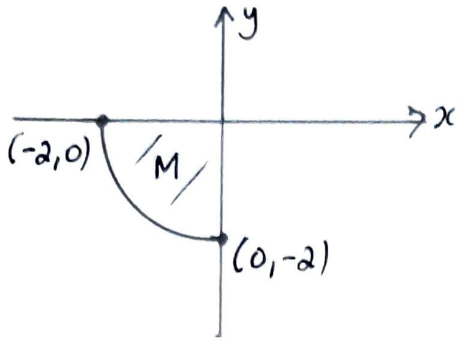
Q1 M: $x^2+y^2 \le 4$, $x \le 0$, $y \le 0$.

(a)



(b) Evaluate $\iint_M (x^2+y^2)^{3/2}\,dx\,dy$.

Use polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow (x^2+y^2)^{3/2} = (r^2)^{3/2} = r^3$

Now $0 \le r \le 2$, $\pi \le \theta \le \frac{3\pi}{2}$

$$\Rightarrow \iint_M (x^2+y^2)^{3/2}\,dx\,dy = \int_0^2 \int_\pi^{3\pi/2} r^3 . r\,d\theta\,dr$$

$$= \int_0^2 \int_\pi^{3\pi/2} r^4\,d\theta\,dr$$

$$= \int_0^2 \left[r^4\theta\right]_{\theta=\pi}^{\theta=\frac{3\pi}{2}} dr$$

$$= \frac{\pi}{2}\int_0^2 r^4\,dr$$

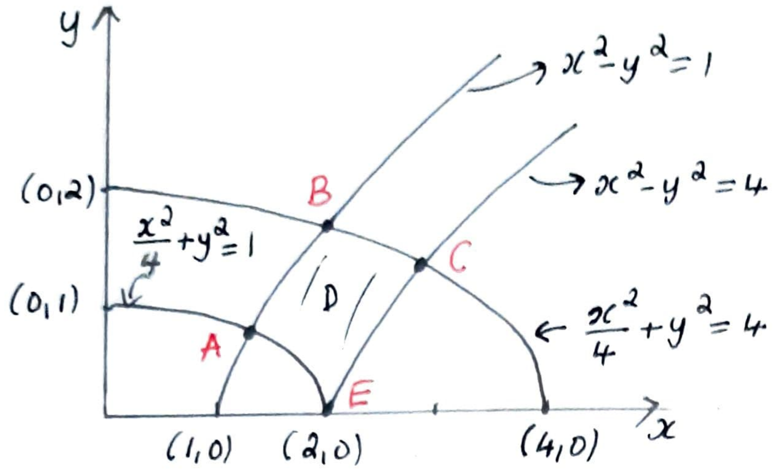
$$= \frac{\pi}{2}\left[\frac{1}{5}r^5\right]_{r=0}^{r=2}$$

$$= \left(\frac{\pi}{2}\right)\left(\frac{32}{5}\right)$$

$$= \frac{16\pi}{5}$$

$\boxed{Q2}$ $D: x^2-y^2=1,\ x^2-y^2=4,\ \frac{x^2}{4}+y^2=1,\ \frac{x^2}{4}+y^2=4$, 1st quadrant.

(a)



Intersections in 1st quadrant

$\boxed{A}$ $x^2-y^2=1$, $\frac{x^2}{4}+y^2=1$ $\Rightarrow \frac{5x^2}{4}=2 \Rightarrow x^2=\frac{8}{5} \Rightarrow x=\sqrt{\frac{8}{5}}=\frac{2}{5}\sqrt{10}$

and $y^2=1-\frac{1}{4}\cdot\frac{8}{5}=\frac{3}{5} \Rightarrow y=\sqrt{\frac{3}{5}}=\frac{1}{5}\sqrt{15}$

