THE UNIVERSITY OF MELBOURNE

Sequential Model

Semester 1, 2021

Ling Luo

Outline

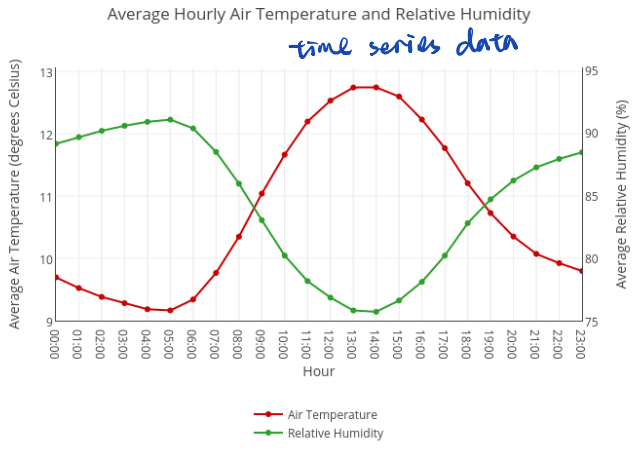
- Introduction
- Hidden Markov Models
- Applications

Structured Classification

sequential structure
hierarchical structure
graph structure

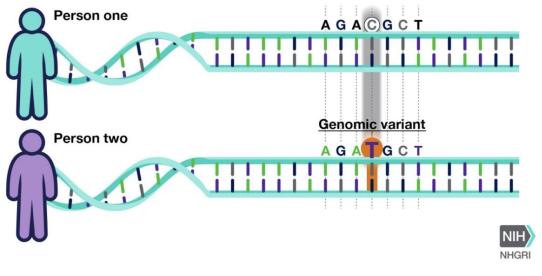
- To date, we have always considered each instance independently, but in many tasks, there is "structure" between instances
  - **Sequential structure**: time series analysis, speech recognition, genomic data

order matters!



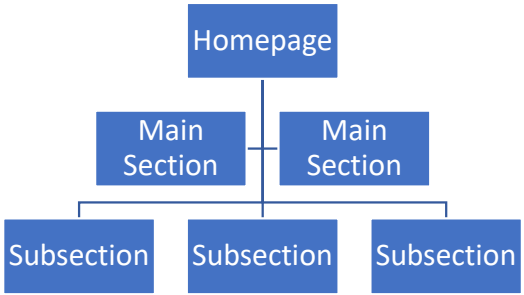
Source: https://chart-studio.plotly.com/~chloecrossman/152/

spatial sequential structure

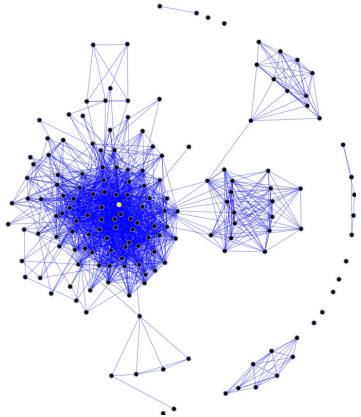


Source: https://www.genome.gov/Health/Genomics-and-Medicine/Polygenic-risk-scores

Structured Classification



Hierarchical structure



Social network graph

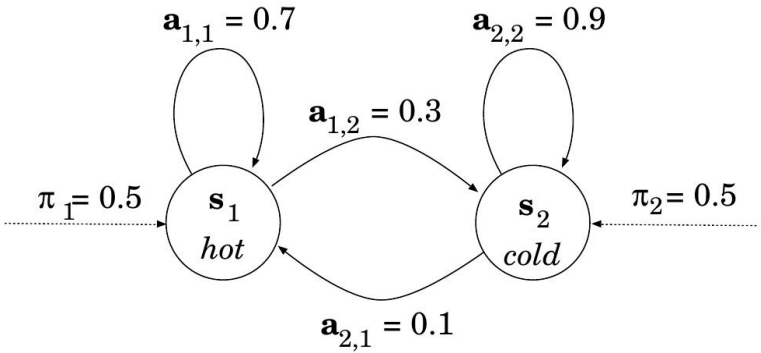
- **Hierarchical structure**: classifying web pages within a website
- **Graph structure**: deriving an influence matrix for a social network

