THE UNIVERSITY OF MELBOURNE

Student Number	

Semester 2 Assessment, 2018

School of Mathematics and Statistics

MAST30001 Stochastic Modelling

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 4 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- Students may bring one double-sided A4 sheet of handwritten notes into the exam room.
- Hand-held electronic scientific (but not graphing) calculators may be used.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- This paper has **6 questions**. Attempt as many questions, or parts of questions, as you can. The number of marks allocated to each question is shown in the brackets after the question statement. There are **85 total marks** available for this examination. Working and/or reasoning must be given to obtain full credit. Clarity, neatness and style count.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

1. A Markov chain $(X_n)_{n\geq 0}$ with state space $S = \{1, 2, 3, 4\}$ has transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 3/5 & 2/5 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

   (a) Find $\mathbb{P}(X_3 = 1, X_1 = 2 | X_0 = 2)$.

   (b) If the initial distribution is uniform on $\{1, 2, 3, 4\}$, find $\mathbb{P}(X_3 = 1, X_1 = 2)$.

   (c) Write down the communication classes of the chain.

   (d) Find the period of each communicating class.

   (e) Determine which classes are essential.

   (f) Classify each essential communicating class as transient or positive recurrent or null recurrent.

   (g) Describe the long run behaviour of the chain (including deriving long run probabilities where appropriate).

   (h) Find the expected number of steps taken for the chain to first reach state 1, given the chain starts at state 4.

[16 marks]

2. A Markov chain $(X_t)_{t\geq 0}$ with state space $S = \{1, 2, 3\}$ has generator

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 2 & -4 & 2 \\ 0 & 1 & -1 \end{pmatrix}.$$

   (a) Given the chain is in state 2 right now, what is the expected amount of time until the chain jumps to a new state?

   (b) Given the chain is in state 2 right now, what is the probability that the next time the chain jumps, it jumps to state 1?

   (c) Find the stationary distribution of the chain.

   (d) Derive a differential equation for $p_{2,2}(t) := P(X_t = 2 | X_0 = 2)$, and show that

$$p_{2,2}(t) = \frac{1}{5} + \frac{4}{5}e^{-5t}.$$

   (e) Calculate $p_{1,2}(t) := P(X_t = 2 | X_0 = 1)$ for $t \geq 0$.

[14 marks]

3. A factory worker is assigned tasks one at a time. The amount of hours it takes to complete tasks are i.i.d. with (gamma) density $100te^{-10t}$, $t > 0$. The worker is given a new task as soon as they finish one.

   (a) State from formulas (or compute) the mean and variance of the time it takes to complete a task.

   (b) On average, about how many tasks does the worker complete over a 40 hour work week?

   (c) Give a symmetric interval around your estimate from (b) that will have a 95% chance of covering the true number of tasks completed over the 40 hour work week. Note that
   $$\frac{1}{\sqrt{2\pi}} \int_{-1.96}^{1.96} e^{-x^2/2} dx = 0.95.$$

   (d) What would you estimate to be the mean of the time to completion of the task the worker is currently working on?

   (e) Now assume that each task is "easy" with probability 1/4 (independently among tasks), in which case it is completed in an exponential with rate 10 (per hour) distributed time. In this new setting, about how many tasks does the worker complete over a 40 hour work week?

   You may find the following formula useful for this problem: for $a, b > 0$
   $$\int_0^\infty t^a e^{-bt} dt = \frac{\Gamma(a+1)}{b^{a+1}}.$$

   [15 marks]

4. Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ, and $(M_t)_{t\geq 0}$ be the "thinned" process where each arrival in N is added to M with probability $p > 0$. Your answers to the questions below should be simple and tidy formulas in terms of λ and p.

   (a) What is the expected time of the first arrival of N that occurs after the tenth arrival of M?

   (b) What is the chance that there are no arrivals of M in the interval $(2, 5)$?

   (c) What is the expected time until the first arrival of either M or $(N - M)$?

   (d) What is the expected time between the first arrivals of N and M?

   (e) What is the probability that in the time interval $(0, 1)$, the number of arrivals of N and M are the same?

   (f) Given there are 10 total arrivals of N in the interval $(0, 1)$, what is the chance that exactly 5 of these are in M?

   (g) Given that $N_{10} = 3$, what is the chance that $N_3 = 1$?

   (h) Given that $N_{10} = 3$, what is the chance that $M_3 = 1$?

   (i) What is the variance of $N_1 + M_1$?

