EM-mixture component

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$K=2 \quad \theta = (\pi_1, \mu_1, \mu_2, \sigma_1^2, \sigma_2^2)$

$$Q(\theta,\theta^\circ) = E_{Z|X,\theta^\circ}\left[\log P(X_1,\cdots X_n, Z_1 \cdots Z_n \mid \theta)\right]$$

$$= \sum_{i=1}^{n}\Big[ P(Z_i=1 \mid X,\theta^\circ)\Big(-\log\sqrt{2\pi\sigma_1^2} - \frac{1}{2}\frac{(x_i-\mu_1)^2}{\sigma_1^2} + \log\pi_1\Big)$$

$$+ P(Z_i=2 \mid X,\theta^\circ)\Big(-\log\sqrt{2\pi\sigma_2^2} - \frac{1}{2}\frac{(x_i-\mu_2)^2}{\sigma_2^2} + \log(1-\pi_1)\Big)\Big]$$

$$\frac{\partial Q(\theta,\theta^\circ)}{\partial \pi_1} = \sum_{i=1}^{n}\left[\frac{P(Z_i=1|X,\theta^\circ)}{\pi_1} - \frac{P(Z_i=2|X,\theta^\circ)}{1-\pi_1}\right] = 0$$

$$\frac{(1-\pi_1)\left[\sum_{i=1}^{n}P(Z_i=1|X,\theta^\circ)\right] - \pi_1\left[\sum_{i=1}^{n}P(Z_i=2|X,\theta^\circ)\right]}{\pi_1(1-\pi_1)} = 0$$

$$\hat{\pi}_1 = \frac{\sum_{i=1}^{n}P(Z_i=1|X,\theta^\circ)}{\sum_{i=1}^{n}\Big[\underbrace{P(Z_i=1|X,\theta^\circ)+P(Z_i=2|X,\theta^\circ)}_{=1}\Big]}$$

$$= \frac{\sum_{i=1}^{n}P(Z_i=1|X,\theta^\circ)}{n}$$

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$K=3. \quad \theta = (\pi_1, \pi_2, \mu_1, \mu_2, \mu_3, \sigma_1^2, \sigma_2^2, \sigma_3^2)$

$$Q(\theta,\theta^\circ) = E_{Z|X,\theta^\circ}\left[\log P(X_1,\cdots X_n, Z_1 \cdots Z_n \mid \theta)\right]$$

$$= \sum_{i=1}^{n}\Big[ P(Z_i=1 \mid X,\theta^\circ)\Big(-\log\sqrt{2\pi\sigma_1^2} - \frac{1}{2}\frac{(x_i-\mu_1)^2}{\sigma_1^2} + \log\pi_1\Big)$$

$$+ P(Z_i=2 \mid X,\theta^\circ)\Big(-\log\sqrt{2\pi\sigma_2^2} - \frac{1}{2}\frac{(x_i-\mu_2)^2}{\sigma_2^2} + \log\pi_2\Big)$$

$$+ P(Z_i=3 \mid X,\theta^\circ)\Big(-\log\sqrt{2\pi\sigma_3^2} - \frac{1}{2}\frac{(x_i-\mu_3)^2}{\sigma_3^2} + \log(1-\pi_1-\pi_2)\Big)\Big]$$

$$\frac{\partial Q(\theta,\theta^\circ)}{\partial \pi_1} = \sum_{i=1}^{n}\left[\frac{P(Z_i=1|X,\theta^\circ)}{\pi_1} - \frac{P(Z_i=3|X,\theta^\circ)}{1-\pi_1-\pi_2}\right]$$

$$\frac{(1-\pi_1-\pi_2)\left[\sum_{i=1}^{n}P(Z_i=1|X,\theta^\circ)\right] - \pi_1\left[\sum_{i=1}^{n}P(Z_i=3|X,\theta^\circ)\right]}{\pi_1(1-\pi_1-\pi_2)} = 0 \qquad (1)$$

