MAST30001 Stochastic Modelling – 2020

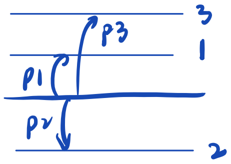
Assignment 2

If you haven't already, please complete the Plagiarism Declaration Form on the LMS before submitting this assignment.

Submission Instructions:

- Typed submissions (ideally using LaTeX) are preferred. For handwritten solutions:
  - Write your answers on blank paper. Write on one side of the paper only. Start each question on a new page. Write the question number at the top of each page.
  - Scan your solutions to a single PDF file with a mobile phone or a scanner. Scan from directly above to avoid any excessive keystone effect. Check that all pages are clearly readable and cropped to the A4 borders of the original page. Poorly scanned submissions may be impossible to mark.
- Upload the PDF file to Gradescope via the LMS. Gradescope will ask you to identify on which of the uploaded pages your answers to each question are located.
- The submission deadline is **5:00pm on Thursday, 29 October, 2020**.

There are 2 questions, and both will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness, and style count.



1. Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ and assume that each arrival has a "type" from the set $\{1, \ldots, K\}$, where K is some positive integer. The probability that a given arrival is of type $j \in \{1, \ldots, K\}$ is p_j, and the types of different arrivals are independent. For $j = 1, \ldots, K$ and $t \geq 0$, let $N_t^{(j)}$ be the number of arrivals of type j in the interval $[0, t]$. base case k=2

   (a) Use the thinning theorem and induction to show that the processes $\left((N_t^{(j)})_{t\geq 0}\right)_{j=1}^{K}$ are independent Poisson processes, and identify their rates.

   (b) Assuming $K \geq 3$, find the chance that there are exactly two arrivals having type 1 or 2 in the interval $[0, 3/4]$, and exactly one arrival of type 3 in the interval $[1/2, 1]$.

   (c) Assuming again $K \geq 3$, given that $N_1 = 3$, find the chance that there are exactly two arrivals having type 1 or 2 in the interval $[0, 3/4]$, and exactly one arrival of type 3 in the interval $[1/2, 1]$.

   (d) For distinct fixed numbers $\alpha_1, \ldots, \alpha_K$, show that the process

   $$\left(\sum_{j=1}^{K} \alpha_j N_t^{(j)}\right)_{t\geq 0}$$

   has the same distribution as some compound Poisson process

   $$\left(\sum_{i=1}^{N_t} X_i\right)_{t\geq 0},$$

   and give the distribution of X_1.

[3 + 2 + 2 + 3 = 10 marks]

2. Customers arrive to an outback auto repair shop according to a Poisson process with rate λ per day. The shop has one mechanic who takes an exponential with rate μ per day amount of time to repair a car. In addition, if there are no cars in the shop, the mechanic will wait an exponential rate ν per day time, and if no car has arrived in that time, the mechanic will leave the shop and take a nap for an exponential rate ν time. If a car arrives while the mechanic is waiting to leave to take a nap, they will begin work on the car. During a nap, cars arriving for repair will move on to the next repair shop.

   (a) Model this system as a continuous time Markov chain and write down its state space and generator.

   (b) Determine values of λ, μ, and ν such that the stationary distribution of the Markov chain exists. For these values, determine the stationary distribution.

   (c) What is the stationary average number of customers waiting for service?

$D = \frac{1}{\mu} + \frac{1}{\mu} \sum_{i \geq 0} i \pi_i$

   (d) Given that a customer is not immediately rejected from the system, what is the average time they spend in the system?

   (e) What proportion of customers are rejected from the system?

[2 + 3 + 2 + 2 + 1 = 10 marks]

1. Let $(N_t)_{t\geq 0}$ be a Poisson process with rate λ and assume that each arrival has a "type" from the set $\{1,\ldots,K\}$, where K is some positive integer. The probability that a given arrival is of type $j \in \{1,\ldots,K\}$ is p_j, and the types of different arrivals are independent. For $j = 1,\ldots,K$ and $t \geq 0$, let $N_t^{(j)}$ be the number of arrivals of type j in the interval $[0,t]$.

