THE UNIVERSITY OF MELBOURNE

SCHOOL OF MATHEMATICS AND STATISTICS

MAST20009

Vector Calculus

Lecture Notes

STUDENT NAME:

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1 Functions of Several Variables

1.1 Limits

We say $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ has the limit L as (x, y) approaches (x_0, y_0):

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if when (x, y) approaches (x_0, y_0) along ANY path in the domain D, $f(x, y)$ gets close to L.

Note

1. The limit can exist if f is undefined at (x_0, y_0).
2. L must be finite.

Laws of Limits

If c, L, M are finite real constants, and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$$

then

1. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [f(x,y) + g(x,y)] = L + M$

2. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [cf(x,y)] = cL$

3. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [f(x,y)g(x,y)] = LM$

4. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{L}{M}$ if $M \neq 0$

Example 1: Evaluate $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x+3}{5xy-y^3}$.

Example 2: Evaluate $\lim_{(x,y)\to(2,1)} \dfrac{x^2-3xy+2y^2}{x-2y}$.

$\left(\text{ has form } \dfrac{0}{0} \text{ at } (2,1)\right)$

<u>Note</u>

If we cannot apply any tricks to evaluate the limit, then examine $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ along different paths approaching (x_0, y_0).

Example 3: Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$.

$\left(\text{ has form } \dfrac{0}{0} \text{ at } (0,0)\right)$

Example 4: Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

$$\left(\text{ has form } \frac{0}{0} \text{ at } (0,0)\right)$$

Example 5: Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{x^2}{\sqrt{x^2+y^2}}$.

$$\left(\text{has form } \frac{0}{0} \text{ at } (0,0) \right)$$

Sandwich Theorem

Suppose for (x, y) near (x_0, y_0) that f, g, and h are continuous and $g(x, y) \leq f(x, y) \leq h(x, y)$, then

$$\text{if} \lim_{(x,y)\to(x_0,y_0)} g(x, y) = L \text{ and } \lim_{(x,y)\to(x_0,y_0)} h(x, y) = L,$$

$$\text{then} \lim_{(x,y)\to(x_0,y_0)} f(x, y) = L.$$

1.2 Continuity

f is continuous at $(x,y) = (x_0, y_0)$ if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

Example 1: Is f, given by $f(x,y) = \dfrac{x^2}{x^2+y^2}$, continuous at (0,0)?

Theorem

The following function types are continuous at every point in their domains: polynomials, trigonometric functions, nth root functions, exponentials, logarithms, hyperbolic functions.

Theorem

If f and g are continuous at (x_0, y_0) and $c \in \mathbb{R}$, then the following functions are continuous at (x_0, y_0):

1. $f + g$
2. cf
3. fg
4. $\dfrac{f}{g}$ if $g(x_0, y_0) \neq 0$
5. $h \circ f$ where h is continuous at $z = f(x_0, y_0)$.
   $[(h \circ f)(x,y) = h(f(x,y))]$

Example 2: Where is f, given by

$$f(x,y) = \log(1 - xy),$$

continuous?

Example 3: Where is f, given by

$$f(x,y) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ \\ 0 & (x,y) = (0,0) \end{cases}$$

continuous?

1.3 Differentiability

f is differentiable at (x_0, y_0) if

1. $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ exist at (x_0, y_0)

2. a tangent plane

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} (x - x_0) \\ &\quad + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} (y - y_0) \end{aligned}$$

exists at (x_0, y_0) and is a good approximation to $f(x,y)$ at (x_0, y_0).

That is, in the limit as $(x,y) \to (x_0, y_0)$,

$$\frac{f(x,y) - f(x_0,y_0) - \boldsymbol{\nabla} f(x_0,y_0) \cdot (x - x_0, y - y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \to 0$$

Theorem

If $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ exist and are continuous at (x_0, y_0) then f is differentiable at (x_0, y_0).

Example 1: Where is f, given by

$$f(x, y) = x^2 + y^2,$$

differentiable?

Example 2: Is f, given by

$$f(x, y) = \begin{cases} \dfrac{x^2}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0), \end{cases}$$

differentiable?

Order C^n

A function is C^n if all of its n^{th} order partial derivatives exist and are continuous.

If a function is C^n, it is automatically $C^1,\ C^2,\ C^3,\ \ldots,\ C^{n-1}$.

Example

f is C^1 if $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist and are continuous.

$\Rightarrow$ All C^1 functions are differentiable.

Note

If a function is C^n then the order of differentiation in the partial derivatives of order n is NOT important.

Example

If f is C^2 then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

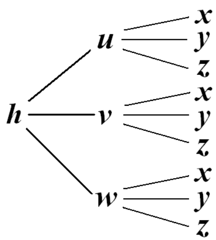
Example 3: Where is

$$f(x,y) = \left(x^2 + y^2\right)^{\frac{3}{4}},$$

C^1?

1.4 Chain Rule for Functions of Several Variables

Example 1: Let $h = f(u, v, w)$ where $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$. Find $\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}$.



Note

For change of variables in multiple integrals we need the Jacobian, the determinant of the Jacobi Matrix.

$$\text{Jacobian} = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

Inverse Function Theorem

Suppose that $\frac{\partial(u,v,w)}{\partial(x,y,z)} \neq 0$.

Then the linear equations relating the changes in coordinates can be inverted and

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Taking determinants, we get

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$$

Derivative Matrix

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable function, then the derivative is an $m \times n$ matrix given by:

$$\boldsymbol{D}f = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

where

$$f(x_1, x_2, \ldots, x_n)$$
$$= \big(f_1(x_1, \cdots, x_n), f_2(x_1, \cdots, x_n), \ldots, f_m(x_1, \cdots, x_n)\big).$$

Example 2: Find the derivative matrix for the function f defined by

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \; f(u, v) = (u + v, u, v^2).$$

Example 3: Find the derivative matrix for the function g defined by

$$g : \mathbb{R}^2 \to \mathbb{R}^2,\ g(x, y) = (x^2 + 1, y^2).$$

Matrix Version of Chain Rule

If $f : \mathbb{R}^m \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable functions and the composition $f \circ g$ is defined then,

$$\boldsymbol{D}(f \circ g) \;=\; \boldsymbol{D}f\, \boldsymbol{D}g$$

Note

$$(f \circ g)(x_1, x_2, \ldots, x_n) \;=\; f\,[g(x_1, x_2, \ldots, x_n)]$$

Example 4: Let $f(u,v) = (u+v, u, v^2)$
and $g(x,y) = (x^2+1, y^2)$.

Compute $\boldsymbol{D}(f \circ g)$ at $(x,y) = (1,1)$.

Check by substitution

Example 5: Let $f(u, v, w) = u^2 + v^2 - w$ and $g(x, y, z) = (x^2 y, y^2, e^{-xz})$.

Compute $\boldsymbol{D}(f \circ g)$ at $(x, y, z) = (0, 1, 2)$.

Example 6: Let $f(x, y) = (x^2, 2x + y, y^3)$ and $g(u, v, w) = (u^2 + 2w, u - v^2)$.

Find the derivative of $f(g[f(x, y)])$ at $(1, 0)$.

1.5 Taylor Polynomials

If f is of order C^n, we can approximate $f(x,y)$ by a polynomial of order n around (a,b).

- Approximate $f(x,y)$ by a linear function $p_1(x,y)$ near (a,b).

$$\begin{aligned} p_1(x,y) &= f(a,b) + \left.\frac{\partial f}{\partial x}\right|_{(a,b)} (x-a) + \left.\frac{\partial f}{\partial y}\right|_{(a,b)} (y-b) \\ &= f(a,b) + (x-a, y-b).\underbrace{\left.\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\right|_{(a,b)}}_{\boldsymbol{\nabla} f(a,b)} \\ &= f(a,b) + \left[(x-a, y-b).\underbrace{\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)}_{\boldsymbol{\nabla}}\right] f|_{(a,b)} \end{aligned}$$

Let $\boldsymbol{x} = (x,y)$ and $\boldsymbol{a} = (a,b)$, then

$$p_1(\boldsymbol{x}) = f(\boldsymbol{a}) + [(\boldsymbol{x}-\boldsymbol{a}).\boldsymbol{\nabla}]\, f|_{\boldsymbol{a}}$$

- In general, the n^{th} order Taylor polynomial for $f(\boldsymbol{x})$ near $\boldsymbol{x} = \boldsymbol{a}$ is

$$\boxed{p_n(\boldsymbol{x}) = \sum_{k=0}^{n} \frac{1}{k!} \left[(\boldsymbol{x} - \boldsymbol{a}).\boldsymbol{\nabla}\right]^k f|_{\boldsymbol{a}}}$$

Expanding the operator gives the third order Taylor polynomial for $f(x, y)$ around (a, b):

$$p_3(x, y) = f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b)$$

$$+ \frac{1}{2}\Big[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b)f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)\Big]$$

$$+ \frac{1}{6}\Big[(x - a)^3 f_{xxx}(a, b) + 3(x - a)^2(y - b)f_{xxy}(a, b) + 3(x - a)(y - b)^2 f_{xyy}(a, b) + (y - b)^3 f_{yyy}(a, b)\Big]$$

Taylor's Theorem for Remainder

The truncation error or remainder when approximating $f(\boldsymbol{x})$ by the n^{th} order Taylor Polynomial about $\boldsymbol{x} = \boldsymbol{a}$ is

$$R_n(\boldsymbol{x}) = f(\boldsymbol{x}) - P_n(\boldsymbol{x})$$

or

$$R_n(\boldsymbol{x}) = \frac{1}{(n+1)!} \left[(\boldsymbol{x} - \boldsymbol{a}).\boldsymbol{\nabla}\right]^{n+1} f|_{(\boldsymbol{a} + \xi(\boldsymbol{x} - \boldsymbol{a}))}$$

where $0 < \xi < 1$.

Example 1: Find the 2^{nd} order Taylor polynomial for $f(x, y) = e^{xy}$ near $(1, 0)$. Hence, approximate $e^{0.11}$.

Example 2: Find the 2^{nd} order Taylor polynomial for $f(x, y) = \sin(x + 2y)$ near $(0, 0)$. Find an upper bound for the error if $|x| < 0.1$ and $|y| < 0.1$.

Example 3: Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x+2y)}{x+2y}$.

1.6 Extrema

The critical points of a function of several variables f occur when

1. $\nabla f = \mathbf{0}$ <u>OR</u>
2. ∇f does not exist.

Near the critical point (a, b), the 2^{nd} order Taylor polynomial reduces to

$$p_2(x,y) = f(a,b) + \frac{1}{2}\Big[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\Big]$$

since $\nabla f(a,b) = \mathbf{0}$.

$$\Rightarrow p_2(x,y) = f(a,b) + \frac{1}{2}\begin{bmatrix} x-a & y-b \end{bmatrix} \underbrace{\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}\Bigg|_{(a,b)}}_{H(a,b)\,=\,\text{Hessian matrix}} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

Then

$$\det(H) = f_{xx}f_{yy} - (f_{xy})^2 .$$

The Hessian determines whether the critical points are maxima, minima or saddle points.

If $(x, y) = (a, b)$ is a critical point then

1. if det $H(a, b) > 0$ then

   (a) if $f_{xx}(a, b) < 0 \ \Rightarrow$ maximum at (a, b)

   (b) if $f_{xx}(a, b) > 0 \ \Rightarrow$ minimum at (a, b)

2. If det $H(a, b) < 0 \ \Rightarrow$ saddle point at (a, b)

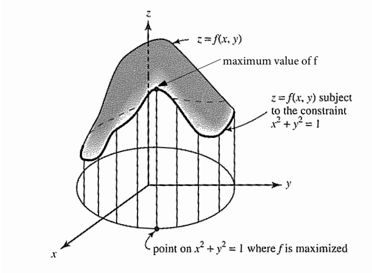
3. If det $H(a, b) = 0 \ \Rightarrow$ test is inconclusive.

Constrained Extrema

Find the maximum and minimum values of a function of several variables subject to one or more constraints.

Example

The geometric meaning of maximising a function of two variables $f(x, y)$ subject to the constraint $x^2 + y^2 = 1$.



1.7 Lagrange Multipliers

Theorem

If $\boldsymbol{a}$ is a maximum and minimum of f subject to the constraint $g(\boldsymbol{x}) = 0$, then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$\boldsymbol{\nabla} f(\boldsymbol{a}) = \lambda \boldsymbol{\nabla} g(\boldsymbol{a}).$$

Geometrically

- $\boldsymbol{\nabla} f(\boldsymbol{a})$ is normal to a level set of f.
- $\boldsymbol{\nabla} g(\boldsymbol{a})$ is normal to $g(\boldsymbol{x}) = 0$.
- At the critical points $\boldsymbol{\nabla} f(\boldsymbol{a})$ and $\boldsymbol{\nabla} g(\boldsymbol{a})$ are parallel.

So the constraint $g(\boldsymbol{x}) = 0$ will be tangent to a level set of f at the critical points.

Guidelines for Extrema

A set D is bounded if there is a finite real number M such that $|x| < M$ for all $x \in D$.

A set is closed if it contains all of its boundary points.

1. If constraint is closed and bounded there exists a maximum and minimum

   If there are only 2 critical points, one is a maximum and one is a minimum.

2. If constraint is open or unbounded then maxima and minima need not exist.

Example 1: Closed, bounded constraint.

$$x^2 + y^2 + z^2 = 1$$

Example 2: Open, bounded constraint.

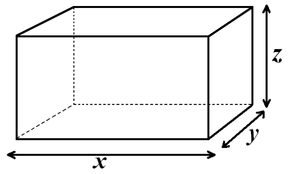
$$x^2 + y^2 + z^2 < 1$$

Example 3: Closed, unbounded constraint.

$$x + y = 1$$

Example 4: An open rectangular box is to be made with fixed volume of $4m^3$.

What dimensions should the box have to minimise the amount of material used to make it?



Theorem

If $\boldsymbol{a}$ is a maximum or minimum of f subject to the constraints

$$g_1(\boldsymbol{x}) = 0 \text{ and } g_2(\boldsymbol{x}) = 0,$$

there exists Lagrange multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\boldsymbol{\nabla} f(\boldsymbol{a}) = \lambda_1 \boldsymbol{\nabla} g_1(\boldsymbol{a}) + \lambda_2 \boldsymbol{\nabla} g_2(\boldsymbol{a}).$$

Example 5: Find the extrema of f given by

$$f(x, y, z) = x + y + z$$

subject to the conditions

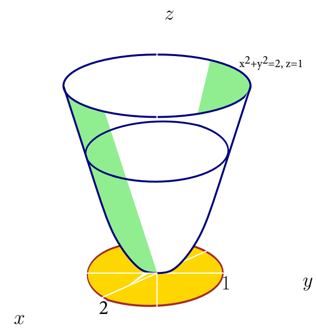
$$x^2 + y^2 = 2 \text{ and } x + z = 1.$$

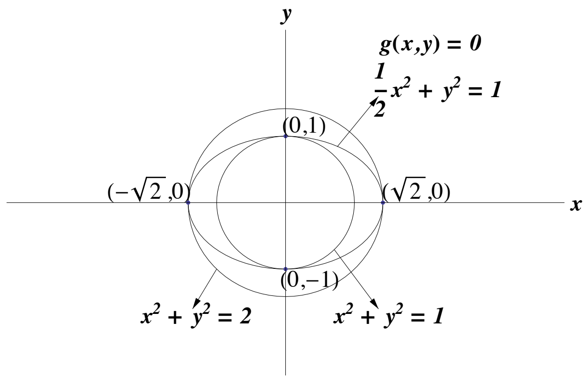
Example 6: Find the absolute maximum and minimum of f, given by

$$f(x,y) = \frac{1}{2}(x^2 + y^2)$$

in the region $\frac{1}{2}x^2 + y^2 \leq 1$.

Geometrical Interpretation





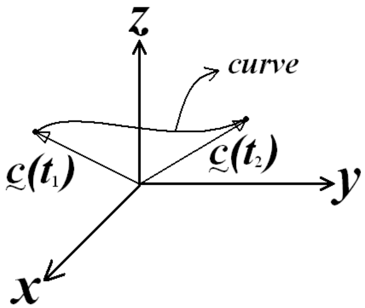
2 Space Curves and Vector Fields

2.1 Parametric Paths

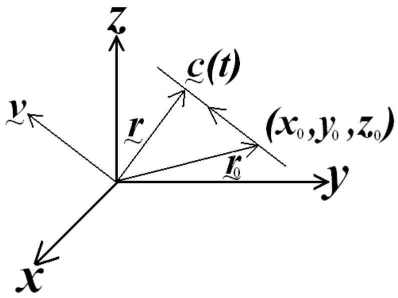
The position at time t of a particle moving in space is given by the path $\boldsymbol{c}$ where

$$\boldsymbol{c}(t) = (x(t), y(t), z(t))\,.$$

$\boldsymbol{c}$ parametrises the curve C, which is traced out by $\boldsymbol{c}(t)$ as t varies.



Example 1: Determine the parametrisation of a line through (x_0, y_0, z_0) in the direction of a vector $\boldsymbol{v}$.



Path Properties

Let $\boldsymbol{c}$ be a differentiable path in $\mathbb{R}^3$.

1. velocity

$$\boldsymbol{v}(t) = \frac{d\boldsymbol{c}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

speed $= |\boldsymbol{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

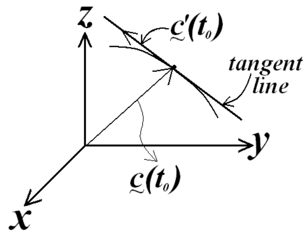
direction of $\boldsymbol{v}(t)$ is tangent to path.

2. acceleration

$$\boldsymbol{a}(t) = \frac{d^2\boldsymbol{c}}{dt^2} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right)$$

3. tangent line to $\boldsymbol{c}$ at $t = t_0$

$$\boldsymbol{\ell}(t) = \boldsymbol{c}(t_0) + (t - t_0)\boldsymbol{c}'(t_0)$$



Example 2: Let $\boldsymbol{c}(t) = (3t + 2, t^2 - 7, t - t^2)$.
Determine the velocity, speed, acceleration and equation of the tangent line at $t = 1$.

