COMP20007 Design of Algorithms

Growth Rate and Algorithm Efficiency

Lars Kulik

Lecture 3

Semester 1, 2020

Assessing Algorithm "Efficiency"

Resources consumed: time and space.

We want to assess efficiency as a function of input size:

- Mathematical vs empirical assessment
- Average case vs worst case

Knowledge about input peculiarities may affect the choice of algorithm.

The right choice of algorithm may also depend on the programming language used for implementation.

Running Time Dependencies

There are many things that a program's running time depends on:

1. The complexity of the algorithms used
2. Input to the program
3. Underlying machine, including memory architecture
4. Language/compiler/operating system

Since we want to compare algorithms, we ignore (3) and (4);
just consider units of time.

Use a natural number n as measure of (2)—size of input.

Express (1) as a function of n.

The RAM Word Model

Assumptions

- Data is represented in words of fixed length in bits (e.g., 64-bit)
- Fundamental operations on words take one unit of time

Basic arithmetic:	$+$, $-$, $\times$, $/$
Memory access:	load, store
Comparisons:	$<$, $>$, $=$, $\neq$, $\geq$, $\leq$
Logical operators:	&&, $\|\|$
Bitwise operators:	&, $\|$

The goal of analysis is to count the operations based on parameters such as input size or problem size.

Estimating Time Consumption

If c is the cost of a basic operation and $g(n)$ is the number of times the operation is performed for input of size n,

then running time $t(n) \approx c \cdot g(n)$.

Examples: Input Size and Basic Operation

Problem	**Size measure**	**Basic operation**
Search in list of *n* items	*n*	Key comparison
Multiply two matrices of floats	Matrix size (rows times columns)	Float multiplication
Graph problem	Number of nodes and edges	Visiting a node

Best, Average, or Worst Case?

The running time $t(n)$ may well depend on more than just n.

Worst-case analysis makes the most adverse assumptions about input. Are they the worst n things your algorithm could see?

Best-case analysis makes optimistic assumptions. Are they the best n things the algorithm could see?

Average-case analysis aims to find the expected running time across all possible input of size n. (Note: This is not an average of the worst and best cases but assumes that your input is drawn randomly from all possible inputs of size n.)

Amortised analysis takes the context of running an algorithm into account and calculates cost spread over many runs.

Average-case Analysis: Sequential Search

```
**function** SequentialSearch($A[0..n-1]$,$K$)
    $i \leftarrow 0$
    **while** $i < n$ and $A[i] \neq K$ **do**
        $i \leftarrow i + 1$
    **if** $i < n$ **then**
        **return** $i$
    **else**
        **return** $-1$
```

If the probability of a successful search is equal to p $(0 \leq p \leq 1)$, then the average number of average number of key comparisons $C_{avg}(n)$ is

$$p \times (n+1)/2 + n \times (1-p).$$

Large Input Is What Matters

Small input does not provide a stress test for an algorithm.

As an alternative to Euclid's algorithm (Lecture 1) we can find the greatest common divisor of m and n by testing each k no greater than the smaller of m and n, to see if it divides both.

For small input (m, n), both these versions of *gcd* are fast.

Only as we let m and n grow large do we witness (big) differences in performance.

