MAST20009 Vector Calculus

Practice Class 3 Questions

> Lagrange Multipliers
>
> If $\boldsymbol{a}$ is an extremum of $f(\boldsymbol{x})$ subject to the constraint $g(\boldsymbol{x}) = 0$, then there exists a real number λ such that
> $$\boldsymbol{\nabla} f = \lambda \boldsymbol{\nabla} g$$
> at $\boldsymbol{x} = \boldsymbol{a}$.

1. Consider the function
$$f(x, y) = x^3 - 3xy^2$$
   (a) Find and classify the extrema of f in the region $x^2 + y^2 < 1$.

   (b) Find and classify the extrema of f subject to the constraint $x^2 + y^2 = 1$.

   (c) Determine the absolute minimum and absolute maximum values of f on the set $S = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$.

> The arclength s of a path $\boldsymbol{c}(t) = (x(t), y(t), z(t))$ for $a \leq t \leq b$ is given by:
> $$s \;=\; \int_a^b \left|\frac{d\boldsymbol{c}}{dt}\right| dt \;=\; \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$
> We can parametrise a path in terms of arclength by defining
> $$s(t) \;=\; \int_a^t \left|\frac{d\boldsymbol{c}}{d\tau}\right| d\tau$$
> to be the length from a point P_0 to any point P on the path.

2. Consider the curve
$$\boldsymbol{c}(t) = (\cos t, \sin t, t)\,.$$
   (a) Find the arclength of the curve from $\boldsymbol{c}(0)$ to $\boldsymbol{c}(2\pi)$.

   (b) Parametrise the curve in terms of the arclength s.

For a curve $\boldsymbol{c}(t)$

Unit tangent vector:	$\boldsymbol{T}(t) = \dfrac{\frac{d\boldsymbol{c}}{dt}}{\left\lvert\frac{d\boldsymbol{c}}{dt}\right\rvert}$
Unit normal vector:	$\boldsymbol{N}(t) = \dfrac{\frac{d\boldsymbol{T}}{dt}}{\left\lvert\frac{d\boldsymbol{T}}{dt}\right\rvert}$
Unit binormal vector:	$\boldsymbol{B}(t) = \boldsymbol{T} \times \boldsymbol{N}$
Curvature:	$\kappa(t) = \dfrac{\left\lvert\frac{d\boldsymbol{T}}{dt}\right\rvert}{\left\lvert\frac{d\boldsymbol{c}}{dt}\right\rvert}$
Torsion:	$\tau(t)$ such that $\dfrac{d\boldsymbol{B}}{ds} = \dfrac{\frac{d\boldsymbol{B}}{dt}}{\left\lvert\frac{d\boldsymbol{c}}{dt}\right\rvert} = -\tau\boldsymbol{N}$

3. Consider the curve
$$\boldsymbol{c}(t) = (e^t \sin t, e^t \cos t, e^t).$$
Find $\boldsymbol{T}(t)$, $\boldsymbol{N}(t)$, $\boldsymbol{B}(t)$, $\kappa(t)$, and $\tau(t)$.

When you have finished the above questions, continue working on the questions in the Vector Calculus Problem Sheet Booklet.