1.6 Vectors in $\mathbb{R}^n$ and applications

Since Descartes, the space we live in is described mathematically as

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\},$$

whereas the inhabitants of Abbott's *Flatland: A Romance of Many Dimensions* dwell in

$$\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}.$$

These have the common (and obvious) generalisation

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_1, x_2, \dots, x_n \in \mathbb{R}\},$$

where n is a nonnegative integer.

We refer to $\mathbb{R}^n$ as *n-space* (or *n-dimensional space*). *Vectors* are elements of $\mathbb{R}^n$ for some n.

(In a few weeks we'll take a more conceptual point of view that will largely liberate vectors of the shackles of coordinates, but for now they are n-tuples of coordinates.)

In $\mathbb{R}^3$, it is customary to single out three particularly simple vectors:

$$\mathbf{i} = (1, 0, 0), \qquad \mathbf{j} = (0, 1, 0), \qquad \mathbf{k} = (0, 0, 1).$$

Part of their mystique is that any vector $\mathbf{v} \in \mathbb{R}^3$ can be written as

$$\mathbf{v} = (v_1, v_2, v_3) =$$

Geometrically, the vector $\mathbf{v}$ is the position vector of the point $P = (v_1, v_2, v_3)$ (relative to the point $O = (0, 0, 0)$).

We define the *length* (or *norm* or *magnitude*) of $\mathbf{v} \in \mathbb{R}^n$ as

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

(You should convince yourself that this definition matches your intuition in $\mathbb{R}^2$ and $\mathbb{R}^3$.)

Example 1.44.

$$\|2\mathbf{i} - \mathbf{j} + 2\mathbf{k}\| =$$

A *unit vector* is a vector of length 1.

1.6.1 Arithmetic operations on vectors

Some of these will look awfully familiar.

Scalar multiplication Given $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we define

$$\lambda\mathbf{v} = (\lambda v_1, \dots, \lambda v_n) \in \mathbb{R}^n.$$

Geometrically:

Two nonzero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are said to be *parallel* if $\mathbf{u} = \lambda\mathbf{v}$ for some scalar $\lambda \in \mathbb{R}$.

Example 1.45. Find a unit vector parallel to $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

let $\vec{u} = \lambda \cdot \vec{v} = \lambda(2\vec{i} - \vec{j} + 2\vec{k})$

$= 2\lambda\vec{i} - \lambda\vec{j} + 2\lambda\vec{k}$

$\|u\| = \sqrt{(2\lambda)^2 + (-\lambda)^2 + (2\lambda)^2}$

$\hat{v} = \frac{1}{\|\vec{v}\|}\vec{v}$

$-\hat{v} = -\frac{1}{\|\vec{v}\|}\vec{v}$

Vector addition Given $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we define

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n) \in \mathbb{R}^n.$$

Geometrically:

Properties of vector arithmetic

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$
- $\|\lambda\mathbf{u}\| = |\lambda|\|\mathbf{u}\|$

Euclidean inner product The *Euclidean inner product* (or *dot product* or *scalar product*) of $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n.$$

Properties of the dot product

- $\mathbf{u} \cdot \mathbf{v}$ is a scalar
- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
- $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$
- $\lambda(\mathbf{u} \cdot \mathbf{v}) = (\lambda \mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\lambda \mathbf{v})$

Geometrically, two non-parallel vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ form a plane $\mathcal{P}$ in $\mathbb{R}^n$. We define the *angle between* $\mathbf{u}$ and $\mathbf{v}$ to be the angle between these vectors as line segments in the plane $\mathcal{P}$ (where this notion is already familiar).