Differentiation Rules

Let $\boldsymbol{b}$, $\boldsymbol{c}$ be differentiable paths in $\mathbb{R}^3$. Then

1. $\dfrac{d}{dt}[\boldsymbol{b}+\boldsymbol{c}] \;=\; \dfrac{d\boldsymbol{b}}{dt}+\dfrac{d\boldsymbol{c}}{dt}$
2. $\dfrac{d}{dt}[\boldsymbol{b}\cdot\boldsymbol{c}] \;=\; \dfrac{d\boldsymbol{b}}{dt}\cdot\boldsymbol{c}+\boldsymbol{b}\cdot\dfrac{d\boldsymbol{c}}{dt}$
3. $\dfrac{d}{dt}[\boldsymbol{b}\times\boldsymbol{c}] \;=\; \dfrac{d\boldsymbol{b}}{dt}\times\boldsymbol{c}+\boldsymbol{b}\times\dfrac{d\boldsymbol{c}}{dt}$

Example 3: Prove differentiation rule 2.

Example 4: If $\boldsymbol{u}(t)$ is differentiable at least three times, evaluate and simplify

$$\frac{d}{dt}\left[(\boldsymbol{u}'' \times \boldsymbol{u}') \cdot \boldsymbol{u}\right].$$

Arclength

The length s (or arclength) of a path $\boldsymbol{c}$, given by

$$\boldsymbol{c}(t) = (x(t), y(t), z(t))$$

for $a \leq t \leq b$, is

$$\begin{aligned} s &= \int_a^b \left|\frac{d\boldsymbol{c}}{dt}\right| \, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt \end{aligned}$$

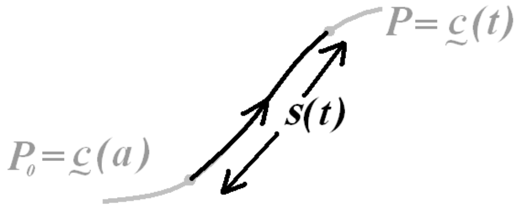
Example 5: Find the arclength of $\boldsymbol{c}$ where $\boldsymbol{c}(t) = (2t, t^2, \log(t))$ for $1 \le t \le 2$.

Arclength Parameter

We can parametrise a path in terms of the arclength s by defining

$$\boxed{s(t) \;=\; \int_a^t \left|\frac{d\boldsymbol{c}}{d\tau}\right| \, d\tau}$$

to be the length from a point P_0 to any point P on the path (where τ is a dummy variable).



Tangent Vector to Path

Now

$$\frac{d\boldsymbol{c}}{ds} = \frac{d\boldsymbol{c}}{dt}\,\frac{dt}{ds} = \frac{\dfrac{d\boldsymbol{c}}{dt}}{\dfrac{ds}{dt}}$$

Since

$$\boxed{\frac{ds}{dt} = \left|\frac{d\boldsymbol{c}}{dt}\right| = \text{speed}}$$

$$\Rightarrow \boxed{\frac{d\boldsymbol{c}}{ds} = \frac{\dfrac{d\boldsymbol{c}}{dt}}{\left|\dfrac{d\boldsymbol{c}}{dt}\right|} = \boldsymbol{T}(t)}$$

where $\boldsymbol{T}(t)$ is the unit tangent vector to the path $\boldsymbol{c}$ at the point $\boldsymbol{c}(t)$.

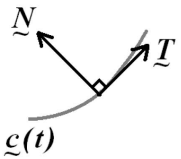
Normal Vectors to Path

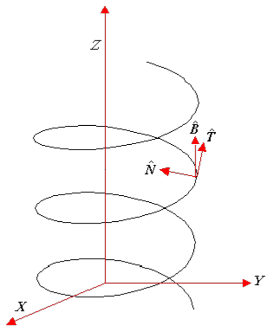
The principal normal vector

$$\boxed{\boldsymbol{N}(t) = \frac{\dfrac{d\boldsymbol{T}}{dt}}{\left|\dfrac{d\boldsymbol{T}}{dt}\right|}}$$

is a unit normal vector to the path $\boldsymbol{c}$ at the point $\boldsymbol{c}(t)$.

Proof





A third unit vector perpendicular to both $\boldsymbol{T}$ and $\boldsymbol{N}$ is the binormal vector $\boldsymbol{B}$.

$$\boxed{\boldsymbol{B}(t) \;=\; \boldsymbol{T}(t) \times \boldsymbol{N}(t)}$$

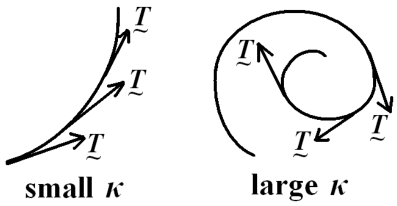
If $\boldsymbol{c}$ lies in page, then $\boldsymbol{B}$ points out of the page.

The vectors $\boldsymbol{T}, \boldsymbol{N}$ and $\boldsymbol{B}$ form a right hand set of axes moving along the curve $\boldsymbol{c}$ with time.

Curvature of Path

The curvature $\kappa(t)$ at a point $\boldsymbol{c}(t)$ on a path $\boldsymbol{c}$ is the angular rate of change of the direction of $\boldsymbol{T}$ per unit change in distance along the path.

$$\boxed{\kappa(t) \;=\; \left|\frac{d\boldsymbol{T}}{ds}\right| \;=\; \frac{\left|\dfrac{d\boldsymbol{T}}{dt}\right|}{\left|\dfrac{ds}{dt}\right|}}$$



Torsion of Path

The torsion $\tau(t)$ measures how fast the path $\boldsymbol{c}$ is twisting out of the plane of $\boldsymbol{T}$ and $\boldsymbol{N}$ at the point $\boldsymbol{c}(t)$:

$$\boxed{\frac{d\boldsymbol{B}}{ds} = -\tau(t)\boldsymbol{N}(t)}$$

If $\boldsymbol{c}$ lies in a plane then $\tau(t) = 0$.

Example 6: Find $\boldsymbol{T}(t), \boldsymbol{N}(t), \boldsymbol{B}(t), \kappa(t), \tau(t)$ for the path $\boldsymbol{c}(t) = (5\cos(3t),\ 6t,\ 5\sin(3t))$.

2.2 Vector Fields

A vector field is a function $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$.

$\boxed{n = 2}$

$$\boldsymbol{F}(x, y) = u(x, y)\,\boldsymbol{i} + v(x, y)\,\boldsymbol{j}$$

- 2 independent variables x, y
- $\boldsymbol{F}(x, y)$ is a vector (u, v) where each component is a scalar function of x, y.

$\boxed{n = 3}$

$$\boldsymbol{F}(x, y, z) = u(x, y, z)\,\boldsymbol{i} + v(x, y, z)\,\boldsymbol{j} + w(x, y, z)\,\boldsymbol{k}$$

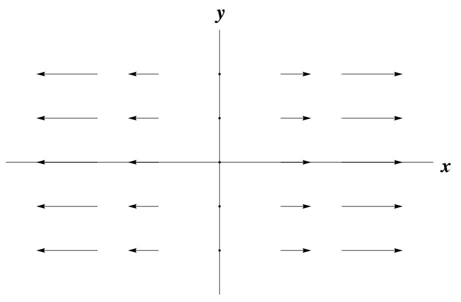
- 3 independent variables x, y, z
- $\boldsymbol{F}(x, y, z)$ is a vector (u, v, w) where each component is a scalar function of x, y, z.

The vector field can be sketched by assigning to each point $\boldsymbol{x}$ a vector $\boldsymbol{F}(\boldsymbol{x})$ represented by an arrow whose tail is at $\boldsymbol{x}$.

The vector field $\boldsymbol{F}(\boldsymbol{x})$ represents a physical vector quantity such as force or velocity at position $\boldsymbol{x}$.

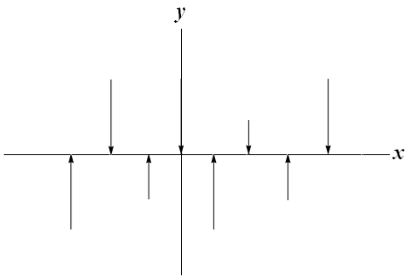
Example 1: Sketch the vector field

$$\boldsymbol{F}(x, y) = x\,\boldsymbol{i} = (x, 0).$$



Example 2: Sketch the vector field

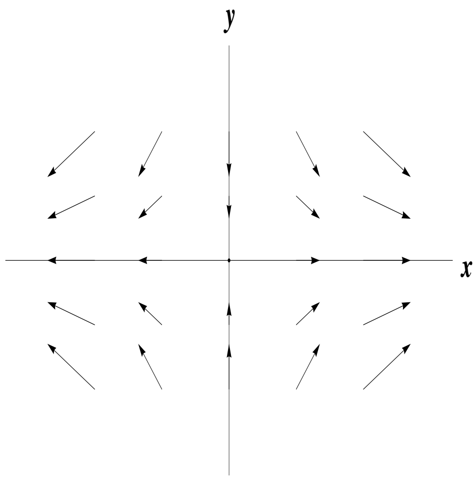
$$\boldsymbol{F}(x,y) = -y\,\boldsymbol{j} = (0,-y).$$



Example 3: Sketch the vector field

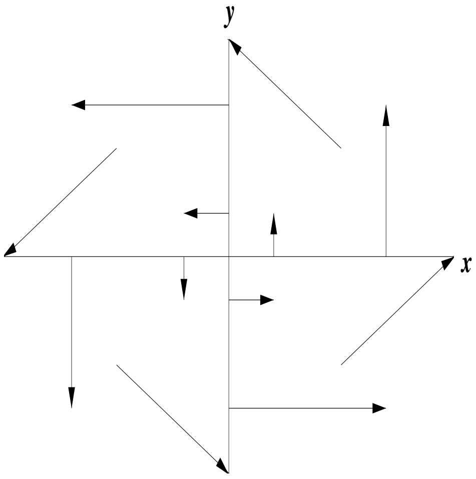
$$\boldsymbol{F}(x,y) = x\,\boldsymbol{i} - y\,\boldsymbol{j} = (x,-y).$$

(length of vectors not drawn to scale)



Example 4: Sketch the vector field

$$\boldsymbol{F}(x,y) = -y\,\boldsymbol{i} + x\,\boldsymbol{j} = (-y, x).$$

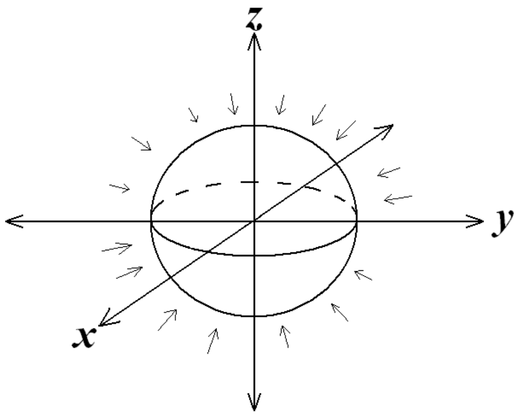


Example 5: Sketch the vector field

$$\boldsymbol{F}(\boldsymbol{r}) = -\frac{GMm}{r^3}\,\boldsymbol{r}$$

where $\boldsymbol{r}(x,y,z) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}\,,\ r = |\boldsymbol{r}|.$

Gravitational force of attraction of the earth (mass M) on a particle (mass m) at position (x,y,z).



(arrows not to scale)

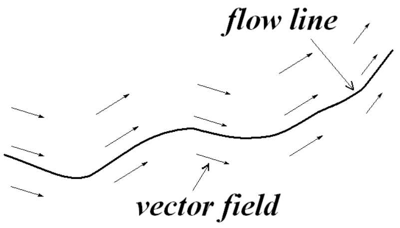
Flow Lines

A path $\boldsymbol{c}$ is a flow line (or streamline) of a vector field $\boldsymbol{F}$ if

$$\boldsymbol{c}'(t) = \boldsymbol{F}[\boldsymbol{c}(t)].$$

Physical Interpretation

Geometrically, a flow line is a curve whose tangent vector coincides with the vector field



Example

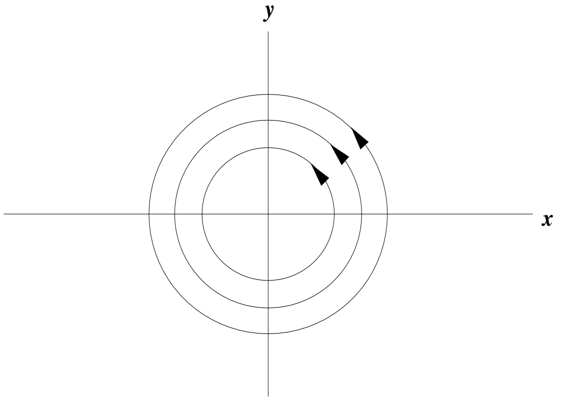
Flow of water through a pipe. Flow line represents the path followed by a particle suspended in the fluid.

Example 6: Show that the path $\boldsymbol{c}$ given by

$$\boldsymbol{c}(t) = (\sin t,\ \cos t,\ 2t)$$

is a flow line of $\boldsymbol{F}(x, y, z) = (y, -x, 2)$.

Example 7: Determine the equation for the flow lines. Hence sketch the flow lines of $\boldsymbol{F}(x, y) = (-y, x)$.



2.3 Differentiation Operators

Grad operator (∇)

In $\mathbb{R}^3$: $\boldsymbol{\nabla} = \boldsymbol{i}\,\frac{\partial}{\partial x} + \boldsymbol{j}\,\frac{\partial}{\partial y} + \boldsymbol{k}\,\frac{\partial}{\partial z}$

- If f is a C^1 scalar function, $\boldsymbol{\nabla} f$ is a vector field in $\mathbb{R}^3$ given by

$$\boldsymbol{\nabla} f = \boldsymbol{i}\,\frac{\partial f}{\partial x} + \boldsymbol{j}\,\frac{\partial f}{\partial y} + \boldsymbol{k}\,\frac{\partial f}{\partial z}.$$

- If $\boldsymbol{F}$ is a C^1 vector field in $\mathbb{R}^3$, $\underline{\underline{\nabla}}\boldsymbol{F}$ is a tensor with 9 components.

  (tensors not covered in this subject)

Divergence of a Vector Field

If $\boldsymbol{F} = F_1\,\boldsymbol{i} + F_2\,\boldsymbol{j} + F_3\,\boldsymbol{k}$ is a C^1 vector field, the divergence of $\boldsymbol{F}$ is a scalar function given by

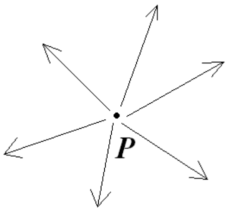
$$\begin{aligned} \operatorname{div} \boldsymbol{F} &= \boldsymbol{\nabla} \cdot \boldsymbol{F} \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_1, F_2, F_3) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \end{aligned}$$

Example 1: If $\boldsymbol{F}(x,y,z) = x^2 y\,\boldsymbol{i} + z\,\boldsymbol{j} + xyz\,\boldsymbol{k}$, find $\boldsymbol{\nabla} \cdot \boldsymbol{F}$.

Physical Interpretation

If $\boldsymbol{F}(x, y, z)$ is the fluid velocity at position (x, y, z), then $\boldsymbol{\nabla} \cdot \boldsymbol{F}$ measures the net transport of fluid in/out of that point.

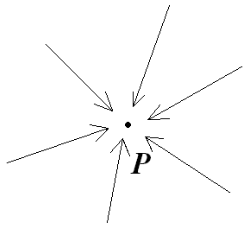
Case 1: $\boldsymbol{\nabla} \cdot \boldsymbol{F} > 0$
More fluid flows out than in, so fluid is expanding.



source at P

Example 2: If $\boldsymbol{F}(x, y) = x\,\boldsymbol{i}$, find $\boldsymbol{\nabla} \cdot \boldsymbol{F}$.

Case 2: $\boldsymbol{\nabla} \cdot \boldsymbol{F} < 0$
More fluid flows in than out, so fluid is compressing.

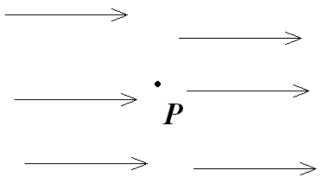


sink at P

Example 3: If $\boldsymbol{F}(x, y) = -y\,\boldsymbol{j}$, find $\boldsymbol{\nabla} \cdot \boldsymbol{F}$.

Case 3: $\boldsymbol{\nabla} \cdot \boldsymbol{F} = 0$

Rate at which fluid flows in equals rate at which fluid flows out.



If $\boldsymbol{\nabla} \cdot \boldsymbol{F} = 0$, then $\boldsymbol{F}$ is an incompressible vector field.

Example 4: If $\boldsymbol{F}(x,y) = x\,\boldsymbol{i} - y\,\boldsymbol{j}$, find $\boldsymbol{\nabla} \cdot \boldsymbol{F}$.

Curl of a Vector Field

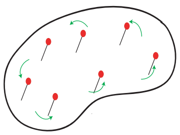
If $\boldsymbol{F} = F_1\,\boldsymbol{i} + F_2\,\boldsymbol{j} + F_3\,\boldsymbol{k}$ is a C^1 vector field, the curl of $\boldsymbol{F}$ is a vector field in $\mathbb{R}^3$ given by

$$\begin{aligned}
\text{curl}\,\boldsymbol{F} &= \boldsymbol{\nabla} \times \boldsymbol{F} \\
&= \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix} \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\boldsymbol{j} \\
&\qquad + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}\,.
\end{aligned}$$

Example 5: If $\boldsymbol{F}(x, y, z) = x^2 y\, \boldsymbol{i} - 2xz\, \boldsymbol{j} + (x + y - z)\, \boldsymbol{k}$, find $\boldsymbol{\nabla} \times \boldsymbol{F}$.

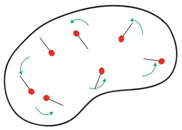
Physical Interpretation

If $\boldsymbol{F}$ is the fluid velocity in a lake. Drop a twig into the lake. Then $\boldsymbol{\nabla} \times \boldsymbol{F}$ measures how quickly and in what orientation the twig rotates as it moves.



twig does not rotate as it travels

$\Rightarrow \quad \boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0}$



Direction of twig changes as it travels

$\Rightarrow \quad \boldsymbol{\nabla} \times \boldsymbol{F} \neq \mathbf{0}$

If $\boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0}$, then $\boldsymbol{F}$ is an irrotational vector field.

Example 6: Is $\boldsymbol{F}$, given by $\boldsymbol{F}(x,y,z) = x\,\boldsymbol{i} - y\,\boldsymbol{j}$, irrotational?

Example 7: Is $\boldsymbol{F}$, given by $\boldsymbol{F}(x,y,z) = -y\,\boldsymbol{i} + x\,\boldsymbol{j}$, irrotational?

Note

If f is a scalar function, $\boldsymbol{\nabla} \cdot f$ and $\boldsymbol{\nabla} \times f$ are NOT defined.

