The University of Melbourne
School of Mathematics and Statistics
Semester 1, 2020

MAST20004
Probability

STUDENT NAME:

EMAIL ADDRESS:

Cover image: Jakob Bernoulli (1654 - 1705, Basel)

(courtesy of mathematik.ch – `https://www.mathematik.ch/mathematiker/jakob_bernoulli.php`)

Jakob Bernoulli was a Swiss mathematician who made significant contributions to algebra, calculus, mechanics, calculus of variations, geometry, infinite series, and probability. His most renowned work was the derivation of the law of large numbers, which put simply, states that the average proportion of times an event occurs approaches the theoretical probability of the event, as the number of trials increases.

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MAST20004 Probabilty
Semester 1, 2020
Subject Organisation

1 *Overview and Learning Outcomes*

This subject offers a thorough grounding in the basic concepts of mathematical probability and probabilistic modelling. Topics covered include random experiments and sample spaces, probability axioms and theorems, discrete and continuous random variables/distributions (including measures of location, spread, and shape), expectations and generating functions, independence of random variables and measures of dependence (covariance and correlation), methods for deriving the distributions of transformations of random variables or approximations for them (including the central limit theorem).

The probability distributions and models discussed in the subject arise frequently in real world applications. These include a number of widely used one- and two-dimensional (particularly the bivariate normal) distributions and also fundamental probability models such as Poisson processes and Markov chains.

On completion of this subject students should

- have a systematic understanding of the basic concepts of probability space, probability distribution, random variable (including the bivariate case) and expectation;
- be able to use conditional expectations, generating functions and other basic techniques taught in the subject;
- be able to interpret a number of important probabilistic models, including simple random processes such as the Poisson process and finite discrete time Markov chains, and appreciate their relevance to real world problems;
- be able to formalise simple real-life situations involving uncertainty in the form of standard probabilistic models and to analyse the latter;
- develop understanding of the relevance of the probabilistic models from the subject to important areas of applications such as statistics and actuarial studies.

2 *Prerequisites*

One of

- MAST10006 Calculus 2 ($\geq$ 60%)
- MAST10009 Accelerated Mathematics 2
- MAST10019 Calculus Extension Studies ($\geq$ 60%)

plus one of

- MAST10007 Linear Algebra
- MAST10008 Accelerated Mathematics 1
- MAST10013 UMEP Mathematics for High Achieving Students
- MAST10018 Linear Algebra Extension Studies

3 *Credit Exclusions*

Students may only gain credit for one of

- MAST20004 Probability
- MAST20006 Probability for Statistics
- MAST30015 Statistics for Mechanical Engineers (prior to 2011)
- ELEN30002 Stochastic Signals and Systems (prior to 2011)

4 *Course Website*

Information regarding the assignments, tutorials, past exam papers, consultation hours, and other important information will be available from the LMS website at

`https://www.lms.unimelb.edu.au/`

5 *Problem Sheets*

There are seven problem sheets corresponding to the seven main topics covered in lectures. Answers are at the back of the problem sheet booklet, but full solutions will **not** be provided.

6 *Lectures and Tutorials*

There are two lecture streams.

- *Stream 1*: Dr Sophie Hautphenne

  Email: sophiemh@unimelb.edu.au
  Office: Room 146, Peter Hall Building

  – Monday 12 noon, Rivett Theatre, Redmond Barry
  – Thursday 10 am, JH Michell Theatre, Peter Hall
  – Friday 12 noon, Lyle Theatre, Redmond Barry

- *Stream 2*: Dr Mark Fackrell (subject coordinator)

  Email: fackrell@unimelb.edu.au
  Office: Room 148, Peter Hall Building

  – Monday 1 pm, Lyle Theatre, Redmond Barry
  – Thursday 2:15 pm, Lyle Theatre, Redmond Barry
  – Friday 1 pm, Lyle Theatre, Redmond Barry

There are 11 one-hour tutorials (one per week). Tutorials start in the second week of semester. Details of your tutorials are given on your personal timetable. Please visit Stop 1 if you are having difficulty accessing your timetable.

During tutorials you will be required to work in groups on the whiteboards. A sheet of questions for discussion will be provided by Thursday of the week before the tutorial (via the LMS), and full solutions will be provided on Friday the week the tutorial is held (via the LMS). The idea is to discuss the questions and their solution, and to learn about mathematics collaboratively with your fellow students.

The computer laboratories are held immediately following the tutorial. These are a critical component of the course; in this day and age every good job requires computational competency. At least one exam question will directly test material that is covered in the computer laboratory sessions.

7 *Lecture Slides*

Partial lecture slides can be downloaded from the LMS. These slides contain the theory and statements of the examples to be covered in lectures - space is left for the completion of the examples during lectures.

8 *Assessment*

The assessment is composed of two parts:

- A **three hour exam** at the end of semester;
- **Four assignments** due as follows:

  Assignment 1 - due 3 pm Melbourne time on Friday 3 April

  Assignment 2 - due 3 pm Melbourne time on Friday 24 April

  Assignment 3 - due 3 pm Melbourne time on Friday 8 May;

  Assignment 4 - due 3 pm Melbourne time on Friday 22 May.

The **Final Mark** in MAST20004 is computed as:

Final Mark = 80% Exam + 20% Assignments (5% per assignment)

Note:

- Assignments will generally consist of 4 or 5 questions, **two** of which will be randomly selected to be marked. Full solutions will be provided to all assignment questions and be made available on the LMS.
- If for any reason you think you will not be able to submit on time you need to see the Tutor Coordinator (Robert Maillardet) - preferably in person. See the Tutor Coordinator as soon as you become aware of any issue and preferably prior to the deadline. In general a medical certificate is required. Note that extensions are only granted in exceptional circumstances and only for a very limited time period.

- ***Do not use the myunimelb student portal to apply for 'special consideration' for any assignments; this online application is for the final exam only.***

9 *Reference Book*

There is no required text for the subject, but the lecture slides refer to the recommended text

Saeed Ghahramani, *Fundamentals of Probability with Stochastic Processes*, 3rd edition.

There are copies of this text on reserve in the ERC Library.

This table gives references to the relevant section(s) of Ghaharamani.

Topic	**Ghahramani**
Axioms of Probability	
Introduction to probability	§1.1
Sample space and events	§1.2
Definition of probability	§1.1, §1.3
Probability axioms	§1.3
Probabilty theorems	§1.4
Conditional Probability and Independence	
Conditional probability	§3.1
Multiplication theorem	§3.2
Independence of events	§3.5
Law of total probability	§3.3
Bayes' formula	§3.4
Random Variables and Distribution Functions	
Random variables	§4.1
Discrete random variables	§4.2, §4.3
Distribution functions	§4.2
Continuous random variables	§4.2, §6.1
Probability density functions	§6.1
Expectation of discrete random variables	§4.4
Expectation of continuous random variables	§6.3
Expectation of functions of random variables	§4.4, 6.3
Variance	§4.5, §6.3
Higher moments of a random variable	§4.5, §11.1
Computing higher moments through tail probabilities	§6.3

Topic	**Ghahramani**
Special Probability Distributions	
Bernoulli random variables, mean and variance	§5.1
Binomial random variables, distribution, mean and variance	§5.1
Geometric random variables, distribution, mean and variance	§5.3
Memoryless property for geometric distributions	pg 216–217
Negative binomial random variables, distribution, mean and variance	§5.3
Extended binomial theorem	none
Hypergeometric random variables, distribution, mean and variance	§5.3
Poisson random variables, distribution, mean and variance	§5.2
Poisson approximation to the binomial	§5.2
Discrete uniform random variables	none
Continuous uniform random variables, distribution, mean and variance	§7.1
Exponential random variables, distribution, mean and variance	§7.3
Memoryless property for exponential distributions	pg 288, 291
Gamma random variables, distribution, mean and variance	§7.4
Beta random variables, distribution, mean and variance	§7.5
Pareto random variables, distribution, mean and variance	none
Normal random variables, distribution, mean and variance	§7.2
Weibull random variables, distribution, mean and variance	none
Cauchy random variables, distribution, mean and variance	§6.3
Lognormal random variables, distribution, mean and variance	§11.1
Transformations of random variables	§6.2
Bivariate Random Variables	
Joint and marginal probability mass functions	§8.1
Conditional probability mass functions	§8.3
Joint and marginal density functions	§8.1
Conditional density functions	§8.3
Bivariate normal distribution	§10.5
Independence of random variables	§8.2
Expectation of a product of random variables	§8.2
Transformations of bivariate random variables	§8.4
Probability distributions of sums of random variables	pg 360–362
Expectation of functions of bivariate random variables	§8.1
Expected values of sums of random variables	§10.1
Variability of sums of random variables	§10.2
Covariance of a linear combination of random variables	§10.2
Relationship between bivariate random variables	§10.2
Correlation	§10.3
Conditioning on random variables	§10.4
Conditional variance	§10.4
Approximating the mean and variance of functions of random variables	none

Topic	**Ghahramani**
Sums of Independent Random Variables and Limit Theorems	
Bienaymé's (Chebyshev's) inequality	§11.3
Probability generating functions	none
Moments of random variables	§4.5, §11.1
Skewness and kurtosis	§11.1
Moment generating functions	§11.1, §11.2
Sums of independent random variables	§11.2
Limiting distributions	§11.5
Law of large numbers	§11.4
Central limit theorem	§11.5
Stochastic Processes	
Introduction to stochastic processes	§12.1
Poisson processes	§5.2, §12.2
Discrete-time Markov chains	§12.3
Transition matrices	§12.3
Equilibrium behaviour of Markov chains	pg 554–559
Solution to the equilibrium equations	pg 554–559
State transition graphs	pg 554–555

For further reading, the following book is also recommended as it develops the subject along with many practical examples.

- Sheldon Ross, *A First Course in Probability*, 8th edition, Pearson, 2010.

For those students who would like to pursue a more sophisticated level of understanding of probability, the following two books are excellent resources.

- William Feller, *An Introduction to Probability Theory and its Applications*, Volume I, 3rd edition, John Wiley & Sons, 1968.
- Geoffrey Grimmett and Dominic Welsh, *Probability*, 2nd edition, Oxford University Press, 2014.

10 *Special Consideration for Exam*

If something major goes wrong during semester or you are sick during the examination period, you should consider applying for ***Special Consideration***.

For information see the Special Consideration link on the website:

`http://ask.unimelb.edu.au/app/home`

11 *Getting Help*

The first source of help is the person beside you in lectures and tutorials, who is doing the same problems as you are and having similar but perhaps not exactly the same difficulties. Remember though, that fellow students have no obligation to help you, nor you to help them. Forming a small study group of two to four people is a preferred mode for "trading secrets" for many students.

Starting in Week 1 the lecturers will be available for consultation.

Your tutor is also a source of help, but be aware that the problems for discussion in the tutorials are the ones on the weekly tutorial sheets, **not** the ones in this problem booklet.

12 *Calculators, Formulae Sheets, and Dictionary*

Students are not permitted to use dictionaries in the end of semester exam.

Students may bring one double-sided A4 sheet of handwritten notes into the exam room.

For MAST20004 Probability the only approved calculator is CasioFX82 (any suffix).
See

`http://ask.unimelb.edu.au/app/answers/detail/a_id/6175/~/calculators-in-examinations`

13 *Lecture-by-Lecture Outline*

This outline is a guide only - material to be covered in each lecture may vary from this lecture-by-lecture outline.

Lecture	Topic
	Axioms of Probability
1	Introduction to probability; random experiments; outcome spaces
2	Events; definition of probability
3	Axioms of probability and theorems; evaluating probabilities
	Conditional Probability and Independence
4	Conditional probability; multiplication theorem; independence of events; independence versus exclusion
5	Law of total probability; Bayes' formula
	Random Variables and Distribution Functions
6	Random variables; Discrete random variables
7	Definition and properties of distribution functions; continuous random variables; definition and properties of probability density functions
8	Interpretation of probability density functions; expectation; expectation of discrete random variables
9	Expectation of continuous random variables; expectation of functions of random variables; variance; higher moments of a random variable; computing higher moments through tail probabilities
	Special Probability Distributions
10	Bernoulli random variables; Bernoulli mean and variance; binomial random variables; binomial distribution, mean and variance; sampling with replacement
11	Geometric random variables; geometric distribution, mean and variance; memoryless property
12	Negative binomial random variables; extended binomial theorem; negative binomial distribution, mean and variance; hypergeometric random variables
13	Hypergeometric distribution, mean and variance; Poisson random variables; Poisson distribution, mean and variance; Poisson approximation to binomial
14	Discrete uniform random variables; discrete uniform mean and variance; continuous uniform random variables; continuous uniform mean and variance; exponential random variables
15	Exponential mean and variance; memoryless property; gamma random variables; gamma mean and variance
16	Beta distribution and its properties; Pareto distribution and its properties; Normal random variables; standardisation
17	Moments of standard normal random variables; normal approximations; Weibull distribution and its properties

Lecture	**Topic**
18	Good Friday - no lecture
19	Transformation of random variables, linear, monotonic, Cauchy distribution; lognormal distribution and its properties; transformation by $\psi(x) = \min(x, M)$; and transformation by square functions
	Bivariate Random Variables
20	Bivariate random variables; Distribution function of a bivariate random variable; joint and marginal probability mass functions; conditional probability mass functions; double integrals
21	Joint and marginal density functions; conditional density functions
22	Bivariate normal distribution
23	Independence of random variables; alternative characterisations; expectation of a product of random variables
24	Transformations of bivariate random variables; probability distribution of sums of random variables; probability distribution of products of random variables; expectation of functions of bivariate random variables
25	Expected values of sums of random variables; variability of sums of random variables; covariance of a linear combination of random variables
26	Relationship between bivariate random variables; correlation; conditioning on a given event; conditional expectation given a discrete random variable; conditional expectation given a continuous random variable; $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X\|Y))$
27	Conditional variance; $V(X) = \mathbb{E}(V(X\|Y)) + V(\mathbb{E}(X\|Y))$; approximations for the mean and variance of functions of random variables
	Sums of Independent Random Variables and Limit Theorems
28	Bienaymé's (Chebyshev's) inequality; generating functions; probability generating functions
29	Properties of probability generating functions; convolution theorem; distribution of sums of independent random variables
30	Moment generating functions; moments of random variables; properties of moment generating functions
31	Cumulant generating function and its properties; Skewness and Kurtosis; Laplace transforms; inverting pgf, mgf and Laplace transform to find the distribution of a random variable; characteristic functions
32	limiting distributions; convergence in distribution; law of large numbers; central limit theorem
	Stochastic Processes
33	Stochastic processes; Poission processes; discrete-time Markov chains
34	Transition matrices; equilibrium behaviour and interpretation of Markov chains
35	Solution of equilibrium equations; state transition graphs
36	Catch-up and/or Revision

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Problem Sheet 1
Axioms of Probability

1. Sample Space

Define a sample space for the experiment of putting three different books on a shelf in random order. If two of these three books are a two-volume dictionary, describe the event that these volumes stand in increasing order side-by-side (i.e., volume I precedes volume II).

