THE UNIVERSITY OF MELBOURNE

Student Number	

Semester 1 Assessment, 2017

School of Mathematics and Statistics

MAST20009 Vector Calculus

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 5 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- Calculators, tablet devices or computers must not be used.
- No handwritten or print materials may be brought into the exam venue.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- There are 11 questions on this exam paper.
- All questions may be attempted.
- Marks for each question are indicated on the exam paper.
- Start each question on a new page.
- Clearly label each page with the number of the question that you are attempting.
- A 5 page formula sheet is appended to the end of this examination paper.
- The total number of marks available is 130.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

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Question 1 (15 marks)

(a) Calculate the following limits, or prove that none exists:

(i)

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$$

(ii)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$$

(b) Find the second order Taylor polynomial for $f(x,y) = \tan\left(\frac{x}{y}\right)$ about the point $(\pi, 4)$. Do not simplify your answer.

Question 2 (10 marks) Using Lagrange Multipliers, determine the maximum and minimum of the function

$$f(x,y,z) = x^2 + y^2 + z^2$$

subject to the constraints

$$x + y + 2z = 2 \quad \text{and} \quad x^2 + y^2 = z.$$

Justify that the points you have found give the maximum and minimum of f.

Question 3 (10 marks)

Consider the path P parameterised by

$$\mathbf{c}(t) = (t^2, \sin t - t\cos t, \cos t + t\sin t)$$

(a) Calculate the speed of a particle moving along P according to the above parameterisation.

(b) Calculate the unit normal vector $\mathbf{N}(t)$ for P.

(c) Find the curvature of P.

Question 4 (5 marks)

Let $\mathbf{r}$ be the position vector of a point in $\mathbb{R}^3$ and let $r = |\mathbf{r}|$. Using the basic vector identities only, calculate $\nabla \cdot \left(\frac{\mathbf{r}}{r^2}\right)$.

Question 5 (15 marks)

Let

$$\mathbf{F}(x, y) = (4x^3 + 9x^2y^2)\mathbf{i} + (6x^3y + 6y^5)\mathbf{j}$$

(a) Show that $\mathbf{F}$ is a conservative vector field in $\mathbb{R}^2$.

(b) Determine a scalar potential ϕ such that $\mathbf{F} = \nabla\phi$.

(c) Determine the work done by $\mathbf{F}$ in moving a particle from $(0, 0)$ to $(1, 2)$ along the straight line segment joining these two points.

Question 6 (15 marks)

Consider the solid region S in the first octant that is bounded by the parabolic cylinder $z = 2 - \frac{1}{2}x^2$ and the planes $z = 0, y = x$ and $y = 0$.

(a) Draw the projection of S onto the plane $z = 0$.

(b) Set up, but do not solve, the triple integral of $f(x, y, z) = 2xyz$ in cylindrical coordinates.

(c) Calculate the triple integral of $f(x, y, z) = 2xyz$ in Cartesian coordinates.

Question 7 (10 marks)

A torus in $\mathbb{R}^3$ is parameterised by

$$\boldsymbol{\Phi}(u, v) = ((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v),$$

where $a > b > 0$, $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$. Find the surface area of a torus in terms of a and b.

Question 8 (15 marks)

Let D be a simple closed region in $\mathbb{R}^2$ with boundary C.

(a) Let $\bar{x}$ be the x-coordinate of the centre of mass of D. Show that $\bar{x} = \dfrac{1}{2A}\displaystyle\int_C x^2 dy$, where A is the area of D.

(b) Consider the quarter disk region $R = \{(x, y) : x^2 + y^2 \le a^2, x \ge 0, y \ge 0\}$ of radius a in the first quadrant of $\mathbb{R}^2$.

(i) Find a parameterisation C for the boundary of R.

(ii) Use your parameterisation and the formula in (a) to calculate $\bar{x}$ for region R.

Question 9 (15 marks)

Consider the integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S},$$

where $\mathbf{F} = (3y, -xz, -yz^2)$ and where S is the portion of the surface $2z = x^2 + y^2$ below the plane $z = 2$.

(a) Calculate the given integral directly using cylindrical coordinates.

(b) Now calculate the given integral using Stokes' theorem.

Question 10 (10 marks)

(a) Let S be the solid determined by

$$1 \leq x^2 + y^2 + z^2 \leq 4$$

and let $\mathbf{F} = x\mathbf{i} + (2y + z)\mathbf{j} + (z + x^2)\mathbf{k}$.