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   has the same distribution as some compound Poisson process

$$\left(\sum_{i=1}^{N_t} X_i\right)_{t\geq 0},$$

   and give the distribution of X_1.

[3 + 2 + 2 + 3 = 10 marks]

(a) base case $k=2$ $\quad ((N_t^{(j)})_{t\geq 0})_{j=1}^{2}$

then by thinning of a Poisson process.

$\{N_t^{j=1} : t\geq 0\}$ $\quad \{N_t^{j=2} : t\geq 0\}$ are independent Poisson process with rate λp_1 and λp_2 respectively

$k=m$

assume $((N_t^{(j)})_{t\geq 0})_{j=1}^{m}$ are independent Poisson processes.

to prove $((N_t^{(j)})_{t\geq 0})_{j=1}^{m+1}$ are independent Poisson process.

is to prove $(N_t^{(m+1)})_{t\geq 0}$ and $((N_t^{(j)})_{t\geq 0})_{j=1}^{m}$ are independent

$$P\left((N_t^{(m+1)})_{t\geq 0} = a,\ ((N_t^{(j)})_{t\geq 0})_{j=1}^{m} = b\right)$$

$$= P\left((N_t^{(m+1)})_{t\geq 0} = a,\ ((N_t^{(j)})_{t\geq 0})_{j=1}^{m+1} = a+b \right)$$

$$= P\left[(N_t^{(m+1)})_{t\geq 0} = a \,\middle|\, ((N_t^{(j)})_{t\geq 0})_{j=1}^{m+1} = a+b \right] P\left[((N_t^{(j)})_{t\geq 0})_{j=1}^{m+1} = a+b \right]$$

$$= \binom{a+b}{a}\left(\frac{p_{m+1}}{\cancel{\sum_{j=1}^{m+1} p_j}}\right)^a\left(\frac{\sum_{j=1}^{m} p_j}{\cancel{\sum_{j=1}^{m+1} p_j}}\right)^b \cdot \frac{e^{-\lambda\left(\sum_{j=1}^{m+1} p_j\right)t}\left(\lambda \cancel{\sum_{j=1}^{m+1} p_j}\right)^{a+b}}{(a+b)!}$$

$$= \frac{\cancel{(a+b)!}}{a!\,b!}\, e^{-\lambda \sum_{j=1}^{m} p_j t} \cdot e^{-\lambda p_{m+1} t}\, \lambda^{a+b} (p_{m+1})^a \left(\sum_{j=1}^{m} p_j\right)^b \frac{1}{\cancel{(a+b)!}}$$

$$= \frac{e^{-\lambda \sum_{j=1}^{m} p_j t}\left(\lambda \sum_{j=1}^{m} p_j\right)^b}{b!} \cdot \frac{e^{-\lambda p_{m+1} t}(\lambda p_{m+1})^a}{a!}$$

$$= P\left((N_t^{(m+1)})_{t\geq 0} = a\right) \cdot P\left(\left((N_t^{(j)})_{t\geq 0}\right)_{j=1}^{m} = b\right)$$

proved.

so $\left((N_t^{(j)})_{t\geq 0}\right)_{j=1}^{k}$ are independent processes

and rate are p_j for $j=1 \cdots k$ respectively

(b) Assuming $K \geq 3$, find the chance that there are exactly two arrivals having type 1 or 2 in the interval $[0, 3/4]$, and exactly one arrival of type 3 in the interval $[1/2, 1]$.