Laplacian Operator - ∇^2

In $\mathbb{R}^3$, the Laplacian operator is given by

$$\begin{aligned}
\nabla^2 &= \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \\
&= \left(\boldsymbol{i}\,\frac{\partial}{\partial x} + \boldsymbol{j}\,\frac{\partial}{\partial y} + \boldsymbol{k}\,\frac{\partial}{\partial z} \right) \cdot \left(\boldsymbol{i}\,\frac{\partial}{\partial x} + \boldsymbol{j}\,\frac{\partial}{\partial y} + \boldsymbol{k}\,\frac{\partial}{\partial z} \right) \\
&= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}$$

- If f is a C^2 scalar function, then $\nabla^2 f$ is a scalar function given by

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

- If $\boldsymbol{F} = u\,\boldsymbol{i} + v\,\boldsymbol{j} + w\,\boldsymbol{k}$ is a C^2 vector field where u, v, and w are scalar functions of (x, y, z), then $\nabla^2 \boldsymbol{F}$ is a vector field in $\mathbb{R}^3$ given by

$$\nabla^2 \boldsymbol{F} \;=\; \nabla^2 u\,\boldsymbol{i} + \nabla^2 v\,\boldsymbol{j} + \nabla^2 w\,\boldsymbol{k}$$

Applications of ∇^2

Example 1

The gravitational potential V of a mass m at (x, y, z) due to a mass M at $(0, 0, 0)$ satisfies Laplace's equation

$$\nabla^2 V \;=\; 0.$$

Example 2

The temperature T of a body at (x, y, z) at time t satisfies the heat equation

$$\kappa \nabla^2 T \;=\; \frac{\partial T}{\partial t},$$

where κ is a positive constant.

Example 8: If $f(x, y, z) = x^2 y + x y^2 + y z^2$, find $\nabla^2 f$.

Example 9: If $\boldsymbol{F}(x,y,z) = x^2y\,\boldsymbol{i} + xy^2z\,\boldsymbol{j} + y^3z^2\,\boldsymbol{k}$, find $\nabla^2\boldsymbol{F}$.

2.4 Basic Identities of Vector Calculus

Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be scalar functions. Let $\boldsymbol{F}$, $\boldsymbol{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be vector fields. Then

1. $\boldsymbol{\nabla}(f+g) = \boldsymbol{\nabla} f + \boldsymbol{\nabla} g$
2. $\boldsymbol{\nabla}(\beta f) = \beta \boldsymbol{\nabla} f$ $\qquad$ (β constant)
3. $\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f$
4. $\boldsymbol{\nabla}\left(\dfrac{f}{g}\right) = \dfrac{g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g}{g^2}$ $\qquad$ provided $g \neq 0$
5. $\boldsymbol{\nabla}\cdot(\boldsymbol{F}+\boldsymbol{G}) = \boldsymbol{\nabla}\cdot\boldsymbol{F} + \boldsymbol{\nabla}\cdot\boldsymbol{G}$
6. $\boldsymbol{\nabla}\times(\boldsymbol{F}+\boldsymbol{G}) = \boldsymbol{\nabla}\times\boldsymbol{F} + \boldsymbol{\nabla}\times\boldsymbol{G}$
7. $\boldsymbol{\nabla}\cdot(f\boldsymbol{F}) = f\boldsymbol{\nabla}\cdot\boldsymbol{F} + \boldsymbol{F}\cdot\boldsymbol{\nabla} f$
8. $\boldsymbol{\nabla}\cdot(\boldsymbol{F}\times\boldsymbol{G}) = \boldsymbol{G}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F}) - \boldsymbol{F}\cdot(\boldsymbol{\nabla}\times\boldsymbol{G})$
9. $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{F}) = 0$
10. $\boldsymbol{\nabla}\times(f\boldsymbol{F}) = f\boldsymbol{\nabla}\times\boldsymbol{F} + \boldsymbol{\nabla} f\times\boldsymbol{F}$

11. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = \mathbf{0}$

12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g$

13. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g) = 0$

14. $\boldsymbol{\nabla} \cdot (f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f) = f\nabla^2 g - g\nabla^2 f$

15. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F}$

Note

Identities $1 - 15$ require f, g, $\boldsymbol{F}$, $\boldsymbol{G}$ to be suitably differentiable, either C^1 or C^2.

Example 1: Prove identity 11. If f is a C^2 scalar function then

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) \;=\; \mathbf{0}.$$

Proof

Application of identity 11

Let $\boldsymbol{V}$ be a C^1 vector field.

If $\boldsymbol{\nabla} \times \boldsymbol{V} = \mathbf{0}$, then $\boldsymbol{V}$ can be represented by the gradient of a scalar function ϕ, so

$$\boldsymbol{V} \;=\; \boldsymbol{\nabla}\phi.$$

Note

1. ϕ is unique up to an unknown constant C.

2. We say that $\boldsymbol{V}$ is a gradient field and ϕ is the scalar potential.

Example 2: Is $\boldsymbol{V}$,
given by $\boldsymbol{V}(x,y,z) = (2xy^2, 2x^2y + 2z^2y, 2y^2z)$,
a gradient field?

So $\boldsymbol{V}$ is a gradient field and we can write
$\boldsymbol{V} = \boldsymbol{\nabla}\phi$ for some scalar function ϕ.

How do we find $\phi(x,y,z)$?

Example 3: Prove identity 9. If $\boldsymbol{F}$ is a C^2 vector field, then

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) \;=\; 0$$

<u>Proof</u>

Application of identity 9

Let $\boldsymbol{V}$ be a C^1 vector field. If $\boldsymbol{\nabla} \cdot \boldsymbol{V} = 0$, then $\boldsymbol{V}$ can be written as the curl of a vector field $\boldsymbol{F}$, so

$$\boldsymbol{V} = \boldsymbol{\nabla} \times \boldsymbol{F}.$$

Note

1. $\boldsymbol{F}$ is not unique.

2. We say that $\boldsymbol{F}$ is the vector potential.

Example 4: Express the vector field $\boldsymbol{V}(x, y, z) = (x^2 + 1, z - 2xy, y)$, as a curl of a vector field $\boldsymbol{F}$.

Example 5: Find $\boldsymbol{\nabla} \cdot \left(\dfrac{\boldsymbol{r}}{r^2}\right)$ where $\boldsymbol{r}(x, y, z) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}\,,\ r = \sqrt{x^2 + y^2 + z^2}$.

Example 6: Find $\nabla^2(r^2 \log(r))$ where $\boldsymbol{r}(x, y, z) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$, $r = \sqrt{x^2 + y^2 + z^2}$.

Example 7: Let f and g be C^2 scalar functions. Using the vector identities, show that:

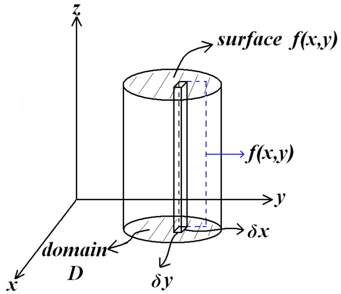
$$\nabla \cdot (f\nabla g - g\nabla f) \;=\; f\nabla^2 g - g\nabla^2 f.$$

3 Double and Triple Integrals

3.1 Double Integrals

If f is a continuous function over a domain D in $\mathbb{R}^2$, we can evaluate the double integral

$$\iint_D f(x,y)\,dA \;=\; \iint_D f(x,y)\,dx\,dy.$$



$$\begin{array}{c}\text{Volume}\\ \text{thin rod}\end{array} = (\text{Area base})\,(\text{height})$$
$$= \delta A\, f(x,y)$$
$$= \delta x\,\delta y\, f(x,y)$$

$$\iint_D f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i,y_i)\delta A_i$$
$$= \iint_D f(x,y)\,dx\,dy$$

Physical Interpretation

The double integral $\iint_D f(x,y)\,dA$ is the volume under the surface $z = f(x,y)$ that lies above the domain D in the xy-plane.

Note

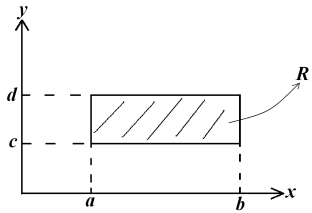
If $f(x,y) = 1$ then

$$\iint_D dA = \iint_D dx\,dy$$

is the total area of domain D.

Double Integrals over Rectangular Domains

1. $R = [a,b] \times [c,d]$ is a rectangular domain defined by $a \le x \le b,\ c \le y \le d$.



2. $$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

   means integrate with respect to x first and then integrate with respect to y.

Fubini's Theorem

Let f be a continuous function over the domain $R = [a,b] \times [c,d]$. Then

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

So the order of integration is NOT important.

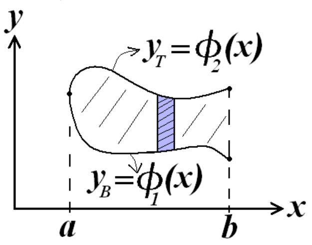
Example 1: Evaluate $\displaystyle\iint_R (x^2+y^2)\,dx\,dy$ if $R = [-1,1] \times [0,1]$.

Double Integrals over General Domains

The domain determines the terminals and order of integration.

Vertical Strips

$$y_B \le y \le y_T\,, \quad a \le x \le b$$

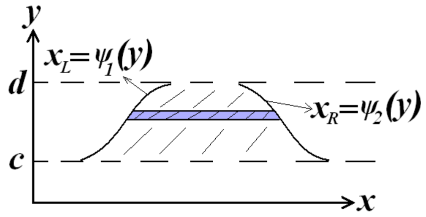


Integrate with respect to y first

$$\iint\limits_D f(x,y)dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$

Horizontal Strips

$$x_L \le x \le x_R\,, \quad c \le y \le d$$



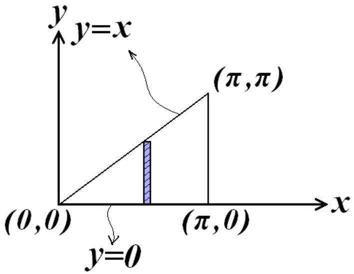
Integrate with respect to x first

$$\iint\limits_D f(x,y)dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$$

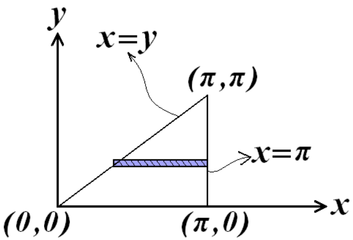
Example 2: Evaluate $\displaystyle\iint_D (x^3 y + \cos x)\, dA$

where D is the triangle with vertices at $(0,0)$, $(\pi, 0)$ and (π, π).

Use vertical strips



Use horizontal strips

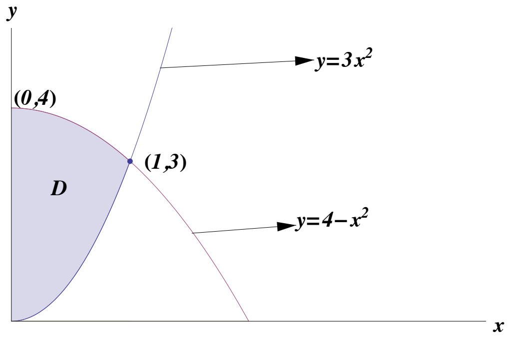


Note

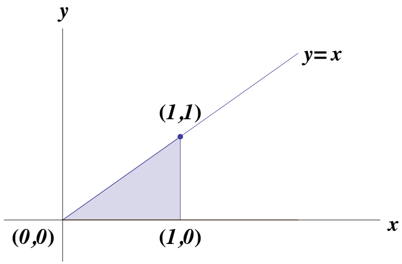
The order of integration is not important if

1. the domain can be divided into horizontal and vertical strips AND
2. f is continuous in the domain.

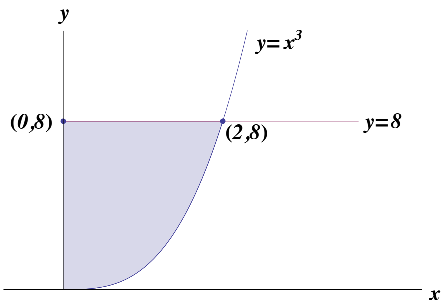
Example 3: Find the area enclosed by $y = 3x^2,\ y = 4 - x^2$ and the y-axis for $x \geq 0$.



Example 4: Evaluate $\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$.



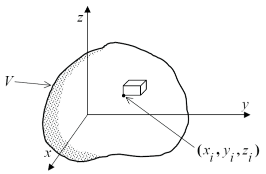
Example 5: Evaluate $\displaystyle\int_0^2 \int_{x^3}^8 \cos\left(y^{\frac{4}{3}}\right)\, dy\, dx.$

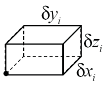


3.2 Triple Integrals

If f is a continuous function over a solid domain D in $\mathbb{R}^3$, we can evaluate the triple integral

$$\iiint_D f(x,y,z)\,dV \;=\; \iiint_D f(x,y,z)\,dx\,dy\,dz$$





Let $\delta V = \delta z \delta y \delta x$ be a small subregion of V. The triple integral of f over V is

$$\iiint_V f(x,y,z)\;dV = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i,y_i,z_i)\;\delta V_i$$
$$= \iiint_V f(x,y,z)\;dz\;dy\;dx$$

Physical Interpretation

The physical meaning of $\displaystyle\iiint_D f(x,y,z)\,dV$ depends on f.

Example 1

If $f(x,y,z)$ is the mass per unit volume of D, then

$$\iiint_D f(x,y,z)\,dV$$

is the total mass of D.

Example 2

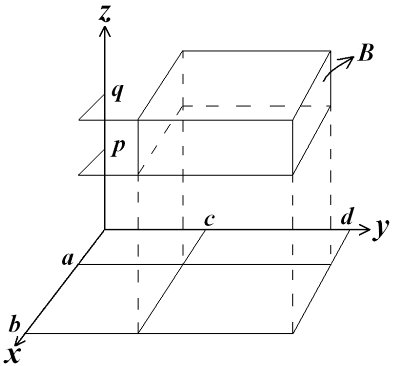
If $f(x,y,z) = 1$ then

$$\iiint_D dV \;=\; \iiint_D dx\,dy\,dz$$

is the volume of D.

Triple Integrals over Rectangular Box Domains

The rectangular box domain B



can be specified as

$$B = [a, b] \times [c, d] \times [p, q]$$

or $a \le x \le b, \quad c \le y \le d, \quad p \le z \le q.$

Fubini's Theorem

Let f be a continuous function over $B = [a, b] \times [c, d] \times [p, q]$, then

$$\begin{aligned}\iiint_B f(x,y,z)\,dV &= \int_p^q \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz \\ &= \int_p^q \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz \\ &= \int_a^b \int_c^d \int_p^q f(x,y,z)\,dz\,dy\,dx\end{aligned}$$

and so on (6 possible combinations)

Example 1: Consider the solid box

$$B = [0, 1] \times \left[-\frac{1}{2}, 0\right] \times \left[0, \frac{1}{3}\right]$$

with mass per unit volume

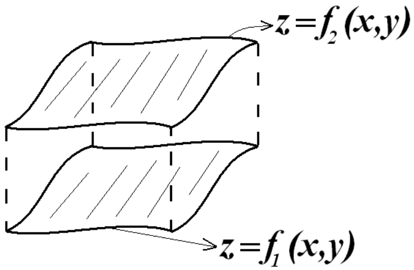
$$\mu(x, y, z) = x + 2y + 3z \quad \text{g/cm}^3.$$

Determine the total mass of B.

Triple Integrals over General Domains

The domain D is an elementary region in $\mathbb{R}^3$ if one variable is bounded by functions of 2 variables, the domains of these functions being described using horizontal or vertical strips.

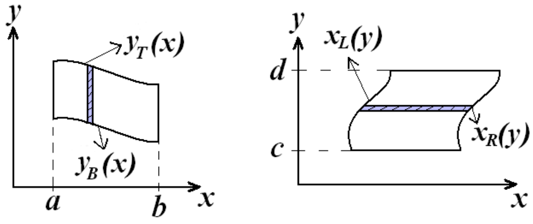
Example 2: If z is bounded by two functions of x and y.



Then

$$f_1(x,y) \leq z \leq f_2(x,y)$$

The projection (or shadow) of D onto the xy plane is



The domain D can be described as

$$f_1(x,y) \leq z \leq f_2(x,y)$$

and either

$y_B(x) \leq y \leq y_T(x), \quad a \leq x \leq b$ (vertical strips)

OR

$x_L(y) \leq x \leq x_R(y), \quad c \leq y \leq d$ (horizontal strips)

Example 3: If the region is oriented so the axis of symmetry is the x axis.

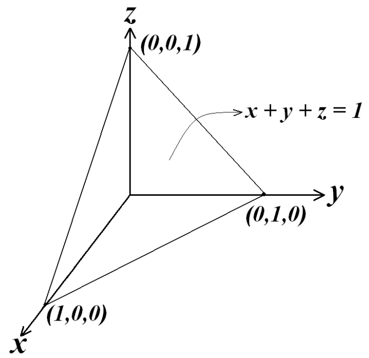
Let x be bounded by two functions of y and z and then project onto the yz plane.

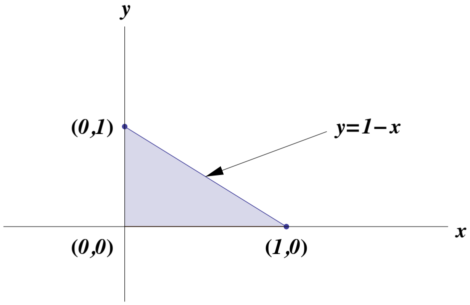
Example 4: If the region is oriented so the axis of symmetry is the y axis.

Let y be bounded by two functions of x and z and then project onto the xz plane.

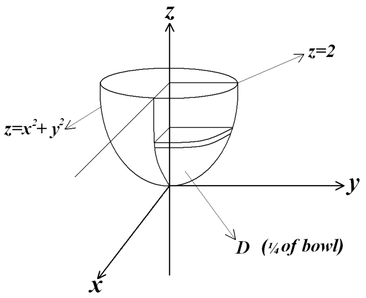
Example 5: Evaluate $\displaystyle\iiint\limits_D xy\,dV$

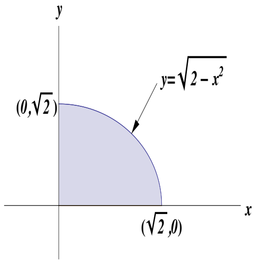
where D is the solid tetrahedron with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.




y
(0,1)
y=1−x
x
(0,0)
(1,0)

Example 6: Evaluate $\iiint_D x\,dV$ where D is the solid region bounded by $x=0, y=0$, $z=2$ and $z=x^2+y^2$. Also $x \geq 0, y \geq 0$.

