Tutorial 5: Vector Spaces

Q1. Are the following sets vector subspaces of the given real vector spaces? If so, prove your answer carefully. Otherwise find a counter example to show why not.

(a) In $\mathbb{R}^2$, $M = \{(t, t^2) : t \in \mathbb{R}\}$.
(b) In $\mathbb{R}^2$, $N = \{(x, y) \in \mathbb{R}^2 : x \geq 0\}$.
(c) In $\mathbb{R}^3$, $P = \{(x, y, z) \in \mathbb{R}^3 : x - 4y + 3z = 0\}$.
(d) In $M_{3,3}$, Q the set of all 3×3 matrices with zero entries along the main diagonal.
(e) In $\mathcal{P}_2$, $S = \{a_0 + a_1x + a_2x^2 \in \mathcal{P}_2 : \ a_1a_2 = 0\}$.
(f) In $\mathcal{F}(\mathbb{R}, \mathbb{R})$, T, the set of all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(4) = 0$.

Q2. Are the following sets vector subspaces of the given vector spaces (with scalars as indicated)? If so, prove your answer carefully. Otherwise find a counter example to show why not.

(a) In $\mathbb{R}^2$ (over $\mathbb{R}$), $W_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 0\}$.
(b) In $\mathbb{F}_2^2$ (over $\mathbb{F}_2$), $W_2 = \{(x, y) \in \mathbb{F}_2^2 : x^2 + y^2 = 0\}$.
(c) In $\mathbb{C}^2$ (over $\mathbb{C}$), $W_3 = \{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 0\}$.
(d) In $\mathbb{C}^2$ (over $\mathbb{C}$), $W_4 = \{(x, y) \in \mathbb{C}^2 : \mathrm{Im}(x) = \mathrm{Im}(y) = 0\} = \mathbb{R}^2$.

Q3. For each of the following, determine whether the vectors (i) span the given vector space, (ii) are linearly independent, (iii) form a basis for the given vector space.

(a) $\{(1, 3, 0, 2, \pi), (-2, 1, 0, -\pi, 1)\} \subseteq \mathbb{R}^5$
(b) $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\} \subseteq \mathbb{R}^3$
(c) $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\} \subseteq \mathbb{Z}_2^3$

Q4. (a) Determine whether $\mathcal{B} = \{2 - x, x^4, 3x^3 - 7x + 1, x^2, 1 - x^4, 2 + x^3 - x^4\}$ forms a basis for $\mathcal{P}_4$.

(b) Determine whether $\mathcal{B}' = \left\{ \begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \right\}$ forms a basis for the vector space of all symmetric 2×2 matrices:

$$S = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} \subseteq M_{2,2}$$

(c) Is the set $\{(1 + i, 3 + i), (3 - i, 5 - 5i)\}$ a basis for $\mathbb{C}^2$?

Q5. Consider the vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$ of all functions from the real numbers to the real numbers.

Vectors $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ are linearly dependent if there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that for all $x \in \mathbb{R}$

$$\alpha_1\mathbf{f}_1(x) + \alpha_2\mathbf{f}_2(x) + \cdots + \alpha_n\mathbf{f}_n(x) = \mathbf{0}(x) = 0 \qquad (1)$$

where at least one of α_i is not zero.

Are the following sets of vectors in $\mathcal{F}(\mathbb{R}, \mathbb{R})$ linearly dependent or independent?

(a) $\{\cos x, \sin x\}$ (Hint: Choose two different values of x and solve equation (1).)
(b) $\{\cos x, \sin x, 1\}$
(c) $\{\cos^2 x, \sin^2 x, 1\}$