MAST20009 Vector Calculus

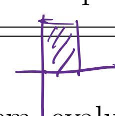
Practice Class 9 Questions

Green's Theorem in the Plane

$$\int_{C=\partial D} P\,dx + Q\,dy \;=\; \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where

- D is a region in the x-y plane bounded by a simple closed curve $C = \partial D$ orientated in the positive direction (anticlockwise).
- $\boldsymbol{F}(x,y) = P(x,y)\boldsymbol{i} + Q(x,y)\boldsymbol{j}$ is a C^1 vector field on D.
- D composed of regions of both vertical and horizontal strips.



1. Let D be the rectangle $[0,1]\times[0,2]$. Using Green's Theorem, evaluate

$$\int_{\partial D} \boldsymbol{F}\cdot d\boldsymbol{s}$$

when $\boldsymbol{F}(x,y) = (\overset{P}{x - y^3}, \overset{Q}{y + x^2})$.

$= \iint_D (\nabla \times \underline{F}) \cdot \underline{k}\, dx\, dy = \int_0^1 \int_0^2 2x + 3y^2\, dy\, dx = 10$

$= \int_0^1 [2xy + y^3]_0^2\, dx$ ($4x+8$)

$= \int_0^1 4x + 8\, dx$

$= [2x^2 + 8x]_0^1$

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x - y^3 & y + x^2 & 0 \end{vmatrix} = \underline{i}\,(0) - \underline{j}\,(0) + \underline{k}\,(2x + 3y^2).$$

Divergence Theorem in the Plane

$$\int_{\partial D} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,ds \;=\; \iint_D \boldsymbol{\nabla}\cdot\boldsymbol{F}\,dx\,dy$$

where

- $D, \partial D$ and $\boldsymbol{F}$ are the same as in Green's Theorem.
- $\hat{\boldsymbol{n}}$ is unit outward normal to ∂D in the x-y plane.

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2. Verify the Divergence Theorem in the Plane when

$$\boldsymbol{F}(x,y) = y^2\boldsymbol{i} + (2x+y)\boldsymbol{j}$$

and D is the disk $x^2 + y^2 \le 4$.

Let ∂D be anticlockwise

then. $\hat{n} = x\underline{i} + y\underline{j}$. since. it is a disk.

∂D. $x = 2\cos t \quad y = 2\sin t$

$\hat{n}\cdot\frac{ds}{dt}\cdot dt$ ($\hat{n} = (\cos t, \sin t)$, $\frac{ds}{dt} \to 2$.)

$\int_0^{2\pi} (4\sin^2 t,\ 4\cos t + 2\sin t)\cdot(2\cos t,\ 2\sin t)\, dt$

$= \int_0^{2\pi} 8\sin^2 t\cos t + 8\sin t\cos t + 4\sin^2 t.\ dt$

$= 4\int_0^{2\pi} \frac{1 - \cos 2t}{2}\, dt$

$= 4\left[\frac{1}{2}t - \frac{\sin 2t}{4}\right]_0^{2\pi}$

$= 2\cdot 2\pi = 4\pi.$

$\iint_D \underline{\nabla}\cdot\underline{F}\, dx\, dy = \iint_D 1\, dx\, dy = \text{Area} = 4\pi.$

$= \int_0^{2\pi} 8\sin^2 t\cos t\,dt + \int_0^{2\pi} 8\sin t\cos t\,dt + \int_0^{2\pi} 4\sin^2 t\,dt$

$= \int_0^{2\pi} 8\sin^2 t\,d(\sin t). \quad = \int_0^{2\pi} 8\sin t\,d(\sin t)$

$\left[\frac{8}{3}\sin^3 t\right]_0^{2\pi} \qquad = \left[4\sin^2 t\right]_0^{2\pi}$

0

Stokes' Theorem

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} \;=\; \int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} \;=\; 0.$$

where

- S is an open oriented surface parametrised by $\Phi(u,v)$; Φ is a C^2 mapping.
- ∂S is the oriented closed boundary of S.
- $\boldsymbol{F}$ is a C^1 vector field on S.
- S and ∂S are oriented so that $\hat{\boldsymbol{n}}$ is the unit outward normal to S.