$A=\left(\frac{2}{5}\sqrt{10}, \frac{1}{5}\sqrt{15}\right)$

$\boxed{B}$ $\frac{x^2}{4}+y^2=4$, $x^2-y^2=1$ $\Rightarrow \frac{5x^2}{4}=5 \Rightarrow x^2=4 \Rightarrow x=2$

and $y^2=4-1=3 \Rightarrow y=\sqrt{3}$

$B=(2,\sqrt{3})$

$\boxed{C}$ $x^2-y^2=4$, $\frac{x^2}{4}+y^2=4$ $\Rightarrow \frac{5x^2}{4}=8 \Rightarrow x^2=\frac{32}{5} \Rightarrow x=\frac{4\sqrt{2}}{\sqrt{5}}=\frac{4}{5}\sqrt{10}$

and $y^2=\frac{32}{5}-4=\frac{12}{5} \Rightarrow y=\sqrt{\frac{12}{5}}=\frac{2}{5}\sqrt{15}$

$C=\left(\frac{4}{5}\sqrt{10}, \frac{2}{5}\sqrt{15}\right)$

(b) *Find $x(u,v)$, $y(u,v)$ if $u=x^2-y^2$, $v=\frac{x^2}{4}+y^2$.

$u=x^2-y^2$, $v=\frac{x^2}{4}+y^2$ $\Rightarrow \frac{5x^2}{4}=u+v \Rightarrow x^2=\frac{4}{5}(u+v)$

and $y^2=x^2-u=\frac{4}{5}(u+v)-u=\frac{-u}{5}+\frac{4v}{5}=\frac{1}{5}(-u+4v)$

As $x,y\geq 0 \Rightarrow x=\frac{2}{\quad}\sqrt{u+v}$, $y=\frac{1}{\quad}\sqrt{-u+4v}$

* Find domain D^*

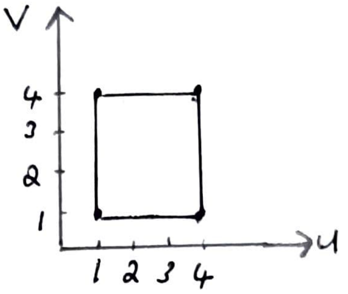
Method 1: Map boundary curves $u = x^2 - y^2,\ v = \frac{x^2}{4} + y^2$

$x^2 - y^2 = 1$	$u = 1$
$x^2 - y^2 = 4$	$u = 4$
$\frac{x^2}{4} + y^2 = 1$	$v = 1$
$\frac{x^2}{4} + y^2 = 4$	$v = 4$

So D^* is a square $1 \le u \le 4,\ 1 \le v \le 4$

Method 2: Map vertices $u = x^2 - y^2,\ v = \frac{x^2}{4} + y^2$

	(x,y)	(u,v)
A	$\left(\frac{2\sqrt{10}}{5}, \frac{\sqrt{15}}{5}\right)$	$(1,1)$
B	$(2, \sqrt{3})$	$(1,4)$
C	$\left(\frac{4\sqrt{10}}{5}, \frac{2\sqrt{15}}{5}\right)$	$(4,4)$
E	$(2,0)$	$(4,1)$



So D^* is a square $1 \le u \le 4,\ 1 \le v \le 4$

* Find Jacobian for mapping

Method 1:

$$\det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{\sqrt{5}\sqrt{u+v}} & \frac{1}{\sqrt{5}\sqrt{u+v}} \\ \frac{-1}{2\sqrt{5}\sqrt{-u+4v}} & \frac{2}{\sqrt{5}\sqrt{-u+4v}} \end{bmatrix}$$

$$= \frac{2}{5} \cdot \frac{1}{\sqrt{u+v}\sqrt{-u+4v}} + \frac{1}{10} \cdot \frac{1}{\sqrt{-u+4v}\sqrt{u+v}}$$

$$= \frac{1}{2} \frac{1}{\sqrt{u+v}\sqrt{-u+4v}}$$

Method 2:

$$\det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \det\begin{bmatrix} 2x & -2y \\ \frac{x}{2} & 2y \end{bmatrix}$$

$$= 4xy + xy$$

$$= 5xy$$

$$\Rightarrow \text{Jacobian} = \frac{1}{5xy}$$

$$= \frac{1}{5} \cdot \frac{\sqrt{5}}{2\sqrt{u+v}} \cdot \frac{\sqrt{5}}{\sqrt{-u+4v}}$$

