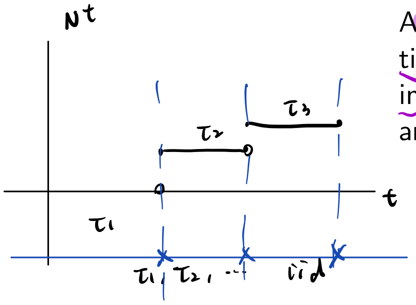
Renewal theory



cdf F

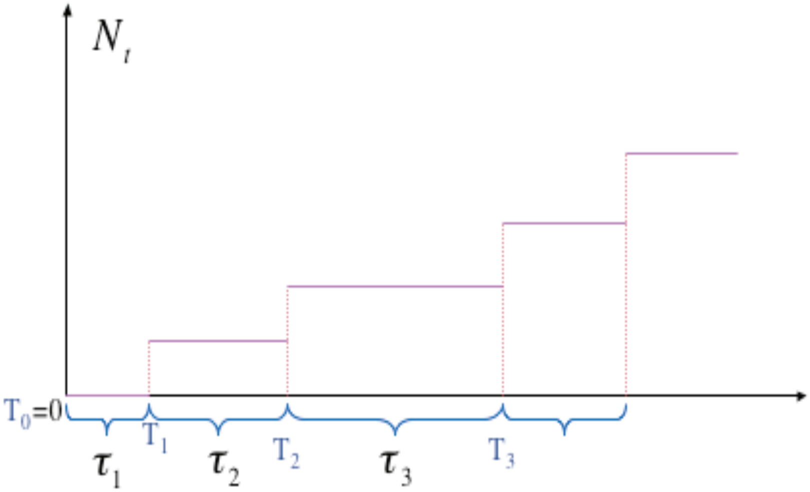
general

A renewal process $\{N_t : \ t \geq 0\}$ is a counting process for which the times $\tau_j \geq 0$ between successive events, called renewals, are independent and identically-distributed random variables with an arbitrary common distribution function F.

- We assume $F(0) < 1$. if $F(0) = 1$, mean $\tau_i = 0$ with probability 1
- A Poisson process is a renewal process, but a renewal process may not be Poisson. usually assume $F(0) = 0$ (no chance $\tau_i = 0$)
- A renewal process that is not a Poisson process is not Markovian.

$\Rightarrow$ if τ_i exponential, then it is Poisson process

Renewal theory



$$T_k = \tau_1 + \ldots + \tau_k,\ N_t = \max\{k : T_k \le t\}$$

↓ # of renewal

Renewal theory

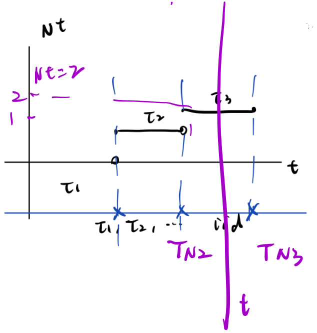
When we looked at the Poisson process, we saw that we could use a counting process description in terms of the number N_t of points in the interval $[0, t]$ or a waiting time description in terms of the time T_n until the nth event. This carries over to the study of renewal processes. Specifically

- $\{N_t \geq n\} = \{T_n \leq t\}$
- $\{N_t < n\} = \{T_n > t\}$
- $\{N_t = n\} = \{T_n \leq t < T_{n+1}\}$
- $T_{N_t} \leq t < T_{N_t+1}$.

$\{N_t \geq n\}$ = "at least n arrivals in $[0, t]$"

= "nth arrival is no greater than t"

= $\{T_n \leq t\}$



Example

Light bulbs have a lifetime that has distribution function F. If a light bulb burns out, it is immediately replaced. Let N_t be the number of bulbs that have failed by time t. Then N_t is a renewal process.

$T_n = \sum_{i=1}^{n} \tau_i$ sum of iid r.v

for large n. LLN $T_n \approx n\mu$.