The Tyranny of Growth Rate

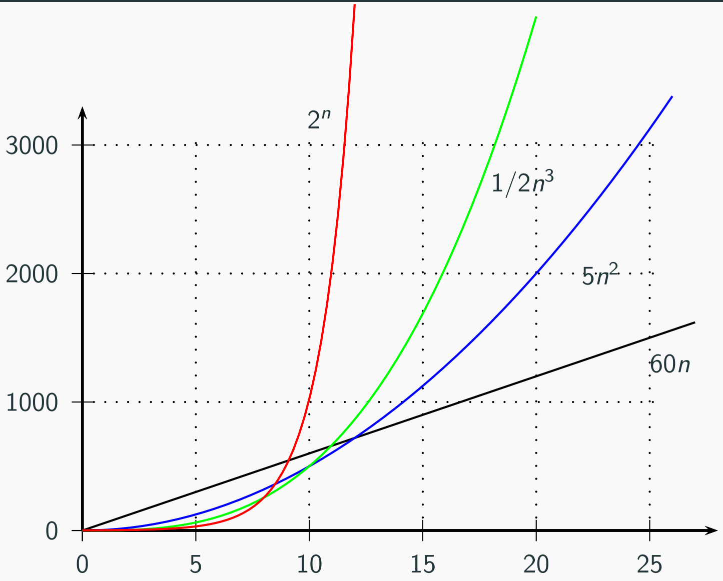
n	$\log_2 n$	n	$n \log_2 n$	n^2	n^3	2^n	$n!$
10^1	3	10^1	$3 \cdot 10^1$	10^2	10^3	10^3	$4 \cdot 10^6$
10^2	7	10^2	$7 \cdot 10^2$	10^4	10^6	10^{30}	$9 \cdot 10^{157}$
10^3	10	10^3	$1 \cdot 10^4$	10^6	10^9	—	—

10^{30} is one thousand times the number of nano-seconds since the Big Bang.

At a rate of a trillion (10^{12}) operations per second, executing 2^{100} operations would take a computer in the order of 10^{10} years.

That is more than the estimated age of the Earth.

The Tyranny of Growth Rate



Functions Often Met in Algorithm Classification

1: Running time independent of input.

$\log n$: Typical for "divide and conquer" solutions, for example, lookup in a balanced search tree.

n: Linear. When each input element must be processed once.

$n \log n$: Each input element processed once and processing involves other elements too, for example, sorting.

n^2, n^3: Quadratic, cubic. Processing all pairs (triples) of elements.

2^n: Exponential. Processing all subsets of elements.

Asymptotic Analysis

We are interested in the growth rate of functions:

- Ignore constant factors
- Ignore small input sizes

Asymptotics

$$f(n) \prec g(n) \text{ iff } \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

That is: g approaches infinity faster than f. For example,

$$1 \prec \log n \prec n^{\epsilon} \prec n^{c} \prec n^{\log n} \prec c^{n} \prec n^{n}$$

where $0 < \epsilon < 1 < c$.

In asymptotic analysis, think big!

For example, $\log n \prec n^{0.0001}$, even though for $n = 10^{100}, 100 > 1.023$.

Big-Oh Notation

$O(g(n))$ denotes the set of functions that grow no faster than g, asymptotically.

We write

$$t(n) \in O(g(n))$$

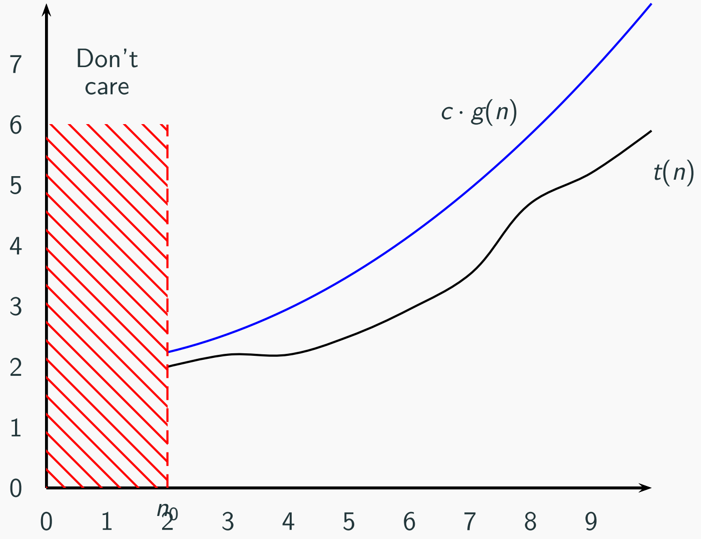
when, for some c and n_0,

$$n > n_0 \Rightarrow t(n) < c \cdot g(n)$$

For example,

$$1 + 2 + \cdots + n \in O(n^2)$$

Big-Oh: What $t(n) \in O(g(n))$ Means



Big-Oh Pitfalls

Levitin's notation $t(n) \in O(g(n))$ is meaningful, but not standard.

