Student Number: ________________

The University of Melbourne

Semester 2 Assessment 2013

Department of Mathematics and Statistics

MAST30001: Stochastic Modelling

Reading Time: 15 minutes

Writing Time: 3 hours

Open Book Status: Closed book

This paper has 5 **pages (including this page).**

Authorised Materials:

☑ The following items are authorised: (list here)
A scientific (but not graphing) calculator is allowed.

OR

☐ Students may have unrestricted access to all materials.

OR

☐ No materials are authorised.

Paper to be held by Baillieu Library: Indicate whether the paper is to be held with the Baillieu Library.

☑ **Yes** ☐ **No**

Instructions to Invigilators:

Students require script books only.
No notes or books are allowed in the examination room.
This paper is to remain in the examination room.

Instructions to Students:

Solve the following 7 problems.
Problems carry marks as indicated.
The exam has 70 marks in total.
Justify all work and give clear, concise explanations, using prose where appropriate.
Clarity, neatness, and style count.

Extra Materials required (please tick & supply)

☐ Graph Paper ☐ Multiple Choice Form ☐ Other (please specify)

1. (6 Marks) A Markov chain has transition matrix

$$\begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 9/10 & 0 & 1/10 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 3/10 & 0 & 7/10 & 0 \\ 3/4 & 0 & 0 & 0 & 1/4 \end{pmatrix}.$$

Analyse the state space (reducibility, periodicity, recurrence, etc), and discuss the chain's long run behavior.

2. (12 Marks) The chance that it rains in Melbourne given that it rained both of the past two days is 3/4. If it rained yesterday but not the day before the chance it rains today is 2/3 and if it rained two days ago but not yesterday then the chance it rains today is 1/2. Finally, if it hasn't rained in the past two days then the chance it rains today is 3/7.

   (a) Model the rain in Melbourne as a Markov chain and write down its transition matrix.

   (b) Analyse the state space of this Markov chain and discuss its long run behaviour.

   (c) Starting from a randomly chosen day of the year, what is the chance of having two consecutive days without rain before there are two consecutive rainy days?

   (d) Given it rained the last two days, what is the expected number of days before it rains twice in a row again? (If it rains tomorrow then it took one day to rain twice in a row again.)

3. (6 Marks) A certain casino game costs 180 dollars to play. The game has two rounds. In the first round a coin is tossed and if it comes up heads you get 280 dollars (that is, 100 dollars plus your 180 dollar fee is returned) and if it comes up tails you get 90 dollars (so overall you're down 90 dollars). At this point you may take your money or you can play one more round where again with even chance you gain 100 dollars or your money is halved (so after playing the second round you'll leave with one of 380, 190, 140, or 45 dollars). What strategy should you use to maximise your expected winnings in this game and what is that expectation? (Note that one potential strategy is not to play at all.)

4. (14 Marks) A yeast culture contains microbes that split and die according to the following rules. Each microbe lives a number of minutes having an exponential distribution with rate 1 and then splits into two microbes with probability p or dies with probability $1 - p$. The lifetime and choice of split or death of a microbe is independent of that of all other microbes. In addition, independent of the process in the culture, yeast microbes from the air outside of the culture float by the culture according to a Poisson process with rate 2 per minute. Each microbe that floats by joins the population of the culture with probability p and with probability $1 - p$ the microbe doesn't join the culture, this choice made independent of all else.

   (a) What is the chance that exactly four outside microbes float by in the first 3 minutes?

   (b) What is the chance that exactly four outside microbes join the culture in the first 3 minutes?

   (c) Given that 7 outside microbes have floated by the culture in first 3 minutes, what is the chance that at least two of the seven join the culture?

   (d) Given that 7 outside microbes have floated by the culture in first 3 minutes, what is the chance that exactly 3 float by in the first 1 minute?

   (e) What is the chance that in the first 3 minutes, exactly four microbes join the culture and 3 float by that don't join the culture?

   (f) Model the number of microbes in the culture as a continuous time Markov chain and define its generator.

   (g) For what values of p is the chain ergodic? Find the limiting distribution in the cases where it's ergodic.

5. (12 Marks) Recall in the $M/M/2$ queue with arrival rate λ per hour and service rate μ per hour with $\lambda < \mu$ and $\rho := \lambda/\mu$, the stationary number of customers in the system has distribution

$$\pi_j = \left(\frac{2-\rho}{2+\rho}\right) \times \begin{cases} \rho^j/j! & \text{for } j = 0, 1, \\ \rho^j/2^{j-1} & \text{for } j \geq 2. \end{cases}$$

And for the same rates in the $M/M/1$ queue the analogous stationary distribution is geometric:

$$\sigma_j = (1-\rho)\rho^j \quad \text{for } j \geq 0.$$

(a) Derive the stationary expected number of customers in the queue for both the $M/M/2$ and the $M/M/1$ queues.

(b) Derive the expected waiting time of an arriving customer for these two queues in their stationary regimes.

(c) Derive the stationary expected number of busy servers in the queue for both the $M/M/2$ and the $M/M/1$ queues.

(d) A company that shreds documents is trying to determine whether to buy one or two shredders. Documents to be shredded arrive in pallets according to a Poisson process with rate 1 per hour. The number of hours it takes a shredder to shred a pallet of paper has an exponential distribution with rate 2 per hour. The cost of storing a pallet of paper is 16 dollars per pallet per hour, the cost of maintaing a shredder is 7 dollars per hour and when a shredder is in use there is an additional cost of electricity of 3 dollars per hour. Over the long run will the company's expenses be lower with one shredder or two?

(e) How would you answer the previous question (d) if the cost of storing a pallet of paper is 14 dollars per pallet per hour, with all other variables the same?

6. (12 Marks) The lifetime in hours of a saw blade in a lumber mill is a random variable with density proportional to $x(2-x)$ on $0 < x < 2$. When the mill opened a new blade was put in the saw. Every time a blade fails it is immediately replaced with a new one. Since the mill is largely automated, it operates 24 hours a day, seven days a week, year round.

   (a) Model the number N_t of saw blades that have been replaced t hours after the mill opened as a renewal process and determine the density, mean, and variance of the random times between renewals.

   (b) On average, about how many replacement blades for the saw will the mill go through in the first 30 days of opening?

   (c) Give an interval around your estimate from (b) that will have a 95% chance of covering the true number of saws needed in the first 30 days.

   (d) If you show up at the mill at the end of day 30, about what is the mean and variance of the amount of time you'll have to wait for the current blade in use to be replaced?

7. (8 Marks) Denote the transition probabilities for a time homogenous Markov chain on $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ by $p_{i,j} := \mathbb{P}(X_1 = j | X_0 = i)$. Let $0 < p < 1$ and assume that for each i in $\mathbb{Z}$,

$$p_{i,i+2} = p, \;\; p_{i,i-1} = 1 - p \;\text{ and } p_{i,j} = 0 \;\text{ for all } j \neq i+2, i-1.$$

For each state i in $\mathbb{Z}$, determine the values of p where i is recurrent and the values where i is transient. You may find it helpful to use Stirling's approximation:

$$1 \leq \frac{n! e^n}{n^n \sqrt{2\pi n}} \leq 2.$$