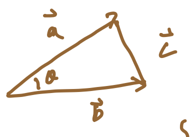
Proposition 1.46. For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ we have

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, \cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Proof. Recall the law of cosines: given a triangle with side lengths a, b, and c, and angle θ between the sides of lengths a and b, we have

$$a^2 + b^2 - 2ab\cos\theta = c^2.$$



$\vec{c} = \vec{b} - \vec{a}$

$|\vec{c}| = \sqrt{(\vec{b} - \vec{a})^2} \Rightarrow \|\vec{c}\|^2 = \|\vec{b}\|^2 + \|\vec{a}\|^2 - 2\vec{a} \cdot \vec{b} = a^2 + b^2 - 2\vec{a}\vec{b}$

$c^2 = a^2 + b^2 - 2ab\cos\theta \quad // \quad \Rightarrow \vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cdot \cos\theta$

Corollary 1.47. Two nonzero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are perpendicular if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

A B

for A only if B.

$A \Leftrightarrow B$

for $A \to B$ (if B then A)

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cdot \cos\theta = 0$

since $\|\vec{u}\|$ & $\|\vec{v}\| \neq 0$

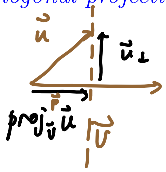
so $\cos\theta = 0$

since $\theta \in (0, \pi)$

so $\theta = \frac{\pi}{2}$

Orthogonal projection Given $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with $\mathbf{v} \neq \mathbf{0}$, the *orthogonal projection of* $\mathbf{u}$ *onto* $\mathbf{v}$ is

$$\mathbf{u}_\parallel = \mathrm{proj}_{\mathbf{v}}\mathbf{u} = (\mathbf{u} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}},$$



where $\hat{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$ denotes the *unit vector in the direction of* $\mathbf{v}$.

The *component of* $\mathbf{u}$ *orthogonal to* $\mathbf{v}$ is

$$\mathbf{u}_\perp = \mathbf{u} - \mathbf{u}_\parallel$$

$\|\vec{p}\| = \|\vec{u}\| \cdot \cos\theta = \|\vec{u}\| \cdot \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}$

$= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$

Here's a picture:

Example 1.48.

(a) Find the orthogonal projection of $\underbrace{\overset{\vec{u}}{(1, 2, 3)} \text{ onto } \overset{\vec{v}}{(1, 0, 0)}}$.

let $\vec{u} = \langle 1, 2, 3 \rangle$
$\vec{v} = \langle 1, 0, 0 \rangle$
$\mathrm{proj}_{\vec{v}}\vec{u} = (\vec{u} \cdot \vec{v}) \cdot \hat{v}$

-W1)

(b) Find the orthogonal projection of $(1, 2, 3)$ onto $(1, 1, 1)$.

(c) Find the component of $(1, 2, 3)$ orthogonal to $(1, 1, 1)$.

$\mathbb{R}^3$ **oddity: The cross product** Given $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, we define the *cross product of* $\mathbf{u}$ *and* $\mathbf{v}$ to be

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$$

$$(u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1) \in \mathbb{R}^3$$

Example 1.49.

$$(1,0,0) \times (0,1,0) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\hat{i} - 0\hat{j} + 1\hat{k}$$

$$(2,3,1) \times (1,1,1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 2\hat{i} - 1\hat{j} + 1k$$

$$(2, -1, -1)$$

Proposition 1.50. For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \text{---}\mathbf{u}\text{---} \\ \text{---}\mathbf{v}\text{---} \\ \text{---}\mathbf{w}\text{---} \end{vmatrix}$$

Proof. $\vec{v} \times \vec{w} \quad \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2w_3 - v_3w_2, w_1v_3 - v_1w_3, v_1w_2 - v_2$

$\vec{u} \cdot (\vec{v} \times \vec{w})$

let $\vec{v}, \vec{w} \in \mathbb{R}^3$, Assume $\vec{v} \nparallel \vec{w}$

let $\vec{n} = \vec{v} \times \vec{w}$ unit vector $\hat{n} = \frac{1}{\|\vec{n}\|} \cdot \vec{n}$

$\hat{n} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} -\hat{n}- \\ -\vec{v}- \\ -\vec{w}- \end{vmatrix}$

Example 1.51. Find a vector that is orthogonal to both $(1, 0, -1)$ and $(1, 2, 1)$.

