THE UNIVERSITY OF MELBOURNE

Student Number

Semester 2 Assessment, 2019

School of Mathematics and Statistics

MAST20009 Vector Calculus

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 5 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- No written or printed materials may be brought into the examination.
- No calculators of any kind may be brought into the examination.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- There are 12 questions on this exam paper.
- All questions may be attempted.
- Marks for each question are indicated on the exam paper.
- Start each question on a new page.
- Clearly label each page with the number of the question that you are attempting.
- There is a separate 3 page formula sheet accompanying the examination paper, which you may use in this examination.
- The total number of marks available is 151.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.
- Please supply **graph paper**.
- Initially students are to receive the exam paper, the 3 page formula sheet, two 11 page script books and two sheets of graph paper.

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Question 1 (12 marks)

Prove that the function

$$f(x,y) = \begin{cases} x^4 \sin\left(\dfrac{1}{x^2+|y|}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous on all of $\mathbb{R}^2$.

Question 2 (9 marks)

For each of the following statements about a given scalar valued function f in two variables x and y, decide whether it is true or false. Give brief justifications for your answers.

(a) If the partial derivatives of f both exist and are continuous, then f is continuous.

(b) Wherever the second derivatives of f are defined, we have

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

(c) If f is C^1 then f is also C^2.

Question 3 (12 marks)

(a) State the matrix chain rule. Be careful to include the conditions under which it holds and to explain each symbol you use.

(b) Let

$$f(u,v,w) = (e^{u-w}, \sin(u+v+w))$$

and let

$$g(x,y) = (e^x, \cos(y-x)).$$

Calculate $g \circ f$ and, using the matrix chain rule, calculate $D(g \circ f)|_{(0,0,0)}$.

Question 4 (20 marks)

(a) Use Lagrange multipliers to determine the maximum of the function $f(x,y) = 4xy + y$ restricted to the line segment given by $x + y = 1$ and $x \geq 0$ and $y \geq 0$.

(b) Prove that the answer you found in part (a) is indeed a maximum.

(c) On the graph paper provided, draw a high quality picture of this situation: the constraint, at least three level curves of f, the gradient of f at three points and the gradient of the constraint function g at three points. Pay attention to labels and scale.

Question 5 (16 marks)

(a) Sketch the path

$$\gamma(t) = \begin{bmatrix} t\cos(t) \\ t\sin(t) \\ t \end{bmatrix} \qquad t \in [0, 6\pi]$$

in $\mathbb{R}^3$ and describe the journey of a particle along this path (interpreting t as time variable) in words.

(b) Compute the tangent, normal and binormal vectors of the path

$$\mathbf{c}(t) = \begin{bmatrix} 2\sin(t) \\ 2 - \sqrt{2}\cos(t) \\ \sqrt{2}\cos(t) \end{bmatrix} \qquad t \in \mathbb{R}.$$

(c) Compute the curvature and the torsion of $\mathbf{c}$.

(d) Interpret your answers to (b) and (c) geometrically.

Question 6 (12 marks)

Consider the vector field

$$\vec{F}(x, y) = \begin{bmatrix} 2y \\ -2x \end{bmatrix}.$$

(a) Use the graph paper provided to draw the vector field at the points $(1, 1)$, $(-1, 2)$, $(2, -1)$, $(-3, -1)$ and $(0, -1)$.

(b) Make an educated guess about the flow lines of $\vec{F}$.

(c) Determine the equation for the flow line of $\vec{F}$ passing through the point $(1, 1)$ in terms of x and y. Show your working.

Question 7 (6 marks)

(a) Let f be a scalar valued function in three variables x, y and z. Under which condition on f does the gradient of f satisfy the vector identity

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \vec{\mathbf{0}}?$$

[**Hint:** It is possible to figure this out.]

(b) Prove the identity in part (a), assuming the condition you found for f.

Question 8 (8 marks)

Consider the double integral
$$\int_0^2 \int_0^x x^2 y \, dy \, dx.$$

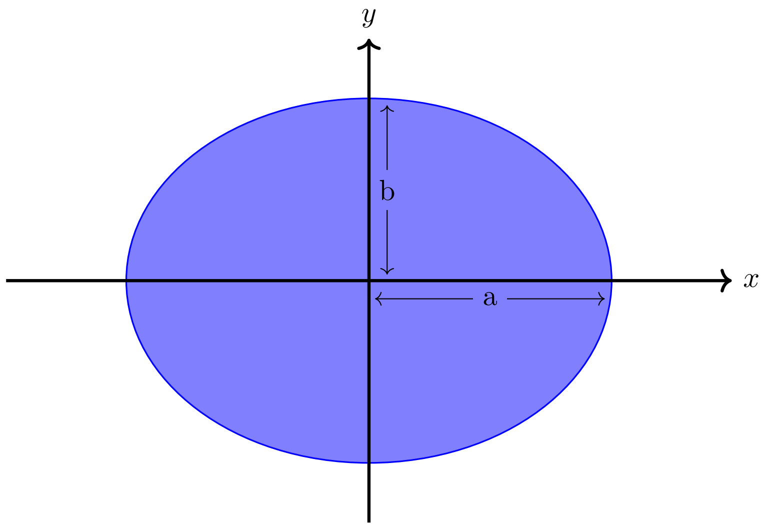
(a) Sketch the region of integration.

(b) Evaluate the integral in the form given.

(c) Change the order of integration and evaluate the integral again.

Question 9 (6 marks)

(a) Use polar coordinates to parametrize the interior of the ellipse with major axis of radius a along the x-axis and minor axis of radius b along the y-axis.



(b) Using double integrals and your parametrization from part (a), prove that the volume of the elliptical cylinder with semi-major axis a, semi-minor axis b, and height h equals πabh.

Question 10 (10 marks)

Let D be the region bounded by the paraboloids $z = x^2 + y^2 - 4$ and $z = 8 - 2x^2 - 2y^2$.

(a) Determine where the paraboloids intersect.

(b) Sketch the region D.

(c) Using cylindrical coordinates, evaluate the triple integral
$$\iiint_D x^2 \; dV.$$

Question 11 (20 marks)

Let $E \subset \mathbb{R}^2$ be the ellipse with main radius equal to 5 and minor radius equal to 3, centred at the origin with the main axis aligned with the x axis.

(a) Parametrize E using the counter-clockwise orientation and starting at the point $(5, 0)$.

(b) Parametrize the rays from $(-4, 0)$ to $(0, 3)$ and from $(-4, 0)$ to $(5\sqrt{2}, 3\sqrt{2})$.

(c) Using Green's theorem in the plane, prove that the area of a closed bounded domain D with boundary ∂D is given by

$$A(D) = \int_{\partial D} x\,dy.$$

(d) Use your result from part (c) to calculate the area of the domain bounded by E and the two rays in part (b).

(e) Using the graph paper provided, draw a high quality picture of the domain D and its boundary, indicating orientations and paying attention to scale and labels.

Question 12 (20 marks)

Let S be the sphere of radius 6, centred at the origin.

(a) Give a parametrization

$$\Psi : [0, \pi] \times [0, 2\pi] \longrightarrow S.$$

(b) Calculate the tangent vectors and the outward normal vector of S.

(c) Let S_+ be the upper hemisphere, i.e., the part of S with $z \geq 0$, oriented by the outward normal vector. Let $\vec{F}$ be the vector field

$$\vec{F}(x, y, z) = \begin{bmatrix} y \\ z \\ 1 \end{bmatrix}$$

on S_+. Verify Stokes' theorem for $\vec{F}$ and S_+.

End of Exam—Total Available Marks = 151