Other authors use $t(n) = O(g(n))$ for the same thing.

As O provides an upper bound, it is correct to say both $3n \in O(n^2)$ and $3n \in O(n)$ (so you can see why using '$=$' is confusing); the latter, $3n \in O(n)$, is of course more precise and useful.

Note that c and n_0 may be large.

Big-Omega and Big-Theta

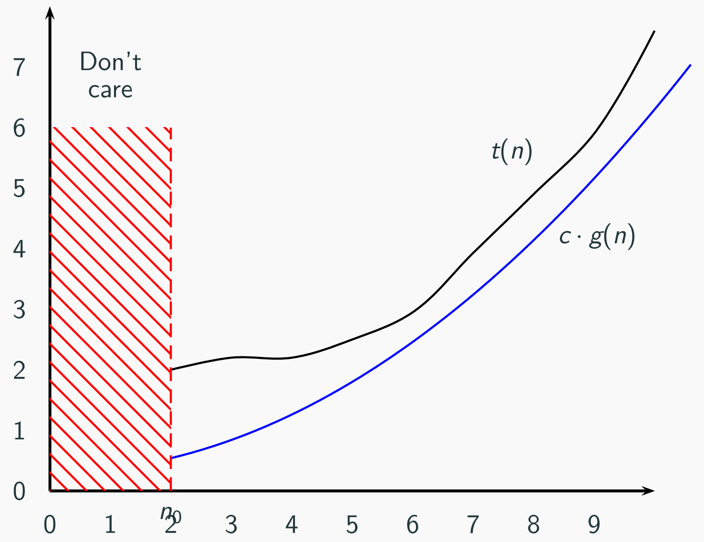
$\Omega(g(n))$ denotes the set of functions that grow no slower than g, asymptotically, so Ω is for lower bounds.

$t(n) \in \Omega(g(n))$ iff $n > n_0 \Rightarrow t(n) > c \cdot g(n)$, for some n_0 and c.

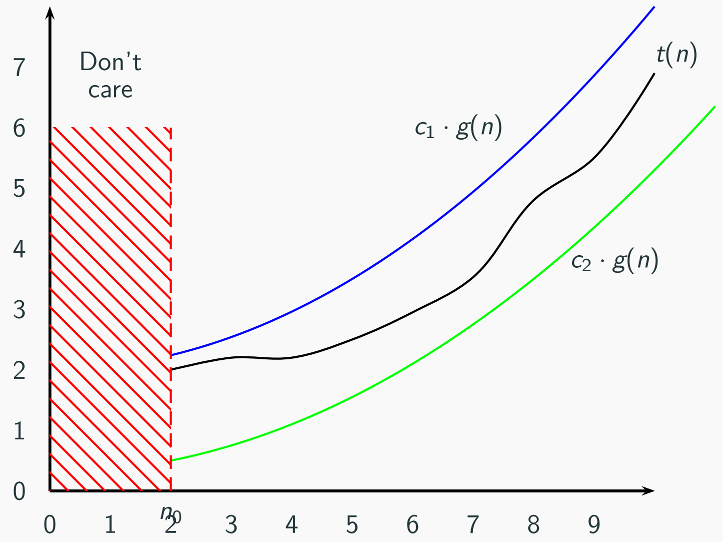
Θ is for exact order of growth.

$t(n) \in \Theta(g(n))$ iff $t(n) \in O(g(n))$ and $t(n) \in \Omega(g(n))$.

Big-Omega: What $t(n) \in \Omega(g(n))$ Means



Big-Theta: What $t(n) \in \Theta(g(n))$ Means



Establishing Growth Rate

We can use the definition of O directly.

$$n > n_0 \Rightarrow t(n) < c \cdot g(n)$$

Exercise: Use this to show that

$$1 + 2 + \cdots + n \in O(n^2)$$

Also show that

$$17n^2 + 85n + 1024 \in O(n^2)$$

✎

Next Up

We go through some examples of time complexity analysis for specific algorithms.