- This calls for **structured classification** models which are able to capture the interaction between instances

Figure source (right): https://commons.wikimedia.org/w/index.php?curid=6057981

Markov Chains

- A Markov chain describes the system that transits from one state to another according to certain probabilistic rules
  - **States**: a set $S = \{s_i\}$
  - **Initial state distribution**: $\Pi = \{\pi_i\}, \sum_i \pi_i = 1$
  - **Transition probability matrix**: $\mathrm{A} = \{a_{ij}\}, \sum_j a_{ij} = 1 \text{ for } \forall i$



$$S = \{\text{hot}, \text{cold}\}$$

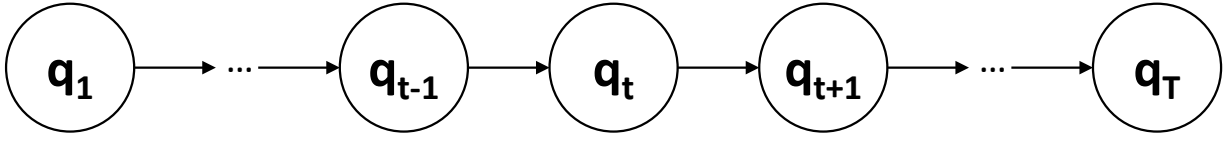
$$\mathrm{A} = \begin{bmatrix} 0.7 & 0.3 \\ 0.1 & 0.9 \end{bmatrix} \quad \pi = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

sum of row = 1

Markov Chains

Markov chains assumption

- For a sequence of states $q_1, q_2, \dots, q_{T-1}, q_T$ at steps $t = 1,2, \dots, T$



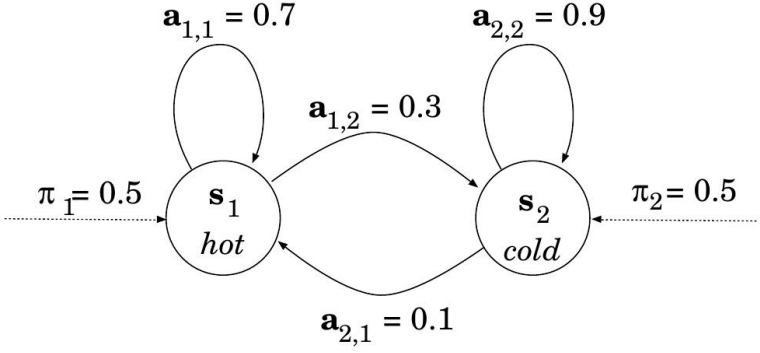
- State q_t *only depends* on the immediately preceding state q_{t-1}

$$P(q_t|q_1, \dots, q_{t-1}) = P(q_t|q_{t-1})$$

memoryless property of Markov chain

Markov Chains

- What is the probability of observing *hot, hot, cold*?



$$S = \{\text{hot}, \text{cold}\}$$

$$A = \begin{bmatrix} 0.7 & 0.3 \\ 0.1 & 0.9 \end{bmatrix} \quad \pi = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$P(q_1 = hot, q_2 = hot, q_3 = cold)$$

