2.6 Inequalities

In addition to its arithmetic operations and their properties, the set of real numbers $\mathbb{R}$ has the structure given by the order relation denoted $a > b$. This satisfies:

(P1) If $a > 0$ and $b > 0$ then $a + b > 0$ and $ab > 0$.

(P2) For every $a \in \mathbb{R}$, exactly one of the following statements is true: $a = 0$, $a < 0$, or $a > 0$.

Example 2.27. The square of any nonzero real number is positive.

Proof. let $x \in \mathbb{R}$, $x \neq 0$

By $x < 0 \Rightarrow -x > 0$ $\quad (-x)\cdot(-x) = x^2 > 0$

$x > 0 \Rightarrow x \cdot x > 0$

Example 2.28. ① Find $n_0 \in \mathbb{N}$ such that ② $n^3 > (n+1)^2$ for all $n \geq n_0$.

[*Hint*: Add $3k^2 + 3k + 1$ to both sides of $k^3 > (k+1)^2$.]

n	n^3	$(n+1)^2$
2	8	9
3	27	16

$n_0 = 3$

$\Downarrow$

$n^3 > (n+1)^2$ for all $n \geq 3$

Base case $n = 3$

$27 > 16$.

$k^3 > (k+1)^2$

$k^3 + 3k^2 + 3k + 1 > (k+1)^2 + 3k^2 + 3k + 1$

$(k+1)^3 > 4k^2 + 5k + 2$

$(4k^2 + 5k + 2) - (k+2)^2$

$4k^2 + 5k + 2 - (k^2 + 4k + 4)$

$3k^2 - k - 2$

$3 \quad +2$
$1 \quad -1$

$(3k+2)(k-1)$.

for $k > 3$

$3k+2 > 0$

$k-1 > 0$

Example 2.29. Find $n_0 \in \mathbb{N}$ such that $n! > n^3$ for all $n \geq n_0$.

[*Hint*: Reduce to the polynomial case by multiplying both sides of $k! > k^3$ by $(k+1)$.]

n	$n!$	n^3
1	1	1
2	2	8
3	6	27
4	24	64
5	120	125
6	720	216

$n_0 = 6$

Assume $k! > k^3$

$(k+1)! = k!\,(k+1)$

$k^3(k+1) = k^4 + k^3$

$(k+1)! > k^3(k+1)$

try to prove

$k^3(k+1) > (k+1)^3$

$k^3 > (k+1)^2$

from example. 2.28.

prove

3 Vector spaces and linear transformations

3.1 Axiomatic definition of vector spaces

If you try to describe properties that are common to the spaces $\mathbb{R}^n$ that we have been working with, you may eventually end up with the following:

A *vector space V over a field* $\mathbb{F}$ is a set with two operations

- addition $V \times V \to V$, $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} + \mathbf{v}$
- scalar multiplication $\mathbb{F} \times V \to V$, $(\lambda, \mathbf{u}) \mapsto \lambda\mathbf{u}$

satisfying the axioms

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(c) There exists an element $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

(d) For all $\mathbf{u} \in V$ there exists an element denoted $-\mathbf{u}$ with the property that $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$.

(e) $(\lambda\mu)\mathbf{v} = \lambda(\mu\mathbf{v})$

(f) $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$

(g) $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$

(h) $1\mathbf{u} = \mathbf{u}$

Example 3.1 (Real n-dimensional space)**.** For any $n \in \mathbb{N}$, the space $\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

So is $\mathbb{R}^0 = \{\mathbf{0}\}$.

$R = \{ (x1, x2, \cdots, xn) \mid x1, x2, \cdots xn \in R \}$

Example 3.2 (Space of $m \times n$ matrices)**.**

$M_{m\times n} = \{ m \times n$ matrices with entries $\mathbb{R}\}$

with addition
$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

scalar multiplication
$$\lambda \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \vdots & \ddots & \vdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{bmatrix}$$

Claim: $M_{m\times n}$ satisfies the axioms for a vector space.

$$0 = \overbrace{\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}}^{n} \Big\} m$$

Example 3.3 (Space of functions)**.** Fix a set S (non empty) and consider
$$\mathcal{F}(S, \mathbb{R}) = \{f : S \to \mathbb{R}\}.$$

addition let $f, g \in \mathcal{F}(S, \mathbb{R})$ so $f: S \to \mathbb{R}$
$g: S \to \mathbb{R}$.