$$\left((N_t^{(j)})_{t \geq 0} \right)_{j=1}^{K}$$

$$P\left(N_{\frac{3}{4}}^{j=1} + N_{\frac{3}{4}}^{j=2} = 2,\ N_{\frac{1}{2}}^{j=3} = 1\right) = P\left(N_{\frac{3}{4}}^{j=1} + N_{\frac{3}{4}}^{j=2} = 2\right) \cdot P\left(N_{\frac{1}{2}}^{j=3} = 1\right).$$

$$P\left(\cancel{(N_{\frac{3}{4}}^{j})_{j=1}^{2} - (N_0^{j})_{j=1}^{2} = 2},\ N_1^{j=3} - N_{\frac{1}{2}}^{j=3} = 1 \right)$$

independent of type 1 or 2 or 3,

$$= P\left((N_{\frac{3}{4}}^{j})_{j=1}^{2} - (N_0^{j})_{j=1}^{2} = 2\right) P\left(N_1^{j=3} - N_{\frac{1}{2}}^{j=3} = 1\right)$$

competing exponential clock

$Poi\left(\lambda(p1+p2)\frac{3}{4}\right)$ ✓ $\qquad \downarrow Poi(\lambda p_3 \frac{1}{2})$.

$$= \frac{e^{-\frac{3}{4}\lambda(p1+p2)}\left(\frac{3}{4}\lambda(p1+p2)\right)^2}{2!} \cdot \frac{e^{-\frac{1}{2}\lambda p3}\left(\frac{1}{2}\lambda p3\right)^1}{1!}$$

$$= \frac{9}{32}\lambda^2 (p1+p2)^2 e^{-\frac{3}{4}\lambda(p1+p2)} \cdot e^{-\frac{1}{2}\lambda p3} \cdot \frac{1}{2}\lambda p3$$

$$= \frac{9}{64}\lambda^3 (p1+p2)^2 p3 \cdot e^{-\frac{3}{4}\lambda(p1+p2+\frac{2}{3}p3)}$$

(c) Assuming again $K \geq 3$, given that $N_1 = 3$, find the chance that there are exactly two arrivals having type 1 or 2 in the interval $[0, 3/4]$, and exactly one arrival of type 3 in the interval $[1/2, 1]$.

拆成4个部分

$$P\left(N_{\frac{3}{4}}^{j=1} + N_{\frac{3}{4}}^{j=2} = 2,\ N_{\frac{1}{2}}^{j=3} = 1 \,\middle|\, N_1 = 3\right)$$

$$= P\left(N_{\frac{3}{4}}^{j=1} + N_{\frac{3}{4}}^{j=2} = 2 \,\middle|\, \underbrace{N_{\frac{1}{2}}^{j=3} = 1}_{\text{independent}},\ N_1 = 3\right) \cdot P\left(N_{\frac{1}{2}}^{j=3} = 1 \,\middle|\, N_1 = 3\right)$$

$$= P\left(N_{\frac{3}{4}}^{j=1} + N_{\frac{3}{4}}^{j=2} = 2 \,\middle|\, N_1 = 3\right) P\left(N_{\frac{1}{2}}^{j=3} = 1 \,\middle|\, N_1 = 3\right)$$

$$= \binom{3}{2}\left[\frac{3}{4}(p_1+p_2)\right]^2\left[1 - \frac{3}{4}(p_1+p_2)\right] \times \binom{3}{1}\left[\frac{1}{2}p_3\right]\left[1 - \frac{1}{2}p_3\right)^2$$

$$= 9 \times \frac{9}{16}(p_1+p_2)^2\left(1 - \frac{3}{4}(p_1+p_2)\right] \cdot \frac{1}{2}p_3\left(1 - \frac{1}{2}p_3\right)^2$$

$$= \frac{81}{32}(p_1+p_2)^2 p_3\left[1 - \frac{3}{4}(p_1+p_2)\right]\left(1 - \frac{1}{2}p_3\right)^2$$

(d) For distinct fixed numbers $\alpha_1, \ldots, \alpha_K$, show that the process

$$\left(\sum_{j=1}^{K} \alpha_j N_t^{(j)}\right)_{t \geq 0}$$

has the same distribution as some compound Poisson process

$$\left(\sum_{i=1}^{N_t} X_i\right)_{t \geq 0},$$

and give the distribution of X_1.