2. Events

Let E, F, and G be three events; explain the meaning of the two relations $E \cup F \cup G = G$ and $E \cap F \cap G = G$.

3. More Events

Prove that the event B is impossible if and only if for every event A,

$$A = (B \cap A^c) \cup (B^c \cap A).$$

4. Cards

In an experiment, cards are drawn, one by one, at random and successively from an ordinary deck of 52 cards. Let A_n be the event that no face card or ace appears on the first $n-1$ draws, and the nth draw is an ace. In terms of A_ns, find an expression for the event that an ace appears before a face card, if

(a) the cards are drawn with replacement;

(b) they are drawn without replacement.

5. Event Identities

Let A and B be two events. Prove the following relations by the elementwise method.

(a) $(A \backslash (A \cap B)) \cup B = A \cup B$;

(b) $(A \cup B) \backslash (A \cap B) = (A \cap B^c) \cup (A^c \cap B)$.

6. Infinite Sequence of Sets

Let $\{A_1, A_2, A_3, \ldots\}$ be a sequence of events of a sample space S. Find a sequence $\{B_1, B_2, B_3, \ldots\}$ of mutually exclusive events such that for all $n \geq 1$, $\cup_{i=1}^{n} A_i = \cup_{i=1}^{n} B_i$.

7. Hiring

A company has only one position with three highly qualified applicants: John, Barbara, and Marty. However, because the company has only a few women employees, Barbara's chance to be hired is 20% higher than John's and 20% higher than Marty's. Find the probability that Barbara will be hired.

8. Probability Statements

Which of the following statements is true? If a statement is true, prove it. If it is false, give a counterexample.

(a) If $P(A) + P(B) + P(C) = 1$, then A, B, and C are mutually exclusive events;

(b) If $P(A \cup B \cup C) = 1$, then A, B, and C are mutually exclusive events.

9. Probability Identity

Let A, B, and C be three events. Prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

10. Random Numbers

A number is selected randomly from the set $\{1, 2, \ldots, 1000\}$. What is the probability that

(a) it is divisible by 3 but not by 5;

(b) it is divisible neither by 3 nor by 5?

11. Voting

For a Democratic candidate to win an election, she must win districts I, II, and III. Polls have shown that the probability of winning I and III is 0.55, losing II but not I is 0.34, and losing II and III but not I is 0.15. Find the probability that this candidate will win all three districts. (Draw a Venn diagram.)

12. Boole's Inequality

Let A_1, A_2, A_3, $\ldots$ be a sequence of events of a sample space. Prove that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n).$$

This is called Boole's inequality.

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Problem Sheet 2

Conditional Probability and Independence

13. Dice

Suppose that two fair dice have been tossed and the total of their top faces is found to be divisible by 5. What is the probability that both of them have landed 5?

14. Movies

The cinemas of a town are showing seven comedies and nine dramas. Brian has seen five of the movies. If the first three movies he has seen are dramas, what is the probability that the last two are comedies? Assume that Brian chooses the shows at random and sees each movie at most once.

15. Conditional Probability Properties

Let Ω be the sample space of an experiment, and let B be an event of Ω with $P(B) > 0$. Prove that

(a) $P(A|B) \geq 0$ for any event $A \subseteq \Omega$;

(b) $P(\Omega|B) = 1$;

(c) If A_1, A_2, $\ldots$ is a sequence of mutually exclusive events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i \,|B\right) = \sum_{i=1}^{\infty} P(A_i|B).$$

16. Defective and Nondefective Items

If eight defective and 12 nondefective items are inspected one by one, at random and without replacement, what is the probability that

(a) the first four items inspected are defective;

(b) from the first three items at least two are defective?

17. Credit Rating

Suppose that 75% of all people with credit records improve their credit ratings within three years. Suppose that 18% of the population at large have poor credit records, and of those only 30% will improve their credit ratings within three years. What percentage of the people who will improve their credit records within the next three years are the ones who currently have good credit ratings?

18. Cards

From an ordinary deck of 52 cards, cards are drawn one by one, at random and without replacement. What is the probability that the fourth heart is drawn on the tenth draw?

Hint: Let F denote the event that in the first nine draws there are exactly three hearts, and E be the event that the tenth draw is a heart. Use $P(F \cap E) = P(F)P(E|F)$.

19. Proof

(a) Show that if $P(A) = 1$, then $P(A \cap B) = P(B)$;

(b) Prove that *any* event A with $P(A) = 0$ or $P(A) = 1$ is independent of *every* event B.

20. More Proofs

(a) Show that if an event A is independent of itself, then $P(A) = 0$ or 1;

(b) Show that if A and B are independent and $A \subseteq B$, then either $P(A) = 0$ or $P(B) = 1$.

21. Matching

A fair die is rolled six times. If on the ith roll, $1 \leq i \leq 6$, the outcome is i, we say that a match has occurred. What is the probability that at least one match occurs?

22. Families

From the set of all families with three children *a family* is selected at random. Let A be the event that "the family has children of both sexes" and B be the event that "there is at most one girl in the family". Are A and B independent?

Answer the same question for families with two children and families with four children. Assume that for any family size all sex distributions have equal probabilities.

23. Independent Events

Let $\{A_1, A_2, \ldots, A_n\}$ be an independent set of events and $P(A_i) = p_i, 1 \leq i \leq n$.

(a) What is the probability that at least one of the events A_1, A_2, $\ldots$, A_n occurs?

(b) What is the probability that none of the events A_1, A_2, $\ldots$, A_n occurs?

24. Urns and Balls

An urn contains two red and four white balls. Balls are drawn from the urn successively, at random and with replacement. What is the probability that exactly three whites occur in the first five trials?

25. *A* Before *B*

Let Ω be the sample space of a repeatable experiment. Let A and B be mutually exclusive events of Ω. Prove that, in independent trials of this experiment, the event A occurs before the event B with probability $P(A)/(P(A)+P(B))$.

26. Guns

A person has six guns. The probability of hitting a target when these guns are properly aimed and fired is 0.6, 0.5, 0.7, 0.9, 0.7, and 0.8, respectively. What is the probability of hitting a target if a gun is selected at random, properly aimed, and fired?

27. Strokes

Of the patients in a hospital, 20% of those with, and 35% of those without myocardial infarction have had strokes. If 40% of the patients have had myocardial infarction, what percent of the patients have had strokes?

28. Lost Spades

(a) One of the cards of an ordinary deck of 52 cards is lost. What is the probability that a random card drawn from this deck is a spade?

(b) Two cards from an ordinary deck of 52 cards are missing. What is the probability that a random card drawn from this deck is a spade?

29. Proof

Let B be an event of a sample space Ω with $P(B) > 0$. For a subset A of Ω, define $Q(A) = P(A|B)$. By Question 15 we know that Q is a probability function. For E and F, events of Ω (with $P(F \cap B) > 0$), show that $Q(E|F) = P(E|F \cap B)$.

30. Another Proof

Let $A, B, C \subseteq \Omega$ with $P(B) > 0$, $P(B \cap C) > 0$, and $P(B \cap C^c) > 0$. Show that

$$P(A|B) = P(A|B \cap C)P(C|B) + P(A|B \cap C^c)P(C^c|B).$$

31. Married on Campus

Suppose that 40% of the students on a campus, who are married to students on the same campus, are female. Moreover, suppose that 30% of those who are married, but not to students at this campus, are also female. If one third of the married students on this campus are married to other students on this campus, what is the probability that a randomly selected married student from this campus is a woman?

32. Batteries

Suppose that 10 good and three dead batteries are mixed up. Jack tests them one by one, at random and without replacement. But before testing the fifth battery he realizes that he does not remember whether the first one tested is good or is dead. All he remembers is that the last three that were tested were all good. What is the probability that the first one is also good?

33. Multiple Choice

On a multiple choice exam with four choices for each question, a student either knows the answer to a question or marks it at random. If the probability that he or she knows the answers is 2/3, what is the probability that an answer that was marked correctly was not marked randomly?

34. Cards

A stack of cards consists of six red and five blue cards. A second stack of cards consists of nine red cards. A stack is selected at random and three of its cards are drawn. If all of them are red, what is the probability that the first stack was selected?

35. More Cards

There are three identical cards that differ only in color. Both sides of one are black, both sides of the second one are red, and one side of the third card is black and its other side is red. These cards are mixed up and one of them is selected at random. If the upper side of this card is red, what is the probability that its other side is black?

36. Horses

There are two stables on a farm, one that houses 20 horses and 13 mules, the other with 25 horses and eight mules. Without any pattern, animals occasionally leave their stables and then return to their stables. Suppose that during a period when all the animals are in their stables, a horse comes out of a stable and then returns. What is the probability that the next animal coming out of the same stable will also be a horse?

37. Chips

An urn contains five red and three blue chips. Suppose that four of these chips are selected at random and transferred to a second urn, which was originally empty. If a random chip from this second urn is blue, what is the probability that two red and two blue chips were transferred from the first urn to the second urn?

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Problem Sheet 3

Random Variables and Distribution Functions

38. More Chips

From an urn that contains five red, five white, and five blue chips, we draw two chips at random. For each blue chip we win \$1, for each white chip we win \$2, but for each red chip we lose \$3. If X represents the amount that we either win or we lose, what are the possible values of X and probabilities associated with them?

39. Plastic Die

The side measurement of a plastic die, manufactured by factory A, is a random number between 1 and 1.25 centimetres. What is the probability that the volume of a randomly selected die manufactured by this company is greater than 1.424? Assume that the die will always be a cube.

40. Families

From families with three children a family is chosen at random. Let X be the number of girls in the family. Calculate and sketch the distribution function of X. Assume that in a three-child family all gender distributions are equally probable.

41. Distribution Functions

Determine if the following are distribution functions.

(a)

$$F_X(t) = \begin{cases} 0, & t < 0 \\ \dfrac{t}{1+t}, & t \geq 0. \end{cases}$$

(b)

$$F_X(t) = \begin{cases} \dfrac{e^t}{2}, & t < 0 \\ 1 - \dfrac{3e^{-t}}{4}, & t \geq 0. \end{cases}$$

42. Random Points

Let X be a randomly selected point from the interval (0, 3). What is the probability that $X^2 - 5X + 6 > 0$?

43. New Car

Let the time until a new car breaks down be denoted by X, and let

$$Y = \begin{cases} X, & \text{if } X \leq 5 \\ 5, & \text{if } X > 5. \end{cases}$$

Then Y is the life of the car, if it lasts less than 5 years, and is 5 if it lasts longer than 5 years. Calculate the distribution function of Y, F_Y, in terms of F_X, the distribution function of X.

44. Minimum Face Value

In the experiment of rolling a fair die twice, let X be the minimum of the two numbers obtained. Determine the probability mass function and the distribution function of X, and sketch their graphs.

45. Probability Mass Function

The distribution function of a random variable X is given by

$$F_X(x) = \begin{cases} 0, & x < -2 \\ 1/2, & -2 \leq x < 2 \\ 3/5, & 2 \leq x < 4 \\ 8/9, & 4 \leq x < 6 \\ 1, & x \geq 6. \end{cases}$$

Determine the probability mass function of X and sketch its graph.

46. Jury

From 18 potential women jurors and 28 potential men jurors, a jury of 12 is chosen at random. Let X be the number of women selected. Find the probability mass function of X.

47. First Six

In successive rolls of a fair die, let X be the number of rolls until the first 6 appears. Determine the probability mass function and the distribution function of X.

48. More Probability Mass Functions

For each of the following, determine the value(s) of k for which p is a probability mass function. Note that in Parts (d) and (e), n is a positive integer.

(a) $p(x) = kx, x = 1, 2, 3, 4, 5;$

(b) $p(x) = k(1 + x)^2, x = -2, 0, 1, 2;$

(c) $p(x) = k(1/9)^x, x = 1, 2, 3, \ldots;$

(d) $p(x) = kx, x = 1, 2, 3, \ldots, n;$

(e) $p(x) = kx^2, x = 1, 2, 3, \ldots, n.$

Hint: Recall that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

49. Soap Opera

The distribution function for the duration of a certain soap opera (in tens of hours) is

$$F(x) = \begin{cases} 1 - \dfrac{16}{x^2}, & x \geq 4 \\ 0, & x < 4. \end{cases}$$

(a) Calculate f, the probability density function of the soap opera.

