PHYC10003 Physics 1

Lecture 17: Angular momentum

Conservation, precession and gyroscopes

Last lecture

- Yo-yo
- Generalization of torque
- Vector cross product
- Angular momentum
- Analogies with linear motion

Torque, angular momentum, inertia

- Note that the torque and angular momentum must be measured relative to the same origin
- If the center of mass is accelerating, then that origin *must* be the center of mass
- We can find the angular momentum of a rigid body through summation:

$$L_z = \sum_{i=1}^{n} \ell_{iz} = \sum_{i=1}^{n} \Delta m_i \, v_i r_{\perp i} = \sum_{i=1}^{n} \Delta m_i (\omega r_{\perp i}) r_{\perp i} = \omega \left(\sum_{i=1}^{n} \Delta m_i \, r_{\perp i}^2 \right). \qquad \textbf{Eq. (11-30)}$$

- The sum is the rotational inertia *I* of the body

Analogies - rigid bodies

- Therefore this simplifies to:

$$L = I\omega \quad \text{(rigid body, fixed axis).}$$

Eq. (11-31)

Table 11-1

Table 11-1 More Corresponding Variables and Relations for Translational and Rotational Motion[a]

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}\ (= \vec{r} \times \vec{F})$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell}\ (= \vec{r} \times \vec{p})$
Linear momentum[b]	$\vec{P}\ (= \Sigma \vec{p}_i)$	Angular momentum[b]	$\vec{L}\ (= \Sigma \vec{\ell}_i)$
Linear momentum[b]	$\vec{P} = M\vec{v}_{com}$	Angular momentum[c]	$L = I\omega$
Newton's second law[b]	$\vec{F}_{net} = \dfrac{d\vec{P}}{dt}$	Newton's second law[b]	$\vec{\tau}_{net} = \dfrac{d\vec{L}}{dt}$
Conservation law[d]	$\vec{P}$ = a constant	Conservation law[d]	$\vec{L}$ = a constant

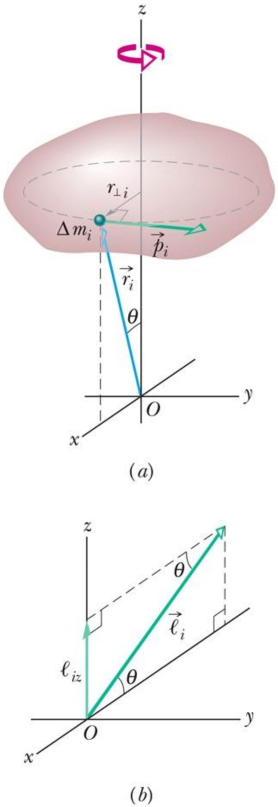


Figure 11-15

Summary

Rolling Bodies

$$v_{com} = \omega R$$ Eq. (11-2)

$$K = \tfrac{1}{2} I_{com}\omega^2 + \tfrac{1}{2} M v_{com}^2.$$ Eq. (11-5)

$$a_{com} = \alpha R$$ Eq. (11-6)

Torque as a Vector

- Direction given by the right-hand rule

$$\vec{\tau} = \vec{r} \times \vec{F}$$ Eq. (11-14)

Angular Momentum of a Particle

$$\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$ Eq. (11-18)

Newton's Second Law in Angular Form

$$\vec{\tau}_{net} = \frac{d\vec{\ell}}{dt}$$ Eq. (11-23)

Conservation of angular momentum

- Since we have a new version of Newton's second law, we also have a new conservation law:

$$\vec{L} = \text{a constant} \qquad \text{(isolated system).}$$

Eq. (11-32)

- The **law of conservation of angular momentum** states that, for an isolated system,

(*net initial angular momentum*) = (*net final angular momentum*)

$$\vec{L}_i = \vec{L}_f \qquad \text{(isolated system).}$$

Eq. (11-33)

Conservation of angular momentum

> If the net external torque acting on a system is zero, the angular momentum $\vec{L}$ of the system remains constant, no matter what changes take place within the system.

- Since these are vector equations, they are equivalent to the three corresponding scalar equations
- This means we can separate axes and write:

> If the component of the net *external* torque on a system along a certain axis is zero, then the component of the angular momentum of the system along that axis cannot change, no matter what changes take place within the system.