$$M_{\left(\sum_{j=1}^{K} \alpha_j N_t^{(j)}\right)_{t \geq 0}}(\theta) = E\left(e^{\theta\left(\sum_{j=1}^{K} \alpha_j N_t^{(j)}\right)_{t \geq 0}}\right)$$

since $(N_t^{(j)})_{t \geq 0}$ are independent

$$= E\left(e^{\theta \alpha_1 (N_t^{(1)})_{t \geq 0}}\right) \cdot E\left(e^{\theta \alpha_2 (N_t^{(2)})_{t \geq 0}}\right)$$

$$\cdots E\left(e^{\theta \alpha_K (N_t^{(K)})_{t \geq 0}}\right)$$

$$= M_{(N_t^{(1)})_{t\geq 0}}(\alpha_1\theta) \cdots M_{(N_t^{(k)})_{t\geq 0}}(\alpha_k\theta)$$

$N_t^{(j)} \sim \mathrm{poi}(\lambda p_j t)$.

$$M_{(N_t^{(1)})_{t\geq 0}}(\alpha_1\theta) = e^{\lambda p_1 t(e^{\alpha_1\theta}-1)}$$

$$\therefore M_{(\sum_{j=1}^{k}\alpha_j N_t^{(j)})_{t\geq 0}}(\theta) = e^{\lambda p_1 t(e^{\alpha_1\theta}-1)} \cdot e^{\lambda p_2 t(e^{\alpha_2\theta}-1)}$$

$$\cdots e^{\lambda p_k t(e^{\alpha_k\theta}-1)}$$

$$= e^{\lambda t(\sum_{j=1}^{k} p_j e^{\alpha_j\theta} - \sum_{j=1}^{k} p_j)}$$

since $\sum_{j=1}^{k} p_j = 1 \Rightarrow = e^{\lambda t(\sum_{j=1}^{k} p_j e^{\alpha_j\theta}-1)}$. ①

Mgf of $(\sum_{i=1}^{N_t} x_i)_{t\geq 0}$

$$M_{(\sum_{i=1}^{N_t} x_i)_{t\geq 0}}(\theta) = E(e^{\theta(\sum_{i=1}^{N_t} x_i)_{t\geq 0}})$$

$$= E\left[E(e^{\theta(\sum_{i=1}^{N_t} x_i)_{t\geq 0}} \mid N_t)\right]$$

$$= E\left[E(e^{\theta x_1} \cdot e^{\theta x_2} \cdots e^{\theta x_{N_t}}\right]$$

$$= E\left[[E(e^{\theta x})]^{N_t}\right]$$

$$= E\left[[\underbrace{M_x(\theta)}_{z}]^{N_t}\right]$$

$= P_{N_t}(M_{X_i}(\theta))$

$N_t \sim Poi(\lambda t)$

$P_{N_t}(M_X(\theta)) = e^{-\lambda t(1-M_X(\theta))}$

$= e^{\lambda t(M_X(\theta)-1)}$ ②

$\therefore$ Let ① = ②

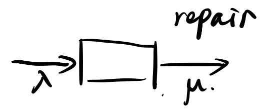
$M_X(\theta) = \sum_{j=1}^{k} p_j e^{\alpha_j \theta}$.

$M_X(\theta) = E(e^{\theta X}) = \sum_{x \in \mathcal{X}} e^{\theta x} P(X=x)$

X_1 has distribution

$P(X=\alpha_j) = p_j$ for $j = 1 \cdots k$

2. Customers arrive to an outback auto repair shop according to a Poisson process with rate λ per day. The shop has one mechanic who takes an exponential with rate μ per day amount of time to repair a car. In addition, if there are no cars in the shop, the mechanic will wait an exponential rate ν per day time, and if no car has arrived in that time, the mechanic will leave the shop and take a nap for an exponential rate ν time. If a car arrives while the mechanic is waiting to leave to take a nap, they will begin work on the car. During a nap, cars arriving for repair will move on to the next repair shop. reject



(a) Model this system as a continuous time Markov chain and write down its state space and generator.

