MAST30001 Stochastic Modelling

Tutorial Sheet 7

1. A two state continuous time Markov chain $(X_t)_{t\geq 0}$ has the following generator with transition rates $\lambda, \mu > 0$:
$$\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix},$$
   (a) Find the time t transition matrix $P^{(t)}$ with $(P^{(t)})_{i,j} = \mathbb{P}(X_t = j | X_0 = i)$.
   (b) Using your answer to part (a) with $\lambda = \mu$, find a simple expression (i.e., not an infinite sum) for the chance that a random variable having the Poisson distribution with mean λ is an even number.

2. If $(X_t^{(1)})_{t\geq 0}, \ldots, (X_t^{(k)})_{t\geq 0}$ are i.i.d. continuous time Markov chains on $\{0, 1\}$ each having generator
$$\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix},$$
then what is the generator for the chain determined by $Y_t = \sum_{i=1}^k X_t^{(i)}$?

3. A workshop has two machines and one repairperson. Each machine is either functional or broken. If the the ith machine ($i = 1, 2$) is functional, then it fails after an exponential rate λ_i time. If the ith machine is broken, it takes the repairperson an exponential rate μ_i amount of time to fix it and once it is fixed, it's good as new. Assume the repairperson begins work the instant a machine breaks down, that only one machine can be repaired at a time, and all lifetime and repair times are independent.
   (a) Construct an appropriate continuous time Markov chain to describe the system and find the generator.
   (b) If $\lambda_i = \mu_i = i$ for $i = 1, 2$, find the stationary distribution of the process.

4. (CTMCs as limits of DTMCs) Let P be a one step transition matrix for a discrete time Markov chain on $0, 1 \ldots$ such that $p_{ii} = 0$ for all i. Also let $0 < \lambda_0, \lambda_1, \ldots$ be such that $\max_{i\geq 0} \lambda_i < N$, with N an integer. Define the discrete time Markov chain $Y_0, Y_1, \ldots$ by
$$\mathbb{P}(Y_{n+1}^{(N)} = i | Y_n^{(N)} = i) = \left(1 - \frac{\lambda_i}{N}\right),$$
and for $i \neq j$
$$\mathbb{P}(Y_{n+1}^{(N)} = j | Y_n^{(N)} = i) = \frac{\lambda_i}{N} p_{ij}.$$
We can think of the discrete jumps of $Y^{(N)}$ occurring at times on the lattice $\{0, 1/N, 2/N \ldots\}$ and make a continuous time process by defining
$$X_t^{(N)} = Y_{\lfloor Nt \rfloor}^{(N)},$$
where $\lfloor a \rfloor$ is the greatest integer not bigger than a.

(a) What does a typical trajectory of $X^{(N)}$ look like? Does it have jumps? At what times? How do jumps correspond to $Y^{(N)}$?

(b) Given $X_0^{(N)} = i$, what is the distribution of the random time

$$T^{(N)}(i) = \min\{t \geq 0 : X_t^{(N)} \neq i\}$$

(c) As $N \to \infty$, to what distribution does that of $(T^{(N)}(i) | X_0^{(N)} = i)$ converge?

(d) Based on the previous two items and comparing to the previous problem, do you think that $X^{(N)}$ converges as $N \to \infty$ to a continuous time Markov chain (not worrying about what exactly convergence means)? What is its generator?

1. two interpretation of CTMC

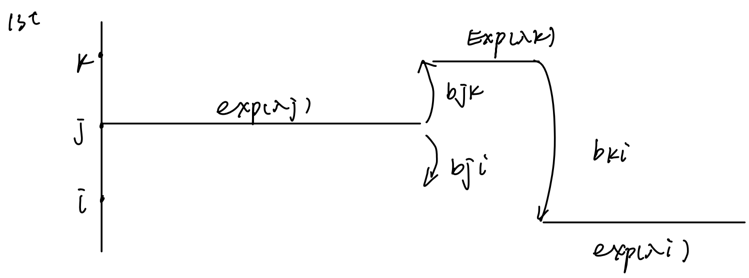
2. Generator

3. Forward equation $\quad (P^{(t)})' = P^{(t)} \cdot A$

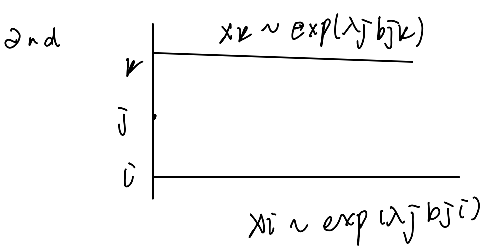
4. Stationary distribution $\quad \pi A = 0 \qquad \pi P^{(t)} = \pi \longrightarrow$ derivative both sides

