4.7 Identifying conic sections

The quadratic equation

$$ax^2 + bxy + cy^2 = d$$

can be rewritten as

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = d$$

$a, b, c, d \in \mathbb{R}$

$d \to D$

$v^T A v = d$

A symmetric $A^T = A$

A is orthogonal diagonalisable

$Q^T A Q = D$

$A = Q D Q^T$

$v^T (Q D Q^T) v = d$

$(Q^T v)^T D (Q^T v) = d$

$w^T D w = d$

$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$w = Q^T v = \begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow \lambda_1 x^2 + \lambda_2 y^2 = d$

$Q = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$

b_1, b_2 are orthonormal eigenvector of A

Ex. $5x^2 - 2xy + 5y^2 = 4$

$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [4]$

A

$\begin{vmatrix} 5-\lambda & -1 \\ -1 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 1 = (4-\lambda)(6-\lambda)$

$\lambda_1 = 4 \quad \lambda_2 = 6$

$\lambda_1 = 4$

$A - 4I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_1 = 6$

$A - 6I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$4X^2 + 6Y^2 = 4$

$X^2 + \frac{3}{2} Y^2 = 1$

5 Introduction to multivariable calculus

5.1 Functions of two variables

A *real-valued function of two variables* is a function $f\colon D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$.

Example 5.1. The volume of a cylinder of radius r and height h is $V(r,h) = \pi r^2 h$

Example 5.2. Bad-mannered individuals will sometimes give you a formula for the function without specifying the domain of definition D. You should then find the largest subset D of $\mathbb{R}^2$ that makes sense.

For instance, if

$$f(x, y) = 9 - xy^2 \qquad f: \mathbb{R}^2 \to \mathbb{R}^2$$

If

$$g(x, y) = \sqrt{1 - x^2 - y^2} \qquad g: D' \to \mathbb{R}$$

D': unit disc

$1 - x^2 - y^2 \geq 0$

$1 \geq x^2 + y^2$

$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} \ \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Quadric surface in R^3

$$3x^2+3z^2+3xy+8xz+4yz=64$$

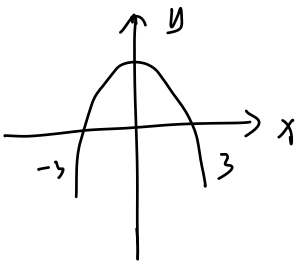
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 64$$

eigenvalue $\lambda = -1, -1, 8$ eigenvector $\begin{cases} u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \\ u_2 = \frac{1}{\sqrt{45}} \begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix} \end{cases}$

$u_3 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$$-X^2 - Y^2 + 8Z^2 = 64$$

$$g(x) = 9 - x^2$$



The *graph of* $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ is the surface

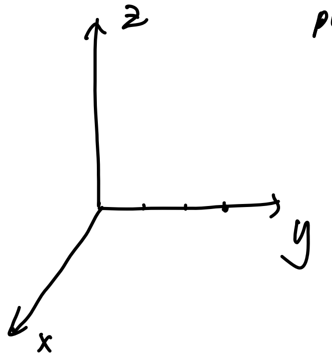
$$\{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y)\}.$$

At a point $(x, y) \in D$, $f(x, y)$ gives the height of the corresponding point on the surface.

Example 5.3. Consider $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 3x - y$.

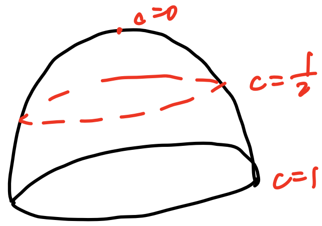
$z = 3x - y$

plane in $\mathbb{R}^3$



Example 5.4. Consider $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 9 - x^2 - y^2$.

The *level curves of $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$* are the subsets of D of the form



$$\{(x, y) \in D \mid f(x, y) = C\}$$

for different values of the constant C.

