

Continuous-Time Markov Chains

CTMC: state space is discrete

Time is continuous

Both the time at a given state in a visit & the state jumped to after visit at random

A non-negative integer valued stochastic process $\{X_t : t \geq 0\}$ in continuous time is said to be a **Continuous-Time Markov Chain (CTMC)** (also known as a jump Markov process) if, for all $k \geq 1$, $t_1 < t_2 < \dots < t_{k+1}$ and non-negative integers i_1, i_2, \dots, i_{k+1} ,

$$\begin{aligned} P(X_{t_{k+1}} = i_{k+1} | X_{t_1} = i_1, \dots, X_{t_k} = i_k) \\ = P(X_{t_{k+1}} = i_{k+1} | X_{t_k} = i_k). \end{aligned}$$

$$P_{jk}^{(t)} = p_{jk}^{(t)}$$

If $P(X_{t+h} = k | X_h = j) = P(X_t = k | X_0 = j) \equiv p_{jk}^{(t)}$ does not depend on h , we say the CTMC is **homogeneous**. We concentrate on homogeneous CTMCs in this subject.

(*) **Assumption:** $P^{(t)}$ is **stochastically (right) continuous** in the sense that $P^{(t+h)} \rightarrow P^{(t)}$ as $h \rightarrow 0+$.

continuous from the right
spend time at each state

Continuous-Time Markov Chains

$(N_t)_{t \geq 0}$ is a process

$$N_{t+k} - N_t = n_k$$

Basic example: Poisson process.

$$P(N_{t+k} = n_k | N_t = n) \text{ is non-decreasing}$$

Most general example we will consider:

If $X_0 = i$, wait an exponential rate λ_i time and then jump to a new state.

When jumping from state i , jump to state $j \neq i$ with probability $b_{i,j}$, where $\sum_{j \neq i} b_{i,j} = 1$.

$$= P(N_{t+k} - N_t = n_k | N_t = n)$$

Why Markovian?

Memoryless property of the exponential + transition probabilities

$b_{i,j}$ only depend on states jumping to and from (and not past states).

moving forward

only depend on $i = X_t$

(Indep incrs $\xrightarrow{\text{imply}} \text{Markov}$)
stoch

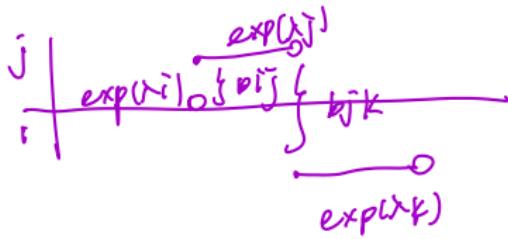
Continuous-Time Markov Chains

Example: Continuous analog of the **simple random walk**.

When in state j wait an exponential rate one time and jump to state $j + 1$ with probability p and state $j - 1$ with probability $1 - p$.

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$$\begin{aligned} & P(X_{t+k} = j \mid X_{t+k-1} = i, \dots, X_0 = t_0) \\ & \geq P(X_{t+k} = j \mid X_{t+k-1} = i) \\ & \quad (0 \leq t_0 < \dots < t_k) \\ & \geq p_{ij}^{(t_k - t_{k-1})} \quad \text{continuous.} \end{aligned}$$



This setup is the most **general** for chains that always spend a positive amount of time at an occupied state:

Suppose $X_0 = j$ and $T(j)$ is the first time the CTMC leaves j . Then

$$\begin{aligned} & P(T(j) > t + s \mid T(j) > s) \\ &= P(X_v = j, 0 \leq v \leq t + s \mid X_u = j, 0 \leq u \leq s) \\ &= P(X_v = j, s < v \leq t + s \mid X_s = j) \text{ (Markov)} \\ &= P(X_v = j, 0 < v \leq t \mid X_0 = j) \text{ (homogeneous)} \\ &= P(T(j) > t) \end{aligned}$$

So $T(j)$ has the memoryless property, and $T(j) \sim \exp(\lambda_j)$ for some λ_j which depends on j only.

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There are more general CTMCs with **instantaneous** states that are only visited for an instant, though we will not talk about these.

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While the **probabilistic** description is intuitive, it isn't always helpful to compute the transitions matrix $P^{(t)}$. For this we need the concept of a **generator** which we will later relate back to the intuitive description.

The Chapman-Kolmogorov equations

Observe that

$$\begin{aligned} p_{ij}^{(s+t)} &= \sum_k P(X_{s+t} = j | X_s = k, X_0 = i) P(X_s = k | X_0 = i) \\ &= \sum_k p_{ik}^{(s)} p_{kj}^{(t)}. \end{aligned}$$

$p_{kj}^{(t)}$ $p_{ik}^{(s)}$

These are the **Chapman-Kolmogorov equations** for a CTMC. In matrix form, we write $P^{(t)} = (p_{jk}^{(t)})$. Then, for $s, t \geq 0$, the Chapman-Kolmogorov equations can be expressed in the form

$$P^{(t+s)} = P^{(t)} P^{(s)}.$$

Continuous-Time Markov Chains

know $P^{(t)} \xrightarrow{t \downarrow 0} \text{Id} = P^{(0)}$

By analogy with the discrete-time case, we might hope that we can write $P^{(t)} = P^t$ for some matrix P .

If $t = m$ (a positive integer), the C-K equations tell us that $P^{(m)} = (P^{(1)})^m$ and our hope is fulfilled, but if $t < 1$?