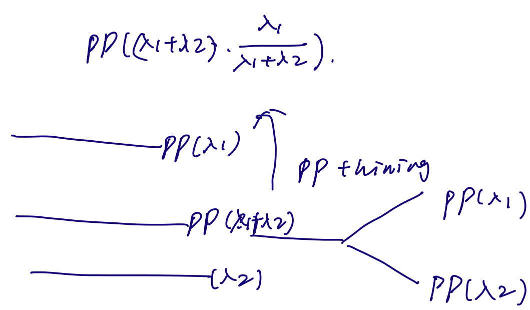
1. intepretation

1st



2nd





① why waiting time $X_j \sim Exp(\lambda_j)$

$X_k \sim exp(\lambda_j b_{jk}) \quad \underline{\text{independent}} \quad X_i \sim exp(\lambda_j b_{ji})$

$X_j \sim \min(X_k, X_i) \sim exp(\lambda_j \underbrace{b_{jk} + \lambda_j b_{ji}}_{=1}) = exp(\lambda_j)$

rigorous superposition of poisson process

order stats

② why jumping prob for j to k is b_{jk}

$= P(X_k < X_i) = \cdots \dfrac{\lambda_j b_{jk}}{\lambda_j b_{jk} + \lambda_j b_{ji}} = \dfrac{\lambda_j b_{jk}}{\lambda_j} = b_{jk}$

which

(think of two clock, ring first → jump

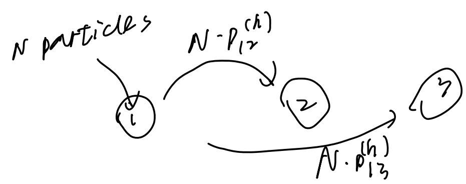
2. Generator

$$A = \frac{P^{(h)} - I}{h} \quad (h \to 0)$$

$$a_{11} = \frac{p_{11}^{(h)} - 1}{h} \quad (h \to 0)$$

$$a_{12} = \frac{p_{12}^{(h)} - 0}{h} \quad (h \to 0)$$

$p_{12}^{(h)}$: proportion of particle in state 1 moving to state 2 after time h



a_{11} : speed of losing particles ⇒ given a short time h, the proportion lost : $h \cdot a_{11}$

⇒ $h \cdot a_{11} N = N \cdot p_{11}^{(h)} - N$

\# particle lost from state 1

一开始 N 在 state 1.

a_{12} : speed of particle from state 1 entering state 2

$N \cdot h \cdot a_{12} = N \cdot p_{12}^{(h)} - 0$

\# particle from 1 to 2 after time h

一开始没有在 state 2

$$a_{12} = \frac{p_{12}^{(h)}}{h}$$

...ing speed ... entering

0

个

$$A = \begin{pmatrix} -\lambda_0 & \lambda_0 & & \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & \\ & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 \\ & & \vdots & \end{pmatrix}$$

↑ losing speed

$$0 \xrightarrow{\text{rate } \lambda_1} 1$$

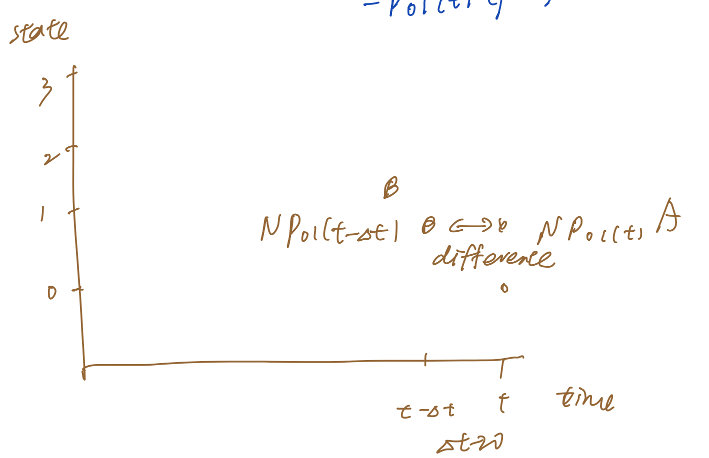
speed

$$p'_{01}(t) = P_{00}(t)\lambda_0 + P_{02}(t)\mu_2 - P_{01}(t)(\lambda_1+\mu_1)$$

