PHYC10003 Physics 1

Lecture 4: Motion

Projectiles, relative and circular motion

Last lecture

- Vectors
- Displacement vectors
- Vector addition and subtraction
- Vectors in one-, two- and three-dimensions
- Unit vectors, $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$.
- Vector multiplication (dot and cross product)

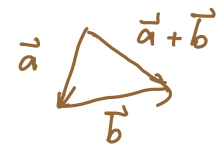
3.1 Vectors – addition and the triangle rule

$$\vec{s} = \vec{a} + \vec{b},$$

Eq. (3-1)

The magnitude of $\vec{s}$ satisfies the triangle inequality

$$|a - b| \leq s \leq (a + b)$$



$|\vec{a}|+|\vec{b}| \geq |\vec{a}+\vec{b}|$

$a+b \geq s$

$|a-b| \leq s$

where $a = |\vec{a}|$, $b = |\vec{b}|$ and $s = |\vec{s}|$

3.3 Vectors- scalar product

- Multiplying two vectors: the **scalar product**
  - Also called the **dot product**
  - Results in a scalar, where *a* and *b* are magnitudes and φ is the angle between the directions of the two vectors:

$$\vec{a}.\vec{b} = ab\cos\varphi \qquad \textbf{Eq. (3-20)}$$

3.3 Unit vectors - properties

- The unit vectors $\hat{\imath}, \hat{\jmath}$, and $\hat{k}$ are described as being *orthonormal*, in the sense that

$$\hat{\imath}.\hat{\imath} = 1, \qquad \hat{\jmath}.\hat{\jmath} = 1, \qquad \hat{k}.\hat{k} = 1$$

(normalisation)

and

$$\hat{\imath}.\hat{\jmath} = 0, \qquad \hat{\jmath}.\hat{k} = 0, \qquad \hat{k}.\hat{\imath} = 0$$

(orthogonality)

3.3 Vectors- scalar product

- The commutative law applies, and we can expand the dot product by identifying components and using properties of the unit vectors:

$$\vec{a} \cdot \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \cdot (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$ **Eq. (3-22)**

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z.$$ **Eq. (3-23)**

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}.$$

3.3 Vectors - projections

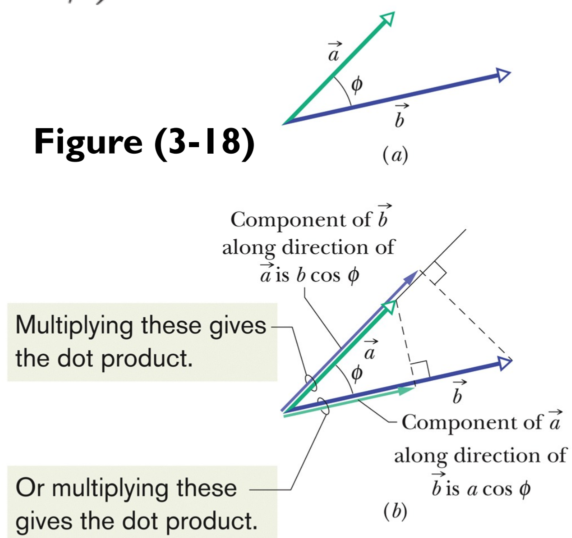
- A dot product is: the product of the magnitude of one vector times the scalar component of the other vector in the direction of the first vector

$$\vec{a} \cdot \vec{b} = (a \cos \phi)(b) = (a)(b \cos \phi).$$

Eq. (3-21)

- Either projection of one vector onto the other can be used
- To multiply a vector by the projection, multiply the magnitudes

Figure (3-18)



3.3 Vectors – cross product

- Multiplying two vectors: the **vector product**
  - The **cross product** of two vectors with magnitudes *a* & *b*, separated by angle φ, produces a vector with magnitude:

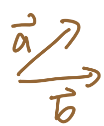
$$c = ab \sin \phi, \qquad \textbf{Eq. (3-24)}$$