(b) Sketch the graphs of F and f.

(c) What is the probability that the soap opera takes at most 50 hours? At least 60 hours? Between 50 and 70 hours? Between 10 and 35 hours?

50. Tyres

The lifetime of a tyre selected randomly from a used tyre shop is 10,000X kilometres, where X is a random variable with the density function

$$f(x) = \begin{cases} \dfrac{2}{x^2}, & 1 < x < 2 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) What percentage of the tyres of this shop last fewer than 15,000 kilometres?

(b) What percentage of those having lifetimes fewer than 15,000 kilometres last between 10,000 and 12,500 kilometres?

51. Another Density Function

Let X be a continuous random variable with density and distribution functions f and F, respectively. Assuming that $\alpha \in \mathbb{R}$ is a point at which $P(X \leq \alpha) < 1$, prove that

$$h(x) = \begin{cases} \dfrac{f(x)}{1 - F(\alpha)}, & x \geq \alpha \\ 0, & x < \alpha \end{cases}$$

is also a probability density function.

52. Investment

Suppose that the loss in a certain investment, in thousands of dollars, is a continuous random variable X that has a density function of the form

$$f(x) = \begin{cases} k(2x - 3x^2), & -1 < x < 0 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Calculate the value of k.

(b) Find the probability that the loss is at most \$500.

53. Convex Combination

Prove that if f and g are two probability density functions, then for $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta = 1$, $\alpha f + \beta g$ is also a probability density function.

54. Fuses

A box contains 20 fuses, of which five are defective. What is the expected number of defective items among three fuses selected randomly?

55. Nonexistant Expected Value

It is well known that $\sum_{x=1}^{\infty} 1/x^2 = \pi^2/6$.

(a) Show that $p(x) = 6/(\pi x)^2, x = 1, 2, 3, \ldots$ is the probability mass function of a random variable X.

(b) Prove that $\mathbb{E}(X)$ does not exist.

56. Distribution Function

The distribution function of a random variable X is given by

$$F_X(x) = \begin{cases} 0, & x < -3 \\ 3/8, & -3 \leq x < 0 \\ 1/2, & 0 \leq x < 3 \\ 3/4, & 3 \leq x < 4 \\ 1, & x \geq 4. \end{cases}$$

Calculate $\mathbb{E}(X)$, $\mathbb{E}(X^2 - 2|X|)$, and $\mathbb{E}(X|X|)$.

57. Children

A newly married couple decides to continue having children until they have one of each sex. If the events of having a boy and a girl are independent and equiprobable, how many children should this couple expect?

Hint: Note that $\sum_{i=1}^{\infty} ir^i = r/(1-r)^2$, $|r| < 1$.

58. Probability Mass Function

(a) Show that

$$p(n) = \frac{1}{n(n+1)}, \quad n \geq 1,$$

is a probability mass function.

(b) Let X be a random variable with probability mass function p given in Part (a). Find $\mathbb{E}(X)$.

59. Variance

Find the variance and the standard deviation of a random variable X with distribution function

$$F(x) = \begin{cases} 0, & x < -3 \\ 3/8, & -3 \leq x < 0 \\ 3/4, & 0 \leq x < 6 \\ 1, & x \geq 6. \end{cases}$$

60. Random Integer

Let X be a random integer from the set $\{1, 2, \ldots, N\}$. Find $\mathbb{E}(X)$, $\mathbb{V}(X)$, and σ_X.

61. Another Variance

Suppose that X is a discrete random variable with $\mathbb{E}(X) = 1$ and $\mathbb{E}(X(X-2)) = 3$. Find $\mathbb{V}(-3X + 5)$.

62. Soap Opera

The distribution function for the duration of a certain soap opera (in tens of hours) is

$$F(x) = \begin{cases} 1 - \dfrac{16}{x^2}, & x \geq 4 \\ 0, & x < 4. \end{cases}$$

(a) Calculate $\mathbb{E}(X)$.

(b) Show that $\mathbb{V}(X)$ does not exist.

63. Aptitude Test

The time it takes for a student to finish an aptitude test (in hours) has the density function

$$f(x) = \begin{cases} 6(x-1)(2-x), & 1 < x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

Determine the mean and standard deviation of the time it takes for a randomly selected student to finish the aptitude test.

64. Expected Value

A random variable X has the density function

$$f(x) = \begin{cases} 3e^{-3x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Calculate $\mathbb{E}(e^X)$.

65. Computer Network

Let Y be a continuous random variable with probability distribution function

$$F(x) = \begin{cases} e^{-k(\alpha - y)/A}, & y \leq \alpha \\ 1, & y > \alpha, \end{cases}$$

where A, k, and α are positive constants. (Such distribution functions arise in the study of local computer network performance.) Find $\mathbb{E}(Y)$.

66. Logarithm of a Random Variable

Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} 2/x^2, & 1 < x < 2 \\ 0, & \text{elsewhere.} \end{cases}$$

Find $\mathbb{E}(\log X)$.

67. Double Exponential Distribution

Let X be a continuous random variable with probability density function

$$f(x) = \frac{e^{-|x|}}{2}, \quad -\infty < x < \infty.$$

Calculate $\mathbb{V}(X)$.

68. Absolute Moment

Let X be a discrete random variable. The *absolute moment of X of order t* is $\mathbb{E}(|X|^t)$. If $0 < s < r$, show that if the absolute moment of order r of X exists, then the absolute moment of order s also exists.

69. Recursive Moments

Let X be a continuous random variable with the probability density function

$$f(x) = \begin{cases} \dfrac{x \sin x}{\pi}, & 0 < x < \pi \\ 0, & \text{otherwise.} \end{cases}$$

Prove that $\mathbb{E}(X^{n+1}) + (n+1)(n+2)\mathbb{E}(X^{n-1}) = \pi^{n+1}$.

70. Computing Moments using Tail Probabilities

If $P(X \leq 0) = 1$ and $\mathbb{E}[X^n] > -\infty$, show that for $n > 0$,

$$\mathbb{E}[X^n] \;=\; -n \int_{-\infty}^{0} x^{n-1} F_X(x) dx.$$

71. Post Office

Suppose that X, the interarrival time between two customers entering a certain post office, satisfies

$$P(X > t) \;=\; \alpha e^{-\lambda t} + \beta e^{-\mu t}, \qquad t \geq 0,$$

where $\alpha + \beta = 1$, $\alpha, \beta \geq 0$, $\lambda, \mu > 0$. Calculate

(a) $\mathbb{E}[X]$;

(b) $\mathbb{V}(X)$.

72. Bounded Random Variable

Let X be a continuous random variable with set of possible values $\{x : 0 < x < \alpha\}$ (where $\alpha > 0$) and distribution function F_X. Using integration by parts, prove that

$$\mathbb{E}[X] \;=\; \int_0^{\alpha} [1 - F_X(x)]\, dx.$$

MAST20004 Probability

Semester 1, 2020

Problem Sheet 4

Special Probability Distributions

73. Number Plates

In a state where license plates contain six digits, what is the probability that the license number of a randomly selected car has two 9s? Assume that each digit of the license number is randomly selected from $\{0, 1, \ldots, 9\}$.

74. Nails

A manufacturer of nails claims that only 3% of its nails are defective. A random sample of 24 nails is selected, and it is found that two of them are defective. Is it fair to reject the manufacturers claim based on this observation?

75. Five Points

From the interval $(0, 1)$, five points are selected at random and independently. What is the probability that

(a) at least two of them are less than $1/3$;

(b) the first decimal point of exactly two of them is 3?

76. Rare Blood Type

A certain rare blood type can be found in only 0.05% of people. If the population of a randomly selected group is 3000, what is the probability that at least two persons in the group have this rare blood type?

77. Children

A woman and her husband want to have at least a 95% chance for at least one boy and at least one girl. What is the minimum number of children that they should plan to have? Assume that the events that a child is a girl and a boy are equiprobable and independent of the gender of other children born in the family.

78. Chuck-a-Luck

A game often played in carnivals and gambling houses is called *chuck-a-luck*, where a player bets on any number 1 through 6. Then three fair dice are tossed. If one, two, or all three land the same number as the players, then he or she receives one, two, or three times the original stake plus his or her original bet, respectively. Otherwise, the player loses his or her stake. Let X be the net gain of the player per unit of stake. First find the probability mass function of X; then determine the expected amount that the player will lose per unit of stake.

79. Aircraft Engines

Suppose that an aircraft engine will fail in flight with probability $1 - p$ independently of the plane's other engines. Also suppose that a plane can complete the journey successfully if at least half of its engines do not fail. Is it true that a four-engine plane is always preferable to a two-engine plane? Explain.

80. Even Number of Successes

What is the probability of an even number of successes in n independent Bernoulli trials? *Hint:* Let r_n be the probability of an even number of successes in n Bernoulli trials. By conditioning on the first trial and using the law of total probability, show that for $n \geq 1$,

$$r_n = p(1 - r_{n-1}) + (1 - p)r_{n-1}.$$

Then prove that $r_n = \frac{1}{2}\left(1 + (1 - 2p)^n\right)$.

81. Targets

The probability is p that Marty hits target M when he fires at it. The probability is q that Alvie hits target A when he fires at it. Marty and Alvie fire one shot each at their targets. If both of them hit their targets, they stop; otherwise, they will continue.

(a) What is the probability that they stop after each has fired r times?

(b) What is the expected value of the number of turns taken before the turn where they both hit the target?

82. Basketball

A certain basketball player makes a foul shot with probability 0.45. What is the probability that

(a) his first basket occurs on the sixth shot?

(b) his first and second baskets occur on his fourth and eighth shots, respectively?

83. Light Bulbs

The probability is p that a randomly chosen light bulb is defective. We screw a bulb into a lamp and switch on the current. If the bulb works, we stop; otherwise, we try another and continue until a good bulb is found. What is the probability that at least n bulbs were defective?

84. Bridge

On average, how many games of bridge are necessary until a player is dealt three aces? A bridge hand is 13 randomly selected cards from an ordinary deck of 52 cards.

85. Senior Citizens

Suppose that 15% of the population of a town are senior citizens. Let X be the number of nonsenior citizens who enter a mall before the tenth senior citizen arrives. Find the probability mass function of X. Assume that each customer who enters the mall is a random person from the entire population.

86. Geometric Distribution

Suppose $X \stackrel{d}{=} \mathrm{G}(p)$. Calculate the probability that X is even.

87. Negative Binomial

For $r = 1, 2, \ldots$, show that,

$$\binom{-r}{z} = (-1)^z \binom{z+r-1}{r-1},$$

using the extended binomial coefficient definition on Slide 166.

88. Negative Binomial Distribution

For $r \in (0, \infty)$ and $0 < p \leq 1$, let $Z \stackrel{d}{=} \mathrm{Nb}(r, p)$. For $z = 0, 1, 2 \ldots$, show that

$$p_Z(z) = \binom{-r}{z} p^r (p-1)^z \geq 0.$$

89. Professors' Cars

Of the 28 professors in a certain department, 18 drive foreign and 10 drive domestic cars. If five of these professors are selected at random, what is the probability that at least three of them drive foreign cars?

90. Bernoulli Trials

Suppose that independent Bernoulli trials with parameter p are performed successively. Let N be the number of trials *until* x successes, and X be the number of successes in the first n trials. Show that

$$P(N = n) = \frac{x}{n} P(X = x).$$

Remark: By this relation, in coin tossing, for example, we can state that the probability of getting a fifth head on the seventh toss is 5/7 of the probability of five heads in seven tosses.

91. Charity

In an annual charity drive, 35% of a population of 560 make contributions. If, in a statistical survey, 15 people are selected at random and without replacement, what is the probability that at least two persons have contributed?

92. Defective Items

The policy of the quality control division of a certain corporation is to reject a shipment if more than 5% of its items are defective. A shipment of 500 items is received, 30 of them are randomly tested, and two have been found defective. Should that shipment be rejected?

93. Trout

To estimate the number of trout in a lake, we caught 50 trout, tagged and returned them. Later we caught 50 trout and found that four of them were tagged. From this experiment estimate n, the total number of trout in the lake.

Hint: Let p_n be the probability of four tagged trout among the 50 trout caught. Find the value of n that maximizes p_n.

94. Annual Income

Suppose that 3% of the families in a large city have an annual income of over \$60,000. What is the probability that, of 60 random families, at most three have an annual income of over \$60,000?

95. Misprints

Misprints in a particular book occur independently, and on average, there are three misprints in every 10 pages. If every chapter of the book contains 35 pages, what is the probability that Chapters 1 and 5 have 10 misprints each?

96. No Crimes

In a certain town, crimes occur at a Poisson rate of five per month. What is the probability of having exactly two months (not necessarily consecutive) with no crimes during the next year?

97. Bookstore

Customers arrive at a bookstore at a Poisson rate of six per hour. Given that the store opens at 9:30am, what is the probability that exactly one customer arrives by 10:00am and 10 customers by noon?

98. Lottery Tickets

Suppose that in Melbourne, on a certain day, N lottery tickets are sold and M win, with M/N very small. To have a probability of at least α of winning on that day, approximately how many tickets should be purchased?

99. Poisson Distribution

Let X be a Poisson random variable with parameter λ. Show that the maximum of $P(X = i)$ occurs at $\lfloor \lambda \rfloor$, where $\lfloor \lambda \rfloor$ is the greatest integer less than or equal to λ.

Hint: Let p_X be the probability mass function of X. Prove that

$$p_X(i) = \frac{\lambda}{i} p_X(i-1).$$

Use this to find the values of i at which p is increasing and the values of i at which it is decreasing.

