Assignment 1

Please complete the Plagiarism Declaration Form (available through the LMS) before submitting this assignment.

Don't forget to staple your solutions, and to print your name, student ID, tutorial time and day, and the subject name and code on the first page (not doing so will forfeit marks). The submission deadline is **Friday, 7 September, 2018 by 4pm** in the appropriate assignment box at the north end of Peter Hall Building (near Wilson Lab).

There are 2 questions, both of which will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness, and style count.

1. A discrete time Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ has the following transition matrix.

$$P = \begin{pmatrix} 0 & 3/10 & 7/10 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 2/3 & 1/3 & 0 & 0 \\ 1/4 & 1/8 & 0 & 0 & 5/8 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

   (a) Write down the communication classes of the chain.

   (b) Find the period of each communicating class.

   (c) Determine which classes are essential.

   (d) Classify each essential communicating class as transient or positive recurrent or null recurrent.

   (e) Describe the long run behaviour of the chain (including deriving long run probabilities where appropriate).

   (f) Find the expected number of steps taken for the chain to first reach state 3, given the chain starts at state 4.

2. Let $0 < \alpha < 1/2$. A Markov chain $(X_n)_{n \geq 0}$ on $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ has transition probabilities given by: for $|i| = 0, 2, 3, 4, \ldots,$

$$p_{i,i+1} = p_{i,i-1} = \alpha, \quad p_{i,0} = 1 - 2\alpha,$$

   and

$$p_{1,2} = \alpha, \quad p_{1,0} = 1 - \alpha,$$
$$p_{-1,-2} = \alpha, \quad p_{-1,0} = 1 - \alpha.$$

   Note that the chain is irreducible.

   (a) For which values of α is the chain transient? Null recurrent? Positive recurrent?

   (b) Describe the long run behaviour of the chain (including deriving long run probabilities where appropriate).

   (c) For $j \in \mathbb{Z}$, define $T(j) = \inf\{n \geq 1 : X_n = j\}$. Find an expression in terms of α for $\mathbb{E}[T(j)|X_0 = j]$.

1. A discrete time Markov chain with state space $S = \{1,2,3,4,5\}$ has the following transition matrix

[handwritten: 各 一下 communicating class]
[handwritten: 每个 是 irreducible DTMC]

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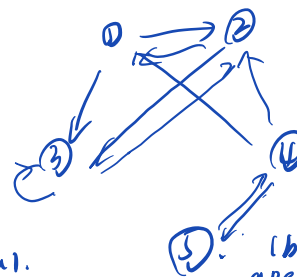
[handwritten diagram of states 1,2,3,4,5 with transitions]

(a) Write down the communication classes of the chain.

(b) Find the period of each communicating class.

(c) Determine which classes are essential.

(d) Classify each essential communicating class as transient or positive recurrent or null recurrent.

(e) Describe the long run behaviour of the chain (including deriving long run probabilities where appropriate).

(f) Find the expected number of steps taken for the chain to first reach state 3, given the chain starts at state 4.

[handwritten annotations:]

(a). $C_1 = \{1,2,3\}$ $C_2 = \{4,5\}$

(b). aperiodic. period $= 1$. (loop) period $= 2$.

(c) C_1 essential C_2 non-essential finite + essential.

(d). C_1 positive recurrent since finite essential. C_2 Transient

(e). $[\pi_1 \quad \pi_2 \quad \pi_3]$