MAST30001 Stochastic Modelling

Tutorial Sheet 3

1. Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with probability mass function given by the following table.

k	0	1	2	3
$P(Y=k)$	0.2	0.2	0.3	0.3

Set $X_0 = 0$ and let $X_n = \max\{Y_1, \ldots, Y_n\}$ be the largest Y_i observed to time n.

(a) Argue that (X_n) is a Markov chain and determine its transition matrix P.

(b) Analyze the state space of the chain: communication classes, essentiality, reducibility.

(c) Describe the long run behavior of the chain. In particular can you determine the matrix $\lim_{n\to\infty} P^n$?

Ans.

(a) We can compute the conditional probabilities given the entire history of the chain as

$$P(X_{n+1} = k | X_n = j, X_{n-1} = x_{n-1}, \ldots, X_1 = x_1, X_0 = 0) = \begin{cases} P(Y = k), & k > j, \\ P(Y \leq j), & k = j, \end{cases}$$

which only depend on j (and not the x_i's) so the chain is Markov and with transitions given by the formula above:

$$P = \begin{bmatrix} 0.2 & 0.2 & 0.3 & 0.3 \\ 0 & 0.4 & 0.3 & 0.3 \\ 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) Each state is its own communicating class, with the classes $\{0\}, \{1\}, \{2\}$ non-essential and $\{3\}$ absorbing and essential.

(c) The chain starts at 0 and increases, eventually landing at the absorbing state 3. Since P^n is the n step transition matrix we expect

$$\lim_{n\to\infty} P^n = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2. Two containers labelled α and ω have $2k$ balls distributed between them. At discrete time steps a ball is uniformly chosen from all $2k$ balls and it is moved from the container it is in to the other container. Let X_n be the number of balls in container α after the nth time step. This is a simple model for molecules diffusing through a membrane.

(a) Is X_n a Markov chain? What are the transition probabilities?

(b) Analyze the state space of the chain: communication classes, essentiality, reducibility.

(c) If X_0 has a binomial distribution with parameters $(2k, 1/2)$ (meaning that X_0 balls are initially put into container α and $2k - X_0$ are put into container ω), what is the distribution of X_{10}? *[Hint: First compute the distribution of X_1.]*

Ans.

(a) Yes, the one step transitions only depend on the current state. The transition probabilities are for $i = 0, \ldots, 2k$

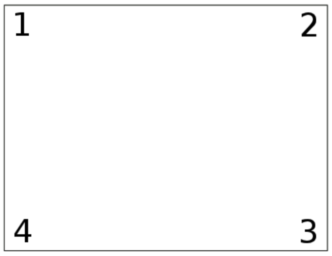
$$p_{i,i+1} = 1 - \frac{i}{2k} \;\; p_{i,i-1} = \frac{i}{2k},$$

and zero otherwise.

(b) All states communicate and are essential so the chain is irreducible.

(c) A calculation (do it!) shows that if $\mathbf{x}$ is the vector of binomial probabilities then $\mathbf{x}P = \mathbf{x}$. So inductively, $\mathbf{x}P^n = \mathbf{x}$ for all $n = 0, 1, 2, \ldots$ so the distribution of X_n is the binomial distribution.

3. A spider lives in a rectangular box with side lengths 3 and 4 centimeters. The spider sits in one of the four corners, marked with numbers $1, 2, 3$ and 4 as illustrated below.



From time to time the spider runs from the corner it's in to another corner, chosen at random with probabilities inversely proportional to the distance from the current corner that the spider occupies. Denote by X_n the number of the corner the spider occupies after the nth change of corner.

(a) Find the transition matrix P for the Markov chain X_n.

(b) Analyze the state space of the chain: communication classes, essentiality, reducibility.

(c) Assume that initially the spider is dropped into the center of the box and chooses a corner uniformly at random. What is $P(X_1 = 1, X_2 = 2, X_4 = 2)$?

Ans.

