Tutorial 3: Cross Product, Solid Geometry, Numbers

Cross Product

Q1. Let $\mathbf{a} = (3, 4, -2)$, $\mathbf{b} = (0, -2, 2)$, and $\mathbf{c} = (-6, -8, 4)$. Then find

(a) $\mathbf{a} \times \mathbf{b}$ (b) $\mathbf{c} \times \mathbf{a}$ (c) The area of the triangle with sides $\mathbf{a}$ and $\mathbf{b}$

Q2. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be defined as in Question 1. Let $\mathbf{d} = (0, 0, 1)$. Calculate, where possible,

(a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{d})$ (b) $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{d}$ (c) $\mathbf{d} \times (\mathbf{a} \times \mathbf{b})$ (d) The volume of the parallelepiped with sides $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{d}$.

Q3. Find vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ such that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}.$$

Q4. Let A be the area of a parallelogram defined by vectors $\mathbf{a}$ and $\mathbf{b}$. Prove that

$$A^2 = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} \end{vmatrix}.$$

Lines and Planes

Q5. Write the line that passes through $(0, 0, -1)$ and $(1, 0, -2)$ in vector and Cartesian form.

Q6. Consider the plane with Cartesian equation

$$2x + 4y - 7z = 5$$

(a) Write down a vector perpendicular to the plane.

(b) Does the point $P(5, 0, 1)$ lie on the plane?

Q7. Write the plane containing the points $A\,(3, 2, 1)$, $B\,(2, -1, 4)$, and $C\,(5, 1, 1)$ in both vector and Cartesian form.

Q8. Find the intersection between the planes $2x - y + 3z = 2$ and $x + z = 1$.

Q9. Consider the lines $\mathcal{L}_1 : (x, y, z) = (0, 1, 1) + t(1, 1, 2)$, $t \in \mathbb{R}$ and $\mathcal{L}_2 : 4 - x = \dfrac{y - 1}{2} = z - 2$.

(a) Are these lines parallel?

(b) What is the shortest distance between the two lines?

(c) Are the lines skew?

Numbers

Q10. How many solutions do the following equations have in: (i) $\mathbb{N}$, (ii) $\mathbb{Z}$, (iii) $\mathbb{F}_2$, (iv) $\mathbb{C}$?

(a) $x^2 = 1$ (b) $x^2 + 1 = 0$ (c) $x^2 + x = 0$ (d) $x^2 + x + 1 = 0$

Q11. Prove that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^3$, no two of which are parallel, then $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane determined by $\mathbf{v}$ and $\mathbf{w}$.