The University of Melbourne

Department of Mathematics and Statistics

620-201 Probability

Semester 1 Exam — June 24, 2010

Exam Duration: 3 Hours

Reading Time: 15 Minutes

This paper has 5 pages

Authorised materials:
Students may bring one double-sided A4 sheet of handwritten notes into the exam room. Hand-held electronic calculators may be used.

Instructions to Invigilators:
Students may take this exam paper with them at the end of the exam.

Instructions to Students:
This paper has **nine** (9) questions.
Attempt as many questions, or parts of questions, as you can.
The approximate number of marks allocated to each question is shown in the brackets after the question statement.
The total number of marks available for this examination is 100.
Working and/or reasoning must be given to obtain full credit.

1. (a) Consider a random experiment with sample space Ω.

    (i) State the axioms that must be satisfied by a probability mapping P defined on the set $\mathcal{A}$ of events of the random experiment.

    (ii) Using these axioms, show that the probability of the empty set $\emptyset$ is zero.

   (b) Now assume that

$$\Omega = \{A, B, C\}$$

and

$$\mathcal{A} = \{\emptyset, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, \Omega\}.$$

Consider the mapping P from $\mathcal{A}$ to $[0, 1]$ defined by

$$\begin{aligned} P(\emptyset) &= 0 \\ P(\{A\}) &= 1/3 \\ P(\{B\}) &= 1/2 \\ P(\{C\}) &= 1/6 \\ P(\{A, B\}) &= 5/6 \\ P(\{A, C\}) &= 1/2 \\ P(\{B, C\}) &= 3/4 \\ P(\Omega) &= 11/12. \end{aligned}$$

    (i) With reference to the axioms you wrote in part (a), give two reasons why the mapping P is not a probability.

    (ii) Redefine the value of the mapping P at two points so that it is a probability.

[10 marks]

2. An employment agency uses a test to decide if applicants are suitable for a particular job. Each applicant is rated either suitable (RS), possibly suitable (RP) or unsuitable (RU). The probabilities of them receiving these ratings if they are actually suitable (S) or unsuitable (U) are given in the following table:

		rating		
		RS	RP	RU
quality	S	0.8	0.1	0.1
	U	0.2	0.2	0.6

   (a) Suppose that the probability that an arbitrary applicant is suitable is 1/25. Use the Law of Total Probability to calculate the probability that an applicant is rated as RS on the test.

   (b) What is the probability that such a person actually is suitable for the job?

   (c) Carefully stating your reasons, explain whether the events RP and S are independent.

   (d) In the light of your answer to part (b), comment on the usefulness of the test in selecting employees.

[8 marks]

3. (a) The random variable $X \stackrel{d}{=} R(-2, 1)$ (that is X has a continuous uniform distribution over the interval $[-2, 1]$). If $Y = X^4$, find both the distribution function $F_Y(y)$ and the probability density function $f_Y(y)$ of Y. (In your answer be careful to state the values of these functions for all values of $y \in \mathbb{R}$).

   (b) In the Oz Lotto draw, each player selects seven numbers from the numbers $1, \ldots, 45$. The lottery is decided when the Lotteries Commission randomly draws seven numbers without replacement, also from the numbers $1, \ldots, 45$.

      (i) Let X be the number of a player's numbers that are drawn by the Lotteries Commission. Giving your reasons, name the distribution of the random variable X and state the value of any parameter(s).

      (ii) For $x = 0, \ldots, 7$, write down the probability mass function of X as a function of x.

      (iii) It is estimated that $100,000,000$ games will be played in the Oz Lotto draw in a particular week. Let M be the number of games that win the division 1 prize, for which all seven of a player's numbers need to be drawn. Assuming that the numbers selected in separate games are independent, name the distribution of the random variable M and state the value of any parameter(s).

      (iv) Write down an expression for $P(M > 0)$.

      (v) Use a Poisson approximation to evaluate this probability.

   NB. The following information might help in answering this question:

$$\binom{45}{7} = 45,379,620,$$
$$\binom{38}{7} = 12,620,256.$$

[18 marks]

4. Consider a two-stage random experiment. In the first stage of the experiment we choose randomly between one of two coins. The first coin has a probability 0.4 of coming up 'heads' and the second coin has probability 0.6 of coming up 'heads'. In the second stage of the random experiment we toss the chosen coin n times and count X, the number of heads. We can think of this experiment as involving a 'randomisation' of the p parameter in a Binomial distribution, effectively replacing it by a discrete random variable R.