$$= \frac{1}{2\sqrt{u+v}\sqrt{-u+4v}}$$

* Rewrite integral

$$\iint_D \frac{x^3 y}{y^2 - x^2}\, dx\, dy$$

$$= \int_1^4 \int_1^4 \frac{\frac{8}{5\sqrt{5}}(u+v)^{3/2} \cdot \frac{1}{\sqrt{5}} \cancel{\sqrt{-u+4v}}}{-u} \cdot \frac{1}{2\sqrt{u+v}\,\cancel{\sqrt{-u+4v}}}\, du\, dv$$

$$= \frac{-8}{50} \int_1^4 \int_1^4 \frac{u+v}{u}\, du\, dv$$

$$= \frac{-4}{25} \int_1^4 \int_1^4 1 + \frac{v}{u}\, du\, dv$$

$$= \frac{-4}{25} \int_1^4 \Big[u + v\log(u) \Big]_{u=1}^{u=4} dv$$

$$= \frac{-4}{25} \int_1^4 (4 + v\log(4)) - (1 + v\underbrace{\cancel{\log(1)}}_{0})\, dv$$

$$= \frac{-4}{25} \int_1^4 3 + v\log(4)\, dv$$

$$= \frac{-4}{25} \Big[3v + \tfrac{1}{2} v^2 \log(4) \Big]_{v=1}^{v=4}$$

$$= \frac{-4}{25} \Big[(12 + 8\log(4)) - (3 + \tfrac{1}{2}\log(4)) \Big]$$

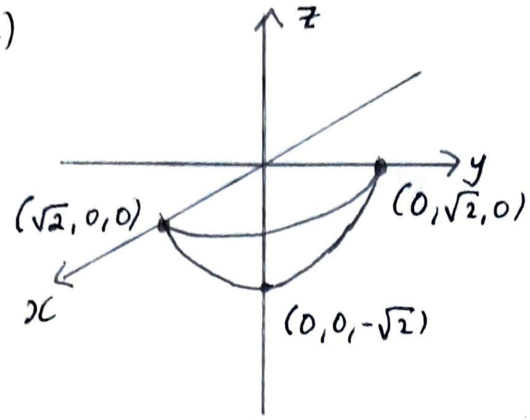
$$= \frac{-4}{25} \Big[9 + \frac{15}{2}\log(4) \Big]$$

$$= \frac{-36}{25} - \frac{6}{5}\log(4)$$

$$= \frac{-36}{25} - \frac{12}{5}\log(2)$$

Q3 B: $z = -\sqrt{2-x^2-y^2}$, xy plane, $x \geq 0$, $y \geq 0$, $z \leq 0$

(a)



* B is octant of sphere radius $\sqrt{2}$ centred at origin

(b) $\text{Mass } B = \iiint_B \rho \, dV$ where $\rho = \dfrac{x^4}{1+(x^2+y^2+z^2)^{7/2}}$ g/cm^3

Use spherical coordinates

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, Jacobian $= r^2\sin\theta$

$\Rightarrow x^2+y^2+z^2 = r^2$

$\Rightarrow \rho = \dfrac{r^4\sin^4\theta\cos^4\phi}{1+r^7}$

Now $0 \leq r \leq \sqrt{2}$, $0 \leq \phi \leq \frac{\pi}{2}$, $\frac{\pi}{2} \leq \theta \leq \pi$

Hence

$$\text{Mass } B = \int_0^{\sqrt{2}} \int_0^{\pi/2} \int_{\pi/2}^{\pi} \frac{r^4\sin^4\theta\cos^4\phi}{1+r^7} \cdot r^2\sin\theta \, d\theta \, d\phi \, dr$$

$$= \left(\int_0^{\sqrt{2}} \frac{r^6}{1+r^7} dr\right)\left(\int_0^{\pi/2} \cos^4\phi \, d\phi\right)\left(\int_{\pi/2}^{\pi} \sin^5\theta \, d\theta\right)$$

