Rank, invertibility, and solvability

The *rank* of a matrix in row echelon form is defined to be the number of leading 1s.

The *rank* of an arbitrary matrix A is the rank of any REF matrix obtained from A via Gaussian elimination.

The rank of the matrix from Example 1.23 is 3, while the rank of the matrix from Example 1.24 is 2.

Theorem 1.25. Let A be an $n \times n$ matrix. The following are equivalent (*TFAE*):

(a) A is invertible.

(b) A has rank n. (We also say it has *full rank*.)

(c) The RREF of A is I_n.

(d) The *homogeneous system* $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$.

(e) Given any $n \times 1$ matrix $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution. (If $\mathbf{b} \neq \mathbf{0}$, we call the system *inhomogeneous*.)

Poof. We show that (a) $\Rightarrow$ (e) $\Rightarrow$ (d) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a).

Example 1.26. Consider the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

1. Find a row echelon form of $[A \mid \mathbf{b}]$.
2. Find the rank of the matrix A.
3. Find all the solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$.
4. For which values of a, b, and c does the system $A\mathbf{x} = \mathbf{b}$ have infinitely many solutions, a unique solution, or no solutions?

In general, given an inhomogeneous system $A\mathbf{x} = \mathbf{b}$, if

$$\text{rank}A < \text{rank}[A \mid \mathbf{b}],$$

then the system has no solutions (it is inconsistent).

Note also, in part 3, that

$$\text{rank}A + \#\text{solution parameters} = \#\text{unknowns} = \#\text{columns}.$$

Example 1.27. Using the result of Example 1.23, solve

$$\begin{cases} x \phantom{{}-y} + z = 0 \\ 2x - y + 3z = 0 \\ 2x + 2y + z = 0 \end{cases}$$

1.5 Determinants

1.5.1 Motivation

We know how to find the inverse of a square matrix (if this inverse exists) by using Gauss–Jordan elimination. In the case of 2×2 matrices, we can give a simple direct formula for the inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

which should be interpreted as saying:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If $ad-bc \neq 0$ the $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible

So the number $ad - bc$ detects whether the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible. We call it the *determinant* of the 2×2 matrix A and denote it $\det A$.

Our aim is to explore this notion beyond the special case of 2×2 matrices.

$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ notation

1.5.2 Axiomatic definition of determinant

Let's write $M_{m\times n}$ for the set of all $m \times n$ matrices.

We will describe the determinant as a function that attaches a number $\det(A)$ to each square matrix $A \in M_{n\times n}$, in a very special way.

It will be easier to do this first on n-tuples of row matrices

$$(A_1, \dots, A_n) \in M_{1\times n} \times \cdots \times M_{1\times n},$$

and then use the identification

$$M_{n\times n} \xrightarrow{\text{rows}} M_{1\times n} \times \cdots \times M_{1\times n}$$
$$\begin{bmatrix} —A_1— \\ —A_2— \\ \vdots \\ —A_n— \end{bmatrix} \longmapsto (A_1, A_2, \dots, A_n)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \mapsto ((1,2,3),(4,5,6),(7,8,9))$$

$S \times T = \{(s,t) \mid s \in S, t \in T\}$

So here is what we will want the function $d\colon M_{1\times n} \times \cdots \times M_{1\times n} \to \mathbb{R}$ to behave like:

(D1) For each $1 \le i \le n$ and each scalar $\lambda \in \mathbb{R}$:

$$d(A_1, \dots, \lambda A_i, \dots, A_n) = \lambda d(A_1, \dots, A_i, \dots, A_n).$$

(D2) For each $1 \le i \le n$:

$$d(A_1, \dots, A_i + A_i', \dots, A_n) = d(A_1, \dots, A_i, \dots, A_n) + d(A_1, \dots, A_i', \dots, A_n).$$

(D3) Swapping two rows changes the sign:

$$d(A_1, \dots, A_j, \dots, A_i, \dots, A_n) = -d(A_1, \dots, A_i, \dots, A_j, \dots, A_n).$$

(D4) For all $1 \le j \le n$, let $I_j = [0, \dots, 1, \dots, 0]$ with 1 in the j-th place only. Then

$$d(I_1, \dots, I_n) = 1.$$

j row

Other properties follow automatically from (D1)–(D4):

Proposition 1.28. Suppose d is a function satisfying axioms (D1)–(D4). Then:

(a) If at least one of $A_1, \ldots, A_n$ is zero, then $d(A_1, \ldots, A_n) = 0$.

