Executive summary Let V be a vector space.

- V has a basis.
- Any two bases of V have the same cardinality. (In the sense we discussed already, i.e. there is a bijection from one basis to the other.) If this cardinality is finite, it is called the *dimension of V*, and V is said to be *finite-dimensional*.
  We will focus almost exclusively on this situation. MAST300?? *Metric and Hilbert spaces* treats infinite-dimensional vector spaces.
- Any spanning set of V contains a basis of V.
- Any linearly independent subset of V can be extended to a basis of V.
- If V has dimension n, then
  - Any subset S with $\#S < n$ is **not** spanning V.
  - Any subset S with $\#S > n$ is **not** linearly independent.

In order to avoid circularity in the logic, we give a slightly different initial definition of finite-dimensionality: A vector space V is *finite-dimensional* if there exists a finite spanning subset S.

Fix a nonzero finite-dimensional vector space V.

Proposition 3.26. Given any linearly independent subset $\mathbf{v}_1, \dots, \mathbf{v}_m$ of V and any finite spanning set $\mathbf{w}_1, \dots, \mathbf{w}_n$ of V, we have $m \leq n$.

Proof.

Proposition 3.27. Every subspace of V is finite-dimensional.

Proof.

Proposition 3.28. There is a basis $\mathcal{B}$ of V.

Proof.

Note that the proof showed something of independent interest, which we record here:

Proposition 3.29. Any finite spanning set S of V contains a basis $\mathcal{B}$ of V.

Proposition 3.30. Any two bases of V have the same size.

Proof.

Example 3.31. The standard basis of the space $M_{m\times n}$ is

Example 3.32. The standard basis of the space $\mathcal{P}_n$ is

3.6 Linear transformations

A guiding principle in mathematics is that, once you have a structure on a set, you should study functions that preserve this structure. We do this now for vector spaces, which are sets with the added structure of an addition and a scalar multiplication.

Let V and W be vector spaces (over the same field of scalars, say $\mathbb{R}$). A function $T\colon V \to W$ is called a *linear transformation* if it satisfies

(a) $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$

(b) $T(\lambda\mathbf{v}) = \lambda T(\mathbf{v})$ for all $\mathbf{v} \in V$ and all scalars λ.

Since linear transformations are functions, they can be injective, surjective, bijective, or none of the above.

A bijective linear transformation is also called an *isomorphism*; we then say that the spaces V and W are *isomorphic*.

We can now clarify the effect of choosing a basis on a vector space.

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis of a vector space V. Choose an ordering on these vectors, say $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$. We will refer to $\mathcal{B}$ as an *ordered basis* of V.

Example 3.33. We know that $\{1, x, x^2\}$ is a linear independent and spanning set for the vector space $\mathcal{P}_2$. So $\mathcal{B} = (1, x, x^2)$ is an ordered basis.

Given any vector $f \in \mathcal{P}_2$, we have

This is a general phenomenon: Let $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an ordered basis of a vector space V.

Given any $\mathbf{w} \in V$, we can write it uniquely as a linear combination

$$\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n,$$

and we let

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

be the *coordinate vector of* $\mathbf{w}$ *with respect to* $\mathcal{B}$.

This defines a function $\varphi_{\mathcal{B}} \colon V \to \mathbb{R}^n$ given by

$$\varphi_{\mathcal{B}}(\mathbf{w}) = [\mathbf{w}]_{\mathcal{B}}$$

Proposition 3.34. The function $\varphi_{\mathcal{B}} \colon V \to \mathbb{R}^n$ is an invertible linear transformation.

So a choice of ordered basis on V defines an isomorphism between V and $\mathbb{R}^n$.

Example 3.35 ($\mathcal{P}_n$)**.**

Example 3.36. Consider the ordered basis $\mathcal{B} = (x^2 + x + 1, x + 1, 1)$ of $\mathcal{P}_2$.

Example 3.37 ($M_{m\times n}$).

Example 3.38. Consider the standard ordered basis $\mathcal{S} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ of $\mathbb{R}^n$.

Example 3.39. Consider the ordered basis $\mathcal{B} = ((1,0),(1,1))$ of $\mathbb{R}^2$.

3.7 Finding bases

A frequent task is finding a basis for a subspace of a vector space. How we go about this depends a lot on the manner in which the subspace is given to us.

$s, t \in \mathbb{R}$

3.7.1 Solution space of a homogeneous system

A homogeneous system with m equations and n variables takes the form

$$A\mathbf{x} = \mathbf{0}, \tag{3.40}$$

where A is an $m \times n$ matrix.