CLT $T_n \approx N(n\mu, n\sigma^2)$

Renewal theory

Three questions:

- Can there be an explosion (that is an infinite number of renewals in a finite time)?
- What is the distribution of N_t?
- What is the average renewal rate? That is, at which rate does $N_t \to \infty$?

Explosion?

in finite time, probability reaches infinity is 0

For any fixed $t < \infty$, $P(N_t = \infty) = 0$. This is true in general, but assuming τ_1 has finite mean the WLLN implies

$$\begin{aligned} P(N_t = \infty) &= \lim_{n\to\infty} P(N_t \geq n) \\ &= \lim_{n\to\infty} P(T_n \leq t) \\ &= 0. \end{aligned}$$

$P(T_n \le t)$

$= P(\frac{T_n}{n} \le \frac{t}{n})$

$= P(\mu \le \frac{t}{n})$

$\mu = E(\tau_1)$

let N large enough, That $\frac{t}{N} < \mu/2$, $\lim_{n\to\infty} \frac{t}{n} = 0$ but $\mu > 0$ contradiction

we can always choose N to let $\frac{t}{N} < \mu/2$

$P(\frac{T_n}{n} \le \frac{t}{n})$

$\therefore \lim_{n\to\infty} P(T_n \le t) = 0$

since $n \to \infty$, so $n \geq N$ hold

$= P(\frac{T_n}{n} \le \frac{\mu}{2})$

WLLN

(from $T_n < t$) $\frac{T_n}{n} \le \frac{t}{n} \le \frac{t}{N} < \frac{\mu}{2}$ for $n \geq N$ $\to 0$

$P(|\frac{T_n}{n} - \mu| > \delta) \to 0$ as $n \to \infty$ for $\forall \delta > 0$

$\delta = \frac{\mu}{2}$

Distribution of N_t

Also, for any fixed n,

as $t \to \infty$, $N_t \to \infty$ with prob 1

$$\begin{aligned} \lim_{t\to\infty} P(N_t \geq n) &= \lim_{t\to\infty} P(T_n \leq t) \\ &= 1. \end{aligned}$$

$\to$ CDF at $t \to \infty$

So, with probability one, $N_t \to \infty$ as $t \to \infty$.

Distribution of N_t

$P(T_n \le t,\ T_{n+1} > t)$

$= P(T_n \le t) - P(T_n \le t, T_{n+1} \le t)$

$= P(T_n \le t) - P(T_{n+1} \le t)$

$$
\begin{aligned}
P(N_t = n) &= P(T_n \le t < T_{n+1}) \\
&= P(T_n \le t) - P(T_n \le t,\, T_{n+1} \le t) \quad = F_n(t) - F_{n+1}(t) \\
&= P(T_n \le t) - P(T_{n+1} \le t) \\
&= F_n(t) - F_{n+1}(t)
\end{aligned}
$$

where F_n is the distribution function of T_n, or n–fold convolution of F.

Distribution of N_t

time for N_t th arrival

Above, we saw that $T_{N_t} \leq t < T_{N_t+1}$. It follows that

$$\frac{N_t}{T_{N_t+1}} < \frac{N_t}{t} \leq \frac{N_t}{T_{N_t}}$$

SLLN : $\frac{T_{N_t}}{N_t} \to \mu$

$\left(\frac{T_n}{n} \to \mu\right)$

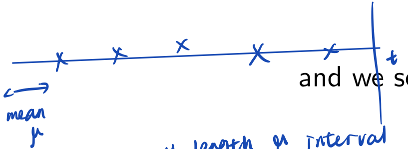
$\frac{N_t}{T_{N_t+1}} = \frac{N_t}{(N_t+1)} \frac{N_t+1}{T_{N_t+1}} \xrightarrow{\text{SLLN}} \frac{1}{\mu}$

tend to 1

Since $N_t \to \infty$ as $t \to \infty$, the Strong Law of Large Numbers tells us that, with probability one, both the first and third terms approach μ^{-1}.