   [19 marks]

5. In a certain computer queuing system, jobs arrive according to a Poisson process with rate 4. There are two servers that process jobs: Server A works at exponential rate 3 and Server B at exponential rate 2. Since Server A is faster than Server B, the system works as follows. When an arriving job finds the system empty, Server A always processes the job. If only one server is free, then an arriving job goes immediately into service with the free server. When both servers are busy, jobs queue in an infinite buffer.

   (a) Model the number of jobs in the system (including those being worked on) as a continuous time Markov chain $(X_t)_{t\geq 0}$ with appropriate state space, and specify its generator.

   (b) Find the stationary distribution of the Markov chain.

   (c) What proportion of time is Server B idle?

   (d) What is the average number of jobs in the system?

   (e) What is the average number of jobs waiting in the queue (that is, in the system but not in service)?

   (f) What is the average amount of time an arriving job waits for service in the queue?

   (g) What is the average amount of time an arriving job is in the system?

   You may find the following formula useful for this problem: for $0 < a < 1$

$$\sum_{j\geq 0} j a^j = \frac{a}{(1-a)^2}.$$

[17 marks]

6. A Markov chain $(X_n)_{n\geq 0}$ on $\{0, 1, 2, \ldots\}$ has transition probabilities given as follows. For $i = 2, 3, 4, \ldots$,

$$p_{i,i+1} = p_{i,i-1} = \frac{1}{4}, \quad p_{i,0} = \frac{1}{2},$$

   and

$$p_{1,2} = p_{1,0} = \frac{1}{2}, \quad p_{0,1} = p_{0,0} = \frac{1}{2}.$$

   Note that the chain is irreducible.

   (a) Define $T(0) = \inf\{n \geq 1 : X_n = 0\}$. For each $j = 0, 1, 2, \ldots$, find $\mathbb{E}[T(0)|X_0 = j]$.

   (b) Show that the chain is positive recurrent and find the long run proportion of time the chain spends at state 0.

[4 marks]

End of Exam

5. In a certain computer queuing system, jobs arrive according to a Poisson process with rate 4. There are two servers that process jobs: Server A works at exponential rate 3 and Server B at exponential rate 2. Since Server A is faster than Server B, the system works as follows. When an arriving job finds the system empty, Server A always processes the job. If only one server is free, then an arriving job goes immediately into service with the free server. When both servers are busy, jobs queue in an infinite buffer.

$\lambda = 4$

$\mu_A = 3 \quad \mu_B = 2$

(a) Model the number of jobs in the system (including those being worked on) as a continuous time Markov chain $(X_t)_{t\geq 0}$ with appropriate state space, and specify its generator.

(b) Find the stationary distribution of the Markov chain.

(c) What proportion of time is Server B idle? $\rightarrow \pi_{00} + \pi_{10} = \frac{75}{113}$

(d) What is the average number of jobs in the system? $\rightarrow L = 0 \cdot \pi_{00} + 1(\pi_{10} + \pi_{01}) + \sum_{i\geq 2} i\pi_i = \frac{20}{113} + \sum_{i\geq 2} i \cdot \left(\frac{4}{5}\right)^{i-2} \cdot \frac{16}{113}$

(e) What is the average number of jobs waiting in the queue (that is, in the system but not in service)?

(f) What is the average amount of time an arriving job waits for service in the queue?

(g) What is the average amount of time an arriving job is in the system?

let $j = i-2$.

$= \frac{16}{113} \sum_{j\geq 0} (j+2) \frac{4}{5}^j$

$= \frac{16}{113} \sum_{j\geq 0} j \left(\frac{4}{5}\right)^j + \frac{32}{113} \sum_{j\geq 0} \left(\frac{4}{5}\right)^j$

$= \frac{16}{113} \times \frac{\frac{4}{5}}{(\frac{1}{5})^2} + \frac{32}{113} \cdot \frac{1}{\frac{1}{5}}$

$= \frac{500}{113}.$

You may find the following formula useful for this problem: for $0 < a < 1$

$$\sum_{j\geq 0} ja^j = \frac{a}{(1-a)^2}.$$

$\Rightarrow \sum_{j\geq 0} j a^j = a \sum_{j\geq 1} j a^{j-1} = a \sum_{j\geq 1} \frac{d}{da}(a^j) = a \frac{d}{da} \sum a^j = a \cdot \frac{d}{da}\left(\frac{1}{1-a}\right) = a \cdot \frac{1}{(1-a)^2} = \frac{a}{(1-a)^2}$

0: idle 1: busy

(a) State space $\{(0,0), (1,0), (0,1), (1,1), 3, 4, \cdots\}$.

