MAST20009 Assign 4 Sem 1, 2020

Q1 Wire W parametrised by

$\underline{c}(t) = t\underline{i} + 2t\underline{j} + \frac{2}{3}t^{3/2}\underline{k}, \quad 0 \le t \le 2$

and mass density $\mu = 3\sqrt{5+z}$

$\Rightarrow \underline{c}'(t) = (1, 2, \sqrt{t})$

$\Rightarrow \frac{ds}{dt} = \|\underline{c}'(t)\| = \sqrt{1+4+t} = \sqrt{5+t}$

and $x(t) = t, \quad y(t) = 2t, \quad z(t) = \frac{2}{3}t^{3/2}$

$\Rightarrow \mu = 3\sqrt{5+t}$

* Mass wire $= \int_{\underline{c}} \mu \, ds$

$= \int_0^2 \mu \frac{ds}{dt} \, dt$

$= \int_0^2 3\sqrt{5+t}\sqrt{5+t} \, dt$

$= 3\int_0^2 5+t \, dt$

$= 3\left[5t + \frac{1}{2}t^2\right]_{t=0}^{t=2}$

$= 3(10+2)$

$= 36$ (units)

* $\int_{\underline{c}} x\mu \, ds = \int_0^2 x\mu \frac{ds}{dt} \, dt$

$= \int_0^2 3t\sqrt{5+t}\sqrt{5+t} \, dt$

$= \int_0^2 3t(5+t) \, dt$

$= \int_0^2 15t + 3t^2 \, dt$

$= \left[\frac{15}{2}t^2 + t^3\right]_{t=0}^{t=2}$

$= 30 + 8$

$= 38$

* $\int_C y\nu \, ds = \int_0^2 y\nu \frac{ds}{dt} \, dt$

$= \int_0^2 (2t)\, 3\sqrt{5+t}\, \sqrt{5+t} \, dt$

$= 2\int_0^2 3t\,(5+t)\, dt$

$= 2 \times 38$ from $\int_C x\nu \, ds$

$= 76$

* $\int_C z\nu \, ds = \int_0^2 z\nu \frac{ds}{dt} \, dt$

$= \int_0^2 \frac{2}{3} t^{3/2} \cdot 3\sqrt{5+t}\, \sqrt{5+t} \, dt$

$= 2\int_0^2 t^{3/2}(5+t)\, dt$

$= 2\int_0^2 5t^{3/2} + t^{5/2} \, dt$

$= 2\left[\cancel{5}\cdot\frac{2}{\cancel{5}} t^{5/2} + \frac{2}{7} t^{7/2}\right]_{t=0}^{t=2}$

$= 4\left[2^{5/2} + \frac{2^{7/2}}{7}\right]$

$= 4\left[4\sqrt{2} + \frac{8\sqrt{2}}{7}\right]$

$= 16\sqrt{2}\,(1 + 2/7)$

$= 16\sqrt{2}\left(\frac{9}{7}\right)$

$= \frac{144\sqrt{2}}{7}$

* Centre of mass of wire is

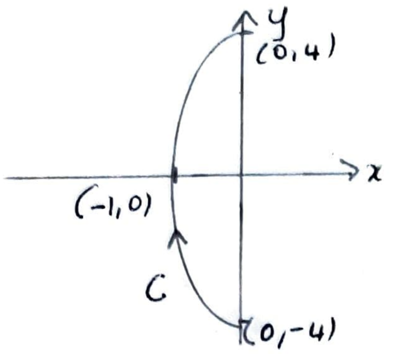
$(x_c, y_c, z_c) = \left(\frac{38}{36}, \frac{76}{36}, \frac{\overset{4}{\cancel{144}}\sqrt{2}}{7(\cancel{36})}\right)$

$= \left(\frac{19}{18}, \frac{38}{18}, \frac{4\sqrt{2}}{7}\right)$

$= \left(\frac{19}{18}, \frac{19}{9}, \frac{4\sqrt{2}}{7}\right)$

$\boxed{Q2}$ $C: \; 16x^2 + y^2 = 16$, $(0,-4)$ to $(0,4)$ clockwise. , $\underline{F} = 2y\,\underline{i} + 3x\,\underline{j}$

(a)



Let $x(t) = \cos t$, $y(t) = -4\sin t$, $\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$

$\Rightarrow \quad \underline{c}(t) = (\cos t, -4\sin t)$, $\frac{\pi}{2} \leq t \leq \frac{3\pi}{2}$