conditional independent

$$= P(q_3 = cold | q_2 = hot, q_1 = hot) P(q_2 = hot, q_1 = hot)$$

$$= P(q_3 = cold | q_2 = hot) P(q_2 = hot | q_1 = hot) P(q_1 = hot)$$

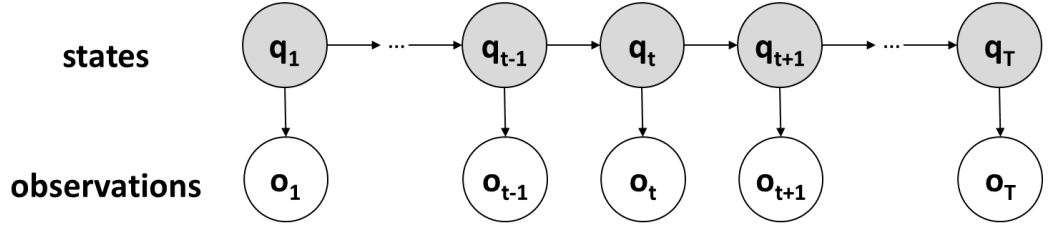
$$= 0.3 \times 0.7 \times 0.5 = 0.105$$

Hidden Markov Models

- Modelling
- Evaluation
- Decoding
- Learning

Hidden Markov Models

- We can see a sequence of observations, but the sequence of states is hidden

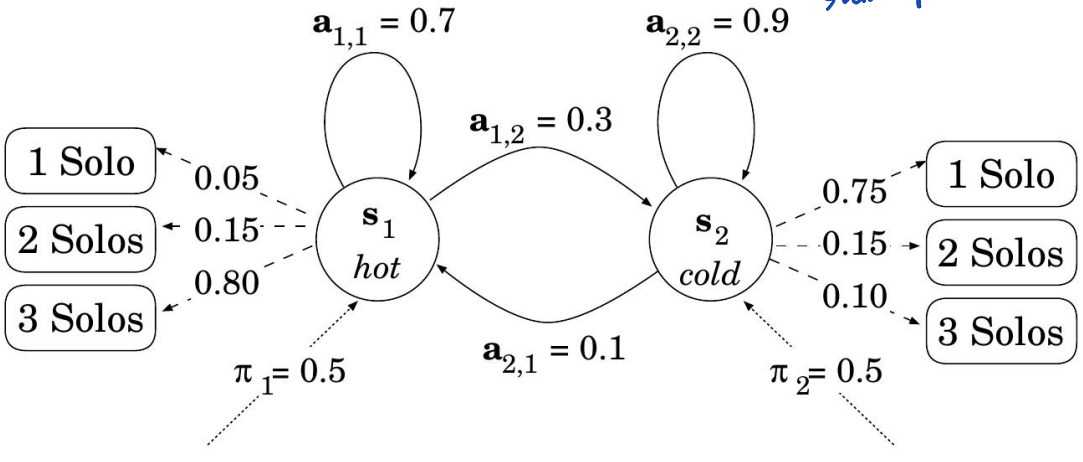


- Hidden Markov Models (HMM): $\mu = (A, B, \Pi)$
  - **States**: a set $S = \{s_i\}$
  - **Observations**: a set $O = \{o_k\}$
  - **Initial state distribution**: $\Pi = \{\pi_i\}, \sum_i \pi_i = 1$
  - **Transition probability matrix**: $A = \{a_{ij}\}, \sum_j a_{ij} = 1 \text{ for } \forall i$
  - **Output probability matrix**: $B = \{b_i(o_k)\}, \sum_k b_i(o_k) = 1 \text{ for } \forall i$

Hidden Markov Models

$$S = \{\text{hot}, \text{cold}\} \qquad A = \begin{bmatrix} 0.7 & 0.3 \\ 0.1 & 0.9 \end{bmatrix} \qquad B = \begin{bmatrix} 0.05 & 0.15 & 0.8 \\ 0.75 & 0.15 & 0.1 \end{bmatrix}$$

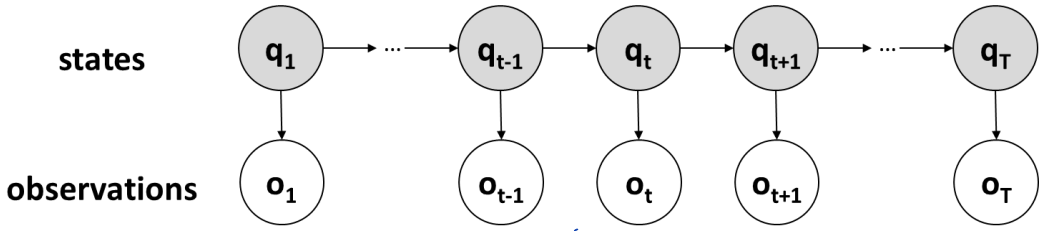
$$\pi = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$



Hidden Markov Models

Assumption
{ Markov assumption
observation only depend on current state.