Define $(f+g): S \to \mathbb{R}$
by $(f+g)(x) = f(x) \underset{\text{scalar in } \mathbb{R}}{+} g(x)$ for all $x \in S$.

scalar multiplication
let $f \in \mathcal{F}(S, \mathbb{R})$ so $f: S \to \mathbb{R}$
$\lambda \in \mathbb{R}$
Define $(\lambda f): S \to \mathbb{R}$.
by $(\lambda f)(x) = \lambda f(x)$ (multiplication in $\mathbb{R}$) for all $x \in S$.

zero vector in $\mathcal{F}(S, \mathbb{R})$ $z: S \to \mathbb{R}$ by $z(x) = 0$ for $\forall x \in S$

want $f + z = f$ for all $f \in \mathcal{F}(S, \mathbb{R})$
ie $(f+z)(x) = f(x)$
$f(x) + z(x) = f(x)$. for all $f \in \mathcal{F}(S, \mathbb{R})$; all $x \in S$.
$\Rightarrow z(x) = 0$ in $\mathbb{R}$.

Example 3.4 (Space of polynomials).

$\mathcal{P} = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid n \geq \mathbb{Z}_{n\geq 0},\ a_0 \cdots a_n \in \mathbb{R}\}$

x is a formal variable

$p = a_0 + a_1 x + \cdots + a_n x^n$

$q = b_0 + b_1 x + \cdots + b_n x^n$

Define

$p + q = (a_0 + b_0) + (a_1 + b_1)x + \cdots (a_n + b_n) x^n$

suppose $m \leq n$

scalar result $\quad p \in \mathcal{P},\ \lambda \in \mathbb{R}$

Def $\lambda p = (\lambda a_0) + (\lambda a_1) x + \cdots + (\lambda a_n) x^n \in \mathcal{P}$

What is zero vector?

it's $0 \in \mathcal{P}$ $\quad (n = 0\ \&\ a_0 = 0)$.

Example 3.5 (Space of polynomials of bounded degree).

Fixed $d \in \mathbb{Z}_{n\geq 0}$

$\mathcal{P}_d = \{a_0 + a_1 x + \cdots + a_d x^d \mid a_0, \cdots, a_d \in \mathbb{R}\}$

"polynomials of degree at most d"

It's a vector space w/ some addition, scalar mult as $\mathcal{P}$

All the examples we have seen have direct analogues over other fields $\mathbb{F}$ such as $\mathbb{Q}$, $\mathbb{C}$, or $\mathbb{F}_2$.

For instance, $\mathbb{C}^n = \{(x_1, x_2, \cdots, x_n) \mid x_1, x_2 \cdots, x_n \in \mathbb{C}\}$

$$M_{n\times n}(\mathbb{C}) = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \;\middle|\; a_{11}, \cdots, a_{mn} \in \mathbb{C} \right\}$$

$$P_3(\mathbb{F}_2) = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{F}_2 \}$$

3.2 Subspaces

Let V be a vector space and let W be a subset of V. We say that W is a *subspace* of V if W is itself a vector space under the operations inherited from V.

The subspace theorem. A subset W of a vector space V is a subspace if and only if it satisfies the three properties

(a) $\mathbf{0} \in W$

(b) if $\mathbf{w}_1, \mathbf{w}_2 \in W$, then $\mathbf{w}_1 + \mathbf{w}_2 \in W$ closure under addition

(c) if $\lambda \in \mathbb{F}$ and $\mathbf{w} \in W$, then $\lambda\mathbf{w} \in W$. closure under scalar multiplication

Proof.

Example 3.6. Let V be a vector space and let $W = \{\mathbf{0}\}$.

claim W is a subspace of V , check the condition in the subspace thm.