We want a single object (like $P = P^{(1)}$ in the discrete case) that encodes the information of the chain.

$$\frac{P^{(t)} - \text{Id}}{t} \xrightarrow{t \downarrow 0} ?$$

$$P(x_0=j \mid x_0=i)$$

$$= \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

Continuous-Time Markov Chains

If t and h are nonnegative real numbers, we can write

$$\begin{aligned}\frac{P^{(t+h)} - P^{(t)}}{h} &= P^{(t)} \left[\frac{P^{(h)} - I}{h} \right] \xrightarrow{\text{Identity matrix}} \\ &= \left[\frac{P^{(h)} - I}{h} \right] P^{(t)}\end{aligned}$$

This suggests that we should investigate the existence of the derivative

$$A \equiv \lim_{h \rightarrow 0^+} \frac{P^{(h)} - I}{h}.$$

Continuous-Time Markov Chains

- ▶ Amazingly, the continuity assumption on $P^{(t)}$ implies the existence of the matrix A , called the matrix or infinitesimal generator of the CTMC.

[If there are instantaneous states (excluded in our study) then some $a_{i,i} = -\infty$.]

$$A_{jk} = \lim_{\substack{j,k \text{ entry} \\ h \rightarrow 0}} \frac{P_{jk}^{(h)} - \delta_{jk}}{h}$$

- ▶ In all cases that we shall discuss, $|a_{j,k}| < \infty$ and $\sum_k a_{jk} = 0$ (inherited from the limit) and we shall assume this henceforth.

$$\delta_{jk} = I_{jk}$$
$$= \begin{cases} 1 & j=k \\ 0 & j \neq k \end{cases}$$

$$\sum_k \lim_{h \rightarrow 0} \frac{P_{jk}^{(h)} - \delta_{jk}}{h}$$

$$= \lim_{h \rightarrow 0} \sum_k \frac{P_{jk}^{(h)} - \delta_{jk}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{l-1}{h} = 0$$

$$\delta_{jk} = j_{jk}$$
$$= \begin{cases} 1 & j=k \\ 0 & j \neq k \end{cases}$$

Continuous-Time Markov Chains

Since

$$\begin{aligned}\frac{P^{(t+h)} - P^{(t)}}{h} &= P^{(t)} \left[\frac{P^{(h)} - I}{h} \right] \\ &= \left[\frac{P^{(h)} - I}{h} \right] P^{(t)}\end{aligned}$$

function of t

we hope that

$$\frac{d}{dt} P^{(t)} = A P^{(t)} \quad (\ddagger) \quad \left(\frac{d}{dt} P^{(t)} \right)_{jk} = \frac{d}{dt} P_j^{(t)} \xrightarrow{k}$$

and, similarly,

$$\begin{aligned}(\mathbf{A} P^{(t)})_{jk} &= \sum_i a_{ji} P_i^{(t)} \\ \frac{d}{dt} P^{(t)} &= P^{(t)} A. \quad (\dagger) \quad (P^{(t)} \mathbf{A})_{jk} = \sum_i P_j^{(t)} a_{ik}\end{aligned}$$

Continuous-Time Markov Chains

We write “hope that” since we need to justify pushing the **limits** through the (possibly infinite) **sums**

$$\lim_{h \rightarrow 0} \sum_{k \in S} p_{i,k}^{(t)} \left[\frac{P^{(h)} - I}{h} \right]_{k,j} = \sum_{k \in S} p_{i,k}^{(t)} \lim_{h \rightarrow 0} \left[\frac{P^{(h)} - I}{h} \right]_{k,j}$$

$$\lim_{h \rightarrow 0} \sum_{k \in S} \left[\frac{P^{(h)} - I}{h} \right]_{i,k} p_{k,j}^{(t)} = \sum_{k \in S} \lim_{h \rightarrow 0} \left[\frac{P^{(h)} - I}{h} \right]_{i,k} p_{k,j}^{(t)}$$

for each $i, j \in S$.

If S is finite, then there is no problem and both (\ddagger) and (\dagger) hold.

swap sum and limit

Continuous-Time Markov Chains

want to show

$$\lim_{h \rightarrow 0} \sum_{i \in S} \frac{p_{ii}^{(h)} - \delta_{ji}}{h} p_{ik}^{(t)}$$

$$= \sum_{i \in S} \left[\lim_{h \rightarrow 0} \frac{p_{ji}^{(h)} - \delta_{ji}}{h} \right] p_{ik}^{(t)}$$

α_{ji}

$$= \sum_{i \in S} \alpha_{ji} p_{ik}^{(t)}$$

Justifying (\dagger)

In fact, we know from Fatou's Lemma that, for $j, k \in S$,

$$\begin{aligned} \liminf_{h \rightarrow 0^+} \frac{p_{jk}^{(t+h)} - p_{jk}^{(t)}}{h} &= \liminf_{h \rightarrow 0^+} \sum_{i \in S} \frac{(p_{ji}^{(h)} - \delta_{ji}) p_{ik}^{(t)}}{h} \\ &\geq \sum_{i \in S} \lim_{h \rightarrow 0^+} \frac{(p_{ji}^{(h)} - \delta_{ji}) p_{ik}^{(t)}}{h} \\ &= \sum_{i \in S} \alpha_{ji} p_{ik}^{(t)}. \end{aligned}$$

should be $\liminf_{h \rightarrow 0}$
but limit exist
 $\liminf = \lim$

Similarly,

$$\liminf_{h \rightarrow 0^+} \frac{p_{jk}^{(t+h)} - p_{jk}^{(t)}}{h} \geq \sum_{i \in S} p_{ji}^{(t)} \alpha_{ik}.$$

Continuous-Time Markov Chains

Justifying (\dagger)

