Department of Finance
FNCE10002 Principles of Finance
Teaching Note 1
Introduction to Financial Mathematics*

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Outline

This teaching note deals with the area of Principles of Finance that are central to the understanding of the investment and financing decisions of investors and companies, namely financial mathematics. After reading this note you should be able to:

- Calculate the future and present values of a series of cash flows
- Calculate the future and present values of ordinary annuities
- Calculate the future and present values of annuities due
- Calculate the present value of ordinary and deferred perpetuities
- Calculate the future and present values of growing annuities

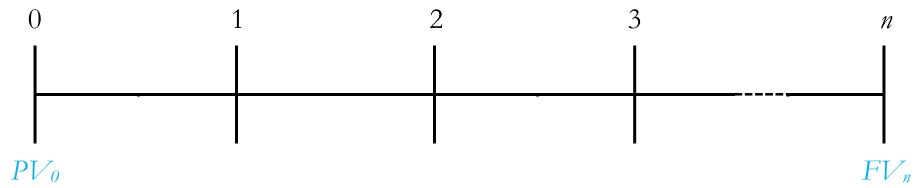
1. Future and present values of a single cash flow

Given the time value of money, a cash flow received today (that is, time 0) is more valuable than the same cash flow received some time in the future. Accordingly, a cash flow of PV_0 today that earns interest at a rate of r percent per period for n periods has a future value of:

$$FV_n = PV_0(1+r)^n, \tag{1}$$

where $(1+r)^n$ is the future dollar value of \$1 today earning an interest rate of r percent per period for n time periods. This amount is then multiplied by PV_0 to obtain its future value at the end of time period n. On a timeline, the cash flow can be represented as follows:

Figure 1: Present and Future Values of Single Cash Flows



Note that in the above timeline we're assuming that the cash flow occurs at the end of a particular period. For example, PV_0 occurs at the end of period 0 while FV_n occurs at the end of period n. This is the convention that we use throughout this subject to simplify the calculations. Where a cash flow does not occur at the end of a period it would need to be specified as such. (For example, in our discussion of annuities due later in this note we will assume that the cash flows occur at the beginning of the period.)

Example 1: Future value of a single cash flow

What is the future value of \$1,000 invested at an interest rate of 10 percent per annum at the end of: (a) 3 years and (b) 20 years?

Solution

a) The future value at the end of 3 years is:

$FV_3 = 1000(1 + 0.10)^3 = \$1,331.00.$

b) The future value at the end of 20 years is:

$FV_{20} = 1000(1 + 0.10)^{20} = \$6,727.50.$

A similar principle applies in the determination of a present value equivalent of an expected cash flow in a future period. Formally, a cash flow of FV_n that is due in n periods and is discounted at a rate of r percent per period has a present value today of:

$$PV_0 = \frac{FV_n}{(1+r)^n}, \tag{2}$$

where $1/(1+r)^n$ is the present dollar value of \$1 to be received n time periods from today at an interest rate of r percent per period. Note that the term $1/(1+r)$ is referred to as the one-period *discount factor*. It is the dollar value today of \$1 occurring at the end of one period discounted at an interest rate of r.

Example 2: Present value of a single cash flow

Calculate the present value today of \$1,331.00 occurring at the end of year 3 assuming an interest rate of 10 percent per annum. Calculate the present value of \$6,727.50 occurring at the end of year 20 assuming an interest rate of 10 percent per annum.

Solution

The present value today of \$1,331 at the end of year 3 is:

PV_0 = 1331/1 + 0.10)^3 = \$1,000.00.

The present value today of \$6,727.50 at the end of year 20 is:

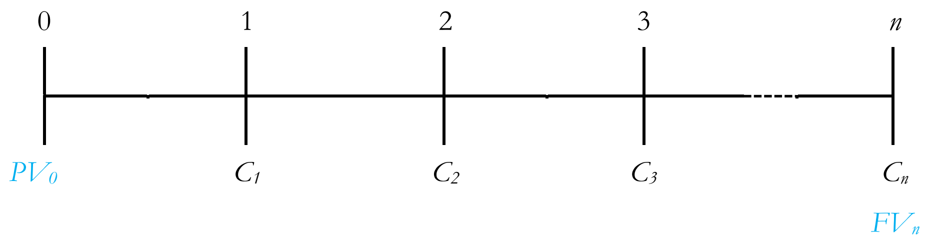
PV_0 = 6727.50/1 + 0.10)^{20} = \$1,000.00.

Note that, as expected, the present values calculated here are the same as those in the previous example.