$\vec{u}$ $\vec{v}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{vmatrix} = -2\hat{i} - 2\hat{j} - 2\hat{k}$$

$(-2, -2, -2)$

$$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$$

1.6.2 Solid geometry

By *solid geometry* we will mean geometry in three-dimensional space $\mathbb{R}^3$.

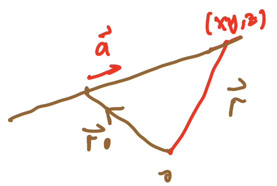
Lines The *vector equation* of a line is

any factor which goes along the line

$$\mathbf{r} = (x, y, z) = \mathbf{r}_0 + t\mathbf{v} = (x_0, y_0, z_0) + t(a, b, c), \qquad t \in \mathbb{R}.$$

The vector $\mathbf{v} = (a, b, c)$ is called a *direction vector* of the line.

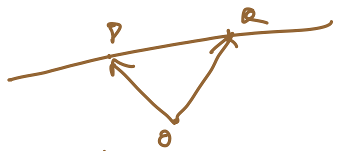
Geometrically:



Example 1.52. Find a vector equation of the line passing through the points $P(-1, 2, 3)$ and $Q(4, -2, 5)$.

$\vec{r_0} = (-1, 2, 3)$

$\vec{PQ} = (5, -4, 2)$



$= r_0 + tv$

$= (-1, 2, 3) + t(5, -4, 2)$

If $abc \neq 0$ then we can eliminate the parameter t in the vector equation to get the *Cartesian equations* of the line:

$$(t =) \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Example 1.53. Find Cartesian equations for the line passing through the points $P(-1, 2, 3)$ and $Q(4, -2, 5)$.

$$\frac{x+1}{4} = \frac{y-2}{-2} = \frac{x-3}{5}$$

Example 1.54. Find a vector equation for the line given by

$$\frac{x + 1}{5} = 3 - y = \frac{z - 4}{2}$$

$$\frac{x+1}{5} = \frac{y-3}{-1} = \frac{z-4}{2}$$

$(-1, 3, 4)$ $(5, -1, 2)$

$(-1, 3, 4) + t(5, -1, 2)$

no intersect

$-1+5s = \frac{3s-3}{3}$

$-\frac{7}{3} = \frac{8}{3}$

Two lines *intersect* if they have (at least) one common point.

Two lines are *parallel* if their direction vectors are parallel.

Some lines are neither intersecting nor parallel; we call them *skew*.

The *angle between two lines* is the angle between their direction vectors.

Example 1.55. Find a vector equation for the line that goes through the point $P(0, 0, 1)$ and is parallel to the line

L1 $$x - 1 = \frac{y+2}{2} = \frac{z-6}{2}$$ L2

$+(-1)\vec{k}$

$$\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-6}{2}$$

direction vector (x_0, y_0, z_0) $(1, -2, 6)$

(a, b, c) $(1, 2, 2)$

vector equation $\hat{r} = (0,0,1) + t(1,2,2)$

Example 1.56. Are the two lines

$$L_1 : x - 1 = \frac{y+2}{2} = \frac{z-6}{2}$$
$$L_2 : \frac{x+1}{5} = 3 - y = \frac{z-4}{2}$$

$\vec{v} = (1,2,2)$

direction $\vec{u} = (5,-1,2)$

intersecting, parallel, or skew?

$\vec{v}$ is not parallel to $\vec{u}$

$\Rightarrow$ two line is not parallel.

