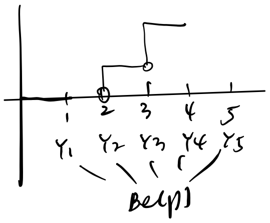
MAST30001 Stochastic Modelling

$\boxed{\text{Tutorial Sheet 5}}$

1. A possum runs from corner to corner along the top of a square fence. Each time he switches corners, he chooses among the two adjacent corners, choosing the corner in the clockwise direction with probability $0 < p < 1$ and the corner in the counter-clockwise direction with probability $1 - p$. Model the possum's movement among the corners of the fence as a Markov chain, analyze its state space (reducibility, periodicity, recurrence, etc), and discuss its long run behaviour.

2. Refer to Tutorial Sheet 3, Problem 3 and now also assume that on any given transition, the spider will not return to the corner it came from on the previous step. Show the sequence of corners occupied by the spider is *not* a Markov chain and suggest a Markov chain model for this new system.

$X_n = \sum_{i=1}^{n} Y_i$



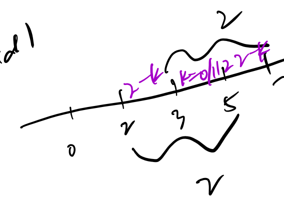
3. (Discrete version of Poisson Process) Let the discrete time Markov chain $(X_n)_{n\geq 0}$ on $\{0, 1, \ldots\}$ have transition probabilities $p_{ii+1} = 1 - p_{ii} = p$ and assume $X_0 = 0$.
   - (a) Draw a picture of a typical trajectory of this process.
   - (b) Show that X_n has the binomial distribution with parameters n and p.
   - (c) Show that for $m < n$, $X_n - X_m$ has the binomial distribution with parameters $n - m$ and p. $X_5 - X_3 \sim Bi(2, p)$
   - (d) Show that $(X_n)_{n\geq 0}$ has the independent increments property: for $0 \leq i < j \leq k < l$, the variables
   $$(X_l - X_k, X_j - X_i)$$
   $(X_5 - X_4, X_3 - X_1)$
   $Y_5 \quad Y_2 + Y_3$

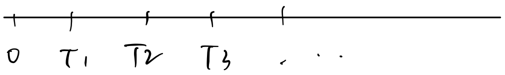
     are independent.
   - (e) Show that the number of steps between "jumps" (times when the chain changes states) has the geometric distribution with parameter p (and started from 1).
   $X_5 > \lim_{i \to 5} X_i = X_4$
   $\uparrow$
   $X_4 + Y_5$
   - (f) Show that given $X_n = 1$, the step number of the first jump is uniform on $\{1, \ldots, n\}$.
   (f) Given $X_5 = 1$
   $P(Y_2 = 1 \mid X_5 = 1)$
   $= \dfrac{P(Y_2=1, X_5=1)}{P(X_5=1)}$
   $= \dfrac{p(1-p)^4}{\binom{5}{1} p(1-p)^4}$
   - (g) More generally, show that given $X_n = k$, the step numbers of the jumps are a uniformly chosen subset of size k from $\{1, \ldots, n\}$.

4. Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ and for each $t \geq 0$, let $X_t = N_{t/\lambda}$. Show that $(X_t)_{t\geq 0}$ is a Poisson process with rate 1.

5. Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ and let $0 < T_1 < T_2 < \cdots$ be the times of "arrivals" or jumps of $(N_t)_{t\geq 0}$. Compute:
   - (a) $P(N_3 \leq 2, N_1 = 1)$, $P(N_1 + N_3 - N_1 \leq 2, N_1 = 1) = P(N_3 - N_1 \leq 1, N_1 = 1) = P(Po(2\lambda) \leq 1) \cdot P(Po(\lambda) = 1)$
   - (b) $P(N_3 \leq 2, N_1 \leq 1)$, $= P(N_3 \leq 2, N_1 = 1) + P(N_3 \leq 2, N_1 = 0) = P(N_3 - N_1 \leq 2, N_1 = 0) = P(Po(2\lambda) \leq 2) P(Po(\lambda) = 0)$
   - (c) $P(N_2 = 2, N_1 = 2, N_{1/2} = 0)$, $= P(N_{\frac{1}{2}} = 0, N_1 - N_{\frac{1}{2}} = 2, N_2 - N_1 = 0)$
   $= P(Po(\frac{\lambda}{2}) = 0) P(Po(\frac{\lambda}{2}) = 2) P(Po(\lambda) = 0)$
   (d) $2-k,\ k = 0,1,2$
   - (d) $P(N_7 - N_3 = 2 \mid N_5 - N_2 = 2)$,
   - (e) the joint distribution function $F(t_1, t_2) = P(T_1 < t_1, T_2 < t_2)$, sol. $t_2 > t_1$ $(t_2 > t_1)$
   - (f) the joint density of (T_1, T_2),
   $= \sum_{k=0}^{2} P(N_3 - N_2 = 2-k, N_5 - N_3 = k, N_7 - N_5 = 2-k)$
   - (g) the distribution of $T_1 | T_2 = t_2$. uniform $(0, t_2)$

$d = \dfrac{P(N_7 - N_3 = 2, N_5 - N_2 = 2)}{P(N_5 - N_2 = 2)}$
$\downarrow$
$(Po(3\lambda) = 2)$



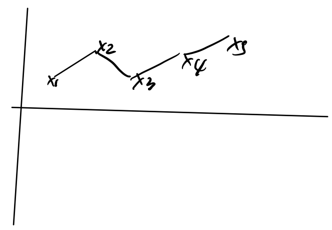


Stationary distribution

① limit $\forall \pi_0$ $\quad \pi_0 P^n \longrightarrow \pi$ (stablize)

② stationary $\quad \pi P = \pi$ (theoretically true)

③ ergodic $\quad \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(x_i = j) \longrightarrow \pi_j = P(\underset{\sim \pi}{X} = j)$



ergodic theory

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \longrightarrow \underset{\text{stationa}}{E(f(x))}$$

let $f = \mathbb{1}$ $\qquad E(\mathbb{1}(X=j)) = P(X=j)$

$= \pi_j$

$$\frac{1}{n}\sum_{i=1}^{n} f(x) \longrightarrow E(F)$$

Time average $\longrightarrow$ space average

$\downarrow$

stationary state

$X_i \sim$ Bernoulli (p) $\xrightarrow{X_1, \dots X_n \text{ (iid)}}$ $Bi(n, p)$

$\downarrow$ # trials until 1st success $\qquad \downarrow$ $n\uparrow$ $p\downarrow$

$Geo(p)$ $\qquad Po(np)$

$n \to \infty$ Normal (CLT

$\downarrow$ waiting time until 1st rare event

$Exp(\lambda)$

$\downarrow$ Σ iid $(Geo(p))$ $\qquad \downarrow$ Σ iid $Exp(\lambda)$

Negative Binominal (discrete of gamma $\qquad$ $Gamma(n, \lambda)$

$$P(Po(\lambda)=i) = e^{-\lambda} \qquad \lambda e^{-\lambda} \qquad \frac{\lambda^2 e^{-\lambda}}{2}$$

$$i=0 \qquad i=1 \qquad i=2$$