HMMs make two independence assumptions



state is determined by previous state

$$P(q_t|q_1, \ldots, q_{t-1}, o_1, \ldots, o_{t-1}) = P(q_t|q_{t-1})$$

$$P(o_t|q_1, \ldots, q_{t-1}, o_1, \ldots, o_{t-1}) = P(o_t|q_t)$$

current observation is only determined by current state

Three Tasks

Evaluation : estimate the likelihood of observation sequence.
HMM μ. observation seq Ω $P(\Omega|\mu)$.
Decoding : estimate the hidden state sequence Q
HMM μ. observation seq Ω $P(Q|\mu,\Omega)$.
Learning : estimate parameters of HMM
given Ω. S. O $\Rightarrow$ learn $\mu=(A,B,\Pi)$

- **Evaluation**: estimate the likelihood of an observation sequence (we don't know the state sequence).

model is given

> Given an HMM μ and observation sequence Ω, determine the likelihood $\mathrm{P}(\Omega|\mu)$

- **Decoding**: find the most probable state sequence

model is given

observation sequence infer state sequence.

> Given an HMM μ and observation sequence Ω, determine the most probable hidden state sequence Q

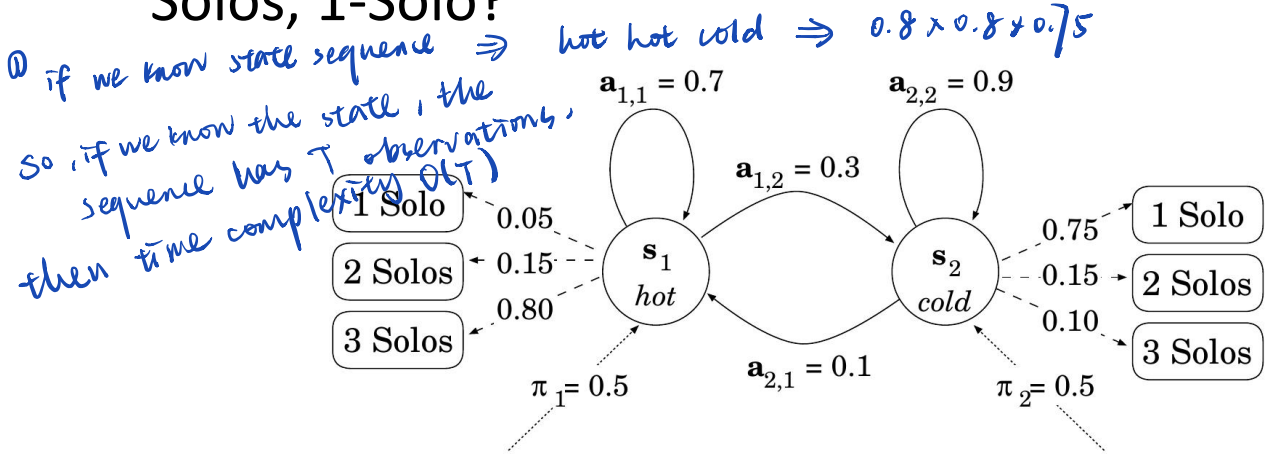
- **Learning**: estimate parameters of HMM

> Given observation sequence Ω, the set of possible states S and observations O in an HMM, learn parameters $\mu = (A, B, \Pi)$

transition, output prob, initial state prob. distribution

Evaluation

- Aim: estimate the likelihood of an observation sequence
- What is the probability of observing 3-Solos, 3-Solos, 1-Solo?