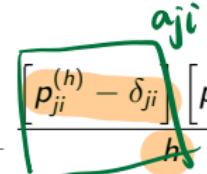
We can show that the inequality in the first expression is, in fact, an equality, as follows. For $N > j$,

$$\begin{aligned} \sum_{i \in S} \frac{[p_{ji}^{(h)} - \delta_{ji}] p_{ik}^{(t)}}{h} &= \sum_{i=1}^N \frac{[p_{ji}^{(h)} - \delta_{ji}] p_{ik}^{(t)}}{h} + \sum_{i=N+1}^{\infty} \frac{p_{ji}^{(h)} p_{ik}^{(t)}}{h} \\ &\leq \sum_{i=1}^N \frac{[p_{ji}^{(h)} - \delta_{ji}] p_{ik}^{(t)}}{h} + \sum_{i=N+1}^{\infty} \frac{p_{ji}^{(h)}}{h} \quad \xrightarrow{\text{since } p_{ik}^{(t)} \leq 1} \\ &= \sum_{i=1}^N \frac{[p_{ji}^{(h)} - \delta_{ji}] p_{ik}^{(t)}}{h} + \underbrace{\left[1 - \sum_{i=1}^N p_{ji}^{(h)}\right]}_{h} \xrightarrow{\text{since } \sum_{i=1}^N p_{ji}^{(h)} = 1} \sum_{i=1}^N \delta_{ji} - p_{ji}^{(h)} \\ &= \sum_{i=1}^N \frac{[p_{ji}^{(h)} - \delta_{ji}] [p_{ik}^{(t)} - 1]}{h}. \end{aligned}$$

Continuous-Time Markov Chains

Justifying (\dagger)

Therefore

$$\begin{aligned} \limsup_{h \rightarrow 0^+} \sum_{i \in S} \frac{[p_{ji}^{(h)} - \delta_{ji}] p_{ik}^{(t)}}{h} &\leq \sum_{i=1}^N \lim_{h \rightarrow 0^+} [p_{ji}^{(h)} - \delta_{ji}] [p_{ik}^{(t)} - 1] \\ &= \sum_{i=1}^N a_{ji} p_{ik}^{(t)} - \sum_{i=1}^N a_{ji}. \end{aligned}$$


Now we let $N \rightarrow \infty$ and use the fact that $\sum_{i=1}^{\infty} a_{ji} = 0$ to derive

$$\limsup_{h \rightarrow 0^+} \frac{p_{jk}^{(t+h)} - p_{jk}^{(t)}}{h} \leq \sum_{i \in S} a_{ji} p_{ik}^{(t)},$$

which proves that $p_{jk}^{(t)}$ is differentiable (since $\liminf \geq \limsup$) and

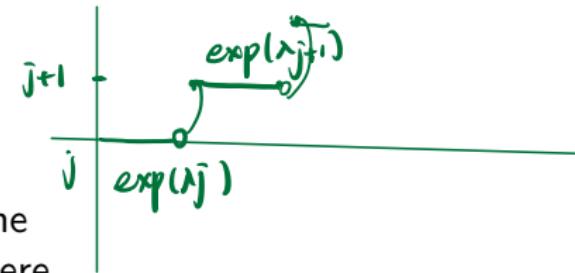
$$\frac{dp_{jk}^{(t)}}{dt} = \sum_{i \in S} a_{ji} p_{ik}^{(t)}.$$

Continuous-Time Markov Chains

So we have justified the interchange leading to (‡). However, the interchange leading to (†) cannot be justified in general, and there is a class of CTMCs, known as **explosive CTMCs** for which equation (†) does not hold.

Example: Explosive CTMC

Consider a CTMC on the nonnegative integers $j = 0, 1, \dots$ with $\lambda_j = \lambda^j$ for some $\lambda > 1$ and which always increases by one; $b_{j,j+1} = 1$. Then the expected total time that the CTMC takes to 'reach infinity' is finite. This is an example of an explosive CTMC.



given $x_0=0$, let

T_j : time chain spent at state j

$$E\left[\sum_{j=0}^{\infty} T_j\right] = \sum_{j=0}^{\infty} E(T_j)$$

$$= \sum_{j=0}^{\infty} \left(\frac{1}{\lambda}\right)^j$$

$$= \frac{1}{1 - \frac{1}{\lambda}} < \infty$$

Continuous-Time Markov Chains

$$\frac{d}{dt} P^{(t)} = A P^{(t)}$$

Equations (\ddagger) are known as the **Kolmogorov backward equations** and equations (\dagger) are known as the **Kolmogorov forward equations**.

$$\frac{d}{dt} P^{(t)} = P^{(t)} A$$

For non-explosive CTMCs, the matrix A determines the transition probability completely by solving the backward or forward equations to get

$$P^{(t)} = \exp(tA) \\ = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k,$$

subject to $P^{(0)} = I$.

