MAST30001 Stochastic Modelling

$\boxed{\text{Tutorial Sheet 5}}$

1. A possum runs from corner to corner along the top of a square fence. Each time it switches corners, it chooses among the two adjacent corners, choosing the corner in the clockwise direction with probability $0 < p < 1$ and the corner in the counter-clockwise direction with probability $1 - p$. Model the possum's movement among the corners of the fence as a Markov chain, analyse its state space (reducibility, periodicity, recurrence, etc), and discuss its long run behaviour.

   Ans. Numbering the fence posts $1, 2, 3, 4$ in a circular way we can model the possum's movements as a Markov chain with transition matrix

$$P = \begin{pmatrix} 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \end{pmatrix}.$$

   The chain is irreducible since all states communicate and it's positive recurrent since the state space is finite. Its period is two, having sub-communicating classes $\{1, 3\}$ and $\{2, 4\}$. Since

$$P^2 = \begin{pmatrix} 2p(1-p) & 0 & p^2 + (1-p)^2 & 0 \\ p^2 + (1-p)^2 & 0 & 2p(1-p) & 0 \\ 0 & 2p(1-p) & 0 & p^2 + (1-p)^2 \\ 0 & p^2 + (1-p)^2 & 0 & 2p(1-p) \end{pmatrix}$$

   so the columns sum to one and the 2-step stationary distributions are uniform, we have

$$P(X_{2n+k} = j | X_0 \in \{1, 3\}) \overset{n\to\infty}{\longrightarrow} \begin{cases} 1/2, & k = 0, j \in \{1, 3\}; \\ 1/2, & k = 1, j \in \{2, 4\}; \\ 0, & \text{else}; \end{cases}$$

   and

$$P(X_{2n+k} = j | X_0 \in \{2, 4\}) \overset{n\to\infty}{\longrightarrow} \begin{cases} 1/2, & k = 0, j \in \{2, 4\}; \\ 1/2, & k = 1, j \in \{1, 3\}; \\ 0, & \text{else}. \end{cases}$$

2. Refer to Tutorial Sheet 3, Problem 3 and now also assume that on any given transition, the spider will not return to the corner it came from on the previous step. Show the sequence of corners occupied by the spider is *not* a Markov chain and suggest a Markov chain model for this new system.

   Ans. The chain is not Markov since, for example,

$$P(X_{n+1} = 2 | X_n = 1, X_{n-1} = 2) = 0 < P(X_{n+1} = 2 | X_n = 1, X_{n-1} = 3).$$

   To make the system Markov, we enlarge the state space to keep track of the current state and the previous one. Then the state space is $S = \{(a, b) : a \neq b, 1 \leq a, b \leq 4\}$ and the transition probabilities are for corners $a \neq c$,

$$p_{(a,b),(b,c)} = \frac{C}{\text{distance between } b \text{ and } c},$$

   where C is an appropriate normalizing constant.

3. (Discrete version of Poisson Process) Let the discrete time Markov chain $(X_n)_{n\geq 0}$ on $\{0, 1, \ldots\}$ have transition probabilities $p_{ii+1} = 1 - p_{ii} = p$ and assume $X_0 = 0$.

   (a) Draw a picture of a typical trajectory of this process.

   (b) Show that X_n has the binomial distribution with parameters n and p.

   (c) Show that for $m < n$, $X_n - X_m$ has the binomial distribution with parameters $n - m$ and p.

   (d) Show that $(X_n)_{n\geq 0}$ has the independent increments property: for $0 \leq i < j \leq k < l$, the variables
   $$(X_l - X_k, X_j - X_i)$$
   are independent.

   (e) Show that the number of steps between "jumps" (times when the chain changes states) has the geometric distribution with parameter p (and started from 1).

   (f) Show that given $X_n = 1$, the step number of the first jump is uniform on $\{1, \ldots, n\}$.

   (g) More generally, show that given $X_n = k$, the step numbers of the jumps are a uniformly chosen subset of size k from $\{1, \ldots, n\}$.

Ans. The key observation is that the chain is completely described by the times of jumps, which can be alternatively defined to occur at a given time n with probability p independent of the occurrence of jumps at other times.

(b) X_n is the number of jumps in the interval $(0, n]$, and the key observation implies the distribution is binomial with parameters n and p.

(c) $X_n - X_m$ is the number of jumps in the interval $(m, n]$, and the key observation implies the distribution is binomial with parameters $n - m$ and p.

(d) $(X_l - X_k, X_j - X_i)$ count the number of jumps in the intervals $(k, l]$ and $(i, j]$, and by key observation, these are functions of independent variables and so are independent.

(e) Again the key is to see the times of the jumps as independent coin tosses with chance of heads p, and so the number of steps between jumps is the number of tosses needed to see the first appearance of a head, which is geometrically distributed.

(f) Conditional on $X_n = 1$, there has been one jump in the first n steps of the chain. By direct calculation the chance it is at the ith step under this conditioning is
$$\frac{p(1-p)^{n-1}}{np(1-p)^{n-1}} = \frac{1}{n},$$
as desired (see the next part for more details).

(g) Similar to (f), conditional on $X_n = k$, there have been k jumps in the first n steps. The chance that k jumps occur at any distinct k positions is the chance of that particular sequence of zeroes and ones:
$$p^k(1-p)^{n-k}$$
and so the conditional probability is this last divided by the chance that $X_n = k$:
$$\frac{p^k(1-p)^{n-k}}{\binom{n}{k}p^k(1-p)^{n-k}} = \frac{1}{\binom{n}{k}}.$$

4. Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ and for each $t \geq 0$, let $X_t = N_{t/\lambda}$. Show that $(X_t)_{t\geq 0}$ is a Poisson process with rate 1.