Evaluation

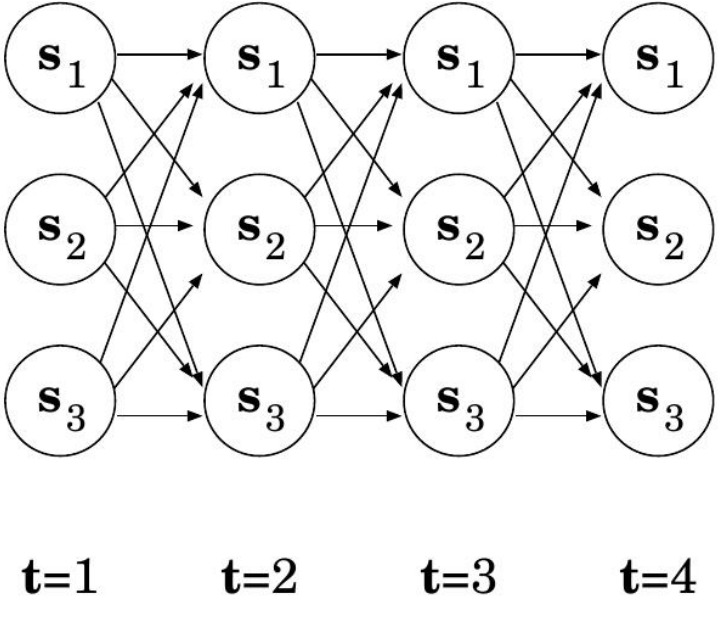
- What is the probability of observing 3-Solos, 3-Solos, 1-Solo?
  - if we know that the states were *hot, hot, cold*
    Easy to calculate: $\mathcal{O}(T)$
  - if we don't know the hidden state sequence
    Harder to calculate: $\mathcal{O}(TN^T)$

    $T = |\Omega|$ is the length of the sequence, and $N = |S|$ is the number of states

N states, T steps
N^T → 确定 state seq
TN^T → map 上 to observation
⇒ each step N state can happen
⇒ the order of T steps is N^T
⇒ also for observation prob
$\mathcal{O}(TN^T)$
↑
T operations
to map from state to observation

Evaluation

- If there are 3 states and 4 time steps



Evaluation

- Probability of the state sequence

$$\mathrm{P}(Q|\mu) = \pi_{q_1} a_{q_1 q_2} a_{q_2 q_3} \dots a_{q_{T-1} q_T}$$

transition prob.

- Probability of observation sequence Ω for state sequence Q:

$$\mathrm{P}(\Omega|Q,\mu) = \prod_{t=1}^{T} P(o_t|q_t,\mu)$$

$T \rightarrow$ length of sequence

- Probability of a given observation sequence Ω considering all possible state sequences:

$$\mathrm{P}(\Omega|\mu) = \sum_{Q} \mathrm{P}(\Omega|Q,\mu)\,\mathrm{P}(Q|\mu)$$

given state sequence

sum up for all state sequences

find the observation sequence

given model to find state sequence

$\mathcal{O}(TN^T)$

The Forward Algorithm

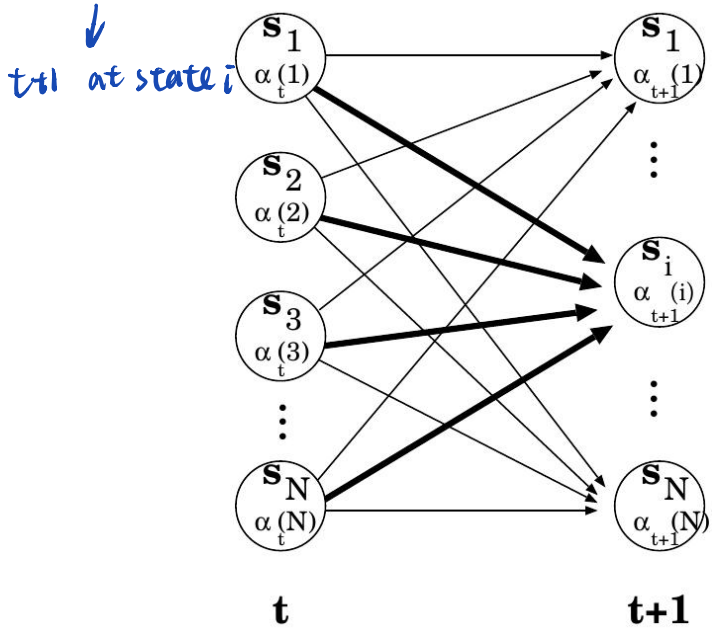
- Efficient computation of total probability $\mathrm{P}(\Omega|\mu)$ through dynamic programming
- Probability of the first t observations is the same for all possible $t+1$ length sequences
- Define **forward probability**: the probability of the partial observation sequence, $o_1, o_2, \ldots, o_t$ and state s_i at time t, given the model μ

