MAST20004 Probability

Tutorial Set 9

1. Let $X \stackrel{d}{=} \mathrm{R}(0,1)$ and $Y \stackrel{d}{=} \mathrm{Bi}(1,0.5)$ be independent random variables, and $Z = X^Y$. For each possible value of y,
   (a) Compute the function $\eta(y) = \mathbb{E}(Z \,|\, Y = y)$.
   (b) Compute the function $\zeta(y) = V(Z \,|\, Y = y)$.
   (c) Hence specify the distribution of the random variables $\mathbb{E}(Z \,|\, Y)$ and $V(Z \,|\, Y)$.
   (d) Use (a) to compute $\mathbb{E}(Z)$.
   (e) Use (a) and (b) to compute $V(Z)$.

   Solution:

   Y can take only the values 0 or 1.

   (a) $\eta(0) = \mathbb{E}(Z \,|\, Y = 0) = \mathbb{E}(1) = 1$.
   $\eta(1) = \mathbb{E}(Z \,|\, Y = 1) = \mathbb{E}(X) = 1/2$.
   (b) $\zeta(0) = V(Z \,|\, Y = 0) = V(1) = 0$.
   $\zeta(1) = V(Z \,|\, Y = 1) = V(X) = 1/12$.
   (c) So $\mathbb{E}(Z \,|\, Y)$ is equal to 1 with probability 1/2 and to 1/2 with probability 1/2, and $V(Z \,|\, Y)$ is equal to 0 with probability 1/2 and to 1/12 with probability 1/2.
   (d) $\mathbb{E}(Z) = \mathbb{E}(\mathbb{E}(Z|Y)) = 1/2 \times 1 + 1/2 \times 1/2 = 3/4$.
   (e)

$$
\begin{aligned}
V(\mathbb{E}(Z|Y)) &= \mathbb{E}((\mathbb{E}(Z|Y) - \mathbb{E}(\mathbb{E}(Z|Y)))^2) \\
&= \frac{1}{2} \times \left(1 - \frac{3}{4}\right)^2 + \frac{1}{2} \times \left(\frac{1}{2} - \frac{3}{4}\right)^2 \\
&= \frac{1}{16},
\end{aligned}
$$

   and

$$
\mathbb{E}(V(Z|Y)) = \frac{1}{2} \times 0 + \frac{1}{2} \times \frac{1}{12} = \frac{1}{24}.
$$

   So, $V(Z) = 1/16 + 1/24 = 5/48$.

2. Let (X, Y) be a uniformly distributed random point in the bounded region between the curves $y = x^2 - 1$ and $y = 1 - x^2$. Write down the joint pdf for (X, Y). Are X and Y correlated or uncorrelated random variables? Are they independent?

   Solution: The region between the two curves lies between the points $(-1, 0)$ and $(1, 0)$. The area of the region is

$$
\int_{-1}^{1} \int_{x^2-1}^{1-x^2} dy dx = \int_{-1}^{1} (2 - 2x^2) dx = \frac{8}{3}.
$$

   Therefore

$$
f_{(X,Y)}(x, y) = \begin{cases} 3/8 & \text{if } x^2 - 1 \le y \le 1 - x^2 \\ 0 & \text{otherwise.} \end{cases}
$$

For $x \in [-1,1]$,

$$f_X(x) = \int_{x^2-1}^{1-x^2} \frac{3}{8} dy = \frac{3-3x^2}{4},$$

and so $\mathbb{E}(X) = 0$.
For $y \in [-1,0]$,

$$f_Y(y) = \int_{-\sqrt{y+1}}^{\sqrt{y+1}} \frac{3}{8} dx = \frac{3\sqrt{y+1}}{4}$$

and, for $y \in [0,1]$,

$$f_Y(y) = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{3}{8} dx = \frac{3\sqrt{1-y}}{4},$$

and so $\mathbb{E}(Y) = 0$. Both of these facts could have been derived using symmetry.