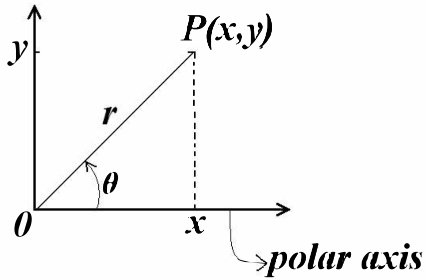




3.3 Orthogonal Curvilinear Coordinates

Polar Coordinates (r, θ)

We can specify a point in the xy plane by Cartesian (x, y) or polar (r, θ) coordinates.



- $r =$ length $\overrightarrow{OP} = \sqrt{x^2 + y^2}$

$$0 \leq r < \infty$$

- $x = r\cos\theta$ and $y = r\sin\theta$

- $\theta =$ angle measured anticlockwise from positive x axis (polar axis) to $\overrightarrow{OP}$

Note

1. The origin is given by $(0, \theta)$.
2. θ is not unique.

   eg. $(x, y) = (1, 1)$ has polar coordinates $\left(\sqrt{2}, \frac{\pi}{4}\right), \left(\sqrt{2}, \frac{9\pi}{4}\right), \left(\sqrt{2}, \frac{-7\pi}{4}\right)$ etc.

3. To find $\theta(x, y)$ we restrict θ to $0 \leq \theta < 2\pi$.

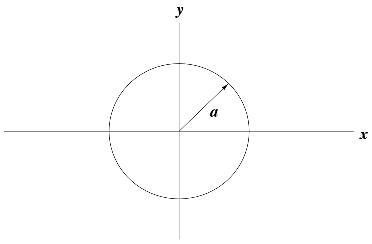
$$
\begin{aligned}
\theta &= \arctan\left(\frac{y}{x}\right) && x > 0,\ y \geq 0 \\
&= \pi + \arctan\left(\frac{y}{x}\right) && x < 0 \\
&= 2\pi + \arctan\left(\frac{y}{x}\right) && x > 0,\ y < 0 \\
&= \frac{\pi}{2} && x = 0,\ y > 0 \\
&= \frac{3\pi}{2} && x = 0,\ y < 0
\end{aligned}
$$

where $-\frac{\pi}{2} < \arctan\left(\frac{y}{x}\right) < \frac{\pi}{2}$

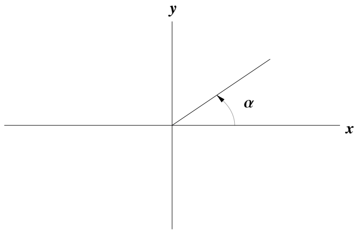
Example 1: Convert $(x, y) = (-1, -1)$ to polar coordinates.

Special Cases

- $\underline{r = a \text{ (constant)}}$

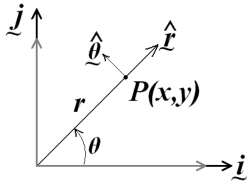


- $\underline{\theta = \alpha \text{ (constant)}}$

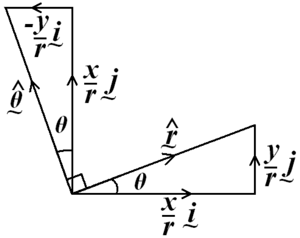


Unit Vectors $\widehat{r}$, $\widehat{\theta}$

We define unit vectors in the direction of increasing coordinate.

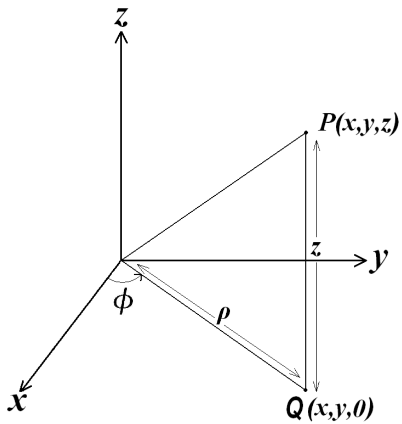


- Directions of $\boldsymbol{i}$, $\boldsymbol{j}$ are fixed.
- Directions of $\widehat{\boldsymbol{r}}, \widehat{\boldsymbol{\theta}}$ change with P.



We can specify a point in 3 dimensional space by Cartesian (x, y, z), cylindrical (ρ, ϕ, z) or spherical (r, θ, ϕ) coordinates.

Cylindrical Coordinates (ρ, ϕ, z)



- $\rho =$ length $\overrightarrow{OQ} = \sqrt{x^2 + y^2}$

$$0 \leq \rho < \infty$$

- $\phi =$ angle measured anticlockwise from positive x axis to $\overrightarrow{OQ}$

$$0 \leq \phi < 2\pi$$

$$\phi = \arctan\left(\frac{y}{x}\right) \qquad x > 0,\ y \geq 0$$

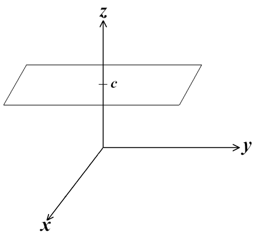
(other x, y values as for θ in polar coordinates)

- Also

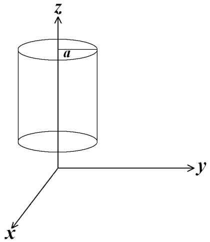
$$x = \rho \cos\phi, \ \ y = \rho \sin\phi, \ \ z = z$$

Special Cases

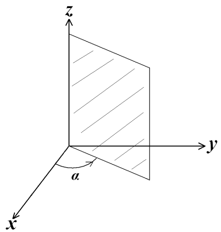
- $z = c$ (constant)



- $\rho = a$ (constant)



- $\phi = \alpha$ (constant)

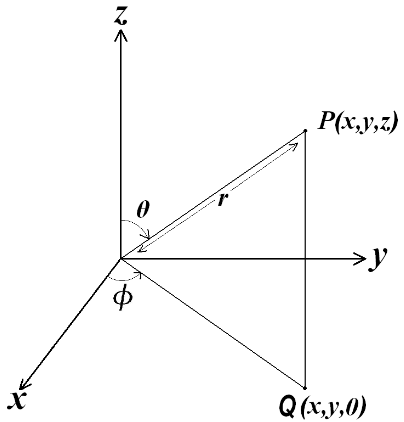


Unit vectors $\widehat{\boldsymbol{\rho}}$, $\widehat{\boldsymbol{\phi}}$, $\widehat{\boldsymbol{z}}$.

- Direction of $\widehat{\boldsymbol{z}}$ is fixed.
- Direction of $\widehat{\boldsymbol{\rho}}$, $\widehat{\boldsymbol{\phi}}$ change with P.

Then, as in polar coordinates,

Spherical Coordinates (r, θ, ϕ)



- r = length of $\overrightarrow{OP} = \sqrt{x^2 + y^2 + z^2}$

$$0 \le r < \infty$$

- ϕ (azimuthal angle) = angle measured anticlockwise from positive x axis to $\overrightarrow{OQ}$

$$0 \le \phi < 2\pi$$

$\phi = \text{arctan}\left(\frac{y}{x}\right) \quad x > 0,\ y \geq 0$
(other x, y values as for θ in polar coordinates)

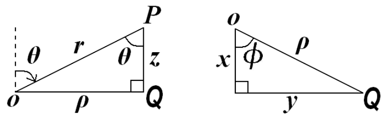
- θ (polar angle) = angle measured from positive z axis to $\overrightarrow{OP}$.

$$0 \leq \theta \leq \pi$$

$$\theta = \text{arccos}\left(\frac{z}{r}\right)$$

- Also

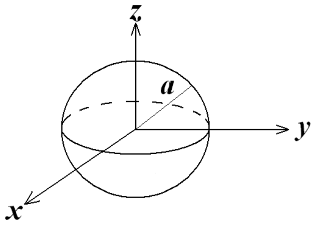
$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$



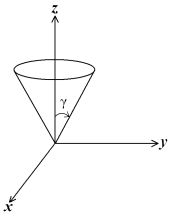
$\Rightarrow \rho = r \sin\theta \qquad \Rightarrow x = \rho \cos\phi$
$\qquad\qquad\qquad\qquad\quad y = \rho \sin\phi$

Special Cases

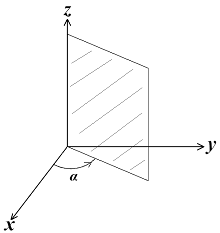
- $r = a$ (constant)



- $\underline{\theta = \gamma \text{ (constant)}}$



- $\underline{\phi = \alpha \text{ (constant)}}$



Unit vectors $\widehat{\boldsymbol{r}}$, $\widehat{\boldsymbol{\theta}}$, $\widehat{\boldsymbol{\phi}}$

- Directions of $\widehat{\boldsymbol{r}}$, $\widehat{\boldsymbol{\theta}}$, $\widehat{\boldsymbol{\phi}}$ change with P

Example 2: Convert $(x, y, z) = (1, 1, \sqrt{3})$ to cylindrical and spherical coordinates.

<u>cylindrical</u>

<u>spherical</u>

Change of Variable Theorem for Double Integrals

Let D, D^* be elementary regions in $\mathbb{R}^2$ and $T : D^* \to D$ be C^1. If T is one-to-one and $D = T(D^*)$

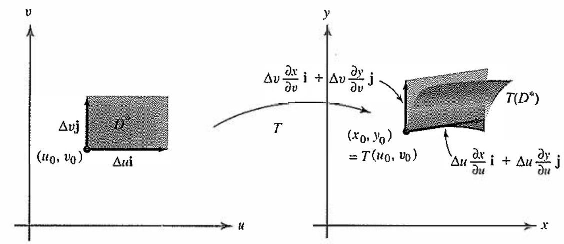
$$\iint_D f(x,y)\,dx\,dy$$

$$= \iint_{D^*} f\left[x(u,v), y(u,v)\right] \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

where the |Jacobian| is

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left| \text{det} \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} \right| .$$

Proof



A rectangle in the u-v plane with sides $\Delta u\,\boldsymbol{i}$ and $\Delta v\,\boldsymbol{j}$ maps to a region in the x-y plane that can be approximated by a parallelogram with sides

$$\Delta u \mathbf{T_u} \approx \Delta u \left(\frac{\partial x}{\partial u}\boldsymbol{i} + \frac{\partial y}{\partial u}\boldsymbol{j} \right)$$

and

$$\Delta v \mathbf{T_v} \approx \Delta v \left(\frac{\partial x}{\partial v}\boldsymbol{i} + \frac{\partial y}{\partial v}\boldsymbol{j} \right).$$

The area of the parallelogram is given by

$$|\Delta u \mathbf{T_u} \times \Delta v \mathbf{T_v}| = |\mathbf{T_u} \times \mathbf{T_v}| \, \Delta u \Delta v$$

$$= \left| \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{bmatrix} \right| \Delta u \Delta v$$

$$= \left| \det \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{bmatrix} \right| \Delta u \Delta v$$

$$= \left| \det \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} \right| \Delta u \Delta v$$

$$= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$$

Change of Variable Theorem for Triple Integrals

Let D, D^* be elementary regions in $\mathbb{R}^3$ and $T : D^* \to D$ be C^1. If T is one-to-one and $D = T(D^*)$

$$\iiint_D f(x,y,z) \, dx \, dy \, dz$$

$$= \iiint_{D^*} f\left[x(u,v,w), y(u,v,w), z(u,v,w)\right] \cdot \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du \, dv \, dw$$

where the |Jacobian| is

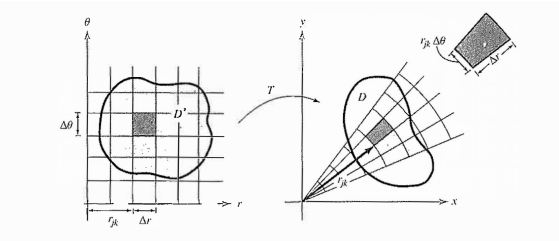
$$\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = \left| \det \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{bmatrix} \right|.$$

Polar Coordinates (r, θ)

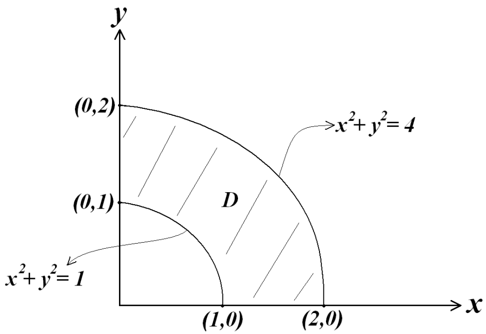
$$x = r\cos\theta, \ y = r\sin\theta \quad (r \geq 0)$$

$$\begin{aligned}
\text{Jacobian} &= \det \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} \\
&= \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\
&= r\cos^2\theta + r\sin^2\theta \\
&= r
\end{aligned}$$

$$\boxed{\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta}$$

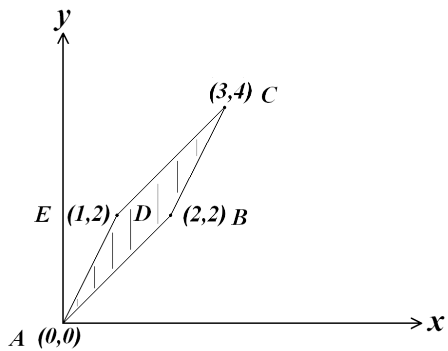


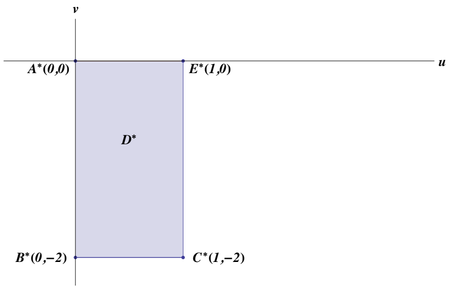
Example 3: Evaluate $\displaystyle\iint_D \log(x^2+y^2)\,dx\,dy$ where D is the region given below.



Example 4: Let D be the parallelogram with vertices $(0,0),(1,2),(2,2),(3,4)$.

Evaluate $\iint\limits_D xy\,dx\,dy$ by making the change of variables $x = u - v,\ y = 2u - v$.




v
u
A*(0,0)
E*(1,0)
D*
B*(0,−2)
C*(1,−2)

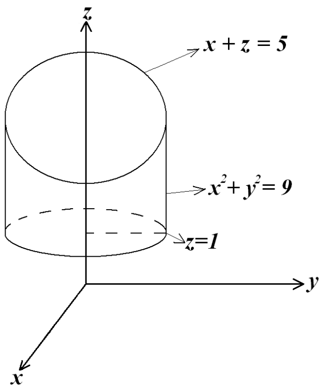
Cylindrical Coordinates (ρ, ϕ, z)

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z \qquad (\rho \geq 0)$$

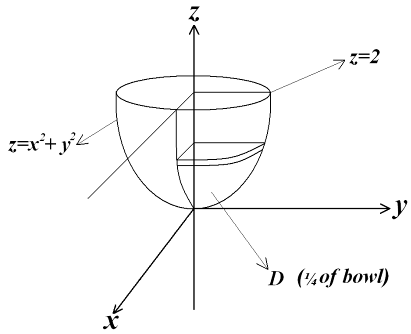
$$\begin{aligned}
\text{Jacobian} &= \det\begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{bmatrix} \\
&= \det\begin{bmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \det\begin{bmatrix} \cos\phi & -\rho\sin\phi \\ \sin\phi & \rho\cos\phi \end{bmatrix} \\
&= \rho\cos^2\phi + \rho\sin^2\phi \\
&= \rho
\end{aligned}$$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz$$
$$= \iiint_{D^*} f(\rho\cos\phi, \rho\sin\phi, z)\rho\,d\rho\,d\phi\,dz$$

Example 5: Find the volume of the solid region enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $z = 1$, $x + z = 5$.



Example 6: Redo Example 6 from triple integrals to find $\iiint\limits_D x\,dV$ for region below.



Spherical Coordinates (r, θ, ϕ)

$$x = r \sin\theta \cos\phi, \;\; y = r \sin\theta \sin\phi,$$

$$z = r \cos\theta \quad (r \geq 0)$$

$$\text{Jacobian} \;\; = \det \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \phi} \end{bmatrix}$$

$$= \det \begin{bmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix}$$

Expand along 3$^{\text{rd}}$ row

$$= \cos\theta \left[r^2 \cos\theta \sin\theta \cos^2\phi + r^2 \sin\theta \cos\theta \sin^2\phi \right]$$
$$+ r \sin\theta \left[r \sin^2\theta \cos^2\phi + r \sin^2\theta \sin^2\phi \right]$$

$$= r^2 \sin\theta \cos^2\theta + r^2 \sin^3\theta$$

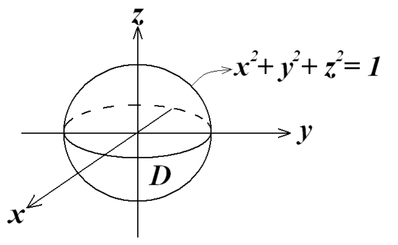
$$= r^2 \sin\theta \qquad (\sin\theta \geq 0)$$

$$\iiint_D f(x,y,z)\, dx\, dy\, dz$$
$$= \iiint_{D^*} F(r,\theta,\phi) r^2 \sin\theta\, dr\, d\theta\, d\phi$$

where
$F(r,\theta,\phi) = f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$

Example 7: Let D be the unit sphere centred at $(0,0,0)$. Determine the mass of D if the mass per unit volume of D is

$$\mu(x,y,z) = \exp\left(\left[x^2 + y^2 + z^2 \right]^{\frac{3}{2}} \right).$$



Note

If terminals are constant and the integrand factorises, ie.

$$g(x,y,z) = g_1(x)g_2(y)g_3(z),$$

then

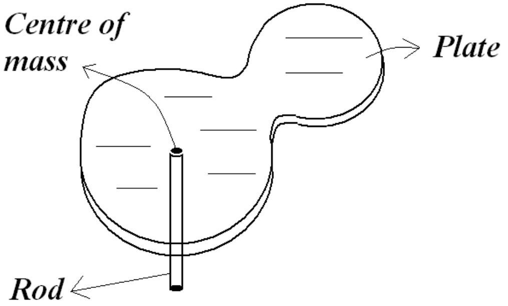
$$\int_a^b \int_c^d \int_e^f g(x,y,z)\,dx\,dy\,dz$$

$$= \left[\int_a^b g_3(z)\,dz\right] \left[\int_c^d g_2(y)\,dy\right] \left[\int_e^f g_1(x)\,dx\right].$$

3.4 Applications of Multiple Integrals

Centre of Mass

The centre of mass of a rigid body is that point at which the body can be supported so it will not experience any unbalanced torques that will cause it to rotate.