2. Future and present values of a series of cash flows

If we are given a series of cash flows that occur over different time periods their future value can be calculated as the sum of the future values of the individual cash flows. This is also known as the *value additivity principle*. On a timeline, the cash flow can be represented as follows:

Figure 2: Present and Future Values of Multiple Cash Flows with no Cash Flow at Time 0

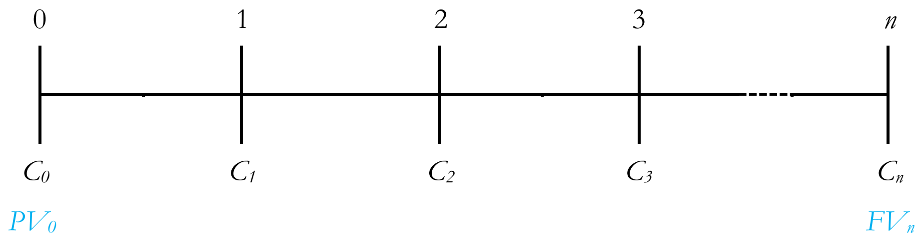


As the future value of a sum of a series of cash flows earning an interest rate of r percent per period at the end of n periods is equal to the sum of their individual future values, we have:

$$FV_n = C_1(1+r)^{n-1} + C_2(1+r)^{n-2} + \ldots + C_n. \tag{3a}$$

Note that in the above expression we assume that the first cash flow occurs at the end of time 1 and the last cash flow occurs at the end of time n. Clearly, the cash flow occurring at the end of time n will not earn any interest. If we assume that the first cash flow occurs at the end of time 0 (that is, today) then the cash flows will look a little different, as follows:

Figure 3: Present and Future Values of Multiple Cash Flows with a Cash Flow at Time 0



The future value at the end of time n will now be:

$$FV_n = C_0(1+r)^n + C_1(1+r)^{n-1} + C_2(1+r)^{n-2} + \ldots + C_n. \tag{3b}$$

This amount is higher than what we got in expression (3a) because there is now one additional cash flow at time 0 (C_0) which earns interest over n time periods.

Example 3: Future value of a series of cash flows

Calculate the future value at the end of year 4 of investing the following cash flows: C_1 = \$1,000, C_2 = \$2,000, C_3 = \$3,500, C_4 = \$3,000. Assume that the applicable interest rate is 10 percent per annum. Recalculate the future value above if an additional cash flow of \$2,000 were invested today (that is, C_0 = \$2,000).

Solution

In the first case, we have:

$FV_4 = 1000(1.10)^3 + 2000(1.10)^2 + 3500(1.10)^1 + 3000.$

$FV_4 = 1331 + 2420 + 3850 + 3000 = \$10{,}601.00.$

In the second case, we have:

$FV_4 = 2000(1.10)^4 + 1000(1.10)^3 + 2000(1.10)^2 + 3500(1.10)^1 + 3000.$

$FV_4 = 2928.20 + 1331 + 2420 + 3850 + 3000 = \$13{,}529.20.$

If we are given a series of cash flows occurring over different time periods their present value can be calculated as the sum of the present values of each individual cash flow. That is, the present value of a sum of a series of cash flows discounted at r percent per period over n periods is equal to the sum of their individual present values, which is:

$$PV_0 = \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2} + \ldots + \frac{C_n}{(1+r)^n}. \tag{4a}$$

Note again that in the above expression we assume that the first cash flow occurs at the end of time 1 and the last cash flow occurs at the end of time n (see the timeline in figure 2). If we assume that the first cash flow occurs at the end of time 0 (that is, today, as in the timeline in figure 3) then the present value today will be:

$$PV_0 = C_0 + \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2} + \ldots + \frac{C_n}{(1+r)^n}. \tag{4b}$$

Note that the difference between the present value in expression (4b) and (4a) is the additional cash flow at time 0 (C_0) which does not need to be discounted as it occurs at time 0.

Example 4: Present value of a series of cash flows

Calculate the present value of the following future cash flows: C_1 = \$1,000, C_2 = \$2,000, C_3 = \$3,500, C_4 = \$3,000. Assume that the applicable interest rate is 10 percent per annum. Recalculate the present value if an additional cash flow of \$2,000 were invested today (that is, C_0 = \$2,000). What is the relation between the future values calculated in each case of the previous example and the present values calculated here?

Solution

In the first case, we have:

$PV_0 = 1000/(1.10)^1 + 2000/(1.10)^2 + 3500/(1.10)^3 + 3000/(1.10)^4$.

$PV_0 = 909.10 + 1652.89 + 2629.60 + 2049.04 = \$7,240.63$.

In the second case, we have:

$PV_0 = 2000 + 1000/(1.10)^1 + 2000/(1.10)^2 + 3500/(1.10)^3 + 3000/(1.10)^4$.

$PV_0 = 2000 + 909.10 + 1652.89 + 2629.60 + 2049.04 = \$9,240.63$.

Note the relation between the future values calculated in each case of the previous example and the present values calculated above. Given the interest rate of 10 percent per annum, if we know the future value at the end of year 4 we can directly calculate the present value today using the future value amount, as follows:

In the first case, we have:

$PV_0 = FV_n/(1 + r)^n = 10601/(1 + 0.10)^4 = \$7,240.63$.

In the second case, we have:

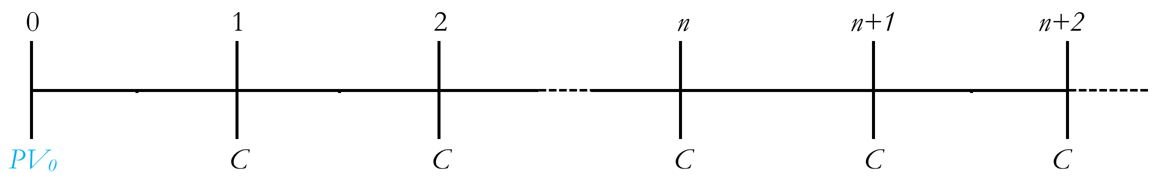
$PV_0 = 13529.20/(1 + 0.10)^4 = \$9,240.63$.

