum_{i=1}^{n} (x_i - \bar{x})}{\sum_{i=1}^{n} (x_i - \mu_0)} \right]^{\frac{n}{2}}$$

- and

$$2 \sum_{i=1}^{n} (x_i - \bar{x})(\bar{x} - \mu_0) = 0.$$

$$\sum_{i=1}^{n}(x_i - \mu_0)^2 = \sum_{i=1}^{n}(x_i - \bar{x} + \bar{x} - \mu_0)^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2$$

- Substitute and rearrange to get

$$\lambda = \left[\frac{1}{1 + \frac{n(\bar{x} - \mu_0)^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}} \right]^{n/2}$$

- Therefore, we have $\lambda \leqslant k$ when,

$$\frac{n(\bar{x}-\mu_0)^2}{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2} \geqslant c$$

$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

- When H_0 is true, $\sqrt{n}(\bar{X}-\mu_0)/\sigma \sim \mathrm{N}(0,1)$ and $\sum_{i=1}^{n}(X_i-\bar{X})^2/\sigma^2 \sim \chi^2_{n-1}$, and is independent of $\bar{X}$.
- Therefore,

$$\begin{aligned} T &= \frac{\sqrt{n}(\bar{X}-\mu_0)/\sigma}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2/\sigma^2}} \\ &= \frac{\sqrt{n}(\bar{X}-\mu_0)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2}} = \frac{\bar{X}-\mu_0}{S/\sqrt{n}} \sim t_{n-1} \end{aligned}$$

- So we reject H_0 when $|T|$ is too large, with the following critical region for a test with significance level α,

$$|T| \geqslant d, \quad \text{where } d \text{ is the } 1-\frac{\alpha}{2} \text{ quantile of } t_{n-1}$$

Remarks

- Usually easy to find the **form** of the test
- What is harder is to find the corresponding sampling distribution
- Manipulating λ until we have something whose distribution we know can be tricky!
- Many of the standard tests arise from the likelihood ratio

Asymptotic distribution & optimality

- The likelihood ratio itself is a statistic and therefore has a sampling distribution.
- For large sample sizes, this approaches a known distribution
- Also, the LRT gives the optimal test
- We will cover this theory later in the semester

Module 8: Challenge problem

You have n Bernoulli trials with success parameter p. Derive the form of the likelihood ratio test (LRT) that compares the null hypothesis $p = 0.2$ against the alternative hypothesis $p = 0.6$.

(In other words, derive a test mathematically equivalent to the LRT but that uses a familiar test statistic.)

$Bi(n, p)$

$L(p) = p^y (1-p)^{n-y}$

under H_0, $p = 0.2$ $\quad L_0 = L(0.2) = 0.2^x (1-0.2)^{n-x}$

H_1 $\quad p = 0.6$ $\quad L_1 = L(0.6) = 0.6^x (1-0.6)^{n-x}$

$$\lambda = \frac{L_0}{L_1} = \frac{0.2^x \, 0.8^{n-x}}{0.6^x \, 0.4^{n-x}} = \left(\frac{1}{3}\right)^x (2)^{n-x} = \frac{2^n}{6^x}$$

$\lambda \le k$

$\frac{2^n}{6^x} \le k$

$6^x \geq \frac{1}{k} \cdot 2^n$

$x \geq \underbrace{\log_6 \left(\frac{1}{k} \cdot 2^n\right)}_{c}$

$X = \sum_{i=1}^{n} y_i \quad y_i \sim Ber(p)$

$X \sim Bi(n, p)$

reject H_0 if $X \geq c$

quantile of $Bi(n, 0.2)$