(b) Determine values of λ, μ, and ν such that the stationary distribution of the Markov chain exists. For these values, determine the stationary distribution.

(c) What is the stationary average number of customers waiting for service?

(d) Given that a customer is not immediately rejected from the system, what is the average time they spend in the system?

(e) What proportion of customers are rejected from the system?

[2 + 3 + 2 + 2 + 1 = 10 marks]

let w be waiting, n be napping

(a). state space $\{(0,w), (0,n), 1, 2, \cdots\}$.

$$A = \begin{array}{c} \\ (0,w) \\ (0,n) \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccccc} (0,w) & (0,n) & 1 & 2 & 3 & 4 & \end{array} \\ \begin{bmatrix} -(\nu+\lambda) & \nu & \lambda & 0 & 0 & 0 & 0 & \cdots \\ \nu & -\nu & 0 & 0 & 0 & 0 & 0 & \cdots \\ \mu & 0 & -(\lambda+\mu) & \lambda & 0 & 0 & 0 & \\ 0 & 0 & \mu & -(\lambda+\mu) & \lambda & 0 & 0 & \\ 0 & 0 & \cdots & \mu & -(\lambda+\mu) & \lambda & 0 & \cdots \end{bmatrix} \end{array}$$

(b). $\pi A = 0$ such that the stationary distribution of Markov chain exist

$$\pi = (\pi_w, \pi_n, \pi_1, \pi_2, \cdots).$$

$$\begin{cases} -(\nu+\lambda)\pi_w + \nu\pi_n + \mu\pi_1 = 0 & ① \\ \nu\pi_w - \nu\pi_n = 0. & ② \\ \lambda\pi_w - (\lambda+\mu)\pi_1 + \mu\pi_2 = 0 & ③ \\ \lambda\pi_1 - (\lambda+\mu)\pi_2 + \mu\pi_3 = 0. & ④ \\ \lambda\pi_i - (\lambda+\mu)\pi_{i+1} + \mu\pi_{i+2} = 0 & \text{for } i \geq 1 \end{cases}$$

for ② $\pi_w = \pi_n$

from ① and $\pi_w = \pi_n$

$$-(\nu+\lambda)\pi_w + \cancel{\nu\pi_w} + \mu\pi_1 = 0$$

$$-\lambda\pi_w + \mu\pi_1 = 0 \qquad \lambda\pi_w = \mu\pi_1$$

$$\pi_n = \pi_w = \frac{\mu}{\lambda}\pi_1 \qquad \pi_w = \frac{\mu}{\lambda}\pi_1$$

from ②

$$\lambda\pi_w - (\lambda+\mu)\pi_1 + \mu\pi_2 = 0$$

$$\cancel{\mu\pi_1} - (\lambda+\cancel{\mu})\pi_1 + \mu\pi_2 = 0$$

$$\lambda\pi_1 = \mu\pi_2$$

$$\pi_2 = \frac{\lambda}{\mu}\pi_1$$

guess

$$\pi_k = \left(\frac{\lambda}{\mu}\right)^{k-1}\pi_1$$

substitute into ⑤

$$LHS = \lambda\left(\frac{\lambda}{\mu}\right)^{i-1}\pi_1 - (\lambda+\mu)\left(\frac{\lambda}{\mu}\right)^{i}\pi_1 + \mu\left(\frac{\lambda}{\mu}\right)^{i+1}\pi_1$$

$$= \pi\left[\left(\frac{\lambda}{\mu}\right)^{i}\cdot\mu - (\lambda+\mu)\cdot\left(\frac{\lambda}{\mu}\right)^{i} + \lambda\left(\frac{\lambda}{\mu}\right)^{i}\right]$$

$$= \pi\left(\frac{\lambda}{\mu}\right)^{i}\left[\mu - (\lambda+\mu) + \lambda\right] = 0. = RHS.$$

$$\therefore \pi_w = \pi_n = \frac{\mu}{\lambda}\pi_1 \Rightarrow \pi_1 = \frac{\lambda}{\mu}\pi_n = \frac{\lambda}{\mu}\pi_w$$

for $i \geq 1$ $\pi_i = \left(\frac{\lambda}{\mu}\right)^{i-1}\pi_1 \Rightarrow \pi_i = \left(\frac{\lambda}{\mu}\right)^{i}\pi_w,\ i \geq 1$

