(continuous time, continuous state space)
$[0, +\infty)$ $\mathbb{R}$

The normal distribution

A random variable Z has the normal distribution with mean μ and variance σ^2 denoted by $Z \sim \mathrm{N}(\mu, \sigma^2)$, if its density is

$$\phi_{\mu,\sigma^2}(z) = \frac{\exp\{-(z-\mu)^2/(2\sigma^2)\}}{\sqrt{2\pi}\sigma},$$

and then its distribution function is

$$\Phi_{\mu,\sigma^2}(z) = \int_{-\infty}^{z} \phi_{\mu,\sigma^2}(t)dt.$$

[We'll drop the subscripts when $\mu = 0$ and $\sigma^2 = 1$]
If $Z \sim \mathrm{N}(0, 1)$, then $(\sigma Z + \mu) \sim \mathrm{N}(\mu, \sigma^2)$.

Brownian motion

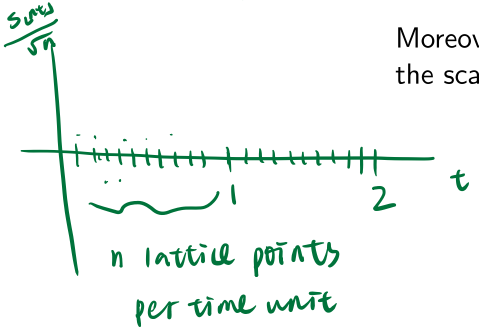
The normal distribution arises as the limit of random walks.

- If $X_1, X_2, \ldots$ are i.i.d. with mean 0 and variance 1, then for $S_n = \sum_{i=1}^n X_i$,

$$\lim_{n\to\infty} P\left(\frac{S_n}{\sqrt{n}} \le z\right) = \Phi(z).$$

t scale in time n

scale in space $\sqrt{n}$



Moreover, we can also sum a different number of terms, but keep the scaling the same: If $t \ge 0$ then

first nt

$$\lim_{n\to\infty} P\left(\frac{S_{\lfloor nt\rfloor}}{\sqrt{n}} \le z\right) = \Phi_{0,t}(z).$$

$$\frac{S_{\lfloor nt\rfloor}}{\sqrt{nt}} \xrightarrow{d} N(0,1)$$

$$\frac{S_{\lfloor nt\rfloor}}{\sqrt{n}} \xrightarrow{d} N(0,t)$$

Brownian motion

Definition:

A continuous time stochastic process $\{B_t : t \geq 0\}$ is a standard Brownian motion if

- it has continuous sample paths,
- it has independent increments on disjoint intervals: for $k \geq 2$ and $0 \leq s_1 < t_1 \leq s_2 < \cdots < t_k$,

$$B_{t_1} - B_{s_1}, \ldots, B_{t_k} - B_{s_k}$$

  are independent variables.
- For each $t \geq 0$, $B_t \sim \mathrm{N}(0, t)$.

$0 \leq k_1 < l_1 \leq k_2 < l_2$ ← integer

$(S_{l_1} - S_{k_1}, S_{l_2} - S_{k_2}$

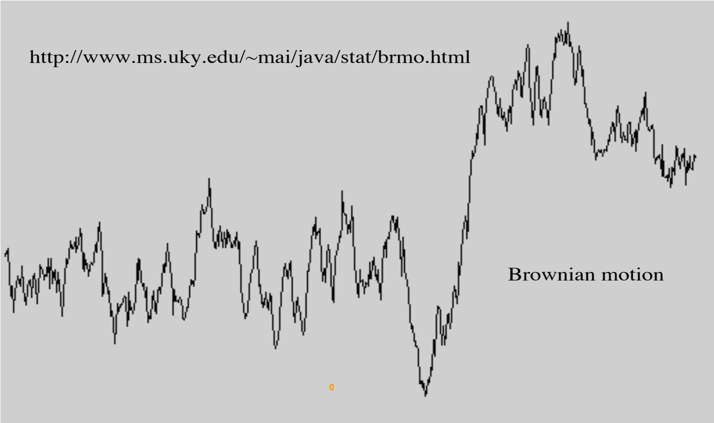
$\sum_{j=k_1+1}^{l_1} x_j$ $\sum_{j=k_2+1}^{l_2} x_i$

IND since sum not overlap

x_i iid

Brownian Motion

A sample path



Simulate:

$(B_0, B_{1/N}, B_{2/N}, \dots B_1)$

$B_{j/N} = \sum_{i=1}^{j} \left(B_{i/N} - B_{(i-1)/N}\right)$

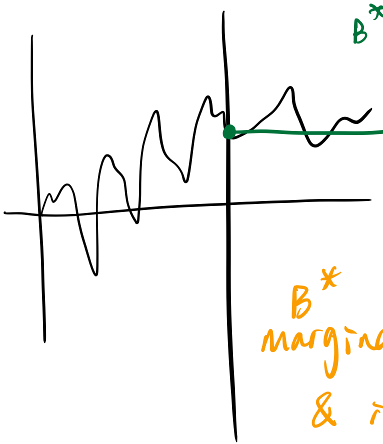
sum of increment

$\overset{d}{=} N(0, \frac{1}{N})$

$Z_i \sim N(0, \frac{1}{N})$

$(0, Z_1, Z_1+Z_2, \dots, Z_1+\dots+Z_N)$

Properties of Brownian motion



- $B_{t+s} - B_t \sim \mathrm{N}(0, s)$.
- Furthermore if, for fixed h, we define $B^*_t = B_{t+h} - B_h$, then B^* is a standard Brownian motion.
- Brownian motion with parameter σ^2 is defined to have the same distribution as $(\sigma B_t)_{t \geq 0}$.

B^* marginal distr are correct & independent increment inherited from B

$B_{t+s} = (B_{t+s} - B_t) + (B_t)$ All normal

IND

LHS:

$E[e^{\theta B_{t+s}}] = e^{\theta^2 (t+s)/2}$ → formula for MGF of normal

variance

RHS.

IND

$E(e^{\theta(B_{t+s} - B_t)} \cdot e^{\theta B_t}]$

$= E[e^{\theta(B_{t+s} - B_t)}] \cdot e^{\theta^2 t/2}$

$\Rightarrow E[e^{\theta(B_{t+s} - B_t)}] = e^{\theta^2 s/2}$

$\Rightarrow$ MGF of $N(0,s)$

Joint distributions of Brownian motion

Let $0 = t_0 < t_1 < \cdots < t_k$. What is the joint distribution of $(B_{t_1}, \ldots, B_{t_k})$?

- Let $Z_i = B_{t_i} - B_{t_{i-1}}$, $i = 1, \ldots, k$. The Z_i are independent normal random variables.
- The B_{t_i} are a linear function of the Z_i:

$$(B_{t_1}, \ldots, B_{t_k}) = \left(Z_1, \sum_{i=1}^{2} Z_i, \ldots, \sum_{i=1}^{k} Z_i \right).$$

The joint distributions of Brownian motion observed at a collection of times are linear functions of independent normal variables. What are these distributions?

Multivariate normal distribution

To define the Multivariate normal distribution we need some facts from linear algebra.

symmetric

Definition

We say the matrix Σ is positive definite if $\Sigma^T = \Sigma$ and for any $x \neq 0$, $\boxed{x^T \Sigma x} > 0$.

+ eigenvalue ? $\Rightarrow$ eigenvalue > 0

Properties of positive definite matrix Σ

- There is a lower triangular matrix R with $\Sigma = RR^T$.
- There is a unique symmetric square root denoted $\Sigma^{1/2}$.
- $\det(\Sigma) > 0$. (In particular Σ is invertible.)

Multivariate normal distribution

Let $Z = (Z_1, \ldots, Z_k)$ be a vector of i.i.d. standard normal variables.

Definition

We say $X = (X_1, \ldots, X_k)$ has the multivariate normal distribution with parameters μ, a k-vector called the mean, and Σ, a $k \times k$ positive definite matrix called the variance or covariance matrix, if

$$X \stackrel{d}{=} \Sigma^{1/2} Z + \mu.$$

iid stand normal

Multivariate normal distribution

Σ sum

Properties

$\underline{X} \overset{d}{=} \Sigma^{\frac{1}{2}} \underline{Z} + \underline{\mu}$

$(\Sigma^{\frac{1}{2}})^T = \Sigma^{\frac{1}{2}}$

$(\Sigma^{\frac{1}{2}})^2 = \Sigma.$

(Note $R = \Sigma^{1/2}$

Then $RR^T = (\Sigma^{1/2})(\Sigma^{1/2})$

$= \Sigma$

1. ▶ $Cov(X_i, X_j) = \Sigma_{i,j}$. [Direct calculation.]

$\underline{X} = \Sigma^{1/2} \underline{Z} + \underline{\mu}$

$X_i = [\sum_k (\Sigma^{1/2})_{ik} Z_k] + \mu_i$

$X_j = [\sum_l (\Sigma^{1/2})_{jl} Z_l] + \mu_j$

from def

$cov(X_i, X_j) = E[(X_i - \mu_i)(X_j - \mu_j)]$

▶ If R is such that $\Sigma = RR^T$ (and there is such a lower triangular matrix R), then

could just take R rather than $\Sigma^{1/2}$

2.
$$X \overset{d}{=} RZ + \mu.$$

[Check densities.]