↓

$$\frac{P_{01}(t) - P_{01}(t-\Delta t)}{\Delta t}$$

$$N P_{01}(t) - N P_{01}(t-\Delta t) = N P_{00}(t)(\lambda_0 \cdot \Delta t) + P_{02}(t)(\mu_2 \cdot \Delta t) N - P_{01}(t)(\lambda_1+\mu_1)\Delta t) N$$



N particle in state 0 on time t

will go to somewhere

at time

A − B = +(particle in state 1 at time t but not ~~x~~ at time t−Δt)

− (particles in state 1 at time t−Δt but not at time t)

$P_{02}(t)\,\Delta t\,\Delta t$

$$= + \Big(\boxed{N \cdot P_{02}(t-\Delta t)} (\mu_2 \cdot \Delta t) + N P_{00}(t-\Delta t)(\lambda_0 \Delta t)$$

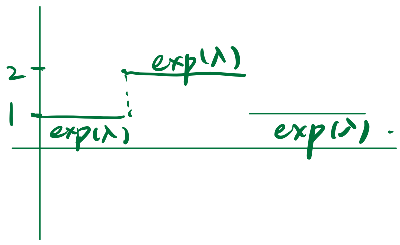
$$-\Big(\underbrace{N P_{01}(t-\Delta t)\cdot\Delta t}_{B}(\lambda_1+\mu_1)\Big)$$

1. A two state continuous time Markov chain $(X_t)_{t\geq 0}$ has the following $\boxed{\text{generator}}$ with transition rates $\lambda, \mu > 0$:

$$\begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \end{array},$$

(a) Find the time t transition matrix $P^{(t)}$ with $(P^{(t)})_{i,j} = \mathbb{P}(X_t = j | X_0 = i)$.

(b) Using your answer to part (a) with $\lambda = \mu$, find a simple expression (i.e., not an infinite sum) for the chance that a random variable having the Poisson distribution with mean λ is an even number.



(a) solve the equation

$$\frac{d}{dt}P^{(t)} = P^{(t)}A$$

$$P_{11}'^{(t)} = \sum_j P_{1j}^{(t)} a_{j1}$$

$$= P_{11}^{(t)} a_{11} + P_{12}^{(t)} a_{21}$$

$$= -\lambda P_{11}^{(t)} + \mu P_{12}^{(t)}$$

since $P_{11}^{(t)} + P_{12}^{(t)} = 1$

so $P_{11}'^{(t)} = -\lambda P_{11}^{(t)} + \mu(1 - P_{11}^{(t)})$

$$= \mu - (\mu+\lambda) P_{11}^{(t)}$$

$$\frac{d}{dt}P_{11}^{(t)} = \mu - (\mu+\lambda)P_{11}^{(t)}$$

$$\frac{d}{dt}P_{11}^{(t)} + (\mu+\lambda)P_{11}^{(t)} = \mu$$

$$I(x) = \exp\left(\int \mu+\lambda\, dt\right) = \exp[(\mu+\lambda)t] = e^{(\mu+\lambda)t}.$$

$$e^{(\mu+\lambda)t}\frac{d}{dt}P_{11}^{(t)} + (\mu+\lambda)e^{(\mu+\lambda)t}P_{11}^{(t)} = \mu\cdot e^{(\mu+\lambda)t}$$

$$\frac{d}{dt}\left[P_{11}^{(t)}\cdot e^{(\mu+\lambda)t}\right] = \mu\cdot e^{(\mu+\lambda)t}$$

$$P_{11}^{(t)}\cdot e^{(\mu+\lambda)t} = \int \mu\cdot e^{(\mu+\lambda)t}\,dt$$

$$P_{11}^{(t)}\cdot e^{(\mu+\lambda)t} = \frac{\mu}{\mu+\lambda}e^{(\mu+\lambda)t} + C.$$

$$P_{11}^{(t)} = \frac{\mu}{\mu+\lambda} + \frac{C}{e^{(\mu+\lambda)t}}$$

since $P_{11}^{(0)} = 1 \Rightarrow \frac{\mu}{\mu+\lambda} + C = 1$

$$C = \frac{\lambda}{\mu+\lambda}.$$

$$\therefore P_{11}^{(t)} = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}e^{-(\mu+\lambda)t}$$

$$P_{12}^{(t)} = \frac{\lambda}{\mu+\lambda} - \frac{\lambda}{\mu+\lambda}e^{-(\mu+\lambda)t}$$