  - Direction is perpendicular to both original vectors

$\vec{a} \times \vec{b}$

- Direction is determined by the **right-hand rule**

move from $\vec{a}$ to $\vec{b}$



3.3 Unit vectors – more properties

Unit vectors and the cross product: the cross product of two vectors $\vec{a}$ and $\vec{b}$ generates a third vector perpendicular to both with magnitude $ab \sin \varphi$

Hence, for unit vectors and $\varphi = \pi/2$:

$$\hat{\imath} \times \hat{\jmath} = \hat{k}, \qquad \hat{\jmath} \times \hat{k} = \hat{\imath}, \qquad \hat{k} \times \hat{\imath} = \hat{\jmath}$$

$$\hat{\imath} \times \hat{\imath} = 0, \qquad \hat{\jmath} \times \hat{\jmath} = 0, \qquad \hat{k} \times \hat{k} = 0$$

and

$$\hat{\imath} \times \hat{\jmath} = -\hat{\jmath} \times \hat{\imath}$$

3.3 Vectors - cross product

- The cross product is not commutative

$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b}).$$

Eq. (3-25)

- To evaluate, we distribute over components:

$$\vec{a} \times \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \times (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$

Eq. (3-26)

$$a_x\hat{\mathrm{i}} \times b_x\hat{\mathrm{i}} = a_x b_x(\hat{\mathrm{i}} \times \hat{\mathrm{i}}) = 0,$$

$$a_x\hat{\mathrm{i}} \times b_y\hat{\mathrm{j}} = a_x b_y(\hat{\mathrm{i}} \times \hat{\mathrm{j}}) = a_x b_y\hat{\mathrm{k}}.$$

- Therefore, by expanding (3-26):

$$\vec{a} \times \vec{b} = (a_y b_z - b_y a_z)\hat{\mathrm{i}} + (a_z b_x - b_z a_x)\hat{\mathrm{j}} + (a_x b_y - b_x a_y)\hat{\mathrm{k}}.$$

Eq. (3-27)

4.1 Position vector

- A **position vector** locates a particle in space
  - Extends from a reference point (origin) to the particle

$$\vec{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}},$$ **Eq. (4-1)**

Example

- Position vector (-3 m, 2 m, 5 m)

$$\vec{r} = (-3\text{ m})\hat{\mathrm{i}} + (2\text{ m})\hat{\mathrm{j}} + (5\text{ m})\hat{\mathrm{k}}$$

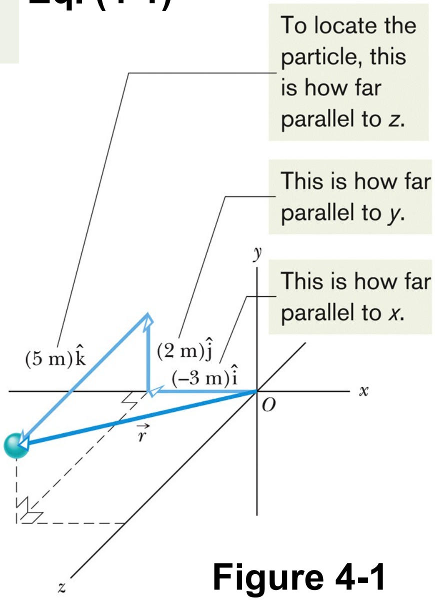


Figure 4-1

4.1 Displacement vector

- Change in position vector is a **displacement**

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1. \qquad \text{Eq. (4-2)}$$

- We can rewrite this as:

$$\Delta\vec{r} = (x_2 - x_1)\hat{\mathrm{i}} + (y_2 - y_1)\hat{\mathrm{j}} + (z_2 - z_1)\hat{\mathrm{k}}, \qquad \text{Eq. (4-3)}$$

- Or express it in terms of changes in each coordinate:

$$\Delta\vec{r} = \Delta x\hat{\mathrm{i}} + \Delta y\hat{\mathrm{j}} + \Delta z\hat{\mathrm{k}}. \qquad \text{Eq. (4-4)}$$

