PHYC10003 Physics 1

Lecture 16: Angular momentum

Vector addition of angular momenta

Last lecture

- Rotational motion and force
- Torque
- Rotational Work
- Rolling- translation and rolling
- Applications to automobiles

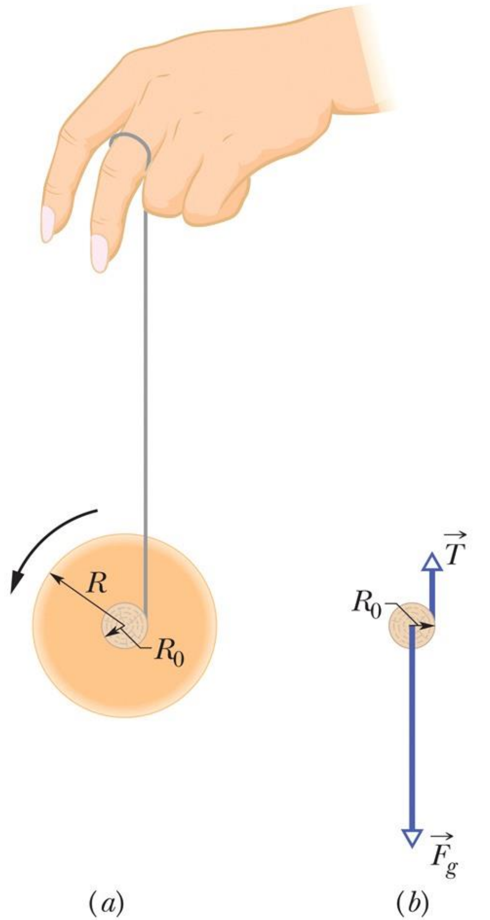
Yo-yo - rolling with friction

Compare the acceleration of a frictionless particle with that of an object with non-zero moment of inertia

$$a_{cm} = \frac{a_{particle}}{1+c} \quad \text{for } c > 0$$

We can apply this to the case of the yo-yo, provided that we are careful about identifying

1. The force that keeps the axle of the yo-yo rolling along the string
2. The effective value of c that is relevant to this problem.



Rotation and the yo-yo

- As a yo-yo moves down a string, it loses potential energy *mgh* but gains rotational and translational kinetic energy
- To find the linear acceleration of a yo-yo accelerating down its string:
  1. Rolls down a "ramp" of angle 90°
  2. Rolls on an axle instead of its outer surface
  3. Slowed by tension *T* rather than friction

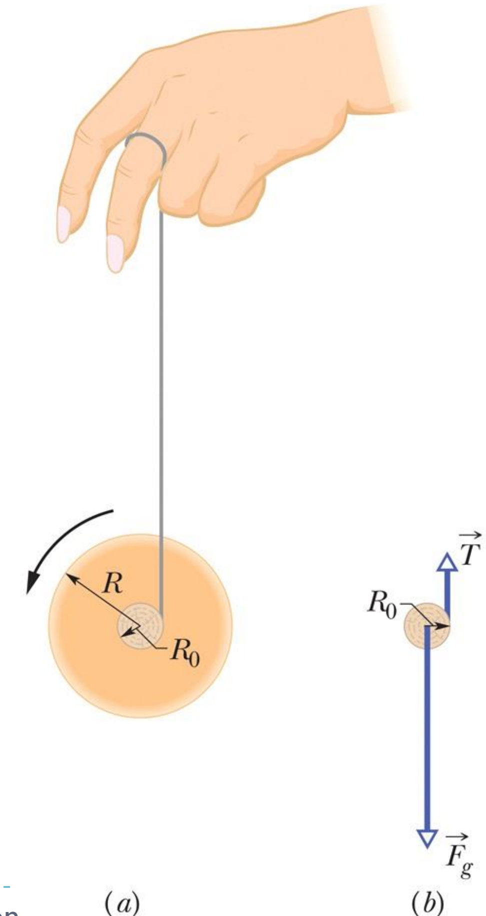
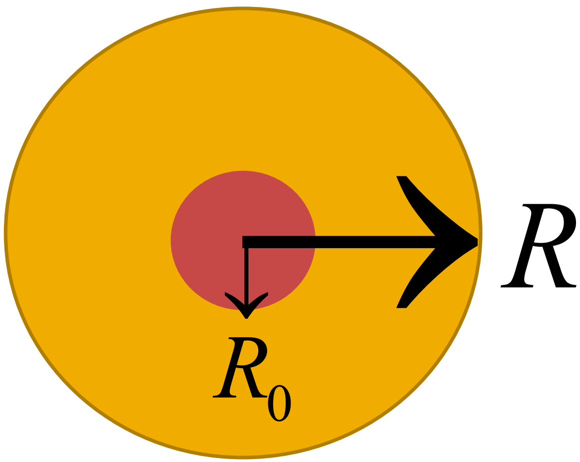