100. Line Segment

A point is selected at random on a line segment of length l. What is the probability that

(a) the longer segment is at least twice as long as the shorter segment?

(b) What is the probability that none of the two segments is smaller than $l/3$?

101. Random Angle

Let Θ be a random number between $-\pi/2$ and $\pi/2$. Find the probability density function of $X = \tan \Theta$

102. Expectation of a Logarithm

Let X be a uniform random variable over the interval $(0, 1)$. Calculate $\mathbb{E}(-\log X)$.

103. Chicken Pen

A farmer who has two pieces of lumber of lengths a and b $(a < b)$ decides to build a pen in the shape of a triangle for his chickens. He sends his foolish son out to cut the longer piece and the boy, without taking any thought as to the ultimate purpose, makes a cut on the lumber of length b, at a point selected randomly. What are the chances that the two resulting pieces and the piece of length a can be used to form a triangular pen?

Hint: Three segments form a triangle if and only if the length of any one of them is less than the sum of the lengths of the remaining two.

104. Heart Attack

The time between the first and second heart attacks for a certain group of people is an exponential random variable. If 50% of those who have had a heart attack will have another one within the next five years, what is the probability that a person who had one heart attack five years ago will not have another one in the next five years?

105. Exponential Random Variable

Let X be an exponential random variable with parameter λ. Find

$$P\left(|X - \mathbb{E}(X)| \geq 2\sigma_X\right).$$

106. Telephone Call

Mr. Jones is waiting to make a phone call at a train station. There are two public telephone booths next to each other, occupied by two persons, say A and B. If the duration of each telephone call is an exponential random variable with $\lambda = 1/8$, what is the probability that among Mr. Jones, A, and B, Mr. Jones will not be the last to finish his call?

107. Double Exponential Random Variable

The random variable X is called double exponentially distributed if its density function is given by

$$f(x) = ce^{-|x|}, \quad -\infty < x < \infty.$$

(a) Find the value of c;

(b) Prove that $\mathbb{E}(X^{2n}) = (2n)!$ and $\mathbb{E}(X^{2n+1}) = 0$.

108. Radio Tubes

Let X, the lifetime (in years) of a radio tube, be exponentially distributed with mean $1/\lambda$. Prove that $\lfloor X \rfloor$, the integer part of X, which is the complete number of years that the tube works, is a geometric random variable.

109. Gamma Distribution

Let $X \stackrel{d}{=} \gamma(r, \alpha)$. If $c > 0$, show that $cX \stackrel{d}{=} \gamma(r, \alpha/c)$.

110. Defective Light Bulbs

A manufacturer produces light bulbs at a Poisson rate of 200 per hour. The probability that a light bulb is defective is 0.015. During production, the light bulbs are tested one by one, and the defective ones are put in a special can that holds up to a maximum of 25 light bulbs. On average, how long does it take until the can is filled?

111. Born on Christmas Day

A small college has 1095 students. What is the approximate probability that more than five students were born on Christmas day? Assume that the birthrates are constant throughout the year and that each year has 365 days.

112. Beta Mean and Variance

Let $X \stackrel{d}{=} \text{Beta}(\alpha, \beta)$. Calculate $\mathbb{E}[X]$ and $\mathbb{V}(X)$.

113. Beta Random Variable

For which value of c is the following a probability density function of some random variable X? Find $\mathbb{E}[X]$ and $\mathbb{V}(X)$.

$$f(x) = \begin{cases} cx^4(1-x)^5, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

114. Blood Pressure Medicines

Suppose that new blood pressure medicines introduced are effective on $100P\%$ of the patients, where P is beta random variable with parameters $\alpha = 20$ and $\beta = 13$. What is the probability that a new blood pressure medicine is effective on at least 60% of the hypertensive population?

115. Resistors

The proportion of resistors a procurement office of an engineering firm orders every month, from a specific vendor, is a beta random variable with mean 1/3 and variance 1/18. What is the probability that next month, the procurement office orders at least 7/12 of its purchase from the vendor?

116. Pareto Random Variable

Let $X \stackrel{d}{=} \text{Pareto}(\alpha, \gamma)$. Find the distribution function F_X, and then calculate $\mathbb{E}[X]$ and $\mathbb{V}(X)$.

117. Pareto Random Variable

Let $X \stackrel{d}{=} \text{Pareto}(\alpha, \gamma)$. Evaluate $\mathbb{E}[X^n]$ and state the values of n for which it is defined.

118. Newspaper Subscribers

The ages of subscribers to a certain newspaper are normally distributed with mean 35.5 years and standard deviation 4.8. What is the probability that the age of a random subscriber is

(a) more than 35.5 years;

(b) between 30 and 40 years?

119. IQ

Suppose that the IQ of a randomly selected student from a university is normal with mean 110 and standard deviation 20. Determine the interval of values that is centered at the mean and includes 50% of the IQs of the students at that university.

120. Manual Dexterity

Suppose that the scores on a certain manual dexterity test are normal with mean 12 and standard deviation 3. If eight randomly selected individuals take the test, what is the probability that none will make a score less than 14?

121. Normal Density Function

Determine the value(s) of k for which the following is the probability density function of a normal random variable.

$$f(x) = \sqrt{k}e^{-k^2x^2-2kx-1}, \qquad -\infty < x < \infty.$$

122. Density Function of Y

Let $X \stackrel{d}{=} \mathrm{N}(0,1)$. Calculate the density function of $Y = \sqrt{|X|}$.

123. Another Normal Density Function

Prove that for some constant k, $f(x) = ka^{-x^2}$, $a \in (1, \infty)$, is a normal probability density function.

124. Skeletons

At an archaeological site 130 skeletons are found and their heights are measured and found to be approximately normal with mean 172 centimetres and standard deviation 9 centimetres. At a nearby site, five skeletons are discovered and it is found that the heights of exactly three of them are above 185 centimetres. Based on this information is it reasonable to assume that the second group of skeletons belongs to the same family as the first group of skeletons?

125. Weibull Mean and Variance

Let $X \stackrel{d}{=} \text{Weibull}(\beta, \gamma)$. Calculate $\mathbb{E}[X]$ and $\mathbb{V}(X)$.

126. Weibull and Exponential

Let $X \stackrel{d}{=} \text{Weibull}(\beta, \gamma)$ and $Y = (X/\beta)^\gamma$, derive the pdf of Y.

127. Exponential of a Random Variable

Let X be a continuous random variable with distribution function F and density function f. Calculate the density function, g, of the random variable $Y = e^X$.

128. Logarithm of a Random Variable

Let X be a continuous random variable with the density function

$$f(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

Find the probability density function of $Y = \log_2 X$.

129. Log of a Pareto Random Variable

Let $X \stackrel{d}{=} \text{Pareto}(\alpha, \gamma)$. Find the distribution and density functions of $Y = \log X$.

130. Square of a Random Variable

Let f be the probability density function of a random variable X. In terms of f, calculate the probability density function of $Y = X^2$, g.

131. Reciprocal of a Random Variable

Let X be a random variable with the probability density function given by

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Let

$$Y = \begin{cases} X, & \text{if } X \leq 1 \\ 1/X, & \text{if } X > 1. \end{cases}$$

Find the probability density function of Y.

132. Transformation by $\psi(x) = \min(x, M)$

Let $X \stackrel{d}{=} \exp(1)$, find the cdf of $Y = \min(X, 5)$ and derive $\mathbb{E}(Y)$ and $\mathbb{V}(Y)$.

133. Lognormal Distribution

Let X be a lognormal random variable with parameters μ and σ^2.

(a) Find $\mathbb{E}(X^r)$;

(b) Calculate $\mathbb{E}(X)$ and $\mathbb{V}(X)$.

134. Large Fires

In 1977 a British researcher demonstrated that if X is the loss from a large fire, then X is a lognormal random variable. Suppose that the expected loss due to fire in the buildings of a certain industry, in thousands of dollars, is 120 with standard deviation 36. What is the probability that the loss from a fire in such an industry is less than \$100,000?

MAST20004 Probability

Semester 1, 2020

Problem Sheet 5

Bivariate Random Variables

135. Joint Distribution

Let the joint probability mass function of discrete random variables X and Y be given by

$$p(x,y) = \begin{cases} c(x+y), & \text{if } x = 1,2,3, \quad y = 1,2 \\ 0, & \text{otherwise.} \end{cases}$$

Determine

(a) the value of the constant c;

(b) the marginal probability mass functions of X and Y;

(c) $P(X \geq 2 | Y = 1)$;

(d) $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

136. Another Joint Distribution

Let the joint probability mass function of discrete random variables X and Y be given by

$$p(x,y) = \begin{cases} \dfrac{x^2+y^2}{25}, & \text{if } x = 1,2, \quad y = 0,1,2 \\ 0, & \text{otherwise.} \end{cases}$$

Find

(a) $P(X > Y)$;

(b) $P(X + Y \leq 2)$;

(c) $P(X + Y = 2)$.

137. Cards

From an ordinary deck of 52 cards, seven cards are drawn at random and without replacement. Let X and Y be the number of hearts and the number of spades drawn, respectively.

(a) Find the joint probability mass function of X and Y;

(b) Calculate $P(X \geq Y)$.

138. Marginal Densities

Let the joint probability density function of random variables X and Y be given by

$$f(x,y) = \begin{cases} 8xy, & \text{if } 0 \le y \le x \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Calculate the marginal probability density functions of X and Y;

(b) Calculate $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

139. Probability Calculations

Let the joint probability density function of random variables X and Y be given by

$$f(x,y) = \begin{cases} 1, & \text{if } 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Calculate

(a) $P(X + Y \le 1/2)$;

(b) $P(X - Y \le 1/2)$;

(c) $P(XY \le 1/4)$;

(d) $P(X^2 + Y^2 \le 1)$.

140. Line Segment

On a line segment AB of length l, two points C and D are placed at random and independently. What is the probability that C is closer to D than to A?

141. Expected Values

Let X and Y be two continuous random variables with finite expectations. Show that if $P(X \le Y) = 1$, then $\mathbb{E}(X) \le \mathbb{E}(Y)$.

142. Cards

From an ordinary deck of 52 cards, eight cards are drawn at random and without replacement. Let X and Y be the number of clubs and spades, respectively. Are X and Y independent?

143. Maximum and Minimum of X and Y

Let X and Y be two independent random variables with distribution functions F and G, respectively. Find the distribution functions of $U = \max(X, Y)$ and $V = \min(X, Y)$.

144. Independent or Not

Let the joint probability density function of random variables X and Y be given by

$$f(x,y) = \begin{cases} 8xy, & \text{if } 0 \leq y \leq x \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Determine if $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

145. Probability Density of X/Y

Let X and Y be two independent random variables with the same probability density function given by

$$f(x) = \begin{cases} e^{-x}, & \text{if } 0 < x < \infty \\ 0, & \text{elsewhere.} \end{cases}$$

Show that g, the probability density function of X/Y, is given by

$$g(t) = \begin{cases} \dfrac{1}{(1+t)^2}, & \text{if } 0 < t < \infty \\ 0, & t \leq 0. \end{cases}$$

146. Expected Value of Maximum and Minimum

Let X and Y be independent random points from the interval $(-1, 1)$. Find $\mathbb{E}(\max(X,Y))$ and $\mathbb{E}(\min(X,Y))$

147. Product of Two Functions

For $x, y \in \mathbb{R}$, let the joint probability density function of two random variables X and Y satisfy

$$f_{(X,Y)}(x,y) = g(x)h(y),$$

where g and h are two functions from $\mathbb{R}$ to $\mathbb{R}$. Show that X and Y are independent.

148. Conditional Density Function

Let the joint probability density function of continuous random variables X and Y be given by

$$f(x,y) = \begin{cases} 2, & \text{if } 0 < x < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Find $f_{X|Y}(x|y)$.

149. Conditional Probability

Let the conditional probability density function of X given that $Y = y$ be given by

$$f_{X|Y}(x|y) = \frac{3(x^2+y^2)}{3y^2+1}, 0 < x < 1, 0 < y < 1.$$

Find $P(1/4 < X < 1/2 | Y = 3/4)$.

150. Another Conditional Density Function

Let the joint probability density function of continuous random variables X and Y be given by

$$f(x,y) = \begin{cases} x+y, & \text{if } 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Find $f_{X|Y}(x|y)$.

151. Random Points

First a point Y is selected at random from the interval $(0,1)$. Then another point X is selected at random from the interval $(Y,1)$. Find the probability density function of X.

152. Conditional Mean and Variance

Let the joint probability density function of continuous random variables X and Y be given by

$$f(x,y) = \begin{cases} ce^{-x}, & \text{if } x \ge 0, \quad |y| < x \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Determine the constant c;

(b) Find $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$;

(c) Calculate $\mathbb{E}(Y|X=x)$ and $\mathbb{V}(Y|X=x)$.

153. Yet Another Conditional Density Function

A point is selected at random and uniformly from the region $R = \{(x,y) : |x| + |y| \le 1\}$. Find the conditional probability density function of X given $Y = y$.

154. Sum of Two Random Numbers

From the interval $(0,1)$, two random numbers are selected independently. Show that the probability density function of their sum is given by

$$g(t) = \begin{cases} t, & \text{if } 0 \le t < 1 \\ 2-t, & \text{if } 1 \le t < 2 \\ 0, & \text{otherwise.} \end{cases}$$

155. Conditional Probability

Let $-1/9 < c < 1/9$ be a constant. Let $p_{(X,Y)}(x, y)$, the joint probability mass function of the random variables X and Y, be given by the following table

	y		
x	-1	0	1
-1	$1/9$	$1/9 - c$	$1/9 + c$
0	$1/9 + c$	$1/9$	$1/9 - c$
1	$1/9 - c$	$1/9 + c$	$1/9$

(a) Show that the probability mass function of $X + Y$ is the convolution function of the probability mass functions of X and Y for all c.