(b) If there exist i and j such that $A_i = A_j$, then $d(A_1, \ldots, A_i, \ldots, A_j, \ldots, A_n) = 0$.

Proof.

(a) $d(\vec{0}, A_2, \cdots, A_n) = d((0)\vec{0}, A_2 \cdots, A_n) = 0\,d(\vec{0}, A_2, \cdots, A_n) = 0$

(b). $d(A_1, \cdots A_i, \cdots A_j, \cdots A_n)$

swap two rows

$= -d(A_i, \cdots A_j, \cdots A_i \cdots A_n)$

$= -d(A_i, \cdots, A_i \cdots, A_j, \cdots A_n)$

$D = -D \Rightarrow D = 0$

so $d(A_1 \cdots, A_i, \cdots, A_j, \cdots, A_n) = 0$

Theorem 1.29. Fix a positive integer n.

(a) There exists a function d satisfying properties (D1)–(D4).

(b) There exists a unique function d satisfying properties (D1)–(D4).

(c) If f is a function satisfying properties (D1)–(D3), then

$$f(A_1, \ldots, A_n) = d(A_1, \ldots, A_n) f(I_1, \ldots, I_n).$$

Proof of the special case $n = 2$.

$A_1 = (a, b)$ $\quad (a, b) = (a, 0) + (0, b)$

$A_2 = (c, d)$ $\quad (c, d) = (c, 0) + (0, d)$

$d(A_1, A_2) = d((a, b), (c, d)) = d((a,0)+(0,b), (c,d))$

$\overset{D2}{=} d((a,0),(c,d)) + d((0,b),(c,d))$

$\overset{D2}{=} d((a,0),(c,0)) + d((a,0),(0,d)) + d((0,b),(c,0)) + d((0,b),(0,d))$

$\overset{D1}{=} ac\,d((1,0),(1,0)) + ad\,d((1,0),(0,1)) + bc\,d((0,1),(1,0)) + bd\,d((0,1),(0,1))$

$\overset{\text{pro 1.28 (b)}}{=} 0 + ad\,d((1,0),(0,1)) + bc\,d((0,1),(1,0)) + 0$

$\overset{(D4)}{=} ad - bc \quad d((1,0),(0,1))$

$\overset{(D3)}{=} ad - bc$

Given a positive integer n, the *determinant* is the function $\det\colon M_{n\times n} \to \mathbb{R}$ that assigns to a matrix A the value of the function d from Theorem 1.29 on the rows of A:

$$\det = d \circ \text{rows}$$

Example 1.30. For a general 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we have

$$\det(A) = d((a,b),(c,d)) = ad - bc$$

Here are some useful properties of the determinant:

Proposition 1.31. Let A be a square matrix.

(a) If A has a zero row, then $\det A = 0$.

(b) If A has two equal rows, then $\det A = 0$.

(c) If A is a *diagonal matrix* (all entries not on the diagonal are zero), then $\det A$ is the product of the entries on the diagonal.

(d) If A is an *upper-triangular matrix* (all entries below the diagonal are zero), then $\det A$ is the product of the entries on the diagonal.

(e) Ditto for any *lower-triangular matrix* A.

Proof. c)

$$\det\left(\begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & a_{33} & \\ 0 & & & \ddots \; a_{nn} \end{bmatrix}\right) = d((a_{11},0,0\ldots0)) \cdots (0,\ldots,0,a_{nn}))$$

$$= a_{11}a_{22}\cdots a_{nn}\, \underbrace{d((1,0,\cdots0)\cdots(0,0,\cdots1))}_{=1}$$

$$= a_{11}a_{22}\cdots a_{nn}$$

1.5.3 Determinants via Gaussian elimination

Part (d) of Proposition 1.31 has a very useful consequence: it allows us to compute the determinant of A by elimination.

