MAST30001 Stochastic Modelling

Tutorial Sheet 5

1. A possum runs from corner to corner along the top of a square fence. Each time he switches corners, he chooses among the two adjacent corners, choosing the corner in the clockwise direction with probability $0 < p < 1$ and the corner in the counter-clockwise direction with probability $1 - p$. Model the possum's movement among the corners of the fence as a Markov chain, analyze its state space (reducibility, periodicity, recurrence, etc), and discuss its long run behaviour.

2. Refer to Tutorial Sheet 3, Problem 3 and now also assume that on any given transition, the spider will not return to the corner it came from on the previous step. Show the sequence of corners occupied by the spider is *not* a Markov chain and suggest a Markov chain model for this new system.

3. (Discrete version of Poisson Process) Let the discrete time Markov chain $(X_n)_{n\geq 0}$ on $\{0, 1, \ldots\}$ have transition probabilities $p_{ii+1} = 1 - p_{ii} = p$ and assume $X_0 = 0$.
   (a) Draw a picture of a typical trajectory of this process.
   (b) Show that X_n has the binomial distribution with parameters n and p.
   (c) Show that for $m < n$, $X_n - X_m$ has the binomial distribution with parameters $n - m$ and p.
   (d) Show that $(X_n)_{n\geq 0}$ has the independent increments property: for $0 \leq i < j \leq k < l$, the variables
   $$(X_l - X_k, X_j - X_i)$$
   are independent.
   (e) Show that the number of steps between "jumps" (times when the chain changes states) has the geometric distribution with parameter p (and started from 1).
   (f) Show that given $X_n = 1$, the step number of the first jump is uniform on $\{1, \ldots, n\}$.
   (g) More generally, show that given $X_n = k$, the step numbers of the jumps are a uniformly chosen subset of size k from $\{1, \ldots, n\}$.

4. Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ and for each $t \geq 0$, let $X_t = N_{t/\lambda}$. Show that $(X_t)_{t\geq 0}$ is a Poisson process with rate 1.

5. Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ and let $0 < T_1 < T_2 < \cdots$ be the times of "arrivals" or jumps of $(N_t)_{t\geq 0}$. Compute:
   (a) $P(N_3 \leq 2, N_1 = 1)$,
   (b) $P(N_3 \leq 2, N_1 \leq 1)$,
   (c) $P(N_2 = 2, N_1 = 2, N_{1/2} = 0)$,
   (d) $P(N_7 - N_3 = 2 | N_5 - N_2 = 2)$,
   (e) the joint distribution function $F(t_1, t_2) = P(T_1 < t_1, T_2 < t_2)$,
   (f) the joint density of (T_1, T_2),
   (g) the distribution of $T_1 | T_2 = t_2$.

3. (Discrete version of Poisson Process) Let the discrete time Markov chain $(X_n)_{n\geq 0}$ on $\{0, 1, \ldots\}$ have transition probabilities $p_{ii+1} = 1 - p_{ii} = p$ and assume $X_0 = 0$.

   (a) Draw a picture of a typical trajectory of this process.

   (b) Show that X_n has the binomial distribution with parameters n and p.

   (c) Show that for $m < n$, $X_n - X_m$ has the binomial distribution with parameters $n - m$ and p.

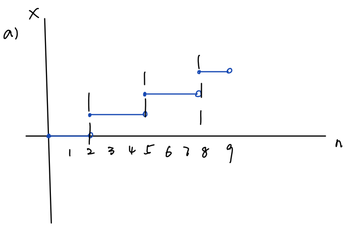
   (d) Show that $(X_n)_{n\geq 0}$ has the independent increments property: for $0 \leq i < j \leq k < l$, the variables
   $$(X_l - X_k, X_j - X_i)$$
   are independent.

   (e) Show that the number of steps between "jumps" (times when the chain changes states) has the geometric distribution with parameter p (and started from 1).

   (f) Show that given $X_n = 1$, the step number of the first jump is uniform on $\{1, \ldots, n\}$.

   (g) More generally, show that given $X_n = k$, the step numbers of the jumps are a uniformly chosen subset of size k from $\{1, \ldots, n\}$.

(a)



(b). X_n = # times of jumps during $[0, n]$

with jumps p, no jump $1-p$.

whether it jumps or not doesn't depend on other time point.

Let event A_n be "jump" at time n

A_n iid $Ber(p)$.

$X_n = \sum_{i=0}^{n} A_i \overset{\text{sum of iid Ber(p)}}{=} Bi(n, p)$.

(c) $X_n - X_m$ = # times of jumps during $[n, m]$

$X_n - X_m = \sum_{i=m}^{n} A_i = Bi(n-m, p)$. if $m < n$.

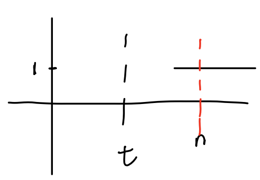
(d). since for each time point $i \in \{0, 1\}$

whether jump or not just depend on itself (not other points)

$$\left.\begin{array}{l} X_l - X_k \sim Bi(l-k, p) \\ X_j - X_i \sim Bi(j-i, p) \end{array}\right\} \Rightarrow \text{independent}$$

(e). $\underset{\text{exact } j \text{ step between jump}}{P(T=j)} = (1-p)^j p^j \sim Geo(p)$.