3. Present and future values of equal, periodic cash flows

3.1 Present value of a perpetuity

The simplest type of equal, periodic cash flow is a perpetuity where the cash flow recurs forever. On a timeline, a perpetual cash flow (C) can be shown as follows:

Figure 4: Present Value of a Perpetuity



The present value of a perpetuity is calculated by discounting each cash flow to time period 0 as follows:

$$PV_0 = C/(1+r) + C/(1+r)^2 + \ldots + C/(1+r)^n + C/(1+r)^{n+1} + \ldots$$

The above expression can be rewritten as:

$PV_0 = C[1/(1+r) + 1/(1+r)^2 + \ldots + 1/(1+r)^n + 1/(1+r)^{n+1} + \ldots]$

As n approaches infinity, the right-hand side expression $[1/(1+r) + 1/(1+r)^2 + \ldots + 1/(1+r)^n + 1/(1+r)^{n+1} + \ldots]$ approaches $1/r$. So, in the limit, the present value of a perpetuity is:

$$PV_0 = \frac{C}{r}. \qquad (5)$$

Note that in the above expression the **first cash flow occurs at the end of time 1**, and not time 0.

Example 5: Present value of a perpetuity

Your company can lease a computer system for equal annual payments of $2,000 forever, or purchase it today for $23,000. The first payment is to be made at the end of year 1 with subsequent payments being made at the end of each year. Ignoring taxes and other complications, what should the company do if the interest rate is 10 percent per annum?

Solution

We need to compare the purchase price today of $23,000 with the present value of equal annual payments of $2000 forever. The present value of this perpetuity is:

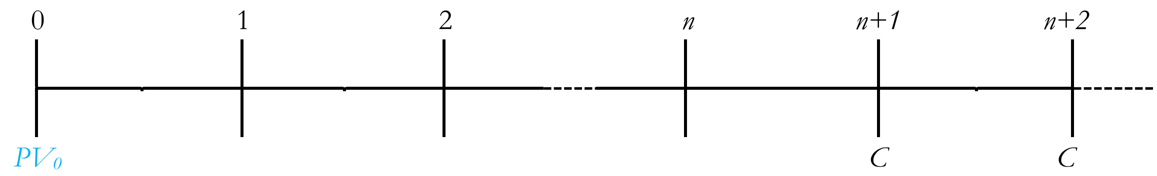
$$PV_0 = \frac{2000}{0.10} = \$20,000.$$

So, the company would prefer to lease the computer system as it has the lower present value of cost.

3.2 Present value of a deferred perpetuity

A deferred perpetuity is a series of equal, periodic cash flows that recur forever but with the first cash flow occurring at some point in the future. For example, the following timeline shows a perpetual cash flow which is *deferred* until the end of year n+1.

Figure 5: Present Value of a Deferred Perpetuity



The present value of a deferred perpetuity can be calculated by first obtaining the present value of the perpetuity at time n using expression (5), as follows:

$$PV_n = \frac{C}{r}.$$

Next, we get the present value of the above cash flow at time 0 by discounting it over n time periods, as follows:

$$PV_0 = \left(\frac{C}{r}\right)\left(\frac{1}{(1+r)^n}\right). \qquad (6)$$

Again, note that in the above expression we assume that the first cash flow of the deferred annuity occurs at the end of time *n + 1*, and not time *n*. The first term in the above expression (*C/r*) is the present value of this deferred perpetuity at the end of time *n*. We then discount this future value over *n* time periods to get the present value at time 0, as shown in expression (6).

Example 6: Present value of a deferred perpetuity

Your company can lease a computer system for equal annual payments of $2,000 forever, or purchase it today for $14,500. The company has been able to enter a deal with the supplier where the first lease payment has been deferred to the end of year 4 with subsequent payments being made at the end of each of the following years forever. Ignoring taxes and other complications, what should the company do if the interest rate is 10 percent per annum?

Solution

We need to compare the purchase price today of $14,500 with the present value of the deferred perpetuity of $2,000 forever where the cash flow is deferred until the end of year 4. The present value of this deferred perpetuity is:

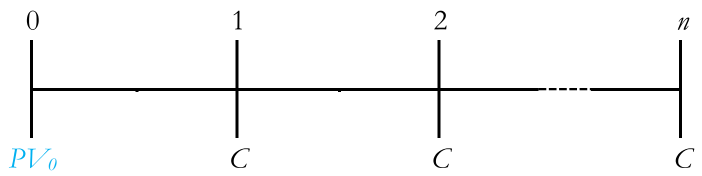
$$PV_0 = \left(\frac{2000}{0.10}\right)\left(\frac{1}{(1+0.10)^3}\right) = \$15,026.30.$$

Again, note that the first amount (2000/0.10 = $20,000) is the present value of the deferred perpetuity at the end of year 3. This amount is then discounted over 3 years to get the present value at the end of year 0. The company would prefer to purchase the computer system as it has the lower present value of cost.