▶ If A is an invertible matrix, then AX is multivariate normal with mean $A\mu$ and covariance matrix $A\Sigma A^T$. [Use second item, checking the covariance matrix is positive definite.]

$= \sum_m \sum_l (\Sigma^{1/2})_{im} (\Sigma^{1/2})_{l} \cdot E[Z_m Z_k]$

$= E\left[\sum_m \sum_l (\Sigma^{1/2})_{im} (\Sigma^{1/2})_{jl} \; E \; Z_m Z_l\right)$

$= \begin{cases} 0 & m \neq l \\ 1 & m = l \end{cases}$

Define $\underline{X} = R\underline{Z} + \underline{\mu}$ Then $\underline{X} \sim MVN(\underline{\mu}, RR^T)$

can

check this is pos def

to prove $\Sigma^{1/2} Z + \cancel{\mu} \overset{d}{=} R Z + \cancel{\mu}$

check this is positive

Multivariate normal distribution

$= \sum_l (\Sigma^{1/2})_{il} (\Sigma^{1/2})_{jl}$ → symmetric $\Sigma^{1/2}$

$= \sum_l (\Sigma^{1/2})_{il} (\Sigma^{1/2})_{lj}$

$= \Sigma_{ij}$ $\quad R_{il} (R^T)_{lj}$

$f_Z = \frac{1}{(\sqrt{2\pi})^k} e^{-(z_1^2 + \dots + z_k^2)/2}$ → iid normal joint density (product of density since iid)

$z^T z$

Look at density of $Y = AZ$ (invertible A)

$z = A^{-1} y$

$f_Y(y) = f_Z(A^{-1} y) \cdot \frac{1}{|\det(A)|}$ → Jacobian

$= \frac{1}{|\det(A)|} \cdot \frac{1}{(\sqrt{2\pi})^k} e^{-(A^{-1}y)^T A^{-1} y/2}$

$= \frac{1}{|\det(A)|} \cdot \frac{1}{2\pi^{k/2}} e^{-y^T (A^{-1})^T A^{-1} y/2}$ $\quad (A^{-1})^T \to (A^T)^{-1}$

$= \frac{1}{|\det(A)|} \frac{1}{(2\pi)^{k/2}} e^{-y^T (AA^T)^{-1} y/2}$

$A = \Sigma^{1/2} \Rightarrow AA^T = \Sigma \qquad A = R \Rightarrow AA^T = \Sigma$

- The third point says that if $X = AZ + \mu$ for a an invertible matrix A then X is multivariate normal with covariance matrix AA^T
- Alternatively the first item says that once it's established that X is multivariate normal (e.g., by recognizing it as a linear function of a multivariate normal vector), then the covariance matrix has (i,j)th entry $Cov(X_i, X_j)$.

since f_Z only depend on $z^T z$

Note.
$\det(AB) = \det(A)\det(B)$
$\Rightarrow \det(R) = \det(\Sigma^{1/2})$
$= \sqrt{\det(\Sigma)}$

Multivariate normal distribution

Density of $MVN(\underline{0}, \Sigma)$ is $\frac{1}{\sqrt{\det(\Sigma)}} \frac{1}{(2\pi)^{k/2}} e^{-y^T \Sigma^{-1} y/2}$

The density of X is

mean $\neq 0$, shift by mean μ

$$f(x) = \frac{1}{(2\pi)^{k/2}\sqrt{\det(\Sigma)}} \exp\left\{-(x-\mu)^T\Sigma^{-1}(x-\mu)/2\right\}.$$

This expression is difficult to compute with in practice so it's best to use a convenient representation as a linear function of independent normal variables.

$A\underline{X} \stackrel{d}{=} AR\underline{Z} + A\underline{\mu}$ $RR^T = \Sigma$

$\Rightarrow A\underline{X} \sim MVN(A\underline{\mu}, AR(AR)^T)$

$AR R^T A^T$
Σ
$= A\Sigma A^T$

Q. 3

$k=1$ $\Sigma = [\sigma^2]$

matrix with one term

Multivariate normal distribution

$\Sigma_{ij} = cov(x_i, x_j) = \rho\, var(x_i)\, var(x_j)$

Example: bivariate normal

(X_1, X_2) are bivariate normal with correlation ρ and means μ_1, μ_2 and variances σ_1^2, σ_2^2.