The matrix A defines a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. The solutions of the system (3.40) are the elements of the set

$$\ker(T) = \{\mathbf{x} \in \mathbb{R}^n \mid T(\mathbf{x}) = \mathbf{0}\},$$

which is called the *kernel (or nullspace)* of T.

$$\left\{ \begin{bmatrix} -3s-2t \\ t \\ 2s \\ s \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$$

subspace of $\mathbb{R}$

More generally, the kernel of a linear transformation $T\colon V \to W$ is

$$\ker(T) = \{\mathbf{v} \in V \mid \boxed{T(\mathbf{v})}_{\to W} = \mathbf{0}\}.$$

Proposition 3.41. $\ker(T)$ is a subspace of V. from set v in V $\Rightarrow$ zero vector in vector space

Proof. ① the 0 vector of V is in $\ker(T)$ $\quad 0_V \in \ker(T)$

$T(0_V) = 0_W$ $\quad T(\vec{0}_V) = T(0 \cdot \vec{v}) = 0 \cdot T(\vec{v}) = \vec{0}_W$

② from axioms

$T(v_1 + v_2) = T(v_1) + T(v_2) \quad v_1, v_2 \in V$

$T(v_1) = \vec{0}_W \quad = \quad T(v_1 + v_2) = \vec{0}_W$

$T(v_2) = \vec{0}_W \quad //$

③ from axioms $\quad v \in V$, $\lambda \in \mathbb{R}$

$T(\lambda v) = \lambda T(v)$

$T(\lambda \vec{v}) = \lambda T(\vec{v}) = \lambda \cdot \vec{0}_W = \vec{0}_W$

Let's get back to the question of finding a basis for the solution space of a system of the form (3.40).

Example 3.42. Find a basis for the solution space of the system

$$\begin{cases} x_1 + 2x_2 + \ \ x_3 + x_4 = 0 \\ 3x_1 + 6x_2 + 4x_3 + x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 3 & 6 & 4 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - 3R_2} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

column no lea

⇓

free param

$x_2 = t$

$x_4 = s$

$x_3 = 2s$

$x_1 = -3s - 2t$

$$\begin{bmatrix} -3s-2t \\ t \\ 2s \\ s \end{bmatrix} = s\begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix} + t\,[$$

is the basi

sub

Note that the dimension of the solution space is equal to the number of free parameters of the system. Things work the same over other fields of scalars than $\mathbb{R}$.

Example 3.43. Find a basis for the solution space of the system over $\mathbb{F}_2$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 \qquad\ \ + x_3 \qquad\ \ = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow -(R_2)} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{bmatrix} s \\ t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad \text{basis} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

3.7.2 Span of a set of vectors

Suppose $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of vectors in $\mathbb{R}^m$ and we are interested in the subspace

$$W = \mathrm{Span}(S) = \{\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n \mid \lambda_i \in \mathbb{R}\}.$$

Note that

$$\underbrace{\begin{bmatrix} | & | & | & | & & | \\ v_1 & v_2 & v_3 & v_4 & \cdots & v_n \\ | & | & | & | & & | \end{bmatrix}}_{=A} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n,$$

where A is the matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Therefore we can identify $\mathrm{Span}(S)$ with the range of the linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\mathbf{v}) = A\mathbf{v}$:

$$\mathrm{im}(T) = \{\mathbf{w} \in \mathbb{R}^m \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in \mathbb{R}^n\} = \mathrm{Span}(S).$$

In algebra, this is more often called the *image of T*.

More generally, if $T\colon V \to W$ is a linear transformation, we define its image to be

$$\mathrm{im}(T) = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}.$$

Proposition 3.44. $\mathrm{im}(T)$ is a subspace of W.

Proof.

① $0_W \in \mathrm{Im}(T)$ $\quad \vec{0}_V \in V \quad T(\vec{0}_V) = \vec{0}_W$

② let $w_1, w_2 \in \mathrm{Im}(T)$.
$\exists v_1 \in V,\ T(v_1) = w_1$
$\exists v_2 \in V,\ T(v_2) = w_2$.
$\exists v_1 + v_2 \in V \quad T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2$

③ Scalar mult
let $w \in \mathrm{Im}\,T$. $\lambda \in \mathbb{R}$
$\exists v \in V \ T(v) = w$
$T(\lambda v) = \lambda T(v) = \lambda w \in W$

Returning to the task of finding a basis for $\mathrm{Span}(S)$, given a subset S of $\mathbb{R}^n$, let's consider the following

Example 3.45. $S = \{(1, 0, 1), (-1, 1, -3), (2, 1, 0), (1, -2, 5)\} \subset \mathbb{R}^3$.