$\pi_n = \pi_w.$

$$\sum_{i\geq 1}\left(\frac{\lambda}{\mu}\right)^{i}\pi_w + \pi_w + \pi_n$$

$$= \sum_{i\geq 0}\left(\frac{\lambda}{\mu}\right)^{i}\pi_w + \pi_w$$

converge need $\frac{\lambda}{\mu} < 1$

$$= \left(\frac{1}{1-\frac{\lambda}{\mu}} + 1\right) \pi w = 1$$

$$\left(\frac{\mu}{\mu-\lambda} + \frac{\mu-\lambda}{\mu-\lambda}\right) \pi w$$

$$= \frac{2\mu-\lambda}{\mu-\lambda} \pi w = 1$$

$$\Rightarrow \quad \pi w = \pi n = \frac{\mu-\lambda}{2\mu-\lambda}$$

$$\pi i = \left(\frac{\lambda}{\mu}\right)^{i} \cdot \frac{\mu-\lambda}{2\mu-\lambda} \quad i \geq 1$$

(c) What is the stationary average number of customers waiting for service?

$$Lq = E(\max(0, X_t - 1))$$

$$= 0 \cdot \pi w + 0 \cdot \pi n + \sum_{k\geq 1} \pi k (k-1)$$

$$= \sum_{k\geq 1} \left(\frac{\lambda}{\mu}\right)^{k} \cdot \frac{\mu-\lambda}{2\mu-\lambda} (k-1)$$

$$= \frac{\mu-\lambda}{2\mu-\lambda} \sum_{k\geq 1} \left(\frac{\lambda}{\mu}\right)^{k} (k-1)$$

Let $j = k-1$

$$= \frac{\mu-\lambda}{2\mu-\lambda} \sum_{j\geq 0} \left(\frac{\lambda}{\mu}\right)^{j+1} j$$

$$= \frac{\mu-\lambda}{2\mu-\lambda} \left(\frac{\lambda}{\mu}\right)^{2} \sum_{j\geq 0} \left(\frac{\lambda}{\mu}\right)^{j-1} j$$

$$= \frac{\mu-\lambda}{2\mu-\lambda} \left(\frac{\lambda}{\mu}\right)^{2} \cdot \frac{1}{\left(1-\frac{\lambda}{\mu}\right)^{2}}$$

$$\sum_{j\geq 0} \left(\frac{\lambda}{\mu}\right)^{j-1} j$$

Let $t = \frac{\lambda}{\mu}$

$$= \sum_{j\geq 0} t^{j-1} j \Big|_{t=\frac{\lambda}{\mu}}$$

$$= \sum_{j\geq 0} \frac{d}{dt}(t^{j}) \Big|_{t=\frac{\lambda}{\mu}}$$

$$= \frac{d}{dt} \sum_{j\geq 0} t^{j} \Big|_{t=\frac{\lambda}{\mu}}$$

$$= \frac{d}{dt}\left(\frac{1}{1-t}\right)\Big|$$

$$= \frac{\cancel{\mu-\lambda}}{2\mu-\lambda} \lambda^2 \frac{1}{(\mu-\lambda)^{\cancel{2}}}$$

$$= \frac{\lambda^2}{(\mu-\lambda)(2\mu-\lambda)}$$

$$\Big|_{t=\frac{\lambda}{\mu}}$$

$$= \frac{1}{(1-t)^2}\Big|_{t=\frac{\lambda}{\mu}}$$

$$= \frac{1}{(1-\frac{\lambda}{\mu})^2}$$

(d) Given that a customer is not immediately rejected from the system, what is the average time they spend in the system?