Therefore, with probability one,

$$\lim_{t \to \infty} \frac{N_t}{t} = \mu^{-1},$$



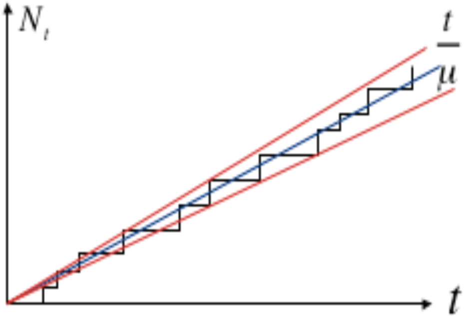
and we see that, for large t, N_t grows like t/μ.

How many length μ interval in interval $[0, t]$ for large t $\approx \frac{t}{\mu}$

same picture to argue proportion of visits to a state of a DTMC $\approx \frac{1}{\text{Expect return time}}$

E[return time]

Renewal theory



Example

Jenny has a tv remote that runs on batteries. When a battery dies, she immediately replaces it with a new (or fully charged) battery. If a new battery's lifetime follows $U(30, 60)$ (months), then at what rate does Jenny have to change batteries? (how many time per month)

- $\mu = E[\tau_1] = 45$, so the rate is

$$\lim_{t \to \infty} \frac{N_t}{t} = \frac{1}{\mu} = \frac{1}{45} \text{ per month.}$$

The $M/G/1/1$ queue

There is no queue: when an arriving customer finds server busy, they do not enter.

Service times are independent and identically-distributed with distribution function G with the mean m_G.

- What is the rate at which customers enter the system?
- What proportion of potential customers actually enter the queue?

Rate customer enter system

$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mu}$

N_t that enter at interval $[0,t]$

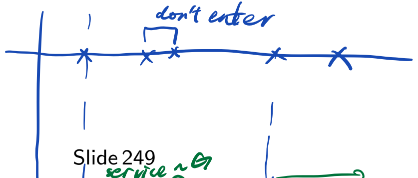
service

for every 1 arrival have Poisson (G) # of rejection

proportion that enter $\frac{1}{1+\lambda m_G}$.

N_t = renewal process driven by τ_i

$\frac{N_t}{t} = \frac{1}{\frac{1}{\lambda} + m_G}$ arrival process



if n blocks $\frac{n}{n+\sum_{i=1}^{n} \text{Poisson}(G_i)} = \frac{1}{1+\frac{1}{n}\sum \text{Poi}(G_i)}$

iid

↳ $E[\text{Poisson } G]$

↳ λM_G

LLN

queue

$\exp(\lambda)$

τ_1

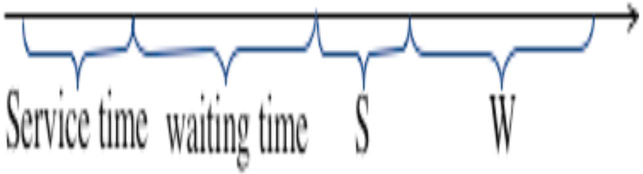
τ_2

why τ_1 ?

The $M/G/1/1$ queue

Let N_t be the number of customers who have been admitted by t. Then the times between successive entries of customers are made up of

- a service time, and then
- a waiting time from the end of service until the next arrival.



Renewal theory

The mean time between renewals is $\underbrace{\mu = 1/\lambda + m_G}$. So that rate at which customers enter is ✓.

$$\frac{1}{\mu} = \frac{1}{1/\lambda + m_G} = \frac{\lambda}{1 + \lambda m_G}.$$ ✓.

Customers arrive at rate λ, and so the proportion that enters the queue is ?

$$\frac{\text{entry rate}}{\text{arrival rate}} = \frac{\lambda/(1 + \lambda m_G)}{\lambda} = \frac{1}{1 + \lambda m_G}.$$ ✓.

If $\lambda = 10$ per hour and $m_G = 0.2$ hours then, on average, only 1 out of 3 customers will actually enter the queue.