(b). $\lambda = \mu$.

$$A = \begin{bmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{bmatrix}$$

the number of times the chain switches states in $[0,t)$ has a Poisson distribution with mean (λt)

if the chain starts at state 1, then the chain being at state 1 at time t

$\Rightarrow$ is same as being even number jump.

$$P_{11}(1) = \frac{\lambda}{\lambda+\lambda} + \frac{\lambda}{\lambda+\lambda}e^{-2\lambda}$$

$$= \frac{1}{2} + \frac{1}{2}e^{-2\lambda}.$$

so by symmetry $p_{22}^{(t)} = \frac{\lambda}{\mu+\lambda} + \frac{\mu}{\mu+\lambda} e^{-(\mu+\lambda)t}$

$p_{21}^{(t)} = \frac{\mu}{\mu+\lambda} - \frac{\mu}{\mu+\lambda} e^{-(\mu+\lambda)t}$

$= \frac{1}{2}(1+e^{-2\lambda})$

or

$$\sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!}$$

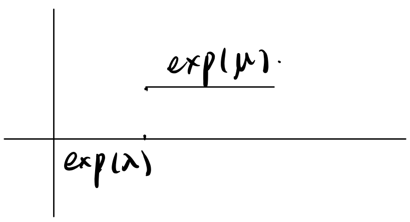
$$+\begin{cases} e^{x} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \end{cases}$$

✓

2. If $(X_t^{(1)})_{t\geq 0}, \ldots, (X_t^{(k)})_{t\geq 0}$ are i.i.d. continuous time Markov chains on $\{0,1\}$ each having generator

$$\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix},$$

then what is the generator for the chain determined by $Y_t = \sum_{i=1}^{k} X_t^{(i)}$?



$Y_t = \sum_{i=1}^{k} X_t^{(i)}$ $\quad X_t^{(i)}$ is iid continuous time Markov chains.

Y_t has state space $\{0, 1, \cdots k\}$.

$Y_t = i$ means exactly i of $X_t^{(j)}$ are 1

$k-i$ of $X_t^{(j)}$ are 0.

⇒. so from this point Y_t increases by one at rate $(k-i)\lambda$. (the minimum of the $k-i$ exponential clocks where $X_t^{(j)}=0$).

decrease by at rate $i\mu$ (the minimum of i exponential clocks where $X_t^{(j)}=1$).

for $i=1\dots k-1$

$a_{i,i+1}=(k-i)\lambda.$

$a_{i,i-1}=i\mu.$

$a_{ii}=-[i\mu+(k-i)\lambda)]=-[i(\mu-\lambda)+k\lambda]$

$i=0$ $a_{00}=-k\lambda.$

$a_{01}=k\lambda.$

$i=k$ $a_{kk}=-k\mu$

$a_{k,k-1}.\ k\mu.$

3. A workshop has two machines and one repairperson. Each machine is either functional or broken. If the the ith machine ($i=1,2$) is functional, then it fails after an exponential rate λ_i time. If the ith machine is broken, it takes the repairperson an exponential rate μ_i amount of time to fix it and once it is fixed, it's good as new. Assume the repairperson begins work the instant a machine breaks down, that only one machine can be repaired at a time, and all lifetime and repair times are independent.

   (a) Construct an appropriate continuous time Markov chain to describe the system and find the generator.

   (b) If $\lambda_i=\mu_i=i$ for $i=1,2$, find the stationary distribution of the process.

state space $\{(1,1),(1,0),(0,1),(0,0)\}$

repare on $i=2$ | repare on $i=1$ | repare $i=1$ repare $i=2$.

