The University of Melbourne

Semester 1 Assessment — June 2009

Department of Mathematics and Statistics

620-301 Stochastic Modelling

Reading Time: 15 Minutes

Writing Time: 3 Hours

This paper has 5 pages

Authorised materials:
Hand-held electronic calculators may be used.

Instructions to Invigilators:
Students require script books only.
No textbooks or lecture notes are allowed in the examination room.
This paper is to remain in the examination room.

Instructions to Students:
This paper has **seven** (7) questions.
Attempt as many questions, or parts of questions, as you can.
Questions carry marks as shown in the brackets after the question statement.
The total of marks available for this examination is 100.
Working and/or reasoning must be given to obtain full credit.
This paper must be handed in with your script books.

This paper is to be lodged with the Baillieu Library

1. Let X_1, X_2, ... be independent and identically distributed integer valued random variables with distribution $\mathbb{P}(X_1 = i) = p_i$ for $i \geq 0$. We say that a record occurs at time $n \geq 1$ if $X_n > \max(X_0, X_1, ..., X_{n-1})$, where $X_0 := -\infty$, and if a record does occur at time n we call X_n the record value. Let R_i denote the ith record value, $i \geq 1$. More precisely, X_1 is the first record, so $R_1 = X_1$, the second record occurs at $T_2 = \min\{n > 1 : \ X_n > X_1\}$ and $R_2 = X_{T_2}$, and for $k \geq 3$, the kth record occurs at $T_k = \min\{n > T_{k-1} : \ X_n > R_{k-1}\}$ and $R_k = X_{T_k}$. Argue that $\{R_i, i \geq 1\}$ is a Markov chain and compute its transition probabilities.

[6 marks]

2. On any given day John is either cheerful (C), so-so (S), or glum (G). If he is cheerful today, then he will be C, S and G tomorrow with respective probabilities 0.5, 0.4 and 0.1. If he is feeling so-so today, then he will be C, S, G tomorrow with probabilities 0.3, 0.4 and 0.3. If he is glum today, then he will be C, S and G with probabilities 0.2, 0.3 and 0.5. Let X_n denote John's mood on the nth day, then $\{X_n, \ n \geq 0\}$ is a three-state Markov chain (state 1=C, state 2=S and state 3=G).

   (a) Find the transition matrix and draw a transition diagram for the Markov chain (MC).

   (b) Find the probability that this week John will be cheerful on Tuesday and Thursday and glum on Saturday given he was so-so on Monday.

   (c) Calculate the probability that this week John is cheerful on Tuesday given he is cheerful on Monday and Wednesday.

   (d) Classify the states of the MC.

   (e) What are conditions for a finite MC to be ergodic? Use the conditions to prove or disprove that the MC $\{X_n\}$ is ergodic. If it is ergodic, compute its stationary distribution. If it is not ergodic, discuss the long run behaviour of the distributions of the MC $\{X_n\}$.

   (f) Assume that John is so-so on day $n = 0$. Let

   $$S_2 = \min\{n > 0 : \ X_n = 1\} \quad \text{and} \quad S_3 = \min\{n > S_2 : \ X_n = 1\}.$$

   That is, S_2 is the waiting time until his first cheerful day and S_3 is the waiting time until his second cheerful day. Prove that S_2 and $S_3 - S_2$ are independent random variables.

[20 marks]

3. Suppose that animals cross at a stretch of road in Tasmania according to a Poisson process with rate 5 per hour and it is known that 0.1% of them are killed by the traffic when they cross the road. Let $N(t)$ be the number of animals that crossed the road in the first t hours.

   (a) What is the probability that two animals will cross the road during a half hour period?

   (b) Compute $\mathbb{P}(N_2 = 7|N_1 = 4)$ and $\mathbb{P}(N_1 = 4|N_2 = 7)$.

   (c) What is the expected time of the tenth animal crossing the road?

   (d) What is the expected time of the first animal being killed at the road?

   (e) What is the probability that no animals will be killed in one day?

   (f) Given that one animal was killed in one day, what is the expected number of all animals that tried to cross the road on that day?

   (g) Given that 50 animals tried to cross the road in the last 8 hours, what is the probability that none of them was killed?

   (h) A tourist travelled along the road in the morning at 10 am and the road was free from dead animals, but when he came back at 2 pm on the same day, he found one dead animal. What is the expected time that the animal was killed?

[20 marks]

4. Martian "amoebae" have the following properties: they live for exponentially long time, with a mean lifetime of 1 hour. At the end of its life, an amoeba either simply dies with probability 0.4 or splits into two amoebae with probability 0.6. Let X_0 be the number of Martian amoebae brought to the Earth by a successful Martian expedition, and X_t the number of amoebae alive in a culture dish at time t (hours after the expedition's arrival). Assume that all amoebae behave independently of each other.

   (a) Find the generator $A = (a_{ij})_{i,j\geq 0}$ of the process $\{X_t,\ t \geq 0\}$ and sketch the transition diagram.