The plate balances when supported at its centre of mass.

- 2D plate with mass per unit area $\mu(x, y)$, the centre of mass is (x_{cm}, y_{cm}) where

$$x_{cm} = \frac{\displaystyle\iint_D x\,\mu(x,y)\,dx\,dy}{\text{mass}}$$

$$y_{cm} = \frac{\displaystyle\iint_D y\,\mu(x,y)\,dx\,dy}{\text{mass}}$$

where

$$\text{mass of plate} = \iint_D \mu(x,y)\,dx\,dy.$$

- 3D body with mass per unit volume $\mu(x, y, z)$, the centre of mass is (x_{cm}, y_{cm}, z_{cm}) where

$$x_{cm} = \frac{\displaystyle\iiint_D x\,\mu(x,y,z)\,dx\,dy\,dz}{\text{mass}}$$

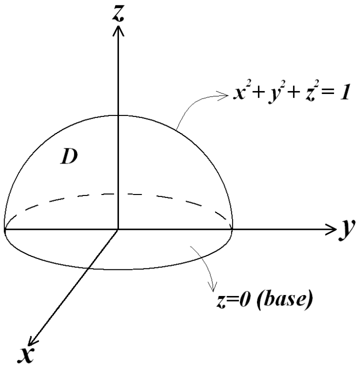
$$y_{cm} = \frac{\displaystyle\iiint_D y\,\mu(x,y,z)\,dx\,dy\,dz}{\text{mass}}$$

$$z_{cm} = \frac{\displaystyle\iiint_D z\,\mu(x,y,z)\,dx\,dy\,dz}{\text{mass}}$$

where

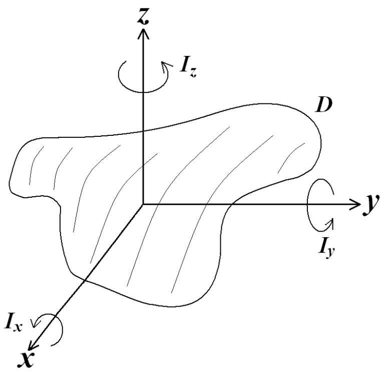
$$\text{mass of body} = \iiint_D \mu(x,y,z)\,dx\,dy\,dz.$$

Example 1: Find the centre of mass of a solid hemisphere radius 1 centred at $(0,0,0)$ for $z \geq 0$, if the mass per unit volume μ is constant.



Moments of Inertia

- I_n is the moment of inertia of a solid body about the n axis.
- I_n measures a body's response to spinning it about the n axis



As I_n increases, it becomes harder to spin the body about the n axis.

- If $\mu(x, y, z)$ is the mass per unit volume of a solid body, then

$$I_x = \iiint_D (y^2 + z^2)\mu(x, y, z)\, dx\, dy\, dz$$

$$I_y = \iiint_D (x^2 + z^2)\mu(x, y, z)\, dx\, dy\, dz$$

$$I_z = \iiint_D (x^2 + y^2)\mu(x, y, z)\, dx\, dy\, dz$$

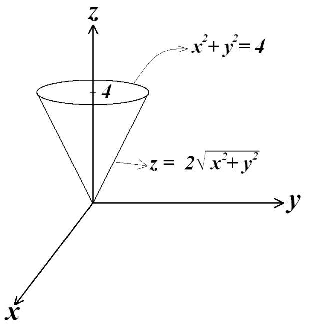
Example 2: If the density of material inside the solid cone

$$z = 2\sqrt{x^2 + y^2}, \;\; z \le 4$$

varies as

$$\mu(x, y, z) = 5 - z,$$

find the moment of inertia of the cone about the z-axis.



4 Integrals over Paths and Surfaces

4.1 Path Integrals

Let f be a continuous scalar function and $\boldsymbol{c}$ be a C^1 path where $\boldsymbol{c}(t) = (x(t), y(t), z(t))$.

The path integral of f along $\boldsymbol{c}$ from $t = a$ to $t = b$ $(a \le t \le b)$ is

$$\int_{\boldsymbol{c}} f \, ds = \int_{\boldsymbol{c}} f(x, y, z) \, ds$$

$$= \int_a^b f\,[x(t), y(t), z(t)] \, \frac{ds}{dt} \, dt$$

$$= \int_a^b f\,[\boldsymbol{c}(t)] \left|\boldsymbol{c}'(t)\right| \, dt.$$

Note

1. When $f = 1$, $\int_{\boldsymbol{c}} f \, ds$ gives the arclength of $\boldsymbol{c}$.

2. $\int_{\boldsymbol{c}} f \, ds$ is independent of how $\boldsymbol{c}$ is parametrised.

Physical Interpretation

Physical interpretation of $\int_{\boldsymbol{c}} f \, ds$ depends on what f represents.

Example 1

If f is the mass per unit length of a cable $\boldsymbol{c}$, $\int_{\boldsymbol{c}} f \, ds$ is the total mass of the cable.

Example 2

If f is the charge per unit length of a cable $\boldsymbol{c}$, $\int_{\boldsymbol{c}} f \, ds$ is the total charge of the cable.

Example 1: Let $\boldsymbol{c}$ be the helix given by

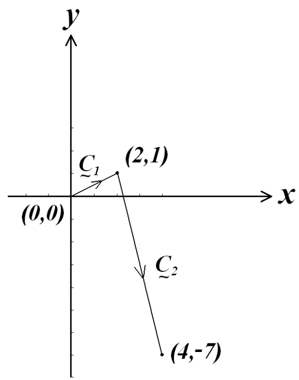
$$\boldsymbol{c}(t) = (\cos t, \sin t, t), \quad 0 \le t \le 2\pi$$

Evaluate $\displaystyle\int_{\boldsymbol{c}} (xy + z)\, ds$.

Example 2: Let $f(x, y) = y - x$ and

$$\boldsymbol{c}(t) = \begin{cases} (2t, t) & 0 \le t \le 1 \\ (t+1, 5-4t) & 1 \le t \le 3 \end{cases}$$

Evaluate $\displaystyle\int_{\boldsymbol{c}} f\, ds$.



Example 3: Find a parametrisation for the following curves with t increasing.

(a) Parabola $y = z^2,\ x = 2$ from

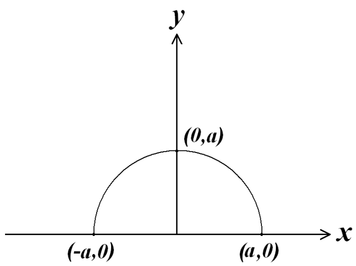
(i) $(2, 0, 0)$ to $(2, 9, 3)$

(ii) $(2, 9, 3)$ to $(2, 0, 0)$

(b) Line joining $(1, 0, 2)$ to $(4, 3, 1)$

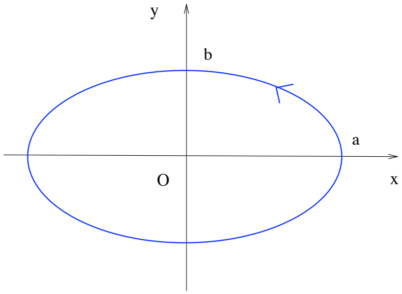
(c) Semicircle $y = \sqrt{a^2 - x^2}$ oriented

(i) anticlockwise (ii) clockwise



(d) Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(i) anticlockwise (ii) clockwise



4.2 Line Integrals

Let $\boldsymbol{F}$ be a continuous vector field and $\boldsymbol{c}$ be a C^1 path where $\boldsymbol{c}(t) = (x(t), y(t), z(t))$.

The line integral of $\boldsymbol{F}$ along $\boldsymbol{c}$ from $t = a$ to $t = b$ $(a \le t \le b)$ is

$$\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_a^b \boldsymbol{F}\left[x(t), y(t), z(t)\right] \cdot \frac{d\boldsymbol{s}}{dt}\, dt$$

$$= \int_a^b \boldsymbol{F}\left[\boldsymbol{c}(t)\right] \cdot \boldsymbol{c}'(t)\, dt.$$

- If $\boldsymbol{F} = u\,\boldsymbol{i} + v\,\boldsymbol{j} + w\,\boldsymbol{k}$ then

$$\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_a^b \left(u\frac{dx}{dt} + v\frac{dy}{dt} + w\frac{dz}{dt} \right) dt$$

$$= \int_{\boldsymbol{c}} \underbrace{u\, dx + v\, dy + w\, dz}_{\text{differential form}}$$

Note

1. $\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s}$ is independent of how $\boldsymbol{c}$ is parametrised.

2. If traverse path $\boldsymbol{c}$ in opposite direction then

$$\int_A^B \boldsymbol{F} \cdot d\boldsymbol{s} = -\int_B^A \boldsymbol{F} \cdot d\boldsymbol{s}$$

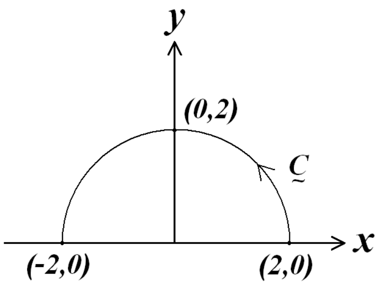
Physical Interpretation

Let $\boldsymbol{F}$ be a force field (eg electric field) that acts on a particle moving along the path $\boldsymbol{c}$.

Then $\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s}$ is the work done by $\boldsymbol{F}$ in moving the particle along $\boldsymbol{c}$.

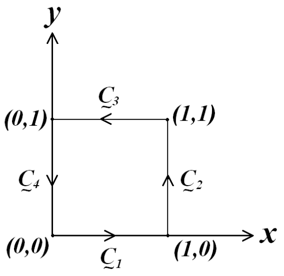
If $\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s} < 0$, then $\boldsymbol{F}$ impedes the movement of the particle along $\boldsymbol{c}$.

Example 1: Determine the work done by the force $\boldsymbol{F}(x, y) = (-y, 0)$ to move a particle around the semicircle $y = \sqrt{4 - x^2}$ from $(2, 0)$ to $(-2, 0)$.



Example 2: Let $\boldsymbol{c}$ be the perimeter of the unit square with vertices at $(0,0),(1,0),(0,1)$ and $(1,1)$, oriented anticlockwise. Evaluate

$$\int_{\boldsymbol{c}} x^2\,dx + xy\,dy$$



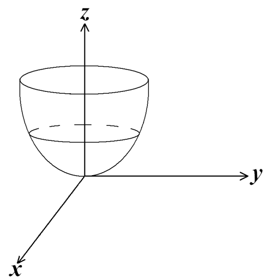
- Choose any parametrisation of $\boldsymbol{c}$

4.3 Parametrised Surfaces

Some surfaces have the form $z = f(x,y)$

Example

$z = x^2 + y^2$



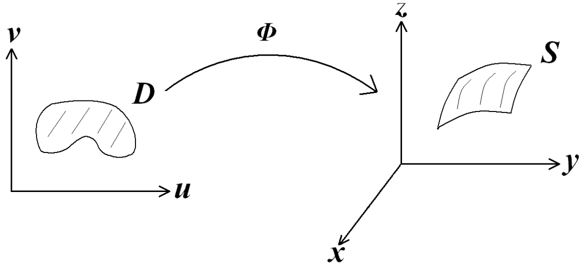
Some surfaces cannot be written in the form $z = f(x,y)$.

Examples

- $x - z + z^3 = 0$
- torus

A parametrised surface is a function Φ that maps a domain D in $\mathbb{R}^2$ to a surface S in $\mathbb{R}^3$, namely

$$\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$$



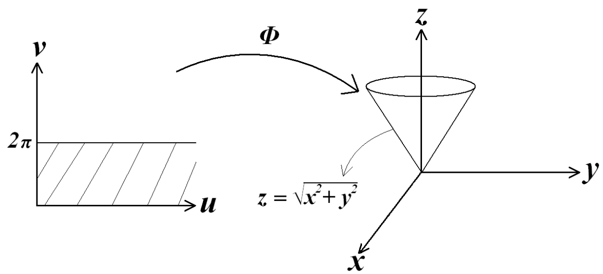
Note

If Φ is C^1, then S is a differentiable or C^1 surface.

Example 1: Let

$$\Phi : x = u \cos v,\ \ y = u \sin v,\ \ z = u$$

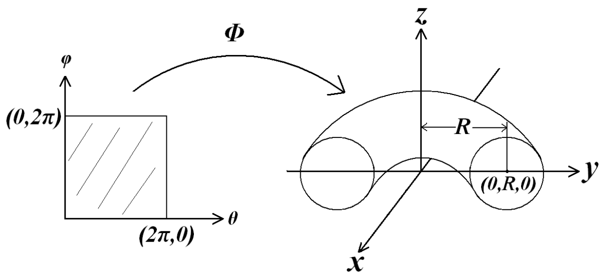
for $0 \leq v \leq 2\pi,\ \ u \geq 0$. Identify the surface.



Example 2: A torus is parametrised by

$$\Phi : x = (R + \cos\phi)\cos\theta$$
$$y = (R + \cos\phi)\sin\theta$$
$$z = \sin\phi$$

$0 \leq \theta \leq 2\pi,\ \ 0 \leq \phi \leq 2\pi,\ \ R > 1$ is fixed.



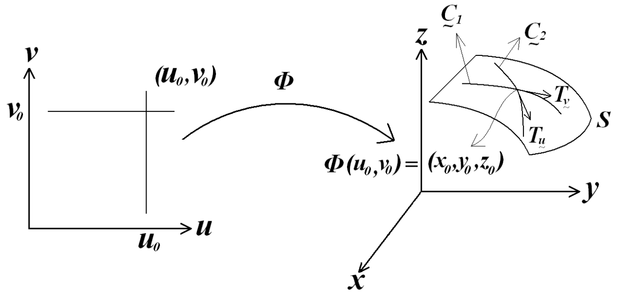
Example 3: Parametrise the surface of the parabolic bowl $z = x^2 + y^2$.

Tangents and Normals to Surfaces

Let S be a differentiable surface.
Consider curves on S given by

$$c_1(v) = \Phi(u_0, v) \quad \text{— } u \text{ is constant}$$

$$c_2(u) = \Phi(u, v_0) \quad \text{— } v \text{ is constant}$$



- $\boldsymbol{T_v}$ is tangent vector to $\boldsymbol{c}_1(v)$ at $\Phi(u_0, v_0)$

$$\boldsymbol{T_v} = \left.\frac{d\boldsymbol{c}_1}{dv}\right|_{v=v_0}$$

$$= \left.\left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)\right|_{(u_0,v_0)}$$

- $\boldsymbol{T_u}$ is tangent vector to $\boldsymbol{c}_2(u)$ at $\Phi(u_0, v_0)$

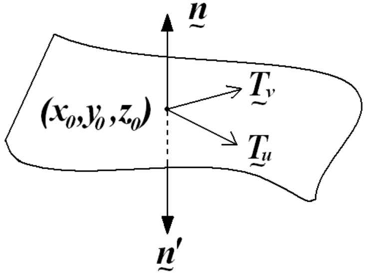
$$\boldsymbol{T_u} = \left.\frac{d\boldsymbol{c}_2}{du}\right|_{u=u_0}$$

$$= \left.\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)\right|_{(u_0,v_0)}$$

Note

$\boldsymbol{T_u}$, $\boldsymbol{T_v}$ change with position (x, y, z). They need not be orthogonal.

- There are 2 normal vectors to the surface at (x_0, y_0, z_0).
  * $\boldsymbol{n} = \boldsymbol{T_u} \times \boldsymbol{T_v}$ OR
  * $\boldsymbol{n}' = -\boldsymbol{n} = \boldsymbol{T_v} \times \boldsymbol{T_u}$



Note

1. $\boldsymbol{n}$ changes with position (x, y, z).

2. If $\boldsymbol{n} \neq \boldsymbol{0}$, the surface is smooth.

Example 4: Find a normal vector to the cone parametrised by

$$x = u \cos v, \;\; y = u \sin v, \;\; z = u$$

$$0 \le v \le 2\pi, \;\; u \ge 0$$

Tangent Plane to Surfaces

If S is a smooth surface at (x_0, y_0, z_0), then the Cartesian equation of the tangent plane to S at (x_0, y_0, z_0) is

$$(x - x_0, y - y_0, z - z_0) \cdot \boldsymbol{n}(u_0, v_0) = 0$$

where $\boldsymbol{n}$ is normal to S.

Example 5: Find the Cartesian equation of the tangent plane to the cone in Example 4 at $(1, 1, \sqrt{2})$.

Surface Area

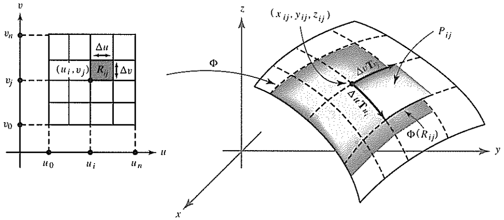
Let S be a smooth (except possibly at a finite number of points) parametrised surface where $\Phi : D \to S$ and

$$\Phi(u, v) = \big(x(u, v), y(u, v), z(u, v)\big).$$

The surface area of S is

$$\iint_S dS = \iint_D |\boldsymbol{T_u} \times \boldsymbol{T_v}| \; du \, dv$$

Proof



The area of a patch on the surface S can be approximated by the area of the parallelogram

$$|\Delta u \mathbf{T_u} \times \Delta v \mathbf{T_v}| = |\mathbf{T_u} \times \mathbf{T_v}| \, \Delta u \Delta v$$

Example 6: Find the surface area of the cone parametrised by

$$x = u \cos v, \;\; y = u \sin v, \;\; z = u$$

$$0 \le v \le 2\pi, \;\; 0 \le u \le 1$$

Example 7: Find an expression for the surface area if $z = f(x, y)$.

4.4 Integrals of Scalar Functions over Surfaces

Let f be a continuous function defined on a smooth, (except possibly at a finite number of points) parametrised surface S. Then

$$\iint\limits_S f\, dS = \iint\limits_D f\,[\Phi(u, v)]\ \ |\boldsymbol{T_u} \times \boldsymbol{T_v}|\ du\, dv$$

where $\Phi : D \to S$ and

$$\Phi(u, v) = \big(x(u, v), y(u, v), z(u, v)\big).$$

Note

1. $\iint_S f\, dS$ is independent of the parametrisation used for S.

2. The surface area of S is a special case of $\iint_S f\, dS$ where $f = 1$.

Physical Interpretation

Physical interpretation of $\iint\limits_S f\,dS$ depends on what f represents.

