PHYC10003 Physics 1

Lecture 3: Motion

Vectors: applications to mechanics

Last lecture

- Position
- Velocity
- Acceleration
- Problems involving constant acceleration
- Free-fall

Vectors

- Physics deals with quantities that have both size and direction
- A **vector** is a mathematical object with size and direction
- A **vector quantity** is a quantity that can be represented by a vector
  - Examples: position, velocity, acceleration
  - Vectors have their own rules for manipulation
- A **scalar** is a quantity that does not have a direction
  - Examples: time, temperature, energy, mass
  - Scalars are manipulated with ordinary algebra

3.1 Displacement vector

- The simplest example is a **displacement vector**
- If a particle changes position from A to B, we represent this by a vector arrow pointing from A to B

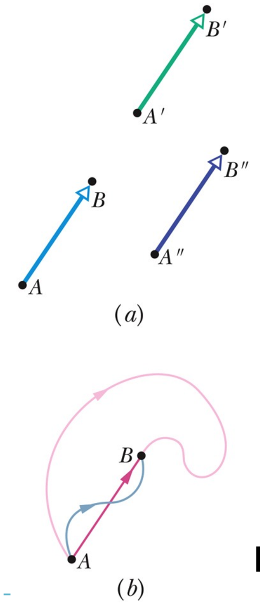


Figure 3-1

- In (a) we see that all three arrows have the same magnitude and direction: they are identical displacement vectors.
- In (b) we see that all three paths correspond to the same displacement vector. The vector tells us nothing about the actual path that was taken between A and B.

3.1 Vector addition

- The **vector sum**, or **resultant**
  - Is the result of performing vector addition
  - Represents the net displacement of two or more displacement vectors

$$\vec{s} = \vec{a} + \vec{b}, \quad \textbf{Eq. (3-1)}$$

  - Can be added graphically as shown:

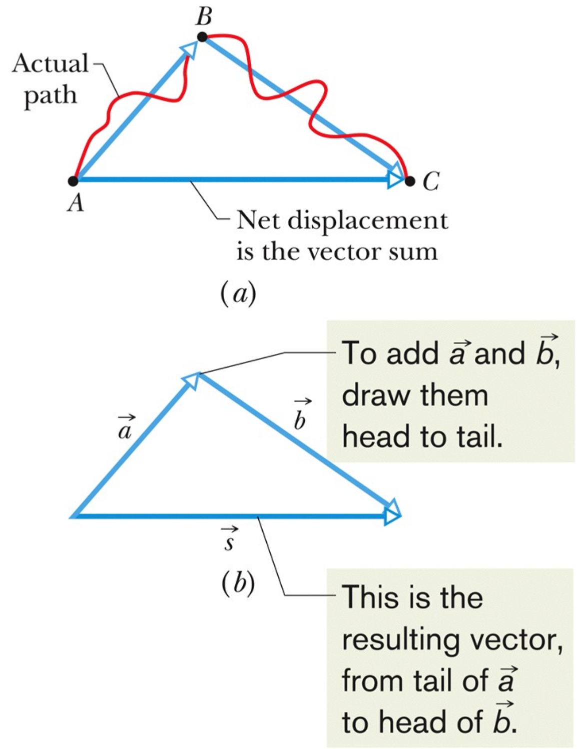


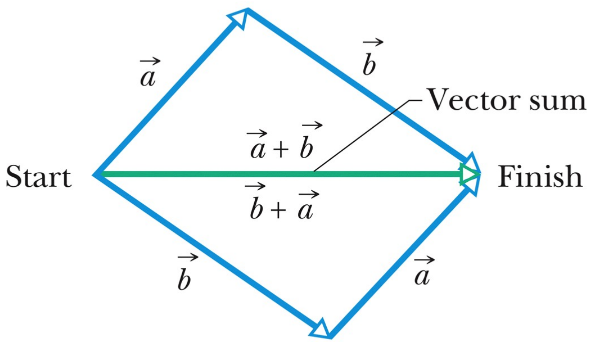
Figure 3-2

3.1 Vector addition

- Vector addition is **commutative**
  - We can add vectors in any order

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \text{(commutative law).}$$

Eq. (3-2)



You get the same vector result for either order of adding vectors.