$(1,1)\xrightarrow{-(\lambda_1+\lambda_2)}(1,1)$

$(1,1)\xrightarrow{\lambda_2}(1,0)$

$(1,1)\xrightarrow{\lambda_1}(0,1)$

$(1,0)\xrightarrow{\mu_2}(1,1)$

$(1,0)\xrightarrow{\lambda_1}(0,0).$

$(1,0)\xrightarrow{-(\mu_2+\lambda_1)}(1,0)$

$(0,1)\xrightarrow{\mu_1}(1,1)$

$(0,1)\xrightarrow{\lambda_2}(0,0)$

$(0,1)\xrightarrow{-(\lambda_2+\mu_1)}(0,1)$

$(0,0)\xrightarrow{\mu_2}(0,1)$

$(0,0)\xrightarrow{\mu_1}(1,0)$

$(0,0)\xrightarrow{-(\mu_2+\mu_1)}(0,0)$

$$
\begin{array}{c|ccccc}
 & (1,1) & (1,0) & (0,1) & (0,0,1) & (0,0,2) \\
\hline
(1,1) & & & & & \\
(1,0) & & & & & \\
(0,1) & & & & & \\
(0,0,1) & & & & & \\
(0,0,2) & & & & &
\end{array}
\qquad
\begin{bmatrix}
-(\lambda_1+\lambda_2) & \lambda_2 & \lambda_1 & 0 & 0 \\
\mu_2 & -(\mu_2+\lambda_1) & 0 & 0 & \lambda_1 \\
\mu_1 & 0 & -(\lambda_2+\mu_1) & \lambda_2 & 0 \\
0 & \mu_1 & 0 & -\mu_1 & 0 \\
0 & 0 & \mu_2 & 0 & -\mu_2
\end{bmatrix}
$$

(b) $\pi A = 0.$

$\pi = [\pi_1 \ \pi_2 \ \pi_3 \ \pi_4 \ \pi_5]$

$$
\begin{cases}
-3\pi_1 + 2\pi_2 + \pi_3 = 0 & \Rightarrow \pi_1 = \frac{2}{3}\pi_2 + \frac{1}{3}\pi_3 \overset{\pi_3 = \frac{1}{2}\pi_4}{=} \frac{2}{3}\pi_2 + \frac{1}{6}\pi_4. \\
2\pi_1 - 3\pi_2 + \pi_4 = 0 & \Rightarrow \pi_1 = \frac{3}{2}\pi_2 - \frac{1}{2}\pi_4 \\
\pi_1 - 3\pi_3 + 2\pi_5 = 0 & \\
2\pi_3 - \pi_4 = 0 & \\
\pi_2 - 2\pi_5 = 0. &
\end{cases}
$$

$$\frac{2}{3}\pi_2 + \frac{1}{6}\pi_4 = \frac{3}{2}\pi_2 - \frac{1}{2}\pi_4.$$

$$\frac{5}{6}\pi_2 = \frac{2}{3}\pi_4$$

$$\underbrace{\pi_2 = \frac{2}{\cancel{3}} \times \frac{\cancel{6}\,2}{5}\pi_4 = \frac{4}{5}\pi_4}$$

$$\pi_1 = \frac{2}{3}\pi_2 + \frac{1}{6} \times \frac{5}{4}\pi_2$$

$$= \frac{\cancel{2}\,16}{\cancel{3}\,24}\pi_2 + \frac{5}{24}\pi_2 = \frac{\cancel{21}\,7}{\cancel{24}\,8}\pi_2.$$

$$\pi_2 = \frac{8}{7}\pi_1$$

$$\pi_4 = \frac{5}{\cancel{4}} \times \frac{\cancel{8}\,2}{7}\pi_1 = \frac{10}{7}\pi_1$$

$$\pi_5 = \frac{1}{2}\pi_2 = \frac{4}{7}\pi_1$$

$$\pi_3 = \frac{1}{2}\pi_4 = \frac{5}{7}\pi_1$$

$$\pi = \left(\frac{7}{34}, \frac{8}{34}, \frac{5}{34}, \frac{10}{34}, \frac{4}{34}\right).$$

$$\pi_1 + \frac{8}{7}\pi_1 + \frac{5}{7}\pi_1 + \frac{10}{7}\pi_1 + \frac{4}{7}\pi_1 = 1$$

$$\frac{34}{7}\pi_1 = 1 \quad \Rightarrow \pi_1 = \frac{7}{34}$$

1. (CTMCs as limits of DTMCs) Let P be a one step transition matrix for a discrete time Markov chain on $0,1\dots$ such that $\boxed{p_{ii}=0}$ for all i. Also let $0<\lambda_0,\lambda_1,\dots$ be such that $\max_{i\geq 0}\lambda_i<N$, with N an integer. Define the discrete time Markov chain $Y_0,Y_1,\dots$ by

