4.1 Inner products

Let V be a vector space with field of scalars $\mathbb{R}$.

An *inner product on V* is a function

$$\langle\ ,\ \rangle\colon V \times V \to \mathbb{R}$$

satisfying

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
2. $\langle \lambda\mathbf{u}, \mathbf{v} \rangle = \lambda\langle \mathbf{u}, \mathbf{v} \rangle$
3. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
4. (a) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$

   (b) $\langle \mathbf{u}, \mathbf{u} \rangle = 0 \Rightarrow \mathbf{u} = \mathbf{0}$.

An *inner product space* is a vector space V together with a choice of inner product.

If W is a subspace of V, then W is itself an inner product space with respect to the inner product of V.

Example 4.1. Consider

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2$$

on $\mathbb{R}^2$.

Example 4.2. Consider

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_1 v_2 + u_2 v_2$$

on $\mathbb{R}^2$.

Example 4.3. Consider

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 - u_2 v_2 + u_3 v_3$$

on $\mathbb{R}^3$.

Example 4.4. Consider

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx$$

on the vector space of continuous functions $\mathcal{C}([0, 1])$.

$\mathcal{C}([0,1]) = \{ f : [0,1] \longrightarrow \mathbb{R} \mid f \text{ continuous} \}$

① $\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx = \int_0^1 g(x) f(x)\, dx = \langle g, f \rangle$

② $\langle \lambda f, g \rangle = \int_0^1 \lambda f(x)\, g(x)\, dx = \lambda \int_0^1 f(x) g(x)\, dx = \lambda \langle f, g \rangle$

③ $\langle f_1 + f_2, g \rangle = \int_0^1 (f_1(x) + f_2(x))\, g(x)\, dx = \int_0^1 f_1 \times g(x)\, dx + \int_0^1 f_2(x) g(x)\, dx$

$= \langle f_1, g \rangle + \langle f_2, g \rangle$

④ $\langle f, f \rangle = \int_0^1 f^2(x)\, dx = \int_0^1 (f(x))^2\, dx \geq 0$

since $f^2(x) \geq 0 \rightarrow$ the integral is greater than 0

⑤ $\langle f, f \rangle = \int_0^1 f^2(x)\, dx = 0$ yes ? 反证 ?

4.2 The Cauchy–Schwarz inequality

Let V be an inner product space.

The *length* of a vector $\mathbf{u} \in V$ is

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}.$$

The *distance* between two vectors $\mathbf{u}, \mathbf{v} \in V$ is

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

To define the angle between two vectors, we need the Cauchy–Schwarz inequality:

Theorem 4.5. Let $\mathbf{u}$, $\mathbf{v}$ be vectors in an inner product space V. Then

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$$

Proof.

if $v = 0$, then both sides are 0

let $\alpha = \frac{\langle u, v \rangle}{\|v\|^2} \in \mathbb{R}$

$$0 \leq \|u - \alpha v\|^2 = \langle u - \alpha v, u - \alpha v \rangle$$

$$= \langle u - \alpha v, u \rangle - \langle u - \alpha v, \alpha v \rangle$$

$$= \langle u, u \rangle - \langle \alpha v, u \rangle - \langle u, \alpha v \rangle + \langle \alpha v, \alpha v \rangle$$

$$= \langle u, u \rangle - 2\alpha \langle v, u \rangle + \alpha^2 \langle v, v \rangle$$

$$= \|u\|^2 + \alpha^2 \|v\|^2 - 2 \cdot \frac{\langle u, v \rangle^2}{\cancel{\|v\|^4} \cdot \|v\|^2}$$

$$2 \frac{\langle u, v \rangle^2}{\|v\|^2}$$

$$= \|u\|^2 + \frac{\langle u, v \rangle^2}{\|v\|^2} - 2 \frac{\langle u, v \rangle}{\qquad}$$

The *angle* θ between two vectors $\mathbf{u}, \mathbf{v} \in V$ is defined by the equation

$$\cos(\theta) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$$

Example 4.6. Consider the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 2 u_2 v_2$$

on $\mathbb{R}^2$.