$$\alpha_t(i) = P(o_1, o_2, \ldots, o_t, q_t = s_i|\mu)$$

- By caching forward probabilities, we can avoid redundant calculations. $\mathcal{O}(TN^2)$.

The Forward Algorithm

- Store forward probabilities $\alpha_t(j)$ and reuse them to compute $\alpha_{t+1}(i)$



The Forward Algorithm

- **Initialisation**: $t = 1$, state $i = 1, \ldots, N$

observation

$$\alpha_1(i) = \pi_i b_i(o_1)$$

- **Induction**: $t = 1, \ldots, T-1$, state $i = 1, \ldots, N$

repeat for T-1 steps

each step N transition

$O(N^2)$

another

from N to N

states states

$$\alpha_{t+1}(i) = \left(\sum_{j=1}^{N} \alpha_t(j) a_{ji} \right) b_i(o_{t+1})$$

$t+1 \ldots T$

N possible transitions

total $O(N^2 T)$

- **Termination**

$\alpha_T(i) = P(o_1 \cdots o_T, q_T = s_i | \mu)$

$$P(\Omega|\mu) = \sum_{i=1}^{N} \alpha_T(i)$$

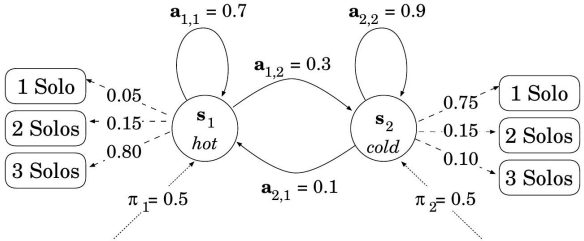
observation sequence

at step T

overall prob

The Forward Algorithm: Example

- P(3−Solos, 3−Solos, 1−Solo|μ)?



- Initialisation/induction: $\alpha_t(i) = P(o_1, o_2, \ldots, o_t, q_t = s_i|\mu)$

	$t = 1$	$t = 2$	$t = 3$
$\alpha_t(hot)$	$0.5 \times 0.8 = 0.4$	$[0.4 \times 0.7 + 0.05 \times 0.1] \times 0.8 = 0.228$	$[0.228 \times 0.7 + 0.0165 \times 0.1] \times 0.05 = 0.0080625$
$\alpha_t(cold)$	$0.5 \times 0.1 = 0.05$	$[0.4 \times 0.3 + 0.05 \times 0.9] \times 0.1 = 0.0165$	$[0.228 \times 0.3 + 0.0165 \times 0.9] \times 0.75 = 0.0624375$

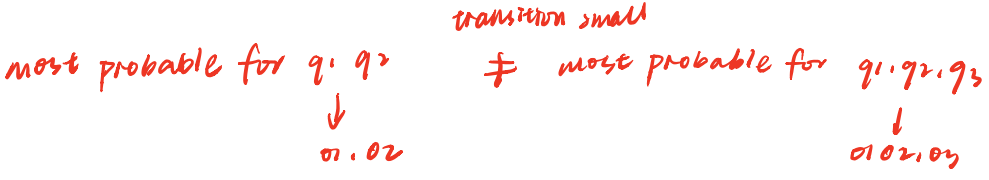
- Termination:

$$P(3-Solos, 3-Solos, 1-Solo|\mu) = 0.0080625 + 0.0624375 = 0.0705$$

Decoding

- Aim: find the most probable state sequence
- Given the observation 3-Solos, 3-Solos, 1-Solo, what is the most probable weather sequence?
  - Use the most probable state for each step?

→ No, cannot guarantee to find the most probable *sequence*, as the transition probability from the most likely state at $t-1$ to the most likely state at t can be small.