since a customer is not immediately rejected from the system, then the system will not in $(0, n)$ state.

then $\pi' = (\pi_w', \pi_1', \pi_2', \cdots)$

where $\pi_w' = \frac{\pi_w}{1-\pi_n}$

$\pi_1' = \frac{\pi_1}{1-\pi_n}$

$\vdots$

$$L' = 0 \cdot \pi_w' + 1 \cdot \pi_1' + 2 \cdot \pi_2' + \cdots + k\pi_k' + \cdots$$

$$= \sum_{k\geq 1} k\pi_k'$$

$$= \sum_{k\geq 0} k \cdot \frac{\pi_k}{1-\pi_n}$$

$$= \sum_{k\geq 0} k \cdot \frac{(\frac{\lambda}{\mu})^k \cdot \frac{\mu-\lambda}{2\mu-\lambda}}{1-\frac{\mu-\lambda}{2\mu-\lambda}}$$

$$= \sum_{k\geq 0} k \cdot \frac{(\frac{\lambda}{\mu})^k \cdot \frac{\mu-\lambda}{\cancel{2\mu-\lambda}}}{\frac{\mu}{\cancel{2\mu-\lambda}}}$$

$\pi_n = \frac{\mu}{\lambda}\pi_1$

$\pi_i = (\frac{\lambda}{\mu})^i \cdot \frac{\mu-\lambda}{2\mu-\lambda}$

$$\pi_n = \frac{\cancel{\mu}}{\cancel{\lambda}} \cdot \frac{\cancel{\lambda}}{\cancel{\mu}} \cdot \frac{\mu-\lambda}{2\mu-\lambda}$$

$$= \frac{\mu-\lambda}{2\mu-\lambda}$$

$$= \sum_{k\geq 0} k \cdot \frac{(\frac{\lambda}{\mu})^k \cdot \mu-\lambda}{\mu}$$

$$= \sum_{k\geq 0} k \left(\frac{\lambda}{\mu}\right)^k \left(1-\frac{\lambda}{\mu}\right)$$

$$= \left(1-\frac{\lambda}{\mu}\right) \sum_{k\geq 0} k \left(\frac{\lambda}{\mu}\right)^k$$

$$\sum_{k\geq 0} k \left(\frac{\lambda}{\mu}\right)^k = \left(\frac{\lambda}{\mu}\right) \sum_{k\geq 0} k \left(\frac{\lambda}{\mu}\right)^{k-1}$$

let $\frac{\lambda}{\mu} = t$

$$= t \sum_{k\geq 0} \frac{d}{dt} t^k \Big|_{t=\frac{\lambda}{\mu}}$$

$$= t \frac{d}{dt} \sum_{k\geq 0} t^k \Big|_{t=\frac{\lambda}{\mu}}$$

$$= t \frac{d}{dt} \left(\frac{1}{1-t}\right) \Big|_{t=\frac{\lambda}{\mu}}$$

$$= t \cdot \frac{1}{(1-t)^2} \Big|_{t=\frac{\lambda}{\mu}}$$

$$= \frac{\lambda}{\mu} \cdot \frac{1}{(1-\frac{\lambda}{\mu})^2}$$

$$\therefore \quad L' = \cancel{\left(1-\frac{\lambda}{\mu}\right)} \cdot \frac{\lambda}{\mu} \cdot \frac{1}{(1-\frac{\lambda}{\mu})^{\cancel{2}}} = \frac{\lambda}{\mu-\lambda}$$

from Little's Law

$$L' = \lambda D'$$

$$D' = \frac{L'}{\lambda} = \frac{1}{\mu-\lambda}$$

(e) What proportion of customers are rejected from the system?

$$\pi_n = \frac{\mu}{\lambda} \pi_1 = \cancel{\frac{\mu}{\lambda}} \cdot \cancel{\frac{\lambda}{\mu}} \cdot \frac{\mu-\lambda}{2\mu-\lambda}$$

$\mu-\lambda$

$$= \frac{\mu-\lambda}{2\mu-\lambda}$$