Suppose B is any row echelon form of A. Then B is upper-triangular, so Proposition 1.31(d) gives an easy way to compute $\det B$.

It remains to relate $\det A$ and $\det B$. For this, we need to investigate the effect of elementary row operations on determinants:

$A \to C \cdots \to B$

- If C is obtained from A by $R \leftarrow \lambda R$ with $\lambda \neq 0$, then

$\det(C) = \begin{vmatrix} \vdots \\ \lambda R \\ \vdots \end{vmatrix} = \lambda \begin{vmatrix} \vdots \\ R \\ \vdots \end{vmatrix} = \lambda \det(A)$

- If C is obtained from A by $R \leftrightarrow S$, then

$\det(C) = \begin{vmatrix} \vdots \\ S \\ R \\ \vdots \end{vmatrix} = - \begin{vmatrix} \vdots \\ R \\ S \\ \vdots \end{vmatrix} = -\det(A)$

- If C is obtained from A by $R \leftarrow R + \mu S$, then

$\det(C) = \begin{vmatrix} \vdots \\ R+\mu S \\ \vdots \\ S \end{vmatrix} = \begin{vmatrix} \vdots \\ R \\ \vdots \\ S \end{vmatrix} + \begin{vmatrix} \vdots \\ \mu S \\ \vdots \\ S \end{vmatrix}$

$= \det(A) + \mu \underbrace{\begin{vmatrix} \vdots \\ S \\ \vdots \\ S \end{vmatrix}}_{\text{prop. 1.28 (b)}}$

$= \det(A) + 0$

$= \det(A).$

Example 1.32. Use Gaussian elimination to compute

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{vmatrix}.$$

$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$

$\xrightarrow{R_2 \leftarrow R_1 + R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} = C$

$\xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = B$

$\det(B) = 1 \cdot 1 \cdot 1 = 1.$

$\det(A) = 1$ (from above)

1.5.4 Determinants via cofactor expansion

Consider an $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

The *(i, j)-submatrix* of A, denoted A_{ij}, is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i-th row and the j-th column:

$$A_{ij} = \begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}$$

The *(i, j)-minor* of A is $\det A_{ij}$, and the *(i, j)-cofactor* of A is $C_{ij} = (-1)^{i+j} \det A_{ij}$.

⇒ $A \setminus$ i-th row and j-th column

Example 1.33. For the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$

- $A_{23} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
- the $(2, 3)$-minor is $\det(A_{23}) = 1$
- the $(2, 3)$-cofactor is $C_{23} =$ $C_{23} = (-1)^5 \det A_{23} = -1$

Cofactor (or Laplace) expansion takes a square matrix and returns a number, in one of the following ways:

(a) along row i:

$$\sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(b) along column j:

$$\sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Theorem 1.34. The result of cofactor expansion of a square matrix A is the determinant of A.

Cofactor expansion has a recursive aspect, whereby the determinant of a larger matrix is written in terms of determinants of smaller matrices (which, in turn, are expressed in terms of determinants of even smaller matrices). This makes statements about cofactor expansion (such as Theorem 1.34) particularly well-suited to a proof technique known as proof by induction, which we will be studying in a few weeks.

Example 1.35. Use cofactor expansion to compute

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{vmatrix}$$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Expansion along 1^{st} row

$$1. \ (+1)\begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} + 2 \times (-1)\begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix} + 1 \cdot (+1)\begin{vmatrix} -1 \\ 0 \end{vmatrix}$$

$$= -3 + 6 - 1 = 1$$

1.5.5 Properties of determinants

$\det(A+B) \neq \det(A) + \det(B)$

Theorem 1.36. If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

Proof. Think of the matrix B as fixed and consider the function $f\colon M_{1\times n} \times \dots M_{1\times n} \to \mathbb{R}$ defined by

$$f(A_1, \dots, A_n) = \det AB, \qquad \text{where } A = \begin{bmatrix} \text{—}A_1\text{—} \\ \text{—}A_2\text{—} \\ \vdots \\ \text{—}A_n\text{—} \end{bmatrix}$$