Example: The Poisson process

The Poisson process can be thought of as a CTMC. For $j, k \geq 0$, the transition probabilities are

$$p_{j,j+k}^{(t)} = \frac{e^{-\lambda t} (\lambda t)^k}{k!}.$$

$$P_{j,j+k}^{(t)} = P(N_t = k)$$

$$= \frac{e^{-\lambda t} (\lambda t)^k}{k!}$$

We can verify that

$$\frac{d}{dt} p_{j,j+k}^{(t)}|_{t=0} = \begin{cases} -\lambda, & \text{for } k = 0, \\ \lambda, & \text{for } k = 1, \\ 0, & \text{for } k \geq 2. \end{cases}$$

$$\frac{d}{dt} = -\lambda e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

$$+ e^{-\lambda t} \frac{k \lambda^{k-1} t^k}{k!}$$

Continuous-Time Markov Chains

$$\frac{\frac{d}{dt} P_{00}^{(ct)}}{P_{00}^{(ct)}} = -\lambda$$

So

$$A = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & \dots \\ 0 & -\lambda & \lambda & 0 & 0 & 0 & \dots \\ 0 & 0 & -\lambda & \lambda & 0 & 0 & \dots \\ 0 & 0 & 0 & -\lambda & \lambda & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

$$\frac{d}{dt} \log(P_{00}^{(t)}) = -\lambda$$

$$\log(P_{00}^{(t)}) = -\lambda t + C$$

$$P_{00}^{(t)} = e^{-\lambda t}$$

$$P_{00}^{(0)} = F e^{-\lambda t}$$

$$P_{00}^{(0)} = 1 \Rightarrow F = 1$$

$$P_{00}^{(t)} = e^{-\lambda t} = P(\exp(\lambda) > t)$$

$$\frac{d}{dt} P_{01}^{(ct)} = -\lambda P_{01}^{(ct)} + \lambda e^{-\lambda t}$$

Continuous-Time Markov Chains

$$e^{\lambda t} \left(\frac{d}{dt} P_{01}^{(t)} \right) + (\lambda e^{\lambda t}) P_{01}^{(t)} = \lambda e^{-\lambda t} \cdot e^{\lambda t}$$

$$\Rightarrow \frac{d}{dt} (e^{\lambda t} P_{01}^{(t)}) = \lambda$$

$$e^{\lambda t} P_{01}^{(t)} = \lambda t + c$$

$$P_{01}^{(c)} = (\lambda t + c) e^{-\lambda t}$$

$$P_{01}^{(0)} = 0 \Rightarrow c = 0$$

Can we derive the transition probabilities from A ? We could

- ▶ Compute $\exp(tA)$.
- ▶ Solve the Kolmogorov backward or forward differential equations.

We solve the forward equations to find $P_{0k}^{(t)}$.

$$\begin{aligned} \frac{d}{dt} P_{0k}^{(t)} &= (P^{(t)} A)_{0k} \\ &= \sum_j P_{0j}^{(t)} a_{j-k} \\ &\downarrow \\ &= 0 \quad \text{for } j > k \\ &\quad j=k-1 \end{aligned}$$

Continuous-Time Markov Chains

$$= \begin{cases} p_{00}^{(t)} (-\lambda) & k=0 \\ p_{0k}^{(t)} (-\lambda) + p_{0k-1}^{(t)} (\lambda) & k \geq 1 \end{cases}$$

Now,

$$\begin{aligned} \frac{d}{dt} p_{00}^{(t)} &= -\lambda p_{00}^{(t)} \\ \implies p_{00}^{(t)} &= e^{-\lambda t}. \end{aligned}$$

Boundary condition

$$p_{0k}^{(0)} = \gamma_{0k}$$

and

$$\begin{aligned} \frac{d}{dt} p_{01}^{(t)} &= \lambda(p_{00}^{(t)} - p_{01}^{(t)}) \\ \implies p_{01}^{(t)} &= \lambda t e^{-\lambda t}. \end{aligned}$$

By mathematical induction, we can show that $p_{0k}^{(t)} = e^{-\lambda t}(\lambda t)^k/k!$.

Continuous-Time Markov Chains

Interpretation of the generator

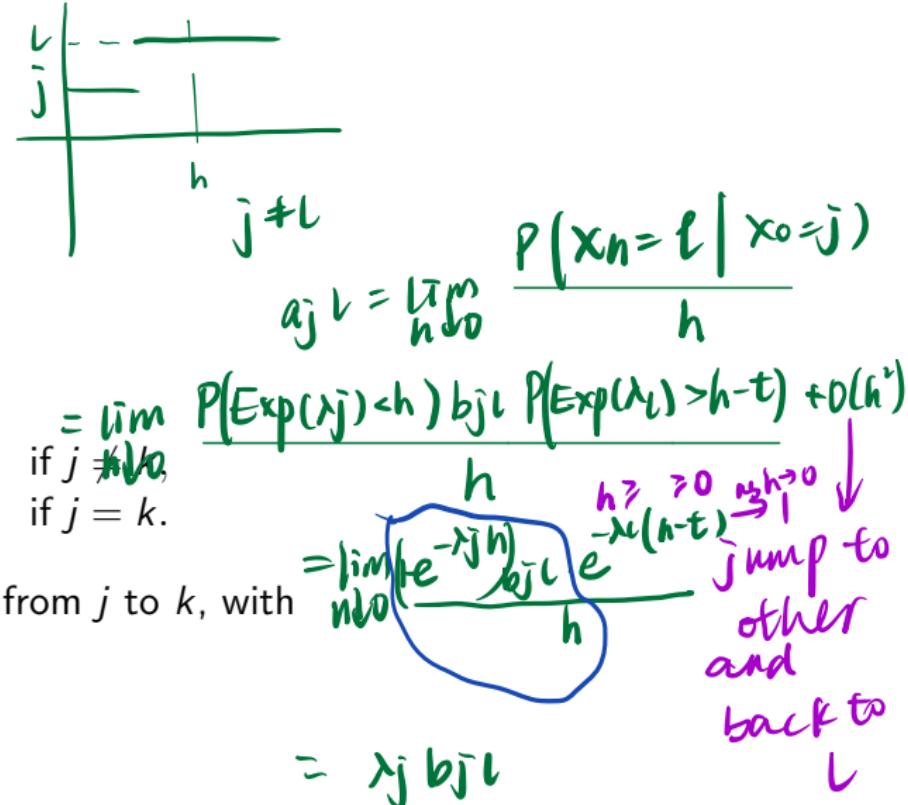
For small h ,

$$\begin{aligned} P(X_{t+h} = k | X_t = j) &= p_{jk}^{(h)} \\ &\approx (I + hA)_{jk} \\ &= \begin{cases} ha_{jk}, & \text{if } j \neq k, \\ 1 + ha_{jj}, & \text{if } j = k. \end{cases} \end{aligned}$$