Example 1

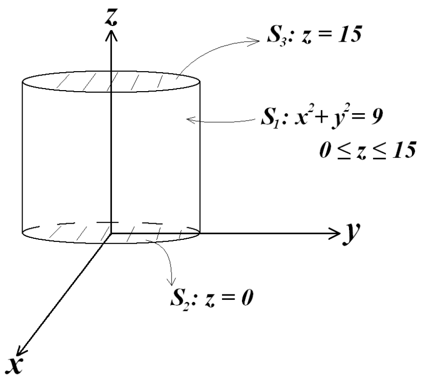
If f is the mass per unit area on S, $\iint\limits_S f\,dS$ is the total mass of the surface S.

Example 2

If f is the charge per unit area on S, $\iint\limits_S f\,dS$ is the total charge on the surface S.

Example 1: Evaluate $\iint\limits_S z^2\,dS$ where S is the unit sphere centred at the origin.

Example 2: Let S be the closed cylinder $x^2 + y^2 = 9,\ z = 0,\ z = 15$. Find $\displaystyle\iint_S z\, dS$



Special Case

If S can be written in the form $z = f(x, y)$, then

$$\iint_S g\, dS$$

$$= \iint_D g[x, y, f(x, y)] \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dx\, dy$$

$$= \iint_D \frac{g[x, y, f(x, y)]}{|\hat{\boldsymbol{n}} \cdot \boldsymbol{k}\,|}\, dx\, dy$$

where $\hat{\boldsymbol{n}}$ is the unit normal to S.

Proof

As $\boldsymbol{n} = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$

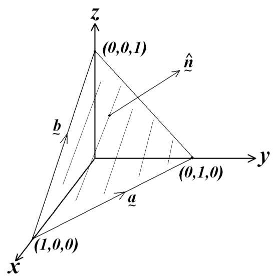
$$\Rightarrow \hat{\boldsymbol{n}} = \frac{\left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}$$

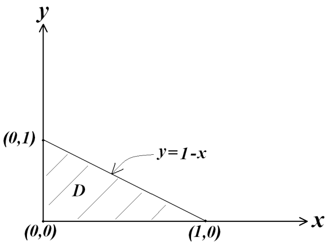
$$\Rightarrow \hat{\boldsymbol{n}} \cdot \boldsymbol{k} = \frac{1}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}$$

$$\Rightarrow \frac{1}{\hat{\boldsymbol{n}} \cdot \boldsymbol{k}} = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

$$\Rightarrow \frac{1}{|\hat{\boldsymbol{n}} \cdot \boldsymbol{k}|} = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$$

Example 3: Find $\iint_S x \, dS$ where S is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.



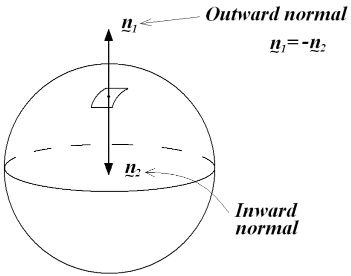


4.5 Oriented Surfaces

An oriented surface is a two sided surface. At each point on S there are 2 normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$.

Example 1

A sphere is an oriented surface.



Example 2

A Möbius band is not an oriented surface. At each point there are 2 normal vectors but the surface has only one side.

Integrals of Vector Fields over Surfaces

Let $\boldsymbol{F}$ be a continuous vector field defined on a smooth (except possibly at a finite number of points), orientable, parametrised surface S. Then

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_S \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,dS$$
$$= \iint_D \boldsymbol{F}\cdot\frac{\boldsymbol{T_u}\times\boldsymbol{T_v}}{|\boldsymbol{T_u}\times\boldsymbol{T_v}|}\;|\boldsymbol{T_u}\times\boldsymbol{T_v}|\;du\,dv$$
$$= \iint_D \boldsymbol{F}\cdot(\boldsymbol{T_u}\times\boldsymbol{T_v})\;du\,dv$$

where $\Phi : D \to S$ such that

$$\Phi(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$$

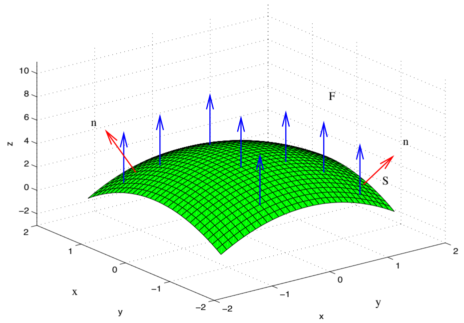
and $\hat{\boldsymbol{n}}$ is the unit outward normal to S.

Physical Interpretation

If $\boldsymbol{F}$ is a vector field then $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$ is the flux of $\boldsymbol{F}$ across S.

Example

If $\boldsymbol{F}$ is the velocity of a fluid, then the flux is the net quantity of fluid to flow across the surface per unit time in the direction of $\hat{\boldsymbol{n}}$. So the flux is the rate of fluid flow in the direction of the outward normal.



Example 1: Determine the flux of

$$\boldsymbol{F}(x, y, z) = (x, y, z)$$

across the unit sphere S centred at $(0, 0, 0)$ in the direction of the outward normal.

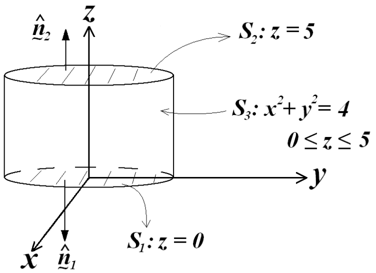
<u>Note</u>

Example 2: Evaluate

$$\iint_S \left(x^3\,\boldsymbol{i} + y^3\,\boldsymbol{j} \right) \cdot \, d\boldsymbol{S}$$

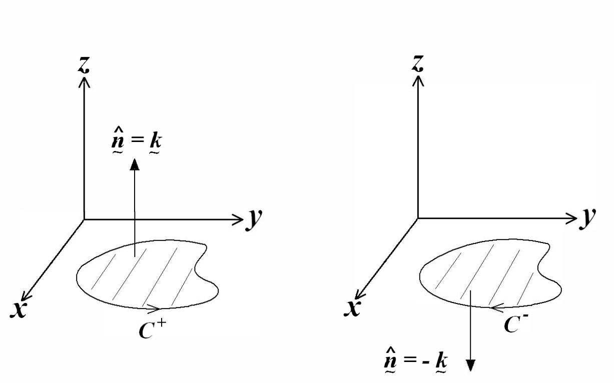
where S is the closed cylinder

$$x^2 + y^2 = 4, \;\; z = 0, \;\; z = 5.$$



5 Integral Theorems

5.1 Oriented Closed Curves in xy Plane



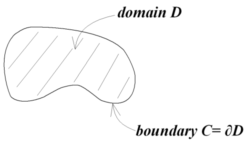
• positive orientation (anticlockwise)

• negative orientation (clockwise)

$\hat{\boldsymbol{n}}$ is the normal to the surface perpendicular to the xy plane.

Note

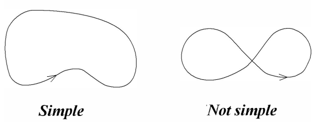
1.



2. $\hat{\boldsymbol{n}}$ and orientation related by right-hand rule.

3. If you walk around the boundary C in the positive orientation then the region D will be on your left.

4. Restricted to simple closed curves (non self intersecting).



Green's Theorem in the Plane

$$\int_{C=\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where

- D is a region in the xy plane bounded by a simple closed curve $C = \partial D$ with positive orientation (anticlockwise).

- $\boldsymbol{F}$ is a C^1 vector field on D where

$$\boldsymbol{F}(x,y) = P(x,y)\,\boldsymbol{i} + Q(x,y)\,\boldsymbol{j}\,.$$

- D is composed of regions of both vertical and horizontal strips.

Vector Form of Green's Theorem

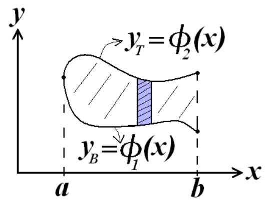
$$\int_{\partial D} \boldsymbol{F} \cdot \, d\boldsymbol{s} = \iint_D (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot \boldsymbol{k} \; dx \, dy$$

Check that double integrals are equivalent.

Proof

Case 1: Vertical Strips

Let D be composed of vertical strips with boundary curve $C = C_1 + B_1 + C_2 + B_2$ oriented in the positive sense.



Since $x = a$ on B_2 and $x = b$ on B_1 then

$$\int_{B_1} P \, dx = 0$$

and

$$\int_{B_2} P \, dx = 0$$

Now C_1 and $-C_2$ can be parametrised by

$$C_1 : x = t, y = \phi_1(t), \;\; a \le t \le b$$

$$-C_2 : x = t, y = \phi_2(t), \;\; a \le t \le b$$

so

$$\iint\limits_D \frac{\partial P}{\partial y}\, dx\, dy = \int\limits_a^b \int\limits_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y}\, dy\, dx$$

$$= \int\limits_a^b P(x, \phi_2(x)) - P(x, \phi_1(x))\ dx$$

$$= -\int_{C_2} P\, dx - \int_{C_1} P\, dx$$

Hence

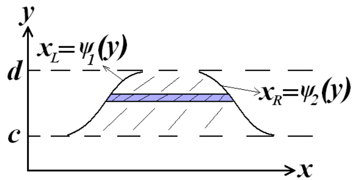
$$\int_C P\, dx$$

$$= \int_{C_1} P\, dx + \int_{B_1} P\, dx + \int_{C_2} P\, dx + \int_{B_2} P\, dx$$

$$= -\iint\limits_D \frac{\partial P}{\partial y}\, dx\, dy$$

Case 2: Horizontal Strips

Let D be composed of horizontal strips with boundary curve $C = C_1 + B_1 + C_2 + B_2$ oriented in the positive sense.



Since $y = c$ on B_1 and $y = d$ on B_2 then

$$\int_{B_1} Q\, dy = 0$$

and

$$\int_{B_2} Q\, dy = 0$$

Now $-C_1$ and C_2 can be parametrised by

$$-C_1 : x = \psi_1(t), y = t,\ \ c \le t \le d$$

$$C_2 : x = \psi_2(t), y = t,\ \ c \le t \le d$$

so

$$\iint\limits_D \frac{\partial Q}{\partial x}\,dx\,dy = \int\limits_c^d \int\limits_{\psi_1(y)}^{\psi_2(y)} \frac{\partial Q}{\partial x}\,dx\,dy$$

$$= \int\limits_c^d Q(\psi_2(y), y) - Q(\psi_1(y), y)\;dy$$

$$= \int_{C_2} Q\,dy + \int_{C_1} Q\,dy$$

Hence

$$\int_C Q\,dy$$

$$= \int_{C_1} Q\,dy + \int_{B_1} Q\,dy + \int_{C_2} Q\,dy + \int_{B_2} Q\,dy$$

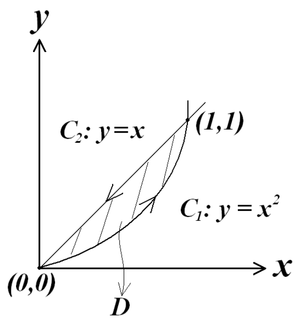
$$= \iint\limits_D \frac{\partial Q}{\partial x}\,dx\,dy$$

Combining case (1) and (2) gives

$$\int_C P\,dx + Q\,dy = \iint\limits_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

Example 1: Let $\boldsymbol{F}(x, y) = (xy^2,\, y + x)$. Verify Green's theorem for the region bounded by

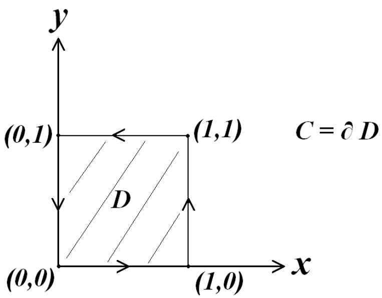
$$y = x^2,\;\; y = x,\;\; (x, y \geq 0).$$



Example 2: Redo Example 2 in line integrals using Green's theorem. Evaluate

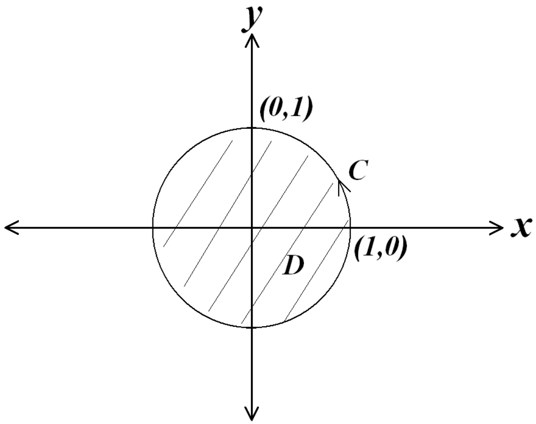
$$\int_C x^2\,dx + xy\,dy$$

over the unit square.



Example 3: Let $\boldsymbol{F}(x,y) = \left(2y + e^x,\, x + \sin(y^2)\right)$

Evaluate $\displaystyle\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$ where C is the unit circle centred at $(0,0)$, traversed anticlockwise.



Note

Calculation of line integral is too hard, so use Green's theorem.

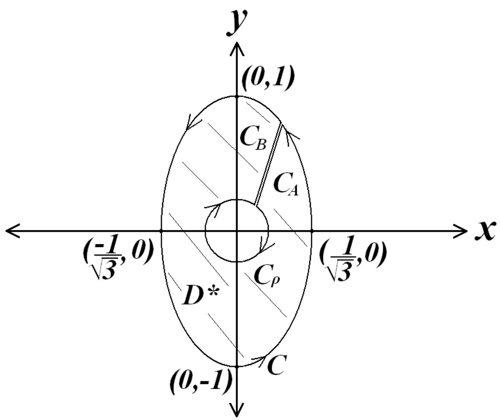
Example 4: Evaluate $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{s}$ where

$$\boldsymbol{F}(x,y) = \left(\frac{-y}{x^2+y^2},\ \frac{x}{x^2+y^2}\right)$$

and C is the curve $3x^2 + y^2 = 1$ oriented anticlockwise.

Note

Calculation of line integral along the ellipse C is too hard.


y
x
(0,1)
C_B
C_A
$(\frac{-1}{\sqrt{3}}, 0)$
$(\frac{1}{\sqrt{3}},0)$
C_ρ
D^*
C
(0,-1)

Area of Region in xy Plane

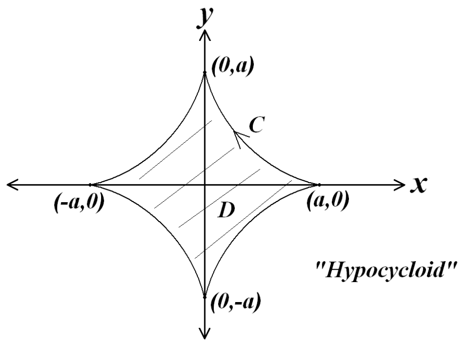
If C is a simple closed curve that bounds a region D, then

$$\boxed{\text{Area of } D = \frac{1}{2}\int_{C=\partial D} x\,dy - y\,dx}$$

Proof

Example 5: Find the area of the region enclosed by

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}, \ (a > 0).$$



5.2 Divergence Theorem in the Plane

$$\boxed{\int\limits_{C=\partial D} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, ds = \iint\limits_{D} \boldsymbol{\nabla} \cdot \boldsymbol{F}\, dx\, dy}$$

where

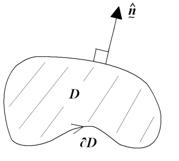
- D is a region in the xy plane bounded by a simple closed curve $C = \partial D$ with positive orientation (anticlockwise).

- $\boldsymbol{F}$ is a C^1 vector field on D where

$$\boldsymbol{F}(x, y) = P(x, y)\, \boldsymbol{i} + Q(x, y)\, \boldsymbol{j}\,.$$

- D is composed of regions of both vertical and horizontal strips.

- $\hat{\boldsymbol{n}}$ is the unit outward normal to ∂D in the xy plane.



Proof

Let

$$\begin{aligned} \boldsymbol{c}(t) &= (x(t), y(t)) \\ \Rightarrow \dot{\boldsymbol{c}}(t) &= (\dot{x}(t), \dot{y}(t)) \\ \Rightarrow |\dot{\boldsymbol{c}}(t)| &= \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} \end{aligned}$$

An outward normal to C is

$$\begin{aligned} \boldsymbol{n}(t) &= (\dot{y}(t), -\dot{x}(t)) \\ \Rightarrow |\boldsymbol{n}(t)| &= \sqrt{\dot{y}(t)^2 + \dot{x}(t)^2} = |\dot{\boldsymbol{c}}(t)| \\ \Rightarrow \widehat{\boldsymbol{n}}(t) &= \frac{1}{|\dot{\boldsymbol{c}}(t)|}(\dot{y}(t), -\dot{x}(t)) \end{aligned}$$

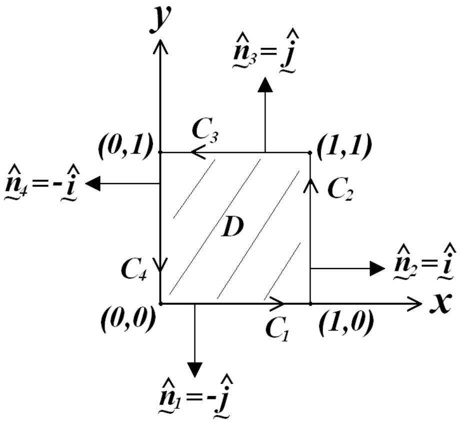
Therefore

$$\begin{aligned} \int_{\partial D} \boldsymbol{F} \cdot \widehat{\boldsymbol{n}}\, ds &= \int_a^b (P, Q) \cdot \frac{1}{|\dot{\boldsymbol{c}}|}(\dot{y}, -\dot{x})\ |\dot{\boldsymbol{c}}| dt \\ &= \int_a^b (P\dot{y} - Q\dot{x})\, dt \\ &= \int_{\partial D} P\, dy - Q\, dx \\ &= \iint_D \frac{\partial P}{\partial x} - \frac{\partial(-Q)}{\partial y}\, dx\, dy \end{aligned}$$

by Green's theorem

$$\begin{aligned} &= \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\, dx\, dy \\ &= \iint_D \boldsymbol{\nabla} \cdot \boldsymbol{F}\, dx\, dy \end{aligned}$$

Example 1: Let $\boldsymbol{F}(x,y) = y^3\,\boldsymbol{i} + x^5\,\boldsymbol{j}$. Verify the Divergence theorem in the plane for the region bounded by the square with vertices $(0,0),(1,0),(1,1),(0,1)$.



