THE UNIVERSITY OF MELBOURNE

Student Number	

Semester 2 Assessment, 2016

School of Mathematics and Statistics

MAST30001 Stochastic Modelling

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 4 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- Students may bring one double-sided A4 sheet of handwritten notes into the exam room.
- Hand-held electronic scientific (but not graphing) calculators may be used.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- This paper has **7 questions**. Attempt as many questions, or parts of questions, as you can. The number of marks allocated to each question is shown in the brackets after the question statement. There are **100 total marks** available for this examination. Working and/or reasoning must be given to obtain full credit. Clarity, neatness and style count.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

1. (a) Analyse the state space $S = \{1, 2, 3, 4\}$ for each of the three Markov chains given by the following transition matrices. That is, classify each state as essential or not, transient or positive recurrent or null recurrent, and find the period if it is periodic.

   i.
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

   ii.
$$\begin{pmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   iii.
$$\begin{pmatrix} 0 & 0 & 1/3 & 2/3 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 1/3 & 2/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

   (b) For the Markov chain given by the transition matrix in part (a) iii above, discuss the long run behaviour of the chain including deriving long run probabilities.

   (c) For the Markov chain given by the transition matrix in part (a) iii above, find the expected number of steps taken for the chain to first reach state 3 given the chain starts at state 4.

[14 marks]

2. Let $(N_t)_{t\geq 0}$ and $(K_t)_{t\geq 0}$ be independent Poisson processes with rates λ and μ. Your answers to the questions below should be simple and tidy formulas in terms of λ and μ.

   (a) What is the expected time of the first arrival of N that occurs after the tenth arrival of K?

   (b) What is the chance that the number of arrivals of K in the interval $(2, 5)$ is at least two?

   (c) What is the expected time until the first arrival of either N or K?

   (d) What is the expected time between the first arrivals of N and K?

   (e) What is the expected number of arrivals of N between the second and third arrival of K?

   (f) Given there are 10 total arrivals of N and K in the interval $(0, 1)$, what is the chance that exactly 5 of these are from N and 5 are from K?

   (g) Given that $N_{10} = 3$, what is the chance that $N_3 = 1$?

   (h) Given that $N_{10} + K_{10} = 3$, what is the chance that $N_3 = 1$?

[20 marks]

3. A company with a subscription to a large cloud computing facility places jobs according to a rate λ Poisson process. If a job arrives and there is at least one server free, then the job is served immediately and the server becomes busy. If a job arrives and there are no servers free, then the job is placed in a queue and served in the order it was received. The subscription only allows for at most three jobs in the queue, and any jobs arriving when the queue is full are rejected. In addition, servers from the facility become free according to a rate μ Poisson process. If there are no customers waiting for service when a server becomes free, then it is placed in a queue of servers waiting for jobs. There is no cap on the number of servers that can queue. Note that it is not possible to have both jobs and servers waiting in queues simultaneously. Let X_t be the number of servers minus the number of jobs in the system at time t.

   (a) Model $(X_t)_{t\geq 0}$ as a continuous time Markov chain: specify its state space and generator matrix A.

   (b) Derive a condition on λ, μ so that the chain is ergodic.

   For the remainder of the problem, assume that $\lambda = 5$ and $\mu = 2$.

   (c) Describe the long run behaviour of the chain.

   (d) What long run proportion of the time are there servers waiting for jobs?

   (e) What long run proportion of jobs are rejected due to the queue being full?

   (f) What is the long run average number of jobs in the queue?

   (g) What is the long run average waiting time for a job arriving in the system?

[22 marks]

4. A certain electrical system requires a single battery. Battery lifetimes in *hours* are independent and have a gamma distribution with density $te^{-t/50}/2500$, $t > 0$. Batteries are replaced as soon as they fail.

   (a) State from formulas (or compute) the mean and variance of battery lifetimes.

   (b) On average, about how many batteries are needed over a 300 day period?

   (c) Give an interval around your estimate from (b) that will have a 95% chance of covering the true number of batteries needed over the 300 day period. Note that

$$\frac{1}{\sqrt{2\pi}}\int_{-1.96}^{1.96} e^{-x^2/2}dx = 0.95.$$

   (d) If you inspect the electrical system at some point, what would you estimate to be the mean and variance of the length of time that the current battery has been in use?

   You may find the following formula useful for this problem: for $a, b > 0$

$$\int_0^\infty t^a e^{-bt}dt = \frac{\Gamma(a+1)}{b^{a+1}}.$$

[13 marks]

5. Let $(X_t)_{t\geq 0}$ be a continuous time Markov chain on $\{0,1,2,3,4\}$ with generator matrix

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

(a) Given the chain is in state 1 right now, what is the expected amount of time until the chain jumps to a new state?

(b) Given the chain is in state 1 right now, what is the probability that the next time the chain jumps, it jumps to state 0?

(c) Find the stationary distribution of the chain.

(d) Find a simple expression for $p_{2,2}(t) := P(X_t = 2|X_0 = 2)$ that holds for all $t \geq 0$.

(e) More generally, assume now that $(Y_t)_{t\geq 0}$ is a continuous time Markov chain on $\{0,1,\ldots,2N\}$ for some $N \geq 2$ with generator matrix A having positive off-diagonal entries given by $a_{i,i+1} = a_{i,i-1} = 1$ for $i = 1,\ldots,2N-1$ and $a_{0,N} = a_{2N,N} = 1$. Find the stationary distribution of $(Y_t)_{t\geq 0}$.

[15 marks]

6. Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{0,1,2,\ldots\}$ with transition probabilities for $i \geq 1$

$$p_{i,i+1} = 1 - p_{i,i-1} = p < 1/2,$$

and $p_{0,1} = 1 - p_{0,0} = p$. Define the random variable $T = \min\{n \geq 0 : X_n = 0\}$.

(a) Compute $E[T|X_0 = i]$ for each $i = 0,1,2,\ldots$, (hint: consider $T' = \min\{n \geq 1 : X_n = 0\}$).

(b) If X_0 has the stationary distribution of the chain, then find $E[T]$.

[7 marks]

7. Let $0 < T_1 < T_2 < \cdots$ be the times of the arrivals of a rate λ Poisson process $(N_t)_{t\geq 0}$. For $t \geq 0$, let $Y_t = T_{N_t+1} - t$ be the time until the next arrival after t and $A_t = t - T_{N_t}$ be the time until the previous arrival before t of the Poisson process $(N_t)_{t\geq 0}$.

(a) Find a simple expression in terms of λ and x for $P(Y_t > x)$, $x > 0$.

(b) Find a simple expression in terms of λ and x for $P(A_t > x)$, $x > 0$.

(c) Hence show the expected length of the inter-arrival interval containing t is

$$\frac{2 - e^{-\lambda t}}{\lambda}.$$

(d) Name the approximate distribution of the length of the inter-arrival interval containing t, for large values of t. (No calculation is required, but instead you can write a sentence or two to justify your answer.)

[9 marks]

End of Exam