So we can think of a_{jk} as the **rate of transition** from j to k , with

$$a_{jk} \begin{cases} \geq 0 \text{ if } k \neq j \\ \leq 0 \text{ if } k = j \end{cases}$$

and, since each row sums to 0, $\sum_{k \neq j} a_{jk} = -a_{jj}$.



Continuous-Time Markov Chains

The total rate of leaving state j is $\sum_{k \neq j} a_{jk} = -a_{jj}$, so $\lambda_j = -a_{jj}$.

To see where the CTMC moves upon leaving state j , observe that, for $k \neq j$,

$$P(X_h = k | X_h \neq j, X_0 = j) = \frac{p_{jk}^{(h)}}{\sum_{l \neq j} p_{jl}^{(h)}} \rightarrow \frac{-a_{jk}}{a_{jj}}$$

as $h \rightarrow 0$.

That is, when the CTMC leaves state j , it has probability $-a_{jk}/a_{jj}$ of moving to state k .

The Evolution of a CTMC

Starting at an initial state $X_0 = j$, the CTMC stays in j for a random time, exponentially distributed with parameter $-a_{jj}$.

Then it jumps to a state $k \neq j$ with probability $-a_{jk}/a_{jj}$,

and stays there for a random time which is exponentially distributed with parameter $-a_{kk}$ and independent of anything that has happened previously,

and then it jumps to ℓ , and so on ...

Continuous-Time Markov Chains

Examples

Determine whether each of the following matrices is the generator of a Markov chain and, if yes, describe how the CTMC evolves (λ and μ are nonnegative):

A is a generator

if $a_{jj} \leq 0$ & $\sum_{k \neq j} a_{jk} = -a_{jj}$

$a_{jk} \geq 0 \quad \forall k$

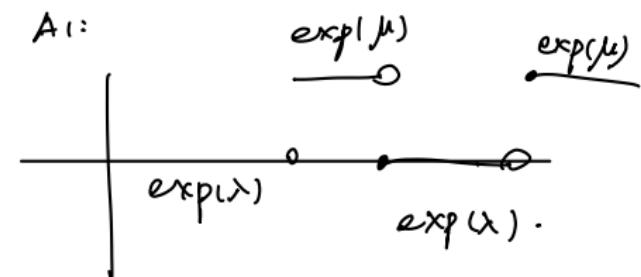
$$b_{jk} = \frac{a_{jk}}{-a_{jj}}$$

$$\sum_k b_{jk} = 1$$

$$A_1 = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}; \quad A_2 = \begin{pmatrix} -2 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & -3 \end{pmatrix};$$

$$A_3 = \begin{pmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 2 & -3 \end{pmatrix}.$$

$$\begin{aligned} b_{12} &= b_{13} = \frac{1}{2} & \lambda_1 &= 2 \\ b_{23} &= 1 & \lambda_2 &= 1 \\ b_{21} &= \frac{1}{3} & \lambda_3 &= 3 \\ b_{32} &= \frac{2}{3} \end{aligned}$$



revision $(x_t)_{t \geq 0}$ is a CTMC Continuous-Time Markov Chains

$$(P^{(t)})_{i,j} = P^{(t)}_{ij} = P(X_t=j | X_0=i)$$

$$\frac{d}{dt} P^{(t)} \Big|_{t=0} = A = a_{ij}$$
$$a_{ij} = \lim_{n \rightarrow \infty} \frac{P_{ij}^{(n)} - \delta_{ij}}{h}$$

Ergodicity

When does $\pi_j = \lim_{t \rightarrow \infty} p_{ij}^{(t)}$ exist, independently of i ?

The conditions are similar to those for DTMCs, but we don't have to deal with periodicity.

- We classify the states and identify whether there is only one class of essential states.
- If yes, and the class is finite, then the CTMC is ergodic.
- If yes, and the class is not finite, then we need to check the mean return times.

Competing exponential clock

given at state j

for each state $l \neq j$

start an $\text{exp}(a_{jl})$ clock

$$T_{jl} = \text{Exp}(a_{jl})$$

when 1st clock rings,
jump to the state of that clock

• Jump at $\min_{l \neq j} T_{jl}$
 $P(\text{jump to } l)$

Continuous-Time Markov Chains

An irreducible CTMC is positive recurrent if and only if there is a stationary distribution satisfying

$$\pi = \pi P^{(t)}$$

with $\sum_j \pi_j = 1$ for any $t \geq 0$.

For non explosive CTMCs, this occurs if and only if π satisfies

*you don't get infinite jump
in finite time* $\pi A = 0,$

with $\sum_j \pi_j = 1$.