The Central Limit Theorem

If $E[\tau_j] = \mu$, $V(\tau_j) = \sigma^2 < \infty$, then

LLN $\Rightarrow$ $\frac{N_t}{t} \to \frac{1}{\mu}$ $\quad N_t \to \frac{t}{\mu}$

$$\frac{N_t - \frac{t}{\mu}}{\sqrt{t\sigma^2/\mu^3}} \xrightarrow{d} N(0,1) \text{ as } t \to \infty.$$

So for each x,

$$\lim_{t\to\infty} P\left(\frac{N_t - \frac{t}{\mu}}{\sqrt{t\sigma^2/\mu^3}} \le x\right) = \Phi(x)$$

where Φ is the normal distribution function.

$P\left(\frac{N_t - \frac{t}{\mu}}{\partial t} \geq x\right)$

$= P(N_t \geq \partial t x + \frac{t}{\mu})$, as an integer

$= P(N_t \geq \lceil \partial t x + \frac{t}{\mu} \rceil) \to k_t$

$= P(T_{k_t} \leq t)$

$= P\left(\frac{T_{k_t} - \mu k_t}{\sqrt{k_t \cdot \sigma^2}} \leq \frac{t - \mu k_t}{\sqrt{k_t \sigma^2}}\right)$

$\downarrow$

$$\approx P\left(Z \le \frac{t-\mu k t}{\sqrt{k t \sigma^2}}\right)$$

$$= P\left(Z \ge \frac{\mu k t - t}{\sqrt{k t \sigma^2}}\right)$$

Proof

Let $Z = \frac{T_i - i\mu}{\sqrt{i\sigma^2}} \overset{d}{\approx} N(0,1)$. Then

$$\begin{aligned} P(N_t \ge i) &= P(T_i \le t) \\ &\approx P\left(Z \le \frac{t - i\mu}{\sqrt{i\sigma^2}}\right) \\ &= P\left(Z \ge \frac{i\mu - t}{\sqrt{i\sigma^2}}\right). \end{aligned}$$

need to choose k_t

such that $\frac{\mu k_t - t}{\sqrt{k_t \sigma^2}} \to x$

$$\frac{\mu[\partial_t x + \frac{t}{\mu}] - t}{\sqrt{[\partial_t x + \frac{t}{\mu}]\sigma^2}} \to x$$

$$= \frac{\mu(\partial_t x + \cancel{\tfrac{t}{\mu}} + \delta_t x) - \cancel{t}}{\sqrt{(\partial_t x + \frac{t}{\mu} + \delta_t x)\sigma^2}} = \frac{\boxed{\mu \partial_t x} + \boxed{\mu \delta_t x}}{\sqrt{\boxed{\frac{t}{\mu}\sigma^2} + \boxed{(\partial_t x + \delta_t x)\sigma^2}}}$$

kills

$$= \frac{\mu \partial_t x}{\sqrt{\frac{t}{\mu}\sigma^2}} \to x$$

need to choose $\partial_t = c\sqrt{t}$

$$\partial_t \cdot \frac{\mu}{\sqrt{\frac{t}{\mu}\sigma^2}} \to 1$$

$$\partial_t = \sqrt{\frac{t\sigma^2}{\mu^3}}$$

Proof

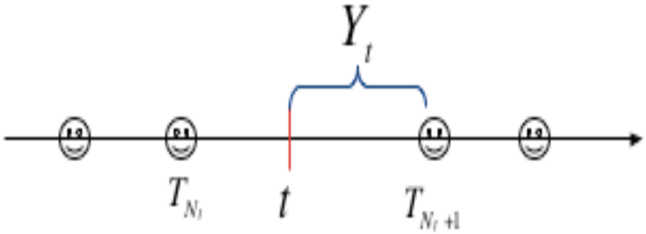
Now, we choose $i(x)$ such that $\frac{i\mu - t}{\sqrt{i\sigma^2}} \approx x$. That is, we put $i(x) \approx \frac{t}{\mu} + x\sqrt{\frac{t}{\mu} \cdot \frac{\sigma^2}{\mu^2}}$.

