MAST30001 Stochastic Modelling

Tutorial Sheet 11

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion.

1. Show that $(X_t)_{t>0} = (tB_{1/t})_{t>0}; X_0 = 0$ and $(Y_t)_{t\geq 0} = (cB_{t/c^2})_{t\geq 0}$ are Brownian motions.

2. Show that $W_t = \exp\{cB_t - c^2t/2\}$ has the property $E[W_t|W_s] = W_s$.

3. Show that for each $0 < u < 1$, $B_u - uB_1$ is independent of B_1.

4. For $0 \leq t_1 < t_2 < t_3$, find $E[B_{t_1}B_{t_2}B_{t_3}]$.

5. The price of a stock at the start of the day is 100 dollars. Suppose that the logarithm of the price of the stock at time t hours after the start of the day is $B_t + 2t + \log(100)$.

   (a) Find the chance that the price of the stock one hour after the start of the day is higher than the starting price.

   (b) If you buy one share of the stock and sell it at after one hour, how much money would you expect to make?

   (c) Given the price of the stock two hours after the start of the day is 100 dollars, what is the chance the price of the stock one hour after the start of the day is higher than the starting price.

6. Show that for $x > 0$, $T_x = \inf\{t : B_t = x\}$ has the same distribution as x^2/Z^2, where Z is standard normal.

7. Let $M_t = \min_{0\leq s\leq t} B_s$.

   (a) Use the reflection principle to show that for $z \leq 0$ and $x \geq z$,

   $$P(M_t \leq z, B_t \geq x) = P(B_t \leq 2z - x).$$

   (b) Find the joint density of (M_t, B_t).

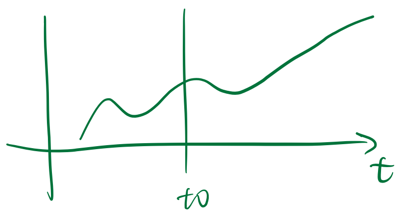
   (c) Show that $B_t - M_t$ has the same distribution as $|B_t|$. [Interesting fact: this distributional identity holds at the process level: $(B_t - M_t)_{t\geq 0}$ has the same distribution as $(|B_t|)_{t\geq 0}$.]

$$\frac{Ga(k,\lambda)}{\underbrace{Ga(k,\lambda)+Ga(\ell,\lambda)}_{\text{ind}}} \sim Beta(k,\ell).$$

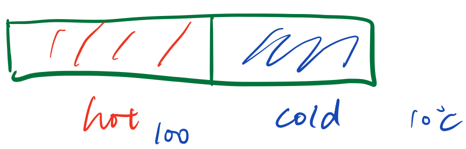
BM

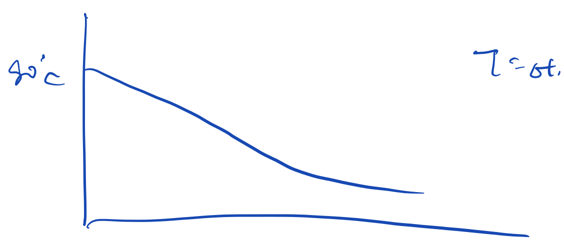
$$\frac{\partial}{\partial t} p_t(x) = \frac{\partial^2}{\partial x^2} p_t(x) \qquad \text{where } p_t(x) = P(W_t = x)$$

heat equation



$$W_{t_0} \sim N(0, t_0)$$

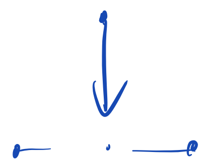


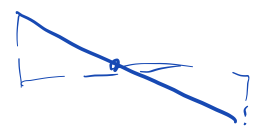


$$\frac{\partial}{\partial t} P_t(x) = \frac{P_{t+\Delta t}(x) - P_t(x)}{\Delta t}$$ fixed t.

$$\approx c \cdot \frac{\partial^2}{\partial x^2} P_t(x)$$

the temperature around you is much smaller than you





$$\frac{\partial}{\partial t} P_t(x) = c \cdot \frac{\partial^2}{\partial x^2} P_t(x)$$

$\downarrow$
speed that x change from t to $t+\Delta t$

average of neighbourhood around x