Figure (3-3)

3.1 Vector addition

- Vector addition is **associative**
  - We can group vector addition however we like

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \qquad \text{(associative law).}$$

Eq. (3-3)

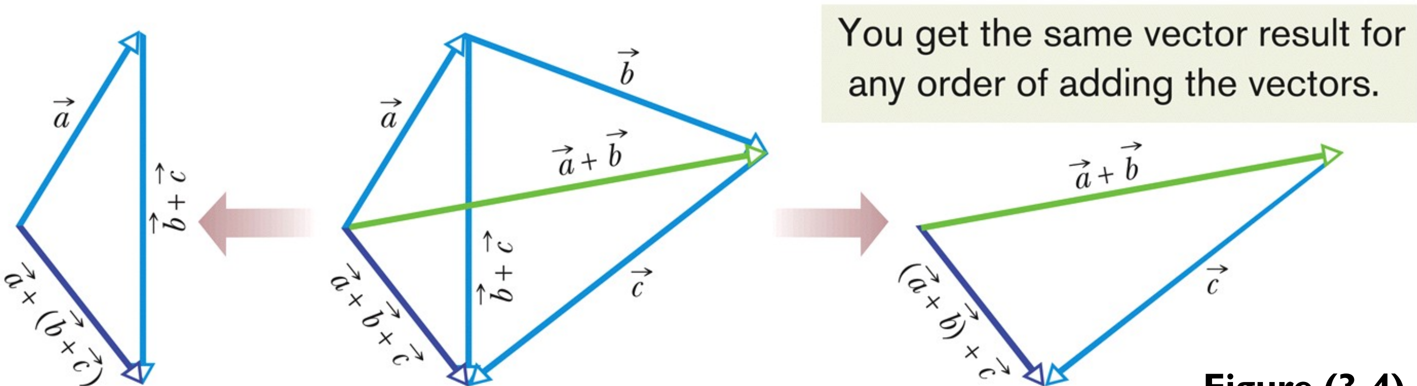


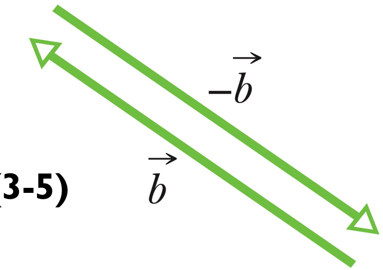
Figure (3-4)

3.1 Vectors; sign and direction

- A negative sign reverses vector direction

$$\vec{b} + (-\vec{b}) = 0.$$

Figure (3-5)



- We use this to define vector subtraction

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

Eq. (3-4)

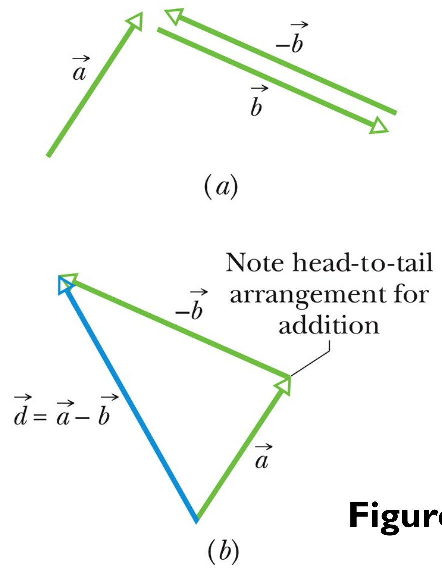


Figure (3-6)

3.1 Vectors

- These rules hold for all vectors, whether they represent displacement, velocity, etc.
- Only vectors of the same kind can be added
  - (*distance*) + (*distance*) makes sense
  - (*distance*) + (*velocity*) does not

Checkpoint 1

The magnitudes of displacements $\vec{a}$ and $\vec{b}$ are 3 m and 4 m, respectively, and $\vec{c} = \vec{a} + \vec{b}$. Considering various orientations of $\vec{a}$ and $\vec{b}$, what are (a) the maximum possible magnitude for $\vec{c}$ and (b) the minimum possible magnitude?

Answer:

(a) 3 m + 4 m = 7 m   (b) 4 m - 3 m = 1 m

3.2 Vectors- addition by components

- Rather than using a graphical method, vectors can be added by **components**
    - A component is the projection of a vector on an axis
- The process of finding components is called **resolving the vector**
- The components of a vector can be positive or negative.
- They are unchanged if the vector is shifted in any direction (but not rotated).

