The University of Melbourne

Department of Mathematics and Statistics

620-201 Probability

Semester 1 Exam — June 24, 2009

Exam Duration: 3 Hours

Reading Time: 15 Minutes

This paper has 5 pages

Authorised materials:
Students may bring one double-sided A4 sheet of handwritten notes into the exam room. Hand-held electronic calculators may be used.

Instructions to Invigilators:
Students may take this exam paper with them at the end of the exam.

Instructions to Students:
This paper has **nine** (9) questions.
Attempt as many questions, or parts of questions, as you can.
The approximate number of marks allocated to each question is shown in the brackets after the question statement.
The total number of marks available for this examination is 100.
Working and/or reasoning must be given to obtain full credit.

1. Consider a random experiment in which the 'spin' of a particle is observed. There are two possibilities: the spin can be positive (+) or negative (-).
   (a) (i) Write down the sample space Ω for this experiment.
   (ii) Let $\mathcal{A}$ be the set of events of the experiment. Write down the elements of $\mathcal{A}$.
   (b) State the axioms that must be satisfied by a probability mapping P from $\mathcal{A}$ to $[0, 1]$.
   (c) Using the axioms that you wrote down in part (b), show that the probability of the empty set $\emptyset$ is zero.

[10 marks]

2. Applications for funding from the Centralian Research Council can be either *poor*, *fair* or *good*. The assessment process is not perfect. Specifically the probabilities of grants being rated poor, fair or good given that they are poor, fair or good are given in the table below.

		rating		
		p	f	g
	p	0.5	0.3	0.2
quality	f	0.3	0.4	0.3
	g	0.2	0.2	0.6

Applications are funded if and only if they are rated as good.
   (a) Assume that an arbitrary application is poor, fair or good with probability 0.2, 0.4 and 0.4 respectively. What is the probability that it will be funded?
   (b) What is the probability that a funded application was actually poor?
   (c) Carefully stating your reasons, explain whether the events (p), that an arbitrary application is poor, and (rf), that it is rated fair, are independent.

[7 marks]

3. (a) Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} 1/4 & \text{if } x \in (-2, 2) \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability density functions of
   (i) $Y = X^3$, and
   (ii) $Z = X^4$.

(i) $P(N \geq 1) = 1 - P(N=0) = 1 - P(T \geq 5)$
no event
$= 1 - (1 - P(T < 5))$
$= P(T < 5)$
$= 1 - e^{-15}$

(ii) ? at least a signal
$N \overset{d}{=} G(1 - e^{-15})$

(iii) $P_N(n) = (e^{-15})^n (1 - e^{-15})$

(iv) $F_{X|A}(x|A) = P(X \leq x | A) = \frac{P(X \leq x, A)}{P(A)} = \frac{1 - e^{-3x}}{1 - e^{-15}}$

(v) at most n-1 period with no signal, the latest time will receive the signal is period n
$P(T < t | N=k) = 1$ for $k \leq n-1$
for $k \geq n+1$ $P(T < t | N=k) = 0$
$k \geq n+1$ at least n+1 on period

(b) Signals are sent from a transmitter at intervals that are exponentially distributed exp(3). with parameter $\lambda = 3$. The receiver that is used to monitor the transmissions is alternately turned on for five seconds and then turned off for three seconds.

(i) What is the probability that at least one signal is received in a five-second 'on period'?

(ii) Let N be the number of 'on periods' in which no signal is received before there is an 'on period' in which a signal is received. Giving your reasons, name the distribution of the random variable N and state the value of any parameter(s).

(iii) Write down the probability mass function of N.

(iv) What is the distribution function of the time until the first receipt of a signal in an 'on period', conditional on at least one signal being received in this 'on period'?

(v) Using your result from part (iv), calculate the distribution function of the total amount T of time taken up in 'on periods' until the first receipt of a signal. (Hint: I suggest that you condition on the value of N).

(vi) Name the distribution of the random variable T. Compare this with the distribution of the intervals between signal transmission.

[20 marks]

4. You walk into a casino and decide to bet \$10 on 'red' (which has a probability of 18/37 of coming up) in roulette. If you win, you make a profit of \$10 while, if you lose, the casino keeps your \$10. Counting a loss as negative, let the random variable X_i be the amount that you gain from the ith bet. You can consider the X_i to be independent.

(a) What is the expected value of X_i?

(b) What is the variance of X_i?

(c) Carefully justifying your reasoning, derive the expected total gain from a sequence of ten bets.