$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ $\qquad \Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$

- What are the multivariate normal parameters μ and Σ in terms of the parameters above?
- Find a lower triangular R such that $RR^T = \Sigma$ and $X = RZ + \mu$.
- Write down the joint density of (X_1, X_2).

$E[\bar{X}] = E(RZ) + \mu$

$E(RZ) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} E[\bar{x}_1] \\ E[\bar{x}_2] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$

$\Sigma_{ij} = cov(x_i, x_j)$

$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$RZ = \begin{bmatrix} az_1 \\ cz_1 + dz_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \mu.$

$= \begin{cases} \sigma_1^2 & i=j=1 \\ \sigma_2^2 & i=j=2 \\ \rho\sigma_1\sigma_2 & i=1, j=2 \\ & j=1, i=2 \end{cases}$

$R = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ $\quad R^T = \begin{bmatrix} a & c \\ 0 & d \end{bmatrix}$.

$RR^T = \Sigma$

decomposition

$\frac{x_2 - \mu_2}{\sigma_2} = \rho\left(\frac{x_1 - \mu_1}{s_1}\right) + \sqrt{1-\rho^2}\, z_2$

$\frac{x_1 - \mu_1}{\sigma_1} = z_1$

Distribution of $x_2 | x_1 = x \Rightarrow \frac{x_2 - \mu_2}{\sigma_2} = \rho\left(\frac{x - \mu_1}{\sigma_1}\right) + \sqrt{1-\rho^2}\, z_2$

$x_1 - \mu_1 = az_1$

$N(0, \sigma_1^2) \Rightarrow a = \sigma_1$

$x_1 = r_{11} z_1 + \mu_1$

$x_2 = r_{21} z_1 + r_{22} z_2 + \mu_2$

$\vdots$

$x_n = \sum_{j=1}^{k} r_{kj} z_j + \mu_k$

$(z_1, \ldots, z_n)$ iid $N(0,1)$

where the coeffs satisfy $RR^T = \Sigma$

$R = (r_{ij})_{1 \le i, j \le n}$ lower triangular

$\Sigma_{ij} = cov(x_i, x_j)$

$\Rightarrow x_2 = \ldots$ $\quad$ $x_2 - \mu_2 = c z_1 + d z_2$

Joint distribution of Brownian motion

$\Rightarrow \mathcal{L}\ N\left(\rho \frac{\sigma_2}{\sigma_1}(x - \mu_1) + \mu_2,\ \sigma_2^2(1-\rho^2)\right)$

- We saw that Brownian motion observed at a collection of times are linear functions of independent normal distributions and so are distributed as multivariate normal.
- The means are zero and the so the distribution is entirely determined by the pairwise covariances.

We can compute the covariance of Brownian motion observed at times $s < t$.

$$\begin{aligned}
Cov(B_t, B_s) &= E[B_t B_s] && (E[B_t]=0) \\
&= E[(B_t - B_s)B_s] + E[B_s^2] \\
&= E[B_t - B_s]E[B_s] + \underline{Var(B_s)} && \text{(ind. incs.)} \\
&= s.
\end{aligned}$$

$x_2 - \mu_2 = c\left(\frac{x_1 - \mu_1}{\sigma_1}\right) + d z_2$

$N(0, \sigma_2^2)$ $\quad$ IND

$Cov(x_1, x_2) = \rho\sigma_1\sigma_2$ II

$var(x_2) = \sigma_2^2$ I

I $\therefore$ $var\left(c \cdot \left(\frac{x_1 - \mu_1}{\sigma_1}\right) + d z_2\right)$

$= c^2 + d^2 = \sigma_2^2$

II $cov\left(x_1, c\left(\frac{x_1 - \mu_1}{\sigma_1}\right) + d z_2\right)$

Joint distribution of Brownian motion

$= \rho\sigma_1\sigma_2$ IND

linearity of covariance

If $0 < t_1 < \cdots < t_k$ then $(B_{t_1}, \ldots, B_{t_k})$ is multivariate normal with mean zero and covariance matrix

$\frac{c}{\cancel{\sigma_1}} \mathrm{cov}(X_1, X_1) = \rho\sigma_1\sigma_2$

σ_1^2

$$\Sigma = \begin{pmatrix} t_1 & t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & t_2 & \cdots & t_2 \\ t_1 & t_2 & t_3 & \cdots & t_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & t_3 & \cdots & t_k \end{pmatrix}.$$

$c\sigma_1 = \rho\sigma_1\sigma_2$

$c = \rho\sigma_2$

$d^2 = (1-\rho)\sigma_2^2$

Finance example

Ex.