Evaluate

$$\iint_{\partial S} \mathbf{F} \cdot d\mathbf{S}.$$

(b) Let W be a solid in $\mathbb{R}^3$ with surface boundary ∂W, and let f be a scalar field in $\mathbb{R}^3$. Prove that

$$\iiint_W (\nabla f) \cdot \mathbf{F} \; dxdydz = \iint_{\partial W} f\mathbf{F} \cdot d\mathbf{S} - \iiint_W f \nabla \cdot \mathbf{F} \; dxdydz$$

Question 11 (10 marks)

Define *prolate spheroidal* coordinates (ϵ, η, ϕ) by

$$x = a\sinh\epsilon\sin\eta\cos\phi, \quad y = a\sinh\epsilon\sin\eta\sin\phi, \quad z = a\cosh\epsilon\cos\eta$$

where $\epsilon \geq 0$, $0 \leq \eta \leq \pi$, $0 \leq \phi < 2\pi$ and a is a positive constant.

(a) Let $\mathbf{r} = (x, y, z)$. Write down expressions for

$$\frac{\partial\mathbf{r}}{\partial\epsilon}, \quad \frac{\partial\mathbf{r}}{\partial\eta} \quad \text{and} \quad \frac{\partial\mathbf{r}}{\partial\phi}.$$

(b) Show that the scale factors are

$$h_\epsilon = h_\eta = a\sqrt{\sinh^2\epsilon + \sin^2\eta}$$

and

$$h_\phi = a\sinh\epsilon\sin\eta.$$

(c) Write down an expression for the element of volume dV.

End of Exam—Total Available Marks = 130

Indefinite Integrals

$$\int \sin x\, dx = -\cos x + C \qquad \int \cos x\, dx = \sin x + C$$

$$\int \sec x\, dx = \log|\sec x + \tan x| + C \qquad \int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$$

$$\int \sec^2 x\, dx = \tan x + C \qquad \int \operatorname{cosec}^2 x\, dx = -\cot x + C$$

$$\int \sinh x\, dx = \cosh x + C \qquad \int \cosh x\, dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x\, dx = \tanh x + C \qquad \int \operatorname{cosech}^2 x\, dx = -\coth x + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \arcsin\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \operatorname{arccosh}\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C$$

where $a > 0$ is constant and C is an arbitrary constant of integration.

Useful Formulae

$\cos^2 x + \sin^2 x = 1$ $\qquad$ $\cosh^2 x - \sinh^2 x = 1$

$1 + \tan^2 x = \sec^2 x$ $\qquad$ $1 - \tanh^2 x = \operatorname{sech}^2 x$

$\cot^2 x + 1 = \operatorname{cosec}^2 x$ $\qquad$ $\coth^2 x - 1 = \operatorname{cosech}^2 x$

$\cos 2x = \cos^2 x - \sin^2 x$ $\qquad$ $\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cos 2x = 2\cos^2 x - 1$ $\qquad$ $\cosh 2x = 2\cosh^2 x - 1$

$\cos 2x = 1 - 2\sin^2 x$ $\qquad$ $\cosh 2x = 1 + 2\sinh^2 x$

$\sin 2x = 2\sin x \cos x$ $\qquad$ $\sinh 2x = 2\sinh x \cosh x$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$ $\qquad$ $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$ $\qquad$ $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$ $\qquad$ $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$

$e^{ix} = \cos x + i \sin x$

$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$ $\qquad$ $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$

$\operatorname{arcsinh} x = \log(x + \sqrt{x^2 + 1})$ $\qquad$ $\operatorname{arccosh} x = \log(x + \sqrt{x^2 - 1})$

$$\operatorname{arctanh} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

Basic Identities of Vector Calculus

Let f and $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be scalar functions, $\boldsymbol{F}$ and $\boldsymbol{G} : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be vector fields, and $\beta \in \mathbb{R}$ be any constant.

