Executive summary Let V be a vector space.

- V has a basis.
- Any two bases of V have the same cardinality. (In the sense we discussed already, i.e. there is a bijection from one basis to the other.) If this cardinality is finite, it is called the *dimension of V*, and V is said to be *finite-dimensional*.
  We will focus almost exclusively on this situation. MAST300?? *Metric and Hilbert spaces* treats infinite-dimensional vector spaces.
- Any spanning set of V contains a basis of V.
- Any linearly independent subset of V can be extended to a basis of V.
- If V has dimension n, then
  - Any subset S with $\#S < n$ is **not** spanning V.
  - Any subset S with $\#S > n$ is **not** linearly independent.

In order to avoid circularity in the logic, we give a slightly different initial definition of finite-dimensionality: A vector space V is *finite-dimensional* if there exists a finite spanning subset S.

Fix a nonzero finite-dimensional vector space V.

Proposition 3.26. Given any linearly independent subset $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of V and any finite spanning set $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of V, we have $m \leq n$.

By contradiction, suppose $m > n$

Proof. The set $\{\underset{v_1}{\wedge} w_1, w_2, \cdots w_n\}$ is linearly dependent

$\mu v_1 + \lambda_1 w_1 + \cdots + \lambda_n w_n = 0$ with $\mu, \lambda_1, \cdots \lambda_n \in R$

At least two coefficient are nonzero $\Rightarrow$ can write w_i in terms of v_1 and the other w's

WLOG w_i is w_n so

$\{v_1, w_1 \cdots w_{n-1}\}$ span V

The set $\{v_1, v_2, w_1, \cdots w_{n-1}\}$ is linear dependent, from the

linear dependent related n, we can express one of the w_i's in terms of v_1, v_2 and the other w's, WLOG this is w_{n-1} so $\{v_1, v_2, w_1, \cdots w_{n-1}\}$

$\vdots$

$\vdots$

$\{v_1, v_2, \dots, v_{n-1}, w_1\}$ is linear independent

$\mu_1 v_1 + \mu_2 v_2 + \dots + \mu_{n-1} v_{n-1} + \mu_n v_n + \lambda w_1 = 0$

$\lambda \neq 0$

Proposition 3.27. Every subspace of V is finite-dimensional.

get w_1 in terms of $v_1, v_2, \dots v_n$

$\Rightarrow \{v_1, v_2, \dots, v_n\}$ span V

$v_{n+1} \in \text{Span}\{v_1, \dots v_n\}$

since $\{v_1, \dots v_{n+1}\}$ is linear independent

$\Rightarrow$ contradiction

Proof.

Since V is finite-dimensional, it has a finite spanning subset $\{w_1, \dots w_n\}$

let U be a subspace of V

if $U = \{0\}$, we're done

otherwise, let $v_1 \in U$, $v_1 \neq 0$

If $U = \text{span}\{v_1\}$, we're done

otherwise, let $v_2 \in U$, $v_2 \notin \text{span}\{v_1\}$

Then, $\{v_1, v_2\}$ is linearly independent

If $U = \text{span}\{v_1, v_2\}$, we're done

otherwise, let $v_3 \in U \setminus \text{span}\{v_1, v_2\}$

Then $\{v_1, v_2, v_3\}$ is linear independent

$\vdots$

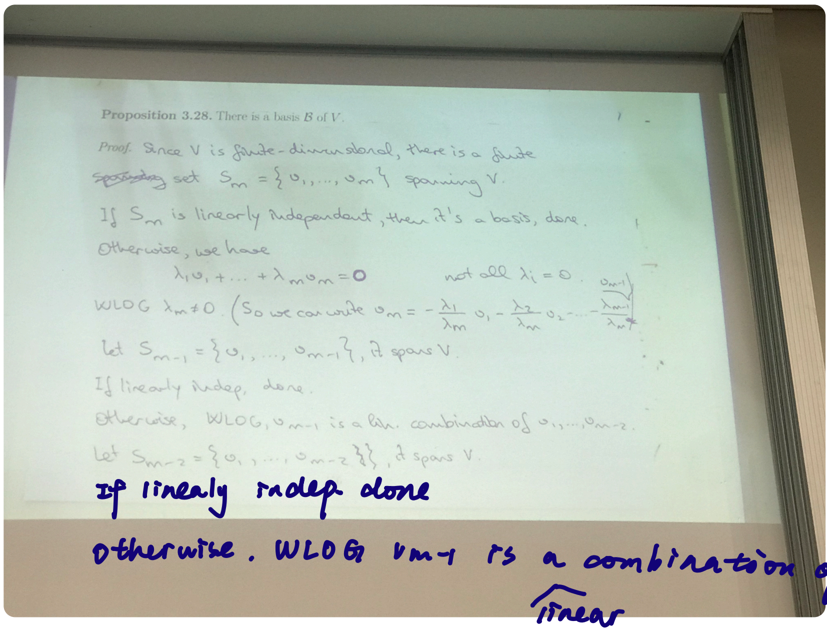
By prop 3.26, the size of any linear subset of U is $\leq n$ so the process must stop after n steps $\{v_1, v_2, \dots v_m\}$

Therefore $U = \text{span}\{v_1, v_2, \dots v_m\}$

$\Rightarrow U$ is finite dimensional

Proposition 3.28. There is a basis $\mathcal{B}$ of V.