Then, reversing the above argument, we have

$$\begin{aligned} P(Z \geq x) &\approx P(N_t \geq i(x)) \\ &\approx P\left(\frac{N_t - \frac{t}{\mu}}{\sqrt{t\sigma^2/\mu^3}} \geq x\right). \end{aligned}$$

Residual lifetime

Since $T_{N_t} \leq t < T_{N_t+1}$, the residual lifetime of the component at time t is $Y_t = T_{N_t+1} - t > 0$.



Renewal theory

limiting "CDF"

$x \to \frac{1}{\mu}\int_0^x (1-F(y))dy$

$\int_0^\infty P(T>y)dy$

$= \int_0^\infty \int_y^\infty f_T(x)\,dx\,dy$

$= \int_0^\infty f_T(x) \int_0^x dy\,dx$

$= \int_0^\infty x f_T(x)\,dx$

$= \mu.$

When the distribution of the τ_j is not arithmetic (that is, it does not concentrate its mass at multiples of a fixed amount), then, for all $x \geq 0$,

$$\lim_{t\to\infty} P(Y_t \leq x) = \frac{1}{\mu}\int_0^x (1 - F(y))dy.$$

Note that, for a non-negative random variable Z,

$$E[Z] = \int_0^\infty (1 - F_Z(z))dz,$$

so $\frac{1-F(y)}{\mu}$, $y \geq 0$, is a probability density function.

$$\frac{\frac{1}{n}\sum_{i=1}^n (\tau_i - x)\,\mathbb{1}(\tau_i > x)}{\frac{1}{n}\sum_{i=1}^n \tau_i}$$

$$\overset{n\to\infty}{=} \frac{E[(\tau - x)\cdot \mathbb{1}(\tau > x)]}{\mu.} \Rightarrow \int_0^\infty P((\tau-x)\mathbb{1}(\tau>x) > y)\,dy$$

$$\overset{y=u-x}{=} \int_x^\infty P((\tau-x)\mathbb{1}(\tau-x) > x \quad)\,du$$

$$= \int_x^\infty P(\tau > u)\,du$$

$$\lim_{t\to\infty} P(Y_t > x) = \frac{\int_x^\infty P(\tau>u)\,du}{x}$$

Sketch of Proof

Consider a period of n renewals. The proportion of time that the residual lifetime is greater than x is, by the strong Law of Large Numbers,

$$\begin{aligned} \frac{\sum_{i=1}^n (\tau_i - x) 1_{[\tau_i > x]}}{\sum_{i=1}^n \tau_i} &= \frac{\frac{1}{n}\sum_{i=1}^n (\tau_i - x) 1_{[\tau_i > x]}}{\frac{1}{n}\sum_{i=1}^n \tau_i} \\ &\to \frac{E\left[(\tau_1 - x) 1_{[\tau_1 > x]}\right]}{E[\tau_1]}. \end{aligned}$$

as n approaches infinity.

Renewal theory

Under the stated conditions, it can also be shown that

$$\frac{\sum_{i=1}^{n}(\tau_i - x)1_{[\tau_i > x]}}{\sum_{i=1}^{n}\tau_i} \to \lim_{t\to\infty} P(Y_t > x).$$

Hence

$$\begin{aligned}
\lim_{t\to\infty} P(Y_t > x) &= \frac{E\left[(\tau_1 - x)1_{[\tau_1 > x]}\right]}{E[\tau_1]} \\
&= \frac{1}{\mu}\int_0^\infty P((\tau_1 - x)1_{[\tau_1 > x]} > y)dy \\
&= \frac{1}{\mu}\int_x^\infty P(\tau_1 > u)du.
\end{aligned}$$

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{4}x - \frac{1}{4} & 1 \le x \le 5 \\ 1 & x \ge 5 \end{cases}$$

Example

A computer receives packets of information whose sizes are uniformly distributed between 1 and 5 GB. It saves them on hard drives of total size 700GB, until the a hard drive is full.

$$1 - F(x) = \begin{cases} 1 & 0 < x < 1 \\ \frac{5-x}{4} & 1 \le x \le 5 \\ 0 & \text{else} \end{cases}$$

- For the first file for which there is not enough space on a hard drive, find the approximate distribution and the mean of the length of the residual part that the hard drive does not have space for.