$S_t = e^{\sigma B_t}$ $t \geq 0$

(geometric BM / Black-Scholes model)

Assume that the logarithm of the (standardized) price of stock t hours into the trading day is given by σB_t for some $\sigma > 0$ and where B_t is a Brownian motion.

- If the stock is worth $e^{4\sigma}$ dollars halfway through the 8 hour trading day, what is the chance the stock will be worth more than its initial price at the end of the day?
- If at the end of the day the stock is worth $e^{4\sigma}$ dollars, what is the chance the stock's price at the middle of the day was greater than its starting price?

$P(S_8 > S \mid S_4 = e^{4\sigma})$

$= P(e^{\sigma B_8} > 1 \mid e^{\sigma B_4} = e^{4\sigma})$

$= P(B_8 > 0 \mid B_4 = 4)$

$= P(B_8 - B_4 > -4 \mid B_4 = 4) = P(B_8 - B_4 > -4)$.

$\leftarrow P((B_8 - B_4) + B_4 > 0)$.

indep increment $= P(N(0,4) > -4)$

$B_t \sim N(0,t)$

$P(2Z > -4)$

$= P(Z > -2)$

$=$

$P(S_4 > 1 \mid S_8 = e^{4\sigma})$

$= P(B_4 > 0 \mid B_8 = 4)$

know (B_8, B_4) $\sim MVN\left(0, \begin{bmatrix} 8 & 4 \\ 4 & 4 \end{bmatrix}\right)$

standardise

$\frac{B_4}{\sqrt{4}} = \rho\left(\frac{B_8}{\sqrt{8}}\right) + \sqrt{1-\rho^2}\, Z_2$

$(B_4 = aB_8 + bZ_2)$ IND

Properties of Brownian motion

$B_{t+s} - B_t \sim N(0, s)$ ← length of interval.

$\begin{bmatrix} B_8 \\ B_4 \end{bmatrix} \sim MVN\left(0, \begin{bmatrix} 8 & 4 \\ 4 & 4 \end{bmatrix}\right)$

$\rho = \frac{cov}{\sqrt{V_1 V_2}} = \frac{4}{\sqrt{8 \times 4}} = \sqrt{\frac{16}{32}} = \frac{1}{\sqrt{2}}$

$\frac{B_8}{\sqrt{8}} \sim Z_1$.

$\frac{B_4}{2} = \rho\left(\frac{B_8}{\sqrt{8}}\right) + \sqrt{1-\rho^2}\, Z_2$

Brownian motion arises as the limit of random walk, inherits definition/properties from there.

- This result is called Donsker's Theorem or the invariance principle.
- Is a good way to approximately simulate Brownian motion.

We can use the ideas around limits of discrete processes and generators to understand this result a bit more.

$B_4 = aB_8 + bZ_2$.

$cov(B_4, B_8) = cov(aB_8 + \underbrace{bZ_2, B_8}_{IND.}) = a8 = 4 \Rightarrow a = \frac{1}{2}$

$cov(B_4, B_4) = cov(aB_8 + bZ_2, aB_8 + bZ_2)$
$= a^2 \cdot 8 + b^2 = 4 \Rightarrow b = \sqrt{2}$

$4 = var(B(4)) = a^2 8 + b^2$

$4 = cov(B_4, B_8) = a8$

$a = 1/2;\ b = \sqrt{2}$

$B_4 = 2 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{8}} B_8 + 2 \cdot \frac{1}{\sqrt{2}} Z_2$

$= \frac{1}{2} B_8 + \sqrt{2} Z_2$.

$P(B_4 > 0 \mid B_8 = 4)$

$= P(\frac{1}{2} B_8 + \sqrt{2} Z_2 > 0 \mid B_8 = 4)$

$= P(2 + \sqrt{2} Z_2 > 0 \mid B_8 = 4)$

$= P(2 + \sqrt{2} Z_2 > 0)$

$= P(Z_2 > -\sqrt{2})$

$Z_2 \sim N(0,1)$.

$\Rightarrow B_4 = \frac{1}{2} B_8 + \sqrt{2} Z_2.$

$P(B_4 > 0 \mid B_8 = \cancel{0}) = P(\frac{1}{2} B_8 + \sqrt{2} Z_2 > 0 \mid B_8 = 2) = P(Z_2 > -\sqrt{2})$

Einstein derivation of Brownian motion

- We'll derive a PDE for $p_t(x)$, the density of the limit of simple random walk (properly scaled).
- A solution to this PDE is

$$p_t(x) = \frac{\exp[-x^2/(2t)]}{\sqrt{2\pi t}},$$

which is the density of $\mathrm{N}(0, t)$.