(a) $0 \in W$, yes by def of W

(b) $0 + 0 \in W$

$0 \in W$. ✓

(c) if $\lambda \in \mathbb{R}$ and $0 \in W$

want $(\lambda 0) \in W$ ← need to prove

$\lambda 0 = 0$

so W is subspace of V

Example 3.7. Let $V = \mathbb{R}^2$ and

$$W = \{(x, y) \in \mathbb{R}^2 \mid y = 2x\}.$$

① $0 = (0,0) \in W$

③ if $\lambda \in \mathbb{R}$ & $(x,y) \in W$, do we have

Example 3.8. Let $V = \mathbb{R}^2$ and

$$W = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}.$$

① $0 = (0, 0) \in W$

③ if $\lambda \in \mathbb{R}$ & $(x, y) \in W$, do we have $\lambda(x, y) \in W$?

$\Downarrow$ $y = x^2$ $\quad\quad$ $\Downarrow$ $(\lambda x)^2 = (\lambda y)$ $\quad$ $\lambda^2 x^2 = \lambda y$ $\quad$ $\lambda = 1$

No, take $w = (1,1) \in W$
$\lambda = 2$

Then $\lambda w = (2, 2) \notin W$

Example 3.9. Let $V = \mathbb{R}^2$ and

$$W = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}.$$

① $\mathbf{0} = (0, 0) \in W$

② $w_1 = (1, 0)$ $\quad w_2 = (0, 1)$. we have $w_1 + w_2 = \underbrace{(1, 1)}_{\Downarrow\ xy = 1 \neq 0} \notin W$.

Example 3.10. Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ and

$$W = \{f \in V \mid f(1) = 0\}.$$

$\mathcal{F}(\mathbb{R},\mathbb{R}) \Rightarrow \{f : \mathbb{R} \to \mathbb{R}\}$

① $\mathbf{0}_V \overset{?}{\in} W$? recall $\vec{0}_V : \mathbb{R} \to \mathbb{R}$

define by $\vec{0}_V(x) = 0$

note $\vec{0}(1) = 0$ so $\vec{0}_V \in W$

② let $f, g \in W$. so $f(1) = 0$ $g(1) = 0$

$f : \mathbb{R} \to \mathbb{R}$ $g : \mathbb{R} \to \mathbb{R}$

recall $(f+g)(x) = f(x) + g(x)$

$(f+g)(1) = f(1) + g(1) = 0 \in W$

Homework ③ let $fg \in W$ so $f(1) = 0$ $g(1) = 0$

$f : \mathbb{R} \to \mathbb{R}$ $g : \mathbb{R} \to \mathbb{R}$

recall $(fg)(x)$

Example 3.11. Let $V = \mathbb{R}^3$ and

$$W = \{(x, y, z) \in V \mid x + 2y + 3z = 3\}.$$

not a subspace since $0 + 2 \cdot 0 + 3 \cdot 0 \neq 3$

$(0, 0, 0)$

subspace my go through the origin

Example 3.12. Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ and

$$W = \{f \in V \mid f \text{ is continuous}\}.$$

(1) $\vec{0}_V$ is continuous ✓

(2) f & g is continuous

$f+g$ is contious

(3) f is continuous, $\lambda \in \mathbb{R}$

then (λf) is continuous

$\mathbb{F}_2 = 0, 1$

Example 3.13. Let $V = \mathbb{F}_2^4$ and

$V = \{(x_1, x_2, x_3, x_4) \mid x_1, x_2, x_3, x_4 \in \mathbb{F}_2\}$

$$W = \{(0,0,0,0), (1,0,0,1), (0,1,1,0), (1,1,1,1)\}.$$

(1) $\vec{0}_V = (0,0,0,0) \in W$ ✓

(2) $w_1 = (1,0,0,1)$

$w_2 = (1,0,0,1)$

$w + w = (0)$

(二进制)

$0+0=0$

$1+1=0.$

in F_2, when we add two thing together, we always get 0

(3) for all $\lambda \in F_2$ & all $w \in W$, have $\lambda w \in W$

for $\lambda = 0 \Rightarrow$ ✓ by (1)

for $\lambda = 1 \Rightarrow$. v don't change $\in W$.