We have $(AB)_i = A_i B$, that is the i-th row of AB is the product of the i-th row of A and the matrix B. So

$$f(A_1, \dots, A_n) = \det AB = d(A_1B, \dots, A_nB).$$

I claim that the function f satisfies the determinant axioms (D1)–(D3):

nal.

he diagonal

ible

(D1) For any i and any $\lambda \in \mathbb{R}$:

$$\begin{aligned} f(A_1, \dots, \lambda A_i, \dots, A_n) &= d(A_1B, \dots, \lambda A_iB, \dots, A_nB) \\ &= \lambda d(A_1B, \dots, A_iB, \dots, A_nB) \\ &= \lambda f(A_1, \dots, A_i, \dots, A_n) \end{aligned}$$

(D2)

(D3)

Therefore, by Theorem 1.29(c), we have

$$\det AB = f(A_1, \dots, A_n) = d(A_1, \dots, A_n) f(I_1, \dots, I_n) = (\det A)(\det B).$$

Corollary 1.37. If A is invertible, then $\det\left(A^{-1}\right) = (\det A)^{-1}$.

Proof. $\det(A^{-1}) \times \det(A) = \det(A^{-1}A) = \det(I) = 1$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Corollary 1.38. A square matrix A is invertible if and only if $\det A \neq 0$.

Proof. when suppose $\det(A) = 0$ and A is invertible

$\det(A^{-1})$ is not defined

$$\det(A) = \frac{1}{\det(A^{-1})} = 0 \quad X.$$

so A is not invertible

if $\det A \neq 0$, then rref (A) has no zero on the diago
since A is a square matrix so only 1 can on t

$\Rightarrow$ rref $(A) = I_n$

$\Rightarrow A$ is invert

Here are a few more facts that we'll, at the moment, state without proof.

Proposition 1.39. For any square matrix A, $\det\left(A^T\right) = \det A$.

Proposition 1.40. Suppose a square matrix A has a *block decomposition*

$$A = \left[\begin{array}{c|c} B & * \\ \hline 0 & C \end{array}\right],$$

where B and C are square matrices (not necessarily of the same size). Then

$$\det A = (\det B)(\det C).$$

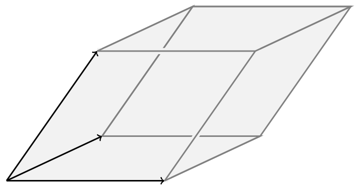
? $\det(BC) = \det A$?

$A = \left[\begin{array}{c|c} B & C \\ \hline D & E \end{array}\right]$)

$\neq \det = \cancel{\det(B)\det(E)}$
$\cancel{-\det(C)\det(D)}$

1.5.6 Determinants as volumes

The rows of an $n \times n$ matrix A form a *parallelepiped* in $\mathbb{R}^n$:



The *oriented volume* of this parallelepiped is $\det A$.

$\neq$ volume

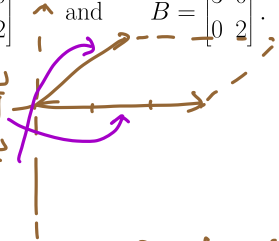
Example 1.41 (2-dimensional)**.** Find the (oriented) areas of the parallelograms corresponding to the matrices

$$A = \begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$

for A

$3\vec{i}+0\vec{j}$

$2\vec{i}+2\vec{j}$

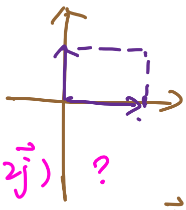


$\det(A) = 3\times 2 - 0\times 2 = 6 = (3\vec{i}+0\vec{j}) \times (2\vec{i}+2\vec{j})$

for B

$3\vec{i}+0\vec{j}$

$0\vec{i}+2\vec{j}$

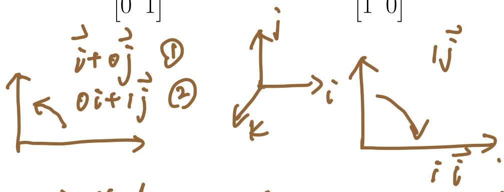


$(3\vec{i}+0\vec{j}) \times (0\vec{i}+2\vec{j})$?