4.1 Average velocity

- **Average velocity** is
  - A displacement divided by its time interval

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t}.$$

Eq. (4-8)

- We can write this in component form:

$$\vec{v}_{avg} = \frac{\Delta x\hat{\mathrm{i}} + \Delta y\hat{\mathrm{j}} + \Delta z\hat{\mathrm{k}}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{\mathrm{i}} + \frac{\Delta y}{\Delta t}\hat{\mathrm{j}} + \frac{\Delta z}{\Delta t}\hat{\mathrm{k}}.$$

Eq. (4-9)

Example

- A particle moves through displacement (12 m)i + (3.0 m)k in 2.0 s:

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{(12\text{ m})\hat{\mathrm{i}} + (3.0\text{ m})\hat{\mathrm{k}}}{2.0\text{ s}} = (6.0\text{ m/s})\hat{\mathrm{i}} + (1.5\text{ m/s})\hat{\mathrm{k}}.$$

4.1 Instantaneous velocity

- **Instantaneous velocity** is
  - The velocity of a particle at a single point in time
  - The limit of avg. velocity
    as the time interval shrinks to 0

$$\vec{v} = \frac{d\vec{r}}{dt}.$$

Eq. (4-10)

- Visualize displacement and instantaneous velocity:

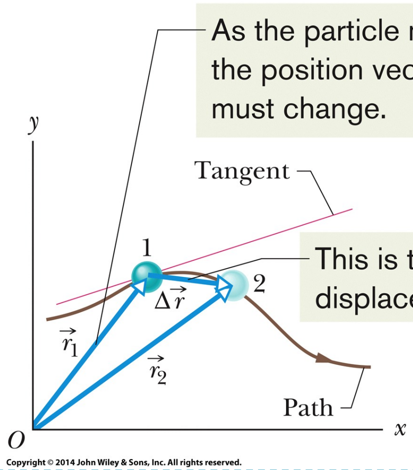


Figure 4-3

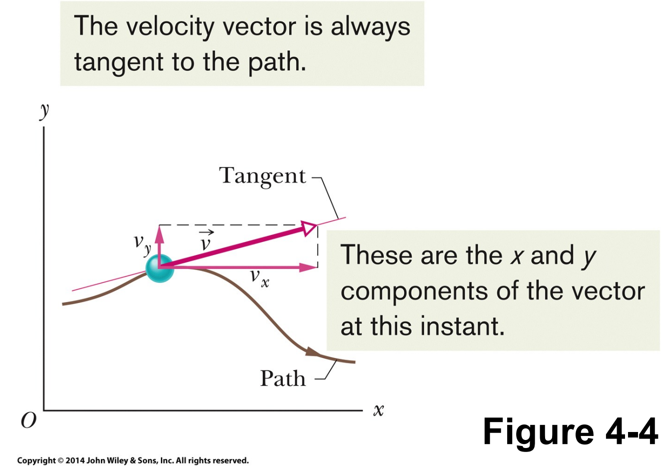


Figure 4-4

4.1 Instantaneous velocity

The direction of the instantaneous velocity $\vec{v}$ of a particle is always tangent to the particle's path at the particle's position.

- In unit-vector form, we write:

$$\vec{v} = \frac{d}{dt}(x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}) = \frac{dx}{dt}\hat{\mathrm{i}} + \frac{dy}{dt}\hat{\mathrm{j}} + \frac{dz}{dt}\hat{\mathrm{k}}.$$

- Which can also be written:

$$\vec{v} = v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}}, \qquad \text{Eq. (4-11)}$$

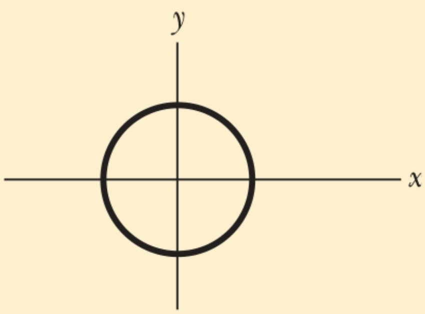
$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}, \quad \text{and} \quad v_z = \frac{dz}{dt}. \qquad \text{Eq. (4-12)}$$

