The University of Melbourne
Semester 2 Assessment 2013

Department of Mathematics and Statistics

MAST20009 Vector Calculus

Reading Time: 15 minutes

Writing Time: 3 hours

Open Book Status: Closed book

This paper has 5 pages including this page.

Authorised materials: No materials are authorised

Paper to be held by Baillieu Library? Yes

Instructions to Invigilators

Initially, students are to receive the exam paper, the 3 page formula sheet, and a 14 page script book.

Students may take this exam paper with them at the end of the examination.

Instructions to Students

There are 12 questions on this exam paper.

All questions may be attempted.

Marks for each question are indicated on the exam paper.

Start each question on a new page.

Clearly label each page with the number of the question that you are attempting.

The total number of marks on the exam paper is 130.

There is a separate 3 page formula sheet accompanying the examination paper, that you may use in this examination.

Extra materials required? None

Question 1 (10 marks)

(a) Evaluate the following limit, if it exists,

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{3x^2+4y^2}.$$

If the limit does not exist, explain why it does not exist.

(b) Consider the function H given by

$$H(x,y) = \begin{cases} \dfrac{9x^2y^2}{y^2+5x^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Is H continuous at $(0,0)$? Justify your answer.

Question 2 (10 marks)

Using *Lagrange Multipliers*, determine the point(s) on the curve $x^2y = 16$ that are closest to the origin. Justify that the point(s) you have found give the minimum distance. What is the minimum distance?

Question 3 (11 marks)

(a) Consider the vector field $\mathbf{F}$ given by

$$\mathbf{F}(x,y) = 2y\mathbf{i} - x\mathbf{j}.$$

Sketch the vector field $\mathbf{F}$ at the points $(1,1), (2,-2), (-2,1)$ on a single set of axes.

(b) Is $\mathbf{c}(t) = \left(e^{2t}, \dfrac{1}{t}, e^{-t}\right)$, $t > 0$ a flow line of the vector field $\mathbf{G}(x,y,z) = (2x, y^2, -z)$? Justify your answer.

(c) Let $\mathbf{T}$ be the unit tangent vector, $\mathbf{N}$ be the unit principal normal vector and $\mathbf{B}$ be the unit binormal vector to a C^3 path. Prove the Frenet-Serret formula

$$\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = \tau\mathbf{B} - \kappa\mathbf{T}.$$

(Hint: Differentiate a suitable cross product with respect to arclength.)

Question 4 (15 marks)

(a) Let $\mathbf{F}$ be a C^2 vector field. Prove the vector identity

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = 0.$$

(b) Consider the vector field $\mathbf{G}$ given by

$$\mathbf{G}(x, y, z) = 3yz^2\mathbf{i} - 2xz\mathbf{j} + 4xy^3\mathbf{k}.$$

(i) Show that $\mathbf{G}$ is an incompressible vector field.

(ii) Give a physical interpretation for an incompressible vector field.

(iii) Since $\mathbf{G}$ is incompressible it can be written as the curl of another vector field. Determine a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ such that $\mathbf{G} = \boldsymbol{\nabla} \times \mathbf{F}$.

Question 5 (6 marks)

Consider the double integral

$$\int_0^3 \int_{y^2}^9 y\,\cos(x^2)\,dxdy.$$

(a) Sketch the region of integration.

(b) Evaluate the double integral.

Question 6 (11 marks)

Let V be the solid region bounded by the paraboloids $z = 10 - x^2 - y^2$ and $z = 2(x^2 + y^2 - 1)$.

(a) Sketch the region V.

(b) Determine the volume of V.

Question 7 (12 marks)

Let S be that part of the sphere $x^2 + y^2 + z^2 = 25$ for $y \geq 0$.

(a) Write down a parametrization for S based on spherical coordinates.

(b) Using part (a), find a normal vector to S. Is the normal you have found pointing inwards or outwards to S?

(c) Determine the Cartesian equation of the tangent plane to the surface at $\left(\frac{5}{\sqrt{2}}, 0, \frac{5}{\sqrt{2}}\right)$.

Question 8 (9 marks)

A metallic surface P is in the shape of a triangular plate

$$3x + 2y + z = 6, \;\; x \geq 0, \;\; y \geq 0, \;\; z \geq 0.$$

Determine the total mass of P if the mass density of the metal per unit area is given by $\mu(x, y, z) = x$.

Question 9 (10 marks)

Let D be the region bounded by $y = \sqrt{1 - x^2}$, $y = 0$, $x = 0$ for $x \leq 0$. Let $\hat{\mathbf{n}}$ be the outward unit normal to the curve ∂D in the x-y plane.

(a) Sketch the region D. Clearly indicate ∂D and $\hat{\mathbf{n}}$.

(b) Let $\mathbf{F}(x, y) = (5\cos^5 y + e^{-4y} + x^3, 4\sinh^2 x - e^{2x} + y^3)$. Evaluate the integral

$$\int_{\partial D} \mathbf{F} \cdot \hat{\mathbf{n}} \, ds$$

where ∂D is traversed in the positive sense.

Question 10 (18 marks)

(a) State Stokes' theorem. Draw a diagram to illustrate your statement and explain all symbols used.

(b) Let $\mathbf{F}(x, y, z) = (3y + z)\mathbf{i} + y\mathbf{j} + (z^3 - x^4)\mathbf{k}$. Let S be the surface of the cone

$$z = 1 + \sqrt{x^2 + y^2} \quad \text{for} \quad z \leq 5.$$

Evaluate the flux integral

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S}$$

using Stokes' theorem and

(i) an appropriate line integral;

(ii) the simplest surface.

Question 11 (10 marks)

(a) Let R be a solid region bounded by an oriented closed surface ∂R. Let f and g be C^2 scalar functions. Let $\hat{\mathbf{n}}$ be the unit outward normal to ∂R. Prove that

$$\iiint_R \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g \, dV = \iint_{\partial R} f \boldsymbol{\nabla} g \cdot d\mathbf{S} - \iiint_R f \nabla^2 g \, dV.$$

(b) Suppose that ∂R is a sphere of radius R_0 centred at the origin. Let $f(r) = r$ and $g(r) = r^2$ where $r = \sqrt{x^2 + y^2 + z^2}$. Using the result in part (a), show that

$$\iiint_R 4r \, dV = \iint_{\partial R} r^2 \, dS.$$

Question 12 (8 marks)

Define *elliptical cylindrical* coordinates (u, v, z) by

$$x = a \cosh u \cos v, \qquad y = a \sinh u \sin v, \qquad z = z$$

where $u \geq 0$, $0 \leq v \leq 2\pi$, $z \in \mathbb{R}$, and a is a non-zero constant.

(a) Determine the scale factors h_u, h_v, h_z.

(b) Show that

$$|\text{Jacobian}| = a^2(\sinh^2 u + \sin^2 v).$$

(c) Determine $\boldsymbol{\nabla}(u^2 + v^3 z)$. Simplify your answer as much as possible.