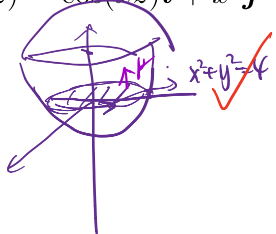
The orientation of ∂S and $\hat{\boldsymbol{n}}$ are related by the right hand rule.

$r=2\sqrt{2}$

3. Let S be that part of the sphere $x^2 + y^2 + (z-2)^2 = 8$ that lies above the x-y plane. Let $\boldsymbol{F}(x,y,z) = \cos(xz)\boldsymbol{i} + x^3\boldsymbol{j} + ye^{-xz}\boldsymbol{k}$.

$z=0. \quad x^2+y^2+4=8 \quad x^2+y^2=4$

Evaluate



$$\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\partial S} F \cdot ds.$$

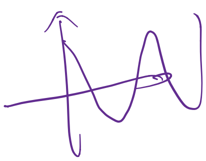
using

traversed anticlockwise.

(a) an appropriate line integral;

anticlockwise. $x^2+y^2=4$. $z=0$.

Let $x=2\cos t$, $y=2\sin t$, $z=0$. $C'(t)=(-2\sin t, 2\cos t, 0)$



(b) the simplest surface.

$\int_0^{2\pi} (1, 8\cos^3 t, 2\sin t)\cdot(-2\sin t, 2\cos t, 0)\,dt$

(b). $\iint_S (\nabla\times F)\,dS$

$= \iint_{S_3} (\nabla\times F)\cdot k\,dS$

$= \iint_{S_3} 3x^2\,dS. \quad = \int_0^{2\pi} -2\sin t + 16\cos^4 t\,dt.$

Area of region in the x-y plane

If C is a simple closed curve that bounds a region D, then

$$\text{Area of } D \;=\; \frac{1}{2}\int_{C=\partial D} x\,dy - y\,dx$$

P Q.

Let $x = r\cos t$
$y = r\sin t$

$r\in[0,2]$
$t\in[0,2\pi]$

$\int_0^{2\pi}\int_0^2 3r^2\cos^2 t\cdot r\,dr\,dt.$

$= \int_0^{2\pi}\int_0^2 3r^3\cos^2 t\,dr\,dt.$ $\quad \frac{1+\cos 2t}{2}$

$= \int_0^2 3r^3\,dr \int_0^{2\pi} \cos^2 t\,dt$

$= \left[\frac{3}{4}r^4\right]_0^2 \left[\frac{t}{2}+\frac{\sin 2t}{4}\right]_0^{2\pi}$

$= \frac{3}{4}\times 16\times[\pi].$

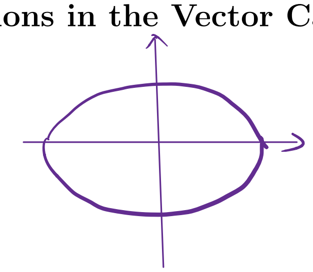
$= 12\pi.$

4. Show that the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a,b>0)$$

is πab.

When you have finished the above questions, continue working on the questions in the Vector Calculus Problem Sheet Booklet.



$x = a\cos t$
$y = b\sin t$

$t\in[0,2\pi]$

$\frac{dy}{dt} = b\cos t$

$\frac{dx}{dt} = -a\sin t.$

Area $D = \frac{1}{2}\int_{\partial D} x\,dy - y\,dx$

$= \frac{1}{2}\int_{\partial D} x\frac{dy}{dt} - y\frac{dx}{dt}\,dt$

$$= \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ba\sin^2 t \, dt$$

$$= \frac{ab}{2}\int \cdot 2\pi = \pi ab$$