3.3 Present value of an ordinary annuity

An annuity is a series of equal, periodic cash flows occurring over *n* periods. *Ordinary annuities* occur at the *end* of each period. In valuing an ordinary annuity, it is assumed that the first cash flow of the annuity occurs at the end of the *first* period, and the last cash flow occurs at the end of period *n*. On a timeline, an *n*-period ordinary annuity is as follows:

Figure 6: Present Value of an Ordinary Annuity



Note that the present value of an *n*-period annuity can be obtained as the difference between the present value of a perpetuity that starts at the end of time period 1, or expression (5) above, and the present value of a deferred perpetuity that starts at the end of time period *n*+1, or expression (6) above. That is:

$$PV_0 = \left(\frac{C}{r}\right) - \left(\frac{C}{r}\right)\left(\frac{1}{(1+r)^n}\right).$$

Simplifying the above expression, we get:

$$PV_0 = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^n}\right). \tag{7}$$

Example 7: Present value of an annuity

You have won a contest and have been given the choice between accepting \$32,000 today or an equal annual cash flow of \$5,000 per year at the end of each of the next 10 years. What should you do if the interest rate is 10 percent per annum?

Solution

We need to compare the lump sum amount of \$32,000 available today with the present value of the ten-year annuity of \$5,000 per year. The present value of this annuity is:

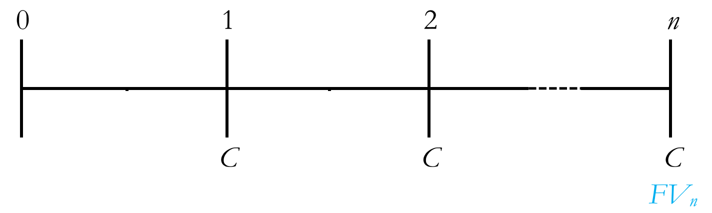
$$PV_0 = \left(\frac{5000}{0.10}\right)\left(1 - \frac{1}{(1+0.10)^{10}}\right) = \$30,722.84.$$

So, you would prefer the lump sum amount today as it has a higher present value compared to the annuity of \$5,000 per year for ten years.

3.4 Future value of an ordinary annuity

The future value at the end of period n of an n-period ordinary annuity is the sum of the future values of each of these n cash flows. On a timeline, the future value of an n-period ordinary annuity can be depicted as follows:

Figure 7: Future Value of an Ordinary Annuity



The future value of an n-period ordinary annuity is easily obtained from its present value in expression (7) by compounding the present value over n time periods, as follows:

$$FV_n = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^n}\right)(1+r)^n.$$

Simplifying the above expression, we get:

$$FV_n = \left(\frac{C}{r}\right)\left[(1+r)^n - 1\right]. \tag{8}$$

Example 8: Future value of an annuity

You have won a contest which pays an equal annual cash flow of \$5,000 per year at the end of each of the next 10 years. What is the future value of this prize if the interest rate is 10 percent per annum?

Solution

Using the future value of an ordinary annuity expression, we get:

$$FV_{10} = \left(\frac{5000}{0.10}\right)\left[(1+010)^{10} - 1\right] = \$79,687.12.$$

Note that we had calculated the present value of this cash flow in the previous example at $30,722.84. So, we could also calculate the future value of this cash flow at the end of ten years as:

$FV_{10} = 30722.84(1 + 0.10)^{10} = \$79,687.12$.

3.5 Present and future values of annuities due

So far, the series of cash flows we have valued have been assumed to occur at the end of each period. In some cases, these cash flows may occur at the beginning, rather than the end, of each period. Such an annuity is referred to as an *annuity due.* Using the convention that we have been using above and treating cash flows as occurring at the end of a particular time period this implies that if a cash flow now occurs at the beginning of time n then that is the same as the cash flow occurring at the end of time $n - 1$. This is because the beginning of time n is the same as the end of time $n - 1$. For example, a cash flow occurring at the beginning of 2017 (that is, 1 January 2017) is the same as if the cash flow occurs at the end of 2016 (that is, 31 December 2016) and so on, as shown below.

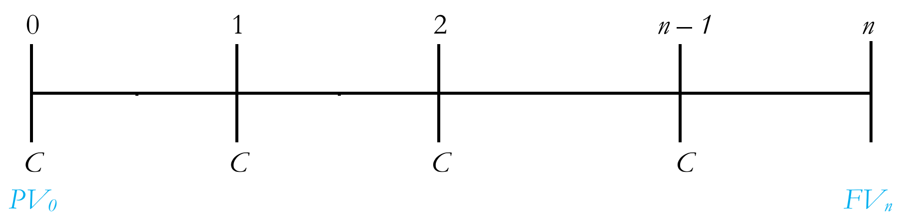
Beginning of year	End of previous year
1 Jan 2015	31 Dec 2014
1 Jan 2016	31 Dec 2015
1 Jan 2017	31 Dec 2016
1 Jan 2018	31 Dec 2017

If the cash flows were monthly, we would have the following equivalence:

Beginning of month	End of previous month
1 Dec 2018	30 Nov 2018
1 Nov 2018	31 Oct 2018
1 Oct 2018	30 Sep 2018
1 Sep 2018	31 Aug 2018

On a timeline, an n-period annuity due can be shown as follows:

Figure 8: Present and Future Values of Annuities Due



As the figure shows, an annuity due is essentially the same as an ordinary annuity with each cash flow "moved back" by one time period. So, the first cash flow occurs at the end of time 0 (which is the same as the beginning of time 1) while the last cash flow occurs at the end of time $n - 1$ (which is the same as the beginning of time n).