(b) Show that X and Y are independent if and only if $c = 0$.

156. Conditional Mean and Variance

Let the joint probability density function of random variables X and Y be given by

$$f(x, y) = \begin{cases} 2e^{-(x+2y)}, & \text{if } x \geq 0, \quad y \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find

(a) $\mathbb{E}(X)$;

(b) $\mathbb{E}(Y)$;

(c) $\mathbb{E}(X^2 + Y^2)$.

157. Letters

An absentminded professor wrote n letters and sealed them in envelopes without writing the addresses on the envelopes. Having forgotten which letter he had put in which envelope, he wrote the n addresses on the envelopes at random. What is the expected number of the letters addressed correctly?

Hint: For $i = 1, 2, \ldots, n$, let

$$X_i = \begin{cases} 1, & \text{if the } i\text{th letter is addressed correctly} \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $\mathbb{E}(X_1 + X_2 + \ldots + X_n)$.

158. Empty Boxes

Suppose that 80 balls are placed into 40 boxes at random and independently. What is the expected number of empty boxes?

Hint: For $i = 1, 2, \ldots, 40$, let

$$X_i = \begin{cases} 1, & \text{if the } i\text{th box is empty} \\ 0, & \text{otherwise.} \end{cases}$$

Calculate $\mathbb{E}(X_1 + X_2 + \ldots + X_{40})$.

159. *X* and *Y*

Let X and Y be nonnegative random variables with an arbitrary joint probability distribution function. Let

$$I(x, y) = \begin{cases} 1, & \text{if } X > x, \quad Y > y \\ 0, & \text{otherwise.} \end{cases}$$

(a) Show that

$$\int_0^\infty \int_0^\infty I(x, y) dx dy = XY.$$

(b) Prove that

$$\mathbb{E}(XY) = \int_0^\infty \int_0^\infty P(X > x, Y > y) dx dy.$$

160. Discrete Random Variable

Let N be a discrete random variable with set of possible values $\{1, 2, 3, \ldots\}$. Prove that

$$\mathbb{E}(N) = \sum_{i=1}^{\infty} P(N \geq i).$$

Hint: For $i = 1, 2, \ldots$, let

$$X_i = \begin{cases} 1, & \text{if } N \geq i \\ 0, & \text{otherwise.} \end{cases}$$

Then show that

$$N = \sum_{i=1}^{\infty} X_i.$$

161. Covariance

Let the joint probability mass function of random variables X and Y be given by

$$p(x, y) = \begin{cases} \dfrac{x(x+y)}{70}, & \text{if } x = 1, 2, 3, \quad y = 3, 4 \\ 0, & \text{elsewhere.} \end{cases}$$

Find $\text{Cov}(X, Y)$.

162. Covariance Proofs

For random variables X, Y, and Z, prove that

(a) $\text{Cov}(X+Y, Z) = \text{Cov}(X, Z) + \text{Cov}(Y, Z)$;

(b) $\text{Cov}(X, Y+Z) = \text{Cov}(X, Y) + \text{Cov}(X, Z)$.

163. More Covariance Proofs

For random variables X, and Y, prove that

(a) $\text{Cov}(X+Y, X-Y) = \mathbb{V}(X) - \mathbb{V}(Y)$;

(b) $\mathbb{V}(X-Y) = \mathbb{V}(X) + \mathbb{V}(Y) - 2\text{Cov}(X, Y)$.

164. Independent and/or Uncorrelated?

Let X and Y be the coordinates of a random point selected uniformly from the unit disk $\{(x, y) : x^2 + y^2 \leq 1\}$. Are X and Y independent? Are they uncorrelated? Why or why not?

165. Investment

Mr. Ingham has invested money in three assets; 18% in the first asset, 40% in the second one, and 42% in the third one. Let r_1, r_2, and r_3 be the annual rate of returns for these three investments, respectively. For $1 \leq i, j \leq 3$, $\text{Cov}(r_i, r_j)$ is the entry in the ith row and the j th column of the following table. (Note that $\mathbb{V}(r_i) = \text{Cov}(r_i, r_i)$.)

	r_1	r_2	r_3
r_1	0.064	0.03	0.015
r_2	0.03	0.0144	0.021
r_3	0.015	0.021	0.01

Find the standard deviation of the annual rate of return for Mr. Inghams total investment.

166. Covariance of Linear Combinations

Let $X_1, X_2, \ldots, X_n$, $Y_1, Y_2, \ldots, Y_m$ be random variables, and $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_m$ be constants. Prove that

$$\text{Cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{i=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n}\sum_{i=1}^{m} a_i b_j \text{Cov}\left(X_i, Y_j\right).$$

167. Variance of a Linear Combination

Let $X_1, X_2, \ldots, X_n$ be random variables, and $a_1, a_2, \ldots, a_n$ be constants. Prove that

$$\mathbb{V}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \mathbb{V}(X_i) + 2\sum_{i=1}^{n}\sum_{j>i} a_i a_j \text{Cov}\left(X_i, X_j\right).$$

168. Correlation

Let X and Y be jointly distributed with $\rho(X,Y) = 1/2$, $\sigma_X = 2$, and $\sigma_Y = 3$. Find $\mathbb{V}(2X - 4Y + 3)$.

169. Broken Stick

A stick of length 1 is broken into two pieces at a random point. Find the correlation coefficient and the covariance of these pieces.

170. Uncorrelated Result

Prove that if $\mathrm{Cov}(X,Y) = 0$, then

$$\rho(X+Y, X-Y) = \frac{\mathbb{V}(X) - \mathbb{V}(Y)}{\mathbb{V}(X) + \mathbb{V}(Y)}.$$

171. No Linear Relation

Show that if the joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} \dfrac{\sin(x+y)}{2}, & \text{if } 0 \le x \le \frac{\pi}{2}, \quad 0 \le y \le \frac{\pi}{2} \\ 0, & \text{otherwise,} \end{cases}$$

then there exists no linear relation between X and Y .

172. Two Successive Tails

A fair coin is tossed until two tails occur successively. Find the expected number of the tosses required.
Hint: Let

$$X = \begin{cases} 1, & \text{if the first toss results in tails} \\ 0, & \text{if the first toss results in heads,} \end{cases}$$

and condition on X.

173. Mixture of Two Random Variables

For given independent random variables Y and Z, let

$$X = \begin{cases} Y, & \text{with probability } p \\ Z, & \text{with probability } 1-p. \end{cases}$$

Find $\mathbb{E}(X)$ in terms of $\mathbb{E}(Y)$ and $\mathbb{E}(Z)$.

174. Verification

Let X and Y be continuous random variables with joint probability density function

$$f(x,y) = \begin{cases} \dfrac{3(x^2+y^2)}{2}, & \text{if } 0 < x < 1, \quad 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Verify that $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$.

175. Eggs Hatching

Suppose an insect lays N eggs where $N \stackrel{d}{=} \mathrm{Pn}(\lambda)$. Suppose each egg, independently, has a probability of p of hatching. Let X be the number of eggs that hatch. Evaluate $\mathbb{E}[X]$ and $\mathbb{V}(X)$, and deduce the distribution of X.

176. Electronic Components

Suppose a machine produces N electronic components each day where $N \stackrel{d}{=} \mathrm{Bi}(m,q)$. Suppose each component, independently, has a probability of p of working. Let Y be the number of components that work. Evaluate $\mathbb{E}[Y]$ and $\mathbb{V}(Y)$, and deduce the distribution of Y.

177. Consecutive Zeros

What is the expected number of random digits that should be generated to obtain three consecutive zeros?

178. Fishing

A fisherman catches fish in a large lake with lots of fish, at a Poisson rate of two per hour. If, on a given day, the fisherman spends randomly anywhere between 3 and 8 hours fishing, find the expected value and the variance of the number of fish he catches.

179. Wallets

Suppose that X and Y represent the amount of money in the wallets of players A and B, respectively. Let X and Y be jointly uniformly distributed on the unit square $[0,1] \times [0,1]$. A and B each places his wallet on the table. Whoever has the smallest amount of money in his wallet wins all the money in the other wallet. Let W_A be the amount of money that player A will win. Show that $E(W_A) = 0$.

180. Heights

Let X be the height of a man and Y the height of his daughter (both in centimetres). Suppose that the joint probability density function of X and Y is bivariate normal with the following parameters: $\mu_X = 180, \mu_Y = 152, \sigma_X = 7.6, \sigma_Y = 6.9$, and $\rho = 0.45$. Find the probability that the height of the daughter, of a man who is 178 centimetres tall, is at least 150 centimetres.

181. Bivariate Normal

The joint probability density function of X and Y is bivariate normal with $\mu_X = \mu_Y = 0, \sigma_X = \sigma_Y = 9$, and $\rho = 0$. Find

(a) $P(X \leq 6, Y \leq 12)$;

(b) $P(X^2 + Y^2 \leq 36)$
Hint: use polar coordinates.

182. Grade Point Average

At a certain university, the joint probability density function of X and Y, the grade point averages of a student in his or her first and second years, respectively, is bivariate normal. From the grades of past years it is known that $\mu_X = 60\%, \mu_Y = 50\%, \sigma_X = 10\%, \sigma_Y = 8\%$, and $\rho = 0.4$. Find the probability that a student with grade point average 70% in his or her first year will earn a grade point average of at least 64% in his or her senior year.

183. Approximating Means and Variances

For each of the following using Taylor series approximate $\mathbb{E}[Y]$ and $\mathbb{V}(Y)$.

(a) Let $X \stackrel{d}{=} R(0, 1)$ and $Y = \sin X$.

(b) Let $X \stackrel{d}{=} \exp(2)$ and $Y = \log X$.

(c) Let $X \stackrel{d}{=} G\left(\frac{1}{3}\right)$ and $Y = X^2$.

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Problem Sheet 6

Sums of Independent Random Variables/Limit Theorems

184. Chebyshev's Inequality

Let X be a nonnegative random variable with $\mathbb{E}(X) = 5$ and $\mathbb{E}(X^2) = 42$. Find an upper bound for $P(X \geq 11)$ using Chebyshev's inequality.

185. Accidents

Suppose that the average number of accidents at an intersection is two per day. Let the variance of the number of accidents be two. Use Chebyshev's inequality to find a bound on the probability that tomorrow at least five accidents will occur.

186. IQ Scores

The average IQ score on a certain campus is 110. If the variance of these scores is 15, what can be said about the percentage of students with an IQ above 140?

187. Chebyshev's Inequality Again

Suppose that X is a random variable with $\mathbb{E}(X) = \mathbb{V}(X) = \mu$. What does Chebyshev's inequality say about $P(X > 2\mu)$?

188. Multiple Choice

In a multiple-choice test with false answers receiving negative scores, the mean of the grades of the students is 0 and its standard deviation is 15. Find an upper bound for the probability that a student's grade is at least 45.

189. Probability Generating Function

The probability generating function for a discrete random variable X is given by

$$P_X(z) = 0.3 + 0.2z + 0.1z^2 + 0.4z^3.$$

(a) Write down the probability mass function for X;

(b) Using the probability generating function, calculate $\mathbb{E}(X)$ and $\mathbb{V}(X)$;

(c) Write down the probability generating function for $Y = X^3$.

190. Sum of Negative Binomial Random Variables

For $i = 1, 2, \ldots, n$, let $X_i \stackrel{d}{=} \mathrm{Nb}(r_i, p)$, where the X_is are independent, the r_is are positive real numbers, and $0 < p < 1$. If

$$X = X_1 + X_2 + \ldots X_n,$$

write down the probability generating function for X.

191. Random Sum

Let $\{X_n\}$ be a sequence of independent and identically-distributed random variables whose values are nonnegative integers, and N a random variable whose values are also nonnegative integers which is independent of $\{X_n\}$. Let

$$S_N = \sum_{j=1}^{N} X_i.$$

Denote by $P_N(z)$ and $P_X(z)$ the probability generating functions of N and the X_is, respectively. Show that the probability generating function of S_N is given by

$$P_{S_N}(z) = P_N(P_X(z)),$$

for $z \in [0, 1]$.

192. Binomial Sum

Let $Y = X_1 + X_2 + \ldots + X_N$ where $N \stackrel{d}{=} Bi(n, p)$, $X_i \stackrel{d}{=} Bi(m, q)$, and $N, X_1, X_2, \ldots$ are independent.

(a) Find $P_{Y|N}(z)$, the conditional probability generating function of Y given N, and state the values of z for which it is defined;

(b) Find $P_Y(z)$, the probability generating function of Y, and state the values of z for which it is defined;

(c) Using $P_Y(z)$, evaluate $\mathbb{E}(Y)$.

193. Continuous Density Function

Let X be a continuous random variable with probability density function $f(x) = 2x$, if $0 \le x \le 1$, zero elsewhere. Find the moment generating function of X.

194. Uniform Random Variable

Let X be a uniform random variable over the interval (a, b). Find the moment generating function of X.

195. Probability Mass Function

Let

$$M_X(t) = \frac{1}{21}\sum_{n=1}^{6} ne^{nt}.$$

Find the probability mass function of X.

196. Linear Function

Let $M_X(t) = 1/(1-t), t < 1$ be the moment generating function of a random variable X. Find the moment generating function of the random variable $Y = 2X + 1$.

197. Moments

Suppose that the moment-generating function of X is given by

$$M_X(t) = \frac{e^t + e^{-t}}{6} + \frac{2}{3}, -\infty < t < \infty.$$

Find $\mathbb{E}(X^r)$, $r \geq 1$.