Compute $\|\mathbf{u}\|$, $d(\mathbf{u}, \mathbf{v})$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ for $\mathbf{u} = (2, 5)$ and $\mathbf{v} = (-1, 3)$.

$$\|\vec{u}\| = \langle u, u \rangle = \sqrt{4 + 50} = \sqrt{54}$$

$$d(u, v) = \|u - v\| = \|\langle 3, 2 \rangle\| = \sqrt{9 + 2 \cdot 4} = \sqrt{17}$$

$$\cos\theta = \frac{\langle u \cdot v \rangle}{\|u\|\|v\|} = \frac{28}{\sqrt{54} \cdot \sqrt{1 + 18}} = \frac{28}{\sqrt{54} \cdot \sqrt{19}}$$

4.3 Orthogonality

Let V be an inner product space.

We say that $\mathbf{u}, \mathbf{v} \in V$ are *orthogonal* if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

We say that a subset $S \subset V$ is *orthogonal* if any two distinct vectors in S are orthogonal.

We say that a subset $S \subset V$ is *orthonormal* if it is orthogonal and every vector in S has length 1.

① ②.

$\{\hat{i}, \hat{j}, \hat{k}\}$ is an orthonormal subset of $\mathbb{R}^3$

Example 4.7. In $\mathbb{R}^2$ with the dot product,

the vectors $(1, 1)$ and $(1, -1)$ are orthogonal

$\langle u, v \rangle = 0$

not orthonormal since $\|\langle u, u \rangle\| = \sqrt{2} \neq 1$

the vectors $(1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$ are

orthonormal

Example 4.8. In $\mathcal{C}([-1,1])$ with inner product

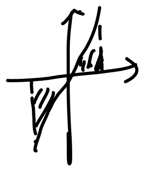
$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx,$$

the functions x^2 and x^3 are

$$\langle x^2, x^3\rangle = \int_{-1}^{1} x^5 dx = \frac{1}{6}x^6 \Big|_{-1}^{1}$$

$$= \frac{1}{6}(1-1) = 0$$

since x^5 odd function $\rightarrow$ symmetric



$$\langle x^2, x^2\rangle = \int_{-1}^{1} x^4 dx = \left[\frac{1}{5}x^5\right]_{-1}^{1} = \frac{2}{5} \neq 1$$

$$\left\| \frac{1}{\sqrt{\frac{2}{5}}} x^2 \right\| = 1$$

Theorem 4.9. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthogonal set of nonzero vectors in an inner product space. Then S is linearly independent.

(We call S an *orthogonal basis* for Span(S). Moreover, if S is an orthonormal set, we call it an *orthonormal basis* for Span(S).)

Suppose linearly relation

$a_1v_1 + \cdots + a_nv_n = \mathbf{0}$ ①

fix v_i $1 \le i \le n$, Take inner product of ① write as

$\langle v_i, a_1v_1 + \cdots + a_nv_n\rangle = \langle v_i, \mathbf{0}\rangle = 0$

$\quad \parallel$

$\langle a_1v_1 + \cdots + a_nv_n, v_i\rangle$

$\quad \parallel$

$\langle a_1v_1, v_i\rangle + \langle a_2v_2, v_i\rangle + \cdots + \langle a_nv_n, v_i\rangle$

$= a_1\langle v_1, v_i\rangle + a_2\langle v_2, v_i\rangle + \cdots + a_n\langle v_n, v_i\rangle$

$0 + \cdots a_i\langle v_i, v_i\rangle + \cdots + 0 \qquad = 0$

$\quad \parallel$

so $\quad a_i\langle v_i, v_i\rangle = 0$

$\Rightarrow a_i = 0$ or $\langle v_i, v_i\rangle = 0$ $\qquad$ so $a_i = 0$

$\downarrow$

non zero vector

non-zero vector

$v_i \neq 0$

If $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthonormal basis for V, then every $\mathbf{v} \in V$ can be written

$\mathbf{v} = a_1 u_1 + \cdots + a_i u_i + \cdots + a_n u_n$ for some $a_1, \cdots, a_n \in R$

$$\langle u_i, v \rangle = \langle u_i, a_1 u_1 + \cdots + a_n u_n \rangle$$
$$= a_i \langle u_i, u_i \rangle = a_i$$

We will soon see that every inner product space has orthonormal bases.