For explosive CTMCs, it is possible to have a solution to

$$\pi A = 0,$$

$$= 0.$$

with $\sum_j \pi_j = 1$ that is not the stationary distribution.

$$= P(T_{jL} = \min_{i \neq j} T_{ji})$$

Exer
check

$$\min_{i \neq j} T_{ji} \stackrel{d}{=} \text{Exp}(-\alpha_{jj})$$

$$P(T_{jL} = \min_{i \neq j} T_{ji}) = \frac{\alpha_{ji}}{-\alpha_{jj}}$$

$$\begin{aligned} \pi A &= \lim_{h \downarrow 0} \frac{\pi^{(h)} - \pi}{h} \xrightarrow{\text{identity}} \\ &= \lim_{h \downarrow 0} \frac{\pi^h - \pi}{h} \end{aligned}$$

Continuous-Time Markov Chains

Example (Explosive CTMC with stationary distribution)

- Take the CTMC with generator with $a_{i,i+1} = \lambda_i p$ for $i \geq 0$, and $a_{i,i-1} = (1-p)\lambda_i$ for $i \geq 1$ and all other off-diagonal entries zero.

$$A = \begin{bmatrix} \lambda_0 p & \lambda_0 p & & \\ \lambda_1(1-p) & \lambda_1 & \lambda_1 p & \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

- If $p > 1 - p$, then embedded DTMC is transient, but choosing $\lambda_i = \lambda^i$ where $\lambda > p/(1-p)$, there is a solution to

$$\pi A = 0$$

sequence of states visited
is same as DTMC SRW
with $p_{i,i+1} = 1 - p_{i,i-1} = p$ with $\sum_{i \geq 0} \pi_i = 1$.

[We know the chain is transient since the discrete time chain is, so it must be explosive if there is a stationary distribution.]

$\pi_{0,1} = 1$
If $p > \frac{1}{2}$, then DTMC "drift" to ∞ transient \Rightarrow CTMC is also transient
BUT: solve $\pi A = 0$

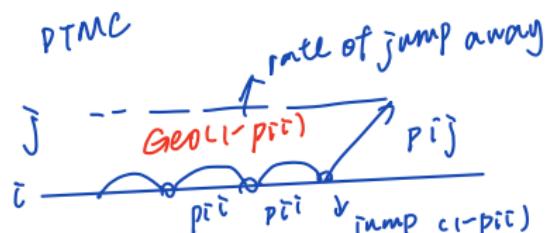
$$\begin{aligned} i \geq 1: \quad -\lambda_i \pi_i + p \pi_{i-1} \pi_{i-1} + (1-p) \lambda_{i+1} \pi_{i+1} &= 0 \quad \Rightarrow \lambda_i = \lambda^i \quad \pi_i = \frac{p}{\lambda(1-p)} \pi_0 \\ i = 0: \quad -\lambda_0 \pi_0 + \lambda_1 (1-p) \pi_1 &= 0 \end{aligned}$$

Continuous-Time Markov Chains

can check $\pi_{\tilde{i}} = \left(\frac{p}{\lambda(1-p)}\right)^{\tilde{i}} \pi_0$

Then prob sol if $\pi_0 \sum_{i \geq 0} \left(\frac{p}{\lambda(1-p)}\right)^i = 1$
 $\downarrow_{<\infty} \Rightarrow \lambda > \frac{p}{1-p}$

DTMC



Summary

Discrete time MC

Unit of time

One step

Basic info

P

Distribution

$P^{(n)} = P^n$

Evolution

geometric times+jumps

Stationarity

$\pi P = \pi$

Cont. time MC

"dt"

A

$\frac{d}{dt} P^{(t)} = P^{(t)} A = AP^{(t)}$ ~~dt~~.

exponential times+jumps

$\pi A = 0$

Analogous to description of trajectory of CTMC

given in state i at time 0,
 visit state j $\text{Geo}(1-p_{ij})$

Given a jump, jump to state $j+i$ with probability p_{ij}
 $\sum_{j \neq i} p_{ij} = 1 - p_{ii}$

Continuous-Time Markov Chains

B-D process, state space = {0, 1, 2, ...}

$$A = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \dots & \dots \\ \mu_1 - (\mu_1 + \lambda_1) & \lambda_1 & 0 & \dots & \dots \\ \mu_2 - (\mu_2 + \lambda_2) & \lambda_2 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Birth and Death Processes

Let

- ▶ X_t be the number of 'people' in a system at time t .
- ▶ Whenever there are n 'people' in the system
 - ▶ new arrivals enter (by birth or immigration) the system at an exponential rate λ_n
 - ▶ 'people' leave (or die from) the system at an exponential rate μ_n
- ▶ arrivals and departures occur independently of one another
- ▶ $\{X(t) : t \geq 0\}$ is a **birth-and-death process** with arrival (or birth) rates $\{\lambda_n\}$ and departure (or death) rates $\{\mu_n\}$.

Continuous-Time Markov Chains

The generator of a birth and death process has the form

$$A = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \ddots \\ \mu_1 & -(\mu_1 + \lambda_1) & \lambda_1 & 0 & 0 & \ddots \\ 0 & \mu_2 & -(\mu_2 + \lambda_2) & \lambda_2 & 0 & \ddots \\ 0 & 0 & \mu_3 & -(\mu_3 + \lambda_3) & \lambda_3 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

The CTMC evolves by remaining in state k for an exponentially distributed time with rate $\lambda_k + \mu_k$,

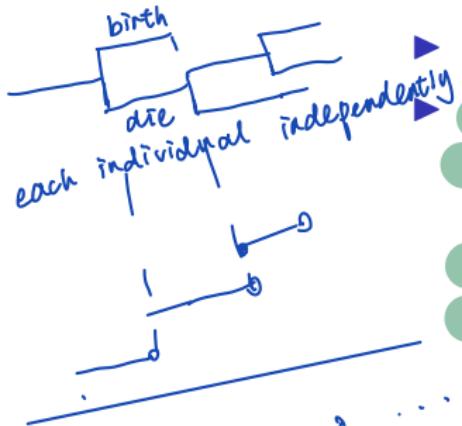


then it moves to state $k + 1$ with probability $\lambda_k / (\lambda_k + \mu_k)$ and state $k - 1$ with probability $\mu_k / (\lambda_k + \mu_k)$, and so on.