- Velocity vector shows ***only*** direction and magnitude

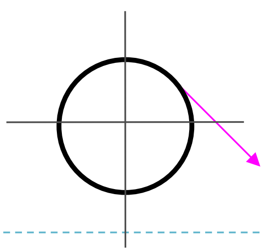
4.2 Velocity and circular motion

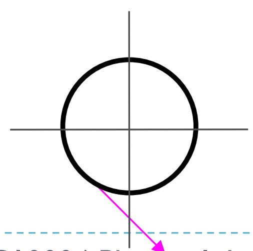
Checkpoint 1

The figure shows a circular path taken by a particle. If the instantaneous velocity of the particle is $\vec{v} = (2\text{ m/s})\hat{\text{i}} - (2\text{ m/s})\hat{\text{j}}$, through which quadrant is the particle moving at that instant if it is traveling (a) clockwise and (b) counterclockwise around the circle? For both cases, draw $\vec{v}$ on the figure.



Answer: (a) Quadrant I   (b) Quadrant III





4.3 Instant and average acceleration

- **Average acceleration** is
  - A change in velocity divided by its time interval

$$\vec{a}_{avg} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta \vec{v}}{\Delta t}. \quad \textbf{Eq. (4-15)}$$

- **Instantaneous acceleration** is again the limit $t \rightarrow 0$:

$$\vec{a} = \frac{d\vec{v}}{dt}. \quad \textbf{Eq. (4-16)}$$

- We can write Eq. 4-16 in unit-vector form:

$$\vec{a} = \frac{d}{dt}(v_x\hat{\mathrm{i}} + v_y\hat{\mathrm{j}} + v_z\hat{\mathrm{k}})$$

$$= \frac{dv_x}{dt}\hat{\mathrm{i}} + \frac{dv_y}{dt}\hat{\mathrm{j}} + \frac{dv_z}{dt}\hat{\mathrm{k}}.$$

4.3 Calculus of Velocity and Acceleration

- We can rewrite as:

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}, \quad \text{Eq. (4-17)}$$

$$a_x = \frac{dv_x}{dt}, \quad a_y = \frac{dv_y}{dt}, \quad \text{and} \quad a_z = \frac{dv_z}{dt}. \quad \text{Eq. (4-18)}$$

- To get the components of acceleration, we differentiate the components of velocity with respect to time

4.3 Acceleration

- Note: as with velocity, an acceleration vector does *not* extend from one point to another, only shows direction and magnitude

Checkpoint 2

Here are four descriptions of the position (in meters) of a puck as it moves in an xy plane:

(1) $x = -3t^2 + 4t - 2$ and $y = 6t^2 - 4t$ (3) $\vec{r} = 2t^2\hat{\mathrm{i}} - (4t + 3)\hat{\mathrm{j}}$

(2) $x = -3t^3 - 4t$ and $y = -5t^2 + 6$ (4) $\vec{r} = (4t^3 - 2t)\hat{\mathrm{i}} + 3\hat{\mathrm{j}}$

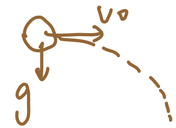
Are the x and y acceleration components constant? Is acceleration $\vec{a}$ constant?

Answer: (1) x:yes, y:yes, a:yes (3) x:yes, y:yes, a:yes

(2) x:no, y:yes, a:no (4) x:no, y:yes, a:no

4.4 Projectile motion

- A **projectile** is
  - A particle moving in the vertical plane
  - With some initial velocity
  - Whose acceleration is always free-fall acceleration (g)



- The motion of a projectile is **projectile motion**
- Launched with an initial velocity v_0

$$\vec{v}_0 = v_{0x}\hat{\mathrm{i}} + v_{0y}\hat{\mathrm{j}}. \qquad \text{Eq. (4-19)}$$

$$v_{0x} = v_0 \cos\theta_0 \quad \text{and} \quad v_{0y} = v_0 \sin\theta_0. \qquad \text{Eq. (4-20)}$$

In projectile motion, the horizontal motion and the vertical motion are independent of each other; that is, neither motion affects the other.