Figure 11-9

Yo-yo - moment of inertia



- Assume that the yo-yo consists of two disks of total mass M and radius R, and a central cylinder of negligible mass and radius R_0.
- The moment of inertia of a disk is $MR^2/2$, It doesn't matter that there are two disks, because only the total mass counts.
- In terms of R_0, the moment of inertia may be written as

$$I = \frac{1}{2}MR^2 = \frac{1}{2}M\left(\frac{R}{R_0}\right)^2 R_0^2$$

$$= cMR_0^2 \quad \text{where } c = \frac{1}{2}\left(\frac{R}{R_0}\right)^2$$

Acceleration of the yo-yo

- Replacing the values in 11-10 leads us to:

$$a_{\text{com}} = -\frac{g}{1 + I_{\text{com}}/MR_0^2}, \quad \textbf{Eq. (11-13)}$$

Example Calculate the acceleration of the yo-yo

- M = 150 grams, R_0 = 3 mm, I_{com} = $MR^2/2$ = 3E-5 kg m^2
- Therefore a_{com} = -9.8 m/s^2 / (1 + 3E-5 / (0.15 * 0.003^2))
  = - .4 m/s^2

Torque – generalization

- Previously, torque was defined only for a rotating body and a fixed axis
- Now we redefine it for an individual particle that moves along any path relative to a fixed point
- The path need not be a circle; torque is now a vector
- Direction determined with right-hand-rule

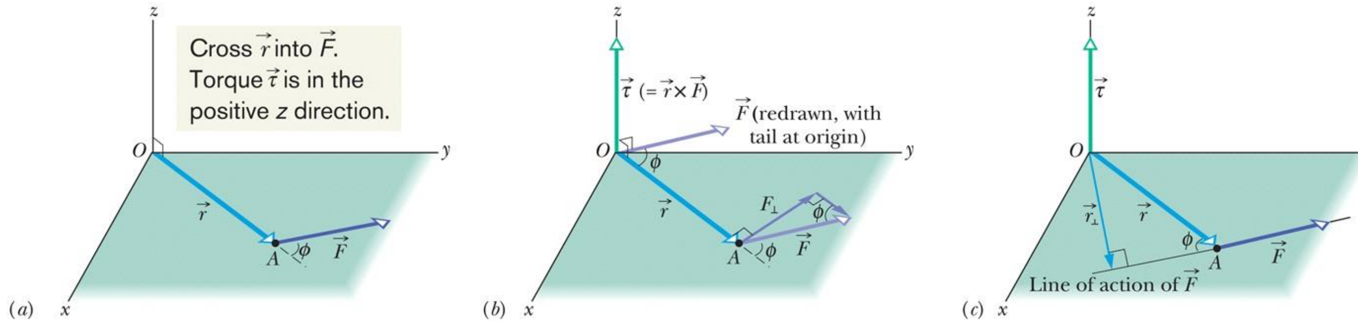


Figure 11-10

Torque - generalization

- The general equation for torque is:

$$\vec{\tau} = \vec{r} \times \vec{F}$$ **Eq. (11-14)**

- We can also write the magnitude as:

$$\tau = rF \sin \phi,$$ **Eq. (11-15)**

- Or, using the perpendicular component of force or the moment arm of *F*:

$$\tau = rF_{\perp},$$ **Eq. (11-16)**

$$\tau = r_{\perp}F,$$ **Eq. (11-17)**

Calculation of the vector product

Example: $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j}$

$$\mathbf{B} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

$$(\mathbf{A}\times\mathbf{B}) = 4(\mathbf{i}\times\mathbf{i}) + 6(\mathbf{i}\times\mathbf{j}) + 4(\mathbf{i}\times\mathbf{k}) + 6(\mathbf{j}\times\mathbf{i}) + 9(\mathbf{j}\times\mathbf{j}) + 6(\mathbf{j}\times\mathbf{k})$$