3.3 Linear combinations and spanning sets

Let V be a vector space and $\mathbf{v}_1, \dots, \mathbf{v}_m \in V$. A *linear combination* of $\mathbf{v}_1, \dots, \mathbf{v}_m$ is a vector of the form

$$\mathbf{w} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_m \mathbf{v}_m$$

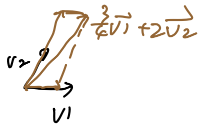
with $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ (or whatever field of scalars V has).

Example 3.14 (Algebraic view)**.** In $\mathbb{R}^3$,

eg. take $\vec{v_1} = (4,0,0)$ $\vec{v_2} = (1,1,1)$

Then $\frac{3}{4}\vec{v_1} + 2\vec{v_2}$ is a linear combination of $\vec{v_1}$ & $\vec{v_2}$

Example 3.15 (Geometric view)**.** In $\mathbb{R}^3$,



? 范围??

The set of **all** linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_m$ is called the *span* of these m vectors:

$$W = \operatorname{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_m\} = \{\lambda_1 \mathbf{v}_1 + \cdots + \lambda_m \mathbf{v}_m \mid \lambda_1, \dots, \lambda_m \in \mathbb{R}\}.$$

For instance, in Example 3.15, the plane is the span of the vectors $\vec{v_1} = \begin{bmatrix}4\\0\\0\end{bmatrix}$ & $\vec{v_2} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

Example 3.16. Is the vector $\begin{bmatrix}0\\1\end{bmatrix}$ in the span of $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$?

find $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\begin{bmatrix}0\\1\end{bmatrix} = \lambda_1 \begin{bmatrix}1\\-1\end{bmatrix} + \lambda_2 \begin{bmatrix}2\\1\end{bmatrix}$$

$$\lambda_1 + 2\lambda_2 = 0$$

$$-\lambda_1 + \lambda_2 = 1$$

$$\lambda_2 = \frac{1}{3} \quad \lambda_1 = \frac{-2}{3}$$

$$\begin{array}{ccc} \vec{v_1} & \vec{v_2} & \vec{w} \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & 1 & 1 \end{array}\right]$$

More generally, if $V = \mathbb{R}^n$ and we want to determine whether $\mathbf{w}$ is in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_m$, we form the augmented matrix

$$\left[\begin{array}{cccc|c} | & | & \cdots & | & | \\ v_1 & v_2 & \cdots & v_n & w \\ | & | & \cdots & | & | \end{array}\right]$$

and see whether the system has at least one solution (then $\mathbf{w}$ is in the span) or no solutions (then $\mathbf{w}$ is not in the span).

What about other vector spaces?

Example 3.17. Consider the space $\mathcal{P}_2$ of real polynomials of degree at most 2. Is $\mathbf{w} = 1 - 2x - x^2$ a linear combination of $\mathbf{v}_1 = 1 + x + x^2$ and $\mathbf{v}_2 = 3 + x^2$?

$$\vec{w} = \lambda_1 \vec{v_1} + \lambda_2 \vec{v_2}$$

$$(1-2x-x^2) = \lambda_1(1+x+x^2) + \lambda_2(3+x^2)$$

$$1-2x-x^2 = (\lambda_1+\lambda_2)x^2 + \lambda_1 x + (\lambda_1 + 3\lambda_2)$$

$$\left\{\begin{array}{l} \lambda_1+\lambda_2 = -1 \\ \lambda_1 = -2 \\ \lambda_1 + 3\lambda_2 = 1 \end{array}\right. \Rightarrow \begin{array}{l} -2+3\lambda_2 = 1 \\ \lambda_2 = 1 \end{array} \quad -2+1=-1 \ \checkmark$$

Proposition 3.18. The span of a set of vectors in a vector space V is a subspace of V.