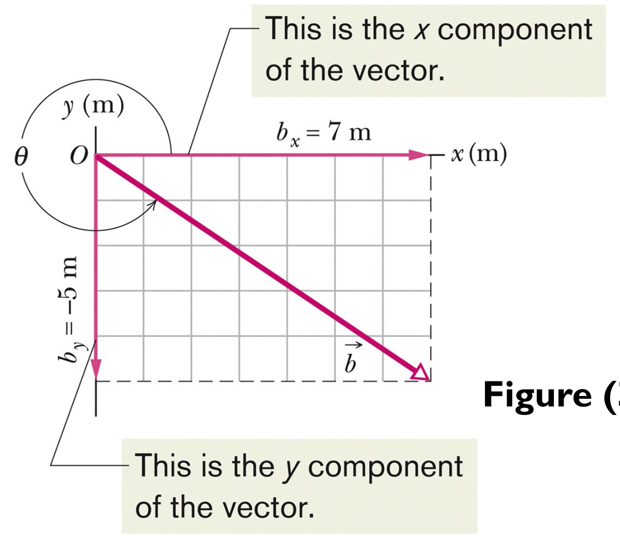


Figure (3-8)

3.2 Vectors – components in 2 dimensions

- Components in two dimensions can be found by:

$$a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta, \qquad \textbf{Eq. (3-5)}$$

- Where θ is the angle the vector makes with the positive *x* axis, and *a* is the vector length
- The length and angle can also be found if the components are known

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\theta = \frac{a_y}{a_x} \qquad \textbf{Eq. (3-6)}$$

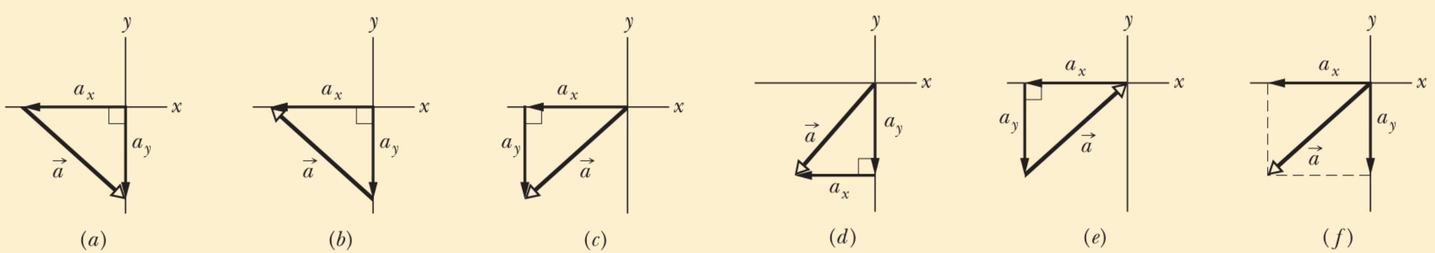
Therefore, components fully define a vector

3.2 Vectors- components in 3 dimensions

- In the 3 dimensional case we need more components to specify a vector
  - (a, θ, φ) or (a_x, a_y, a_z)

Checkpoint 2

In the figure, which of the indicated methods for combining the x and y components of vector $\vec{a}$ are proper to determine that vector?



Answer: choices (c), (d), and (f) show the components properly arranged to form the vector

3.2 Vectors- trigonometry

- Angles may be measured in degrees or radians
- Recall that a full circle is 360°, or 2π rad

$$40^\circ \frac{2\pi \text{ rad}}{360^\circ} = 0.70 \text{ rad}.$$

- Know the three basic trigonometric functions

$$\sin\theta = \frac{\text{leg opposite } \theta}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{leg adjacent to } \theta}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{leg opposite } \theta}{\text{leg adjacent to } \theta}$$

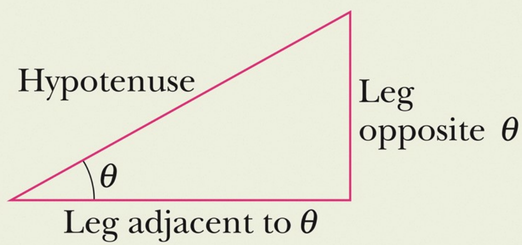


Figure (3-11)

3.2 Vectors- unit vectors and conventions

- A **unit vector**
  - Has magnitude 1
  - Has a particular direction
  - Lacks both dimension and unit
  - Is labeled with a hat: ^

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} \quad \textbf{Eq. (3-7)}$$

$$\vec{b} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}}. \quad \textbf{Eq. (3-8)}$$

- We use a **right-handed coordinate system**
  - Remains right-handed when rotated

The unit vectors point along axes.

