MAST30001 Stochastic Modelling

$\boxed{\textbf{Tutorial Sheet 8}}$

1. Consider a population consisting of particles arriving from outside according to a Poisson process with rate λ. The lifetime of each particle (after it arrives) is exponential with rate α and the lifetimes are all independent.

   (a) Model the system as a birth-death process and find the birth and death rates.

   (b) Show that the process is ergodic and find its stationary distribution.

   (c) What is the expected number of living particles in the population in stationary?

   Ans.

   (a) If X_t is the number of particles alive at time t, then X_t is a birth-death process with birth rates $\lambda_i = \lambda$ (due to arrivals) and death rates $\mu_i = \alpha i$ since if there are currently i particles, then the time until the next death is the minimum of i independent exponential rate α variables.

   (b) Birth death processes are ergodic if and only if for $K_j = \prod_{\ell=1}^{j} \frac{\lambda_{\ell-1}}{\mu_\ell}$,

   $$\sum_{j\geq 0} K_j < \infty$$

   and in this case the stationary distribution π has $\pi_i = K_i / \sum_{j\geq 0} K_j$. For this example, $K_j = (\lambda/\alpha)^j/j!$ and the sum above is $e^{\lambda/\alpha}$ and so the stationary distribution is Poisson with mean λ/α.

   (c) From part (b), the average number of particles in stationary is λ/α.

2. A system has N particles each of which at any given time are in one of the two energy states α or β. The particles switch between states α and β according to the following rules. When a particle enters state α, it switches to state β after an exponentially distributed with rate $\mu > 0$ amount of time, independent of the other particles' behaviour and the time the particle entered state α. Similarly, when a particle enters state β, it switches to state α after an exponentially distributed with rate $\lambda > 0$ amount of time, independent of the other particles' behaviour and the time the particle entered state β.

   (a) Model the number of particles in the energy state α as a continuous time Markov chain and define its generator.

   (b) Describe the long run behaviour of the chain.

   (c) If the chain starts with N particles in the α energy state and X_t is the number of α particles at time t, find the mean and variance of X_t as $t \to \infty$. Your answer should be a tidy formula.

   Ans.

   (a) Given the system has k particles in energy state α and thus $N - k$ particles in energy state β, one of two things can happen. Either a β particle switches to an α particle or vice versa. Since all the exponential clocks are independent, the α

particles' clocks ring at rate $k\mu$ and the β particles' clocks ring at rate $(N-k)\lambda$. From this description we see the generator A has entries

$$\begin{aligned} a_{k,k+1} &= (N-k)\lambda \\ a_{k,k-1} &= k\mu \\ a_{k,k} &= -(k\mu + (N-k)\lambda). \end{aligned}$$

Note these formulas are correct for $k = 0$ and $k = N$.

(b) We have a birth-death chain on $\{0, \dots, N\}$ with birth rates $\lambda_k = a_{k,k+1}$ and death rates $\mu_k = a_{k,k-1}$. The process is ergodic with long run distribution π satisfying $\pi A = 0$. The usual arguments imply this system of equations has unique probability distribution solution

$$\pi_k = \binom{N}{k} \left(\frac{\lambda}{\mu+\lambda}\right)^k \left(\frac{\mu}{\mu+\lambda}\right)^{N-k};$$

that is π has a binomial distribution with parameters N and $\lambda/(\mu+\lambda)$.

(c) Since the distribution of X_t tends to the stationary distribution π as t tends to infinity, we expect the mean and variance to do the same. Since the stationary distribution is binomial we have the formulas

$$\lim_{t\to\infty} \mathbb{E}X_t = \frac{N\lambda}{\lambda+\mu}, \quad \lim_{t\to\infty} \mathrm{Var}(X_t) = \frac{N\lambda\mu}{(\lambda+\mu)^2}.$$

3. The following continuous time Markov chain is used to model population growth without death. The basic assumption of the model is that every member of the population gives birth to a new member with rate λ (that is, at times with distribution exponential with rate λ), independently of the other members of the population. Let X_t be the size of the population at time t.

   (a) What is $\mathbb{P}(X_t = n | X_0 = 1)$?

   (b) If U is uniform on the interval $(0, 1)$, independent of X_t, find the distribution of $X_U | X_0 = 1$.

   Ans.

   (a) The first thing to notice is that since the minimum of i independent exponential variables is exponential and the rates add, the generator A of the chain has $a_{ii+1} = i\lambda$ and $a_{ii} = -i\lambda$. If $P_n(t) = \mathbb{P}(X_t = n | X_0 = 1)$, then according to forward equation, for $n \geq 1$

   $$P_n'(t) = -\lambda n P_n(t) + \lambda(n-1)P_{n-1}(t),$$

   and note $P_n(0) = 0$ if $n \neq 1$, $P_1(0) = 1$, and $P_0(t) = 0$. So taking $n = 1$ in the ODE above we have

   $$P_1'(t) = -\lambda n P_1(t)$$

   and the initial condition $P_1(0) = 1$ implies that $P_1(t) = e^{-\lambda t}$. Moving forward we see find taking $n = 2$ in the ODE above

   $$P_2'(t) = -2\lambda P_2(t) + \lambda e^{-\lambda t};$$

or

$$\frac{d}{dt}(e^{2\lambda t}P_2(t)) = \lambda e^{\lambda t},$$

so

$$P_2(t) = e^{-\lambda t} + Ce^{-2\lambda t}.$$

Using the initial conditions shows that $C = -1$ and so we have

$$P_2(t) = e^{-\lambda t}(1 - e^{-\lambda t}),$$

and we now guess the formula

$$P_n(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}.$$

To check this is correct we see it satisfies the initial conditions and then check it satisfies the ODE above which it does. So X_t given $X_0 = 1$ is geometric $e^{-\lambda t}$.

(b) For $n = 1, 2, \ldots,$

$$\mathbb{P}(X_U = n | X_0 = 1) = \int_0^1 e^{-\lambda t}(1 - e^{-\lambda t})^{n-1} dt = \frac{(1 - e^{-\lambda})^n}{n\lambda}.$$

4. Show that in in $M/M/1$ queue with arrival rate λ and service rate $\mu > \lambda$, the expected lengths of the idle and busy periods are $1/\lambda$ and $1/(\mu - \lambda)$, respectively. *[Hint: the proportion of time the server is idle is equal to the stationary chance the system is empty.]*

   Ans. Since the arrivals follow a Poisson process (using in particular the memoryless property of the exponential), the time between the moment the system clears and the next arrival is exponential rate λ and so the expected length of an idle period is the expectation of this exponential, that is, $1/\lambda$. If ℓ is the expected length of a busy period and $\pi_0 = 1 - \lambda/\mu$ is the long run proportion of time the system is empty, then

$$\pi_0 = \frac{1/\lambda}{1/\lambda + \ell},$$

or $\ell = 1/(\mu - \lambda)$.