1. $\boldsymbol{\nabla}(f+g) = \boldsymbol{\nabla} f + \boldsymbol{\nabla} g$
2. $\boldsymbol{\nabla}(\beta f) = \beta \boldsymbol{\nabla} f$
3. $\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f$
4. $\boldsymbol{\nabla}\left(\dfrac{f}{g}\right) = \dfrac{g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g}{g^2}$ provided $g \neq 0$.
5. $\boldsymbol{\nabla} \cdot (\boldsymbol{F} + \boldsymbol{G}) = \boldsymbol{\nabla} \cdot \boldsymbol{F} + \boldsymbol{\nabla} \cdot \boldsymbol{G}$
6. $\boldsymbol{\nabla} \times (\boldsymbol{F} + \boldsymbol{G}) = \boldsymbol{\nabla} \times \boldsymbol{F} + \boldsymbol{\nabla} \times \boldsymbol{G}$
7. $\boldsymbol{\nabla} \cdot (f\boldsymbol{F}) = f\boldsymbol{\nabla} \cdot \boldsymbol{F} + \boldsymbol{F} \cdot \boldsymbol{\nabla} f$
8. $\boldsymbol{\nabla} \cdot (\boldsymbol{F} \times \boldsymbol{G}) = \boldsymbol{G} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) - \boldsymbol{F} \cdot (\boldsymbol{\nabla} \times \boldsymbol{G})$
9. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) = 0$
10. $\boldsymbol{\nabla} \times (f\boldsymbol{F}) = f\boldsymbol{\nabla} \times \boldsymbol{F} + \boldsymbol{\nabla} f \times \boldsymbol{F}$
11. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \mathbf{0}$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g$
13. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g) = 0$
14. $\boldsymbol{\nabla} \cdot (f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f) = f\nabla^2 g - g\nabla^2 f$
15. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F}$

Note:

The identities require $f, g, \boldsymbol{F}$ and $\boldsymbol{G}$ to be suitably differentiable, either order C^1 or C^2.

Grad, Div, Curl, and Laplacian in Orthogonal Curvilinear Coordinates

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a C^2 scalar function and $\boldsymbol{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a C^1 vector field where

$$\boldsymbol{F}(u_1, u_2, u_3) = F_1(u_1, u_2, u_3)\boldsymbol{e}_1 + F_2(u_1, u_2, u_3)\boldsymbol{e}_2 + F_3(u_1, u_2, u_3)\boldsymbol{e}_3.$$

Then

1. $\boldsymbol{\nabla} f = \dfrac{1}{h_1}\dfrac{\partial f}{\partial u_1}\,\boldsymbol{e}_1 + \dfrac{1}{h_2}\dfrac{\partial f}{\partial u_2}\,\boldsymbol{e}_2 + \dfrac{1}{h_3}\dfrac{\partial f}{\partial u_3}\,\mathbf{e_3}$

2. $\boldsymbol{\nabla} \cdot \boldsymbol{F} = \dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial\left(h_2 h_3 F_1\right)}{\partial u_1} + \dfrac{\partial\left(h_1 h_3 F_2\right)}{\partial u_2} + \dfrac{\partial\left(h_1 h_2 F_3\right)}{\partial u_3}\right]$

3. $\boldsymbol{\nabla} \times \boldsymbol{F} = \dfrac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$

4. $\nabla^2 f = \dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial}{\partial u_1}\left(\dfrac{h_2 h_3}{h_1}\dfrac{\partial f}{\partial u_1}\right) + \dfrac{\partial}{\partial u_2}\left(\dfrac{h_1 h_3}{h_2}\dfrac{\partial f}{\partial u_2}\right) + \dfrac{\partial}{\partial u_3}\left(\dfrac{h_1 h_2}{h_3}\dfrac{\partial f}{\partial u_3}\right)\right]$

Note: Equations 1-4 reduce to the usual expressions for cartesian coordinates if

$$h_1 = h_2 = h_3 = 1; \quad \boldsymbol{e}_1 = \boldsymbol{i}, \boldsymbol{e}_2 = \boldsymbol{j}, \boldsymbol{e}_3 = \boldsymbol{k}; \quad (u_1, u_2, u_3) = (x, y, z).$$

Cylindrical Coordinates

Cylindrical coordinates (ρ, ϕ, z) are defined by

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$$

where $\rho \geq 0$, $0 \leq \phi \leq 2\pi$. Then $(u_1, u_2, u_3) = (\rho, \phi, z)$ and $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$. Equations 1-4 reduce to:

1. $\boldsymbol{\nabla} f = \dfrac{\partial f}{\partial \rho}\,\hat{\boldsymbol{\rho}} + \dfrac{1}{\rho}\dfrac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}} + \dfrac{\partial f}{\partial z}\,\hat{\boldsymbol{z}}$

