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The University of Melbourne

Department of Mathematics and Statistics

MAST20004 Probability

Semester 1 Exam — June 27, 2013

Exam Duration: 3 Hours

Reading Time: 15 Minutes

This paper has 5 pages

Authorised materials:
Students may bring one double-sided A4 sheet of handwritten notes into the exam room. Hand-held electronic calculators may be used.

Instructions to Invigilators:
Students may take this exam paper with them at the end of the exam.

Instructions to Students:
This paper has **nine** (9) questions.
Attempt as many questions, or parts of questions, as you can.
The approximate number of marks allocated to each question is shown in the brackets after the question statement.
The total number of marks available for this examination is 100.
Working and/or reasoning must be given to obtain full credit.

1. (a) Consider a random experiment with sample space Ω. State the axioms that must be satisfied by a probability mapping P defined on the set $\mathcal{A}$ of events of the random experiment.

   (b) Now assume that

$$\Omega = \{a, b, c, d\}$$

   and $\mathcal{A}$ is the set of subsets of Ω.

   (i) How many elements does the set $\mathcal{A}$ have?

   (ii) Consider the mapping P from $\mathcal{A}$ to $[0, 1]$ which is such that

$$P(E) = \begin{cases} 1/4 & \text{if } E = \{a\} \\ 1/2 & \text{if } E = \{a, b\} \\ 2/3 & \text{if } E = \{a, c\}. \end{cases}$$

   With reference to the axioms you wrote in part (a), give the values of $P(E)$ for all the other sets $E \in \mathcal{A}$.

[9 marks]

2. (a) Let A and B be events in a random experiment. State Bayes' Formula for the probability $P(B|A)$.

   (b) Consider a test for the presence of a particular disease. Suppose that if a person has the disease, then there is a probability of .95 that the test is positive. If a person does not have the disease, there is still a .05 probability that the test is positive. Suppose also that the probability that a randomly-chosen individual from the population has the disease is .002.

   (i) What is the probability that a randomly-chosen individual who tests positive actually has the disease?

   (ii) Giving your reasons, state whether such a test likely to be useful in diagnosing cases of the disease?

[6 marks]

3. In the Oz Lotto draw, each player selects seven numbers from the numbers $1, \ldots, 45$. The lottery is decided when the Lotteries Commission randomly draws seven numbers without replacement, also from the numbers $1, \ldots, 45$.

   (a) Let X be the number of a player's numbers that are drawn by the Lotteries Commission. Giving your reasons, name the distribution of the random variable X and state the value of any parameter(s).

   (b) For $x = 0, \ldots, 7$, write down the probability mass function of X as a function of x.

(c) Suppose that 80,000,000 games will be played in the Oz Lotto draw during a given week. Let M be the number of games that win the division 1 prize, for which all seven of a player's numbers need to be drawn. Assuming that the numbers picked in separate games are independent, name the distribution of the random variable M and state the value of any parameter(s).

(d) Write down an expression for $P(M > 0)$.

(e) Use a Poisson approximation to evaluate this probability.

(f) Let the random variable N be the number of weeks that Oz Lotto 'jackpots', that is the number of weeks where the division 1 prize is not won, before it is eventually won. Assuming that the number of games played each week is the same as in part (c), name the distribution of the random variable N and state the value of any parameter(s).

(g) What are the values of $E(N)$ and $V(N)$?

[16 marks]

4. Let $X \stackrel{d}{=} \mathrm{R}(0,1)$ and $Y = e^X$.

(a) Write down $E(X)$ and $V(X)$.

(b) Describe the set S_Y of possible values of the random variable Y.

(c) Write down the distribution function $F_Y(y)$ and density function $f_Y(y)$ of Y.

(d) Calculate $E(Y)$,

(i) by evaluating $\int_{S_X} e^x f_X(x)dx$, and

(ii) by evaluating $\int_{S_Y} y f_Y(y)dy$.

(e) Calculate $V(Y)$.

(f) Using appropriate Taylor series expansions of e^x, give approximations for $E(Y)$ and $V(Y)$. Compare this with the exact values that you calculated in parts (d) and (e).

[14 marks]

5. (a) Let X and Y have joint probability density function of the form

$$f_{(X,Y)}(x,y) = \begin{cases} kxy & \text{if } 0 \le x \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Giving your reasons, state whether X and Y are dependent or independent.

(ii) What is the value of k?