$$\begin{aligned}
\mathbb{E}(XY) &= \int_{-1}^{1} \int_{x^2-1}^{1-x^2} \frac{3xy}{8} dy dx \\
&= \int_{-1}^{1} \left[\frac{3xy^2}{16}\right]_{x^2-1}^{1-x^2} dx \\
&= \int_{-1}^{1} \left(\frac{3x(1-x^2)^2}{16} - \frac{3x(x^2-1)^2}{16}\right) dx \\
&= 0.
\end{aligned}$$

So X and Y are uncorrelated.
The random variables X and Y are clearly not independent. For example $f_{(X,Y)}(3/4, 3/4) = 0$ but $f_X(3/4)$ and $f_Y(3/4)$ are both positive, so $f_{(X,Y)}(3/4,3/4) \neq f_X(3/4) f_Y(3/4)$.

3. Let $X \stackrel{d}{=} \text{Pn}(\lambda)$, $Y_1 = \min(X,1)$, $Y_2 = \min(X, 1.5)$.
   (a) Find the pmf, cdf and mean of Y_1.
   (b) Derive the pmf, cdf and mean of Y_2.
   (c) Compute the conditional distribution and conditional expectation of Y_1 given $X \leq 1$.
   (d) Obtain the conditional distribution and conditional expectation of Y_2 given $X \leq 1.5$.
   (e) Explain why $\mathbb{E}(Y_1)$ and $\mathbb{E}(Y_2)$ are different while the conditional expectations in (c) and (d) are the same.

Solution:

(a) The possible values of Y_1 are 0 and 1, with

$$\mathbb{P}(Y_1 = 0) = \mathbb{P}(X = 0) = e^{-\lambda}, \quad \mathbb{P}(Y_1 = 1) = \mathbb{P}(X \geq 1) = 1 - e^{-\lambda},$$

hence $\mathbb{E}(Y_1) = 0 \times e^{-\lambda} + 1 \times (1 - e^{-\lambda}) = 1 - e^{-\lambda}$ and the cdf of Y_1 is

$$F_{Y_1}(x) = \begin{cases} 0, & \text{if } x < 0, \\ e^{-\lambda}, & \text{if } 0 \leq x < 1, \\ 1, & \text{if } x \geq 1. \end{cases}$$

(b) The possible values of Y_2 are 0, 1 and 1.5, with

$$\mathbb{P}(Y_2 = 0) = \mathbb{P}(X = 0) = e^{-\lambda}, \quad \mathbb{P}(Y_2 = 1) = \mathbb{P}(X = 1) = \lambda e^{-\lambda},$$
$$\mathbb{P}(Y_2 = 1.5) = \mathbb{P}(X \geq 1.5) = 1 - e^{-\lambda} - \lambda e^{-\lambda},$$

hence

$$\mathbb{E}(Y_2) = 0 \times e^{-\lambda} + 1 \times \lambda e^{-\lambda} + 1.5 \times (1 - e^{-\lambda} - \lambda e^{-\lambda}) = 1.5 - e^{-\lambda}(1.5 + 0.5\lambda);$$

and the cdf of Y_2 is

$$F_{Y_2}(x) = \begin{cases} 0, & \text{if } x < 0, \\ e^{-\lambda}, & \text{if } 0 \le x < 1, \\ (1+\lambda)e^{-\lambda}, & \text{if } 1 \le x < 1.5, \\ 1, & \text{if } x \ge 1.5. \end{cases}$$

(c) Given $X \le 1$, Y_1 takes values 0 and 1, with

$$\mathbb{P}(Y_1 = 0 | X \le 1) = \mathbb{P}(X = 0 | X \le 1) = \frac{1}{1+\lambda},$$
$$\mathbb{P}(Y_1 = 1 | X \le 1) = \mathbb{P}(X \ge 1 | X \le 1) = \mathbb{P}(X = 1 | X \le 1) = \frac{\lambda}{1+\lambda},$$

hence $\mathbb{E}(Y_1 | X \le 1) = 0 \times \frac{1}{1+\lambda} + 1 \times \frac{\lambda}{1+\lambda} = \frac{\lambda}{1+\lambda}$.