4.4 Projectile motion

- Therefore we can decompose two-dimensional motion into 2 one-dimensional problems

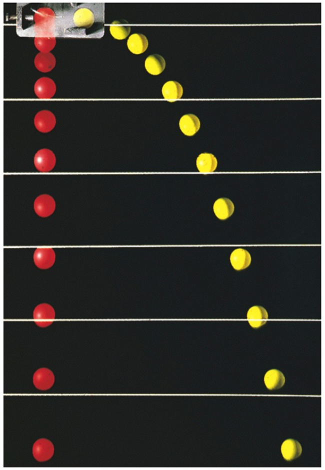


Figure 4-10

Richard Megna/Fundamental Photographs

Checkpoint 3

At a certain instant, a fly ball has velocity $\vec{v} = 25\hat{\mathrm{i}} - 4.9\hat{\mathrm{j}}$ (the x axis is horizontal, the y axis is upward, and $\vec{v}$ is in meters per second). Has the ball passed its highest point?

Answer: Yes. The y-velocity is negative, so the ball is now falling.

4.4 Projectile motion - components

- Horizontal motion:
  - No acceleration, so velocity is constant (recall Eq. 2-15):

$$x - x_0 = v_{0x}t.$$

$$x - x_0 = (v_0 \cos \theta_0)t. \qquad \textbf{Eq. (4-21)}$$

- Vertical motion:
  - Acceleration is always -g (recall Eqs. 2-15, 2-11, 2-16):

$$y - y_0 = v_{0y}t - \tfrac{1}{2}gt^2$$

$$= (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2, \qquad \textbf{Eq. (4-22)}$$

$$v_y = v_0 \sin \theta_0 - gt \qquad \textbf{Eq. (4-23)}$$

$$v_y^2 = (v_0 \sin \theta_0)^2 - 2g(y - y_0). \qquad \textbf{Eq. (4-24)}$$

4.4 Projectile motion – trajectory and range

- The projectile's **trajectory** is
  - Its path through space (traces a parabola)
  - Found by eliminating time between Eqs. 4-21 and 4-22:

$$y = (\tan \theta_0)x - \frac{gx^2}{2(v_0 \cos \theta_0)^2} \quad \textbf{Eq. (4-25)}$$

- The **horizontal range** is:
  - The distance the projectile travels in *x* by the time it returns to its initial height

$$R = \frac{v_0^2}{g} \sin 2\theta_0. \quad \textbf{Eq. (4-26)}$$

The horizontal range R is maximum for a launch angle of 45°.

4.4 Projectile motion

- In these calculations we assume air resistance is negligible
- In many situations this is a poor assumption:

Table 4-1 Two Fly Balls[a]

	Path I (Air)	Path II (Vacuum)
Range	98.5 m	177 m
Maximum height	53.0 m	76.8 m
Time of flight	6.6 s	7.9 s

[a]See Fig. 4-13. The launch angle is 60° and the launch speed is 44.7 m/s.