Decoding

- Aim: find the most probable state sequence
- Given the observation 3-Solos, 3-Solos, 1-Solo, what is the most probable weather sequence?
  - Brute-force: enumerate the probabilities of all hidden state sequences and sort, $\mathcal{O}(TN^T + N^T log N^T)$

enumerate: TN^T map to observation prob

sort $N^T \log N^T$

compute probabilities

sort

  - More efficient method: Viterbi algorithm

$O(TN^2)$

The Viterbi Algorithm

- Preliminaries: define some variables
  - The maximum probability for a partial sequence $(q_1, q_2, \dots, q_{t-1}, q_t = s_i)$ along a single path:

$$\delta_t(i) = \max_{q_1, q_2, \dots, q_{t-1}} P(q_1, q_2, \dots, q_{t-1}, o_1, o_2, \dots, o_t, q_t = s_i | \mu)$$

  - Keep track of the path: for the most probable partial sequence $(q_1, q_2, \dots, q_{t-1}, q_t = s_i)$, the hidden state at $t-1$ is $\psi_t(i)$

The Viterbi Algorithm

- **Initialisation**: $t = 1$, state $i = 1, \ldots, N$

$$\delta_1(i) = \pi_i b_i(o_1)$$

no preceding value

$$\psi_1(i) = 0$$

- **Induction**: $t = 1, \ldots, T-1$, state $i = 1, \ldots, N$

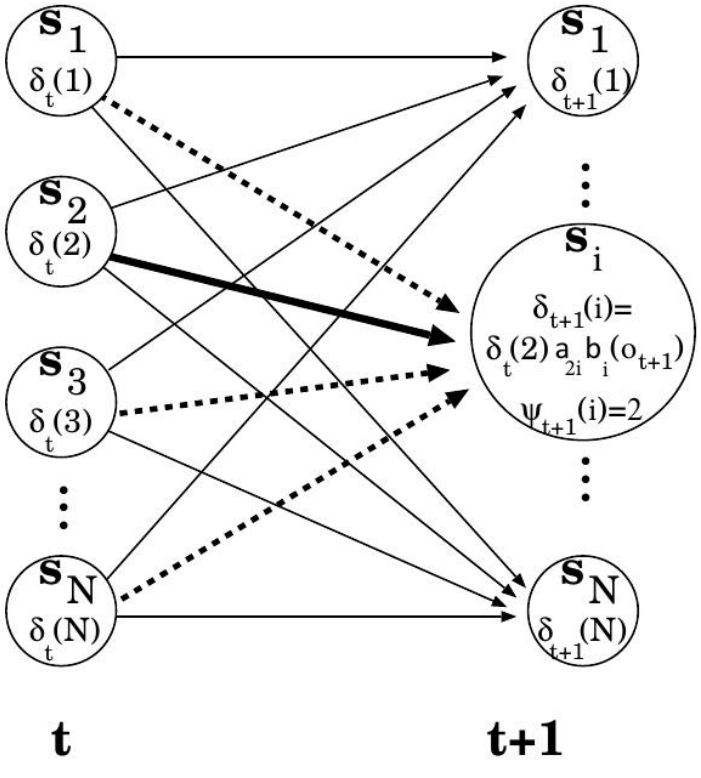
$$\delta_{t+1}(i) = \max_{1 \le j \le N} (\delta_t(j) a_{ji})\, b_i(o_{t+1})$$

$$\psi_{t+1}(i) = \arg\max_{1 \le j \le N} (\delta_t(j) a_{ji})$$

save the previous argmax state.