Proof.



Proposition 3.28. There is a basis $\mathcal{B}$ of V.

Proof. Since V is finite-dimensional, there is a finite ~~spanning~~ set $S_m = \{v_1, \dots, v_m\}$ spanning V.

If S_m is linearly independent, then it's a basis, done.

Otherwise, we have

$$\lambda_1 v_1 + \dots + \lambda_m v_m = 0 \qquad \text{not all } \lambda_i = 0.$$

WLOG $\lambda_m \neq 0$. (So we can write $v_m = -\frac{\lambda_1}{\lambda_m} v_1 - \frac{\lambda_2}{\lambda_m} v_2 - \dots - \frac{\lambda_{m-1}}{\lambda_m} v_{m-1}$)

Let $S_{m-1} = \{v_1, \dots, v_{m-1}\}$, it spans V.

If linearly indep, done.

Otherwise, WLOG, v_{m-1} is a lin. combination of $v_1, \dots, v_{m-2}$.

Let $S_{m-2} = \{v_1, \dots, v_{m-2}\}$, it spans V.

If linealy indep done

otherwise, WLOG v_{m-1} is a linear combination of $v_1, \dots v_{m-2}$.

Note that the proof showed something of independent interest, which we record here:

Proposition 3.29. Any finite spanning set S of V contains a basis $\mathcal{B}$ of V.

let $\mathcal{B}_1$, $\mathcal{B}_2$ be basis of V

B_1 lin indep & B_2 spans V $\overset{3.26}{\Longrightarrow}$ size $B_1 \leq$ size B_2

B_2 lin indep & B_1 spans V $\overset{3.26}{\Longrightarrow}$ size $B_2 \leq$ size B_1

so size B_1 = size B_2

Define the dimension of V ~~to~~ be the cardinality of any basis of V

Proposition 3.30. Any two bases of V have the same size.

Proof.

Example 3.31. The standard basis of the space $M_{m\times n}$ is

$$M_{m\times n} := \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \middle| \ a_{ij} \in R \right\}$$

$$E_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ except } a_{ij} = 1 \text{ (ij entry) and the other } = 0$$

$\downarrow$

get $m \times n$ matrince

get $M_{m\times n} = mn$

Example 3.32. The standard basis of the space $\mathcal{P}_n$ is

$$\mathcal{P}_n = \{ a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in R \}$$

standard basis

$$\{ 1, x, x^2, x^3 \ldots x^n \}$$

$$\text{dimen } \mathcal{P}_n = n+1$$

3.6 Linear transformations

A guiding principle in mathematics is that, once you have a structure on a set, you should study functions that preserve this structure. We do this now for vector spaces, which are sets with the added structure of an addition and a scalar multiplication.

Let V and W be vector spaces (over the same field of scalars, say $\mathbb{R}$). A function $T\colon V \to W$ is called a *linear transformation* if it satisfies

(a) $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$

(b) $T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$ for all $\mathbf{v} \in V$ and all scalars λ.

f' $\int$ are linear transformations

$T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 T(v_1) + \lambda_2 T(v_2)$

Since linear transformations are functions, they can be injective, surjective, bijective, or none of the above.

A bijective linear transformation is also called an *isomorphism*; we then say that the spaces V and W are *isomorphic*.

= equal structure

同构的

Example:

① $id_V : V \to V$ function $id(v) = v$

② $0 : V \to V$

$0(v) = 0_V \quad \forall v \in V$

③ $\pi : \mathbb{R}^n \to \mathbb{R}$ $\pi(x_1, \cdots x_n) = x_1$

④ $v_1 = (x_1 \cdots x_n)$

$v_2 = (y_1 \cdots y_n)$

$\pi(v_1 + v_2) = \pi(x_1 + y_1, \cdots x_n + y_n) = x_1 + y_1$

$\pi(v_1) + \pi(v_2) = x_1 + y_1$

$\pi(\lambda v) = \pi(\lambda x_1, \cdots \lambda x_n) = \lambda x_1 = \lambda \pi(v)$

⑤ $T : \mathbb{R}^3 \to \mathbb{R}^2$

(a) $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y+z \\ y-z \end{bmatrix}$

$T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix} = \begin{bmatrix} \cdots \\ \cdots \end{bmatrix}$

$= [\ ] + [\ ]$

We can now clarify the effect of choosing a basis on a vector space.

Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis of a vector space V. Choose an ordering on these vectors, say $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$. We will refer to $\mathcal{B}$ as an *ordered basis* of V.

$(1,0) \neq (0,1)$ for tuple, sequence matter

$\{0,1\} = \{1,0\}$ for set, don't matter

Example 3.33. We know that $\{1, x, x^2\}$ is a linear independent and spanning set for the vector space $\mathcal{P}_2$. So $\mathcal{B} = (1, x, x^2)$ is an ordered basis.