Example 5.5. Consider $f\colon \mathbb{D}^1 \to \mathbb{R}$ given by $f(x, y) = \sqrt{1 - x^2 - y^2}$.

for fixed $C = f(x,y) = \sqrt{1-x^2-y^2}$ $\quad C^2 = 1 - x^2 - y^2$

$x^2 + y^2 = 1 - C^2$

$C = 1$ point $(0,0)$

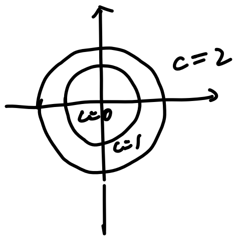
$C = 0$ $\quad x^2 + y^2 = 1$

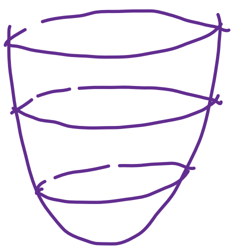
$0 < C < 1$ $\quad x^2 + y^2 = 1 - r^2$ with radius $\sqrt{1-r^2}$

$C > 1$ nothing

Example 5.6. Consider $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + y^2$.

$c = f(x,y) = x^2 + y^2$





Example 5.7. Consider $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 - y^2$.

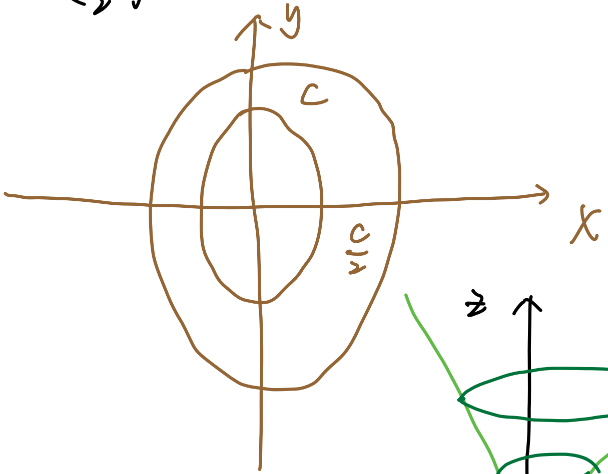
$$c = x^2 - y^2$$

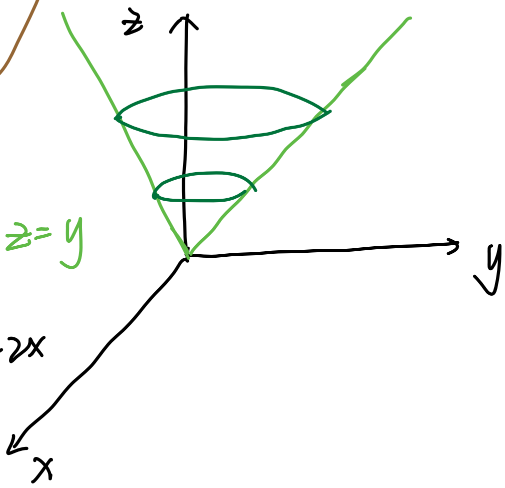
$$\frac{1}{c}x^2 - \frac{1}{c}y^2 = 1$$

Example 5.8. Sketch the graph of $z = \sqrt{4x^2 + y^2}$.

if $c \neq 0$

$$\frac{x^2}{(\frac{c}{2})^2} + \frac{y^2}{c^2} = 1$$





$x = 0$ $\quad z = \sqrt{y^2}$ $\quad z = \pm y$ $\quad z > 0$ $\quad z = y$

$y = 0$ $\quad z = \sqrt{4x^2}$ $\quad z = \pm 2x$ $\quad z > 0 \to z = 2x$

5.2 Limits, continuity, and partial derivatives

We say that *$f: D \subset \mathbb{R}^2 \to \mathbb{R}$ has limit $L \in \mathbb{R}$ as (x, y) approaches (x_0, y_0)* and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if, whenever (x, y) approaches (x_0, y_0) along any path in D, the values $f(x, y)$ become arbitrarily close to L.