Table 4-1

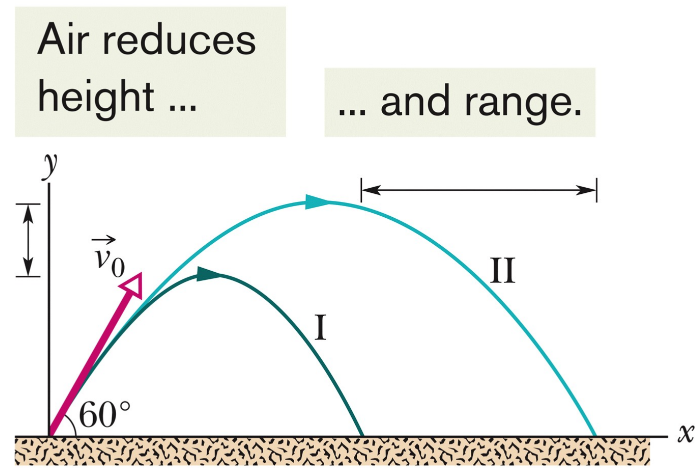


Figure 4-13

4.4 Projectile motion

Checkpoint 4

A fly ball is hit to the outfield. During its flight (ignore the effects of the air), what happens to its (a) horizontal and (b) vertical components of velocity? What are the (c) horizontal and (d) vertical components of its acceleration during ascent, during descent, and at the topmost point of its flight?

Answer: (a) is unchanged (b) decreases (becomes negative)

(c) 0 at all times (d) -g (-9.8 m/s^2) at all times

4.5 Uniform circular motion

- A particle is in **uniform circular motion** if
  - It travels around a circle or circular arc
  - At a constant speed
- Since the velocity changes, the particle is accelerating!
- Velocity and acceleration have:
  - Constant magnitude
  - Changing direction

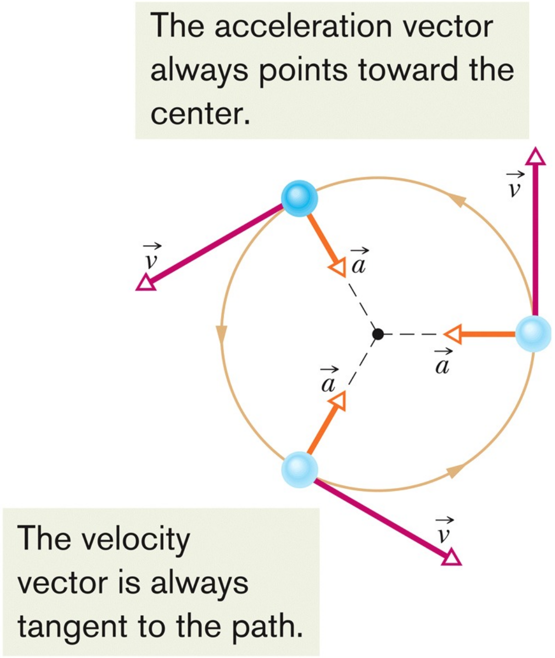


Figure 4-16

4.5 Circular motion – centripetal acceleration

- Acceleration is called **centripetal acceleration**
  - Means "centre seeking"
  - Directed radially inward

$$a = \frac{v^2}{r}$$

Eq. (4-34)

- The **period of revolution** is:
  - The time it takes for the particle go around the closed path exactly once

$$T = \frac{2\pi r}{v}$$

Eq. (4-35)

4.5 Circular motion

Checkpoint 5

An object moves at constant speed along a circular path in a horizontal xy plane, with the center at the origin. When the object is at $x = -2$ m, its velocity is $-(4 \text{ m/s})\hat{\text{j}}$. Give the object's (a) velocity and (b) acceleration at $y = 2$ m.

Answer: (a) -(4 m/s)i (b) -(8 m/s^2)j

4.6 Relative motion

- Measures of position and velocity depend on the **reference frame** of the measurer
  - How is the observer moving?
  - Our usual reference frame is that of the ground

- Read subscripts “PA”, “PB”, and “BA” as “P as measured by A”, “P as measured by B”, and “B as measured by A"