$$\mathbb{P}(Y_{n+1}^{(N)}=i|Y_n^{(N)}=i)=\left(1-\frac{\lambda_i}{N}\right),$$

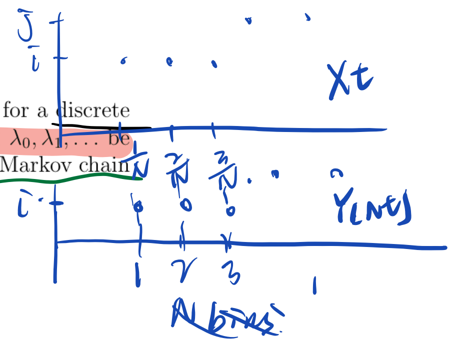
and for $i\neq j$

$$\mathbb{P}(Y_{n+1}^{(N)}=j|Y_n^{(N)}=i)=\frac{\lambda_i}{N}p_{ij}.$$

We can think of the discrete jumps of $Y^{(N)}$ occurring at times on the lattice $\{0,1/N,2/N\dots\}$ and make a continuous time process by defining

$$X_t^{(N)}=Y_{\lfloor Nt\rfloor}^{(N)},$$

where $\lfloor a\rfloor$ is the greatest integer not bigger than a.



Q ①. if $\frac{k}{N}<1$ ✓.
never be integer until $k=N$

② $Y^{(N)}$ meaning

(a) What does a typical trajectory of $X^{(N)}$ look like? Does it have jumps? At what times? How do jumps correspond to $Y^{(N)}$?

only have jump at $\frac{k}{N}$ where k is integer

so the chain is in state i, it stays there

$\frac{\lambda_i}{N}$ number of $\frac{1}{N}$ time units

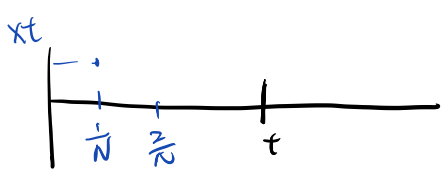
and jump to one step transition matrix P.

The number of these time unit are number of integer time unit between jumps in $Y^{(N)}$ chain

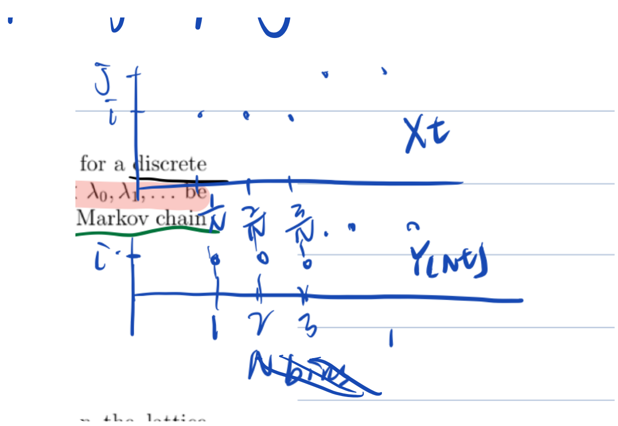
(b) Given $X_0^{(N)}=i$, what is the distribution of the random time

$$T^{(N)}(i)=\min\{t\geq 0: X_t^{(N)}\neq i\}$$

$X_0^{(N)}=i$



$Y^{(N)}$ chain stay at i for a geometric ($\frac{\lambda_i}{N}$) number of time units before jumping



first time jump from i $\quad \frac{3}{N}.$

prob $\frac{\lambda_3}{N}.$

$$T^{(N)}(i) = \frac{1}{N} \times \inf\{n \geq 1 : Y_n \neq i\}$$

$\downarrow$

$\mathrm{Geo}_1\left(\frac{\lambda_i}{N}\right)$

$\downarrow$

success para.

$$P\left(T^{(N)}(i) = \frac{k}{N}\right) = \left(1 - \frac{\lambda_i}{N}\right)^{k-1} \frac{\lambda_i}{N}$$

(c) As $N \to \infty$, to what distribution does that of $(T^{(N)}(i) | X_0^{(N)} = i)$ converge?

(d) Based on the previous two items and comparing to the previous problem, do you think that $X^{(N)}$ converges as $N \to \infty$ to a continuous time Markov chain (not worrying about what exactly convergence means)? What is its generator?

$$T^{(N)}(i) \overset{d}{=} \frac{1}{N} \mathrm{Geo}_1\left(\frac{\lambda_i}{N}\right)$$

$$p\, \mathrm{Geo}_1(p) \xrightarrow[p \downarrow 0]{} \mathrm{Exp}(1).$$