(f) √.



suppose first jump at time $t \in \{1, \cdots n\}$

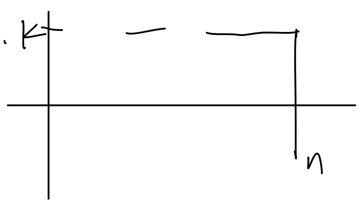
$P(T_1 = t) = (1-p)^{t-1} p (1-p)^{n-t}$

$= (1-p)^{n-1} p$.

$P(T_1 = t \mid X_n = 1) = \dfrac{P(T_1 = t, X_n = 1)}{P(X_n = 1)} = \dfrac{(1-p)^{n-1} p}{\binom{n}{1} p (1-p)^{n-1}}$

$X_n \sim Bi(n, p)$ $= \dfrac{1}{n}$

(g).



$P(T_1 = t_1, T_2 = t_2, \cdots T_k = t_k \mid X_n = k)$

$$= \frac{P(T_1=t_1, \cdots T_k=t_k, X_n=k)}{P(X_n=k)}.$$

$$= \cancel{\frac{(1-p)^{t_1-1}\, p\,(1-p)^{t_2-t_1}\, p \cdots (1-p)^{t_k-t_{k-1}}\, p}{\binom{n}{k} p^k (1-p)^{n-k}}}$$

$$= \frac{1}{\binom{n}{k}}$$

4. Let $\boxed{(N_t)_{t\geq 0} \text{ be a Poisson process}}$ with rate λ and for each $t \geq 0$, let $X_t = N_{t/\lambda}$. Show that $(X_t)_{t\geq 0}$ is a Poisson process with rate 1.

$N_t \sim PProc(\lambda t)$ $\quad X_t = N_{t/\lambda} \sim PProc(t)$

to prove a Poisson process, need to prove ① independent increment

② $N_{t/\lambda}$ is Poisson with rate t.

for ① since N_t is a Poisson process

so with $0 \leq s_1 < t_1 \leq s_2 < t_2 < \cdots < t_k$

$N_{t_1} - N_{s_1}, \cdots, N_{t_k} - N_{s_k}$ are independent variables.

so with $0 \leq s_1\lambda < t_1\lambda \leq s_2\lambda < t_2\lambda < \cdots < t_k\lambda$.

$N_{t_1\lambda/\lambda} - N_{s_1\lambda/\lambda}, \cdots, N_{t_k\lambda/\lambda} - N_{s_k\lambda/\lambda}$ are independent variable

so $X_{t_1} - X_{s_1}, \cdots X_{t_k} - X_{s_k}$ are independent variables.

for ② . for each $t \geq 0$ $\quad N_t = Po(\lambda t). = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$

$$X_t = N_{t/\lambda} = \frac{e^{-\lambda(t/\lambda)}(\lambda t/\lambda)^k}{k!}$$

$$= \frac{e^{-t} t^k}{k!} \sim Po(t).$$

so $(X_t)_{t\geq 0}$ is a Poisson process with rate 1

5. Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ and let $0 < T_1 < T_2 < \cdots$ be the times of "arrivals" or jumps of $(N_t)_{t\geq 0}$. Compute:

(a) $P(N_3 \leq 2, N_1 = 1)$,

(b) $P(N_3 \leq 2, N_1 \leq 1)$,

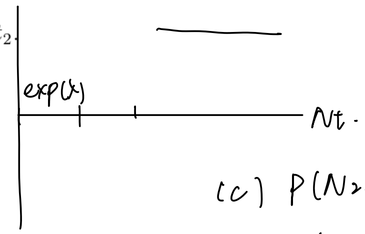
(c) $P(N_2 = 2, N_1 = 2, N_{1/2} = 0)$,

(d) $P(N_7 - N_3 = 2 | N_5 - N_2 = 2)$,

(e) the joint distribution function $F(t_1, t_2) = P(T_1 < t_1, T_2 < t_2)$,

(f) the joint density of (T_1, T_2),

(g) the distribution of $T_1 | T_2 = t_2$.