We say that *$f: D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous at (x_0, y_0)* if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0).$$

Example. $f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2+y^2} & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$

$(x,y) \to (0,0)$

Along x axis $\quad y=0 \quad f(x,y) = \dfrac{x^2}{x^2} = 1 \quad$ lim(f(x,y)) = 1 along x axis

Along y axis $\quad x=0 \quad f(x,y) = -1 \quad$ lim(f(x,y)) = -1 along y axis

Example. $g(x,y) = \dfrac{xy^2}{x^2+y^4}$ $\quad$ if $(x,y) \neq 0$

fix $a \in R$ along y axis $y = ax$

$g(x,y) = \dfrac{a^2x^3}{x^2+a^4x^4} = \dfrac{a^2x}{1+a^4x^2} \to 0$

along $y=x^2$

$g(x,y)=\frac{x^5}{x^2+y^8}=\frac{x^3}{1+x^5}\to(0,0)$

Let $f\colon D\subset\mathbb{R}^2\to\mathbb{R}$ and $(x_0,y_0)\in D$.

The partial derivative of f with respect to x at (x_0,y_0) is

along $x=y^2$

in two variable

$$\boxed{\frac{\partial f}{\partial x}(x_0,y_0)=\lim_{t\to 0}\frac{f(x_0+t,y_0)-f(x_0,y_0)}{t},}$$

$g(x,y)=\frac{y^4}{2y^4}=\frac{1}{2}\to\frac{1}{2}$

as $(x,y)\to(0,0)$

if this limit exists.

first find the derivative for one variable

Similarly,

with respect to y

$$\frac{\partial f}{\partial y}(x_0,y_0)=\lim_{t\to 0}\frac{f(x_0,y_0+t)-f(x_0,y_0)}{t}$$

in one variable $F:\mathbb{R}\to\mathbb{R}$

Derivative of F at x_0 is

$$\frac{\delta F}{\delta x}=\lim_{t\to 0}\frac{F(x_0+t)-F(x_0)}{t}$$

Example 5.9. Consider $f\colon\mathbb{R}\times(0,\infty)\to\mathbb{R}$ given by $f(x,y)=x\log(y)+xy$.

domain(x) domain(y)

$$f_x=\frac{\partial f}{\partial x}=\log(y)+y$$

partial derivative

$$f_y=\frac{\partial f}{\partial y}=x\cdot\frac{1}{y}+x=\frac{x}{y}+x$$

We can of course (try to) differentiate more than once: second derivative

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(f) = f_x$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

which close to f, do first

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

If the second partial derivatives are continuous, then the mixed partials agree:

$$\boxed{f_{xy} = f_{yx}.}$$

Example: $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

$(f_{xy})(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -1 \qquad (f_{yx})(0,0) = 1$

Example 5.10. Find the second partial derivatives of $f\colon \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$ given by

$$f(x,y) = x^3 e^{-2y} + y^{-2}\cos(x).$$

$f_x = 3x^2 \cdot e^{-2y} - y^{-2}\sin x$

$f_y = x^3 \cdot (-2) \cdot e^{-2y} - 2y^{-3}\cos(x)$

$f_{xx} = 6x \cdot e^{-2y} - y^{-2}\cos x \qquad = 6x e^{-2y} - y^{-2}\cos(x)$

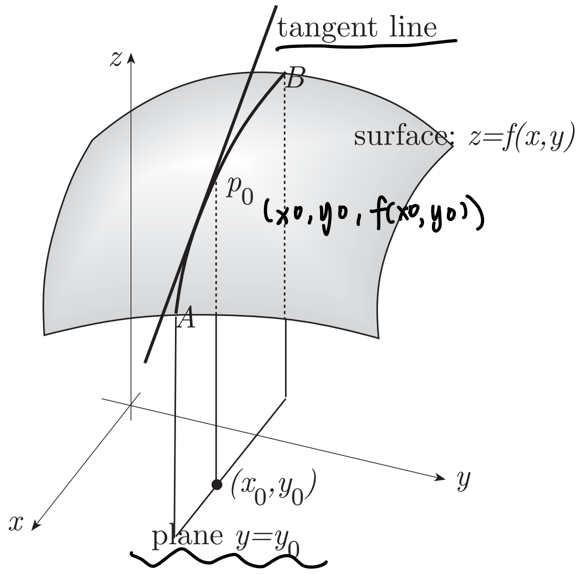
$f_{xy} = 3x^2 \cdot (-2) \cdot e^{-2y} + 2y^{-3}\sin x \qquad = -6x^2 e^{-2y} + 2y^{-3}\sin(x)$