I claim that the vectors in S corresponding to the columns with leading ones in the RREF form a basis for $\mathrm{Span}(S)$:

This leads to *the column method* for finding a basis of $\mathrm{Span}(S)$, for finite $S \subset \mathbb{R}^n$:

(a) Construct a matrix A whose columns are the elements of S.

(b) Compute a row echelon form of A.

(c) The vectors of S corresponding to the columns with leading ones form a basis.

Note that the basis we obtain is a subset of the S we started with.

3.7.3 Span of a set of vectors, take two

I'll mention this one without delving into the theoretical underpinnings, which if done properly involve dual vector spaces.

Instead of building a matrix A whose **columns** are the vectors in S, we build a matrix B whose **rows** are the vectors in S.

Example 3.46. $S = \{(1,0,1), (-1,1,-3), (2,1,0), (1,-2,5)\} \subset \mathbb{R}^3$.

I claim that the nonzero rows in the REF of B form a basis for $\mathrm{Span}(S)$.

This leads to *the row method* for finding a basis of $\mathrm{Span}(S)$, for finite $S \subset \mathbb{R}^n$:

(a) Construct a matrix B whose columns are the elements of S.

(b) Compute a row echelon form of B.

(c) The nonzero rows of B form a basis of $\mathrm{Span}(S)$.

Note that the basis we obtain is in most cases **not a subset** of the S we started with.

Let C be an $m \times n$ matrix.

The subspace of $\mathbb{R}^m$ spanned by the columns of C is called *the column space of C*.

The subspace of $\mathbb{R}^n$ spanned by the rows of C is called *the row space of C*.

Proposition 3.47. $\dim \text{row space}(C) = \dim \text{column space}(C) = \text{rank}(C)$.

Proof.

Rank-nullity Theorem, Version I. For any matrix C,

$$\dim \text{column space}(C) + \dim \ker(C) = \text{number of columns of } C.$$

Proof.

Example 3.48. Given the matrix

$$
\begin{array}{cc}
 & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\
C = & \begin{bmatrix} 1 & -1 & 2 & -2 \\ 2 & 0 & 1 & 0 \\ 5 & -3 & 7 & -6 \\ 1 & 1 & -1 & 3 \end{bmatrix},
\end{array}
$$

find a basis and the dimension of the

(a) column space of C

(b) row space of C

(c) solution space of C.

$$\xrightarrow[\substack{R_3 \leftarrow R_3 - 5R_1 \\ R_4 \leftarrow R_4 - R_1}]{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 2 & -3 & 4 \\ 0 & 2 & -3 & 4 \\ 0 & 2 & -3 & 5 \end{bmatrix}$$

$$\xrightarrow[R_4 \leftarrow R_4 - R_2]{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \Leftrightarrow R_4} \begin{array}{cc} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 2 & -3 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} w_1 \\ w_2 \\ w_3 \\ \\ \end{array} \end{array} \xrightarrow{R_2 \leftarrow \frac{1}{2}R_2} \begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & -\frac{3}{2} & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_2 \leftarrow R_2 - 2R_3]{R_1 \leftarrow R_1 + R_2} \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

free parameter

$x_3 = t \quad x_4 = 0$

$x_2 = +\frac{3}{2}t \quad x_1 = \ldots$

(a) $\{v_1, v_2, v_4\}$

(b) $\{w_1, w_2, w_3\}$

(c) basis for ker(c)

$$\begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \longleftarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}t \\ \frac{3}{2}t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} t$$

3.8 Change of basis

We continue our investigation of the benefits of having bases for vector spaces and look into the effects of changing basis.

Recall that an ordered basis $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of a vector space V gives rise to an isomorphism

$$\varphi_{\mathcal{B}} \colon V \to \mathbb{R}^n$$

从 vector space $V \to R$

defined by taking coordinates with respect to $\mathcal{B}$: $\varphi_{\mathcal{B}}(\mathbf{w}) = [\mathbf{w}]_{\mathcal{B}}$.

3.8.1 Effect of change of basis on coordinates

Suppose we are given a second ordered basis $\mathcal{C} = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$ for V.