- Frames A and B are each watching the movement of object P

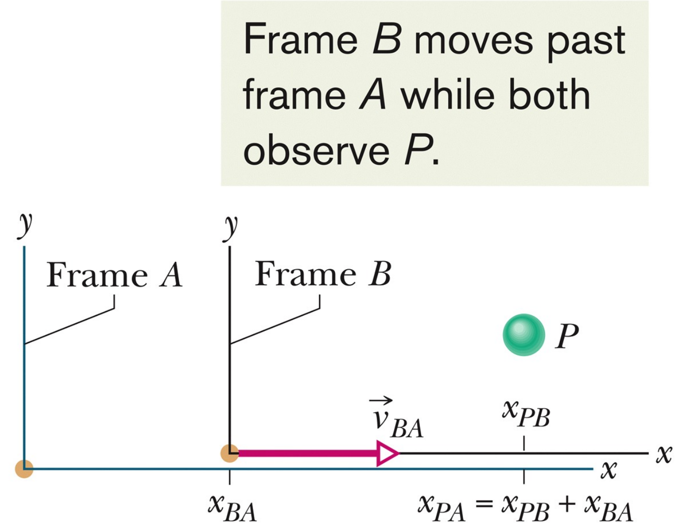


Figure 4-18

4.6 Relative motion

- Positions in different frames are related by:

$$x_{PA} = x_{PB} + x_{BA}.$$ **Eq. (4-40)**

- Taking the derivative, we see velocities are related by:

$$\frac{d}{dt}(x_{PA}) = \frac{d}{dt}(x_{PB}) + \frac{d}{dt}(x_{BA}).$$

$$v_{PA} = v_{PB} + v_{BA}.$$ **Eq. (4-41)**

- But accelerations (for non-accelerating reference frames, $a_{BA} = 0$) are related by

$$\frac{d}{dt}(v_{PA}) = \frac{d}{dt}(v_{PB}) + \frac{d}{dt}(v_{BA}).$$

$$a_{PA} = a_{PB}.$$ **Eq. (4-42)**

4.6 Relative motion

Observers on different frames of reference that move at constant velocity relative to each other will measure the same acceleration for a moving particle.

Example

Frame A: $x = 2$ m, $v = 4$ m/s

Frame B: $x = 3$ m, $v = -2$ m/s

P as measured by A: $x_{PA} = 5$ m, $v_{PA} = 2$ m/s, $a = 1$ m/s^2

So P as measured by B:

- $x_{PB} = x_{PA} + x_{AB} = 5\text{ m} + (2\text{m} – 3\text{m}) = 4\text{ m}$
- $v_{PB} = v_{PA} + v_{AB} = 2\text{ m/s} + (4\text{ m/s} – -2\text{m/s}) = 8\text{ m/s}$
- $a = 1\text{ m/s}^2$

4.7 Relative motion – frames of reference

- The same as in one dimension, but now with vectors:
- Positions in different frames are related by:

$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}.$$

Eq. (4-43)

- Velocities:

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}.$$

Eq. (4-44)

- Accelerations (for non-accelerating reference frames):

$$\vec{a}_{PA} = \vec{a}_{PB}.$$

Eq. (4-45)