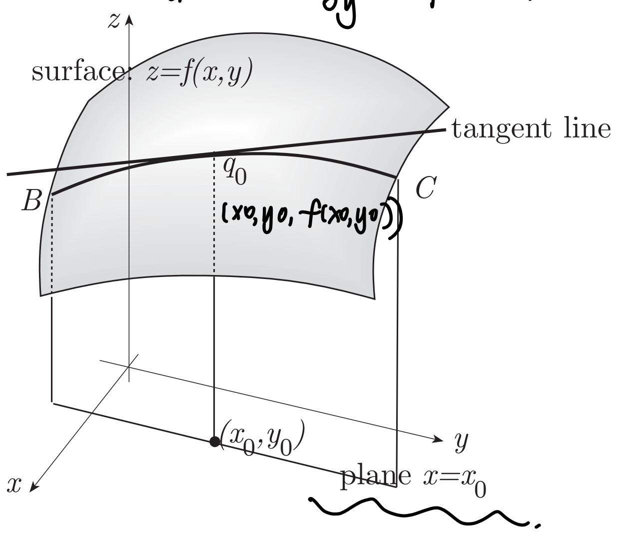
$f_{yx} = 3x^2 \cdot (-2) \cdot e^{-2y} + 2y^{-3}\sin(x) \qquad = -6x^2 e^{-2y} + 2y^{-3}\sin(x)$

$f_{yy} = x^3 \cdot 4 e^{-2y} + 6y^{-4}\cos(x). \qquad = 4x^3 \cdot e^{-2y} + 6y^{-4}\cos(x)$

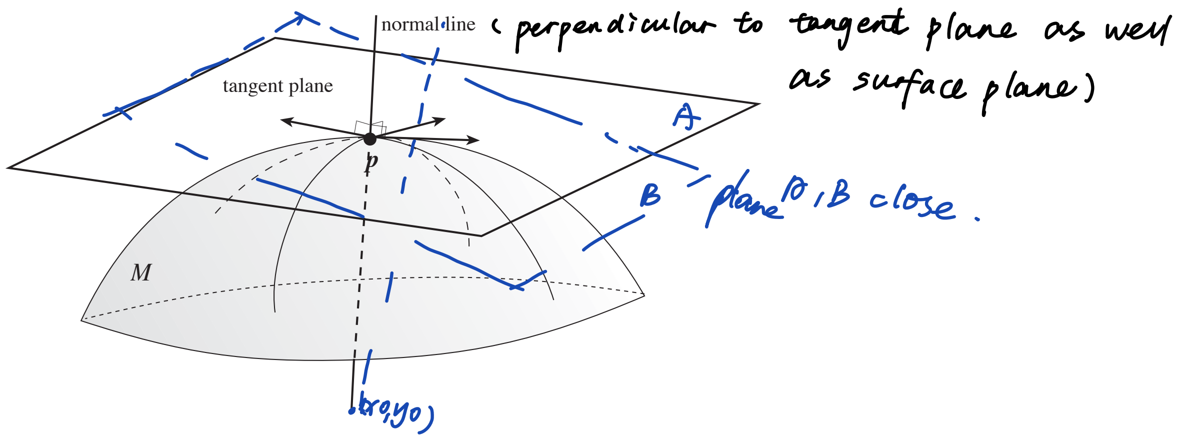
5.2.1 Geometric meaning of partial derivatives → $\frac{\partial f}{\partial x}$ → we fix $y = y_0$

similar $\frac{\partial f}{\partial y}$ → fix $x = x_0$





The slope of the tangent line on the left is $f_x(x_0, y_0)$; on the right, $f_y(x_0, y_0)$.



We say that *$f: D \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at a point (x_0, y_0) in the interior of D* if the tangent lines to all the curves on the surface $z = f(x, y)$ passing through the point (x_0, y_0) form a plane.

This is called the *tangent plane to the surface at (x_0, y_0, z_0)*, where $z_0 = f(x_0, y_0)$.

The line orthogonal to the tangent plane and passing through (x_0, y_0, z_0) is called the *normal line to the surface at (x_0, y_0, z_0)*.