(d) Again carefully justifying your reasoning, derive the variance in the total gain from a sequence of ten bets.

(e) Instead of placing ten bets of \$10, you decide to place one bet of \$100. How do the mean and variance of your total gain compare with those that you calculated in parts (b) and (c)?

[10 marks]

5. Two random variables X and Y have a joint probability density function of the form

$$f_{(X,Y)}(x,y) = \begin{cases} cx^2y & \text{if } 0 \le x \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Draw a diagram identifying the region in the xy plane where the joint probability density function is non-zero.

(b) Calculate the value of the constant c.

(c) Find the marginal probability density function of X.

(d) Verify that the expression that you derived in part (c) integrates to one over the appropriate range.

(e) For $0 < x < 1$, find the conditional probability density function of Y given $X = x$.

(f) Calculate the covariance of X and Y.

[12 marks]

6. For $k = 1, 2, \ldots$, let X_k be independent and identically-distributed random variables with $E(X_k) = \mu$ and $V(X_k) = \sigma^2$ and let N be independent of the X_k with mean λ and variance λ^2. Define

$$T = \sum_{k=1}^{N} X_k$$

(a) By conditioning on N, find $E(T)$.

(b) Similarly find $V(T)$.

(c) How would the variance of T be reduced if the variance of N was equal to zero instead of λ^2?

[9 marks]

7. In this question, you may use the fact that, for $s > 0$, the gamma function is defined by

$$\Gamma(s) = \int_0^\infty e^{-u} u^{s-1} du. \qquad (*)$$

For non-negative integer d, a random variable X that has a probability density function

$$f(x) = \begin{cases} \dfrac{x^{d/2-1} e^{-x/2}}{\Gamma(d/2) 2^{d/2}} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

is known as a χ^2-random variable with d degrees of freedom.

(a) For what values of t is the moment generating function $M_X(t)$ of such a random variable X defined?

(b) Derive $M_X(t)$. (Hint: You will need to use an appropriate integral substitution to reduce the integral to something that looks like the right hand side of (*)).

(c) Hence calculate the mean and variance of X.

[10 marks]

8. (a) Let $X_1, X_2, \ldots$ be independent and identically-distributed random variables with $E(X_i) = \mu$ and let $S_n = \sum_{i=1}^{n} X_i$. Assume that the moment generating function $M_{X_i}(t)$ of X_i exists for some $t \neq 0$. The (Weak) Law of Large Numbers states that

$$Z_n = \frac{S_n}{n}$$

converges in distribution to a random variable which takes the constant value μ with probability one.

Prove the Law of Large Numbers by showing that the moment generating function $M_{Z_n}(t)$ converges to $e^{\mu t}$ as $n \to \infty$.

(b) Let $X_1, \ldots, X_{100}$ be independent Bernoulli random variables with $P(X_i = 1) = 1 - P(X_i = 0) = 0.2$ and let $S_{100} = \sum_{i=1}^{100} X_i$.

(i) Write down $E(X_i)$ and $V(X_i)$.

(ii) Use the Central Limit Theorem to derive an approximation for the probability that $16 \leq S_{100} \leq 24$.
(You may use the following information about a standard normal random variable Z: $P(Z \leq 0) = .5000$, $P(Z \leq 1) = .8413$, $P(Z \leq 2) = .9772$, $P(Z \leq 3) = .9987$).

(iii) Physically, what does the random variable S_{100} represent? Name its distribution, giving the values of any parameters.

(iv) Let Y_n be a $Bi(n, p)$ random variable. Motivated by your answer to part (iii), explain how the Central Limit Theorem can be used to derive the fact that Y_n converges in distribution to a normally distributed random variable with mean np and variance $np(1-p)$ as $n \to \infty$.

[14 marks]

9. Consider the Branching Process $\{X_n, n = 0, 1, 2, 3, \ldots\}$ where X_n is the population size at the nth generation. Assume $P(X_0 = 1) = 1$ and that the probability generating function of the common offspring distribution

$$A(z) = \frac{1}{5 - 4z},$$

which is defined for $0 \leq z < 5/4$.

(a) Express $A(z)$ as a power series and hence find the probability that an individual has four offspring.

(b) Find the expected value and the variance of the number of offspring that an individual has.

(c) If $q_n = P(X_n = 0)$ for $n = 0, 1, \ldots$, write down an equation relating q_{n+1} and q_n. Hence, or otherwise, evaluate q_n for $n = 0, 1, 2$.

(d) Find the extinction probability $q = \lim_{n \to \infty} q_n$.

[8 marks]

End of the exam