   Ans. We need to check that $(X_t)_{t\geq 0}$ has independent increments and that X_t is Poisson, mean t. X_t inherits independent increments from N_t: If $0 \leq s_1 < t_1 < \cdots \leq s_k < t_k$ then

$$(X_{t_1} - X_{s_1}, \ldots, X_{t_k} - X_{s_k}) = (N_{t_1/\lambda} - N_{s_1/\lambda}, \ldots, N_{t_k/\lambda} - N_{s_k/\lambda}),$$

   and the right hand side are independent variables since N is a Poisson process and $0 \leq s_1/\lambda < t_1/\lambda < \cdots \leq s_k/\lambda < t_k/\lambda$.

   To check that X_t has a Poisson distribution with mean t, note that by the definition of Poisson process, $N_{t/\lambda}(= X_t)$ is Poisson with mean $\lambda(t/\lambda) = t$.

5. Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ and let $0 < T_1 < T_2 < \cdots$ be the times of "arrivals" or jumps of $(N_t)_{t\geq 0}$. Compute:

   (a) $P(N_3 \leq 2, N_1 = 1)$,
   (b) $P(N_3 \leq 2, N_1 \leq 1)$,
   (c) $P(N_2 = 2, N_1 = 2, N_{1/2} = 0)$,
   (d) $P(N_7 - N_3 = 2 | N_5 - N_2 = 2)$,
   (e) the joint distribution function $F(t_1, t_2) = P(T_1 < t_1, T_2 < t_2)$,
   (f) the joint density of (T_1, T_2),
   (g) the distribution of $T_1 | T_2 = t_2$.

   Ans.

   The name of the game for all of these problems is to recast the event you want to compute in terms of independent variables or events which are amenable to the Poisson process description.

   (a) $(N_1, N_3 - N_1)$ are independent, Poisson with respective means λ and 2λ, so that

$$\begin{aligned} P(N_3 \leq 2, N_1 = 1) &= P(N_3 - N_1 \leq 1, N_1 = 1) \overset{\text{ind}}{=} P(N_3 - N_1 \leq 1)P(N_1 = 1) \\ &= \left(e^{-2\lambda} + 2\lambda e^{-2\lambda}\right) \lambda e^{-\lambda}. \end{aligned}$$

   (b) Similar to (a),

$$\begin{aligned} P(N_3 \leq 2, N_1 \leq 1) &= P(N_3 \leq 2, N_1 = 1) + P(N_3 \leq 2, N_1 = 0) \\ &= [\text{Ans. to (a)}] + e^{-2\lambda}\left(1 + 2\lambda + (2\lambda)^2/2\right) e^{-\lambda}. \end{aligned}$$

   (c) Use that $(N_2 - N_1, N_1 - N_{1/2}, N_{1/2})$ are independent and Poisson and that

$$P(N_2 = 2, N_1 = 2, N_{1/2} = 0) = P(N_2 - N_1 = 0, N_1 - N_{1/2} = 2, N_{1/2} = 0)$$

   to find

$$P(N_2 = 2, N_1 = 2, N_{1/2} = 0) = e^{-\lambda} \frac{(\lambda/2)^2}{2} e^{-\lambda/2} e^{-\lambda/2}.$$

(d) Again, $(N_7 - N_5, N_5 - N_3, N_3 - N - 2)$ are independent and Poisson and

$$P(N_7 - N_3 = 2 | N_5 - N_2 = 2) = \frac{P(N_7 - N_3 = 2, N_5 - N_2 = 2)}{P(N_5 - N_2 = 2)}.$$

The numerator of the fraction on the RHS equals

$$\sum_{i=0}^{2} P(N_7 - N_5 = 2 - i, N_5 - N_3 = i, N_3 - N_2 = 2 - i)$$
$$= \sum_{i=0}^{2} \frac{(2\lambda)^{2-i}}{(2-i)!} e^{-2\lambda} \frac{(2\lambda)^i}{(i)!} e^{-2\lambda} \frac{(\lambda)^{2-i}}{(2-i)!} e^{-\lambda}.$$

(e) We know marginally that T_1 is exponential rate λ and T_2 is gamma, parameter 2, rate λ. It's easiest to compute

$$P(T_1 < t_1, T_2 > t_2) = P(N_{t_1} = 1, N_{t_2} - N_{t_1} = 0) = \lambda t_1 e^{-\lambda t_1} e^{-\lambda(t_2 - t_1)} = \lambda t_1 e^{-\lambda t_2}.$$

And the law of total probability implies that for $0 < t_1 < t_2$,

$$P(T_1 < t_1, T_2 < t_2) = P(T_1 < t_1) - P(T_1 < t_1, T_2 > t_2) = 1 - e^{-\lambda t_1} - \lambda t_1 e^{-\lambda t_2}.$$

(f) To obtain the joint density, we differentiate the joint distribution function of part (e) with respect to t_1 and t_2 to find for $0 < t_1 < t_2$,

$$f(t_1, t_2) = \lambda^2 e^{-\lambda t_2}.$$

(g) The conditional density is the joint density f from part (f) divided by the marginal gamma density $f_2(t_2) = \lambda^2 t_2 e^{-t_2}$, yielding for $0 < t_1 < t_2$,

$$f_{T_1|T_2}(t_1|t_2) = \frac{\lambda^2 e^{-\lambda t_2}}{\lambda^2 t_2 e^{-t_2}} = \frac{1}{t_2},$$

so T_1 is uniformly distributed on $(0, T_2)$.