Einstein derivation of Brownian motion

Basic r.w $\overset{+1}{\nearrow}$ wp $\frac{1}{2}$

$\overset{-1}{\searrow}$ wp. $\frac{1}{2}$

N is a (large) integer.

- Let $X_1^{(N)}, X_2^{(N)}, \ldots$ be i.i.d. with
$$P\left(X_i^{(N)} = \frac{\pm 1}{\sqrt{N}}\right) = 1/2.$$

At time t $\approx$ Nt step of walk

$\Rightarrow$ sp at time t is $\sqrt{N}$

$p_t(x)$ density of B_t at x

- Simple random walk with jumps at times $1/N, 2/N, \ldots$:
$$S_t^{(N)} = \sum_{i=1}^{\lfloor Nt \rfloor} X_i^{(N)}.$$

$\sum_{i=1}^{N} x_i$ mean 0 var $= N$

CLT: $\frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i \to N(0,1)$

$\frac{1}{\sqrt{N}} \sum_{i=1}^{Nt} x_i \to N(0,t)$

$x_i^{(N)} = \frac{x_i}{\sqrt{N}}$

$Q_{\frac{n}{N}}(\frac{k}{\sqrt{n}}) = P(S_{\frac{n}{N}}^{(N)} \le \frac{k}{\sqrt{N}})$

$- (S_{\frac{n}{N}}^{(N)} \le \frac{k}{\sqrt{N}} - \frac{1}{\sqrt{N}})$

- $Q_{n/N}(k/\sqrt{N}) = P\left(S_{n/N}^{(N)} = k/\sqrt{N}\right)$.

$\approx P(B_t \le x) - P(B_t \le x - \frac{1}{\sqrt{N}})$

$\approx p_t(x) \cdot \frac{1}{\sqrt{N}}$ density $\times$ width of interval

(small n)

rescale time by N

$(F(x+h) - F(x) \approx f(x) \cdot h$

space by $\sqrt{N}$

Einstein derivation of Brownian motion

- $Q_{n/N}(k/\sqrt{N}) = P\left(S^{(N)}_{n/N} = k/\sqrt{N}\right)$.

If we let n and k grow with N such that as $N \to \infty$,

$$n/N \to t > 0 \quad \text{and} \quad k/\sqrt{N} \to x,$$

then, by the CLT,

$$\sqrt{N} Q_{n/N}(k/\sqrt{N}) \to p_t(x).$$

Einstein derivation of Brownian motion

By the law of total probability:

$$Q_{\frac{n}{N}}\left(\frac{k}{\sqrt{N}}\right) = \frac{1}{2} Q_{\frac{n-1}{N}}\left(\frac{k+1}{\sqrt{N}}\right) + \frac{1}{2} Q_{\frac{n-1}{N}}\left(\frac{k-1}{\sqrt{N}}\right),$$

and so

$$N\left[Q_{\frac{n}{N}}\left(\frac{k}{\sqrt{N}}\right) - Q_{\frac{n-1}{N}}\left(\frac{k}{\sqrt{N}}\right)\right]$$
$$= \frac{N}{2}\left[Q_{\frac{n-1}{N}}\left(\frac{k+1}{\sqrt{N}}\right) - Q_{\frac{n-1}{N}}\left(\frac{k}{\sqrt{N}}\right)\right]$$
$$- \frac{N}{2}\left[Q_{\frac{n-1}{N}}\left(\frac{k}{\sqrt{N}}\right) - Q_{\frac{n-1}{N}}\left(\frac{k-1}{\sqrt{N}}\right)\right].$$

As $N \to \infty$, remembering that $k/\sqrt{N} \to x$ and $n/N \to t$,

- LHS$\to \frac{\partial}{\partial t} p_t(x)$,
- RHS$\to \frac{1}{2}\frac{\partial^2}{\partial x^2} p_t(x)$.

C-K equ for DTMC

$P(\sum_{i=1}^{n} X_i = k)$

$= \frac{1}{2} P(\sum_{i=1}^{n-1} X_i = k+1)$

$+ \frac{1}{2} P(\sum_{i=1}^{n-1} X_i = k-1)$

RHS.