Proof. check the condition of subspace

let $\vec{w}$ be the span

① $\vec{0_v} \in W$ (let all coefficient be 0) ✓

② closure under addition

$\vec{w_1} \in W = \lambda_1 \vec{v_1} + \cdots \lambda_m \vec{v_m}$

$\vec{w_2} \in W = \mu_1 \vec{v_1} + \cdots \mu_m \vec{v_m}$

$\vec{w_1} + \vec{w_2} = (\lambda_1 + \mu_1)\vec{v_1} + \cdots + (\lambda_m + \mu_m)\vec{v_m} \in W$ ✓

③ if $\alpha \in \mathbb{R}$, w in W

$\alpha \vec{w} = \alpha(\lambda_1 \vec{v_1} + \cdots \lambda_m \vec{v_m})$

$= (\alpha \lambda_1 \vec{v_1} + \cdots \alpha \lambda_m \vec{v_m}) \in W$ ✓

so prove

Let $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$. We say that these vectors *span V* if

$$\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} = V,$$

in other words every vector in V is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_m$. This set of vectors is then called a *spanning set of V*.

Example 3.19 ($\mathbb{R}^n$).

$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ $e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \to$ 1 in position n

Given $0 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_3 \end{bmatrix} \in \mathbb{R}^n$

$0 = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$

so $\{e_1, e_2, \cdots e_n\}$ is a spanning set of $\mathbb{R}^n$

Example 3.20. Do the following vectors span $\mathbb{R}^3$?

$$\mathbf{v}_1 = \begin{bmatrix}1\\1\\2\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}2\\1\\3\end{bmatrix}$$

$\text{span}\{a_1, a_2, a_3\} \subseteq R^3$

let $w = \begin{bmatrix}a\\b\\c\end{bmatrix} \subseteq R^3$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{array}\right] \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

does this have solution or not

$$\xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & -1 & -1 & c-2a \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & 0 & 0 & c-b-a \end{array}\right]$$

Take $a=0$, $b=0$, $c=1$

$\begin{bmatrix}0\\0\\1\end{bmatrix} \notin \text{span}\{a_1, a_2, a_3\}$

3.4 Linear (in)dependence

A vector space has many spanning sets. For instance $\mathbb{R}^3$ can be spanned any of the following:

- $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$
- $\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\1\\4\end{bmatrix}\right\}$
- $\{\mathbf{i}, \mathbf{j}, \mathbf{k}, \underbrace{\mathbf{i}, -\mathbf{j}, \mathbf{i} + 3\mathbf{k}}_{\text{waste}}\}$
- $\mathbb{R}^3$
- and many others

Clearly some are more economical than others. To make this precise, we want to define the notion of redundancy in a set of vectors. This is called linear dependence.

A set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ is *linear dependent* if there exist scalars $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$, not all zero, such that

$$\lambda_1 \mathbf{v}_1 + \ldots \lambda_m \mathbf{v}_m = \mathbf{0}.$$

at least one notion that λ is not 0

(We say that there is a nontrivial linear relation between the vectors.)

A set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ is *linear independent* if it is not linearly dependent. More precisely, if the only scalars $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that

$$\lambda_1 \mathbf{v}_1 + \ldots \lambda_m \mathbf{v}_m = \mathbf{0}$$

are $\lambda_1 = \cdots = \lambda_m = 0$.

Example 3.21.

- $\begin{bmatrix}1\\-2\end{bmatrix}$ and $\begin{bmatrix}3\\6\end{bmatrix}$

$\lambda_1 \begin{bmatrix}1\\-2\end{bmatrix} + \lambda_2 \begin{bmatrix}3\\6\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ try to think whether it has $\Rightarrow$ unique solutions $\lambda_1=\lambda_2=0$

$\left[\begin{array}{cc|c}1 & 3 & 0\\-2 & 6 & 0\end{array}\right]$ $\rightarrow$ always 0

$\rightarrow$ so we do row operation, find the rank.

for this case, if rank is 2 $\rightarrow$ unique

$\begin{bmatrix}1&3\\-2&6\end{bmatrix} = \begin{bmatrix}1&3\\0&12\end{bmatrix} \Rightarrow$ rank 2

so unique solution $\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow$ linear independent