(b) Now

$\underline{c}'(t) = (-\sin t, -4\cos t)$

and $F[\underline{c}(t)] = (-8\sin t, 3\cos t)$

Work done $= \int_C \underline{F} \cdot d\underline{s}$

$= \int_{\pi/2}^{3\pi/2} F[\underline{c}(t)] \cdot \underline{c}'(t)\, dt$

$= \int_{\pi/2}^{3\pi/2} (-8\sin t, 3\cos t) \cdot (-\sin t, -4\cos t)\, dt$

$= \int_{\pi/2}^{3\pi/2} 8\sin^2 t - 12\cos^2 t\, dt$

Now $\cos(2t) = 2\cos^2 t - 1$, $\cos(2t) = 1 - 2\sin^2 t$

$= \int_{\pi/2}^{3\pi/2} 4(1 - \cos(2t)) - 6(\cos(2t) + 1)\, dt$

$= \int_{\pi/2}^{3\pi/2} -2 - 10\cos(2t)\, dt$

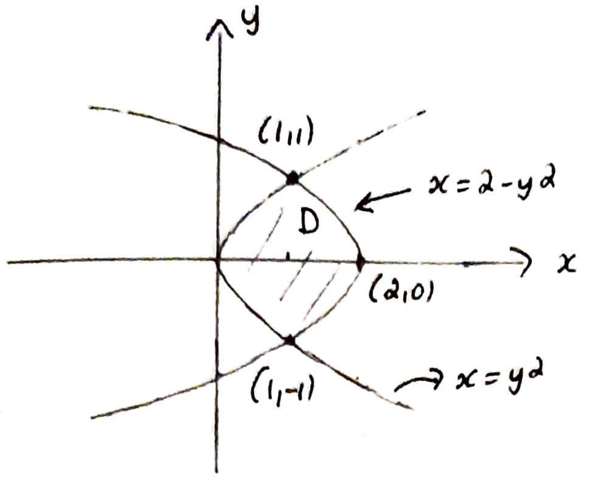
$= \left[-2t - \underbrace{5\sin(2t)}_{\to 0}\right]_{t=\pi/2}^{t=3\pi/2}$

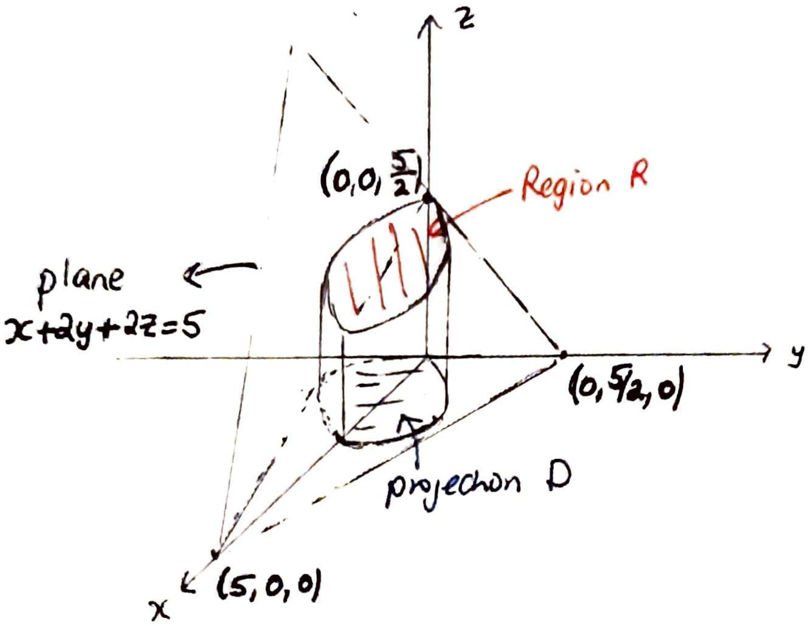
$= -2\left(\frac{3\pi}{2} - \frac{\pi}{2}\right)$

$= -2\pi$

$\boxed{Q3}$ R: part of $x+2y+2z=5$ cut by cylinder with walls $x=y^2$ and $x=2-y^2$.

(a) D: Cross section of cylinder in xy plane.





(b) <u>Method 1</u>

Parametrise R by

$$x=u, \quad y=v, \quad z=\frac{5-x-2y}{2}=\frac{5}{2}-\frac{u}{2}-v$$

A normal to plane is

$$\underline{T_u}\times\underline{T_v}=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & -1/2 \\ 0 & 1 & -1 \end{vmatrix}$$

$$=\underline{i}(1/2)-\underline{j}(-1)+\underline{k}(1)$$

$\Rightarrow |\underline{T_u} \times \underline{T_v}| = \sqrt{\frac{1}{4}+1+1} = \sqrt{\frac{9}{4}} = \frac{3}{2}$

So $\underset{R}{\text{Area}} = \iint_R 1 \, dS$

$= \iint_D |\underline{T_u} \times \underline{T_v}| \, du dv$

Where D is projection into xy plane shown in part (a).