$\frac{1}{\sqrt{N}}\{\frac{N}{2}[p_t(x+\frac{1}{\sqrt{N}}) - p_t(x)] - \frac{N}{2}[p_t(x) - p_t(x-\frac{1}{\sqrt{N}})]\}$

$\frac{\partial}{\partial x} p_t(x+\frac{1}{\sqrt{N}}) \cdot \frac{1}{\sqrt{N}}$

$\frac{\partial}{\partial x} p_t(x) \cdot \frac{1}{\sqrt{N}}$

$\frac{1}{\sqrt{N}}\{\frac{\sqrt{N}}{2}[\frac{\partial}{\partial x} p_t(x+\frac{1}{\sqrt{N}}) - \frac{\partial}{\partial x} p_t(x)]\}$

$Q_{n/N}(\frac{k}{\sqrt{N}}) \approx \frac{p_t(x)}{\sqrt{N}}$

$\frac{n}{N} \to t \quad \frac{k}{\sqrt{N}} \to x$

LHS.

$N[Q_{\frac{n}{N}}(\frac{k}{\sqrt{N}}) - Q_{\frac{n}{N}-\frac{1}{N}}(\frac{k}{\sqrt{N}})]$

$\approx \frac{1}{\sqrt{N}}(\cancel{N}\underbrace{[p_t(x) - p_{t-\frac{1}{N}}(x)]}_{(\frac{1}{N})})$

$\approx \frac{1}{\sqrt{N}} \frac{\partial}{\partial t} p_t(x)$

$$\approx \frac{1}{\sqrt{N}} \cdot \frac{1}{2} \frac{\partial^2}{\partial x^2} p_t(x).$$

Einstein derivation of Brownian motion

So the limiting stochastic process should have density $p_t(x)$ at time t satisfying the PDE

$$\frac{\partial}{\partial t} p_t(x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} p_t(x).$$

This PDE is called the heat equation and under appropriate boundary conditions the unique solution is

$$p_t(x) = \frac{\exp[-x^2/(2t)]}{\sqrt{2\pi t}}.$$

$N(0, t)$

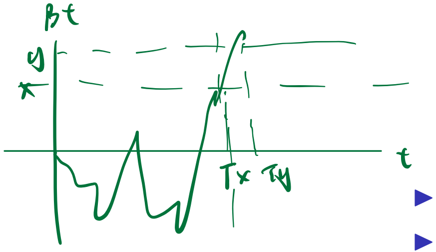
We can think of the heat equation for Brownian motion as the continuous state space analog of the forward equation for CTMCs:

$$\frac{\partial}{\partial t} p_t = \mathcal{A}(p_t),$$

where $\mathcal{A}$ is the linear operator on twice differentiable functions with

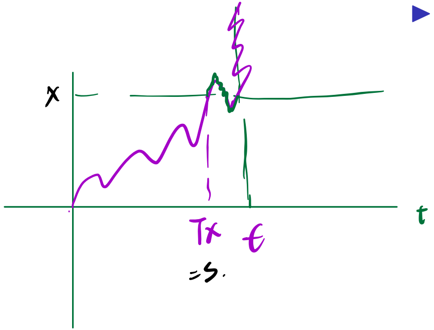
$$\mathcal{A}f(x) = \frac{1}{2} f''(x).$$

Hitting times of Brownian motion



- Define the hitting time of level x by $T_x = \inf\{t : B_t = x\}$.
- Brownian motion is continuous so if $0 < x < y$, then $T_x < T_y$.
- Since simple symmetric random walk is recurrent, T_x is finite.

$\frac{1}{2}$ $\frac{1}{2}$



Dist of B_t given $T_x = s < t$

$\{T_x = s\} \Rightarrow \{B_s = x\} \cap \{$don't hit level x before $s\}$

$B_t \mid T_x = s \sim N(x, t-s)$.

$\Rightarrow$ by indept increment

$B_t - B_s = B_t - x$ given $T_x = s$.

$\sim N(0, (t-s))$

Hitting times of Brownian motion

We derive the distribution of T_x for $x > 0$ by using the relation:

$$\begin{aligned} \underbrace{P(B_t \geq x)}_{N(0,t)} &= P(B_t \geq x, T_x \leq t) \\ &= P(B_t \geq x | T_x \leq t) P(T_x \leq t). \end{aligned}$$

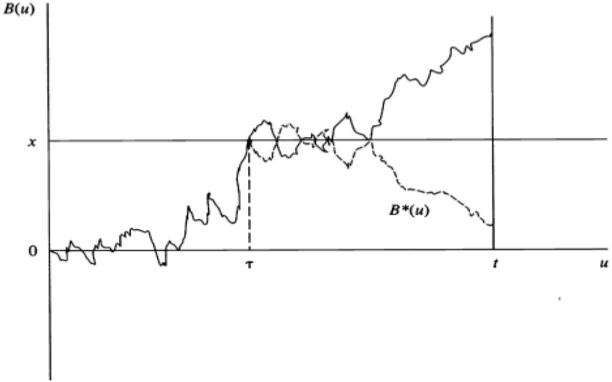
We only need to determine $P(B_t \geq x | T_x \leq t)$ since we know the distribution of B_t and hence the LHS.

