MAST20009 Vector Calculus

Practice Class 2 Questions

<u>Matrix version of chain rule</u>

If $f : \mathbb{R}^m \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable functions, and the composition $f \circ g$ is defined, then

$$\boldsymbol{D}(f \circ g) \;=\; \boldsymbol{D}f \; \boldsymbol{D}g$$

Note that $f \circ g \;=\; f\,(g(x_1, x_2, \ldots, x_n))$.

1. Let $f(u, v, w) = (u + v + w, uvw)$ and $g(x, y) = (x + y, x^2y, xy^2)$.
   Evaluate $\boldsymbol{D}(g \circ f \circ g)$ at $(1, -1)$ using the matrix version of the chain rule.

For change of variables from (x, y) to (u, v) in multiple integrals we need to compute the Jacobian, which is the determinant of the Jacobi matrix.

$$\text{Jacobian } = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

2. The relation between Cartesian coordinates (x, y) and polar coordinates (r, θ) is given by

   $$x = r\cos\theta, \qquad y = r\sin\theta,$$

   where $r \geq 0$ and $0 \leq \theta < 2\pi$.

   (a) Find the Jacobian for this transformation.

   (b) Where possible solve these equations for r and θ as functions of x and y.

   (c) Calculate the partial derivatives $\dfrac{\partial r}{\partial x}$ and $\dfrac{\partial \theta}{\partial y}$ in terms of r and θ.

   (d) Is $\dfrac{\partial r}{\partial x} = \left(\dfrac{\partial x}{\partial r}\right)^{-1}$? Is $\dfrac{\partial \theta}{\partial y} = \left(\dfrac{\partial y}{\partial \theta}\right)^{-1}$?

The n^{th} order *Taylor polynomial* for $f(\boldsymbol{x})$ near $\boldsymbol{x} = \boldsymbol{a}$ is

$$p_n(\boldsymbol{x}) \;=\; \sum_{k=0}^{n} \frac{1}{k!} \left[(\boldsymbol{x}-\boldsymbol{a}) \cdot \nabla\right]^k f\big|_{\boldsymbol{a}}$$

Taylor's formula for the remainder near $\boldsymbol{x} = \boldsymbol{a}$ is given by

$$R_n(\boldsymbol{x}) = \frac{1}{(n+1)!} \left[(\boldsymbol{x}-\boldsymbol{a}).\nabla\right]^{n+1} f(\boldsymbol{a} + \xi(\boldsymbol{x}-\boldsymbol{a}))$$

for $0 < \xi < 1$.

3. Consider the function

$$f(x, y) = x^2 e^{2y-1}.$$

   (a) Determine the second order Taylor polynomial for f about the point $(1, \frac{1}{2})$.

   (b) Use the first order Taylor polynomial for f to approximate $f(0.9, 0.4)$.

   (c) Using the remainder formula, determine an upper bound for the error in part (b).

   (d) Using a calculator, determine the exact error in part (b). How does the exact error compare to your answer in part (c)?

When you have finished the above questions, continue working on the questions in the Vector Calculus Problem Sheet Booklet.