   (b) For a small positive number $h > 0$, use the fact that $p_{ij}^{(h)} \approx p_{ij}^{(0)} + a_{ij}h$, $i, j \geq 0$, to approximate $\mathbb{E}X_{t+h} = \mathbb{E}[\mathbb{E}(X_{t+h}|X_t)]$ and then derive a differential equation for $\frac{d\mathbb{E}X_t}{dt}$.

   (c) Let $X_0 = 100$, find the expected number $\mathbb{E}X_t$ of amoebae alive at time t?

[10 marks]

5. Customers arrive at a bank according to a Poisson process with the rate $\lambda = 1$ customers per minute. The service times are independent and identically distributed exponential random variables with mean 90 seconds. The bank follows this policy: when there are fewer than three customers in the bank, only one teller is active; for three to five customers, there are two tellers, and beyond five customers there are three tellers. When a customer arrives and sees the teller(s) busy, the customer joins the queue; otherwise his/her service starts immediately. We model the queuing system by setting a continuous time Markov process (MP) $X_t =$ number of customers in the bank at time t and assume that the first-in-first-out rule applies.

   (a) Give the state space of the process and find the generator of the MP.

   (b) Explain why the MP is ergodic and calculate its steady-state probability distribution.

   (c) Find the expected number of customers in the bank in the steady-state regime.

   (d) What is the long-run proportion of the time that all tellers are busy?

   (e) Calculate the expected delay time for a newly arrived customer at the bank in the steady-state regime.

   (f) Compute the expected waiting time for a newly arrived customer at the bank in the steady-state regime.

   (g) Calculate, in the steady-state regime, the expected number of customers in the queue in the bank.

[18 marks]

6. The following is a sample of five random numbers from the uniform distribution on $(0, 1)$:

$$0.3526,\ 0.1032,\ 0.3883,\ 0.9369,\ 0.5303.$$

   (a) Describe the Box-Muller algorithm and use the above random sample to simulate four independent normally distributed random variables with the common mean $\mu = 1$ and standard deviation $\sigma = 3$.

   (b) Use the same sample to simulate an initial segment of the trajectory of a birth-and-death process $\{X_t,\ t \geq 0\}$ starting at $X_0 = 2$ and having birth rates $\lambda_j = j+1$, $j = 0,\ 1,\ 2,\ \ldots$ and death rates $\mu_j = 1$, $j = 1,\ 2,\ \ldots$ You have enough random numbers to simulate its behaviour till the time of the third transition.

[10 marks]

7. The lifetime of a machine is a random variable with the triangular probability density function $f(x) = 2x$ for $0 < x < 1$. Initially, the machine is new. If it fails, we repair it in a fixed time $r = 0.1$. After repairs, the machine will be as good as a new one and immediately starts working again.

   (a) Define an appropriate renewal process $\{N_t, t \geq 0\}$ to model the situation and find the cumulative distribution function of the random time between successive renewals. Calculate the mean and variance of the random time between successive renewals.

   (b) Determine an asymptotic expression for N_t when t is large. Give an interval such that with approximate probability 90% the interval captures the values of N_{1000}.

   (c) Give an asymptotic distribution for the residual time Y_t of the renewal process at t when t is large.

   (d) Give an asymptotic expression for the joint distribution of the residual time and the age of the renewal process at time t when t is large.

   (e) Explain why $N_t + 1$ is a stopping time in terms of the inter-renewal times $\{\tau_1, \tau_2, ...\}$ while N_t is not a stopping time.

   (f) Give the definition of the renewal function and use Wald's identity to compute $\mathbb{E}\sum_{i=1}^{N_t+1} \tau_i$ in terms of the renewal function.

(*Hint*: if $Z \sim N(0,1)$, then $\mathbb{P}(Z < 1.645) \approx 0.95$ and $\mathbb{P}(Z < 1.96) \approx 0.975$.)

[16 marks]

Total marks = 100

End of Questions

Useful formulae

(a) $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$ for $|x| < 1$.

(b) $\sum_{i=1}^{\infty} ix^{i-1} = (1-x)^{-2}$ for $|x| < 1$.

(c) For a birth-death process with birth rates $\{\lambda_i : i \geq 0\}$ and death rates $\{\mu_i : i \geq 1\}$, let $K_i = \frac{\lambda_0\lambda_1\cdots\lambda_{i-1}}{\mu_1\mu_2\cdots\mu_i}$ for $i \geq 1$. If the stationary distribution $\{\pi_i : i \geq 0\}$ exists, then $\pi_i = K_i\pi_0$, $i \geq 1$.

(d) If $\{N_t : t \geq 0\}$ is a renewal process with inter-renewal times having mean μ and variance $\sigma^2 > 0$, then

$$\frac{N_t - \frac{t}{\mu}}{\sqrt{t\sigma^2/\mu^3}} \xrightarrow{d} N(0,1) \quad \text{as } t \to \infty$$

End of Examination