(d) Notice that $\{X \le 1.5\} = \{X \le 1\}$, so given $X \le 1.5$, Y_2 takes values 0 and 1, with

$$\mathbb{P}(Y_2 = 0 | X \le 1.5) = \mathbb{P}(X = 0 | X \le 1.5) = \frac{1}{1+\lambda},$$
$$\mathbb{P}(Y_2 = 1 | X \le 1.5) = \mathbb{P}(X = 1 | X \le 1) = \frac{\lambda}{1+\lambda},$$

hence $\mathbb{E}(Y_2 | X \le 1.5) = 0 \times \frac{1}{1+\lambda} + 1 \times \frac{\lambda}{1+\lambda} = \frac{\lambda}{1+\lambda}$.

(e) $\mathbb{E}(Y_1) < \mathbb{E}(Y_2)$ because $Y_1 \le Y_2$ and 1.5 is a possible value for Y_2 but not for Y_1. However, given $X \le 1.5$, Y_2 can't take the value 1.5 anymore, resulting in the identical conditional distribution.

4. Let $N \ge 0$ be an integer-valued random variable with $\mathbb{E}(N) = a, V(N) = b^2$ and $X_1, X_2, \cdots$ be independent random variables, also independent of N, with $\mathbb{E}(X_j) = \mu$ and $V(X_j) = \sigma^2$. Using conditional expectations, compute $\mathrm{Cov}(S_N, N)$, where $S_N = \sum_{j=1}^{N} X_j$.

Solution:

$\mathrm{Cov}(S_N, N) = \mathbb{E}(S_N N) - \mathbb{E}(S_N)\mathbb{E}(N)$. First $\mathbb{E}(S_N | N = n) = \mathbb{E}(S_n | N = n) = \mathbb{E}(S_n) = n\mu$, where the second last equality is due to the fact that S_n and N are independent. Therefore $E(S_N | N) = N\mu$. Now

$$\begin{aligned} \mathbb{E}(S_N) &= \mathbb{E}(\mathbb{E}(S_N | N)) \\ &= \mathbb{E}(N\mu) \\ &= a\mu. \end{aligned}$$

Also

$$\begin{aligned} \mathbb{E}(S_N N | N = n) &= \mathbb{E}(n S_n | N = n) \\ &= n\mathbb{E}(S_n | N = n) \\ &= n\mathbb{E}(S_n) \\ &= n^2\mu, \end{aligned}$$

where, again, the second last equality is from the independence between S_n and N. Therefore $\mathbb{E}(S_N N|N) = N^2\mu$. Now

$$\begin{aligned}
\mathbb{E}(S_N N) &= \mathbb{E}(\mathbb{E}(S_N N|N)) \\
&= \mathbb{E}(N^2\mu) \\
&= \mu\left(V(N) + \mathbb{E}(N)^2\right) \\
&= \mu\left(b^2 + a^2\right).
\end{aligned}$$

Therefore $\mathrm{Cov}(S_N, N) = \mu\left(b^2 + a^2\right) - a^2\mu = \mu b^2$.

5. Let the random variable X have the probability generating function

$$P_X(z) = c + 0.1(1+z)^3 + 0.3z^5.$$

(a) Find the constant c.

(b) Give the distribution of X.

(c) Use the $P_X(z)$ to compute $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$.

(d) Compute the probability generating function of $Y = X + 2$.

Solution:

(a) $P_X(1) = 1$ implies that $c + 0.1(2)^3 + 0.3 = 1$ and so $c = -0.1$.

(b) $P_X(z) = 0.3z + 0.3z^2 + 0.1z^3 + 0.3z^5$, so

$$p_X(x) = \begin{cases} 0.3 & \text{if } x = 1 \\ 0.3 & \text{if } x = 2 \\ 0.1 & \text{if } x = 3 \\ 0.3 & \text{if } x = 5 \\ 0 & \text{otherwise} \end{cases}$$

(c) $P_X'(z) = 0.3(1+z)^2 + 1.5z^4$,
$P_X''(z) = 0.6(1+z) + 6z^3$,
and so $\mathbb{E}(X) = P_X'(1) = 2.7$ and $\mathbb{E}(X^2) = P_X''(1) + P_X'(1) = 9.9$.