$$A = \begin{array}{c|cccccc} & (0,0) & (1,0) & (0,1) & (1,1) & 3 & 4 \; \cdots \\ \hline (0,0) & -\lambda & \lambda & 0 & 0 & 0 & 0 \\ (1,0) & \mu_A & -(\mu_A+\lambda) & 0 & \lambda & 0 & 0 \\ (0,1) & \mu_B & 0 & -(\mu_B+\lambda) & \lambda & 0 & 0 \\ (1,1) & 0 & \mu_B & \mu_A & -(\mu_A+\mu_B+\lambda) & \lambda & 0 \\ 3 & 0 & 0 & 0 & \mu_A+\mu_B & -(\mu_A+\mu_B+\lambda) & \lambda \\ 4 & 0 & 0 & 0 & 0 & \mu_A+\mu_B & -(\mu_A+\mu_B+\lambda) \quad \lambda \end{array}$$

(b) $\underline{\pi} A = 0 \quad \Rightarrow$

$\lambda \pi_{00} = \mu_A \pi_{10} + \mu_B \pi_{01}$

$(\mu_A + \lambda)\pi_{10} = \lambda \pi_{00} + \mu_B \pi_2$

$(\mu_B + \lambda)\pi_{01} = \mu_A \pi_2$

$(\mu_A + \mu_B + \lambda)\pi_2 = \lambda \pi_{10} + \lambda \pi_{01} + (\mu_A + \mu_B)\pi_3.$

$(\mu_A + \mu_B + \lambda)\pi_2 = \lambda \pi_{i-1} + (\mu_A + \mu_B)\pi_{i+1} \qquad i \geq 3$

$\lambda = 4 \qquad \mu_A = 3 \qquad \mu_B = 2$

$$4\pi_{00} = 3\pi_{10} + 2\pi_{01} \Rightarrow 4\pi_{00} = \frac{9}{4}\pi_2 + \pi_2 = \frac{13}{4}\pi_2 \Rightarrow \pi_{00} = \frac{13}{16}\pi_2$$

$$7\pi_{10} = 4\pi_{00} + 2\pi_2 \Rightarrow 7\pi_{10} = 3\pi_{10} + 2\pi_{01} + 2\pi_2 = 3\pi_{10} + 3\pi_2.$$

$$4\pi_{10} = 3\pi_2$$

$$6\pi_{01} = 3\pi_2 \Rightarrow \underline{2\pi_{01} = \pi_2} = \frac{1}{2}\pi_2 \qquad \pi_{10} = \frac{3}{4}\pi_2$$

$$9\pi_2 = 4(\pi_{10} + \pi_{01}) + 5\pi_3.$$

$$\vdots \quad \pi_i = \left(\frac{\lambda}{\mu_A + \mu_B}\right)^{i-2}\pi_2 = \left(\frac{4}{5}\right)^{i-2}\pi_2$$

since $\sum_i \pi_i = 1$

$$= \pi_{00} + \pi_{10} + \pi_{01} + \sum_{i \geq 2} \pi_i = 1$$

$$\frac{13}{16}\pi_2 + \frac{3}{4}\pi_2 + \frac{1}{2}\pi_2 + \frac{\pi_2}{1 - \frac{4}{5}} = 1$$

$$\frac{13}{16}\pi_2 + \frac{12}{16}\pi_2 + \frac{8}{16}\pi_2 + 5\pi_2 = 1$$

$$\frac{113}{16}\pi_2 = 1$$

$$\pi_2 = \frac{16}{113}$$

$$\pi_{00} = \frac{13}{113} \quad \pi_{10} = \frac{12}{113} \quad \pi_{01} = \frac{8}{113} \quad \pi_i = \frac{16}{113}\left(\frac{4}{5}\right)^{i-2}, \ i \geq 2$$

(e) What is the average number of jobs waiting in the queue (that is, in the system but not in service)?

$$L_q = \sum_{i \geq 2} (i-2)\pi_i$$

$$= \sum_{i \geq 2} (i-2) \frac{16}{113}\left(\frac{4}{5}\right)^{i-2}$$

$j = i-2$

$$= \sum_{j \geq 0} j \frac{16}{113}\left(\frac{4}{5}\right)^j$$

$$= \frac{16}{113} \times \sum_{j \geq 0} j\left(\frac{4}{5}\right)^j$$

$$= \frac{16}{113} \times \frac{\frac{4}{5}}{\left(\frac{1}{5}\right)^2}$$

$$= \frac{320}{113}$$

(f) What is the average amount of time an arriving job waits for service in the queue?

use Little's Law

$E(W) = \frac{L_q}{\lambda} = \frac{80}{113}$

(g) What is the average amount of time an arriving job is in the system?

$D = \frac{L}{\lambda} = \frac{125}{113}$