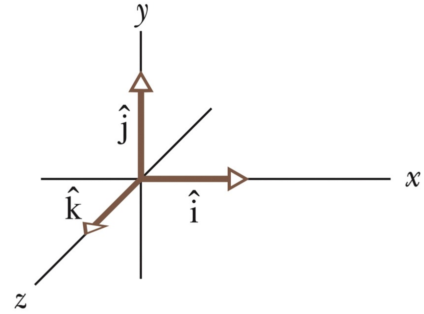


Figure (3-13)

3.2 Vectors- unit vectors and components

- The quantities a_x**i** and a_y**j** are **vector components**

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}}$$ **Eq. (3-7)**

$$\vec{b} = b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}}.$$ **Eq. (3-8)**

- The quantities a_x and a_y alone are **scalar components**
  - Or just "components" as before

- Vectors can be added using components

Eq. (3-9) $$\vec{r} = \vec{a} + \vec{b}, \quad \rightarrow$$

$$r_x = a_x + b_x$$ **Eq. (3-10)**

$$r_y = a_y + b_y$$ **Eq. (3-11)**

$$r_z = a_z + b_z.$$ **Eq. (3-12)**

3.2 Vectors - subtraction

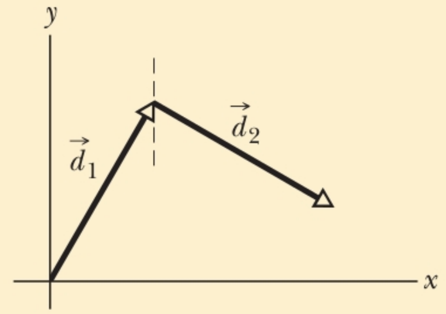
- To subtract two vectors, we subtract components

$$d_x = a_x - b_x, \quad d_y = a_y - b_y, \quad \text{and} \quad d_z = a_z - b_z, \qquad \textbf{Eq. (3-13)}$$

$$\vec{d} = d_x\hat{\mathrm{i}} + d_y\hat{\mathrm{j}} + d_z\hat{\mathrm{k}}.$$

Checkpoint 3

(a) In the figure here, what are the signs of the x components of $\vec{d}_1$ and $\vec{d}_2$? (b) What are the signs of the y components of $\vec{d}_1$ and $\vec{d}_2$? (c) What are the signs of the x and y components of $\vec{d}_1 + \vec{d}_2$?



Answer: (a) positive, positive (b) positive, negative

(c) positive, positive

3.2 Vectors – behaviour under rotation

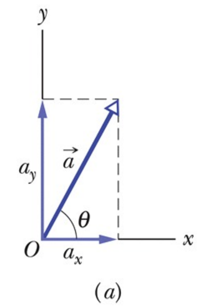
- Vectors are independent of the coordinate system used to measure them

- We can rotate the coordinate system, without rotating the vector, and the vector remains the same

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{a_x'^2 + a_y'^2} \quad \textbf{Eq. (3-14)}$$

$$\theta = \theta' + \phi. \quad \textbf{Eq. (3-15)}$$

- All such coordinate systems are equally valid



Rotating the axes changes the components but not the vector.

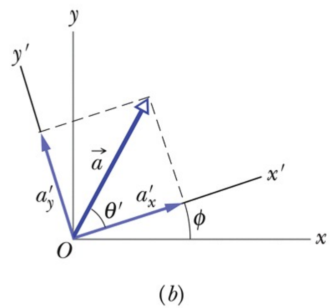


Figure (3-15)

3.2 Vectors- multiplication by a scalar

- Multiplying a vector **z** by a scalar *c*
  - Results in a new vector
  - Its magnitude is the magnitude of vector **z** times |*c*|
  - Its direction is the same as vector **z**, or opposite if *c* is negative
  - To achieve this, we can simply multiply each of the components of vector **z** by *c*
- To divide a vector by a scalar we multiply by 1/*c*

Example Multiply vector **z** by 5

- **z** = -3 **i** + 5 **j**
- 5 **z** = -15 **i** + 25 **j**

3.3 Vectors- scalar product

- Multiplying two vectors: the **scalar product**
  - Also called the **dot product**
  - Results in a scalar, where *a* and *b* are magnitudes and φ is the angle between the directions of the two vectors:

$$\vec{a} \cdot \vec{b} = ab \cos \phi, \qquad \textbf{Eq. (3-20)}$$

- The commutative law applies, and we can do the dot product in component form

$$\vec{a} \cdot \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \cdot (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}), \qquad \textbf{Eq. (3-22)}$$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}. \quad \vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z. \qquad \textbf{Eq. (3-23)}$$