$\det(A) = 6 \neq (3\vec{i}+0\vec{j})(0\vec{i}+2\vec{j})$

Example 1.42 (Orientation in 2D)**.** Find the oriented areas of the parallelograms corresponding to the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$



oriented area +

$(\vec{i} + 0\vec{j}) \times (0\vec{i} + 1\vec{j})$

$= \vec{i} \times \vec{j}$

$(0\vec{i} + 1\vec{j}) \times (\vec{i} + 0\vec{j})$

$= 1\vec{j} \times 1\vec{i}$

right hand rule

Example 1.43 (Orientation in 3D)**.** Find the oriented volumes of the parallelepipeds corresponding to the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

1.6 Vectors in $\mathbb{R}^n$ and applications

Since Descartes, the space we live in is described mathematically as

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\},$$

whereas the inhabitants of Abbott's *Flatland: A Romance of Many Dimensions* dwell in

$$\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}.$$

These have the common (and obvious) generalisation

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in \mathbb{R}\},$$

where n is a nonnegative integer.

We refer to $\mathbb{R}^n$ as *n-space* (or *n-dimensional space*). *Vectors* are elements of $\mathbb{R}^n$ for some n.

(In a few weeks we'll take a more conceptual point of view that will largely liberate vectors of the shackles of coordinates, but for now they are n-tuples of coordinates.)

In $\mathbb{R}^3$, it is customary to single out three particularly simple vectors:

$$\mathbf{i} = (1,0,0), \qquad \mathbf{j} = (0,1,0), \qquad \mathbf{k} = (0,0,1).$$

Part of their mystique is that any vector $\mathbf{v} \in \mathbb{R}^3$ can be written as

$$\mathbf{v} = (v_1, v_2, v_3) =$$

Geometrically, the vector $\mathbf{v}$ is the position vector of the point $P = (v_1, v_2, v_3)$ (relative to the point $O = (0,0,0)$).

We define the *length* (or *norm* or *magnitude*) of $\mathbf{v} \in \mathbb{R}^n$ as

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

(You should convince yourself that this definition matches your intuition in $\mathbb{R}^2$ and $\mathbb{R}^3$.)

Example 1.44.

$$\|2\mathbf{i} - \mathbf{j} + 2\mathbf{k}\| =$$

A *unit vector* is a vector of length 1.

1.6.1 Arithmetic operations on vectors

Some of these will look awfully familiar.

Scalar multiplication Given $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we define

$$\lambda\mathbf{v} = (\lambda v_1, \ldots, \lambda v_n) \in \mathbb{R}^n.$$

Geometrically:

Two nonzero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are said to be *parallel* if $\mathbf{u} = \lambda\mathbf{v}$ for some scalar $\lambda \in \mathbb{R}$.

Example 1.45. Find a unit vector parallel to $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Vector addition Given $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$ and $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$, we define

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, \dots, u_n + v_n) \in \mathbb{R}^n.$$

Geometrically:

Properties of vector arithmetic

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$
- $\|\lambda\mathbf{u}\| = |\lambda|\|\mathbf{u}\|$

Euclidean inner product The *Euclidean inner product* (or *dot product* or *scalar product*) of $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n.$$

Properties of the dot product

- $\mathbf{u} \cdot \mathbf{v}$ is a scalar
- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$
- $\lambda(\mathbf{u} \cdot \mathbf{v}) = (\lambda \mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\lambda \mathbf{v})$

Geometrically, two non-parallel vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ form a plane $\mathcal{P}$ in $\mathbb{R}^n$. We define the *angle between* $\mathbf{u}$ and $\mathbf{v}$ to be the angle between these vectors as line segments in the plane $\mathcal{P}$ (where this notion is already familiar).

Proposition 1.46. For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ we have

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, \cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Proof. Recall the law of cosines: given a triangle with side lengths a, b, and c, and angle θ between the sides of lengths a and b, we have

$$a^2 + b^2 - 2ab\cos\theta = c^2.$$

Corollary 1.47. Two nonzero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.