This gives rise to another isomorphism $\varphi_{\mathcal{C}}$ from V to $\mathbb{R}^n$, which we can fit into a diagram

$$\begin{tikzcd} V \arrow[r, equal, "id_V"] \arrow[d, "\varphi_B"'] & V \arrow[d, "\varphi_C"] \\ \mathbb{R}^n \arrow[r] & \mathbb{R}^n \end{tikzcd}$$

v_1 v_1 id_V φ_C e_1 $[v_1]_C$

since the other are linear transformation

so $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is linear transformation

transformation

So we end up with a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$, which we know corresponds to an $n \times n$ matrix.

We denote this matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and call it *the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$*.

The change of basis matrix is straightforward to compute:

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ [v_1]_C & [v_2]_C & \cdots & [v_n]_C \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

Example 3.49. In $V = \mathbb{R}^2$, write down the change of basis matrix from $\mathcal{B}$ to $\mathcal{S}$, where $\mathcal{B} = ((1,1),(1,-1))$ and $\mathcal{S} = ((1,0),(0,1))$, and use it to compute $[\mathbf{v}]_{\mathcal{S}}$, given that $[\mathbf{v}]_{\mathcal{B}} = (1,1)$.

$$P_{S\leftarrow B} = \begin{bmatrix} \vdots & \vdots \\ [(1,1)]_S & [(1,-1)]_S \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V \arrow[d] \\ B \arrow[r] & S \end{tikzcd}$$

suppose $v \in \mathbb{R}^2$ satisfying $[v]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$[v]_S = P_{S\leftarrow B}[v]_B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$[v]_B \cdot B = [v]_S \cdot S$

$S = \underbrace{[v]_S' \cdot [v]_B}_{P_{S\leftarrow B}} \cdot B$

$[v]_S = I$

$I' = I$

$S = I \cdot [v]_B \cdot B$

3.8.2 Matrix representation of a linear transformation

Now suppose we have a vector space V with ordered basis $\mathcal{B}$, a vector space W with ordered basis $\mathcal{C}$, and a linear transformation $T\colon V \to W$.

This gives rise to a diagram

$$
\begin{tikzcd}
v \in V \arrow[r, "T"] \arrow[d, "\varphi_B"', "n"] & W \ni T(v) \arrow[d, "\varphi_C"', "m"] \\
\mathbb{R}^n \arrow[r, "{[T]_{C\leftarrow B}}"] & \mathbb{R}^m
\end{tikzcd}
$$

$v_1 \quad T(v_1)$

$\#B = n$

$\#C = m$

$e_1 \quad [v]_B \longrightarrow [T(w)]_m = [T]_{C\leftarrow B}[v]_B \quad [T(v_1)]_C$

So we end up with a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$, which we know corresponds to an $m \times n$ matrix.

We denote this matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ and call it *the matrix of T with respect to the ordered bases $\mathcal{B}$ and $\mathcal{C}$*.

The matrix representation of T is straightforward to compute:

$$[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ [T(v_1)]_C & [T(v_2)]_C & \cdots & [T(v_n)]_C \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

Example 3.50. Consider the linear transformation $T\colon \mathcal{P}_1 \to \mathcal{P}_2$, $T(f) = (x+2)f$. Find the matrix of T with respect to the ordered bases $(1, x)$ and $(1, x, x^2)$.

$T(2+3x) = (x+2)(2+3x) = 3x^2 + 8x + 4$

$[T]_{C\leftarrow B} = \begin{bmatrix} \vdots & \vdots \\ [T(1)]_C & [T(x)]_C \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}$

when $x = 1$ $\quad T(1) = x+2$ based on basis of $(1, x, x^2)$ $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$x = x$ $\quad T(x) = x^2 + 2x \Rightarrow \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$P_{C\leftarrow B}$ & $P^{-1}_{B\leftarrow C}$

$$
\begin{tikzcd}
V \arrow[r, phantom, "="] \arrow[d, "\varphi_B"] & V \arrow[r, phantom, "="] \arrow[d, "\varphi_C"] & V \arrow[d, "\varphi_D"] \\
\mathbb{R}^n \arrow[r, "P_{C\leftarrow B}"] & \mathbb{R}^n \arrow[r, "P_{D\leftarrow C}"] & \mathbb{R}^n
\end{tikzcd}
$$

$[v]_B$

$[v]_B \longrightarrow P_{C\leftarrow B}[v]_B \longrightarrow P_{D\leftarrow C} P_{C\leftarrow B}[v]_B$

$\Rightarrow P_{D\leftarrow B}$