Assume that we need the present value of an annuity due at the end of time 0 and the future value at the end of time n, as shown in the timeline above. Since we have already calculated the present and future values of an ordinary annuity we can obtain the present and future values of an annuity due by simply compounding these values by a factor of $(1 + r)$. So, the present and future values of an annuity due are obtained as:

$$PV_0 = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^n}\right)(1+r), \text{ and} \tag{9a}$$

$$FV_n = \left(\frac{C}{r}\right)\left[(1+r)^n - 1\right](1+r). \tag{9b}$$

The reason this simple adjustment works is because the present value *without* the adjustment would occur at the end of "time –1" and not time 0. So, compounding this present value by a factor of $(1 + r)$ moves the present value to the correct point in time, which is the end of time 0. Similarly, *without* the adjustment, the future value is calculated at the end of time $n - 1$ and the adjustment moves the future value to the correct point in time, which is the end of time n.

Example 9: Present and future values of an annuity due

You have won a contest and have been given the choice of between accepting $32,000 today or an equal annual cash flow of $5,000 per year at the *beginning* of each of the next 10 years. What should you do if the interest rate is 10 percent per annum? What is the future value of this prize at the end of ten years?

Solution

We need to compare the lump sum amount available today with the present value of the ten-year annuity due of $5,000 per year. The present value of this annuity due is:

$$PV_0 = \left(\frac{5000}{0.10}\right)\left(1 - \frac{1}{(1+0.10)^{10}}\right)(1+0.10) = \$33,795.12.$$

So, you would prefer the annuity due of $5,000 per year since it has a higher present value than the lump sum of $32,000 today.

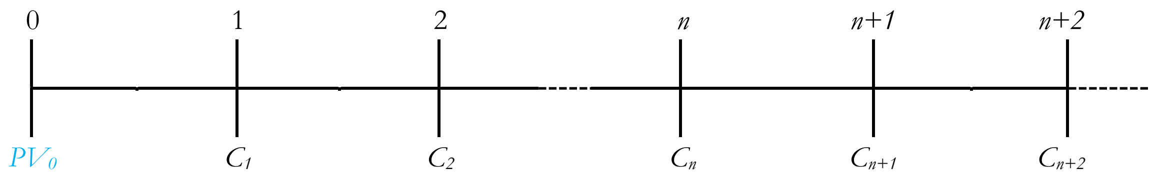
The future value of this annuity due is:

$$FV_n = \left(\frac{5000}{0.10}\right)\left[(1+0.10)^{10} - 1\right](1+0.10) = \$87,655.84.$$

3.6 Present value of a growing perpetuity

A growing perpetuity is a series of periodic cash flows occurring at the end of each period and growing at a constant rate forever. On a timeline, a growing perpetual cash flow can be shown as follows:

Figure 9: Present Value of Growing Perpetuity



A constant growth rate of g percent per annum implies the following cash flows over time:

$C_2 = C_1(1 + g)$.

$C_3 = C_2(1 + g) = C_1(1 + g)^2$.

$C_n = C_1(1 + g)^{n-1}$.

Note that the first cash flow is assumed to occur at the end of time 1, not time 0. That is, we assume that the present value at time 0 of a growing perpetuity does not include time 0's cash flows. The present value of this growing perpetuity is:

$$PV_0 = C(1 + g)/(1 + r) + C(1 + g)^2/(1 + r)^2 + \ldots + C(1 + g)^n/(1 + r)^n + \ldots$$

The above expression can be rewritten as:

$$PV_0 = C([1 + g)/(1 + r) + (1 + g)^2/(1 + r)^2 + \ldots + (1 + g)^n/(1 + r)^n + \ldots]$$

As n approaches infinity, the right-hand side expression $[1 + g)/(1 + r) + (1 + g)^2/(1 + r)^2 + \ldots + (1 + g)^n/(1 + r)^n + \ldots]$ approaches $1/(r - g)$. So, in the limit, the present value of a growing perpetuity is:

$$PV_0 = \frac{C_1}{r - g}. \qquad (10)$$

Note that we've added the subscript 1 to highlight the fact that the first cash flow occurs at the end of time 1, not time 0. Also note that for the above expression to be correctly defined we also need to assume that $r > g$. The growth rate itself can be positive or negative (that is, declining rather than growing cash flows). Finally, if the growth rate of the cash flows is zero we revert to the simple perpetual cash flow case discussed earlier.

