# MAST10008 Assignment 4

Due Thursday 18 April at 12pm in your tutor's assignment box

1. Let $V$ be a vector space with scalars $\mathbb{R}$. In each one of the following parts, you can only use axioms of vector spaces, as well as the statements from the previous parts. Make sure you label clearly which axiom or statement you use at each step.

   (a) Show that
   $$0\,\mathbf{v} = \mathbf{0}$$
   for all $\mathbf{v} \in V$, where $\mathbf{0} \in V$ is the zero vector.

   (b) Let $\mathbf{v} \in V$. Suppose $\mathbf{w} \in V$ is such that
   $$\mathbf{v} + \mathbf{w} = \mathbf{0}.$$
   Prove that $\mathbf{w} = -\mathbf{v}$. (In other words, given $\mathbf{v} \in V$, the additive inverse $-\mathbf{v}$ is unique.)

   (c) Show that
   $$(-1)\,\mathbf{v} = -\mathbf{v}$$
   for all $\mathbf{v} \in V$, where $-\mathbf{v}$ denotes the additive inverse of $\mathbf{v}$ as per the axioms, while $(-1)\,\mathbf{v}$ is the scalar product of the scalar $-1 \in \mathbb{R}$ with the vector $\mathbf{v}$.

2. Consider the set of functions
   $$\mathcal{C}^1(\mathbb{R},\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable and } f' \text{ is continuous}\}.$$
   You may assume, without proof, that $\mathcal{C}^1(\mathbb{R},\mathbb{R})$ is a subspace of the vector space $\mathcal{F}(\mathbb{R},\mathbb{R})$ of all functions $\mathbb{R} \to \mathbb{R}$.
   Given $f, g \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$, define $W : \mathbb{R} \to \mathbb{R}$ by
   $$W(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$$

   (a) Prove that if the set $\{f, g\}$ is linearly dependent, then $W = \mathbf{0}$, the constant function zero.

   (b) Prove that if $W \neq \mathbf{0}$, then the set $\{f, g\}$ is linearly independent.

   (c) Is the set $\{e^x, e^{2x}\}$ linearly independent?





$w_1+w_2 \in W$ $\lambda w \in W$
$0_u \in U$ $0_v \in W$ $w_1, u_2 \in U$ $\lambda u \in U$
$u_1+u_2 \in U$

3. Suppose $U$ and $W$ are subspaces of a vector space $V$, with the property that $U \cup W$ is a subspace of $V$. Prove that $U \subset W$ or $W \subset U$.

$a_1 \in U \cup W \to a_1 \in W$, $a_1 \in U$
$b_1 \in U \cup W \to b_1 \in W$, $b_1 \in U$
$a_1 + b_1 \in U \cup W$
$\lambda a_1, \lambda b_1 \in U \cup W$
if

4. Is the set $\{(1,1,0),(1,0,1),(0,1,1)\}$ linearly independent in

   (a) $\mathbb{R}$?

   (b) $\mathbb{F}_2$?

5. Let $B \in M_{n\times n}$ and consider the set
   $$W = \{A \in M_{n\times n} \mid [A,B] = 0\},$$
   where $[A,B] = AB - BA$.

   (a) Prove that $W$ is a subspace of $M_{n\times n}$.

   (b) Consider the special case $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \in M_{2\times 2}$. Find the dimension of the corresponding subspace $W$.

spanning set of $U \Rightarrow U = \text{span}\{u_1, \dots u_m\}$
$W = \text{span}\{w_1, w_2 \dots w_n\}$
$U \cup W$