Theorem 5.11. If there exists an open ball $B \subset D$ containing (x_0, y_0) such that f_x and f_y exist and are continuous at all the points of B, then f is differentiable at (x_0, y_0).

In this case, the equation of the tangent plane is

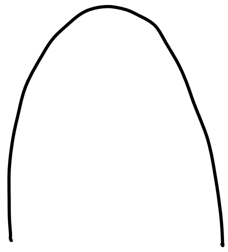
$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The plane passes through $(x_0, y_0, f(x_0, y_0))$ and has normal vector $(f_x(x_0, y_0), f_y(x_0, y_0), -1)$.

Example 5.12. Find the Cartesian equation of the tangent plane to the surface

$$z = 1 - x^2 - y^2$$

at the point $(\overset{x_0}{1}, \overset{y_0}{2}, -4)$.



$f(x,y) = 1 - x^2 - y^2$

$f(1,2) = -4 = f(x_0, y_0)$.

$f_x = -2x = -2$

$f_y = -2y = -4$

$z = -4 \times (-2)(x-1) + (-4)(y-2)$

$= -2x - 4y + 6$ | $\boxed{2x + 4y + z = 6}$

If f is differentiable at (x_0, y_0), then at points (x, y) close to (x_0, y_0) we can estimate the value $z = f(x, y)$ using the *linear approximation to f near (x_0, y_0)*

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Example 5.13. Estimate the value $f(0.01, 0.02)$, where

$$f(x, y) = \sqrt{1 - x + 2y}.$$

$(0.01, 0.02) \approx (0, 0)$

$f_x = \dfrac{1}{2\sqrt{1-x+2y}} \times (-1)$

$= \dfrac{-1}{2\sqrt{1-x+2y}}$ $\quad f_x(0,0) = -\frac{1}{2}$

$f_y = \dfrac{2}{2\sqrt{1-x+2y}} = \dfrac{1}{\sqrt{1-x+2y}}$ $\quad f_y(0,0) = 1$

$f(0,0) \approx 1 + -\frac{1}{2}(x) + 1(y)$

$= 1 + (-\frac{1}{2}) \times (0.01) + 1 \times 0.02 = 1 - 0.005 + 0.02$

$= 1.015$

5.2.2 Chain rule

Suppose $z = f(x, y)$ is a differentiable function of two variables, $x = g(t)$ and $y = h(t)$ are differentiable functions of a single variable t.

Then the function of one variable $z = f(g(t), h(t))$ is differentiable and its derivative is given by the chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Example 5.14. If $z = x^2 - y^2$, $x = \sin(t)$, and $y = \cos(t)$, find $\frac{dz}{dt}$ at $t = \frac{\pi}{3}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= 2\sin t\cdot\cos t + (-2\cdot\cos(t))\cdot(-\sin t)$$

$$= 4\sin t\cos t = 2\sin 2t \overset{t=\frac{\pi}{3}}{=} 2\sin\frac{2\pi}{3} = \sqrt{3}$$

5.3 Directional derivative and gradient

It is useful to know the rate of change of a function in a certain direction.

The *directional derivative of f in the direction of* $\mathbf{u} = (u_1, u_2)$ *at the point* $P_0 = (x_0, y_0)$ is

$$(D_{\mathbf{u}} f)(x_0, y_0) = \frac{d}{dt} f(P_0 + t\mathbf{u})\,|_{t=0}$$

$$= \frac{d}{dt} f(x_0 + tu_1, y_0 + tu_2)\Big|_{t=0}$$

$x(t) = x_0 + tu_1$

$y(t) = y_0 + tu_2$

$$= f\left(\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}\right)\Big|_{t=0} + f\left(\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}\right)\Big|_{t=0}$$

$$= \left(\frac{\partial f}{\partial x}\cdot u_1\right)\Big|_{t=0} + \left(\frac{\partial f}{\partial y}\cdot u_2\right)\Big|_{t=0}$$

$$= \frac{\partial f}{\partial x}\Big|_{t=0}\cdot u_1 + \frac{\partial f}{\partial y}\Big|_{t=0}\cdot u_2 = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\cdot(P_0)\cdot u$$

$= \quad x_0 u_1 + y_0 u_2 \quad = (\nabla f)(P_0) \cdot u$

The *gradient of f* is

$$\nabla f = (f_x, f_y).$$

So the result of the calculation we performed above can be written

$$(D_{\mathbf{u}} f)(P_0) = (\nabla f)(P_0) \cdot \mathbf{u}.$$

Example 5.15. Find the rate of change of the function $f\colon \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = 1 - \frac{x^2}{4} - \frac{y^2}{4}$$

at the point $(1,0)$ in the direction of the vectors

(a) $\mathbf{a} =$ the unit vector making an angle of $\pi/4$ with the x-axis

(b) $\mathbf{b} = (0,1)$.