- Give an approximate interval to which, with probability 0.95, the total number of saved files belongs.

N_{700} = # saved files

Solution

- The limiting distribution of the residual part has density

$$\frac{1}{\mu}(1 - F(x)) = \begin{cases} \frac{1}{3} & \text{if } x \in [0, 1) \\ \frac{5-x}{12} & \text{if } x \in [1, 5]. \end{cases}$$

- The mean of the residual part is $31/18$, which is greater than half of the mean interval length, which is $3/2$.
- We have

$$\frac{N_t - \frac{t}{\mu}}{\sqrt{\frac{t}{\mu} \times \frac{\sigma^2}{\mu^2}}} \overset{appr}{\sim} N(0, 1).$$

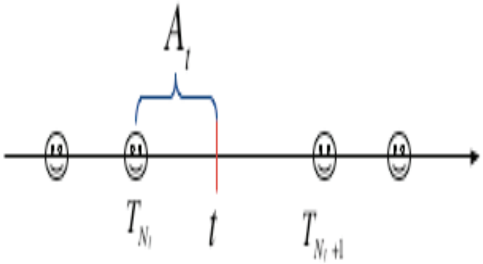
With $t = 700$, $\mu = 3$, $\sigma^2 = 4/3$, the desired (symmetric) interval is $233.33 \pm 5.88 \times 1.96 = (221.81, 244.85)$.

Y_t residual lifetime

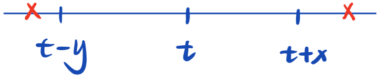
A_t age

The limiting distribution of age

The age of the component at time t is $A_t = t - T_{N_t}$,



Joint dist of Y_t & A_t



$\{A_t > y,\ Y_t > x\}$

$= \{Y_{t-y} > x+y\}$

Renewal theory

Now the event $\{Y_t > x, A_t > y\}$ is the same as $\{Y_{t-y} > x + y\}$, so

$$\begin{aligned}
\lim_{t\to\infty} P(Y_t > x, A_t > y) &= \lim_{t\to\infty} P(Y_{t-y} > x + y) \\
&= \frac{1}{\mu}\int_{x+y}^{\infty} [1 - F(z)]dz,
\end{aligned}$$

which, putting $x = 0$, implies that

$$\begin{aligned}
\lim_{t\to\infty} P(A_t \le y) &= \frac{1}{\mu}\int_{0}^{y} [1 - F(z)]dz \\
&= \lim_{t\to\infty} P(Y_t \le y).
\end{aligned}$$

Why? Some Intuition

- Consider the process after it has been in operation for a long time.
- When we look backwards in time, the times between successive renewals are still independent and identically-distributed with distribution F.
- Looking backwards, the residual lifetime at t is exactly the age at t of the original process.

Example (continued)

For large t, find the joint probability density function of (Y_t, A_t) in the previous example.

- First,

$$P(A_t \le x, Y_t \le y) = P(A_t \le x) - P(Y_t > y) + P(A_t > x, Y_t > y),$$

so

$$\frac{\partial^2 P(A_t \le x, Y_t \le y)}{\partial x \partial y} = \frac{\partial^2 P(A_t > x, Y_t > y)}{\partial x \partial y}.$$

- When t is large, $P(A_t > y, Y_t > x) \approx \int_{x+y}^{\infty} \frac{1-F(z)}{\mu} dz$.
- Hence, the joint pdf is $1/12$ if $1 < x + y < 5$ and 0 otherwise.

Example

Suppose $\{N_t, t \geq 0\}$ is a Poisson process with rate λ, find the distributions of Y_t, A_t and (Y_t, A_t) when t is large. What is the expected duration of the inter-event time $T_{N_t+1} - T_{N_t}$?