MAST30001 Stochastic Modelling

Tutorial Sheet 1

1. Let $A_0, \dots, A_m$ be events such that $\mathbb{P}(\cap_{j=0}^{i} A_j) > 0$ for all $i = 0, \dots, m-1$. Show that
$$\mathbb{P}(\cap_{j=0}^{m} A_j) = \mathbb{P}(A_0) \prod_{i=1}^{m} \mathbb{P}(A_i | \cap_{j=0}^{i-1} A_j).$$
Ans. Compute the right hand side using the definition of conditional expectation:
$$\mathbb{P}(A_0) \prod_{i=1}^{m} \mathbb{P}(A_i | \cap_{j=0}^{i-1} A_j) = \mathbb{P}(A_0) \prod_{i=1}^{m} \frac{\mathbb{P}(\cap_{j=0}^{i} A_j)}{\mathbb{P}(\cap_{j=0}^{i-1} A_j)} = \mathbb{P}(A_0) \frac{\prod_{i=1}^{m} \mathbb{P}(\cap_{j=0}^{i} A_j)}{\prod_{i=0}^{m-1} \mathbb{P}(\cap_{j=0}^{i} A_j)}.$$
Cancelling the common terms in the numerator and denominator of the fraction shows the result.

2. Let $X_1, X_2, \dots, X_n$ be independent and have exponential distributions with common rate λ (recall this means they each have density $\lambda e^{-\lambda x}$ for $x > 0$). What is the density of $X = \min\{X_1, \dots, X_n\}$?
Ans. Use $\mathbb{P}(X > t) = \mathbb{P}(X_1 > t, \dots, X_n > t)$ and then differentiate. X is exponential with rate $n\lambda$.

3. Bob and Doug are playing the following game. Bob starts by rolling two fair dice; if the sum of his dice is six, then he wins the game. If not, then Doug rolls the dice, and if the sum of his rolls is seven, then he wins the game. If neither player wins the game during the first round, then they repeat the process (with Bob going first) until someone wins a round. What is the probability that Bob wins this game? Is he more or less likely than Doug to win?
Ans. Use the law of total probability across the trial in which Bob wins. The infinite series is geometric and the chance that Bob wins is
$$\frac{5}{5 + 31/6},$$
which is slightly less than a half.

4. An investment bank rates the performance of stocks in each quarter as Good (G), Satisfactory (S), or Unsatisfactory (U). The historical ratings of stock performance in Quarter 2 (Q2) of the financial year given the stock's Quarter 1 (Q1) rating is encoded in the following table:

	Q2 G	Q2 S	Q2 U
Q1 G	50%	40%	10%
S	30%	40%	30%
U	10%	10%	80%

Assume that in 2012, the percentage of stocks rated G, S and U in Q1 were respectively 20%, 30% and 50%.

(a) What is the chance a randomly chosen stock will be rated G in Q2 of 2012?

(b) Given a stock receives a G rating in Q2 of 2012, what is the chance it was rated U in Q1?

(c) Are the events "rated U in Q1 of 2012" and "rated G in Q2 of 2012" independent?

Ans. Denote $G1$ the event that a stock gets rated good in Q1, $G2$ the event that a stock gets rated good in Q2, etc.

(a) Using the law of total probability,

$$P(G2) = P(G2|G1)P(G1) + P(G2|S1)P(S1) + P(G2|U1)P(U1) = 0.24.$$

(b) Using Bayes' formula (really the definition of conditional probability),

$$P(U1|G2) = \frac{P(G2|U1)P(U1)}{P(G2)} = 5/24.$$

(c) No. From (a), $P(G2) = 0.24$, but from the table, $P(G2|U1) = 0.1$.

5. Let N be geometric with parameter $0 < p < 1$; that is

$$P(N = n) = (1-p)^n p, \quad n = 0, 1, 2, \dots$$

and has probability generating function

$$P_N(s) = \frac{p}{1 - s(1-p)}.$$

Let X be such that the density of $X|N = n$ is given by

$$f_{X|N}(x|n) = \frac{x^n e^{-x}}{n!}, \quad x > 0.$$

(a) Derive the conditional moment generating function of $X|N$.

(b) Use the answer to part (a) to derive the moment generating function of X and identify the distribution by name.

(c) Use the moment generating function in part (b) to derive a formula for the moments of X.

(d) Find the conditional probability mass function of N given $X = x$ and identify this distribution by name.

Ans.

(a) $X|N$ is standard gamma $N + 1$ and so has mgf

$$M_{X|N}(t) = (1-t)^{-(N+1)},$$

defined for $t < 1$.