L1: $(1,-2,6) + t(1,2,2)$ $t \in \mathbb{R}$

L2: $(-1,3,4) + s(5,-1,2)$ $s \in \mathbb{R}$

$\Rightarrow$ to solve intersection

$(1,-2,6) + t(1,2,2) = (-1,3,4) + s(5,-1,2)$

for s and t

$$\begin{cases} 1+t = -1+5s \\ -2+2t = 3-s \\ 6+2t = 4+2s \end{cases}$$

$8 = 1+3s$

$s = \frac{7}{3}$

$1 + \frac{4}{3} = \frac{7}{3}$

then skew

The *(shortest) distance* between two non-parallel lines

$$L_1\colon \mathbf{u}_1 = \mathbf{r}_1 + t_1\mathbf{v}_1$$
$$L_2\colon \mathbf{u}_2 = \mathbf{r}_2 + t_2\mathbf{v}_2$$

is

$$? \quad \left|(\mathbf{r}_2-\mathbf{r}_1)\cdot \widehat{(\mathbf{v}_1\times\mathbf{v}_2)}\right| \rightarrow \text{unit vector}$$

Example 1.57. Find the distance between the lines in Example 1.56.

$L1: \underbrace{(1,-2,6)}_{\vec{r_1}} + t\,\underbrace{(1,2,2)}_{\vec{v_1}}$

$L2: \underbrace{(-1,3,4)}_{\vec{r_2}} + s\,\underbrace{(5,-1,2)}_{\vec{v_2}}$

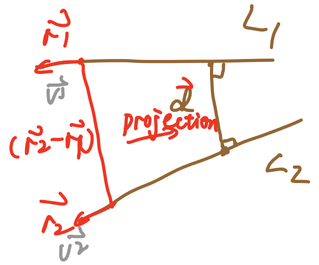
$$\begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 5 & -1 & 2 \end{vmatrix} = 6\vec{i} + 8\vec{j}$$

$$|(\vec{r_2}-\vec{r_1})\cdot(\vec{v_1}\times\vec{v_2})|$$

$$= |(-2,5,-2)\cdot(6,\cancel{8},-11)|$$

$$\|\vec{v_1}\times\vec{v_2}\| = \sqrt{36+64+121} = \sqrt{221}$$

$=$

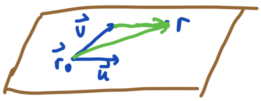


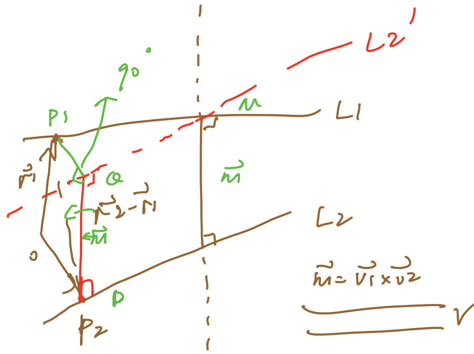
Planes The *vector equation* of a plane is

$$\mathbf{r} = (x,y,z) = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v}, \qquad s,t\in\mathbb{R},$$

where $\mathbf{r}_0, \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ are fixed vectors such that $\mathbf{u}$ is not parallel to $\mathbf{v}$.

Geometrically:





$PQ \perp$ Area QP_1M. $\quad L_2' \parallel L_2$

$\Rightarrow PQ \perp P_1Q$

P_2, $\vec{r_2}-\vec{r_1}$, Q, $\vec{n}$, P_1

Example 1.58. Find a vector equation for the plane passing through the points $P(-1,2,3)$, $Q(4,-2,5)$, and $R(1,0,1)$.

$\vec{r_0} = (-1,2,3)$

$\vec{u} = \vec{PQ} = (5,-4,2)$

$\vec{v} = \vec{PR} = (2,-2,-2) = 2(1,-1,1)$

$(x,y,t) = (-1,2,3) + t(5,-4,2) + s(1,-1,1)$

The *Cartesian equation* of a plane is

$$ax + by + cz = d,$$

where $a,b,c,d \in \mathbb{R}$ are fixed and at least one of a,b,c is nonzero.