Given any vector $f \in \mathcal{P}_2$, we have

by definition of P_2

$f = a_0 + a_1 x + a_2 x^2$

$= a_0(1) + a_1(x) + a_2(x^2)$

$[f]_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$ eg $f = 3 - 5x$

$[f]_B = \begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}$

let $B' = \{x^2, x, 1\}$ $f = 3 - 5x$

$[f]_{B'} = \begin{bmatrix} 0 \\ -5 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \neq \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$

not the same thing

(b) scalar multi ...

Any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is of the form

$T(v) = Av$ from some matrix $A \in M_{m \times n}$

How to get A?

Take standard basis $e_1, e_2 \cdots e_n$ for $\mathbb{R}^n$

$T(e_1) \in \mathbb{R}^m$

$T(e_2) \in \mathbb{R}^m$

$\vdots$

$T(e_n) \in \mathbb{R}^m$

⑥. Take a matrix $A \in M_{m \times n}$

Define $T : \mathbb{R}^n \to \mathbb{R}^m$

by $T(v) = Av$

v is column vector.

$T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

(a) if $v_1, v_2 \in \mathbb{R}^n$, then

$T(v_1 + v_2) = A(v_1 + v_2)$

$= Av_1 + Av_2$

$= T(v_1) + T(v_2)$

(b) if $v \in \mathbb{R}^n$

$A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}$

$T(\lambda v) = A(\lambda \cdot v) = A((\lambda I) \cdot v)$
$= (A(\lambda I)) \cdot v$
$= ((\lambda I) A) v$
$= \lambda A v$
$= \lambda T(v)$

This is a general phenomenon: Let $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ be an ordered basis of a vector space V.

Given any $\mathbf{w} \in V$, we can write it uniquely as a linear combination

$$\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n,$$

and we let

$$[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

be the *coordinate vector of* $\mathbf{w}$ *with respect to* $\mathcal{B}$.

This defines a function $\varphi_{\mathcal{B}} \colon V \to \mathbb{R}^n$ given by

since we can do this for each w.
we get a function

$$\varphi_{\mathcal{B}}(\mathbf{w}) = [\mathbf{w}]_{\mathcal{B}}$$

Proposition 3.34. The function $\varphi_{\mathcal{B}} \colon V \to \mathbb{R}^n$ is an invertible linear transformation.

check linear transformation

(a) $w_1, w_2 \in V$

$\varphi_B(w_1 + w_2) = [w_1 + w_2]_B = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$

$\varphi_B(w_1) = \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{= [w_1]_B} = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$

$\varphi_B(w_2) = [w_2]_B = b_1 v_1 + \cdots + b_n v_n$

$\varphi_B(w_1) + \varphi_B(w_2) = (a_1 + b_1) v_1 + \cdots + (a_n + b_n) v_1$

(b) let $\lambda \in \mathbb{R}$ $w \in V$

$\varphi_B(\lambda w) = [\lambda w]_B$

Define $\phi_B : \mathbb{R}^n \to V$

$\phi_B\left(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right) = a_1 v_1 + \cdots + a_n v_n$

$\varphi_B = \psi_B = id_B$ $\qquad\qquad$ $\phi_B \circ \psi_B = id(\mathbb{R}^n)$ ✓

So a choice of ordered basis on V defines an isomorphism between V and $\mathbb{R}^n$.

Example 3.35 ($\mathcal{P}_n$).

$\mathcal{B} = (1, x, x^2 \dots x^n)$ is an ordered basis of $\mathcal{P}_n$

Get $\quad \varphi_B = \mathcal{P}_n \longrightarrow \mathbb{R}^{n+1}$ invertible linear transformation

$$\varphi_B(a_0 + a_1 x + \dots + a_n x^n) = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Example 3.36. Consider the ordered basis $\mathcal{B} = (x^2 + x + 1, x + 1, 1)$ of $\mathcal{P}_2$.

$\varphi_B : \mathcal{P}_2 \longrightarrow \mathbb{R}^3$

let $a + bx + cx^2 \in \mathcal{P}_2$

want $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$a + bx + cx^2 = \lambda_1(x^2 + x + 1) + \lambda_2(x+1) + \lambda_3(1)$$

$$\lambda_1 = c$$

$$\lambda_1 + \lambda_2 = b$$

$$\lambda_2 = b - c$$

$$\lambda_1 + \lambda_2 + \lambda_3 = a$$

$$\lambda_3 = a - b.$$

$$[a + bx + cx^2]_B = \begin{bmatrix} c \\ b-c \\ a-b \end{bmatrix}$$

$$\varphi_B(a + bx + cx^2) = \begin{bmatrix} c \\ b-c \\ a-b \end{bmatrix}$$

Example 3.37 ($M_{m\times n}$).

Consider the standard basis

Example 3.38. Consider the ordered basis $\mathcal{B} = ((1,0),(1,1))$ of $\mathbb{R}^2$.