MAST20009 Vector Calculus

Practice Class 11 Questions

General Curvilinear Coordinates

For each point P with Cartesian coordinates (x, y, z), associate a unique set of *curvilinear coordinates* (u_1, u_2, u_3) where $x = f_1(u_1, u_2, u_3)$, $y = f_2(u_1, u_2, u_3)$, and $z = f_3(u_1, u_2, u_3)$.

Unit vectors

Let $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ be the position vector of P, then $\boldsymbol{r} = \boldsymbol{f}(u_1, u_2, u_3)$.

- A tangent vector at P for u_2, u_3 constant is $\dfrac{\partial \boldsymbol{r}}{\partial u_1}$

  A unit tangent vector in the direction of u_1 increasing is:

$$\boldsymbol{e}_1 = \frac{\partial \boldsymbol{r}}{\partial u_1} \Big/ \left|\frac{\partial \boldsymbol{r}}{\partial u_1}\right| \text{ so } \frac{\partial \boldsymbol{r}}{\partial u_1} = \left|\frac{\partial \boldsymbol{r}}{\partial u_1}\right| \boldsymbol{e}_1 = h_1 \boldsymbol{e}_1$$

- Similarly, tangent vectors at P in the directions of u_2, u_3 increasing are:

$$\frac{\partial \boldsymbol{r}}{\partial u_2} = \left|\frac{\partial \boldsymbol{r}}{\partial u_2}\right| \boldsymbol{e}_2 = h_2 \boldsymbol{e}_2, \qquad \frac{\partial \boldsymbol{r}}{\partial u_3} = \left|\frac{\partial \boldsymbol{r}}{\partial u_3}\right| \boldsymbol{e}_3 = h_3 \boldsymbol{e}_3$$

Note:

- $h_1,\ h_2,\ h_3$ are called *scale factors*.
- The curvilinear coordinate system is *orthogonal* if: $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = 0$ for $i \neq j$.
- The volume element dV is $h_1 h_2 h_3\, du_1\, du_2\, du_3$.

Grad, Div, Curl, and Laplacian in Orthogonal Curvilinear Coordinates

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^2 scalar function and $\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field where

$$\boldsymbol{F}(u_1, u_2, u_3) = F_1(u_1, u_2, u_3)\boldsymbol{e}_1 + F_2(u_1, u_2, u_3)\boldsymbol{e}_2 + F_3(u_1, u_2, u_3)\boldsymbol{e}_3.$$

- $\boldsymbol{\nabla} f = \dfrac{1}{h_1}\dfrac{\partial f}{\partial u_1}\boldsymbol{e}_1 + \dfrac{1}{h_2}\dfrac{\partial f}{\partial u_2}\boldsymbol{e}_2 + \dfrac{1}{h_3}\dfrac{\partial f}{\partial u_3}\mathbf{e_3}$
- $\boldsymbol{\nabla} \cdot \boldsymbol{F} = \dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial (h_2 h_3 F_1)}{\partial u_1} + \dfrac{\partial (h_1 h_3 F_2)}{\partial u_2} + \dfrac{\partial (h_1 h_2 F_3)}{\partial u_3}\right]$
- $\boldsymbol{\nabla} \times \boldsymbol{F} = \dfrac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$
- $\nabla^2 f = \dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial}{\partial u_1}\left(\dfrac{h_2 h_3}{h_1}\dfrac{\partial f}{\partial u_1}\right) + \dfrac{\partial}{\partial u_2}\left(\dfrac{h_1 h_3}{h_2}\dfrac{\partial f}{\partial u_2}\right) + \dfrac{\partial}{\partial u_3}\left(\dfrac{h_1 h_2}{h_3}\dfrac{\partial f}{\partial u_3}\right)\right]$

1. Define parabolic coordinates (u, v, θ) by

$$x = uv\cos\theta,\ \ y = uv\sin\theta,\ \ z = \frac{1}{2}\left(u^2 - v^2\right),$$

Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} = \cdots$ be the position vector of a point.

where $u > 0$, $v > 0$, and $0 \le \theta < 2\pi$.

(a) Find the scale factors h_u, h_v, h_θ.

(b) Find the unit vectors $\boldsymbol{e}_u$, $\boldsymbol{e}_v$, $\boldsymbol{e}_\theta$.

(c) Show that the parabolic coordinate system is orthogonal.