3.3 Vectors - projections

- A dot product is the product of the magnitude of one vector times the scalar component of the other vector in the direction of the first vector

$$\vec{a}\cdot\vec{b} = (a\cos\phi)(b) = (a)(b\cos\phi).$$

Eq. (3-21)

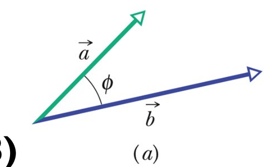
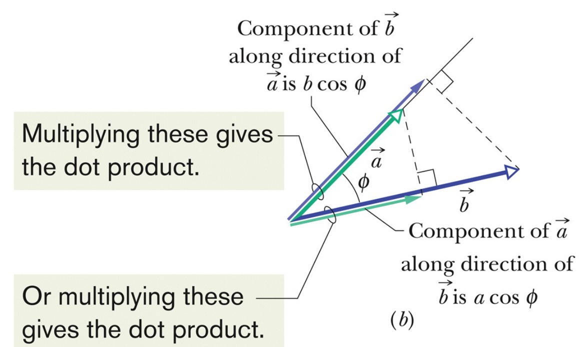


Figure (3-18)



- Either projection of one vector onto the other can be used

- To multiply a vector by the projection, multiply the magnitudes

Vectors

If the angle ϕ between two vectors is $0°$, the component of one vector along the other is maximum, and so also is the dot product of the vectors. If, instead, ϕ is $90°$, the component of one vector along the other is zero, and so is the dot product.

Checkpoint 4

Vectors $\vec{C}$ and $\vec{D}$ have magnitudes of 3 units and 4 units, respectively. What is the angle between the directions of $\vec{C}$ and $\vec{D}$ if $\vec{C} \cdot \vec{D}$ equals (a) zero, (b) 12 units, and (c) -12 units?

Answer: (a) 90 degrees (b) 0 degrees (c) 180 degrees

3.3 Vectors – cross product

- Multiplying two vectors: the **vector product**
  - The **cross product** of two vectors with magnitudes *a* & *b*, separated by angle φ, produces a vector with magnitude:

$$c = ab \sin \phi, \qquad \textbf{Eq. (3-24)}$$

  - And a direction perpendicular to both original vectors
- Direction is determined by the **right-hand rule**
- Place vectors tail-to-tail, sweep fingers from the first to the second, and thumb points in the direction of the resultant

If $\vec{a}$ and $\vec{b}$ are parallel or antiparallel, $\vec{a} \times \vec{b} = 0$. The magnitude of $\vec{a} \times \vec{b}$, which can be written as $|\vec{a} \times \vec{b}|$, is maximum when $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