Unit vector properties

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$$

$$\mathbf{i}\times\mathbf{j} = -\mathbf{j}\times\mathbf{i} = \mathbf{k} \text{ with cyclic permutation}$$

Hence

$$\mathbf{A}\times\mathbf{B} = 6\mathbf{i} - 4\mathbf{j}$$

Torque-vector product (general result)

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= (\mathrm{yF}_z - zF_y)\mathbf{i} + (zF_x - xF_z)\mathbf{j} + (xF_y - yF_x)\mathbf{k}$$

- Can always calculate the torque directly from the components of **r** (*x,y,z*) and the components of **F** (F_x, F_y, F_z), in terms of orthogonal unit vectors **i**, **j**, **k**.
- The torque is always a vector that is perpendicular to both r and F, with a direction determined by the right-hand rule.

Angular momentum

- Here we investigate the angular counterpart to linear momentum
- We write:

$$\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$ **Eq. (11-18)**

- Note that the particle need not rotate around *O* to have angular momentum around it
- The unit of angular momentum is kg m^2/s, or J s

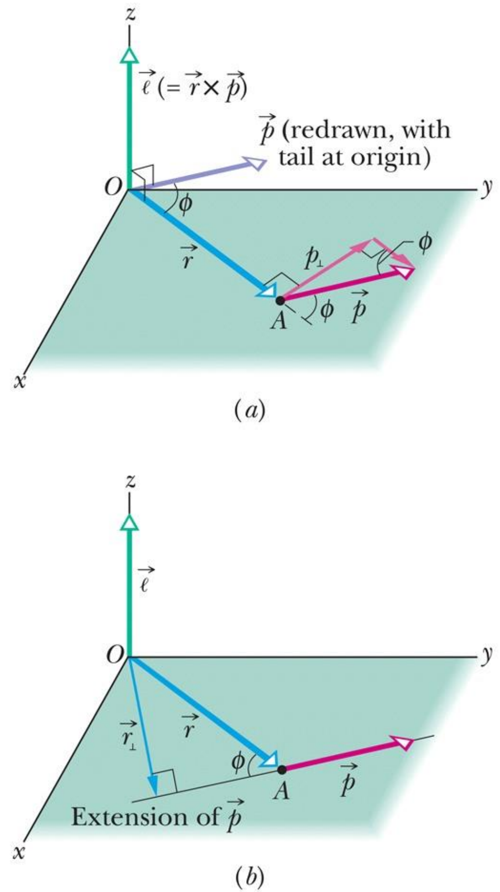


Figure 11-12

Angular momentum – magnitude and direction

- To find the direction of angular momentum, use the right-hand rule to relate *r* and *v* to the result
- To find the magnitude, use the equation for the magnitude of a cross product:

$$\ell = rmv \sin \phi, \qquad \text{Eq. (11-19)}$$

- Which can also be written as:

$$\ell = rp_{\perp} = rmv_{\perp}, \qquad \text{Eq. (11-20)}$$

$$\ell = r_{\perp}p = r_{\perp}mv, \qquad \text{Eq. (11-21)}$$

Angular momentum

- Angular momentum has meaning only with respect to a specified origin
- It is always perpendicular to the plane formed by the position and linear momentum vectors

Angular momentum – Newton's second law

- We rewrite Newton's second law as:

$$\vec{\tau}_{\text{net}} = \frac{d\vec{\ell}}{dt} \quad \text{(single particle).}$$

Eq. (11-23)

- The torque and the angular momentum must be defined with respect to the same point (usually the origin)

- Note the similarity to the linear form:

$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \quad \text{(single particle)}$$

Eq. (11-22)

Addition of angular momenta

- We sum the angular momenta of the particles to find the angular momentum of a system of particles:

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \vec{\ell}_3 + \cdots + \vec{\ell}_n = \sum_{i=1}^{n} \vec{\ell}_i. \quad \textbf{Eq. (11-26)}$$

- The rate of change of the net angular momentum is:

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \vec{\tau}_{\text{net},i}. \quad \textbf{Eq. (11-28)}$$

- In other words, the net torque is defined by this change:

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} \quad \text{(system of particles)}, \quad \textbf{Eq. (11-29)}$$

Torque and angular momentum

The (vector) sum of all the torques acting on a particle is equal to the time rate of change of the angular momentum of that particle.

The net external torque $\vec{\tau}_{net}$ acting on a system of particles is equal to the time rate of change of the system's total angular momentum $\vec{L}$.

Torque, angular momentum, inertia

- Note that the torque and angular momentum must be measured relative to the same origin
- If the center of mass is accelerating, then that origin *must* be the center of mass
- We can find the angular momentum of a rigid body through summation:

$$L_z = \sum_{i=1}^{n} \ell_{iz} = \sum_{i=1}^{n} \Delta m_i \, v_i r_{\perp i} = \sum_{i=1}^{n} \Delta m_i (\omega r_{\perp i}) r_{\perp i}$$

$$= \omega \left(\sum_{i=1}^{n} \Delta m_i \, r_{\perp i}^2 \right).$$

Eq. (11-30)

- The sum is the rotational inertia *I* of the body

Analogies - rigid bodies

- Therefore this simplifies to:

$$L = I\omega \qquad \text{(rigid body, fixed axis).}$$

Eq. (11-31)

Table 11-1

Table 11-1 More Corresponding Variables and Relations for Translational and Rotational Motion[a]

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}\ (= \vec{r} \times \vec{F})$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell}\ (= \vec{r} \times \vec{p})$
Linear momentum[b]	$\vec{P}\ (= \Sigma \vec{p}_i)$	Angular momentum[b]	$\vec{L}\ (= \Sigma \vec{\ell}_i)$
Linear momentum[b]	$\vec{P} = M\vec{v}_{com}$	Angular momentum[c]	$L = I\omega$
Newton's second law[b]	$\vec{F}_{net} = \dfrac{d\vec{P}}{dt}$	Newton's second law[b]	$\vec{\tau}_{net} = \dfrac{d\vec{L}}{dt}$
Conservation law[d]	$\vec{P}$ = a constant	Conservation law[d]	$\vec{L}$ = a constant

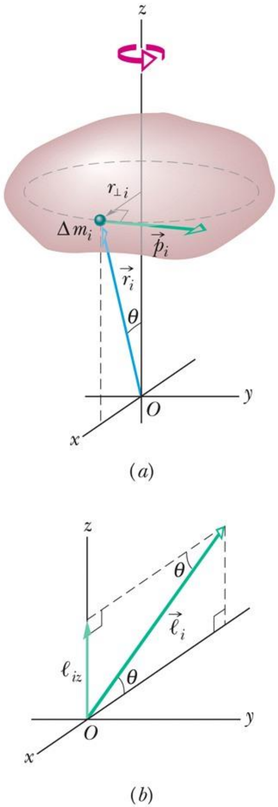


Figure 11-15

Summary

Rolling Bodies

$$v_{com} = \omega R$$ **Eq. (11-2)**

$$K = \tfrac{1}{2} I_{com}\omega^2 + \tfrac{1}{2} M v_{com}^2.$$ **Eq. (11-5)**

$$a_{com} = \alpha R$$ **Eq. (11-6)**

Torque as a Vector

- Direction given by the right-hand rule

$$\vec{\tau} = \vec{r} \times \vec{F}$$ **Eq. (11-14)**

Angular Momentum of a Particle

$$\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

Eq. (11-18)

Newton's Second Law in Angular Form

$$\vec{\tau}_{net} = \frac{d\vec{\ell}}{dt}$$ **Eq. (11-23)**