• Evaluate $\displaystyle\int_{\partial D} \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\,ds$

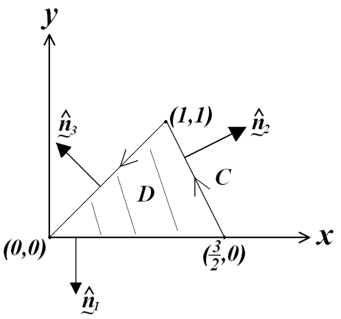
From example 2 on line integrals, we can parametrise each path by letting $0 \le t \le 1$ and

$$\begin{aligned}
C_1 &: x = t,\ \ y = 0 && \Rightarrow\ \dot{x} = 1,\ \dot{y} = 0\\
C_2 &: x = 1,\ \ y = t && \Rightarrow\ \dot{x} = 0,\ \dot{y} = 1\\
C_3 &: x = 1-t,\ \ y = 1 && \Rightarrow\ \dot{x} = -1,\ \dot{y} = 0\\
C_4 &: x = 0,\ \ y = 1-t && \Rightarrow\ \dot{x} = 0,\ \dot{y} = -1
\end{aligned}$$

Example 2: Let C be the triangle with vertices $(0,0)$, $(1,1)$, $\left(\frac{3}{2},0\right)$ traversed anticlockwise.

Evaluate $\displaystyle\int_C \boldsymbol{F}\cdot\widehat{\boldsymbol{n}}\,ds$ where

$\boldsymbol{F}(x,y) = (2x^2y - 3x + \sin(5y), 5y - 2xy^2 - \cos^3(4x))$.



<u>Note</u>

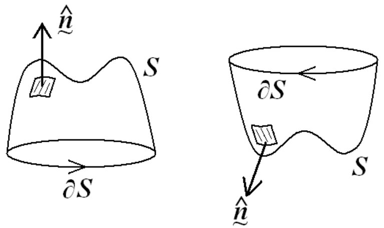
Calculation of the path integral around C is too hard, so use the Divergence theorem in the plane.

5.3 Stokes' Theorem

$$\boxed{\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s}}$$

where

- S is an open, oriented surface parametrised by the C^2 mapping $\Phi(u,v)$.
- ∂S is the oriented closed boundary of S.
- $\boldsymbol{F}$ is a C^1 vector field on S.
- S and ∂S are oriented so that $\hat{\boldsymbol{n}}$ is the unit outward normal to S.
- The orientation of ∂S and $\hat{\boldsymbol{n}}$ are related by the right hand rule.



Think of S as a fishing net and ∂S as its rim.

General Definition of Orientation

Walk along boundary ∂S with the normal as your upright direction. You are moving in the positive direction if the surface S is on your left.

Note

Green's theorem is a special case of Stokes' theorem where S and ∂S are confined to the xy-plane.

Proof

Special case $z = f(x,y)$

Let $\boldsymbol{F}(x,y,z) = (u(x,y,z), v(x,y,z), w(x,y,z))$ and $\boldsymbol{c}(t) = (x(t), y(t), z(t))\,, a \le t \le b.$

If $z = f(x,y)$ then $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}.$

Hence

$$\int_{\partial S} \boldsymbol{F} \cdot\, d\boldsymbol{s}$$

$$= \int_a^b u\frac{dx}{dt} + v\frac{dy}{dt} + w\frac{dz}{dt}\, dt$$

$$= \int_a^b \left(u + w\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(v + w\frac{\partial z}{\partial y}\right)\frac{dy}{dt}\, dt$$

$$= \int_C \left(u + w\frac{\partial z}{\partial x}\right)\, dx + \left(v + w\frac{\partial z}{\partial y}\right)\, dy$$

Using Green's theorem

$$\int_{\partial S} \boldsymbol{F} \cdot\, d\boldsymbol{s}$$

$$= \iint_D \frac{\partial}{\partial x}\left(v + w\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(u + w\frac{\partial z}{\partial x}\right)\, dx\, dy$$

$$= \iint_D \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + w\frac{\partial^2 z}{\partial x \partial y}$$

$$- \frac{\partial u}{\partial y} - \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} - \frac{\partial w}{\partial y}\frac{\partial z}{\partial x} - \frac{\partial w}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} - w\frac{\partial^2 z}{\partial y \partial x}\, dx\, dy$$

$$= \iint_D \frac{\partial v}{\partial z}\frac{\partial z}{\partial x} - \frac{\partial w}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial z}{\partial y} - \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} dxdy$$

$$= \iint_D \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

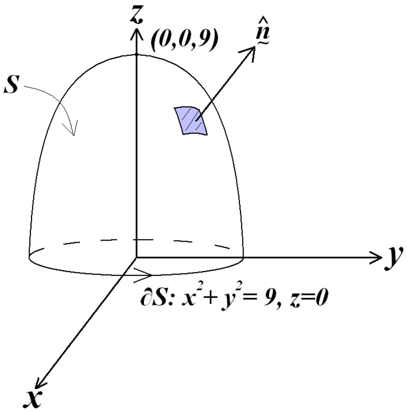
$$\cdot \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right)\, dxdy$$

$$= \iint_D (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot \mathbf{n}\, dxdy$$

$$= \iint_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot\, d\boldsymbol{S}$$

Example 1: Verify Stokes' theorem for the paraboloid $z = 9 - x^2 - y^2,\ z \geq 0$ if

$\boldsymbol{F}(x, y, z) = (2z - y)\,\boldsymbol{i} + (x + z)\,\boldsymbol{j} + (3x - 2y)\,\boldsymbol{k}\,.$



Example 2: Evaluate

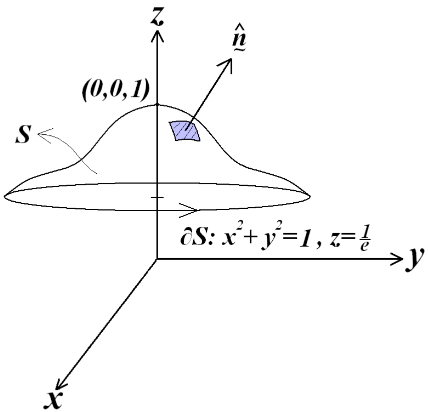
$$\iint_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{S}$$

where

$$\boldsymbol{F}(x, y, z) = \left(e^{y+z} - 2y,\ xe^{y+z},\ e^{x+y}\right)$$

and S is the surface of the bell

$$z = e^{-(x^2+y^2)} \text{ for } z \geq \frac{1}{e}.$$

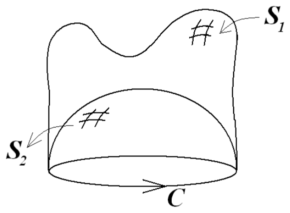


- Calculation of surface integral is too hard.
- Apply Stokes' theorem and evaluate line integral.

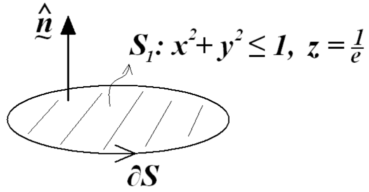
Note

For any two surfaces S_1, S_2 with the same boundary C, Stokes' theorem implies that

$$\iint_{S_1} (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \iint_{S_2} (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{S}$$
$$= \int_C \boldsymbol{F} \cdot d\boldsymbol{s}$$



• Apply Stokes' theorem and use simplest surface with consistent orientation of C and $\hat{\boldsymbol{n}}$.

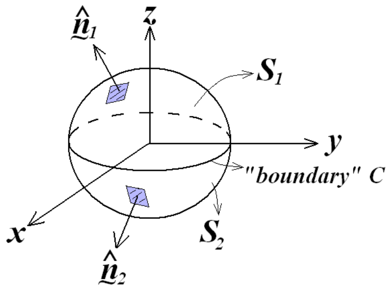


Note

If surface is closed, Stokes' theorem implies

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \, d\boldsymbol{S} = 0.$$

Example 3: Show that $\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = 0$ if S is a sphere of radius a, centred at $(0, 0, 0)$.



5.4 Conservative Fields

Let $\boldsymbol{F}$ be a C^1 vector field defined on $\mathbb{R}^2$ or $\mathbb{R}^3$.

The following conditions are all equivalent.

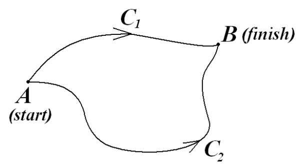
1. For any oriented simple closed curve C,

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = 0.$$

2. For any two oriented simple curves C_1 and C_2 with the same endpoints,

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{s}$$

(integral is path independent)



3. $\boldsymbol{F} = \boldsymbol{\nabla}\phi$ for some scalar function ϕ ($\boldsymbol{F}$ is a gradient field).

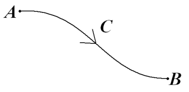
4. $\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$ ($\boldsymbol{F}$ is an irrotational field).

A vector field satisfying one (and hence all) of the four conditions is called a *conservative* vector field.

If $\boldsymbol{F}$ is conservative, $\displaystyle\int\limits_C \boldsymbol{F}\cdot d\boldsymbol{s}$ depends only on the endpoints of C.

Proof

Let $\boldsymbol{c}(t) = (x(t), y(t), z(t))\,,\ a \le t \le b$.



Since $\boldsymbol{F}$ is conservative, $\boldsymbol{F} = \boldsymbol{\nabla}\phi$ and

$$
\begin{aligned}
\int\limits_C \boldsymbol{F}\cdot\, d\boldsymbol{s} = \int\limits_C \boldsymbol{\nabla}\phi\cdot\, d\boldsymbol{s}
&= \int_a^b \boldsymbol{\nabla}\phi\,[\boldsymbol{c}(t)]\cdot \boldsymbol{c}'(t)\,dt \\
&= \int_a^b \frac{d}{dt}\,[\phi(\boldsymbol{c}(t))]\ dt \\
&= \phi\,[\boldsymbol{c}(b)] - \phi\,[\boldsymbol{c}(a)] \\
&= \phi(B) - \phi(A)
\end{aligned}
$$

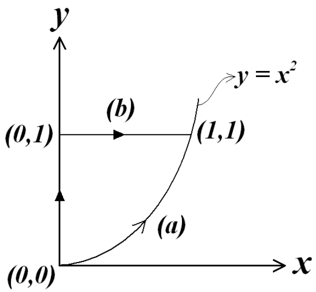
Example 1: Let $\boldsymbol{F}(x,y) = x\,\boldsymbol{i} + y\,\boldsymbol{j}$.

1. Determine the scalar potential ϕ.

2. Evaluate $\displaystyle\int_C \boldsymbol{F}\cdot\, d\boldsymbol{s}$ along

(a) $y = x^2$ from $(0,0)$ to $(1,1)$;

(b) line segments joining $(0,0)$ to $(0,1)$ to $(1,1)$.



3. Evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$ if C is the unit circle centred at $(0, 0)$.

Example 2: Determine the work done by

$$\boldsymbol{F}(x, y, z) = (x^2,\ \cos y \sin z,\ \sin y \cos z)$$

to move a particle along the path

$$\boldsymbol{c}(t) = (t^2 + 1,\ e^t,\ e^{2t}),\ \ 0 \leq t \leq 1.$$

<u>Note</u>

Calculation of the line integral is too hard, so use conservative fields.

5.5 Gauss' Divergence Theorem

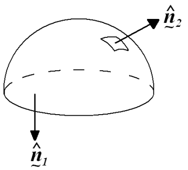
$$\iiint\limits_{\Omega} \boldsymbol{\nabla} \cdot \boldsymbol{F}\, dV = \iint\limits_{\partial\Omega} \boldsymbol{F} \cdot d\boldsymbol{S}$$

where

- Ω is a closed solid region in $\mathbb{R}^3$.
- $\partial\Omega$ is the oriented closed surface that bounds Ω.
- $\boldsymbol{F}$ is a C^1 vector field on Ω.
- Orientation is defined by the unit outward normal $\hat{\boldsymbol{n}}$ to $\partial\Omega$.

Example

solid hemisphere (2 surfaces)



Proof

Let $\boldsymbol{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$. Divide region into rectangular boxes V with surface $S = \partial V$ comprising six faces S_1 to S_6.

Surface S_1

Parametrise S_1 by $x = x_0 + a, y = y, z = z$ where $y_0 \le y \le y_0 + b, z_0 \le z \le z_0 + c$. Then $\hat{\boldsymbol{n}}_1 = \boldsymbol{i}$ and $|\mathbf{T_y} \times \mathbf{T_z}| = 1$. Hence

$$\iint_{S_1} \boldsymbol{F} \cdot d\boldsymbol{S}$$
$$= \int_{z_0}^{z_0+c} \int_{y_0}^{y_0+b} \boldsymbol{F} \cdot \boldsymbol{i}\, |\mathbf{T_y} \times \mathbf{T_z}|\, dy\, dz$$
$$= \int_{z_0}^{z_0+c} \int_{y_0}^{y_0+b} P(x_0 + a, y, z)\, dy\, dz$$

Surface S_2

Parametrise S_2 by $x = x_0, y = y, z = z$ where $y_0 \le y \le y_0 + b, z_0 \le z \le z_0 + c$. Then $\hat{\boldsymbol{n}}_2 = -\boldsymbol{i}$

and $|\mathbf{T_y} \times \mathbf{T_z}| = 1$. Hence

$$\iint_{S_2} \boldsymbol{F} \cdot d\boldsymbol{S}$$
$$= \int_{z_0}^{z_0+c} \int_{y_0}^{y_0+b} \boldsymbol{F} \cdot (-\boldsymbol{i})|\mathbf{T_y} \times \mathbf{T_z}|\, dy\, dz$$
$$= \int_{z_0}^{z_0+c} \int_{y_0}^{y_0+b} -P(x_0, y, z)\, dy\, dz$$

Hence

$$\iint_{S_1} \boldsymbol{F} \cdot d\boldsymbol{S} + \iint_{S_2} \boldsymbol{F} \cdot d\boldsymbol{S}$$
$$= \int_{z_0}^{z_0+c} \int_{y_0}^{y_0+b} P(x_0 + a, y, z) - P(x_0, y, z)\, dy\, dz$$
$$= \int_{z_0}^{z_0+c} \int_{y_0}^{y_0+b} \int_{x_0}^{x_0+a} \frac{\partial P}{\partial x}\, dx\, dy\, dz$$
$$= \iiint_V \frac{\partial P}{\partial x}\, dV$$

Similarly for surface S_3 and S_4

$S_3 : x = x, y = y_0 + b, z = z$

$S_4 : x = x, y = y_0, z = z$

$$\iint_{S_3} \boldsymbol{F} \cdot d\boldsymbol{S} + \iint_{S_4} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_V \frac{\partial Q}{\partial y} \, dV$$

Similarly for surface S_5 and S_6

$S_5 : x = x, y = y, z = z_0 + c$

$S_6 : x = x, y = y, z = z_0$

$$\iint_{S_5} \boldsymbol{F} \cdot d\boldsymbol{S} + \iint_{S_6} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_V \frac{\partial R}{\partial z} \, dV$$

Combining all 6 faces gives

$$\begin{aligned}\iint_{\partial V} \boldsymbol{F} \cdot d\boldsymbol{S} &= \sum_{i=1}^{6} \iint_{S_i} \boldsymbol{F} \cdot d\boldsymbol{S} \\ &= \iiint_V \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \, dV \\ &= \iiint_V \boldsymbol{\nabla} \cdot \boldsymbol{F} \, dV\end{aligned}$$

For the general region Ω consisting of rectangular boxes V with surface ∂V

$$\iiint_\Omega \boldsymbol{\nabla} \cdot \boldsymbol{F} \, dV = \sum_{all\,boxes} \iiint_V \boldsymbol{\nabla} \cdot \boldsymbol{F} \, dV$$

$$\iint_{\partial\Omega} \boldsymbol{F} \cdot d\boldsymbol{S} = \sum_{all\,boxes} \iint_{\partial V} \boldsymbol{F} \cdot d\boldsymbol{S}$$

since the surface integrals $\iint_{\partial V} \boldsymbol{F} \cdot d\boldsymbol{S}$ from adjacent faces will cancel because the outwards normals are in opposite directions.

Hence

$$\iiint\limits_{\Omega} \boldsymbol{\nabla} \cdot \boldsymbol{F} \, dV = \iint\limits_{\partial\Omega} \boldsymbol{F} \cdot d\boldsymbol{S}$$

Example 1: Redo Example 1 of surface integrals using the Divergence theorem:

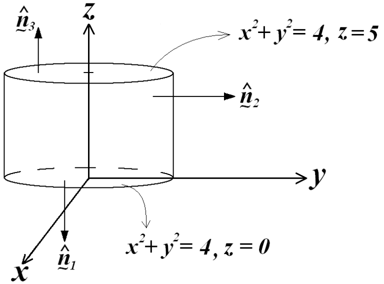
$$\iint\limits_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

where $\boldsymbol{F}(x, y, z) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$ and S is the unit sphere centred at origin.

Example 2: Redo Example 2 of surface integrals using the Divergence theorem.

$$\iint\limits_S (x^3\,\boldsymbol{i} + y^3\,\boldsymbol{j}\,) \cdot d\boldsymbol{S}$$

where S is the closed cylinder $x^2 + y^2 = 4$, $z = 0,\ z = 5$.

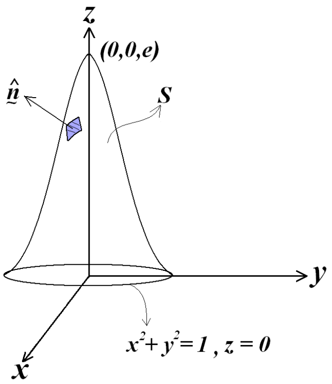


Example 3: Evaluate $\displaystyle\iint\limits_S \boldsymbol{F} \cdot d\boldsymbol{S}$ where S is the surface of the bell

$$z = (1 - x^2 - y^2)e^{(1-x^2-3y^2)}$$

for $z \geq 0$, and

$$\boldsymbol{F}(x, y, z) = \left(e^y \cos z,\ \sqrt{x^3+1}\sin z,\ x^2+y^2+3\right).$$



Note

Cannot do the surface integral over the bell directly.