(d) Write down the expression for the volume element dV.

(e) Let $f(u, v, \theta) = u^4 v^3$ and $\mathbf{F}(u, v, \theta) = \cos 2\theta\, \boldsymbol{e}_\theta$. Find expressions for

 (i) $\boldsymbol{\nabla} f$;

 (ii) $\boldsymbol{\nabla} \cdot \mathbf{F}$;

 (iii) $\boldsymbol{\nabla} \times \mathbf{F}$;

 (iv) $\nabla^2 f$

in terms of u, v, and θ.

(a) $\frac{\partial \underline{r}}{\partial u} = (v\cos\theta,\ v\sin\theta,\ u)$. $h_u = \left|\frac{\partial \underline{r}}{\partial u}\right| = \sqrt{v^2+u^2}$

$\frac{\partial \underline{r}}{\partial v} = (u\cos\theta,\ u\sin\theta,\ -v)$ $h_v = \left|\frac{\partial \underline{r}}{\partial v}\right| = \sqrt{v^2+u^2}$

$\frac{\partial \underline{r}}{\partial \theta} = (-uv\sin\theta,\ uv\cos\theta,\ 0)$ $h_\theta = uv$.

$\frac{\partial f}{\partial u} = 4u^3v^3$

$\frac{\partial f}{\partial v} = 3u^4v^2$

$\frac{\partial f}{\partial \theta} = 0$

(b) $\underline{e_u} = \frac{1}{\sqrt{v^2+u^2}}(v\cos\theta,\ v\sin\theta,\ u)$

$\underline{e_v} = \frac{1}{\sqrt{v^2+u^2}}(u\cos\theta,\ u\sin\theta,\ -v)$

$\underline{e_\theta} = (-\sin\theta,\ \cos\theta,\ 0)$.

(d). $dV = h_u h_v h_\theta\, du\, dv\, d\theta$

$= (v^2+u^2)uv\, du\, dv\, d\theta$.

(u^3v+uv^3) u.

(c). $\underline{e_u}\cdot\underline{e_v} = \frac{1}{v^2+u^2}(uv - uv) = 0$

$\underline{e_v}\cdot\underline{e_\theta} = -u\sin\theta + uv\sin\theta = 0$

$\underline{e_u}\cdot\underline{e_\theta} = -v\sin\theta\cos\theta + v\sin\theta\cos\theta = 0$

(e). (i) $\nabla f = \frac{1}{\sqrt{u^2+v^2}} 4u^3v^3\, \hat{\underline{u}} + \frac{1}{\sqrt{u^2+v^2}} 3v^2u^4\, \hat{\underline{v}}$

(ii). $\underline{\nabla}\cdot\underline{F} = \frac{1}{(u^2+v^2)uv}\left[\frac{\partial\left((u^2+v^2)\cos 2\theta\right)}{\partial\theta}\right]$

$= \frac{1}{\cancel{(u^2+v^2)}uv} -\cancel{(u^2+v^2)}\sin 2\theta\cdot 2$

$= -\frac{2\sin 2\theta}{uv}$

(iii) $\underline{\nabla}\times\underline{F} = \frac{1}{(u^2+v^2)uv}\begin{vmatrix} \sqrt{u^2+v^2}\,\hat{\underline{u}} & \sqrt{u^2+v^2}\,\hat{\underline{v}} & uv\,\hat{\underline{\theta}} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial\theta} \\ 0 & 0 & uv\cos 2\theta \end{vmatrix}$

$u\cos 2\theta\, \hat{\underline{u}} - v\cos 2\theta\, \hat{\underline{v}}$).

$= \frac{1}{(u^2+v^2)uv}\left(\cancel{0\hat{\underline{u}} + 0\hat{\underline{v}} + 0\hat{\underline{\theta}}}\right) = \underline{0}$.

$\cancel{4u^4v^4}$ $\cancel{3u^5v^3}$

(iv) $\nabla^2 f = \frac{1}{(u^2+v^2)uv}\left[\frac{\partial}{\partial u}\left(uv\, 4u^3v^3\right) + \frac{\partial}{\partial v}\left(uv\, 3u^4v^2\right)\right]$

$= \frac{1}{(u^2+v^2)uv}\left[16u^3v^4 + 9v^2u^5\right] = \frac{16u^2v^3 + 9vu^4}{u^2+v^2}$