(d)

$$\begin{aligned}
P_Y(z) &= \mathbb{E}(z^Y) \\
&= \mathbb{E}(z^{X+2}) \\
&= z^2\mathbb{E}(z^X) \\
&= z^2 P_X(z) \\
&= 0.3z^3 + 0.3z^4 + 0.1z^5 + 0.3z^7.
\end{aligned}$$

MAST20004 Probability

Computer Lab 9

In this lab you

- investigate the bivariate distribution in Tutorial 8, Question 3.
- investigate various distributions in the unit square.

In both exercises you will contrast bivariate distributions with dependent X and Y with distributions which utilise the same marginal distributions but assume independence.

Exercise A - Petrol station question - Tutorial 8, Question 3

Suitably modified, the **incomplete** Matlab m-file **Lab9ExA.m** will simulate the bivariate distribution from this question. You will need to add a few lines to the program to generate the required distributions.

1. We use the formula
$$f_{(X,Y)}(x,y) = f_X(x) \times f_{Y|X}(y|x)$$
from part (a) of the solutions above to generate our bivariate observations. Add the necessary code to do this, remembering that the command **u=rand(npts,1)** generates 'npts' observations on a uniform distribution. I suggest that you work in units of 'thousands of litres', so that for example X takes values in the interval $[10, 20]$.

2. As a check on the accuracy of your simulation, uncomment the section of code calculating an estimate for $\mathbb{P}(Y < 15)$ and check your result against what you get using the formula for $f_Y(y)$ in the solution to part (c). [Hint: $\int_0^x log(u)du = xlog(x) - x$].

3. In this problem X and Y are clearly not independent as you cannot sell more than you have stocked for the week. Try to change the program to generate bivariate observations with the same marginal distributions as X and Y in this problem but which are independent. [Hint: It is numerically too difficult to generate observations on Y using the inverse transformation method - instead simply take the Y coordinate only of the bivariate points (X, Y) which your program already generates. Then generate independent uniform observations on X. Contrast the two joint densities and compare the values for $\mathbb{P}(Y < 15)$.

Exercise B - Various bivariate distributions within the unit square

1. Consider the random experiment of observing a uniform random point in the unit square $S = [0, 1] \times [0, 1]$. In this case the X and Y coordinates are both $R(0, 1)$ and independent. The Matlab m-file **Lab9ExB.m** simulates this distribution.

2. Run **Lab9ExB.m** and observe the bivariate observations and the marginals. Note the 'cobweb' like apparent patterns which appear within the totally random points, that is, there are 'clusters', 'holes' and apparent 'lines' in the realisations. If asked to draw random points manually people often draw points very evenly spaced out, with the natural 'clumping' evident in the simulation much reduced.

3. Now consider the random point (U, V), where $U = \min(X, Y)$ and $V = \max(X, Y)$. Input the appropriate lines into the program to generate observations on (U, V). You will still need to store the results in the vectors x and y so the plotting commands will work. From the simulated observations try to guess the joint pdf for (U, V).

4. From a theoretical point of view, derive the distribution functions and hence the pdf's for U and V. Check them against the simulated marginals. Uncomment the section of code which prints out estimates of $\mathbb{P}(U < \frac{1}{2})$ and $\mathbb{P}(V < \frac{1}{2})$ and check the results against your theoretical marginals. Why do these two probabilities add to one?

5. Assume that you mistakenly tried to simulate the joint distribution for (U, V) using the marginals and assuming independence. Add code to generate the resulting distribution using the inverse transformation method to generate the observations on U and V. Compare the resulting distribution with the correct one for (U, V). In particular compare $\mathbb{P}(U < V)$ under these two models.

6. Check your guess for the joint pdf of (U, V) by deriving the marginals and comparing with the results for the pdfs of U and V that you derived above.