Therefore,

$$v = \langle u_1, v \rangle u_1 + \langle u_2, v \rangle u_2 + \cdots \langle u_n, v \rangle u_n$$

Proposition 4.10. Let $\mathbf{u}, \mathbf{v}$ be vectors in an inner product space V, with $\mathbf{u} \neq \mathbf{0}$. Then the vector

$$\mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$$

is orthogonal to $\mathbf{u}$.

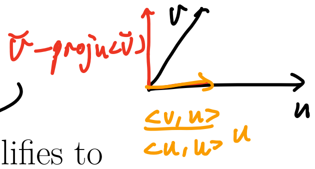
$$\langle u, v - \frac{\langle v, u \rangle u}{\langle u, u \rangle} \rangle$$
$$= \langle u, v \rangle - \langle u, \frac{\langle v, u \rangle}{\langle u, u \rangle} u \rangle$$

scalar

$$= \langle u, v \rangle - \frac{\langle v, u \rangle}{\cancel{\langle u, u \rangle}} \cancel{\langle u, u \rangle}$$
$$= 0$$

This motivates the definition: the *orthogonal projection of* $\mathbf{v}$ *onto* $\mathbf{u}$ is

$$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}.$$



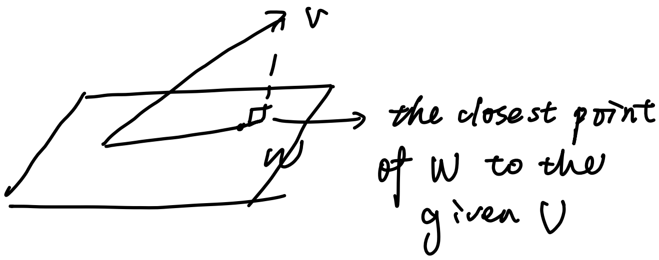
If $\mathbf{u}$ happens to be a unit vector, then the formula simplifies to

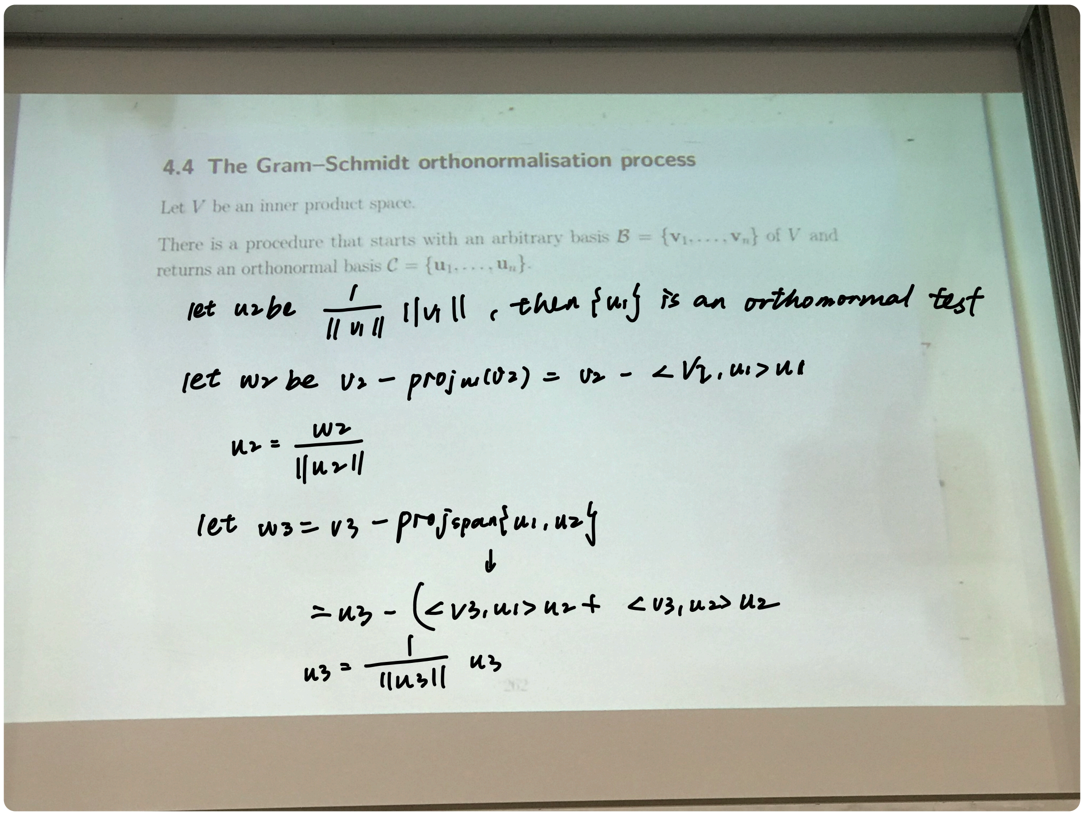
$$\operatorname{proj}_{\mathbf{u}}(\mathbf{v}) = \langle \mathbf{v}, \mathbf{u} \rangle \mathbf{u}.$$