- $\begin{bmatrix}1\\-2\end{bmatrix}$ and $\begin{bmatrix}-3\\6\end{bmatrix}$

$3\begin{bmatrix}1\\-2\end{bmatrix} + \begin{bmatrix}-3\\6\end{bmatrix} = \vec{0}$

linear dependent

- $\begin{bmatrix}1\\-2\end{bmatrix}$ and $\begin{bmatrix}0\\0\end{bmatrix}$

$0\begin{bmatrix}1\\-2\end{bmatrix} + \underset{\text{any number}}{\underset{\downarrow}{\lambda}} \begin{bmatrix}0\\0\end{bmatrix} = \vec{0}$

linear dependent

More generally, if $V = \mathbb{R}^n$ and we want to determine whether $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is linearly independent, we form the augmented matrix

$$\begin{bmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_m \\ \vdots & & & \\ | & | & \cdots & | \end{bmatrix}$$ and find the rank

and see whether the system has a unique solution (then the vectors are linearly independent) or infinitely many solutions (then the vectors are linearly dependent).

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + R_1} \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

not full rank

$\Rightarrow$ infinitely many solutions

$\Rightarrow$ vector are linear dependent

non-trivial ?

Example 3.22. Are the vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ linearly independent?

Example 3.23. Are the polynomials

$$x(x-1), \qquad (x-1)(x-2), \qquad x^2-1$$

linearly independent in $\mathcal{P}_2$?

Does the constant polynomial 1 lie in their span?

$\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 = 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & 2 & -1 & -1 \end{array}\right]$$

$\lambda_1 (x(x-1)) + \lambda_2 ((x-1)(x-2)) + \lambda_3 (x^2-1) = 0$

$\Rightarrow \lambda_1 (x(x-1)) + \lambda_2 ((x-1)(x-2)) + \lambda_3 (x+1)(x-1) = 0$

constant polynomial zero

when $x=1$ no matter what $\lambda_1 \dots \lambda_3$

$(\lambda_1+\lambda_2+\lambda_3)x^2 + (-\lambda_1-3\lambda_2)x + (2\lambda_2-\lambda_3) = 0$

$$\begin{cases} \lambda_1+\lambda_2+\lambda_3 = 0 \\ -\lambda_1-3\lambda_2 = 0 \\ 2\lambda_2-\lambda_3 = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $2 < 3$

$(\lambda_1, \lambda_2, \lambda_3) \neq (0,0,0)$

linear dependent

3.5 Bases and dimension

Let V be a vector space. A subset S of V is a *basis of V* if it spans V and is linearly independent.

Example 3.24 (Standard basis of $\mathbb{R}^n$).

$\{e_1, e_2, \cdots, e_n\}$

Example 3.25. Is $\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$ a basis of $\mathbb{R}^2$? Yes

$$\begin{bmatrix} a \\ b \end{bmatrix} \in \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \left(\frac{a}{2}\right)\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \left(\frac{b}{3}\right)\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

linear dependent? $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ has rank 2 ✓

Executive summary Let V be a vector space.

- V has a basis.
- Any two bases of V have the same cardinality. (In the sense we discussed already, i.e. there is a bijection from one basis to the other.) If this cardinality is finite, it is called the *dimension of V*, and V is said to be *finite-dimensional*.
  We will focus almost exclusively on this situation. MAST300?? *Metric and Hilbert spaces* treats infinite-dimensional vector spaces.
- Any spanning set of V contains a basis of V.
- Any linearly independent subset of V can be extended to a basis of V.
- If V has dimension n, then
  - Any subset S with $\#S < n$ is **not** spanning V.
  - Any subset S with $\#S > n$ is **not** linearly independent.

In order to avoid circularity in the logic, we give a slightly different initial definition of finite-dimensionality: A vector space V is *finite-dimensional* if there exists a finite spanning subset S.

Fix a nonzero finite-dimensional vector space V.

Proposition 3.26. Given any linearly independent subset $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of V and any finite spanning set $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of V, we have $m \leq n$.

Proof.

Proposition 3.27. Every subspace of V is finite-dimensional.

Proof.

Proposition 3.28. There is a basis $\mathcal{B}$ of V.

Proof.

Note that the proof showed something of independent interest, which we record here:

Proposition 3.29. Any finite spanning set S of V contains a basis $\mathcal{B}$ of V.

Proposition 3.30. Any two bases of V have the same size.

Proof.