- If the distribution of mass changes with no external torque, we have:

$$I_i\omega_i = I_f\omega_f. \qquad \textbf{Eq. (11-34)}$$

Conservation - examples

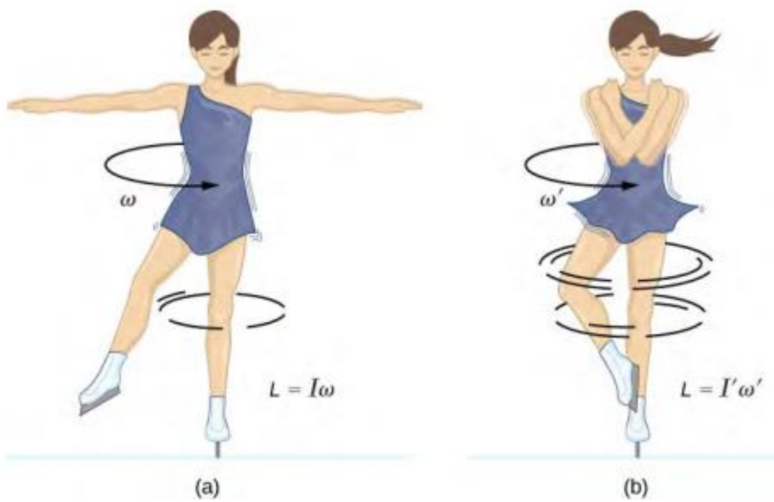
- A student spinning on a stool: rotation speeds up when arms are brought in, slows down when arms are extended
- A springboard diver: rotational speed is controlled by tucking her arms and legs in, which reduces rotational inertia and increases rotational speed
- A long jumper: the angular momentum caused by the torque during the initial jump can be transferred to the rotation of the arms, by windmilling them, keeping the jumper upright

Figure 11-18

Conservation of angular momentum

The change in the angular momentum is achieved by the skater changing her moment of inertia.

By pulling in her arms and right leg, I' < I and, therefore ω'>ω to maintain constant L.



$$I\omega = I'\omega'$$

Conservation of rotational energy?

- The initial rotational energy is $K=I\omega^2/2$ and the final rotational energy is $K'=I'\omega'^2/2$.
- But $\omega'=\omega(I/I')$ by conservation of angular momentum.
- So $K'=K(I/I')$: the final rotational energy is greater than the initial.

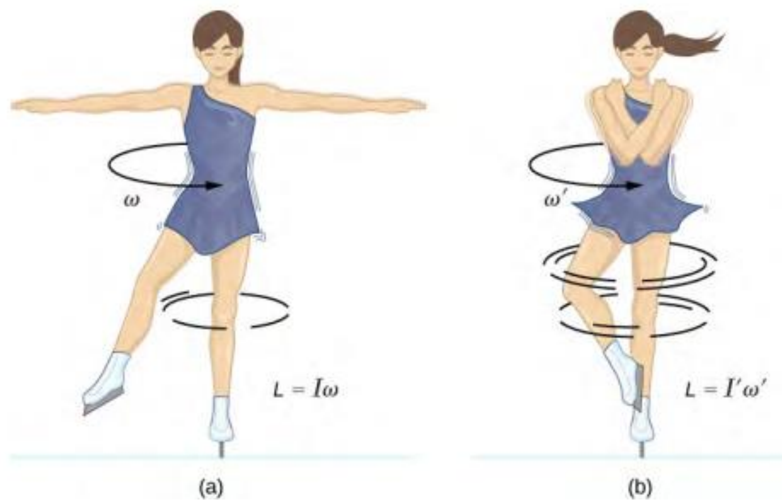


Image: phys.libretexts.org

The skater has to do work to reduce her moment of inertia; this work is converted into an increased rotational kinetic energy. Ignoring friction and air resistance, energy is conserved!.

Conservation of angular momentum

- The solar system formed from a large rotating gas cloud
- It progressively formed structures that conserved the original angular momentum.
- Solar system is a collection of rigid bodies (planets, moons, satellites, comets) that possess rotational angular momentum with respect to the centre of mass of the sun.