Example 10: Present value of a growing perpetuity

Your company can lease a computer system for an annual lease payment of \$2,000 next year with lease payments increasing at a constant annual rate of 2 percent forever, or purchase it today for \$23,000. Assume an interest rate of 10 percent per annum and end of the year cash flows. Ignoring taxes and other complications, what should the company do? How does your answer change if the lease payments were \$2,400 next year but *declining* at an annual rate of 1 percent forever?

Solution

The cash flow at the end of year 1 is \$2,000 which is expected to grow at 2% per annum. The present value of this growing perpetuity is:

$$PV_0 = \frac{2000}{0.10 - 0.02} = \$25,000.$$

So, the company would prefer to purchase the computer system because the cost (in present value terms) is lower compared with leasing the system.

In the second case, we a have higher lease payment but with that cash flow declining at a constant rate over time (that is, g = –1% p.a.):

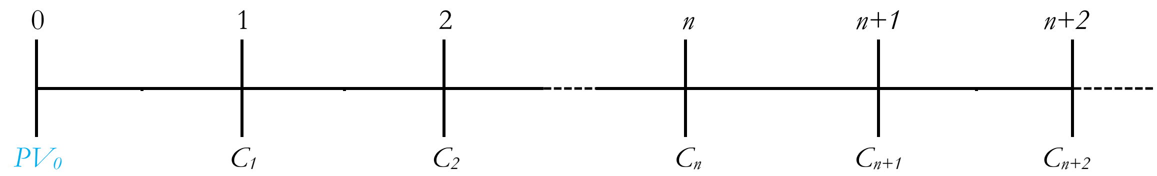
$$PV_0 = \frac{2400}{0.10 - (-0.01)} = \$21,818.18.$$

The company would now prefer to lease the computer system because the leasing cost is lower in present value terms.

3.7 Present and future values of a growing annuities

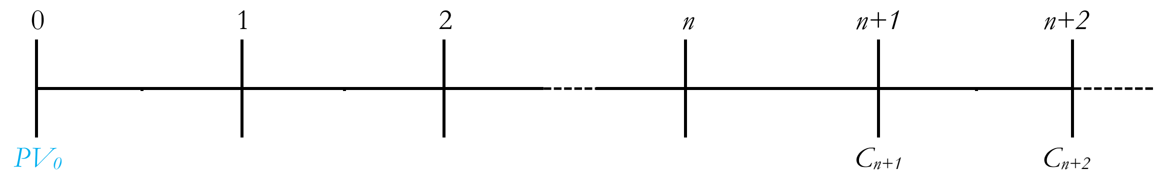
Another common type of cash flow is a growing ordinary annuity which is a series of periodic cash flows occurring at the end of each period and lasting for n periods where the cash flows grow at a constant rate g percent until time n. We can calculate the present and future values of such cash flows using the same method used to calculate the present and future values of ordinary annuities. The difference, of course, is that in this case the cash flows are growing over time. Graphically, what we're doing is taking the difference of a growing perpetual cash flow with the first cash flow occurring at the end of time 1 and another growing perpetual cash flow where the first cash flows is *deferred* until the end of time n + 1. Recall that on a timeline, a growing perpetual cash flow with the first cash flow occurring at the end of time 1 was as follows:

Figure 9: Present Value of Growing Perpetuity



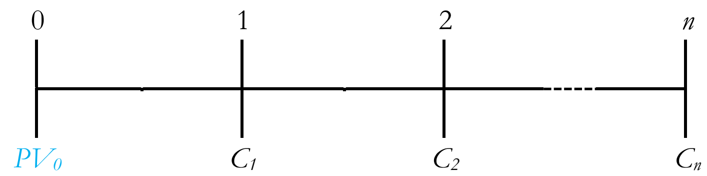
A growing perpetual cash flow with the first cash flow *deferred* until at the end of time n + 1 is as follows:

Figure 10: Present Value of Growing Deferred Perpetuity



The difference between the first and the second set of cash flows is a growing ordinary annuity, as follows:

Figure 11: Present Value of Growing Annuity



We have already obtained the present value of a growing perpetuity as:

$$PV_0 = \frac{C_1}{r-g}. \tag{10}$$

For the growing perpetuity deferred until the end of time n + 1 we first get its present value at the end of time n as:

$$PV_n = \frac{C_{n+1}}{r-g} = \frac{C_1(1+g)^n}{r-g}.$$

The above expression shows that the cash flow at the end of time $n + 1$ is equal to the cash flow at time 1 growing at g percent per period over n periods, that is, $C_{n+1} = C_1(1 + g)^n$.