2. $\boldsymbol{\nabla} \cdot \boldsymbol{F} = \dfrac{1}{\rho}\left[\dfrac{\partial(\rho F_1)}{\partial \rho} + \dfrac{\partial F_2}{\partial \phi} + \dfrac{\partial(\rho F_3)}{\partial z}\right] = \dfrac{1}{\rho}\left(F_1 + \rho\dfrac{\partial F_1}{\partial \rho}\right) + \dfrac{1}{\rho}\dfrac{\partial F_2}{\partial \phi} + \dfrac{\partial F_3}{\partial z}$

3. $$\boldsymbol{\nabla} \times \boldsymbol{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ F_1 & \rho F_2 & F_3 \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\left(\frac{\partial F_3}{\partial \phi} - \frac{\partial(\rho F_2)}{\partial z}\right)\hat{\boldsymbol{\rho}} - \left(\frac{\partial F_3}{\partial \rho} - \frac{\partial F_1}{\partial z}\right)\rho\hat{\boldsymbol{\phi}} + \left(\frac{\partial(\rho F_2)}{\partial \rho} - \frac{\partial F_1}{\partial \phi}\right)\hat{\boldsymbol{z}}\right]$$

4. $\nabla^2 f = \dfrac{1}{\rho}\left[\dfrac{\partial}{\partial \rho}\left(\rho\dfrac{\partial f}{\partial \rho}\right) + \dfrac{\partial}{\partial \phi}\left(\dfrac{1}{\rho}\dfrac{\partial f}{\partial \phi}\right) + \dfrac{\partial}{\partial z}\left(\rho\dfrac{\partial f}{\partial z}\right)\right] = \dfrac{1}{\rho}\dfrac{\partial f}{\partial \rho} + \dfrac{\partial^2 f}{\partial \rho^2} + \dfrac{1}{\rho^2}\dfrac{\partial^2 f}{\partial \phi^2} + \dfrac{\partial^2 f}{\partial z^2}$

Spherical Coordinates

Spherical coordinates (r, θ, ϕ) are defined by

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta$$

where $r \geq 0,\ 0 \leq \theta \leq \pi,\ 0 \leq \phi \leq 2\pi$. Then $(u_1, u_2, u_3) = (r, \theta, \phi)$ and $h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$. Equations 1-4 reduce to:

1. $\boldsymbol{\nabla} f = \dfrac{\partial f}{\partial r}\,\hat{\boldsymbol{r}} + \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}} + \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial \phi}\,\hat{\boldsymbol{\phi}}$

2. $\boldsymbol{\nabla} \cdot \boldsymbol{F} = \dfrac{1}{r^2\sin\theta}\left[\dfrac{\partial\left(r^2\sin\theta F_1\right)}{\partial r} + \dfrac{\partial\left(r\sin\theta F_2\right)}{\partial \theta} + \dfrac{\partial\left(rF_3\right)}{\partial \phi}\right]$

$$= \frac{1}{r^2}\frac{\partial\left(r^2F_1\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta F_2\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_3}{\partial\phi}$$

3. $\boldsymbol{\nabla} \times \boldsymbol{F} = \dfrac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_1 & rF_2 & r\sin\theta F_3 \end{vmatrix}$

$$= \frac{1}{r^2\sin\theta}\left[\left(\frac{\partial\left(r\sin\theta F_3\right)}{\partial\theta} - \frac{\partial\left(rF_2\right)}{\partial\phi}\right)\hat{\boldsymbol{r}} - \left(\frac{\partial\left(r\sin\theta F_3\right)}{\partial r} - \frac{\partial F_1}{\partial\phi}\right)r\hat{\boldsymbol{\theta}} + \left(\frac{\partial\left(rF_2\right)}{\partial r} - \frac{\partial F_1}{\partial\theta}\right)r\sin\theta\hat{\boldsymbol{\phi}}\right]$$

4. $\nabla^2 f = \dfrac{1}{r^2\sin\theta}\left[\dfrac{\partial}{\partial r}\left(r^2\sin\theta\,\dfrac{\partial f}{\partial r}\right) + \dfrac{\partial}{\partial\theta}\left(\sin\theta\,\dfrac{\partial f}{\partial\theta}\right) + \dfrac{\partial}{\partial\phi}\left(\dfrac{1}{\sin\theta}\dfrac{\partial f}{\partial\phi}\right)\right]$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$