THE UNIVERSITY OF MELBOURNE

Student Number

Semester 2 Assessment, 2018

School of Mathematics and Statistics

MAST20009 Vector Calculus

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 4 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- Calculators, tablet devices or computers must not be used.
- No handwritten or print materials may be brought into the exam venue.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- You should attempt all questions.
- Start each question on a new page.
- Clearly label each page with the number of the question you are attempting.
- There is a separate 3 page formula sheet accompanying the examination paper, that you may use in this examination.
- There are 11 questions with marks as shown. The total number of marks available is 110.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.
- Initially students are to receive the exam paper, the 3 page formula sheet, and two 14 page script books.

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Question 1 (10 marks) Consider the following function:

$$g(x,y) = \begin{cases} ye^{-1/x^2}, & \text{for } (x,y) \text{ with } x \neq 0, \\ y, & \text{for } (x,y) \text{ with } x = 0. \end{cases}$$

(a) Calculate $\lim_{(x,y)\to(0,0)} g(x,y)$.

(b) Determine where g is continuous. Justify your answer, referring to any theorems you use.

(c) Using the definition of the partial derivative, calculate $\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$ at $(x,y) = (0,0)$.

Question 2 (10 marks)

(a) Use a Lagrange multiplier to find the point (x,y,z) closest to the origin on the graph of the function $z = x + y - 3$. (Hint: to simplify work, take $f(x,y,z) = \frac{1}{2}(x^2 + y^2 + z^2)$ as the function to be minimised.)

(b) Consider the general problem of finding points (x,y,z) which minimise the distance from the origin on the graph of a function $z = g(x,y)$. Give a system of equations in terms of $x, y, g, \dfrac{\partial g}{\partial x}$, and $\dfrac{\partial g}{\partial y}$ whose solutions will give such points.

Question 3 (10 marks) A curve C has the parametric equations:

$$x = 2t, \quad y = t^2, \quad z = \log t, \qquad \text{for} \quad 0 < t < \infty.$$

(a) Find the acceleration $\mathbf{a}(t)$ and the unit tangent vector $\mathbf{T}(t)$ to C.

(b) Find the curvature of C at the point where $t = 1$.

(c) Let $\mathbf{N}(t)$ be the principal normal vector to C. The acceleration at the point $t = 1$ can be written as

$$\mathbf{a}(1) = a_T\mathbf{T}(1) + a_N\mathbf{N}(1). \qquad \text{(You do not need to prove this fact.)}$$

Show that $|\mathbf{a}(1)| = \sqrt{a_T^2 + a_N^2}$.

(d) Calculate a_T and a_N.

Question 4 (10 marks) Let f be a scalar function of order C^2 on $\mathbb{R}^3$ and let $\mathbf{F}$ be a vector field of order C^2 on $\mathbb{R}^3$.

(a) Just using the definitions (i.e. without using the identities on the formula sheet), prove that $\text{curl}(\text{grad}(f)) = \mathbf{0}$.

(b) Just using the definitions (i.e. without using the identities on the formula sheet), prove that $\text{div}(\text{curl}(\mathbf{F})) = 0$.

Question 5 (10 marks) A triangle has vertices $(0, 0, 0)$, $(0, 1, -1)$ and $(0, 1, 1)$. The plane of the triangle is rotated about the z-axis, and the moving triangle forms a solid (which is a cylinder from which two parts of a cone have been removed).

(a) Set up a multiple integral in *spherical coordinates* to calculate the volume of this solid.

(b) Find the volume.

Question 6 (10 marks) Use multiple integration to find the moment of inertia about its axis of symmetry for a cylinder of radius a, height h, constant density μ and total mass M. Express your answer in terms of a and M.

Question 7 (10 marks) Let C be the curve

$$\mathbf{c}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + t\,\mathbf{k}, \qquad \text{for} \quad 0 \le t \le 2\pi,$$

and let

$$\mathbf{F}(x, y, z) = 2x\,\mathbf{i} - 4yz^2\,\mathbf{j} - (4y^2z - 1)\,\mathbf{k}.$$

(a) Find f such that $\mathbf{F} = \boldsymbol{\nabla} f$.

(b) Calculate the work done by the vector field $\mathbf{F}$ to move a particle along the curve C in the direction of increasing t.

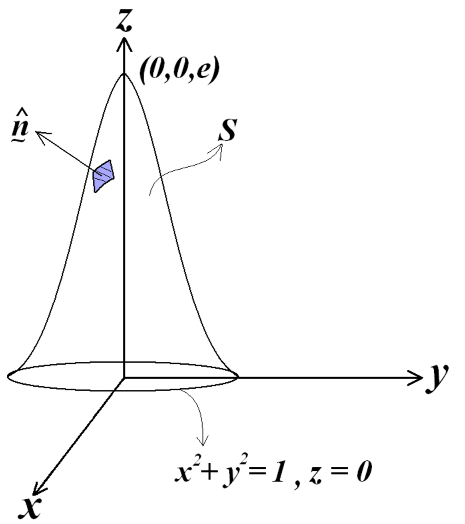
Question 8 (10 marks) Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where S is the surface of the bell

$$z = (1 - x^2 - y^2)e^{1-x^2-y^2} \qquad \text{for } z \ge 0,$$

and

$$\mathbf{F}(x, y, z) = (e^y \cos z, (x^3 + 1)^{\frac{1}{2}} \sin z, x^2 + y^2 + 3).$$

(Hint: Use the divergence theorem.)



Question 9 (10 marks) Let the surface S be the disk $x^2 + y^2 \leq 9$ in the plane $z = 2$. The normal to S is directed upwards. Let $\mathbf{F}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$.

(a) Evaluate

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S},$$

without using Stokes' theorem.

(b) Stokes' theorem asserts that the value of this surface integral is equal to the value of a certain line integral. Set up and evaluate the line integral.

Question 10 (10 marks)

(a) Sketch the region enclosed by the curves $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $y = 0$ and $2y = x$, if $x > 0$.

(b) Use the change of variables

$$u = \frac{y}{x} \qquad \text{and} \qquad v = x^2 - y^2$$

to find the area of the region. (Hint: Note that $\dfrac{1}{1-u^2} = \dfrac{1}{2}\Big(\dfrac{1}{1+u} + \dfrac{1}{1-u}\Big)$.)

Question 11 (10 marks) Use *spherical coordinates* for this question.

(a) Compute the scale factors h_r, h_θ, and h_φ for spherical coordinates.

(b) Show that the coordinate system is orthogonal.

(c) Find an expression for $\boldsymbol{\nabla}\theta$ in terms of r, θ, φ and $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$.

(d) Find an expression for $\boldsymbol{\nabla} \cdot (\sin\theta\, \hat{\boldsymbol{\theta}})$ in terms of r, θ, φ and $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$.

End of Exam—Total Available Marks = 110