Moreover, we can project onto a subspace W of V as follows: let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be an orthonormal basis for W. The *orthogonal projection of* $\mathbf{v}$ *onto* W is

$$\begin{aligned} \operatorname{proj}_W(\mathbf{v}) &= \operatorname{proj}_{\mathbf{u}_1}(\mathbf{v}) + \cdots + \operatorname{proj}_{\mathbf{u}_m}(\mathbf{v}) \\ &= \langle \mathbf{v}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \cdots + \langle \mathbf{v}, \mathbf{u}_m \rangle \mathbf{u}_m. \end{aligned}$$

Note that this defines a linear transformation $\operatorname{proj}_W \colon V \to V$ with image W.





4.4 The Gram–Schmidt orthonormalisation process

Let V be an inner product space.

There is a procedure that starts with an arbitrary basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V and returns an orthonormal basis $\mathcal{C} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$.

let u_1 be $\frac{1}{\|v_1\|} \|v_1\|$, then $\{u_1\}$ is an orthonormal test

let w_2 be $v_2 - \operatorname{proj}_w(v_2) = v_2 - \langle v_2, u_1 \rangle u_1$

$u_2 = \frac{w_2}{\|u_2\|}$

let $w_3 = v_3 - \operatorname{proj}_{\operatorname{span}\{u_1, u_2\}}$

$= u_3 - (\langle v_3, u_1 \rangle u_2 + \langle v_3, u_2 \rangle u_2$

$u_3 = \frac{1}{\|u_3\|} u_3$

Example 4.11. Let $W = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \subset \mathbb{R}^4$ with the dot product, where

$$\mathbf{v}_1 = (1,1,1,1) \qquad \mathbf{v}_2 = (2,4,2,4) \qquad \mathbf{v}_3 = (1,5,-1,3).$$

(a) Find an orthonormal basis for W.

(b) Find the point of W closest to the point $\mathbf{v} = (1,2,3,0)$.

(a) $u_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{(1,1,1,1)}{2} = \frac{1}{2}(1,1,1,1)$

$v_2 = v_2 - \mathrm{proj}_{(u_1)}(v_2) \rightarrow \langle u_1, v_2 \rangle \cdot u_1$

$= (2,4,2,4) - \frac{1}{4} \times 12(1,1,1,1) = (2,4,2,4) - (3,3,3,3)$

$(-1,1,-1,1)$

$u_2 = \frac{1}{2}(-1,1,-1,1)$

$w_3 = v_3 - \langle u_1, v_3 \rangle \cdot u_1 - \langle u_2, v_3 \rangle \cdot u_2$

$= (1,5,-1,3) - \cancel{\frac{1}{4}} \cdot \cancel{8}^{2}(1,1,1,1) - \cancel{8}^{2} \cdot \cancel{\frac{1}{4}}(-1,1,-1,1)$

$= (1,1,-1,-1)$