$f_x = -\frac{x}{2}$

$f_y = -\frac{y}{2}$

$(\nabla f) \cdot (1,0) = (-\frac{1}{2}, 0)$

(a). $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

$(D_a f)(1,0) = (-\frac{1}{2}, 0) \cdot (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}).$

$= -\frac{\sqrt{2}}{4}$

(b) $(0,1)$

$D_b f(0,1) = (-\frac{1}{2}, 0) \cdot (0,1)$

$= 0$

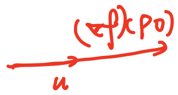
If $\mathbf{u}$ is a unit vector, then

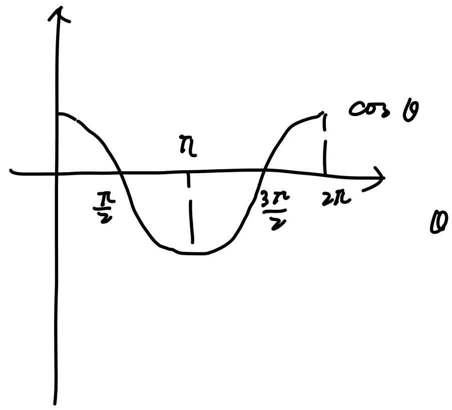
$$(D_{\mathbf{u}}f)(P) = (\nabla f)\cdot(P)\cdot u = \|(\nabla f)(P)\|\cdot \underbrace{u}_{=1} \cdot \cos\theta$$

θ is the angle between $(\nabla f)(P)$ & u

What is the direction $\mathbf{u}$ in which the derivative is

- largest $\rightarrow$ when $\cos\theta = 1$





- smallest $\rightarrow$ when $\cos\theta = -1$

Example 5.16. In which directions does $f = xy^2$ increase, resp. decrease, most rapidly at $(1, 2)$?

$f_x = y^2$

$f_y = 2xy$

$(\nabla f)(1,2) = (4, 4)$

direction of steepest increase $(4,4)$

" " " " " decrease $(-4,-4)$

Consider a level curve of f:

$$f(x(t), y(t)) = C.$$

Apply the chain rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = C' = 0$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\left(\frac{dx}{dt} \cdot \frac{dy}{dt}\right)$$

$$= (\nabla f)\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = 0$$

gradient is perpendular to the vector tangent to the curve
切线

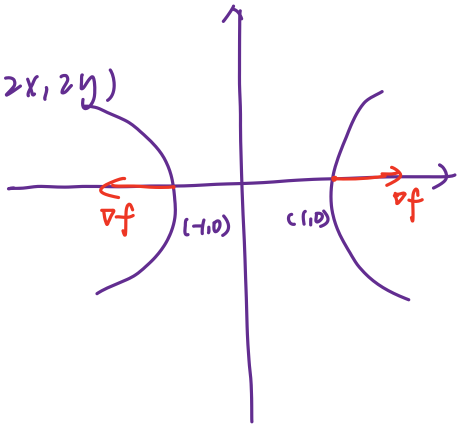
gradient is normal to the curve

So $(\nabla f)(x_0, y_0)$ is normal to the level curve of f at (x_0, y_0) and points in the direction in which f is increasing most rapidly.

Example 5.17. For the level curve $x^2 - y^2 = 1$, sketch the gradient vector at the points $(1, 0)$ and $(-1, 0)$.

$f_x = 2x$ $\quad \nabla f(x,y) = (2x, 2y)$

$f_y = 2y$



$\nabla f(1,0) = (2,0)$

$\nabla f(-1,0) = (-2,0)$