198. Constant Random Variable

Suppose that for a random variable X, $\mathbb{E}(X^n) = 2^n$, $n = 1, 2, 3, \ldots$. Calculate the moment generating function and the probability mass function of X.

199. Cumulant generating function for Poisson

Calculate the cumulant generating function of $X \stackrel{d}{=} \text{Poisson}(\lambda)$ and then find its skewness coefficient and kurtosis coefficient.

200. Cumulant generating function for exponential

Derive the cumulant generating function of $X \stackrel{d}{=} \exp(\lambda)$ and use the cgf to compute its skewness coefficient and kurtosis coefficient.

201. Inversion formula for Laplace transform

For random variable X having pdf of the form $f_X(x) = \begin{cases} 0.5e^{-x} + e^{-2x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$ calculate the Laplace transform of X and use the inversion formula for Laplace transforms to derive the cdf of X.

202. Recognising Distributions

In each of the following cases $M_X(t)$, the moment generating function of X, is given. Determine the distribution of X.

(a) $M_X(t) = \left(\frac{e^t + 3}{4}\right)^7$;

(b) $M_X(t) = \dfrac{1}{2-e^t}$;

(c) $M_X(t) = \left(\dfrac{2}{2-t}\right)^r$;

(d) $M_X(t) = \exp\left(3e^t - 3\right)$.

203. Poisson Probability

Let X, Y, and Z be three independent Poisson random variables with parameters λ_1, λ_2, and λ_3, respectively. For $y = 0, 1, 2, \ldots, t$, calculate $P(Y = y | X + Y + Z = t)$.

204. Linear Combination of Normals

For $i = 1, 2, \ldots, n$, let $X_1, X_2, \ldots, X_n$ be independent random variables such that $X_i \stackrel{d}{=} N(\mu_i, \sigma_i^2)$. Then for constants $\alpha_1, \alpha_2, \ldots, \alpha_n$, show that

$$\sum_{i=1}^{n} \alpha_i X_i \stackrel{d}{=} N\left(\sum_{i=1}^{n} \alpha_i \mu_i, \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2\right)$$

205. Independent Normals

Let $X \stackrel{d}{=} N(1, 2)$ and $Y \stackrel{d}{=} N(4, 7)$ be independent random variables. Find the probability of the following events

(a) $X + Y > 0$;

(b) $X - Y < 2$;

(c) $3X + 4Y > 20$.

206. IQ

The distribution of the IQ of a randomly selected student from a certain college is $N(110, 16)$. What is the probability that the average of the IQs of 10 randomly selected students from this college is at least 112?

207. Achievement Test

For the scores on an achievement test given to a certain population of students, the expected value is 500 and the standard deviation is 100. Let $\bar{X}$ be the mean of the scores of a random sample of 35 students from the population. Estimate $P(460 < \bar{X} < 540)$.

208. Credit Card

Each time that Jim charges an item to his credit card, he rounds the amount to the nearest dollar in his records. If he has used his credit card 300 times in the last 12 months, what is the probability that his record differs from the total expenditure by, at most, 10 dollars?

Hint: For $i = 1, 2, \ldots, 300$, let X_i be the be the amount of the ith expenditure minus Jim's ith record. Then $X_i \stackrel{d}{\approx} R\left(-\frac{1}{2}, \frac{1}{2}\right)$.

209. Party Guests

Suppose that, whenever invited to a party, the probability that a person attends with his or her guest is 1/3, attends alone is 1/3, and does not attend is 1/3. A company has invited all 300 of its employees and their guests to a Christmas party. What is the probability that at least 320 will attend?

Hint: For $i = 1, 2, \ldots, 300$, let $X_i = 0$ if employee i does not attend; $X_i = 1$ if employee i attends alone; $X_i = 2$ if employee i attends with a guest.

210. Heads Before Tails

A fair coin is tossed successively. Using the central limit theorem, find an approximation for the probability of obtaining at least 25 heads before 50 tails.

Hint: For $i = 1, 2, \ldots, 50$, let X_i be the number of heads between the $(i-1)$th and the ith tails. Then $X_i \stackrel{d}{=} \mathrm{G}\left(\frac{1}{2}\right)$.

211. Poisson Identity

Let $\{X_1, X_2, \ldots\}$ be a sequence of independent Poisson random variables, each with parameter 1. By applying the central limit theorem to this sequence, prove that

$$\lim_{n\to\infty} \frac{1}{e^n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

Hint: Let $Y_n = \sum_{i=0}^{\infty} X_i$. Write down the exact expression for $P(Y_n \leq n)$, and the approximate expression using a normal approximation. They will be equal in the limit by the central limit theorem.

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Problem Sheet 7

Stochastic Processes

212. Wire

A wire manufacturing company has inspectors to examine the wire for fractures as it comes out of a machine. The number of fractures is distributed in accordance with a Poisson process, having one fracture on the average for every 60 metres of wire. One day an inspector has to take an emergency phone call and is missing from his post for ten minutes. If the machine turns out 7 metres of wire per minute, what is the probability that the inspector will miss more than one fracture?

213. Even and Odd Numbers of Events

Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. What is the probability of an

(a) an even number of events in $(t, t+\alpha)$;

(b) odd number of events in $(t, t+\alpha)$?

214. Trees

In a forest, the number of trees that grow in a region of area R has a Poisson distribution with mean λR, where λ is a given positive number.

(a) Find the probability that the distance from a certain tree to the nearest tree is more than d.

(b) Find the probability that the distance from a certain tree to the nth nearest tree is more than d.

215. Poisson Process

For a Poisson process with parameter λ, show that, for all $\epsilon > 0$,

$$P\left(\left|\frac{N(t)}{t} - \lambda\right| \geq \epsilon\right) \to 0,$$

as $t \to \infty$. This shows that, for a large t, $N(t)/t$ is a good estimate for λ.

216. Bank

Customers arrive at a bank at a Poisson rate of λ. Let $M(t)$ be the number of customers who enter the bank by time t only to make deposits to their accounts. Suppose that, independent of other customers, the probability is p that a customer enters the bank only to make a deposit. Show that $\{M(t) : t \geq 0\}$ is a Poisson process with parameter λp.

217. Absorbing Markov Chain

Let $\{X_n : n = 0, 1, \ldots\}$ be a Markov chain with state space $\{1, 2, 3\}$ and transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 2/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Starting from state 1, what is the probability that the process never enters state 2?

218. Die

A fair die is tossed repeatedly. The maximum of the first n outcomes is denoted by X_n. Is $\{X_n : n = 1, 2, \ldots\}$ a Markov chain? Why or why not? If it is a Markov chain, calculate its transition probability matrix.

219. Trout

An observer at a lake notices that when fish are caught, only 1 out of 9 trout is caught after another trout, with no other fish between, whereas 10 out of 11 nontrout are caught following nontrout, with no trout between. Assuming that all fish are equally likely to be caught, what fraction of fish in the lake is trout?

220. Emmett

On a given day, Emmett drives to work (state 1), takes the train (state 2), or hails a taxi (state 3). Let $X_n = 1$ if he drives to work on day n, $X_n = 2$ if he takes the train on day n, and $X_n = 3$ if he hails a taxi on that day. Suppose that $\{X_n : n = 1, 2, \ldots\}$ is a Markov chain, and depending on how Emmett went to work the previous day, the probability of choosing any one of the means of transportation is given by the transition probability matrix

$$P = \begin{pmatrix} 1/6 & 2/3 & 1/6 \\ 1/2 & 1/3 & 1/6 \\ 2/5 & 1/2 & 1/10 \end{pmatrix}.$$

(a) Given that Emmett took the train today and every day in the last five days, what is the probability that he will not take the train to work tomorrow?

(b) If Emmett took the train to work today, what is the probability that he will not take the train to work tomorrow and the day after tomorrow?

221. Product of Transition Matrices

Show that if $\boldsymbol{P}$ and $\boldsymbol{Q}$ are two transition probability matrices with the same number of rows, and hence columns, then $\boldsymbol{PQ}$ is also a transition probability matrix. Note that this implies that if $\boldsymbol{P}$ is a transition probability matrix, then so is $\boldsymbol{P}^n$ for any positive integer n.

222. Vacation

On a given vacation day, a sportsman either goes horseback riding (activity 1), or sailing (activity 2), or scuba diving (activity 3). For $1 \leq i \leq 3$, let $X_n = i$, if the sportsman devotes vacation day n to activity i. Suppose that $\{X_n : n = 1, 2, \ldots\}$ is a Markov chain, and depending on which of these activities the sportsman chooses on a certain vacation day, the probability of engagement in any one of the activities on the next vacation day is given by the transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 0.20 & 0.30 & 0.50 \\ 0.32 & 0.15 & 0.53 \\ 0.60 & 0.13 & 0.27 \end{pmatrix}.$$

We know that the sportsman did not go scuba diving on the first day of his vacation, and it was equally likely that he went either riding or sailing on the first day. What is the probability that he did not go scuba diving on the second and third vacation days either?

223. Another Vacation

On a given vacation day, Francesco either plays golf (activity 1) or tennis (activity 2). For $i = 1, 2$, let $X_n = i$, if Francesco devotes vacation day n to activity i. Suppose that $\{X_n : n = 1, 2, \ldots\}$ is a Markov chain, and depending on which of the two activities he chooses on a certain vacation day, the probability of engagement in any one of the activities on the next vacation day is given by the transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 0.30 & 0.70 \\ 0.58 & 0.42 \end{pmatrix}.$$

Find the long-run probability that, on a randomly selected vacation day, Francesco plays tennis.

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Problem Sheet 1 Answers
Axioms of Probability

1. Sample Space

Denote the dictionaries by d_1, d_2; the third book by a.
The answers are $\{d_1d_2a, d_1ad_2, d_2d_1a, d_2ad_1, ad_1d_2, ad_2d_1\}$ and $\{d_1d_2a, ad_1d_2\}$.

2. Events

If E or F occurs, then G occurs; If G occurs, then E and F occur.

3. More Events

If $B = \emptyset$, the relation is obvious; To show the reverse implication, let $A = \Omega$ to show that $B = \emptyset$.

4. Cards

(a) $\displaystyle\bigcup_{n=1}^{\infty} A_n$;

(b) $\displaystyle\bigcup_{n=1}^{37} A_n$.

5. Event Identities

(a) Let $x \in (A\backslash(A\cap B))\cup B$ and show that $x \in A\cup B$, which shows $(A\backslash(A\cap B))\cup B \subseteq A\cup B$. Similarly show that $(A\backslash(A\cap B))\cup B \supseteq A\cup B$

(b) Use the method explained for Part (a).

6. Infinite Sequence of Sets

Let $B_1 = A_1$, $B_2 = A_2\backslash A_1$, $B_3 = A_3\backslash(A_1 \cup A_2)$, $\ldots$, $B_n = A_n\backslash \cup_{i=1}^{n-1} A_i$, $\ldots$.

7. Hiring

7/15 or 3/8, depending on your interpretation of the question.

8. Probability Statements

(a) False. Consider rolling a die to give a counterexample;

(b) False. Consider rolling a die to give a counterexample.

9. Probability Identity

Apply property (9) on Slide 37 twice.

10. Random Numbers

(a) 267/1000;

(b) 533/1000.

11. Voting

A Venn diagram shows that the answer is 0.36.

12. Boole's Inequality

Define the sequence of mutually exclusive events B_1, B_2, B_3, $\ldots$ as in the solution to Question 6, and use the fact, for $n \geq 1$, $B_n \subseteq A_n$.

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Problem Sheet 2 Answers

Conditional Probability and Independence

13. Dice

1/7

14. Movies

$\binom{7}{2}/\binom{13}{2} = 0.269$

15. Conditional Probability Properties

(a) Use the definition of conditional probability;

(b) Use the definition of conditional probability;

(c) Use the definition of conditional probability and Axiom 3.

16. Defective and Nondefective Items

(a) 0.0144;

(b) 0.344.

17. Credit Rating

92.8%

18. Cards

$\frac{\binom{13}{3}\binom{39}{6}}{\binom{52}{9}} \times \frac{10}{43} = 0.059$

19. Proof

(a) Use Property (9) on Slide 37;

(b) For the case when $P(A) = 0$ use the definition of independence; For the case when $P(A) = 1$ use Part (a) and the definition of independence.

20. More Proofs

(a) Use the definition of independence;

(b) Recognise that if $A \subseteq B$, then $P(A \cap B) = P(A)$. Then use the definition of independence.

21. Matching

$1 - (5/6)^6 = 0.6651$

22. Families

A and B are independent when there are three children, but not independent when there are two or four children.

23. Independent Events

(a) $1 - (1 - p_1)(1 - p_2) \ldots (1 - p_n)$;

(b) $(1 - p_1)(1 - p_2) \ldots (1 - p_n)$.

24. Urns and Balls

$\binom{5}{3} \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^2 = 0.329$

25. *A* Before *B*

Let $P(A) = p$ and $P(B) = q$. Define A_n to be the event that neither A nor B occurs in the first n trials, but A occurs on the nth trial. Calculate $P\left(\cup_{n=1}^{\infty} A_n\right)$.

26. Guns

0.7

27. Strokes

29%

28. Lost Spades

(a) Condition on the two events: a spade is missing, a spade is not missing. The answer is 1/4;

(b) Condition on the three events: two spades are missing, one spade is missing, no spade is missing. The answer is 1/4;

29. Proof

Use the definition of conditional probability.

30. Another Proof

Use the definition of conditional probability and the law of total probability.

31. Married on Campus

Use the result from Question 31. The answer is 1/3.