(iii) Carefully stating the regions in which your expressions apply, find the marginal probability density functions of X and Y.

(iv) Find the conditional probability density function of Y given $X = \frac{1}{2}$.

(v) Find $P(Y \le \frac{3}{4} | X \le \frac{1}{2})$.

(b) Consider independent continuous random variables X and Y with probability distribution functions $F_X(x)$ and $F_Y(y)$ and probability density functions $f_X(x)$ and $f_Y(y)$ respectively. Let S_X be the set of possible values of X.

  (i) Justifying all your steps, derive the expression

$$F_Z(z) = \int_{x \in S_X} F_Y(z - x) f_X(x) dx$$

  for the probability distribution function $F_Z(z)$ of $Z = X + Y$.

  (ii) Hence, or otherwise, derive the probability distribution function $F_Z(z)$ for $Z = X + Y$ where $Y \stackrel{d}{=} \exp(\alpha)$ and $X \stackrel{d}{=} \exp(\alpha)$.

[16 marks]

6. Let $X \stackrel{d}{=} \mathrm{R}(0,1)$ and the random variable Y be independent of X and equal to 1 with probability 1/2 and 2 with probability 1/2. Define $Z = X^Y$.
   (a) For $y = 1$ and $y = 2$, compute the function $\eta(y) = E(Z \mid Y = y)$.
   (b) Similarly, compute the function $\zeta(y) = \mathrm{V}(Z \mid Y = y)$.
   (c) Hence specify the probability mass function of the random variables $E(Z \mid Y)$ and $\mathrm{V}(Z \mid Y)$.
   (d) Compute $E(Z)$.
   (e) Compute $\mathrm{V}(Z)$.

[10 marks]

7. Consider random variables R and Θ that have the joint probability density function

$$f_{(R,\Theta)}(r, \theta) = \frac{re^{-\frac{r^2}{2}}}{2\pi},$$

   for $r > 0$ and $0 < \theta \leq 2\pi$.
   (a) Write down the marginal probability density functions $f_R(r)$ and $f_\Theta(\theta)$ of R and Θ.
   (b) Write down the marginal probability distribution functions $F_R(r)$ and $F_\Theta(\theta)$ of R and Θ.
   (c) Justifying your reasoning, state whether R and Θ are independent.
   (d) Assume that you have a means of generating random variables U that have a continuous uniform distribution on the interval $(0, 1)$.
      (i) Explain how you would transform such a random variable U_1 to generate a random variable R having the marginal distribution $F_R(r)$ that you derived in part (b).
      (ii) Explain how you would transform such a random variable U_2 to generate a random variable Θ having the marginal distribution $F_\Theta(\theta)$ that you derived in part (b).
   (e) Let $X = R\cos\Theta$ and $Y = R\sin\Theta$. Name the distribution of the bivariate random variable (X, Y) (NB, You do not have to provide any working to justify your statement).

[9 marks]

8. Let Z be a standard normal random variable.
   (a) Derive the moment generating function $M_Z(t)$ of Z.
   (b) Use the fact that a random variable $X \stackrel{d}{=} N(\mu, \sigma^2)$ has the same distribution as $\mu + \sigma Z$ to write down the moment generating function of X.
   (c) For $i = 1, 2, \ldots, n$, let $X_i \stackrel{d}{=} N(\mu_i, \sigma_i^2)$ be mutually independent. Use moment generating functions to derive the distribution of the sum

$$S_n = \sum_{i=1}^{n} X_i.$$

   (d) Hence derive the distribution of the sample average

$$A_n = \frac{S_n}{n} = \frac{\sum_{i=1}^{n} X_i}{n}.$$

   (e) Now assume that $\mu_i = \mu$ and $\sigma_i^2 = \sigma^2$ for all i. What is the distribution of A_n?

[14 marks]

9. Consider the Branching Process $\{X_n, n = 0, 1, 2, 3, \ldots\}$ where X_n is the population size at the nth generation. Assume $P(X_0 = 1) = 1$ and that the probability generating function of the common offspring distribution is

$$A(z) = \frac{1}{10}(z^3 + 3z^2 + 2z + 4).$$

   (a) If $q_n = P(X_n = 0)$ for $n = 0, 1, \ldots$, write down an equation relating q_{n+1} and q_n. Hence, or otherwise, evaluate q_n for $n = 0, 1, 2$.
   (b) Find the extinction probability $q = \lim_{n\to\infty} q_n$.

[6 marks]

End of the exam