3.3 Vectors – cross product

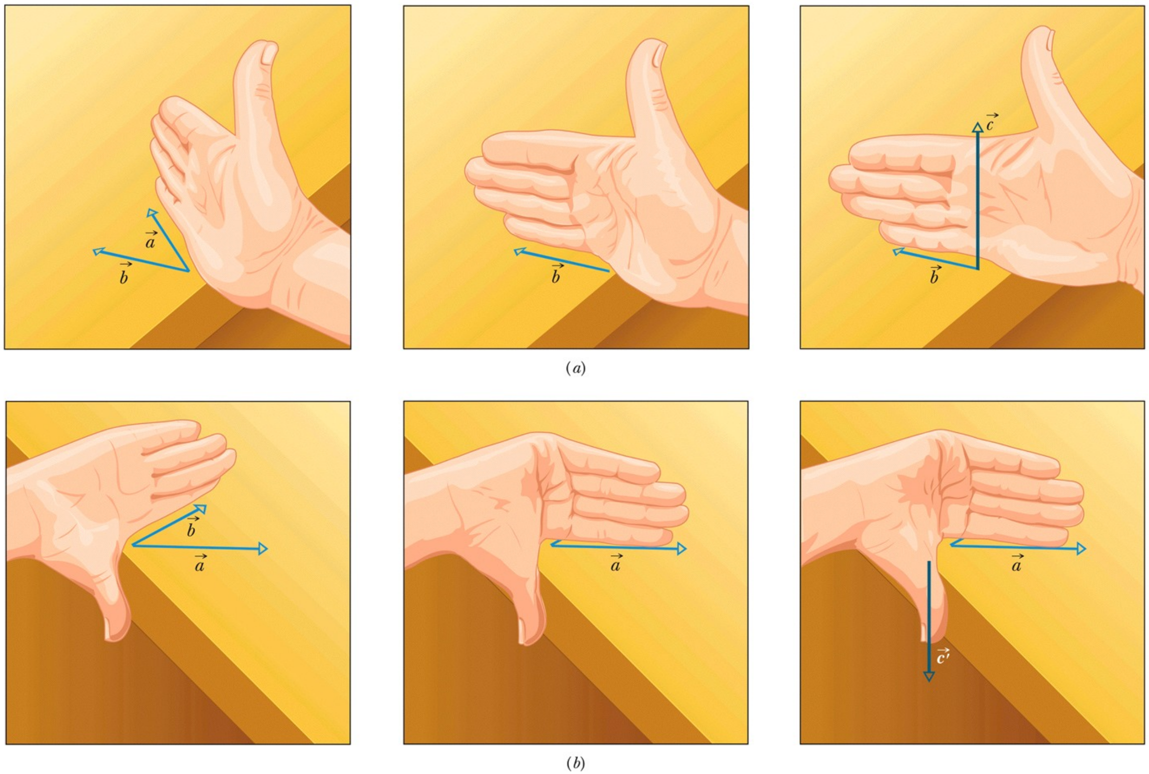


Figure (3-19)

The upper shows vector *a* cross vector *b*, the lower shows vector *b* cross vector *a*

3.3 Vectors - cross product

- The cross product is not commutative

$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b}).$$

Eq. (3-25)

- To evaluate, we distribute over components:

$$\vec{a} \times \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \times (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$

Eq. (3-26)

$$a_x\hat{\mathrm{i}} \times b_x\hat{\mathrm{i}} = a_x b_x(\hat{\mathrm{i}} \times \hat{\mathrm{i}}) = 0,$$

$$a_x\hat{\mathrm{i}} \times b_y\hat{\mathrm{j}} = a_x b_y(\hat{\mathrm{i}} \times \hat{\mathrm{j}}) = a_x b_y\hat{\mathrm{k}}.$$

- Therefore, by expanding (3-26):

$$\vec{a} \times \vec{b} = (a_y b_z - b_y a_z)\hat{\mathrm{i}} + (a_z b_x - b_z a_x)\hat{\mathrm{j}} + (a_x b_y - b_x a_y)\hat{\mathrm{k}}.$$

Eq. (3-27)

Vectors

Checkpoint 5

Vectors $\vec{C}$ and $\vec{D}$ have magnitudes of 3 units and 4 units, respectively. What is the angle between the directions of $\vec{C}$ and $\vec{D}$ if the magnitude of the vector product $\vec{C} \times \vec{D}$ is (a) zero and (b) 12 units?

Answer: (a) 0 degrees (b) 90 degrees

Summary

Scalars and Vectors

- Scalars have magnitude only
- Vectors have magnitude and direction
- Both have units!

Adding Geometrically

- Obeys commutative and associative laws

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \textbf{Eq. (3-2)}$$

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}). \qquad \textbf{Eq. (3-3)}$$

Vector Components

- Given by

$$a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta, \qquad \textbf{Eq. (3-5)}$$

- Related back by

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\theta = \frac{a_y}{a_x} \qquad \textbf{Eq. (3-6)}$$

Unit Vector Notation

- We can write vectors in terms of unit vectors

$$\vec{a} = a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}, \qquad \textbf{Eq. (3-7)}$$

Summary

Adding by Components

- Add component-by-component

$$r_x = a_x + b_x$$

$$r_y = a_y + b_y$$

Eqs. (3-10) - (3-12) $$r_z = a_z + b_z.$$

Scalar Times a Vector

- Product is a new vector
- Magnitude is multiplied by scalar
- Direction is same or opposite

Scalar Product

- Dot product

$$\vec{a} \cdot \vec{b} = ab \cos \phi,$$

Eq. (3-20)

$$\vec{a} \cdot \vec{b} = (a_x\hat{\mathrm{i}} + a_y\hat{\mathrm{j}} + a_z\hat{\mathrm{k}}) \cdot (b_x\hat{\mathrm{i}} + b_y\hat{\mathrm{j}} + b_z\hat{\mathrm{k}}),$$

Eq. (3-22)

Cross Product

- Produces a new vector in perpendicular direction
- Direction determined by right-hand rule

$$c = ab \sin \phi,$$

Eq. (3-24)

Preparation for the next lecture

1. Read 4.1-4.7 of the text
2. You will find short answers to the odd-numbered problems in each chapter at the back of the book and further resources on LMS. You should try a few of the simple odd numbered problems from each section (the simple questions have one or two dots next to the question number).