THE UNIVERSITY OF MELBOURNE

Semester 2 Assessment, 2014

Department of Mathematics and Statistics

MAST30001 Stochastic Modelling

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam.

This paper consists of 4 pages (including this page)

Authorised materials:

- Students may bring one double-sided A4 sheet of handwritten notes into the exam room.
- Hand-held electronic scientific (but not graphing) calculators may be used.

Instructions to Students

- You may remove this question paper at the conclusion of the examination.
- This paper has 8 questions. Attempt as many questions, or parts of questions, as you can. The number of marks allocated to each question is shown in the brackets after the question statement. The total number of marks available for this examination is 66. Working and/or reasoning must be given to obtain full credit. Clarity, neatness and style count.

Instructions to Invigilators

- Students may remove this question paper at the conclusion of the examination.

This paper may be held in the Baillieu Library

1. Let X_n be a Markov chain with transition matrix

$$P = \begin{pmatrix} 1/6 & 1/3 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}.$$

Define

$$Y_n = \begin{cases} 1, & X_n \in \{1,2\}, \\ 2, & X_n = 3, \\ 3, & X_n = 4. \end{cases}$$

Is $(Y_n)_{n\geq 0}$ a Markov chain? If so, find its transition matrix and if not, carefully explain why.

[4 marks]

2. A Markov chain has transition matrix

$$\begin{pmatrix} 1/3 & 0 & 0 & 2/3 & 0 \\ 1/7 & 2/7 & 2/7 & 2/7 & 0 \\ 0 & 3/7 & 4/7 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 4/9 & 0 & 0 & 5/9 \end{pmatrix}.$$

(a) What is $P(X_4 = 1, X_2 = 2 | X_0 = 5)$?

(b) Analyze the state space of the Markov chain: communicating classes, reducibility, periodicity and recurrence.

(c) Describe the long run behavior of the chain.

[8 marks]

3. Four fair coins are lying on a table. We perform the following procedure: a coin is selected uniformly at random and then tossed and placed back on the table. Let X_n be the number of heads showing among the four coins after performing this procedure n times.

(a) Model X_n as a Markov chain and write down its transition probabilities.

(b) If initially there are 2 heads among the four coins lying on the table and then we perform this procedure indefinitely, what is the chance that all four coins show tails before they all show heads?

(c) If initially all of the coins show tails and then we perform this procedure indefinitely, what is the chance that all four coins show tails again before they all show heads?

(d) Now assume the coins are biased with chance of heads equal to $p \in (0,1)$. If initially there are 2 heads among the four coins lying on the table and then we perform this procedure indefinitely, what is the chance that all four coins show tails before they all show heads? (You don't need to simplify your answer past an expression entirely in terms of p.)

[8 marks]

4. Let $p \in (0,1)$. A Markov chain on $\{0, 1, 2, \dots\}$ has transition probabilities

$$p_{i,i+1} = p, \qquad p_{i,j} = (1-p)/(i+1) \ \text{ for } \ j = 0, \dots, i.$$

(a) Determine the values of p for which a stationary distribution exists and for these values find the stationary distribution.

(b) Determine the values of p where the chain is transient, null recurrent, positive recurrent.

[6 marks]

5. A shop has two machines that operate independently and occasionally break. At the beginning of the day, each machine is in perfect working order and the times until failure of each machine are independent and exponentially distributed with rate μ. When a machine breaks, service is immediately started on it and it is repaired in an exponential rate λ amount of time; service times are independent of each other and failure times.

(a) Model the number of working machines as a continuous time Markov chain and write down its generator.

(b) Argue the Markov chain is ergodic and find its steady state distribution.

(c) Assume $\lambda = 1$. What is the maximum rate μ so that the chance at least one machine is working is at least 95%?

(d) Assume now that $\lambda = \mu$. Derive the transition probability $p_{0,1}(t)$ and verify that the limiting probability as $t \to \infty$ matches that of part (b). [9 marks]

6. Let $(B_t)_{t \geq 0}$ be a standard Brownian motion.

(a) If the price of a stock at time t hours into a trading day is given by

$$S_t = 25 \exp\{2B_t - t\},$$

(i) What is the chance that the stock price is higher than it started 4 hours into the day given that it was equal to $25e^2$ dollars two hours into the day?

(ii) Given that the stock's price four hours into the day was 25 dollars, what is the chance the price was less than $25e^2$ dollars 2 hours into the day?

(b) Let $0 < t_1 < t_2$. Find the expected value of $M = \max\{B_s : t_1 \leq s \leq t_2\}$. (You may want to use the fact that $\max\{B_s : 0 \leq s \leq t\} \stackrel{d}{=} |B_t|$.)

[7 marks]

Tables of the Normal Distribution

Probability Content from -oo to Z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015

7. A boutique dress shop has two entrances, one on A street and the other on B street. Customers enter the shop from A street according to a rate one Poisson process and from B street according to a rate two Poisson process independent of the flow of customers from A street (units are in hours). Customers who enter the shop buy something with probability $1/3$, independent of other customers' behavior.

   (a) What is the chance that exactly 3 people enter the shop from A street between noon and 1pm?
   (b) What is the distribution of time until the first person enters the shop after it opens?
   (c) What is the chance that exactly 4 people enter the shop (total from A and B street) between noon and 1pm?
   (d) Given 2 people entered the shop between noon and 1pm, what is the chance that exactly one person entered the shop from A street between noon and 12:30pm?
   (e) What is the chance that exactly 3 people buy something between noon and 1pm?
   (f) What is the chance that between 1pm and 2pm, exactly 2 people who entered from A street buy something and exactly 1 person who entered from B street buys something?

   Customers can purchase a dress in advance and then drop in at the shop to have it fitted. Assume that these customers arrive according to a rate one Poisson process and that there are two tailors to fit dresses. The service times for each tailor are independent and both distributed as exponential with rate $3/4$.

   (g) Model the number of customers in the shop who purchased a dress in advance to be fitted as a queuing system and determine its steady state distribution.
   (h) What is the average amount of time a customer who purchased a dress in advance to be fitted spends in the store?
   (i) What is the average amount of time a customer who purchased a dress in advance waits to be served?

   [12 marks]

8. A rat has a cage with a tunnel loop that he runs through periodically. The rat stays in the cage for a random amount of time and then runs through the loop, and then starts the process anew. Assume that both the amount of time it takes the rat to run through the loop and the amount of time the rat stays in the cage between times when it runs through the loop are independent and (continuously) uniformly distributed between zero and five minutes.

   (a) Model the number N_t of times the rat returns to the cage from the loop up to t minutes into its day as a renewal process and determine the density, mean, and variance of the random times between renewals.
   (b) On average, about how many times does the rat return to its cage from the loop in the first 4 hours of its day?
   (c) Give an interval around your estimate from (b) that will have a 95% chance of covering the true number of times the rat returned to its cage from the loop in the first 4 hours of its day.
   (d) If you go visit the rat's cage at exactly 4 hours into its day, about what is the mean and variance of the time since the rat last returned from the loop?

   [12 marks]

End of Exam