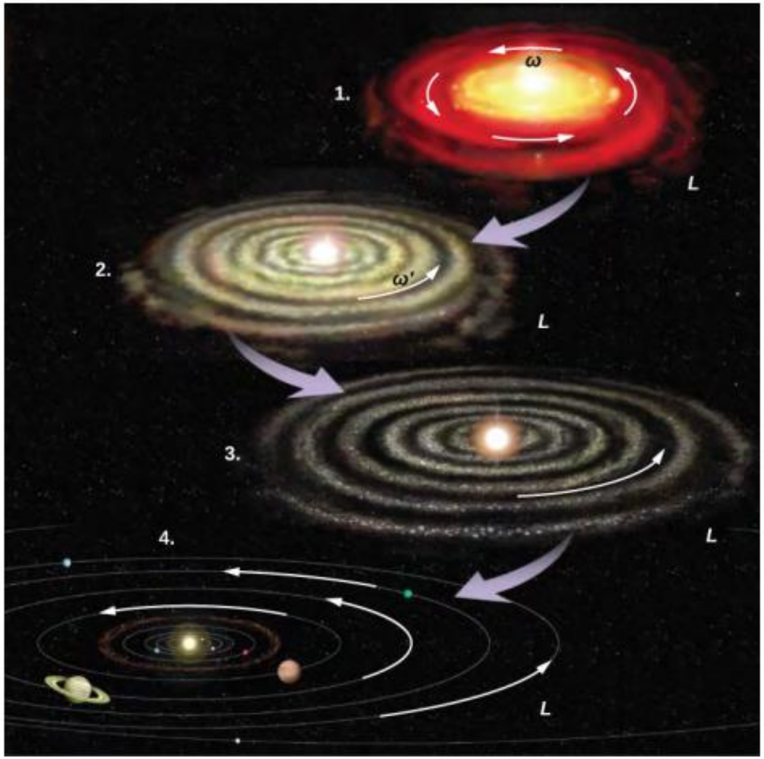
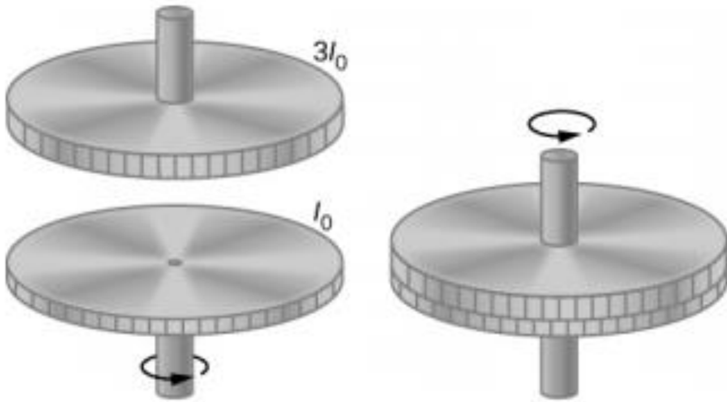


Image: NASA

Conservation of angular momentum

Coupled flywheels



The bottom flywheel, moment of inertia I_0, rotates with angular velocity ω_0. The upper flywheel ($I=3I_0$) is dropped onto the lower one. They rapidly start spinning together at the same angular frequency, ω.

$$I_0\omega_0 = (I_0 + 3I_0)\omega \qquad \omega = \frac{\omega_0}{4}$$

$$K_i = \frac{1}{2}I_0\omega_0^2$$

$$K_f = \frac{1}{2}(4I_0)\left(\frac{\omega_0}{4}\right)^2$$

$$\frac{K_f}{K_i} = \frac{\frac{1}{8}I_0\omega_0^2}{\frac{1}{2}I_0\omega_0^2} = \frac{1}{4}$$

¾ of the rotational energy is lost when the flywheels are coupled.

Gyroscopes - precession

- Over short times (hundreds of years) the orientation of the earth's rotation relative to its orbital plane may be regarded as essentially fixed.
- The rotation of the earth causes precession around an axis that causes an exchange in the equinox between northern and southern hemispheres every 25772 years.
- "Precession of the equinoxes"

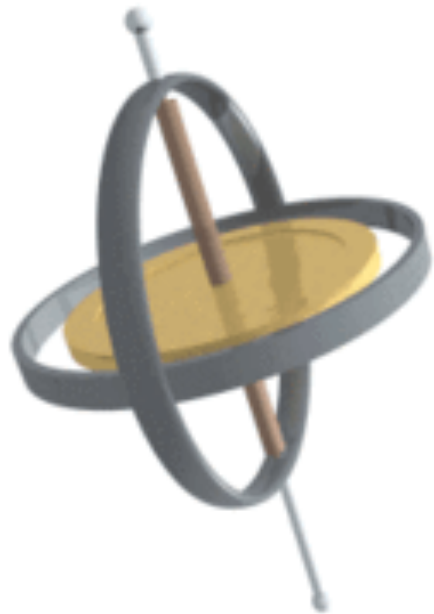


Image: Wikipedia

Gyroscopes - precession

- A non-spinning gyroscope, as attached in 11-22 (a), falls
- A spinning gyroscope (b) instead rotates around a vertical axis
- This rotation is called **precession**