Next, we calculate the present value of this cash flow at time 0 by discounting what is now a future value equal to PV_n by discounting it at r percent over n time periods as:

$$PV_0 = \left[\left(\frac{C_1(1+g)^n}{r-g}\right) \Big/ (1+r)^n\right]. \tag{11}$$

Finally, the present value of an annuity growing at g percent per period over n periods can be calculated as the difference between expressions (10) and (11) as:

$$PV_0(GA) = \frac{C_1}{r-g} - \left[\left(\frac{C_1(1+g)^n}{r-g}\right) \Big/ (1+r)^n\right].$$

Simplifying the above expression, we get the present value of a growing annuity as:

$$PV_0(GA) = \left(\frac{C_1}{r-g}\right)\left(1 - \left(\frac{1+g}{1+r}\right)^n\right). \tag{12}$$

Note that the only time the above expression cannot be used is when $r = g$. In that case, one would need to value the cash flows individually. The future value of a growing annuity is simply calculated as the future value at r percent over n time periods of the right-hand side in expression (12) as:

$$FV_n(GA) = \left(\frac{C_1}{r-g}\right)\left(1 - \left(\frac{1+g}{1+r}\right)^n\right)(1+r)^n. \tag{13}$$

Note that in the above expressions if we set the growth rate equal to 0 we revert to ordinary annuities and get expressions (7) and (8) for the present and future values respectively. Also, in expression (12), if the growth in cash flows is perpetual, that is n approaches infinity, the second term converges to 1 giving us expression (10) which is the present value of a growing perpetuity.

Example 11: Present and future value of a growing annuity

You have won a contest and have been given the choice between accepting \$32,000 today or \$5,000 in year 1 which will then grow at 2 percent per annum until the end of year 10. What should you do if the interest rate is 10 percent per annum? What is the future value of this prize at the end of year 10?

Solution

We need to compare the lump sum amount of \$32,000 available today with the present value of the ten-year growing annuity of \$5,000. The present value of this growing annuity is:

$$PV_0(GA) = \left(\frac{5000}{0.10-0.02}\right)\left(1 - \left(\frac{1+0.02}{1+0.10}\right)^{10}\right) = \$33{,}126.56.$$

Note that in example 7 we had the same cash flows but with zero growth. The difference between the present value in that example and this one is \$2,403.72. This is the present value that we can attach to the 2 percent growth in cash flows over the ten-year period. As a result of this growth in cash flows, you would now prefer the growing cash flows over time rather than the lump sum today.

The value of the prize at the end of year 10 is the future value of this growing annuity, which is:

$$FV_{10}(GA) = \left(\frac{5000}{0.10-0.02}\right)\left(1-\left(\frac{1+0.02}{1+0.10}\right)^{10}\right)(1+0.10)^{10} = \$85,921.75.$$

Note that we could have also calculated the future value using the present value already calculated, as:

$FV_{10} = 33126.56(1 + 0.10)^{10} = \$85,921.75.$

Practice Problem 1

You have just won a lottery and have been offered the following alternative ways of receiving the prize money. Assume that each alternative is risk free and the interest rate is 8 percent per annum.

a) $110,000 immediately.

b) $140,000 at the end of year 3.

c) $28,000 at the end of each of the next 5 years.

d) $9,000 at the end of each year (starting at the end of year 1) in perpetuity.

e) $6,500 at the end of the first year growing at an annual rate of 2 percent in perpetuity.

Assuming end of the year cash flows, which is the *best* way to receive the prize money?

Practice Problem 2

Assume that the interest rate is 10 percent per annum and that all cash flows occur at the end of each year. *Round off your final answers to the nearest dollar.*

a) Suppose you decide to invest $50,000 today. Calculate the future value of your investment at the end of 10 years.

b) Your friend also decides to invest $50,000 but plans to do so in installments. Specifically, she will invest $5,000 now, $10,000 at the end of year 1, $15,000 at the end of year 2, and $20,000 at the end of year 3. Calculate the future value of her investment at the end of 10 years.

c) Another friend of yours also decides to invest $50,000 but to defer the investment until the end of year 3. Calculate the future value of his investment at the end of 10 years.

d) What factor(s) account for the differences in the future values in parts (a) through (c)?

e) Now assume that you and your friends have an investment time horizon of 10 years. What *additional* amount would your friends need to invest *today* so that, at the end of year 10, the future value of their respective investments is the same as the future value of your investment?

f) Suppose that instead of investing the additional amounts today in part (e) your friends decide to invest funds in equal annual amounts over the ten-year period. Calculate the equal annual amounts your two friends would now need to invest.

g) Now suppose that you and your friends decide to invest funds in equal annual amounts over the ten-year period so each of you has the same total value at the end of ten years as calculated in part (a). What would this equal annual amount be?

Practice Problem 3

You are running a hot internet company. Market analysts estimate that its earnings will grow at 30% per annum for the next five years (that is, years 1 – 5). After that, as competition increases, earnings growth is expected to slow down to 2% per annum and continue at that level forever. Your company has just announced earnings of $1 million today. Assuming end of the year cash flows, what is the present value of all **future** earnings if the interest rate is 8 percent per annum?

4. Suggested answers to practice problems

Practice Problem 1

We can calculate and compare the present values of each prize as follows:

b) $$PV_0 = \frac{140000}{(1+0.08)^3} = \$111{,}137.$$

c) $$PV_0 = \left(\frac{28000}{0.08}\right)\left(1 - \frac{1}{(1+0.08)^5}\right) = \$111{,}796.$$

d) $$PV_0 = \left(\frac{9000}{0.08}\right) = \$112{,}500.$$

e) $$PV_0 = \left(\frac{6500}{0.08-0.02}\right) = \$108{,}333.$$

Based on the present values of the prizes you would prefer the prize of \$9,000 at the end of each year in perpetuity.

