THE UNIVERSITY OF MELBOURNE

Student Number	

Semester 1 Assessment, 2019

School of Mathematics and Statistics

MAST20009 Vector Calculus

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 5 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- No written or printed materials may be brought into the examination.
- No calculators of any kind may be brought into the examination.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- There are 11 questions on this exam paper.
- All questions may be attempted.
- Marks for each question are indicated on the exam paper.
- Start each question on a new page.
- Clearly label each page with the number of the question that you are attempting.
- There is a separate 3 page formula sheet accompanying the examination paper, which you may use in this examination.
- The total number of marks available is 120.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.
- Initially students are to receive the exam paper, the 3 page formula sheet, and two 14 page script books.

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Question 1 (9 marks)

Consider the function

$$f(x,y) = \begin{cases} \dfrac{x^2 - 3y^3}{5x^2 + 2y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

(a) Calculate $\dfrac{\partial f}{\partial y}$ if $(x,y) \neq (0,0)$.

(b) Calculate $\dfrac{\partial f}{\partial y}$ if $(x,y) = (0,0)$.

(c) Evaluate

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}$$

if it exists. If the limit does not exist, explain why it does not exist.

(d) Is $\dfrac{\partial f}{\partial y}$ continuous at $(0,0)$? Explain briefly.

(e) Is f of order C^1 at $(0,0)$? Explain briefly.

Question 2 (8 marks)

Consider the function

$$g(x,y) = \log\left(4 - 2x + y\right).$$

(a) Determine the first order Taylor polynomial for g about the point $(1,-1)$.

(b) Using part (a), approximate $g(1.1, -0.9)$.

(c) Determine an upper bound for the error in your approximation in part (b).

Question 3 (12 marks)

(a) Consider the vector field

$$\mathbf{F}(x,y) = x^3\mathbf{i} - y^3\mathbf{j}.$$

(i) Sketch the vector field at the points $(1,1)$, $(-1,2)$ and $(0,-1)$.

(ii) Determine the equation for the flow line of $\mathbf{F}$ passing through the point $(2,1)$ in terms of x and y.

(b) Let $\mathbf{T}$ be the unit tangent vector, $\mathbf{N}$ be the unit principal normal vector and $\mathbf{B}$ be the unit binormal vector to a C^3 path. Prove the *Frenet-Serret* formula

$$\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}s} = \tau\mathbf{B} - \kappa\mathbf{T}$$

where κ is the curvature of the path and τ is the torsion of the path.

Hint: Differentiate $\mathbf{B} \times \mathbf{T}$ with respect to arclength.

Question 4 (15 marks)

(a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be C^1 non-zero scalar functions. Prove the vector identity

$$\boldsymbol{\nabla}\left(\frac{f}{g}\right) = \frac{g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g}{g^2}.$$

(b) Consider the vector field $\mathbf{G}$ given by

$$\mathbf{G}(x,y,z) = 2y^2z\mathbf{i} - 3xz^2\mathbf{j} + 4xy^4\mathbf{k}.$$

(i) Show that $\mathbf{G}$ is an incompressible vector field.

(ii) Give a physical interpretation for an incompressible vector field.

(iii) Determine a vector field

$$\mathbf{F}(x,y,z) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j}$$

such that

$$\mathbf{G} = \boldsymbol{\nabla} \times \mathbf{F}.$$

Question 5 (11 marks)

Consider the double integral

$$\int_0^2 \int_{\frac{x}{2}}^{1+\frac{x}{2}} (2x-1)^5(2y-x)^2 \cosh[(2y-x)^3]\, dydx.$$

(a) Sketch the region of integration, clearly labelling any vertices.

(b) Evaluate the double integral by making the substitutions $u = 2x$ and $v = 2y - x$.

Question 6 (7 marks)

Let V be the solid region bounded by the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 2$.

Find the total mass of V if the mass density is $\mu = (x^2+y^2+z^2)^{\frac{5}{2}}$ grams per unit volume.

Question 7 (11 marks)

Let S be the cone

$$z = 1 + \sqrt{x^2+y^2} \quad \text{for} \quad 1 \le z \le 5.$$

(a) Sketch S.

(b) Write down a parametrisation for S based on cylindrical coordinates.

(c) Using part (b), find an outward normal vector to S.

(d) Determine the cartesian equation of the tangent plane to S at $(1,0,2)$.

Question 8 (10 marks)

Let D be the region bounded by the curves $y = \sqrt{2 - x^2}$ and $y = |x|$. Let C be the boundary of D, traversed anticlockwise. Let $\hat{\mathbf{n}}$ be the outward unit normal to C in the x-y plane.

(a) Sketch C, indicating the direction of $\hat{\mathbf{n}}$ on each arc of C.

(b) Let $\mathbf{F}(x, y) = (2x^3 y + \sin^4(2y) + xy^2, x^5 \cos^3(2x) - 3x^2y^2)$. Evaluate the path integral

$$\int_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds.$$

Question 9 (18 marks)

(a) State Stokes' theorem. Explain all symbols used and any required conditions.

(b) Let S be the surface of the paraboloid

$$z = x^2 + y^2 + 3 \quad \text{for} \quad z \leq 7.$$

Sketch S.

(c) Let $\mathbf{F}(x, y, z) = (3y + z)\mathbf{i} + y\mathbf{j} + (z^2 - x^4)\mathbf{k}$ and S be the surface in part (b).

Evaluate the surface integral

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S}$$

using

(i) Stokes' theorem and a line integral;

(ii) a surface integral over the simplest surface.

Question 10 (10 marks)

(a) Let R be a solid region bounded by an oriented closed surface ∂R. Let $f(x, y, z)$ and $g(x, y, z)$ be C^2 scalar functions. Let $\hat{\mathbf{n}}$ be the unit outward normal to ∂R. Show that

$$\iiint_R \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g\, dV = \iint_{\partial R} f \boldsymbol{\nabla} g \cdot d\mathbf{S} - \iiint_R f \nabla^2 g\, dV.$$

(b) Suppose that ∂R is a sphere of radius R_0 centred at the origin. Let $f(r) = r$ and $g(r) = r^2$ where $r = \sqrt{x^2 + y^2 + z^2}$.

(i) Find $\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g$ and $\nabla^2 g$.

(ii) Using parts (a) and (i), show that

$$\iiint_R 4r\, dV = \iint_{\partial R} r^2\, dS.$$

Question 11 (9 marks)

Define *oblate spheroidal* coordinates (u, θ, ϕ) by

$$x = \cosh u \cos\theta \cos\phi, \quad y = \cosh u \cos\theta \sin\phi, \quad z = \sinh u \sin\theta$$

where $u \geq 0$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $0 \leq \phi < 2\pi$.

(a) Let $\boldsymbol{r} = (x, y, z)$. Find $\dfrac{\partial \boldsymbol{r}}{\partial u}$, $\dfrac{\partial \boldsymbol{r}}{\partial \theta}$ and $\dfrac{\partial \boldsymbol{r}}{\partial \phi}$.

(b) Show that the scale factors are

$$\begin{aligned} h_u &= h_\theta = \sqrt{\sinh^2 u + \sin^2\theta}, \\ h_\phi &= \cosh u \cos\theta. \end{aligned}$$

(c) Find an expression for

$$\boldsymbol{\nabla}(u^2\theta^3 + \phi^4)$$

in terms of u, θ and ϕ.

End of Exam—Total Available Marks = 120