32. Batteries

7/10

33. Multiple Choice

8/9

34. Cards

4/37

35. More Cards

1/3

36. Horses

$205/297 = 0.69$

37. Chips

$4/7 = 0.571$

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Problem Sheet 3 Answers

Random Variables and Distribution Functions

38. More Chips

x	-6	-2	-1	2	3	4
$p_X(x)$	0.095	0.238	0.238	0.095	0.238	0.095

39. Plastic Die

1/2

40. Families

$$F_X(t) = \begin{cases} 0, & t < 0 \\ 1/8, & 0 \le t < 1 \\ 1/2, & 1 \le t < 2 \\ 7/8, & 2 \le t < 3 \\ 1, & t \ge 3. \end{cases}$$

41. Distribution Functions

(a) F is a distribution function. $F(-\infty) = 0$; $F(\infty) = 1$; F is right continuous; and since, for $t > 0$, $F'(t) > 0$, it is nondecreasing.

(b) F is not a distribution function. $\lim_{t\to 0^-} F(t) = 1/2 > \lim_{t\to 0^+} F(t) = 1/4$, and so F is not nondecreasing.

42. Random Points

2/3

43. New Car

$$F_Y(t) = \begin{cases} F_X(t), & t < 5 \\ 1, & t \ge 5. \end{cases}$$

44. Minimum Face Value

x	1	2	3	4	5	6
$p_X(x)$	11/36	9/36	7/36	5/36	3/36	1/36

$$F_X(x) = \begin{cases} 0, & x < 1 \\ 11/36, & 1 \le x < 2 \\ 20/36, & 2 \le x < 3 \\ 27/36, & 3 \le x < 4 \\ 32/36, & 4 \le x < 5 \\ 35/36, & 5 \le x < 6 \\ 1, & x \ge 6. \end{cases}$$

45. Probability Mass Function

x	-2	2	4	6
$p_X(x)$	1/2	1/10	13/45	1/9

46. Jury

$$p_X(i) = P(X = i) = \frac{\binom{18}{i}\binom{28}{12-i}}{\binom{46}{12}}, \qquad i = 0, 1, 2, \ldots, 12.$$

47. First Six

For $i = 1, 2, \ldots,$

$$p_X(i) = \left(\frac{5}{6}\right)^{i-1} \frac{1}{6}.$$

$$F_X(x) = \begin{cases} 0, & x < 1 \\ 1 - \left(\frac{5}{6}\right)^{\lfloor x \rfloor}, & x \ge 1. \end{cases}$$

48. More Probability Mass Functions

(a) $k = 1/15$;

(b) $k = 1/15$;

(c) $k = 8$;

(d) $k = \dfrac{2}{n(n+1)}$;

(e) $k = \dfrac{6}{n(n+1)(2n+1)}$.

49. Soap Opera

(a)

$$f(x) = \begin{cases} \dfrac{32}{x^3}, & x \geq 4 \\ 0, & x < 4. \end{cases}$$

(b) Graphs required.

(c) 9/25; 4/9; 0.313; 0

50. Tyres

(a) 66.67%

(b) 60%

51. Another Density Function

Show, for $x \in \mathbb{R}$, $h(x) \geq 0$, and $\displaystyle\int_{-\infty}^{\infty} h(x)dx = 1$.

52. Investment

(a) $k = -1/2$.

(b) 3/16

53. Convex Combination

Show, for $x \in \mathbb{R}$, $\alpha f(x) + \beta g(x) \geq 0$, and $\displaystyle\int_{-\infty}^{\infty} (\alpha f(x) + \beta g(x))\, dx = 1$.

54. Fuses

0.75

55. Nonexistant Expected Value

(a) Show that sum of the probabilities equals 1.

(b) Show that the sum diverges.

56. Distribution Function

5/8; 31/8; 23/8

57. Children

3

58. Probability Mass Function

(a) Show that sum of the probabilities equals 1 by expressing the series as a telescoping series.

(b) $\mathbb{E}(X)$ does not exist as the sum diverges.

59. Variance

$\mathbb{V}(X) = 12.234$; $\mathrm{sd}(X) = 3.498$

60. Random Integer

$\mathbb{E}(X) = (N+1)/2$; $\mathbb{V}(X) = (N^2-1)/12$; $\sigma_X = \sqrt{(N^2-1)/12}$

61. Another Variance

36

62. Soap Opera

(a) 8

(b) Show that $\mathbb{E}(X^2)$ does not exist.

63. Aptitude Test

$\mathbb{E}(X) = 3/2$; $\mathrm{sd}(X) = 1/\sqrt{20}$

64. Expected Value

3/2

65. Computer Network

$\alpha - A/k$

66. Logarithm of a Random Variable

$1 - \log 2$

67. Double Exponential Distribution

2

68. Absolute Moment

Use $|x|^s \leq \max(1, |x|^r) \leq 1 + |x|^r$, $x \in \mathbb{R}$, and show that $\mathbb{E}(|x|^s) < \infty$.

69. Recursive Moments

Use integration by parts twice.

70. Computing Moments using Tail Probabilities

See Slides 149–152.

71. Post Office

(a) $\mathbb{E}[X] = \dfrac{\alpha}{\lambda} + \dfrac{\beta}{\mu}$;

(b) $\mathbb{V}(X) = \dfrac{2\alpha - \alpha^2}{\lambda^2} + \dfrac{2\beta - \beta^2}{\mu^2} - \dfrac{2\alpha\beta}{\lambda\mu}$.

72. Bounded Random Variable

Proof required.

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Problem Sheet 4 Answers

Special Probability Distributions

73. Number Plates

$\binom{6}{2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^4 = 0.098$

74. Nails

Let X be the number of defective nails in the sample of 24. $P(X = 2) = 0.127$ which is relatively high, and so it is unfair to reject the company's claim. If $P(X = 2) < 0.05$ then it would be reasonable to question the manufacturer's claim.

75. Five Points

(a) 0.539

(b) 0.073

76. Rare Blood Type

0.442

77. Children

6

78. Chuck-a-Luck

$P(X = -1) = 125/216$; $P(X = 1) = 75/216$; $P(X = 2) = 15/216$; $P(X = 3) = 1/216$; $\mathbb{E}(X) = -0.08$

79. Aircraft Engines

No. A four-engine plane is preferable to a two-engine plane if and only if $p > 2/3$.

80. Even Number of Successes

To find r_n, use the recursive relation, $r_0 = 1$, and induction.

81. Targets

(a) $(1-pq)^{r-1}pq$

(b) $(1-pq)/pq$

82. Basketball

(a) $(0.55)^5(0.45) = 0.023$

(b) $(0.55)^6(0.45)^2 = 0.0056$

83. Light Bulbs

p^n

84. Bridge

The probability a bridge hand contains three aces is $p = \frac{\binom{4}{3}\binom{48}{10}}{\binom{52}{13}}$. Therefore, the average number of hands until one has three aces is $1/p = 24.27$.

85. Senior Citizens

$X \stackrel{d}{=} \mathrm{Nb}(10, 0.15)$, therefore, for $i = 0, 1, 2, \ldots$, $p_X(i) = \binom{i+9}{9}(0.85)^i(0.15)^{10}$.

86. Geometric Distribution

$1/(2-p)$

87. Negative Binomial

Proof required.

88. Negative Binomial Distribution

See Slides 166–169.

89. Professors' Cars

0.772

90. Bernoulli Trials

Note that, if $M \stackrel{d}{=} \mathrm{Nb}(x, p)$, then $N = M + x$.

91. Charity

0.987

92. Defective Items

No. If 5% of items are defective then the shipment should contain 25 defective items. The probability that there are 2 defective items is $\binom{25}{2}\binom{475}{28}/\binom{500}{30} = 0.268$, which is quite high.

93. Trout

The probability of four tagged trout were among the second 50 caught is

$$p_n = \frac{\binom{50}{4}\binom{n-50}{46}}{\binom{n}{50}}.$$

Now,

$$\frac{p_n}{p_{n-1}} = \frac{(n-50)^2}{n(n-96)},$$

and $p_n \geq p_{n-1}$ if and only if $n \leq 625$. So the estimated trout population size is 625.

94. Annual Income

0.8913

95. Misprints

0.0154

96. No Crimes

0.0028

97. Bookstore

0.013

98. Lottery Tickets

If X is the number of wins in a day then $X \stackrel{d}{=} \mathrm{Pn}(M/N)$. The answer is the least integer greater than or equal to $-N \log(1-\alpha)/M$.

99. Poisson Distribution

Proof required.

100. Line Segment

(a) 2/3

(b) 1/3

101. Random Angle

$f_X(x) = 1/(\pi(x^2+1))$

102. Expectation of a Logarithm

1

103. Chicken Pen

a/b

104. Heart Attack

1/2. Use the memoryless property.

105. Exponential Random Variable

e^{-3}

106. Telephone Call

1/2. Use the memoryless property.

107. Double Exponential Random Variable

(a) $c = 1/2$

(b) Note that $x^{2n+1}e^{-|x|}$ is an odd function, and $x^{2n}e^{-|x|}$ is an even function. Use the gamma function defined on Slide 239.

108. Radio Tubes

$\lfloor X \rfloor \stackrel{d}{=} \mathrm{G}(1 - e^{-\lambda})$

109. Gamma Distribution

Proof required.

110. Defective Light Bulbs

Let X be the time until 25 defective light bulbs are produced. Then $X \stackrel{d}{=} \gamma(25, 200 \times 0.015 = 3)$ and $\mathbb{E}(X) = 25/3$.

111. Born on Christmas Day

$X \stackrel{d}{=} \mathrm{Bi}(1095, 1/365)$ and $X \stackrel{d}{\approx} \mathrm{Pn}(3)$. The answer is 0.0892.

112. Beta Mean and Variance

$\mathbb{E}[X] = \dfrac{\alpha}{\alpha + \beta}$; $\mathbb{V}(X) = \dfrac{\alpha\beta}{(\alpha + \beta + 1)(\alpha + \beta)^2}$

113. Beta Random Variable

$X \stackrel{d}{=}$ Beta(5,6); $c = 1,260$; $\mathbb{E}[X] = 5/11$; $\mathbb{V}(X) = 5/242$

114. Blood Pressure Medicines

0.538

115. Resistors

$\alpha = 1$; $\beta = 2$; $P(X \geq 7/12) = 0.174$

116. Pareto Random Variable

$$F_X(x) = \begin{cases} 1 - \left(\dfrac{\alpha}{x}\right)^{\gamma}, & x \geq \alpha \\ 0, & \text{otherwise} \end{cases}; \ \mathbb{E}[X] = \frac{\gamma\alpha}{\gamma - 1}, \gamma > 1; \ \mathbb{V}(X) = \frac{\gamma\alpha^2}{(\gamma-1)^2)(\gamma-2)}, \gamma > 2$$

117. Pareto Random Variable

$\mathbb{E}[X^n] = \dfrac{\gamma\alpha^n}{\gamma - n}$ which is defined for $n < \gamma$.

118. Newspaper Subscribers

(a) 0.5

(b) 0.7013

119. IQ

$(96.6, 123.4)$

120. Manual Dexterity

0.000016

121. Normal Density Function

$k = \pi$ and f is the probability function of $X \stackrel{d}{=} \mathrm{N}\left(-1/\pi, 1/\left(2\pi^2\right)\right)$.

122. Density Function of $\boldsymbol{Y}$

$$f(y) = \begin{cases} \dfrac{4y}{\sqrt{2\pi}} e^{-y^4/2}, & y \geq 0 \\ 0, & y < 0. \end{cases}$$

123. Another Normal Density Function

$k = \sqrt{\log a/\pi}$ and f is the density function of $X \stackrel{d}{=} \mathrm{N}\left(0, \sqrt{1/(2\log a)}\right)$.

124. Skeletons

Let X be the height of a randomly selected skeleton from the first group. Then $P(X > 185) = 0.0749$. Let Y be the number in the second group that are taller than 185 centimetres. If the second group belongs to the same family as the first, then $Y \stackrel{d}{=} \mathrm{Bi}(5, 0.0749)$, and $P(Y = 3) = 0.0036$. Therefore, the chance of finding exactly three or more skeletons taller than 185 centimetres is very small. Thus, it is unlikely that the second group belonged to the same family as the first.

125. Weibull Mean and Variance

$\mathbb{E}[X] = \beta\Gamma\left(\frac{\gamma+1}{\gamma}\right)$; $\mathbb{V}(X) = \beta^2\Gamma\left(\frac{\gamma+2}{\gamma}\right) - \left(\beta\Gamma\left(\frac{\gamma+1}{\gamma}\right)\right)^2$

126. Weibull and exponential

$\exp(1)$

127. Exponential of a Random Variable

$$g(y) = \begin{cases} \dfrac{f(\log y)}{y}, & y > 0 \\ 0, & y \leq 0. \end{cases}$$

128. Logarithm of a Random Variable

$f(y) = (3\log 2)2^y e^{-3(2^y)}, y \in \mathbb{R}.$

129. Log of a Pareto Random Variable

$$F_Y(y) = \begin{cases} 1 - \left(\dfrac{\alpha}{e^y}\right)^\gamma, & y \geq \log\alpha \\ 0, & \text{otherwise} \end{cases}; \; f_Y(y) = \begin{cases} \gamma\alpha^\gamma e^{-\gamma y}, & y \geq \log\alpha \\ 0, & \text{otherwise} \end{cases}$$

130. Square of a Random Variable

$$g(y) = \begin{cases} \dfrac{1}{2\sqrt{y}}\left(f(\sqrt{y}) + f(-\sqrt{y})\right), & y > 0 \\ 0, & y < 0. \end{cases}$$

131. Reciprocal of a Random Variable

Let F_Y and f_Y be distribution and density functions of Y, respectively. Then use the law of total probability to get $F_Y(y) = P(Y \leq y) = P(Y \leq y, X \leq 1) + P(Y \leq y, X > 1)$.