Practice Problem 2

a) The future value at the end of *n* years of your investment is given by:

$FV_n = PV_0(1 + r)^n$.

So, $FV_{10} = 50000(1 + 0.10)^{10} = \$129{,}687$.

b) The future value at the end of 10 years of this friend's investment can be calculated as the sum of the future values of the individual cash flows. The first cash flow will earn interest over 10 years, the second over 9 years and so on, as follows:

$FV_{10} = 5000(1.10)^{10} + 10000(1.10)^9 + 15000(1.10)^8 + 20000(1.10)^7 = \$107{,}676$.

c) In this case your friend earns no return for the first 3 years and the value of the \$50,000 invested over the remaining 7 years is as follows:

$FV_{10} = 50000(1.10)^7 = \$97{,}436$.

d) As expected, the future values in parts (b) and (c) are lower than in part (a). This is because in part (b) all the funds are not being invested immediately so the funds that are invested later earn a lower compounded return over the time horizon. In part (c) all the funds are invested at the same time, but this investment is deferred until year 4 so no compounded return is earned over the first three years.

e) From part (a), the future value of your investment at the end of 10 years is \$129,687.

Let the *additional* amounts invested today by your friends be $\$X$ and $\$Y$, respectively.

We need the future value of the first friend's investment to be worth \$129,687 once the additional amount of $\$X$ invested today is included in the answer from part (b) above. That is:

$129687 = 107676 + X(1.10)^{10}$.

So, $X = (129687 - 107676)/1.10^{10} = \$8,486$.

Similarly, for the second friend, we need the future value of his investment to be worth \$129,687 once the additional amount of \$$Y$ invested today is included in the answer from part (c) above. That is:

$129687 = 97436 + Y(1.10)^{10}$.

So, $Y = (129687 - 97436)/1.10^{10} = \$12,434$.

f) From part (e) we know that the amounts your two friends need to invest today are \$8,486 and 12,434, respectively. To obtain the equal annual amounts that they should invest rather than these lump sum amounts we need to convert these amounts today into annuities using the present value of an annuity expression, which is:

$$PV_0 = \left(\frac{C}{r}\right)\left(1 - \frac{1}{(1+r)^n}\right).$$

Rewriting the above expression in terms of the unknow cash flow (C), we get:

$$C = \frac{r \times PV_0}{\left(1 - \frac{1}{(1+r)^n}\right)}.$$

So, the equal annual amounts that the first friend needs to invest can be calculated as:

$$C_{Mate1} = \frac{r \times PV_0}{\left(1 - \frac{1}{(1+r)^n}\right)}.$$

$$C_{Mate1} = \frac{0.10 \times 8486}{\left(1 - \frac{1}{(1+0.10)^{10}}\right)} = \$1,381.$$

Similarly, the equal annual amounts that the second friend needs to invest can be calculated as:

$$C_{Mate2} = \frac{0.10 \times 12434}{\left(1 - \frac{1}{(1+0.10)^{10}}\right)} = \$2,024.$$

g) The future value at the end of 10 years that we need is \$129,687 from part (a). To obtain the equal annual amounts that should be invested over the ten-year period we need to use the future value of an annuity, which is:

$$FV_n = \left(\frac{C}{r}\right)\left[(1+r)^n - 1\right].$$

So, the equal annual amounts that needs to be invested (C) can be calculated as:

$$FV_{10} = 129687 = \left(\frac{C}{0.10}\right)\left[(1+0.10)^{10} - 1\right].$$

$$C = \frac{0.10 \times 129687}{\left[(1+0.10)^{10} - 1\right]} = \$8{,}137.$$

Practice Problem 3

The timeline for this scenario is as follows where the \$1 million earnings today are expected to grow by 30% per annum for five years and then at 2% per annum forever.

0	1	2	3	4	5	6	7
	$\$1(1.3)$	$\$1(1.3)^2$	$\$1(1.3)^3$	$\$1(1.3)^4$	$\$1(1.3)^5$	$\$1(1.3)^5(1.02)$	$\$1(1.3)^5(1.02)^2$

This scenario consists of two parts:

1. A growing annuity over years 1 – 5.

2. A growing perpetuity after 5 years (that is, year 6 onwards).

We calculate the present value of part 1 as:

$$PV(\text{GA}) = \left(\frac{1(1+0.30)}{0.08-0.30}\right)\left(1-\left(\frac{1.30}{1.08}\right)^5\right) = 9.02\,million.$$

Next, we calculate the present value of part 2. The cash flow at the end of year 6 is $\$1(1.3)^5(1.02)$. So, the present value at the end of year 5 of this growing perpetuity is:

$$PV_5 = \frac{1(1.30)^5(1+0.02)}{0.08-0.02} = \$63.12\,million.$$

The present value of this amount at time 0 is:

$$PV_0 = \frac{63.12}{(1.08)^5} = \$42.96\,million.$$

Finally, adding the present value of parts 1 and 2 together gives the present value of future earnings as:

$PV_0 = 9.02 + 42.96 = 51.98$ million.