- Again, observers in different frames will see the same acceleration

4.7 Relative motion – Frames of reference

- Frames A and B are both observing the motion of P

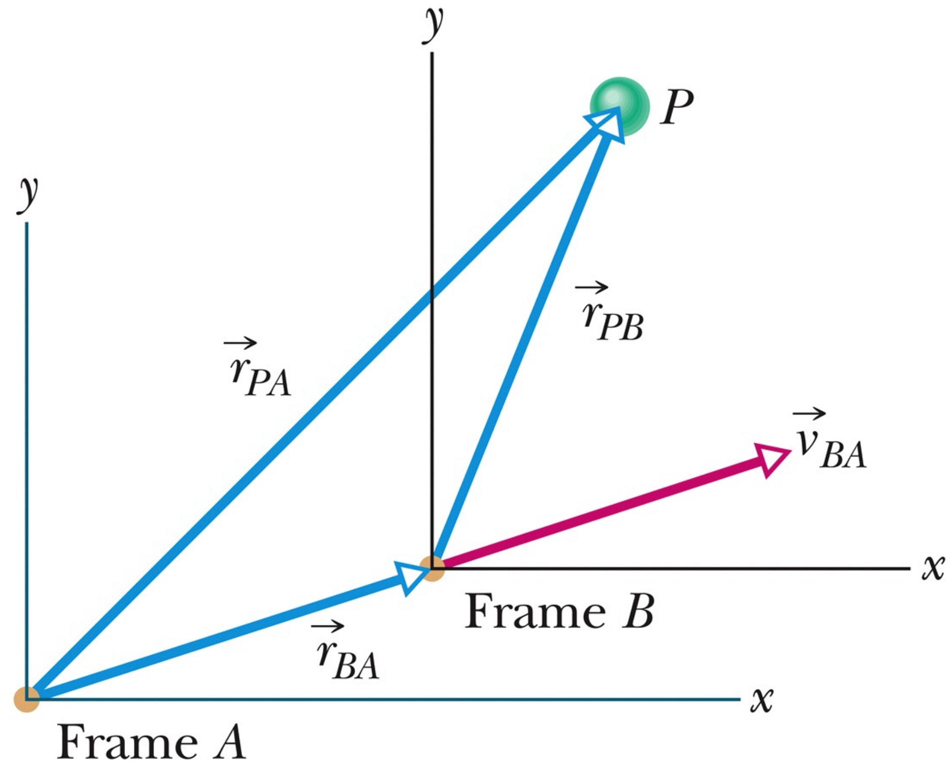


Figure 4-19

Summary

Position Vector

- Locates a particle in 3-space

$$\vec{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}},$$

Eq. (4-1)

Displacement

- Change in position vector

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1.$$

Eq. (4-2)

$$\Delta\vec{r} = (x_2 - x_1)\hat{\mathrm{i}} + (y_2 - y_1)\hat{\mathrm{j}} + (z_2 - z_1)\hat{\mathrm{k}},$$

Eq. (4-3)

$$\Delta\vec{r} = \Delta x\hat{\mathrm{i}} + \Delta y\hat{\mathrm{j}} + \Delta z\hat{\mathrm{k}}.$$

Eq. (4-4)

Average and Instantaneous Velocity

$$\vec{v}_{\text{avg}} = \frac{\Delta\vec{r}}{\Delta t}.$$

Eq. (4-8)

$$\vec{v} = \frac{d\vec{r}}{dt}.$$

Eq. (4-10)

Average and Instantaneous Accel.

$$\vec{a}_{\text{avg}} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta\vec{v}}{\Delta t}.$$

Eq. (4-15)

$$\vec{a} = \frac{d\vec{v}}{dt}.$$

Eq. (4-16)

Summary

Projectile Motion

- Flight of particle subject only to free-fall acceleration (g)

$$y - y_0 = v_{0y}t - \tfrac{1}{2}gt^2 = (v_0 \sin \theta_0)t - \tfrac{1}{2}gt^2,$$ **Eq. (4-22)**

$$v_y = v_0 \sin \theta_0 - gt$$ **Eq. (4-23)**

- Trajectory is parabolic path

$$y = (\tan \theta_0)x - \frac{gx^2}{2(v_0 \cos \theta_0)^2}$$ **Eq. (4-25)**

- Horizontal range:

$$R = \frac{v_0^2}{g} \sin 2\theta_0.$$ **Eq. (4-26)**

Uniform Circular Motion

- Magnitude of acceleration:

$$a = \frac{v^2}{r}$$ **Eq. (4-34)**

- Time to complete a circle:

$$T = \frac{2\pi r}{v}$$ **Eq. (4-35)**

Relative Motion

- For non-accelerating reference frames

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}.$$ **Eq. (4-44)**

$$\vec{a}_{PA} = \vec{a}_{PB}.$$ **Eq. (4-45)**

Preparation for the next lecture

1. Read 5.1-5.3 and 2.6 of the text
2. You will find short answers to the odd-numbered problems in each chapter at the back of the book and further resources on LMS. You should try a few of the simple odd numbered problems from each section (the simple questions have one or two dots next to the question number).