Evaluating integrals separately

* $\displaystyle\int_0^{\sqrt{2}} \frac{r^6}{1+r^7} dr$

put $u = 1+r^7 \Rightarrow \frac{du}{dr} = 7r^6$

$$= \int_1^{1+8\sqrt{2}} \frac{1}{7u} du$$

$$= \left[\frac{1}{7}\log(u)\right]_{u=1}^{u=1+8\sqrt{2}}$$

$$* \int_0^{\pi/2} \cos^4\phi \, d\phi$$

$$= \int_0^{\pi/2} \left(\frac{1+\cos(2\phi)}{2}\right)^2 d\phi \qquad \cos(2\phi) = 2\cos^2\phi - 1$$

$$= \frac{1}{4}\int_0^{\pi/2} 1 + 2\cos(2\phi) + \cos^2(2\phi) \, d\phi$$

$$= \frac{1}{4}\int_0^{\pi/2} 1 + 2\cos(2\phi) + \frac{1}{2}(1+\cos(4\phi)) \, d\phi$$

$$= \frac{1}{4}\int_0^{\pi/2} \frac{3}{2} + 2\cos(2\phi) + \frac{1}{2}\cos(4\phi) \, d\phi$$

$$= \frac{1}{4}\left[\frac{3}{2}\phi + \underbrace{\sin(2\phi)}_{0} + \frac{1}{8}\underbrace{\sin(4\phi)}_{0}\right]_{\phi=0}^{\phi=\pi/2}$$

$$= \frac{1}{4}\left[\frac{3}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{3\pi}{16}$$

$$* \int_{\pi/2}^{\pi} \sin^5\theta \, d\theta$$

$$= \int_{\pi/2}^{\pi} \sin\theta \, (1-\cos^2\theta)^2 \, d\theta$$

Let $u = \cos\theta \Rightarrow \frac{du}{d\theta} = -\sin\theta$

$\theta = \pi/2 \Rightarrow u = 0, \quad \theta = \pi \Rightarrow u = -1$

$$= \int_0^{-1} -(1-u^2)^2 \, du$$

$$= \int_{-1}^{0} 1 - 2u^2 + u^4 \, du$$

$$= \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_{u=-1}^{u=0}$$

$$= -\left(-1 + \frac{2}{3} - \frac{1}{5}\right)$$

$$= -\left(\frac{-15+10-3}{15}\right)$$

$$= \frac{8}{15}$$

Hence

$$\text{Mass } B = \frac{1}{7}\log(1+8\sqrt{2}) \cdot \frac{\cancel{3}\pi}{\cancel{16}_2} \cdot \frac{\cancel{8}^1}{\cancel{15}_5}$$