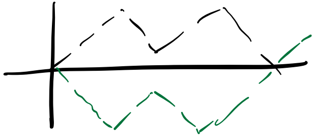
$\Rightarrow P(B_t \geq x \mid T_x = s)$ $\quad s < t.$

$= \frac{1}{2}$

Reflection principle

If $\tau < t$, then $P(B_t - x > 0 | T_x = \tau) = 1/2$.





same prob

$P(B_t \geq x \,|\, T_x \leq t)$

$= P(B_t \leq x \,|\, T_x \leq t)$

But they also sum to 1

$\Rightarrow$ each $p = \frac{1}{2}$

Hitting times of Brownian motion

Combining the last two slides we have the distribution function

$$
\begin{aligned}
P(T_x \le t) &= 2P(B_t > x) \\
&= \sqrt{\frac{2}{\pi t}} \int_x^\infty \exp[-u^2/(2t)]du \\
&= \sqrt{\frac{2}{\pi}} \int_{x/\sqrt{t}}^\infty \exp[-u^2/2]du.
\end{aligned}
$$

$$\frac{\partial}{\partial t} P(T_x \le t)$$

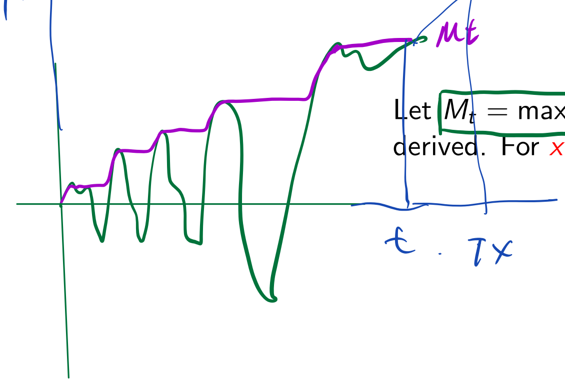
$$= \exp[-(\tfrac{x}{\sqrt{t}})^2/2] \times \frac{d}{dt}(\tfrac{x}{\sqrt{t}}) \times (-1)$$

bottom

So T_x has density

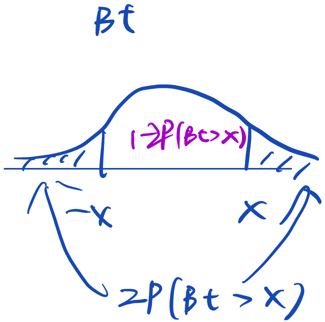
$$f_{T_x}(t) = \frac{x t^{-3/2}}{\sqrt{2\pi}} \exp[-x^2/(2t)].$$

The Maximum of Brownian motion on an interval



Let $\boxed{M_t = \max_{0 \le s \le t} B_s}$. The distribution of M_t is now easily derived. For $x > 0$,

$$\begin{aligned} P(M_t \le x) &= P(T_x > t) \\ &= 1 - 2P(B_t > x) \\ &= P(-x \le B_t \le x) \\ &= P(|B_t| \le x). \end{aligned}$$

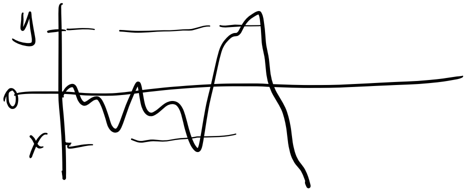


So for each fixed t, $M_t \stackrel{d}{=} |B_t|$ (but not as processes!), and the maximum of Brownian motion is distributed as the absolute value of a normal distribution.

Gambler's ruin via the invariance principle

Let $x < 0 < y$. What is $P(T_x < T_y)$?

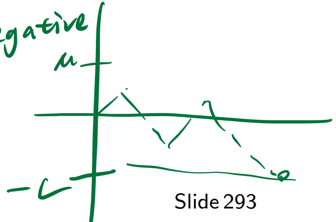
- Gambler's ruin from DTMC slides says that the chance simple symmetric random walk hits $-L$ before hitting M is $M/(L+M)$.
- Using the approximation from before, we set $-L = \lfloor \sqrt{N} x \rfloor$ and $M = \lfloor \sqrt{N} y \rfloor$ and take the limit as $N \to \infty$.



$$P(T_x < T_y) = \frac{y}{y - x}.$$

x is negative

for random walk



$$\frac{M}{\sqrt{N}} \to y$$

$\frac{-L}{\sqrt{N}} \to x$