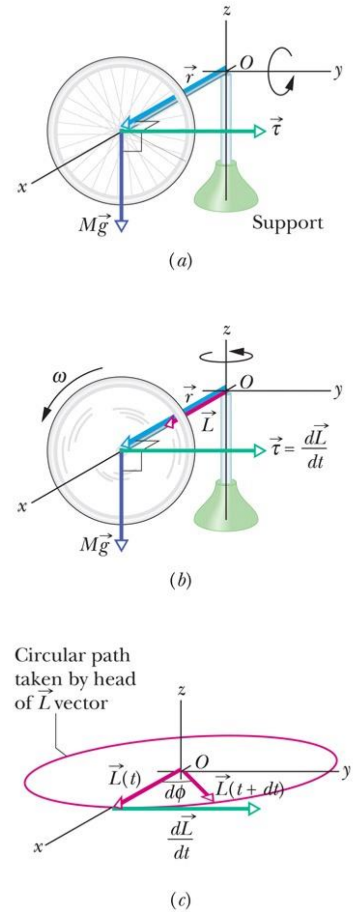


Figure 11-22

Gyroscopes- precession

- The angular momentum of a (*rapidly spinning*) gyroscope is:

$$L = I\omega, \qquad \textbf{Eq. (11-43)}$$

- The torque can only change the direction of *L*, not its magnitude, because of (11-43)

$$d\vec{L} = \vec{\tau}\,dt. \qquad \textbf{Eq. (11-44)}$$

- The only way its direction can change along the direction of the torque without its magnitude changing is if it rotates around the central axis

- Therefore it precesses instead of toppling over

Gyroscope-precession

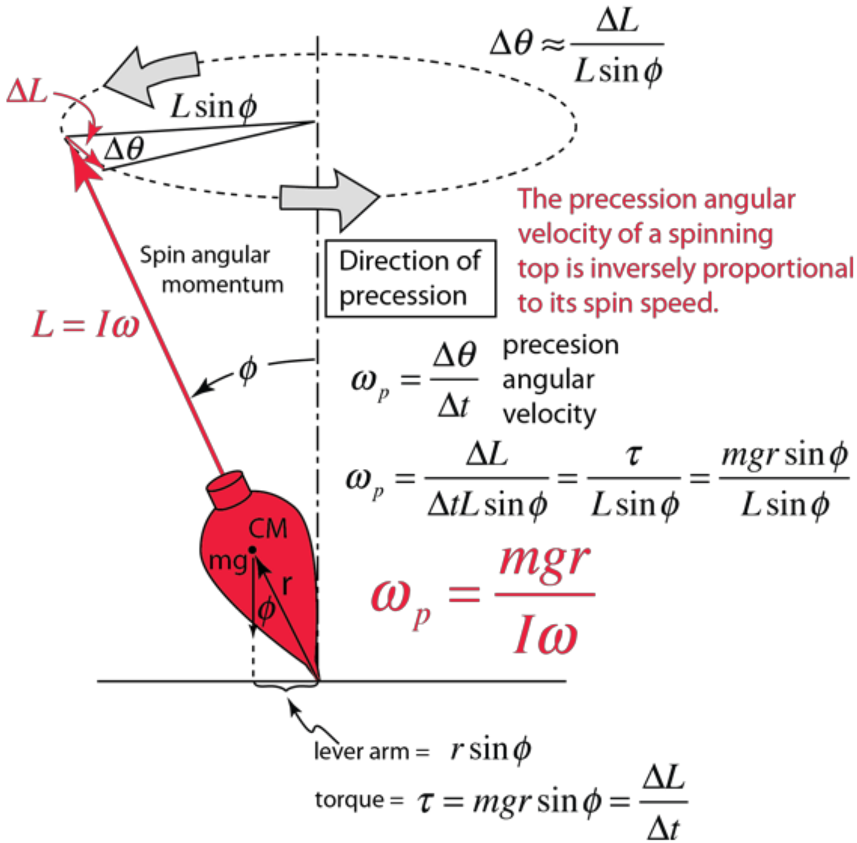


Image: hyperphysics.phy-astr.gsu.edu

Δθ comes from the formula for arc-length:

$$\Delta L = r_{\perp} . \Delta\theta$$

$$r_{\perp} = L\sin\phi$$

Precession

- The **precession rate** is given by:

$$\Omega = \frac{Mgr}{I\omega}$$

Eq. (11-46)

- True for a sufficiently rapid spin rate
- Independent of mass, (*I* is proportional to *M*) but does depend on *g*
- *Independent of ϕ*
- Valid for a gyroscope at an angle to the horizontal as well (a top for instance)

Summary

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$$\vec{\tau}_{net} = \frac{d\vec{\ell}}{dt}$$ **Eq. (11-23)**

Summary

Angular Momentum of a System of Particles

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \vec{\ell}_3 + \cdots + \vec{\ell}_n = \sum_{i=1}^{n} \vec{\ell}_i.$$

Eq. (11-26)

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}$$

Eq. (11-29)

Angular Momentum of a Rigid Body

$$L = I\omega$$

Eq. (11-31)

Conservation of Angular Momentum

$$\vec{L} = \text{a constant}$$

Eq. (11-32)

$$\vec{L}_i = \vec{L}_f$$

Eq. (11-33)

Precession of a Gyroscope

$$\Omega = \frac{Mgr}{I\omega}$$

Eq. (11-46)