$= \frac{3}{2} \iint_D du \, dv$

$= \frac{3}{2} \iint_D dx \, dy \quad (x=u, \ v=y)$

<u>Method 2</u>

Special case formula where $z = f(x,y) = \frac{5-x-2y}{2} = \frac{5}{2} - \frac{x}{2} - y$

Now normal to plane is $\underline{n} = (1, 2, 2)$

$\Rightarrow |\underline{n}| = \sqrt{1+4+4} = \sqrt{9} = 3$

$\Rightarrow \hat{\underline{n}} = \frac{1}{3}(1, 2, 2)$

$\Rightarrow \hat{\underline{n}} \cdot \underline{k} = \frac{2}{3}$

So Area $R = \iint_R 1 \, dS$

$= \iint_D \frac{1}{|\hat{\underline{n}} \cdot \underline{k}|} \, dx \, dy$

$= \frac{3}{2} \iint_D dx \, dy$

Where D is projection into xy plane shown in part (a)

To evaluate integral use horizontal strips to describe D

$$y^2 \leq x \leq 2-y^2$$
$$-1 \leq y \leq 1$$

So Area $R = \frac{3}{2}\int_{-1}^{1}\int_{y^2}^{2-y^2} dx\, dy$

$$= \frac{3}{2}\int_{-1}^{1}\left[x\right]_{x=y^2}^{x=2-y^2} dy$$

$$= \frac{3}{2}\int_{-1}^{1} 2-y^2-y^2\, dy$$

$$= \frac{3}{2}\int_{-1}^{1} 2-2y^2\, dy$$

$$= 3\int_{-1}^{1} 1-y^2\, dy$$

$$= 3\left[y-\frac{1}{3}y^3\right]_{y=-1}^{y=1}$$

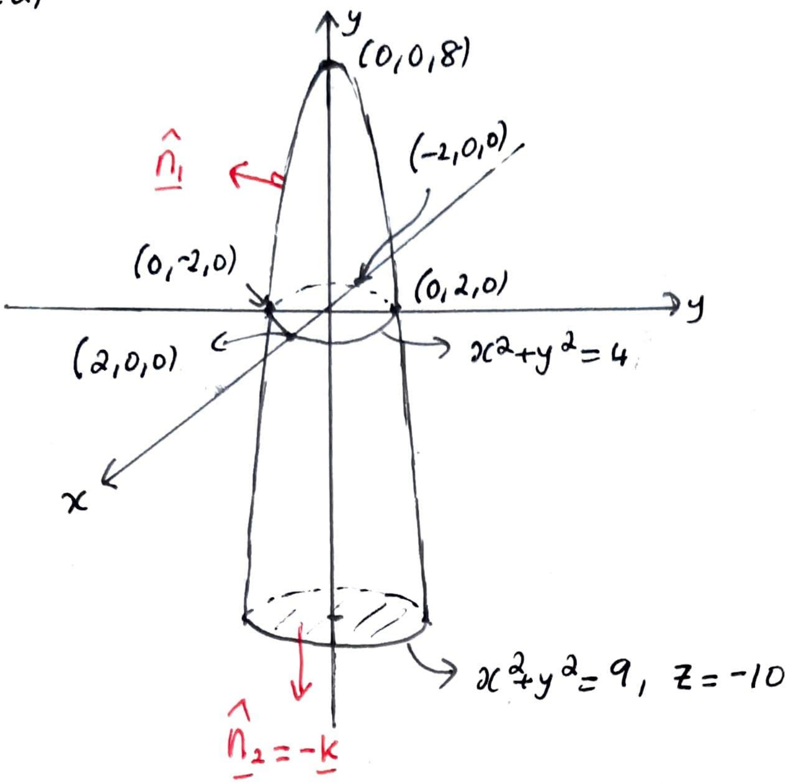
$$= 3\left[\left(1-1/3\right)-\left(-1+\frac{1}{3}\right)\right]$$

$$= 3\left(\frac{2}{3}+\frac{2}{3}\right)$$

$$= 4 \quad (\text{units})^2$$

$\boxed{Q4}$ S: $z = 8 - 2x^2 - 2y^2$ and base at $z = -10$. , outward normal.