Example 1.59. Find a vector equation for the plane with Cartesian equation

$$x + y + z = 1.$$

find line!!

$\vec{r_0}$ $(1,0,0)$

$\vec{v} = (-1,1,0)$ $\quad (0,1,0) \quad$ $\vec{u} = (-1,0,1)$

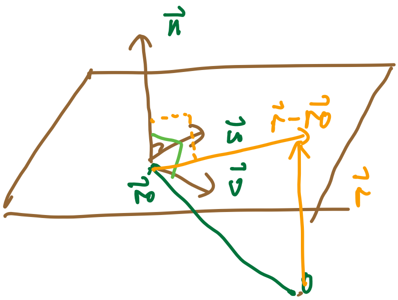
$(0,0,1)$

Consider a plane given by the vector equation

$$\mathbf{r} = \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v}.$$

The vector $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ is a *normal vector* to the plane, that is $\mathbf{n}$ is orthogonal to any vector in the plane.

Since $\mathbf{r} - \mathbf{r}_0$ is a vector in the plane, we have



$\vec{n} \times (\vec{r} - \vec{r_0}) = 0$

$\vec{r_0} = (x_0, y_0, z_0)$

$\vec{r} = (x, y, z)$

Example 1.60. Find the Cartesian equation of the plane passing through the points $P(2, 1, -1)$, $Q(3, 0, 1)$, and $R(-1, 1, -1)$.

$P\ (2, 1, -1)$

$\vec{u} = \vec{PQ} = (1, -1, 2)$

$\vec{v} = \vec{PR} = (-3, 0, 0).$

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ -3 & 0 & 0 \end{vmatrix} = 0\hat{i} - 6\hat{j} - 3\hat{k} = (0, -6, -3)$$

$$= -3(0, 2, 1)$$

$\vec{m} = (0, 2, 1)$

$(x, y, z) \times (0, 2, 1) = (2, 1, -1) \cdot (0, 2, 1)$

$2y + z = 1$

Example 1.61 (Intersection of a line and a plane)**.** Find the point(s) of intersection between the line

$$(x, y, z) = \quad \mathbf{r} = (0, -1, 3) + t(-1, 2, 1)$$

and the plane

$$x + y + z = 1.$$

$$\begin{cases} x = -t \\ y = -1 + 2t \\ z = 3 + t \end{cases}$$

$$(-t) + (-1 + 2t) + (3 + t) = 1$$

$$t = -\frac{1}{2}$$

$$\begin{cases} x = \frac{1}{2} \\ y = -2 \\ z = \frac{5}{2} \end{cases}$$ point of intersection

Example 1.62 (Intersection between two planes)**.** Find the line of intersection between the plane

$$x + y + z = 1$$

and the plane passing through the points $P(2, 1, -1)$, $Q(3, 0, 1)$, and $R(-1, 1, -1)$.

$$\begin{cases} x + y + z = 1 \\ 2y + z = 1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{1}{2} R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

for $z = t$

$$\begin{cases} x = \frac{1}{2} - \frac{1}{2} t \\ y = \frac{1}{2} - \frac{1}{2} t \end{cases}$$

$$\underline{\underline{x = y}}$$

The *angle between two planes* is, by definition, the angle between their normal vectors.

Example 1.63 (The angle between two planes)**.** Find the angle between the plane

$$x + y + z = 1$$

and the plane passing through the points $P(2,1,-1)$, $Q(3,0,1)$, and $R(-1,1,-1)$.