   (a) Write down the probability mass function for R and calculate $E(R)$ and $\mathrm{V}(R)$.

   (b) Carefully explaining your reasoning, write down the probability mass function for X.

   (c) Hence or otherwise, calculate $E(X)$ and $\mathrm{V}(X)$. You may use, without proof, anything you know about the moments of a binomial random variable.

   (d) Give an expression in terms of an integral for the probability mass function of X when R is uniformly and continuously distributed on $[0, 1]$.

[10 marks]

5. Consider the random variable (X, Y) which is uniformly distributed over the triangle $T = \{(x, y) : x > 0, y > 0, x + y < 4\}$.
   (a) Write down $f_{(X,Y)}(x, y)$, the joint pdf of (X, Y), and indicate on a graph the triangle T where it is non-zero.
   (b) Explain why X and Y are dependent (no calculations should be required).
   (c) Find $f_Y(y)$, the marginal pdf of Y. Check that it is a probability density.
   (d) For a fixed $y \in (0, 4)$ write down the set of values of x for which $f_{X|Y=y}(x|y)$ is non-zero.
   (e) Find $f_{X|Y=y}(x|y)$ over the range that you derived in part (d).
   (f) Calculate $Cov(X, Y)$.

[11 marks]

6. Let $X \stackrel{d}{=} \mathrm{R}(0, 1)$ and $Y \stackrel{d}{=} \mathrm{Bi}(1, 0.5)$ be independent random variables and $Z = X^Y$.
   (a) What are the possible values that Y can take?
   (b) For each y that you gave in answer to part (a), compute the function $\eta(y) = E(Z \mid Y = y)$.
   (c) For each y that you gave in answer to part (a), compute the function $\zeta(y) = \mathrm{V}(Z \mid Y = y)$.
   (d) Hence specify the random variables $E(Z \mid Y)$ and $\mathrm{V}(Z \mid Y)$.
   (e) Compute $E(Z)$.
   (f) Compute $\mathrm{V}(Z)$.

[12 marks]

7. In this question, you may use the fact that the probability density function for a standard normal random variable Z is

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right)$$

   (a) Derive the moment generating function $M_Z(t)$ of Z. (Hint: You will need to use the method of 'completing the square' to evaluate the integral).
   (b) Hence or otherwise, write down the moment generating function for $X \stackrel{d}{=} N(\mu, \sigma^2)$.
   (c) Let $X_i \stackrel{d}{=} N(\mu_i, \sigma_i^2)$ for $i = 1, 2, \ldots, n$ be mutually independent. Use moment generating functions to derive the distribution of the linear combination

$$L = \sum_{i=1}^{n} a_i X_i$$

   where $\{a_i\}$ are constants.

[11 marks]

8. (a) Let $X_1, X_2, \ldots$ be independent and identically-distributed random variables with $E(X_i) = 0$ and $V(X_i) = 1$, and let $S_n = \sum_{i=1}^{n} X_i$. Assume that the moment

generating function $M_{X_i}(t)$ of X_i exists for some $t \neq 0$. The Central Limit Theorem states that

$$Z_n = \frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0,1), \quad \text{as } n \to \infty.$$

Prove the Central Limit Theorem by showing that the moment generating function $M_{Z_n}(t)$ converges as $n \to \infty$ to the function that you derived in Question 7, part (a).

(b) Let $Y_1, \ldots, Y_{100}$ be independent random variables with mean 20 and variance 4. Use the Central Limit Theorem to derive an approximation for the probability that $19.6 \leq \sum_{i=1}^{100} Y_i/100 \leq 20.4$. (You may use the following information about a standard normal random variable Z: $P(Z \leq 0) = .5000$, $P(Z \leq 1) = .8413$, $P(Z \leq 2) = .9772$, $P(Z \leq 3) = .9987$).

[12 marks]

9. Consider the Branching Process $\{X_n, n = 0, 1, 2, 3, \ldots\}$ where X_n is the population size at the nth generation. Assume $P(X_0 = 1) = 1$ and that the probability generating function of a random variable N with the common offspring distribution is

$$A(z) = \frac{1}{12}(3 + 5z + 3z^2 + z^3).$$

(a) If $q_n = P(X_n = 0)$ for $n = 0, 1, \ldots$, write down an equation relating q_{n+1} and q_n. Hence, or otherwise, evaluate q_n for $n = 0, 1, 2$.

(b) Find the extinction probability $q = \lim_{n \to \infty} q_n$.

[8 marks]

End of the exam