(a)



(b) If $T = x^2 + y^2 + 3(z-2)^2$

$$\Rightarrow \underline{H} = -k\underline{\nabla T} = -k\,(2x,\ 2y,\ 6(z-2))$$

If $k=1$ on dome and $k=3$ on floor then

$$\underline{H} = \begin{cases} (-2x,\ -2y,\ -6z+12) & \text{on dome} \\ (-6x,\ -6y,\ -18z+36) & \text{on floor} \end{cases}$$

Consider surfaces of dome and floor separately

* $\boxed{S_1 - \text{dome},\ z = 8 - 2x^2 - 2y^2}$

Let $x = \rho\cos\phi$, $y = \rho\sin\phi$

$$\Rightarrow z = 8 - 2\rho^2\cos^2\phi - 2\rho^2\sin^2\phi = 8 - 2\rho^2$$

so $\Phi(\rho,\phi) = (\rho\cos\phi,\ \rho\sin\phi,\ 8-2\rho^2)$

where $0 \le \rho \le 3$, $0 \le \phi \le 2\pi$

Now $\underline{T_\rho} \times \underline{T_\phi} = \begin{vmatrix} \underset{\sim}{i} \oplus & \underset{\sim}{j} \ominus & \underset{\sim}{k} \oplus \\ \cos\phi & \sin\phi & -4\rho \\ -\rho\sin\phi & \rho\cos\phi & 0 \end{vmatrix}$

$= \underset{\sim}{i}\,(4\rho^2\cos\phi) - \underset{\sim}{j}\,(-4\rho^2\sin\phi) + \underset{\sim}{k}\,(\rho\cos^2\phi + \rho\sin^2\phi)$

$= \underset{\sim}{i}\,(4\rho^2\cos\phi) + \underset{\sim}{j}\,(4\rho^2\sin\phi) + \underset{\sim}{k}\,\rho$

From picture outward normal has positive $\underset{\sim}{k}$ component.

Since $\rho \geq 0$ then $\underline{T_\rho} \times \underline{T_\phi}$ is <u>outward</u> normal to dome.

So heat flux across S_1 (dome) is

$$\iint_{S_1} \underline{H} \cdot d\underline{S} = \int_0^3 \int_0^{2\pi} \underline{H} \cdot (\underline{T_\rho} \times \underline{T_\phi})\, d\phi\, d\rho$$

$$= \int_0^3 \int_0^{2\pi} (-2\rho\cos\phi, -2\rho\sin\phi, -48 + 12\rho^2 + 12) \cdot (4\rho^2\cos\phi, 4\rho^2\sin\phi, \rho)\, d\phi\, d\rho$$

$$= \int_0^3 \int_0^{2\pi} -8\rho^3\cos^2\phi - 8\rho^3\sin^2\phi - 36\rho + 12\rho^3\, d\phi\, d\rho$$

$$= \int_0^3 \int_0^{2\pi} -8\rho^3 - 36\rho + 12\rho^3\, d\phi\, d\rho$$

$$= \int_0^3 \int_0^{2\pi} 4\rho^3 - 36\rho\, d\phi\, d\rho$$

$$= 2\pi \int_0^3 4\rho^3 - 36\rho\, d\rho$$

$$= 2\pi \left[\rho^4 - 18\rho^2\right]_{\rho=0}^{\rho=3}$$

$$= 2\pi\,(81 - 162)$$

$$= -162\pi$$

$\boxed{S_2 - \text{floor}, \quad z=-10 \quad x^2+y^2 \leq 9}$

On floor $\hat{\underline{n}} = -\underline{k}$

So heat flux across S_2 (disk) is

$$\iint_{S_2} \underline{H} \cdot d\underline{S} = \iint_{S_2} \underline{H} \cdot \hat{\underline{n}}\, dS$$

$$= \iint_{S_2} (-6x, -6y, -18z+36) \cdot (0,0,-1)\, dS$$

$$= \iint_{S_2} 18z - 36\, dS$$

On S_2 $z=-10 \Rightarrow 18z-36 = -180-36 = -216$

$$= \iint_{S_2} -216\, dS$$

$$= -216 \text{ area}(S_2)$$

As floor is a disk of radius 3, its area is 9π

$$= (-216)(9\pi)$$

$$= -1944\pi$$

$\boxed{S - \text{dome} + \text{floor}}$

Combining the heat flux across S is

$$\iint_S \underline{H} \cdot d\underline{S} = \iint_{S_1} \underline{H} \cdot d\underline{S} + \iint_{S_2} \underline{H} \cdot d\underline{S}$$

$$= -162\pi - 1944\pi$$

$$= -2106\pi$$