for 1st plane $(x,y,z)\cdot\vec{n} = 1$ → normal vector

$\vec{n_1} = (1,1,1)$

for 2nd plane

$3 = \vec{n_1}\cdot\vec{n_2} = \|\vec{n_1}\|\,\|\vec{n_2}\| \cos\theta$

$= \sqrt{3}\cdot\sqrt{3}\cos\theta$

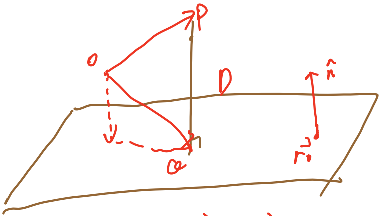
$\cos\theta = \frac{3}{5}$

The *(shortest) distance* between the point $P(x_0, y_0, z_0)$ and the plane with Cartesian equation $ax + by + cz = d$ is

$$D = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}}$$

Example 1.64. Find the distance between the point $P(2,1,-1)$ and the plane

$$x + y + z = 1.$$



$D = \frac{|2+1-1-1|}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\vec{PQ} = t\vec{n}$

$\vec{PQ} = \vec{OQ} - \vec{OP}$

$\vec{OQ} = \vec{PQ} + \vec{OP} = t\vec{n} + \vec{u}$

$(\vec{OQ} - \vec{r})\cdot\vec{n} = 0$

$(t\vec{n} + \vec{u} - \vec{r_0})\,\vec{n} = 0$

$t\|\vec{n}\|^2 + \vec{u}\cdot\vec{n} - \underbrace{\vec{r_0}\cdot\vec{n}}_{d} = 0$

$t\|\vec{n}\| = \frac{d - \vec{u}\cdot\vec{n}}{\|\vec{n}\|}$

$D = \|\vec{PQ}\|$

2 An introduction to rigorous mathematics

2.1 Sets and number systems

It is surprisingly subtle to give good axiomatic presentations of the concepts of set and element, which is why set theory is still an active branch of research in the 21-st century.

We will instead rely on our naive grasp of these concepts. In short, a *set* S is a collection of things that we call the *elements* of S.

The notation $s \in S$ signifies that s is an element of S.

Example 2.1.

$$\begin{aligned} \emptyset &= \{\} \\ \text{Lecturers} &= \{\text{Alex Ghitza}\} \\ \text{Students} &= \{\text{Aaron, Abhiram}, \ldots, \text{Zi, Ziyuan}\} \\ \text{Suits} &= \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\} \end{aligned}$$

We say that a set S is a *subset* of a set T if every element of S is also an element of T, that is

$$\text{for all } s \in S \text{ we have } s \in T.$$

This is written as $S \subset T$.

We say that a set S is *equal to* a set T if $S \subset T$ and $T \subset S$, or equivalently if

$$s \in S \qquad \Longleftrightarrow \qquad s \in T.$$

Given set X, the subset of elements satisfying property P is

$S = \{ x \in X \mid x \text{ has property } P \}$

eg. $\text{Even} = \{ x \in \mathbb{Z} \mid x \text{ is even} \}$

2.1.1 Operations on sets

Here we will assume that all the sets we consider are subsets of a fixed set X.

Union If A and B are sets, then their *union* is

$$A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}.$$

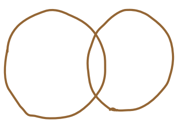
Intersection If A and B are sets, then their *intersection* is

$$A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}.$$

Difference If A and B are sets, then the *difference* $A \setminus B = A - B$ is

$$A \setminus B = \{x \in X \mid x \in A \text{ and } x \notin B\}.$$

Example 2.2 (Everybody loves Venn diagrams)**.**



(Cartesian) product Let A and B be sets. Their *Cartesian product* is the set of all pairs:

$$A \times B = \{(a, b) \mid a \in A, b \in B\}. \qquad \text{pair}$$

Example 2.3. If $A = \{1, 3, 5\}$ and $B = \{\alpha, \beta\}$, then

$$A \times B = \{(1,\alpha), (1,\beta), (3,\alpha), (3,\beta), (5,\alpha), (5,\beta)\}$$

Example 2.4. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(a,b) \mid a \in \mathbb{R}, b \in \mathbb{R}\}$

This can be extended to the product of more than two sets:

$$S_1 \times \cdots \times S_n = \{(x_1, x_2, \cdots x_n) \mid x_1 \in S_1, x_2 \in S_2 \cdots x_n \in S_n\}$$

2.1.2 Numbers

Kronecker: "God created the natural numbers; all the rest is the work of man."