(a). $P(N_3 \leq 2, N_1 = 1)$

$= P(N_3 - N_1 \leq 1, N_1 = 1)$

independent increment

$= P(N_3 - N_1 \leq 1)\, P(N_1 = 1)$

$= P(Poi(2\lambda) \leq 1) \cdot P(Poi(\lambda) = 1).$

(c) $P(N_2 = 2, N_1 = 2, N_{1/2} = 0).$

$= P(N_2 - N_1 = 0, N_1 - N_{1/2} = 2, N_{1/2} = 0)$

$= P(N_2 - N_1 = 0) \cdot P(N_1 - N_{1/2} = 2) \cdot P(N_{1/2} = 0)$

$= P(Poi(\lambda) = 0)\, P(Poi(\frac{1}{2}\lambda) = 2)$

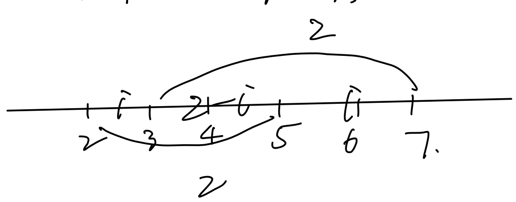
$P(Poi(\frac{\lambda}{2}) = 0)$

(b). $P(N_3 \leq 2, N_1 \leq 1)$

$= P(N_3 \leq 2, N_1 = 1) + P(N_3 \leq 2, N_1 = 0)$

$= P(N_3 - N_1 \leq 1, N_1 = 1) + P(N_3 - N_1 \leq 2, N_1 = 0)$

$= P(Poi(2\lambda) \leq 1) \cdot P(Poi(\lambda) = 1) + P(Poi(2\lambda) \leq 2) \cdot P(Poi(\lambda) = 0).$



(d) $P(N_7 - N_3 = 2 | N_5 - N_2 = 2)$

$= \dfrac{P(N_7 - N_3 = 2, N_5 - N_2 = 2)}{P(N_5 - N_2 = 2)}$

$= \dfrac{\sum_i P(N_3 - N_2 = i, N_5 - N_2 = 2 - i, N_7 - N_5 = i)}{P(N_5 - N_2 = 2)}$

$\overset{\text{indep}}{=} \dfrac{\sum_i P(N_3 - N_2 = i) \cdot P(N_5 - N_2 = 2 - i)\, P(N_7 - N_5 = i)}{P(N_5 - N_2 = 2).}$

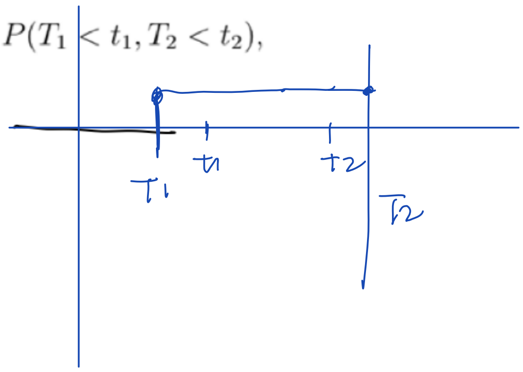
$= \dfrac{\sum_{i=0}^{2} P(Poi(\lambda) = i)\, P(Poi(3\lambda) = 2 - i)\, P(Poi(2\lambda) = i)}{\ }$

$P(Poi(3\lambda)=2)$

(e) the joint distribution function $F(t_1,t_2)=P(T_1<t_1,T_2<t_2)$,

(f) the joint density of (T_1,T_2),

(g) the distribution of $T_1|T_2=t_2$



(e) $F(t_1,t_2)=P(T_1<t_1,\ T_2<t_2)$

$$
\begin{aligned}
\Rightarrow\quad & P(T_1<t_1,\ T_2>t_2)\\
= \quad & P(N_{t_1}=1,\ N_{t_2}-N_{t_1}=0)\\
= \quad & P(N_{t_1}=1)\,P(N_{t_2}-N_{t_1}=0)\\
= \quad & P\big(Poi(\lambda t_1)=1\big)\ P\big(Poi[\lambda(t_2-t_1)]=0\big)\\
= \quad & (e^{-\lambda t_1}\lambda t_1)\ e^{-\lambda(t_2-t_1)}\\
= \quad & e^{-\lambda t_2}\lambda t_1
\end{aligned}
$$

$$
\begin{aligned}
F(t_1,t_2) &= P(T_1<t_1) - P(T_1<t_1,\ T_2<t_2)\\
&= P(N_{t_1}\geq 1) - e^{-\lambda t_2}\lambda t_1\\
&= 1-P(N_{t_1}=0) - e^{-\lambda t_2}\lambda t_1\\
&= 1 - e^{-\lambda t_1} - e^{-\lambda t_2}\lambda t_1
\end{aligned}
$$

(f). $$
\begin{aligned}
f_{(T_1,T_2)} &= \frac{d}{dt_1 dt_2}\left(1-e^{-\lambda t_1}-e^{-\lambda t_2}\lambda t_1\right)\\
&= \frac{d}{dt_1}\left(\lambda^2 t_1 e^{-\lambda t_2}\right)\\
&= \lambda^2 e^{-\lambda t_2}
\end{aligned}
$$

(g) $$f_{T_1|T_2} = P(T_1=t_1\,|\,T_2=t_2) = \frac{f(T_1,T_2)}{\underbrace{f(T_2)}_{gamma(2,\lambda)}} = \frac{\cancel{\lambda^2} e^{\cancel{-\lambda t_2}}}{\cancel{\lambda^2} t_2\, \cancel{e^{-\lambda t_2}}} = \frac{1}{t_2}$$

$$\therefore f_{T_1|T_2} = \frac{1}{T_2} \sim Unif(0,\ T_2).$$