The density function of Y is $f_Y(y) = \begin{cases} e^{-y} + \dfrac{e^{-1/y}}{y^2}, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$

132. Transformation by $\psi(x) = \min(x, M)$

$F_Y(y) = \begin{cases} 0, & y < 0, \\ 1 - e^{-y}, & 0 \leq y < 5, \\ 1, & y \geq 5; \end{cases}$ using the formula of moments via tail probabilities, $\mathbb{E}(Y) = 1 - e^{-5}$, $\mathbb{E}(Y^2) = 2(1 - 6e^{-5})$, $\mathbb{V}(Y) = 1 - 10e^{-5} - e^{-10}$.

133. Lognormal Distribution

(a) $\mathbb{E}(X^r) = e^{\mu r + \frac{1}{2}\sigma^2 r^2}$

(b) $\mathbb{E}(X) = e^{\mu + \sigma^2/2}$ and $\mathbb{V}(X) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$.

134. Large Fires

0.3192

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Problem Sheet 5 Answers

Bivariate Random Variables

135. Joint Distribution

(a) $c = 1/21$

(b) $p_X(x) = (2x+3)/21,\ x = 1,2,3;\ p_Y(y) = (3y+6)/21,\ y = 1,2$

(c) $7/9$

(d) $\mathbb{E}(X) = 46/21; \mathbb{E}(Y) = 11/7.$

136. Another Joint Distribution

(a) $2/5$

(b) $7/25$

(c) $6/25$

137. Cards

(a) For $0 \le x \le 7,\ 0 \le y \le 7,\ 0 \le x+y \le 7$,

$$p_{(X,Y)}(x,y) = \frac{\binom{13}{x}\binom{13}{y}\binom{26}{7-x-y}}{\binom{52}{7}}.$$

(b) $P(X \ge Y) = \sum_{y=0}^{3} \sum_{x=y}^{7-y} p_{(X,Y)}(x,y) = 0.61107$

138. Marginal Densities

(a) For $0 \le x \le 1,\ f_X(x) = 4x^3$; For $0 \le y \le 1,\ f_Y(y) = 4y(1-y^2)$

(b) $\mathbb{E}(X) = 4/5;\ \mathbb{E}(Y) = 8/15$

139. Probability Calculations

Note that if $A \in [0,1] \times [0,1]$, then $P((X,Y) \in A) = \text{Area}(A)$.

(a) $1/8$

(b) $7/8$

(c) 0.597

(d) $\pi/4$

140. Line Segment

$3/4$

141. Expected Values

Proof required. Note that, for $x > y$, $f_{(X,Y)}(x,y) = 0$.

142. Cards

X and Y are not independent.

143. Maximum and Minimum of X and Y

$F_U(u) = F(u)G(u)$; $F_V(v) = F(v) + G(v) - F(v)G(v)$

144. Independent or Not

$\mathbb{E}(XY) = 4/9 \neq \mathbb{E}(X)\mathbb{E}(Y) = 32/75$.

145. Probability Density of X/Y

Proof required.

146. Expected Value of Maximum and Minimum

$\mathbb{E}(\max(X,Y)) = 1/3$; $\mathbb{E}(\min(X,Y)) = -1/3$

147. Product of Two Functions

Show that $f_{(X,Y)}(x,y) = f_X(x)f_Y(y)$.

148. Conditional Density Function

$f_{X|Y}(x|y) = 1/y, 0 < x < y, 0 < y < 1$

149. Conditional Probability

$f_{X|Y}(x|3/4) = \left(48x^2 + 27\right)/43$ and $P(1/4 < X < 1/2|Y = 3/4) = 17/86$.

150. Another Conditional Density Function

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{x+y}{1/2+y}, & \text{if } 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

151. Random Points

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{1-y}, & \text{if } 0 < y < 1, \quad y < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

$$f_X(x) = \begin{cases} -\log(1-x), & \text{if } 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

152. Conditional Mean and Variance

(a) $c = 1/2$

(b) $f_{X|Y}(x|y) = e^{-x+|y|}, x > |y|$ and $f_{Y|X}(y|x) = 1/(2x), |y| < x$

(c) $\mathbb{E}(Y|X = x) = 0$ and $\mathbb{V}(Y|X = x) = x^2/3$.

153. Yet Another Conditional Density Function

$$f_{X,Y}(x,y) = \begin{cases} 1/2, & \text{if } |x| + |y| \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

$f_Y(y) = 1 - |y|, \quad -1 \le y \le 1$

$$f_{X|Y}(x|y) = \frac{1}{2(1-|y|)}, \quad -1 + |y| \le x \le 1 - |y|, \quad -1 \le y \le 1$$

154. Sum of Two Random Numbers

Proof required. Use the convolution formula.

155. Conditional Probability

(a) Verify the convolution formula

(b) Show that $p_{(X,Y)}(x,y) = p_X(x)p_Y(y)$ if and only if $c = 0$.

156. Conditional Mean and Variance

Note that X and Y are independent.

(a) $\mathbb{E}(X) = 1$

(b) $\mathbb{E}(Y) = 1/2$

(c) $\mathbb{E}(X^2 + Y^2) = 5/2$

157. Letters

1

158. Empty Boxes

5.28

159. *X* and *Y*

Proofs required.

160. Discrete Random Variable

Proof required.

161. Covariance

$-1/245$

162. Covariance Proofs

Proofs required.

163. More Covariance Proofs

Proofs required.

164. Independent and/or Uncorrelated?

X and Y are not independent but are uncorrelated.

165. Investment

0.1407

166. Covariance of Linear Combinations

Use, if $X_0 = X - \mu_X$ and $Y_0 = Y - \mu_Y$, then $\mathrm{Cov}(X, Y) = \mathbb{E}(X_0, Y_0)$.

167. Variance of a Linear Combination

Use the result from Question 153

168. Correlation

112

169. Broken Stick

$-1/12$

170. Uncorrelated Result

Proof required.

171. No Linear Relation

$\rho(X, Y) = -0.245 \neq \pm 1$, so there exists no linear relation between X and Y.

172. Two Successive Tails

6

173. Mixture of Two Random Variables

$\mathbb{E}(X) = p\mathbb{E}(Y) + (1-p)\mathbb{E}(Z)$.

174. Verification

$\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}\left(\dfrac{6Y^2+3}{12Y^2+4}\right) = \mathbb{E}(X) = 5/8.$

175. Eggs Hatching

$\mathbb{E}[X] = \lambda p$; $\mathbb{V}(X) = \lambda p$; $X \stackrel{d}{=} \mathrm{Pn}(\lambda p)$

176. Electronic Components

$\mathbb{E}[X] = mpq$; $\mathbb{V}(Y) = mpq(1-pq)$; $X \stackrel{d}{=} \mathrm{Bi}(m, pq)$

177. Consecutive Zeros

1110

178. Fishing

The expected value is 11 and the variance is 19.33

179. Wallets

Suppose Player A carries \$$x$ in his wallet. First show that $\eta(x) = \mathbb{E}(W_A|X = x) = \frac{1}{2} - \frac{3}{2}x^2$. Then use $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(W_A|X))$.

180. Heights

0.5753

181. Bivariate Normal

(a) 0.6799

(b) 0.1993

182. Grade Point Average

0.0708

183. Approximating Means and Variances

(a) $\mathbb{E}[Y] \approx 0.4594$; $\mathbb{V}(Y) \approx 0.0648$

(b) $\mathbb{E}[Y] \approx -1.1931$; $\mathbb{V}(Y) \approx 1$

(c) $\mathbb{E}[Y] \approx 10$; $\mathbb{V}(Y) \approx 96$

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Problem Sheet 6 Answers

Sums of Independent Random Variables/Limit Theorems

184. Chebyshev's Inequality

0.472

185. Accidents

0.222

186. IQ Scores

Less than 1.7% of students have IQ scores above 140.

187. Chebyshev's Inequality Again

$P(X > 2\mu) \leq 1/\mu$

188. Multiple Choice

1/9

189. Probability Generating Function

(a)

$$p_X(x) = \begin{cases} 0.3, & x = 0 \\ 0.2, & x = 1 \\ 0.1, & x = 2 \\ 0.4, & x = 3. \end{cases}$$

(b) $P_X'(z) = 0.2 + 0.2z + 1.2z^2$; $\mathbb{E}(X) = P_X'(1) = 1.6$
$P''(z) = 0.2 + 2.4z$; $\mathbb{V}(X) = P_X''(1) + P_X'(1) - P_X'(1)^2 = 1.64$

(c) $P_Y(z) = 0.3 + 0.2z + 0.1z^8 + 0.4z^{27}$

190. Sum of Negative Binomial Random Variables

$P_X(z) = p^{r_1+r_2+\ldots r_n}\,(1-(1-p)z)^{-(r_1+r_2+\ldots r_n)}$

191. Random Sum

Use the expression for conditional probability generating functions.

192. Binomial Sum

(a) For $z \in \mathbb{R}$, $P_{Y|N}(z) = (1 - q + qz)^{mN}$.

(b) For $z \in \mathbb{R}$, $P_Y(z) \left(1 - p + p\left(1 - q + qz\right)^m\right)^n$.

(c) $\mathbb{E}(Y) = mnpq$

193. Continuous Density Function

$M_X(t) = \dfrac{2e^t}{t} - \dfrac{2e^t}{t^2} + \dfrac{2}{t^2}$

194. Uniform Random Variable

$$M_X(t) = \begin{cases} \dfrac{e^{tb} - e^{ta}}{(b-a)t}, & \text{if } t \neq 0 \\ 1, & \text{if } t = 0. \end{cases}$$

195. Probability Mass Function

$p_X(x) = x/21,\ x = 1, 2, 3, 4, 5, 6$

196. Linear Function

$M_Y(t) = e^t/(1 - 2t)$, $t < 1/2$

197. Moments

If r is odd, $\mathbb{E}(X^r) = 0$; If r is even, $\mathbb{E}(X^r) = 1/3$

198. Constant Random Variable

$M_X(t) = e^{2t}$; $P(X = 2) = 1$

199. Cumulant generating function for Poisson

$K_X(t) = \lambda(e^t - 1)$; $\text{Skew}(X) = \lambda^{-1/2}$; $\text{Kurt}(X) = 1/\lambda$

200. Cumulant generating function for exponential

$K_X(t) = -\ln(1 - t/\lambda)$; $\text{Skew}(X) = 2$; $\text{Kurt}(X) = 6$

201. Inversion formula for Laplace transform

$L_X(t) = \frac{1}{2(t+1)} + \frac{1}{t+2}$; the cdf of X is $F_X(x) = \frac{1}{2}(1 - e^{-x}) + \frac{1}{2}(1 - e^{-2x}),\ x \geq 0.$

202. Recognising Distributions

(a) $X \stackrel{d}{=} \mathrm{Bi}(7, \frac{1}{4})$

(b) $X \stackrel{d}{=} \mathrm{G}(\frac{1}{2})$

(c) $X \stackrel{d}{=} \gamma(r, 2)$

(d) $X \stackrel{d}{=} \mathrm{Pn}(3)$

203. Poisson Probability

$\binom{t}{y}\left(\frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}\right)^y\left(\frac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+\lambda_3}\right)^{t-y}$

204. Linear Combination of Normals

Use the moment generating function for normal random variables.

205. Independent Normals

(a) $X + Y \stackrel{d}{=} N(5, 9)$; 0.9525

(b) $X - Y \stackrel{d}{=} N(-3, 9)$; 0.9525

(c) $3X + 4Y \stackrel{d}{=} N(19, 130)$; 0.4641

206. IQ

0.0571

207. Achievement Test

0.9822

208. Credit Card

0.9544

209. Party Guests

0.0793

210. Heads Before Tails

0.9938

211. Poisson Identity

Proof required.

MAST20004 Probability

Semester 1, 2020

Problem Sheet 7 Answers

Stochastic Processes

212. Wire

0.325

213. Even and Odd Numbers of Events

(a) $\left(1+e^{-2\lambda\alpha}\right)/2$

(b) $\left(1-e^{-2\lambda\alpha}\right)/2$

214. Trees

(a) $e^{-\lambda\pi d^2}$

(b) $\sum_{i=1}^{n-1} e^{-\lambda\pi d^2}\left(\lambda\pi d^2\right)^i/i!$

215. Poisson Process

Use $\mathbb{E}(N(t)) = \mathbb{V}(N(t)) = \lambda t$ and apply Chebyshev's inequality.

216. Bank

Condition on the event $N(t) = n$ and use the law of total probability.

217. Absorbing Markov Chain

1/2

218. Die

Let Z_n be the outcome of the nth toss. Then $X_{n+1} = \max(X_n, Z_{n+1})$. $\{X_n : n = 1, 2, \ldots\}$ is a Markov chain since X_{n+1} only depends on X_n. The state space is $\{1, 2, 3, 4, 5, 6\}$ and the transition probability matrix is

$$\boldsymbol{P} = \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

219. Trout

$P = \begin{pmatrix} 10/11 & 1/11 \\ 8/9 & 1/9 \end{pmatrix}$ and $\boldsymbol{\pi} = \begin{pmatrix} 88/97 & 9/97 \end{pmatrix}$. So the proportion of trout is approximately 9.3%

220. Emmett

(a) 2/3

(b) 1/4

221. Product of Transition Matrices

Consider the ij entry of $\boldsymbol{PQ}$ and then show that the sum of the entries in the ith row sum to one.

222. Vacation

0.2358

223. Another Vacation

0.5469