(a) Starting with corner 1, the problem specifies that for some constant C,

$$p_{1,2} = C/4, \;\; p_{1,3} = C/5, \;\; p_{1,4} = C/3.$$

Since these probabilities must sum to one, we find $C = (3 \cdot 4 \cdot 5)/47$ and so the transition matrix is

$$P = \begin{bmatrix} 0 & 15/47 & 12/47 & 20/47 \\ 15/47 & 0 & 20/47 & 12/47 \\ 12/47 & 20/47 & 0 & 15/47 \\ 20/47 & 12/47 & 15/47 & 0 \end{bmatrix}.$$

(b) Since all states communicate, the chain is irreducible; that is, having one essential communicating class.

(c) Setting $\pi^0 = (1/4, 1/4, 1/4, 1/4)$, we want

$$(\pi^0 P)_1 p_{1,2} (P^2)_{2,2} = \left(\frac{1}{4} \sum_{i=1}^{4} p_{i,1} \right) \frac{15}{47} \left(\sum_{i=1}^{4} p_{2,i} p_{i,2} \right) = \left(\frac{1}{4} \right) \left(\frac{15}{47} \right) \frac{15^2 + 20^2 + 12^2}{47^2}.$$

4. A time *inhomogeneous* Markov chain (X_n) has one step transition matrix at the nth step given by $P(n)$:

$$(P(n))_{i,j} = P(X_n = j | X_{n-1} = i).$$

Show that

$$P(X_{n+m} = j | X_n = i) = (P(n+1)P(n+2) \cdots P(n+m))_{i,j}. \qquad (1)$$

Ans. Fix n and use induction on m. For $m = 1$, (1) is true by definition. Assume (1) is true for some $m \geq 1$ and we show it's also true for $m + 1$. The Chapman-Kolmogorov equations (or really just the law of total probability plus the Markov property) imply

$$P(X_{n+m+1} = j | X_n = i) = \sum_{k \in S} P(X_{n+m} = k | X_n = i) P(X_{n+m+1} = j | X_{n+m} = k),$$

and now using the induction hypothesis (1) and the definition of $P(n)$ we simplify

$$\begin{aligned} P(X_{n+m+1} = j | X_n = i) &= \sum_{k \in S} (P(n+1)P(n+2) \cdots P(n+m))_{i,k} (P(n+m+1))_{k,j} \\ &= (P(n+1)P(n+2) \cdots P(n+m)P(n+m+1))_{i,j}; \end{aligned}$$

the last equality is just the formula for matrix multiplication and this is what we wanted to show.

5. Let $(X_n)_{n \geq 1}$ be a Markov chain with state space $\{1, \ldots, k\}$ for some $k \geq 1$. Show that if i and j communicate, then the probability that the chain started in state i reaches state j in k steps or fewer is greater than 0.

Ans. We show that there is a path $i \to i_1 \to \cdots \to i_l \to j$ of distinct states such that $p_{ii_1} p_{i_1 i_2} \cdots p_{i_l j} > 0$ and $l < k - 1$; this quantity is the chance the chain travels this path from i to j. This implies the theorem since the event the chain takes this path is a subset of the event the chain goes from i to j in k steps or fewer and so the product of transition probabilities $p_{ii_1} \cdots p_{i_l j}$ lower bounds the probability that the chain started in state i reaches state j in k steps or fewer.

Since i and j communicate, there is a path $i \to j_1 \to \cdots \to j_m \to j$ such that $p_{ij_1} \cdots p_{j_m j} > 0$. If $j_s = j_t$ for some $s < t$, then we can shorten the path and remove this loop and still find $p_{ij_1} \cdots p_{j_{s-1} j_s} p_{j_t j_{t+1}} \cdots p_{j_m j} > 0$, a path with positive probability. Continuing in this way, we remove all loops from the original path and end up with a path where each state appears at most once, yielding a path $i \to i_1 \to \cdots \to i_l \to j$ with $l < k - 1$ having the properties desired above.