There is actually an axiomatic treatment of *natural numbers* (due to Peano), but we will take them as given:

$$\mathbb{N} = \{1, 2, 3, 4, 5, \dots, 196785219, 196785220, \dots\}$$

A natural operation on $\mathbb{N}$ is addition. This leads one to consider equations of the form

$$x + 3 = 5 \qquad x = 2 \in \mathbb{N}$$
$$x + 6 = 5$$

We can resolve this difficulty by enlarging our notion of number. Define the *set of integers* $\mathbb{Z}$ as pairs

$$\mathbb{Z} = \{(a,b) | a \in \mathbb{N}, b \in \mathbb{N}\}/\sim, \quad \text{where we declare} \quad (a,b) \sim (c,d) \quad \text{if } a+d = b+c.$$

We will think of the integer $(a,b) \in \mathbb{Z}$ in more conventional terms as $a-b$, and therefore write

$$\mathbb{Z} = \{\ldots, -196785220, -196785219, \ldots, -2, -1, 0, 1, 2, \ldots, 196785219, 196785220, \ldots\}.$$

Now we can solve **any** equation of the form

$$x + b = a \qquad \text{given } a, b \in \mathbb{Z}.$$

But $\mathbb{Z}$ also has another natural operation, namely multiplication. This leads one to consider equations of the form

$$3x = -6$$
$$3x = -5$$

We can enlarge our notion of number yet again. Define the *set of rational numbers* $\mathbb{Q}$ as pairs

$$\mathbb{Q} = \{(a,b) | a \in \mathbb{Z}, b \in \mathbb{Z} - \{0\}\}/\sim, \quad \text{where we declare} \quad (a,b) \overset{\text{similar}}{\sim} (c,d) \quad \text{if } ad = bc.$$

We will think of the rational number $(a,b) \in \mathbb{Q}$ in more conventional terms as the fraction $\frac{a}{b}$, and therefore write

$$\mathbb{Q} = \left\{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z} - \{0\}\right\}.$$

$\frac{a}{b} = \frac{c}{d}$ if $ad = bc$

Now we can solve any equation of the form

$$qx = p \qquad \text{given } p \in \mathbb{Q}, q \in \mathbb{Q} - \{0\}.$$

But multiplication also leads to equations of the form

$$x^2 = \frac{9}{4}$$ has two solution $x = \frac{3}{2}$ $x = -\frac{3}{2}$

$$x^2 = 2$$ has no solution in $\mathbb{Q}$

We introduce more numbers to fix this issue. The *set of real numbers* $\mathbb{R}$ is defined as the set of all (finite) limits of convergent sequences of rational numbers.

This allows us to solve any equation of the form

$$x^2 = a \qquad \text{given } a \in \mathbb{R}_{\geq 0},$$

but not equations such as

$$x^2 = -1.$$

This gets sorted by introducing a formal solution i of this equation, giving rise to the *set of complex numbers* $\mathbb{C}$ (more about these later).

The upshot is a sequence of larger and larger sets:

$$\mathbb{N} \subset \mathbb{Z} \subset \underbrace{\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}}_{\text{field.}},$$

each of which has arithmetic operations.

There are other number systems that are obtained by different mechanisms.

We single out one of these, the *set of integers modulo 2*:

$$\mathbb{F}_2 = \{0, 1\},$$

with addition and multiplication defined by

+	0	1
0	0	0
1	0	1

×	0	1
0	0	0
1	0	1

(One can do similar tricks modulo other prime numbers p.)

$F_5 = \{0, 1, 2, 3, 4\}$