(b) $M_X(t) = E[M_{X|N}(t)] = P_{N+1}((1-t)^{-1})$, where P_{N+1} is the probability generating function of $N + 1$, a geometric variable started from one. And so

$$M_X(t) = \frac{p}{p-t},$$

for $t < p$, which means X is exponential rate p.

(c) $M_x^{(k)}(t) = pk!/(p-t)^{k+1}$ and so

$$E[X^k] = M_X^{(k)}(0) = \frac{k!}{p^k}.$$

(d) Using the information above and formulas derived therein, for $x > 0$ and $n = 0, 1, 2, \ldots,$

$$P(N = n|X = x) = \frac{f_{X|N}(x|n)P(N = n)}{f_X(x)} = \frac{x^n e^{-x}(1-p)^n p}{pe^{-px}n!} = \frac{(x(1-p))^n e^{-x(1-p)}}{n!},$$

and so $N|X = x$ is Poisson with mean $(1-p)x$.

6. The standard "pass" bet in craps has some of the best odds found in a casino on a simple bet. If a player wagers D dollars, then they win D dollars with probability 0.493 and lose D dollars otherwise.

   (a) If a player wagers D dollars in a pass bet, what is the mean of their winnings?

   (b) If a player wagers D dollars in a pass bet, what is the variance of their winnings?

   You go into a casino with \$100 dollars in your pocket and consider two betting strategies. You either make one craps pass bet for all \$100 or you make 100 one dollar pass bets in a row. Let Y be your winnings using the first strategy and W be your winnings using the second strategy.

   (c) What is the mean and variance of Y?

   (d) What is the mean and variance W?

   (e) Compute $P(Y > 0)$ and approximate $P(W > 0)$.

   $\frac{X_i+1}{2} \sim Ber(0.493)$

   $\sum_{i=1}^{100} \frac{X_i+1}{2} \sim Bi(100, 0.493)$

   $\Rightarrow \sum_{i=1}^{100} X_i \sim 2Bi(100, 0.493) - 100$

   [A normal distribution table will be useful for this problem.]

   Ans.

   (a) Expected winnings are $D(.493 - .507) = -0.014D$.

   (b) Variance of winnings is $D^2(1 - (.014)^2) = D^2(0.999804)$.

   (c) The expected value and variance of Y are respectively (a) and (b) with $D = 100$; $E[Y] = -1.4$ and $V(Y) = 9998$.

   (d) $W = \sum_{i=1}^{100} X_i$, where the X_i are independent and each is ± 1 depending on the outcome of the ith bet and using parts (a) and (b) with $D = 1$, $E[X_i] = -0.14$ and $V(X_i) = 0.9804$. So

   $$E[W] = -1.4, \quad V(W) = 99.98.$$

   (e) Since $Y > 0 \iff Y = D$, $P(Y > 0) = 0.493$. On the other hand, using the CLT, W is roughly normal with mean and variance as in (d) so if Z is standard normal,

   $$P(W > 0) \approx P(Z > 1.4/\sqrt{99.98}) = P(Z > .1400) \approx 0.4443.$$

7. For fixed $n \geq 1$, let X be the number of heads in n independent tosses of a fair coin and let Y be the number of tails in the same n tosses.

(a) What is the covariance and correlation of X and Y?

Assume that X is the number of heads and Y is the number of tails in N independent tosses of a fair coin, where now $N \geq 1$ is an integer valued random variable having mean μ and variance σ^2. Find expressions in terms of μ and σ^2 for

(b) the expected value of both X and Y.

(c) the variance of both X and Y.

(d) the covariance of X and Y.

(e) the correlation of X and Y.

Ans.

(a) $Y = n - X$, so $Cov(X,Y) = -V(X) = -n/4$. $Corr(X,Y) = -1$.

(b) $E[X] = E\{E[X|N]\} = EN/2 = \mu/2$.

(c)

$$V(X) = V(E[X|N]) + E[V(X|N)] = \frac{\sigma^2 + \mu}{4}.$$

And the same formulas hold with Y replacing X since $X \overset{d}{=} Y$.

(d) Since $Y = N - X$:

$$Cov(X,Y) = Cov(X,N) - Cov(X,X) = Cov(X,N) - (\sigma^2 + \mu)/4.$$

To compute $Cov(X,N) = E[XN] - \mu^2/2$, we condition:

$$E[XN] = E\{NE[X|N]\} = E[N^2]/2 = (\sigma^2 + \mu^2)/2,$$

and so $Cov(X,N) = \sigma^2/2$, and putting everything together we get

$$Cov(X,Y) = \frac{\sigma^2 - \mu}{4}.$$

(e)

$$Corr(X,Y) = \frac{(d)}{(c)} = \frac{\sigma^2 - \mu}{\sigma^2 + \mu^2}.$$