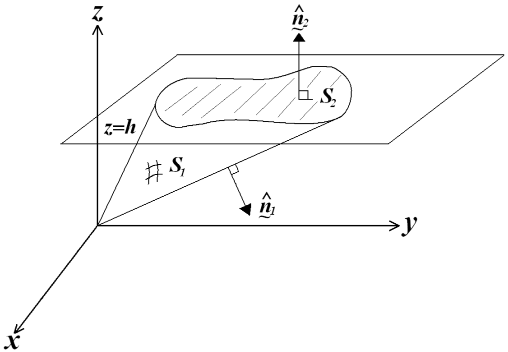
Volume via Surface Integrals

If $\boldsymbol{F}(x,y,z) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$ and Ω is a region to which Gauss' theorem applies, then

$$\boxed{\text{volume of } \Omega = \frac{1}{3}\iint\limits_{\partial\Omega} \boldsymbol{F} \cdot d\boldsymbol{S}}$$

Proof

Example 4: Find the volume of a solid cone with base area A and height h.



5.6 Applications of Integral Theorems

Example 1: Gauss' Law

Let Ω be an elementary region in $\mathbb{R}^3$ bounded by the closed surface $\partial\Omega$. Then if $\mathbf{0} \notin \partial\Omega$

$$\iint\limits_{\partial\Omega} \frac{\boldsymbol{r}}{r^3} \cdot d\boldsymbol{S} = \begin{cases} 0, & \mathbf{0} \notin \Omega \\ 4\pi, & \mathbf{0} \in \Omega \end{cases}$$

where $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$ and $r = \sqrt{x^2 + y^2 + z^2}$.

Example 2: Continuity Equation for Fluid Flow.

A fluid of density $\rho(\boldsymbol{r}, t)$ moves with velocity $\boldsymbol{v}(\boldsymbol{r}, t)$. If there are no sources or sinks of fluid, show that

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \boldsymbol{J}$$

where $\boldsymbol{J} = \rho \boldsymbol{v}$ is the fluid current.

Note

This is consistent with our interpretation of $\boldsymbol{\nabla}\cdot\boldsymbol{J}$ earlier. If $\boldsymbol{\nabla}\cdot\boldsymbol{J} \neq 0$ then the fluid density must change if no sources or sinks.

Example 3: Maxwell's Equations for Electromagnetic Fields.

Define the following quantities

- electric charge density – $\rho(\boldsymbol{r}, t)$
- electric current – $\boldsymbol{J}(\boldsymbol{r}, t)$
- vector field for magnetic force – $\boldsymbol{B}(\boldsymbol{r}, t)$
- vector field for electric force – $\boldsymbol{E}(\boldsymbol{r}, t)$
- permittivity of free space – ε_0
- permeability of free space – μ_0

In S.I. units, Maxwell's equations can be written as

(a) $\boldsymbol{\nabla} \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$ (Faraday's Law)
(if $\boldsymbol{B}$ changes with time an electric field is produced)

(b) $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \dfrac{\rho}{\varepsilon_0}$ (Gauss' Law)
(charges present make $\boldsymbol{\nabla} \cdot \boldsymbol{E} \neq 0$)

(c) $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \varepsilon_0 \mu_0 \dfrac{\partial \boldsymbol{E}}{\partial t}$ (Ampere's Law)
(if $\boldsymbol{\nabla} \times \boldsymbol{B} \neq \boldsymbol{0}$ then currents or $\boldsymbol{E}$ changes with time)

(d) $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$
($\boldsymbol{B}$ is always incompressible, no magnetic sources)

Consequences of Maxwell's equations are

1. If $\boldsymbol{B}$ is constant in time so $\boldsymbol{B}(\boldsymbol{r})$ only.

2. If $\boldsymbol{E}$ is constant in time so $\boldsymbol{E}(\boldsymbol{r})$ only.

3. If there are no currents or charges.

This is the wave equation in a vacuum.

Electromagnetic wave travels with speed

$$\frac{1}{\sqrt{\varepsilon_0\mu_0}} \approx 300,000 \text{ km/sec}.$$

This was predicted by Maxwell 20 years before experimentally observed by Hertz.

6 General Curvilinear Coordinates

6.1 Curvilinear Coordinate Systems

For each point P with Cartesian coordinates (x, y, z), associate a unique set of curvilinear coordinates (u_1, u_2, u_3) where

1. $x = f_1(u_1, u_2, u_3),$

   $y = f_2(u_1, u_2, u_3),$

   $z = f_3(u_1, u_2, u_3).$

2. $u_1 = g_1(x, y, z),$

   $u_2 = g_2(x, y, z),$

   $u_3 = g_3(x, y, z).$

Examples

cylindrical coordinates

spherical coordinates

Unit Vectors

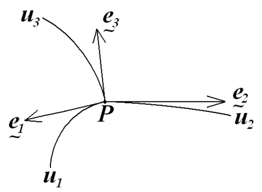
Let $\boldsymbol{r}(x, y, z) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$ be the position vector of P, then $\boldsymbol{r}(u_1, u_2, u_3) = \boldsymbol{f}(u_1, u_2, u_3)$.

- A tangent vector at P for u_2, u_3 constant is $\dfrac{\partial \boldsymbol{r}}{\partial u_1}$

  A unit tangent vector in the direction of u_1 increasing is

$$\boldsymbol{e}_1 = \frac{\dfrac{\partial \boldsymbol{r}}{\partial u_1}}{\left|\dfrac{\partial \boldsymbol{r}}{\partial u_1}\right|}$$

$$\Rightarrow \quad \frac{\partial \boldsymbol{r}}{\partial u_1} = \left|\frac{\partial \boldsymbol{r}}{\partial u_1}\right| \boldsymbol{e}_1 = h_1\,\boldsymbol{e}_1$$



- Similarly, tangent vectors at P in the direction of u_2 and u_3 increasing are

$$\frac{\partial \boldsymbol{r}}{\partial u_2} = \left|\frac{\partial \boldsymbol{r}}{\partial u_2}\right| \boldsymbol{e}_2 = h_2\, \boldsymbol{e}_2$$

$$\frac{\partial \boldsymbol{r}}{\partial u_3} = \left|\frac{\partial \boldsymbol{r}}{\partial u_3}\right| \boldsymbol{e}_3 = h_3\, \boldsymbol{e}_3$$

Note

1. $h_1,\ h_2,\ h_3$ are called scale factors.

2. The curvilinear coordinate system is orthogonal if

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = 0 \text{ for } i \neq j.$$

Example 1: Cylindrical coordinates (ρ, ϕ, z)

$$x = \rho \cos\phi,\ \ y = \rho \sin\phi,\ \ z = z$$

$$0 \leq \phi \leq 2\pi,\ \ \rho \geq 0,\ \ z \in \mathbb{R}$$

Find the scale factors and unit tangent vectors.

Example 2: Spherical Coordinates (r, θ, ϕ)

$$x = r \sin\theta \cos\phi, \;\; y = r \sin\theta \sin\phi, \;\; z = r \cos\theta$$

$$0 \leq \phi \leq 2\pi, \;\; 0 \leq \theta \leq \pi, \;\; r \geq 0$$

Find the scale factors and unit tangent vectors.

Tangents to Curves

Let u_1, u_2, u_3 be curvilinear coordinates. Consider the parametrised curve

$$\boldsymbol{r}(t) = \boldsymbol{r}\left(u_1(t), u_2(t), u_3(t)\right).$$

Using the chain rule, the tangent to $\boldsymbol{r}(t)$ is

$$\frac{d\boldsymbol{r}}{dt} = \frac{\partial \boldsymbol{r}}{\partial u_1}\frac{du_1}{dt} + \frac{\partial \boldsymbol{r}}{\partial u_2}\frac{du_2}{dt} + \frac{\partial \boldsymbol{r}}{\partial u_3}\frac{du_3}{dt}$$

$$\Rightarrow \frac{d\boldsymbol{r}}{dt} = h_1\boldsymbol{e}_1\frac{du_1}{dt} + h_2\boldsymbol{e}_2\frac{du_2}{dt} + h_3\boldsymbol{e}_3\frac{du_3}{dt}.$$

Example 3: If

$$\boldsymbol{r}(t) = 2\cos(3t)\,\boldsymbol{i} + 2\sin(3t)\,\boldsymbol{j} + \boldsymbol{k}\,,$$

write $\dfrac{d\boldsymbol{r}}{dt}$ in cylindrical coordinates.

6.2 Grad, Div, Curl, and Laplacian in Orthogonal Curvilinear Coordinates

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^2 scalar function and $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field where

$$\boldsymbol{F}(u_1, u_2, u_3) = F_1\boldsymbol{e}_1 + F_2\boldsymbol{e}_2 + F_3\boldsymbol{e}_3.$$

1. $\boldsymbol{\nabla} f = \dfrac{1}{h_1}\dfrac{\partial f}{\partial u_1}\boldsymbol{e}_1 + \dfrac{1}{h_2}\dfrac{\partial f}{\partial u_2}\boldsymbol{e}_2 + \dfrac{1}{h_3}\dfrac{\partial f}{\partial u_3}\boldsymbol{e}_3$

2. $\boldsymbol{\nabla}\cdot\boldsymbol{F} = \dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial\,(h_2h_3F_1)}{\partial u_1} + \dfrac{\partial\,(h_1h_3F_2)}{\partial u_2} + \dfrac{\partial\,(h_1h_2F_3)}{\partial u_3}\right]$

3. $\boldsymbol{\nabla}\times\boldsymbol{F} = \dfrac{1}{h_1h_2h_3}\begin{vmatrix} h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}$

4. $\nabla^2 f = \dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial}{\partial u_1}\left(\dfrac{h_2h_3}{h_1}\dfrac{\partial f}{\partial u_1}\right) + \dfrac{\partial}{\partial u_2}\left(\dfrac{h_1h_3}{h_2}\dfrac{\partial f}{\partial u_2}\right) + \dfrac{\partial}{\partial u_3}\left(\dfrac{h_1h_2}{h_3}\dfrac{\partial f}{\partial u_3}\right)\right]$

Note

Equations 1-4 reduce to the usual expressions for cartesian coordinates if

$$h_1 = h_2 = h_3 = 1, \quad (u_1, u_2, u_3) = (x, y, z),$$
$$\boldsymbol{e}_1 = \boldsymbol{i}\,,\ \boldsymbol{e}_2 = \boldsymbol{j}\,,\ \boldsymbol{e}_3 = \boldsymbol{k}\,.$$

Cylindrical Coordinates

Cylindrical coordinates (ρ, ϕ, z) are defined by

$$x = \rho \cos\phi, \quad y = \rho \sin\phi, \quad z = z$$

where $\rho \geq 0$, $0 \leq \phi \leq 2\pi$.

Then $(u_1, u_2, u_3) = (\rho, \phi, z)$ and $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$.

Equations 1-4 reduce to

1. $$\boldsymbol{\nabla} f = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$$

2. $$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{1}{\rho}\left[\frac{\partial(\rho F_1)}{\partial \rho} + \frac{\partial F_2}{\partial \phi} + \frac{\partial(\rho F_3)}{\partial z}\right]$$
$$= \frac{1}{\rho}\left(F_1 + \rho\frac{\partial F_1}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial F_2}{\partial \phi} + \frac{\partial F_3}{\partial z}$$

3. $$\boldsymbol{\nabla} \times \boldsymbol{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ F_1 & \rho F_2 & F_3 \end{vmatrix}$$

4. $$\nabla^2 f = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial f}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial f}{\partial z}\right)\right]$$
$$= \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Coordinates

Spherical coordinates (r, θ, ϕ) are defined by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

where $r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$.

Then $(u_1, u_2, u_3) = (r, \theta, \phi)$ and $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$.

Equations 1-4 reduce to

1. $$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$$

2. $$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{1}{r^2\sin\theta}\left[\frac{\partial(r^2\sin\theta F_1)}{\partial r} + \frac{\partial(r\sin\theta F_2)}{\partial \theta} + \frac{\partial(rF_3)}{\partial \phi}\right]$$
$$= \frac{1}{r^2}\frac{\partial(r^2 F_1)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta F_2)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_3}{\partial \phi}$$

3. $$\boldsymbol{\nabla} \times \boldsymbol{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_1 & rF_2 & r\sin\theta F_3 \end{vmatrix}$$

4. $\nabla^2 f$
$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial \phi}\right)\right]$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Proof of ∇f result

Let $\nabla f = F_1 \boldsymbol{e}_1 + F_2 \boldsymbol{e}_2 + F_3 \boldsymbol{e}_3$.

To determine F_1, F_2, F_3 differentiate along the curve $\boldsymbol{r}(t)$ in two ways.

- Consider $f\,(x(t), y(t), z(t))$. Then chain rule gives

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$= \nabla f \cdot \frac{d\boldsymbol{r}}{dt}$$

$$= (F_1 \boldsymbol{e}_1 + F_2 \boldsymbol{e}_2 + F_3 \boldsymbol{e}_3) \cdot \left(h_1 \boldsymbol{e}_1 \frac{du_1}{dt} + h_2 \boldsymbol{e}_2 \frac{du_2}{dt} + h_3 \boldsymbol{e}_3 \frac{du_3}{dt}\right)$$

$$= F_1 h_1 \frac{du_1}{dt} + F_2 h_2 \frac{du_2}{dt} + F_3 h_3 \frac{du_3}{dt} \quad (1)$$

- Consider $f\,(u_1(t), u_2(t), u_3(t))$. Then chain rule gives

$$\frac{df}{dt} = \frac{\partial f}{\partial u_1}\frac{du_1}{dt} + \frac{\partial f}{\partial u_2}\frac{du_2}{dt} + \frac{\partial f}{\partial u_3}\frac{du_3}{dt} \qquad (2)$$

Comparing equations (1) and (2) gives

$$\frac{\partial f}{\partial u_1} = F_1 h_1 \quad \Rightarrow \quad F_1 = \frac{1}{h_1}\frac{\partial f}{\partial u_1}$$

$$\frac{\partial f}{\partial u_2} = F_2 h_2 \quad \Rightarrow \quad F_2 = \frac{1}{h_2}\frac{\partial f}{\partial u_2}$$

$$\frac{\partial f}{\partial u_3} = F_3 h_3 \quad \Rightarrow \quad F_3 = \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

Hence

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\boldsymbol{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\boldsymbol{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\boldsymbol{e}_3$$

Example 1: Let $f(r,\theta,\phi) = r\theta\phi$,
$\boldsymbol{F}(r,\theta,\phi) = \sin\theta\,\hat{\boldsymbol{r}} + r\,\hat{\boldsymbol{\phi}}$

Find $\boldsymbol{\nabla} f$, $\boldsymbol{\nabla}\cdot\boldsymbol{F}$, $\boldsymbol{\nabla}\times\boldsymbol{F}$, $\nabla^2 f$ in spherical co-ordinates.

Example 2: Express the heat equation

$$\frac{\partial u}{\partial t} = k\nabla^2 u$$

in spherical coordinates if u depends only on r and t.

Example 3: Redo Example 5 of basic identities of vector calculus.

Find $\boldsymbol{\nabla} \cdot \left(\dfrac{\boldsymbol{r}}{r^2}\right)$ where
$\boldsymbol{r}(x,y,z) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}\,,\ r = \sqrt{x^2+y^2+z^2}$.

Example 4: Redo Example 6 of basic identities of vector calculus.

Find $\nabla^2(r^2\,\log(r))$ where
$\boldsymbol{r}(x,y,z) = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}\,,\ r = \sqrt{x^2+y^2+z^2}$.

6.3 Volume and Surface Area Elements

Volume Element

When we change variables from (x, y, z) to (u_1, u_2, u_3), the Jacobian is

$$\det \begin{bmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} & \frac{\partial x}{\partial u_3} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} & \frac{\partial y}{\partial u_3} \\ \frac{\partial z}{\partial u_1} & \frac{\partial z}{\partial u_2} & \frac{\partial z}{\partial u_3} \end{bmatrix}$$

$$= \det \begin{bmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{bmatrix}$$

$$= \left(\frac{\partial x}{\partial u_1}, \frac{\partial y}{\partial u_1}, \frac{\partial z}{\partial u_1}\right) \cdot \left[\left(\frac{\partial x}{\partial u_2}, \frac{\partial y}{\partial u_2}, \frac{\partial z}{\partial u_2}\right) \times \left(\frac{\partial x}{\partial u_3}, \frac{\partial y}{\partial u_3}, \frac{\partial z}{\partial u_3}\right)\right]$$

$$\Rightarrow \text{Jacobian} = \frac{\partial \boldsymbol{r}}{\partial u_1} \cdot \left(\frac{\partial \boldsymbol{r}}{\partial u_2} \times \frac{\partial \boldsymbol{r}}{\partial u_3}\right)$$

$$= (h_1\, \boldsymbol{e}_1) \cdot ((h_2\, \boldsymbol{e}_2) \times (h_3\, \boldsymbol{e}_3))$$

$$= h_1 h_2 h_3\ \boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)$$

$$\Rightarrow |\text{Jacobian}| = h_1 h_2 h_3\ \ |\boldsymbol{e}_1 \cdot (\boldsymbol{e}_2 \times \boldsymbol{e}_3)|$$

- If $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are orthogonal unit vectors then $|\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 \times \boldsymbol{e}_3| = 1$, so

$$\boxed{\begin{aligned} |\text{Jacobian}| &= h_1 h_2 h_3 \\ dV = dx\, dy\, dz &= h_1 h_2 h_3\, du_1\, du_2\, du_3 \end{aligned}}$$

- cylindrical coordinates

$$|\text{Jacobian}| = h_\rho h_\phi h_z = \rho$$

- spherical coordinates

$$|\text{Jacobian}| = h_r h_\theta h_\phi = r^2 \sin\theta$$

Surface Area Element

If S is a surface parametrised by

$x = f(u,v),\ y = g(u,v),\ z = h(u,v)$ then

$$dS = |\boldsymbol{T_u} \times \boldsymbol{T_v}|\ du\, dv$$

$$\text{Now } |\boldsymbol{T_u} \times \boldsymbol{T_v}| = \left|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right|$$

$$= |(h_u \boldsymbol{e}_u) \times (h_v \boldsymbol{e}_v)|$$

$$= h_u h_v\, |\boldsymbol{e}_u \times \boldsymbol{e}_v|$$

- If $\boldsymbol{e}_u,\ \boldsymbol{e}_v$ are orthogonal unit vectors then $|\boldsymbol{e}_u \times \boldsymbol{e}_v| = 1$, so

$$\boxed{\begin{aligned} |\boldsymbol{T_u} \times \boldsymbol{T_v}| &= h_u h_v \\ dS &= h_u h_v\, du\, dv \end{aligned}}$$

Example 1: Evaluate $\iint\limits_S z\,dS$ where S is the cylinder $x^2 + y^2 = 9,\ 0 \leq z \leq 5$.

Example 2: Evaluate $\iint\limits_S z\,dS$ where S is the hemisphere $x^2 + y^2 + z^2 = 4,\ \ z \geq 0$.