Semester 2, 2014 MAST20009 Vector Calculus Exam Answers

1. (a) Approach $(0,0)$ along $y = kx$ $(k \in \mathbb{R})$ to show the two variable limit does not exist.

   (b) $\boldsymbol{D}(f \circ g)(1,0) = \begin{pmatrix} 12 & 2 \\ -2 & -1 \end{pmatrix}$.

2. (a) Find critical points of $f(x,y) = 3x^2 + xy + 3y^2$ subject to $x^2 + y^2 = 8$. Maximum at $(2,2), (-2,-2)$ of $f = 28$. Minimum at $(2,-2), (-2,2)$ of $f = 20$.

   (b) Find critical points of $f(x,y) = 3x^2 + xy + 3y^2$. Minimum at $(0,0)$ of $f = 0$. Combining results: absolute minimum of $f = 0$ at $(0,0)$ and absolute maximum of $f = 28$ at $(2,2), (-2,-2)$.

3. (a) Helix winding around cylinder $x^2 + z^2 = 4$ starting at $(2,0,0)$ and finishing at $(2, 36\pi, 0)$.

   (b) $\boldsymbol{T} = \frac{1}{5}(-4\sin 4t, 3, -4\cos 4t)$

   $\boldsymbol{N} = (-\cos 4t, 0, \sin 4t)$

   $\boldsymbol{B} = \frac{1}{5}(3\sin 4t, 4, 3\cos 4t)$

   (c) $\kappa = \dfrac{8}{25}$, $\tau = \dfrac{6}{25}$.

4. (a) Proof required. Let $F(x,y,z) = (F_1, F_2, F_3)$.

   $\boldsymbol{\nabla} \cdot (fF_1, fF_2, fF_3) = f(F_{1x} + F_{2y} + F_{3z}) + (f_x F_1 + f_y F_2 + f_z F_3)$.

   (b) Use identities 6, 10 and 11 to get $\mathbf{0}$

   (c) Use identities or calculate directly to get $\dfrac{2}{r}\sinh r + \cosh r$

5. (a) Proof required. Expand derivative of dot product first and then expand derivative of cross product.

   (b) Sketch required. Interchange the order of integration to give

   $$\int_0^2 \int_0^{3x} \frac{x^2}{x^2 + y^2}\, dy\, dx = 2 \arctan 3.$$

6. Use cylindrical coordinates.

   $$\text{Mass} = \int_0^2 \int_0^{2\pi} \int_{-\sqrt{9-\rho^2}}^{\sqrt{9-\rho^2}} z^2 \rho\, dz\, d\phi\, d\rho = \frac{4\pi}{15}\left(9^{\frac{5}{2}} - 5^{\frac{5}{2}}\right).$$

7. (a) $\boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0}$.

   (b) Let $\boldsymbol{F}$ be the velocity field of a fluid. An object in the fluid will not rotate as it moves since $\boldsymbol{F}$ is irrotational.

   (c) $\phi = x^2 z^3 + \sin x \cos y + C$.

   (d) work done $= \phi\left(\frac{\pi}{2}, \frac{\pi}{6}, 2\right) - \phi(0,3,1) = 2\pi^2 + \frac{\sqrt{3}}{2}$.

8. (a) $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = 1 - \rho^2$, $0 \le \rho \le 1$, $0 \le \phi \le 2\pi$.

   (b) normal $= (2\rho^2\cos\phi, 2\rho^2\sin\phi, \rho)$

(c) $y + z = \frac{5}{4}$.

(d) Charge $= \displaystyle\int_0^1 \int_0^{2\pi} \mu\rho\sqrt{4\rho^2+1}\, d\phi\, d\rho = \frac{\pi\mu}{6}\left(5\sqrt{5}-1\right)$.

9. (a) See lecture notes.

   (b) Proof required. Apply Green's theorem with $P(x,y) = -y, Q(x,y) = x$.

   (c) Area $= \displaystyle\frac{1}{2}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos^2(2t)\, dt = \frac{\pi}{8}$.

10. Use Gauss' Divergence theorem and spherical coordinates to get

    Flux$= \displaystyle\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^3 2r^3 \sin^2\theta \sin\phi + 3r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{189\pi}{8}$

11. Use Stokes' theorem and the simplest surface $D : x^2 + y^2 \leq 4,\ z = 1$ to get

    $\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{S} = 3 \iint_D dS = 12\pi$

12. (a) $\dfrac{\partial \boldsymbol{r}}{\partial u} = (a\sinh u \cos\theta \cos\phi, a\sinh u \cos\theta \sin\phi, a\cosh u \sin\theta)$

    $\dfrac{\partial \boldsymbol{r}}{\partial \theta} = (-a\cosh u \sin\theta \cos\phi, -a\cosh u \sin\theta \sin\phi, a\sinh u \cos\theta)$

    $\dfrac{\partial \boldsymbol{r}}{\partial \phi} = (-a\cosh u \cos\theta \sin\phi, a\cosh u \cos\theta \cos\phi, 0)$

    (b) Proof required. Use identities to simplify scale factors.

    (c) $\dfrac{2u\theta^3}{a\sqrt{\sinh^2 u + \sin^2\theta}}\mathbf{e}_u + \dfrac{3u^2\theta^2}{a\sqrt{\sinh^2 u + \sin^2\theta}}\mathbf{e}_\theta + \dfrac{4\phi^3}{a\cosh u \cos\theta}\mathbf{e}_\phi$