$$= \frac{1}{70}\pi\log(1+8\sqrt{2}) \text{ grams.}$$

$\boxed{Q4}$ $R: \ z = 4 - \sqrt{x^2+y^2}, \quad z = \frac{1}{9}(x^2+y^2)$

(a) <u>Intersection</u>

Let $\rho = \sqrt{x^2+y^2}$

$\Rightarrow \ z = 4 - \rho = \frac{1}{9}\rho^2$

$\Rightarrow \ 36 - 9\rho = \rho^2$

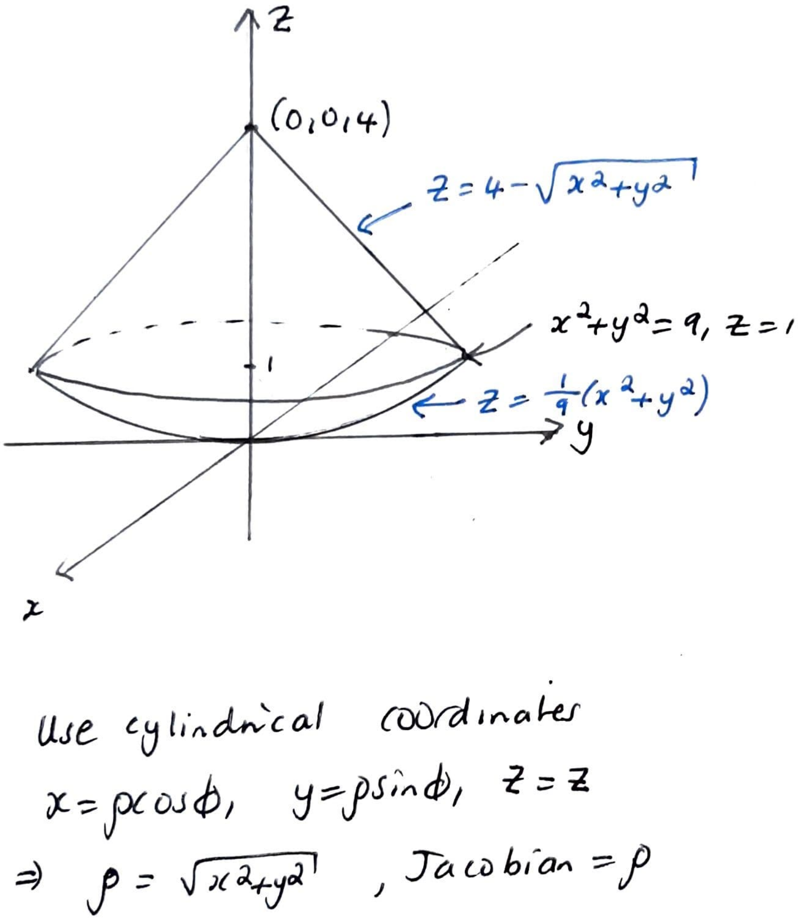
$\Rightarrow \ \rho^2 + 9\rho - 36 = 0$

$\Rightarrow \ (\rho+12)(\rho-3) = 0$

$\Rightarrow \ \rho = \underset{\times}{-12}, 3 \qquad \Rightarrow \ \rho = 3$

So intersection occurs at $\sqrt{x^2+y^2} = 3$

$\Rightarrow \ x^2+y^2 = 9$ and $z = 1$.



(b) Use cylindrical coordinates

$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z$

$\Rightarrow \ \rho = \sqrt{x^2+y^2}$, Jacobian $= \rho$

Now $\quad \frac{1}{9}(x^2+y^2) \le z \le 4 - \sqrt{x^2+y^2} \ \Rightarrow \ \frac{\rho^2}{9} \le z \le 4 - \rho$

$0 \le \phi \le 2\pi$

$0 \le \rho \le 3$

Find Moment of inertia I_z if $\mu = y + z$

$$I_z = \iiint_R (x^2+y^2)\,\mu(x,y,z)\,dV$$

$$= \int_0^3 \int_0^{2\pi} \int_{\rho^2/9}^{4-\rho} \rho^2(\rho\sin\phi + z)\,\rho \; dz\,d\phi\,d\rho$$

$$= \int_0^3 \int_0^{2\pi} \int_{\rho^2/9}^{4-\rho} \rho^4\sin\phi + \rho^3 z \; dz\,d\phi\,d\rho$$

$$= \int_0^3 \int_0^{2\pi} \left[\rho^4\sin\phi\, z + \frac{\rho^3 z^2}{2}\right]_{z=\rho^2/9}^{z=4-\rho} d\phi\,d\rho$$

$$= \int_0^3 \int_0^{2\pi} \left[\rho^4(4-\rho)\sin\phi + \frac{\rho^3}{2}(4-\rho)^2 - \frac{\rho^6}{9}\sin\phi - \frac{\rho^7}{162}\right] d\phi\,d\rho$$

$$= \int_0^3 \left[\underbrace{-\rho^4(4-\rho)\cos\phi}_{\text{cancels (0)}} + \frac{\rho^3}{2}(4-\rho)^2\phi + \underbrace{\frac{\rho^6}{9}\cos\phi}_{\text{cancels (0)}} - \frac{\rho^7\phi}{162}\right]_{\phi=0}^{\phi=2\pi} d\rho$$

$$= \int_0^3 \pi\rho^3(4-\rho)^2 - \frac{\rho^7\pi}{81}\,d\rho$$

$$= \pi\int_0^3 \rho^3(16-8\rho+\rho^2) - \frac{\rho^7}{81}\,d\rho$$

$$= \pi\int_0^3 16\rho^3 - 8\rho^4 + \rho^5 - \frac{\rho^7}{81}\,d\rho$$

$$= \pi\left[4\rho^4 - \frac{8}{5}\rho^5 + \frac{1}{6}\rho^6 - \frac{\rho^8}{648}\right]_{\rho=0}^{\rho=3}$$

$$= \pi\left[324 - \frac{1944}{5} + \frac{729}{6} - \frac{6561}{648}\right]$$

$$= \pi\left[324 - \frac{1944}{5} + \frac{243}{2} - \frac{81}{8}\right]$$

$$= \pi\left[\frac{12960 - 15552 + 4860 - 405}{40}\right]$$

$$= \frac{1863\pi}{40}$$