Continuous-Time Markov Chains

Examples

- The Poisson process is a **pure birth process** with constant birth rates. Thus $\lambda_n = \lambda$ and $\mu_n = 0$ for all n .
- If $\lambda_n = 0$ for all n , then the CTMC is a **pure death process**.



A binary continuous time branching process is a birth and death process: each particle lives for an exponentially distributed time with parameter z , then either splits into two new particles, with probability p or just dies, with probability $q = 1 - p$. All particles behave independently.

The rates are $\lambda_n = znp$ and $\mu_n = znq$ for all n .

$$A = \begin{pmatrix} 0 & 0 & 0 & \dots \\ (-p) & z-pz & \dots \\ 0 & znpz & zp^2 & \dots \\ 0 & znp^2 & zp^3 & \dots \\ & zp^3 & zp^4 & \dots \end{pmatrix}$$

$$\lambda_n = znp ; \mu_n = zn(1-p)$$

(2)

embedded DTMC is sRW with boundary at zero & absorbing at 0

$p \leq q \Rightarrow$ chain recurrent

Continuous-Time Markov Chains

\Rightarrow absorption at zero is guaranteed

(Given $x_0 \geq k > 0$ & $T = \inf\{X_n : X_n = 0\}$)

$$(X_n)_{0 \leq n \leq T} \stackrel{d}{=} (Y_n)_{0 \leq n \leq T}$$

The generator for the binary continuous-time branching process is

SRW with boundary

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ qz & -z & pz & 0 & 0 & \dots \\ 0 & 2qz & -2z & 2pz & 0 & \dots \\ 0 & 0 & 3qz & -3z & 3pz & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$p > q$, $(Y_n)_{n \geq 0}$ is transient
 $P(T = \infty) > 0$

$\Rightarrow (X_n)_{n \geq 0}$ is also transient

If $p \leq q$, then eventual extinction is guaranteed. If $p > q$ then extinction occurs with probability less than one.

Continuous-Time Markov Chains

From generator to distribution

For a birth and death process with a given initial distribution, one can find $p_j(t) = P(X_t = j)$ by solving the system of differential equations

$$\frac{d}{dt} \mathbf{p}^{(t)} = \mathbf{p}^{(t)} A,$$

that is

$$\frac{d}{dt} p_0(t) = -\lambda_0 p_0(t) + \mu_1 p_1(t)$$

$$\frac{d}{dt} p_k(t) = \lambda_{k-1} p_{k-1}(t) - (\lambda_k + \mu_k) p_k(t) + \mu_{k+1} p_{k+1}(t), \quad k \geq 1.$$

This system of equations governs the 'redistribution' of 'probability mass' as time passes. For finite state space birth and death process, it can be solved numerically. For infinite state space CTMCs there are techniques of solution that involve **orthogonal polynomials**.

Generator A has
 $a_{n,n+1} = \lambda_n \quad ; \quad a_{n,n-1} = \mu_n$
 $\hookrightarrow n \geq 0$

$$a_{n,n} = \begin{cases} -\lambda_0 & n=0 \\ -(\lambda_n + \mu_n) & n \geq 1 \end{cases}$$

state space = $\{0, 1, \dots, g\}$

$$p_0(t) = \sum_i p_{0i} e^{(t)} \quad \text{any } t \in [0, \infty)$$

$$= -\lambda_0 p_0(t) + \mu_1 p_1(t)$$

Continuous-Time Markov Chains

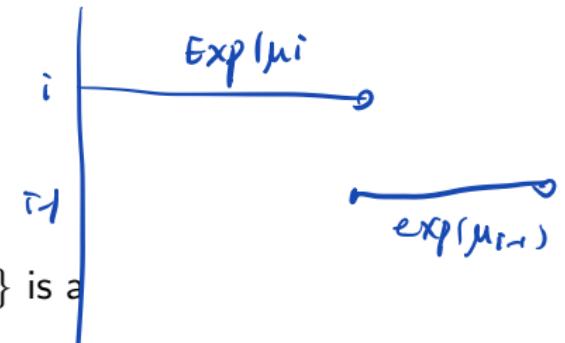
Example

Consider a population in which there are no births ($\lambda_i = 0$ for $i \geq 0$) and, for $i \geq 1$, the death rates are μ_i .

Suppose X_t is the population size at time t and so $\{X_t, t \geq 0\}$ is a CTMC. Define $p_{ij}(t) = P(X_{t+s} = j | X_s = i)$.

1. Find expressions for $p_{ii}(t)$ and $p_{i,i-1}(t)$ for $i \geq 1$.
2. Given that the population size is two at a particular time, calculate the probability that no more than one death will occur within the next one unit of time.