$$\frac{\partial Q(\theta,\theta^\circ)}{\partial \pi_2} = \sum_{i=1}^{n}\left[\frac{P(Z_i=2|X,\theta^\circ)}{\pi_2} - \frac{P(Z_i=3|X,\theta^\circ)}{1-\pi_1-\pi_2}\right]$$

$$\frac{(1-\pi_1-\pi_2)\left[\sum_{i=1}^{n}P(Z_i=2|X,\theta^\circ)\right] - \pi_2\left[\sum_{i=1}^{n}P(Z_i=3|X,\theta^\circ)\right]}{\pi_2(1-\pi_1-\pi_2)} = 0 \qquad (2)$$

from (1) $\Rightarrow$ $(1-\pi_1-\pi_2)\left[\sum_{i=1}^{n}P(Z_i=1|X,\theta^\circ)\right] = \pi_1\left[\sum_{i=1}^{n}P(Z_i=3|X,\theta^\circ)\right]$ $\quad (1)'$

from (2) $\Rightarrow$ $(1-\pi_1-\pi_2)\left[\sum_{i=1}^{n}P(Z_i=2|X,\theta^\circ)\right] = \pi_2\left[\sum_{i=1}^{n}P(Z_i=3|X,\theta^\circ)\right]$ $\quad (2)'$

$(1)'+(2)' \Rightarrow$ $(1-\pi_1-\pi_2)\left[\sum_{i=1}^{n}\big(P(Z_i=1|X,\theta^\circ)+P(Z_i=2|X,\theta^\circ)\big)\right]$

$$= \underbrace{(\pi_1+\pi_2)}_{1-(1-\pi_1-\pi_2)}\left[\sum_{i=1}^{n}P(Z_i=3|X,\theta^\circ)\right]$$

$$1-\pi_1-\pi_2 = \frac{\sum_{i=1}^{n}P(Z_i=3|X,\theta^\circ)}{\sum_{i=1}^{n}\Big(\underbrace{P(Z_i=1|X,\theta^\circ)+P(Z_i=2|X,\theta^\circ)+P(Z_i=3|X,\theta^\circ)}_{=1}\Big)}$$

$$= \frac{\sum_{i=1}^{n}P(Z_i=3|X,\theta^\circ)}{n}$$

from $(1)'$ $\Rightarrow$ $\hat{\pi}_1 = (1-\pi_1-\pi_2)\dfrac{\left[\sum_{i=1}^{n}P(Z_i=1|X,\theta^\circ)\right]}{\left[\sum_{i=1}^{n}P(Z_i=3|X,\theta^\circ)\right]}$

$$= \frac{\sum_{i=1}^{n}\cancel{P(Z_i=3|X,\theta^\circ)}}{n}\cdot\frac{\left[\sum_{i=1}^{n}P(Z_i=1|X,\theta^\circ)\right]}{\cancel{\left[\sum_{i=1}^{n}P(Z_i=3|X,\theta^\circ)\right]}}$$

$$= \frac{\sum_{i=1}^{n}P(Z_i=1|X,\theta^\circ)}{n}$$

from $(2)'$ $\Rightarrow$ $\hat{\pi}_2 = (1-\pi_1-\pi_2)\dfrac{\left[\sum_{i=1}^{n}P(Z_i=2|X,\theta^\circ)\right]}{\left[\sum_{i=1}^{n}P(Z_i=3|X,\theta^\circ)\right]}$

$$= \frac{\sum_{i=1}^{n}\cancel{P(Z_i=3|X,\theta^\circ)}}{n}\cdot\frac{\left[\sum_{i=1}^{n}P(Z_i=2|X,\theta^\circ)\right]}{\cancel{\left[\sum_{i=1}^{n}P(Z_i=3|X,\theta^\circ)\right]}}$$

$$= \frac{\sum_{i=1}^{n}P(Z_i=2|X,\theta^\circ)}{n}$$