2. $P(\text{no more than 1 death in } [t, t+1])$
 $\quad \quad \quad X_0=2$
 $= p_{2,1}(1) \times p_{2,2}(1)$



$$\begin{aligned} p_{ii}(t) &= P(\text{Exp}(\mu_i) > t) \\ &= e^{-\mu_i t} \end{aligned}$$

Exer:
use the forward equ.
 $\frac{d}{dt} p_{i,i-1}(t) = -\mu_{i-1} p_{i,i-1}(t)$

$$p_{i,i-1}(t) = P(T_{t+s} > t, T \leq t)$$
$$T \sim \text{Exp}(\mu_i) \quad s \sim \text{Exp}(\mu_{i-1})$$

indep

Continuous-Time Markov Chains

$$\begin{aligned}
 & \frac{\pi_i e^{-\mu_i t} \frac{\pi_i}{\mu_i}}{e^{-\mu_i t}} \\
 &= \iint_{0 \leq s \leq t} \mu_i e^{-\mu_i s} \cdot \mu_i e^{-\mu_i(t-s)} ds du \\
 &= \int_0^t \int_{s \leq u} \mu_i e^{-\mu_i s} \cdot \mu_i e^{-\mu_i(t-u)} du \\
 &= \int_0^t \mu_i e^{-\mu_i u} e^{-\mu_i(t-u)} du \\
 &= \mu_i e^{-\mu_i t} + \int_0^t e^{-\mu_i u + \mu_i u} du \\
 &= \mu_i e^{-\mu_i t} \left(\frac{1 - e^{-c(\mu_i - \mu_i)}}{\mu_i - \mu_i} \right)
 \end{aligned}$$

Stationary distribution of a Birth and Death process

Assume $\lambda_i > 0$ and $\mu_i > 0$ for all i . We derive the stationary distribution (if it exists) by solving $\pi A = 0$. This leads to

$$\begin{aligned}
 0 &= -\lambda_0 \pi_0 + \mu_1 \pi_1 \quad \xrightarrow{\text{jump away from } k} \\
 0 &= \underbrace{\lambda_{k-1} \pi_{k-1}}_{k-1 \rightarrow k} - \underbrace{(\lambda_k + \mu_k) \pi_k}_{k+1 \rightarrow k} + \mu_{k+1} \pi_{k+1}, \quad k \geq 1.
 \end{aligned}$$

It turns out that this is equivalent to

$$\lambda_k \pi_k = \mu_{k+1} \pi_{k+1}$$

for $k \geq 0$.

$$\begin{aligned}
 k=0: \quad \pi_1 &= \pi_0 \left(\frac{\lambda_0}{\mu_1} \right) \\
 k=1: \quad \pi_2 &= \pi_1 \left(\frac{\lambda_1}{\mu_2} \right) = \lambda_0 \pi_0 + \mu_2 \pi_2
 \end{aligned}$$

$$\frac{\lambda_0}{\mu_1} \pi_0 \pi_0 = \mu_2 \pi_2$$

$$\cancel{(\lambda_0 + \mu_1) \left(\frac{\lambda_0}{\mu_1} \right) \pi_0} = \cancel{\lambda_0 \pi_0 + \mu_1 \pi_1} \quad \pi_1 \pi_1 = \mu_2 \pi_2$$

$$\pi_1 \pi_1 = \frac{\lambda_0 \pi_1}{\mu_1 \mu_2} \pi_2$$

$$\text{Gives: } \pi_k = \prod_{i=1}^k \frac{\lambda_{i-1}}{\mu_i} \pi_0 \quad \text{for } k \geq 0.$$

Continuous-Time Markov Chains

The first-order recursion has solution

$$\pi_k = \pi_0 \prod_{\ell=1}^k \frac{\lambda_{\ell-1}}{\mu_\ell}.$$

So a stationary distribution exists if and only if

$$\sum_{k=0}^{\infty} \prod_{\ell=1}^k \frac{\lambda_{\ell-1}}{\mu_\ell} < \infty$$

in which case

$$\pi_0 = \left(\sum_{k=0}^{\infty} \prod_{\ell=1}^k \frac{\lambda_{\ell-1}}{\mu_\ell} \right)^{-1}.$$

Continuous-Time Markov Chains

Ex.

$$\begin{cases} \text{EXP}(\lambda + \mu) & \lambda + \mu = p \\ 0 & \mu / (\lambda + \mu) = 1 - p \end{cases}$$

discrete time skeleton

↑ like a SRW with
prob p .

[reflection at 0]

pos recurr if $p < \lambda < \mu$

sol to $\underline{\pi} A = 0$

Examples

- ▶ Compute the stationary distribution of an 'M/M/1 queue', which is a birth and death process with $\lambda_i = \lambda$, $i \geq 0$, and $\mu_i = \mu$, $i \geq 1$.
- ▶ Compute the stationary distribution of a birth and death process with constant birth rate $\lambda_i = \lambda$, $i \geq 0$, and unit per capita death rate $\mu_i = i$, $i \geq 1$.

Ex 2

individual die indep at state i
& new arrival come according
to p prob (rate λ)

$$\underline{\pi} A = 0$$

$$\Rightarrow \pi_k = \pi_0 \left(\prod_{i=1}^k \frac{\lambda_{i-1}}{\mu_i} \right)$$

\uparrow_{i-1}

$$= \pi_0 \frac{\lambda^k}{k!}$$

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^\lambda$$

$$\pi_0 = e^{-\lambda}$$

$$\pi_k = \frac{e^{-\lambda} \lambda^k}{k!}$$

$\pi_k = \left(\frac{\mu}{\lambda} \frac{\lambda^{k+1}}{k!} \right) \pi_0 = \pi_0 \left(\frac{\lambda}{\mu} \right)^k$ This is summable when $\frac{\lambda}{\mu} < 1$.

$$\text{if } \lambda > \mu, \pi_k = \left(\frac{\lambda}{\mu} \right)^k \left(1 - \frac{\lambda}{\mu} \right)$$
$$\pi_0 = \frac{1}{\sum_{k \geq 0} \left(\frac{\lambda}{\mu} \right)^k}$$
$$\downarrow$$
$$\frac{1}{1 - \frac{\lambda}{\mu}}$$