The Viterbi Algorithm

- **Induction**



The Viterbi Algorithm

- **Termination**

$$P_{best} = \max_{1 \le i \le N} \delta_T(i)$$

$$q_T^* = \arg\max_{1 \le i \le N} \delta_T(i)$$

- **Backtrack** to establish the best path

$$q_t^* = \psi_{t+1}(q_{t+1}^*) \text{ for } t = T-1, T-2, \ldots, 1$$

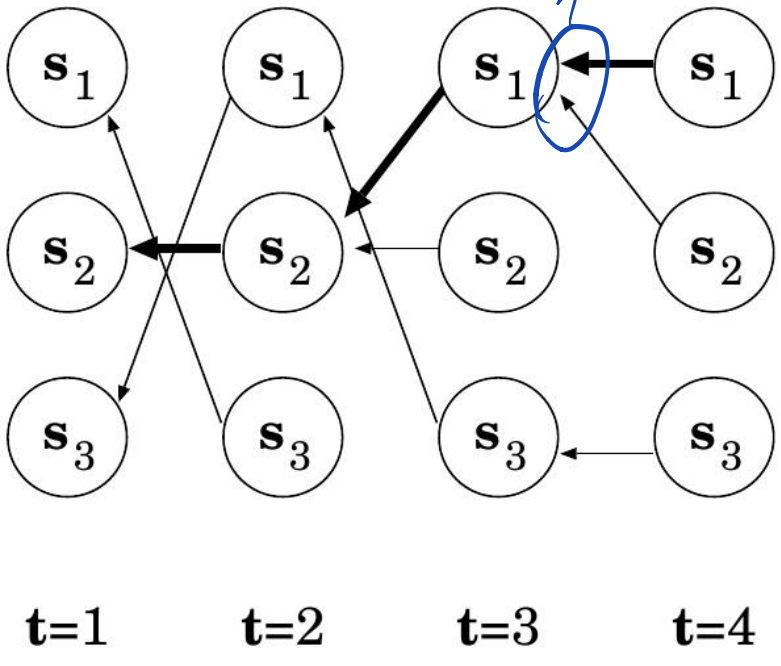
since $\psi_{t+1}(q^*_{t+1})$ records the best selection at t

so we backtrack

The Viterbi Algorithm

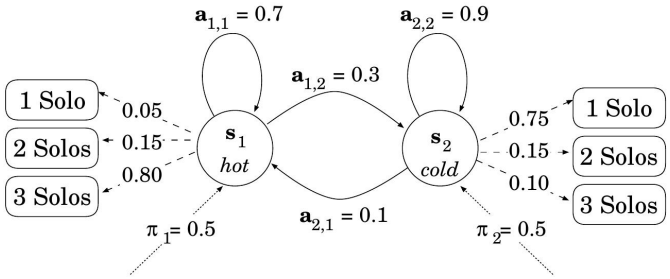
- **Backtrack**

two states can have the same preceding state



The Viterbi Algorithm: Example

- Most probable state sequence, given the observation sequence: 3−Solos, 3−Solos, 1−Solo



- Initialisation/induction:

	$t = 1$	$t = 2$	$t = 3$
$\delta_t(hot)$	$0.5 \times 0.8 = 0.4$	$\max(0.4 \times 0.7, 0.05 \times 0.1) \times 0.8 = 0.224$	$\max(0.224 \times 0.7, 0.012 \times 0.1) \times 0.05 = 0.00784$
$\psi_t(hot)$	0	$\leftarrow hot$	$\leftarrow hot$
$\delta_t(cold)$	$0.5 \times 0.1 = 0.05$	$\max(0.4 \times 0.3, 0.05 \times 0.9) \times 0.1 = 0.012$	$\max(0.224 \times 0.3, 0.012 \times 0.9) \times 0.75 = 0.0504$
$\psi_t(cold)$	0	$\nwarrow hot$	$\nwarrow hot$

hot → hot cold → hot

Final step

larger backtrace

The Viterbi Algorithm: Example

- Termination/backtracking

$$P_{best} = 0.0504$$

$$q_T^* = cold$$

$$q_{T-1}^* = hot$$

$$q_{T-2}^* = hot$$

- The most probable sequence of hidden states for the observation sequence (3-Solos, 3-Solos, 1-Solo) is *hot, hot, cold*

Learning HMMs

? $P(y|w,s)$?
state, observation have been labelled

- **Supervised** case: assume we have labelled data, it is possible to use simple MLE to learn the parameters of our model.

$$a_{ij} = P(s_j|s_i) = \frac{freq(s_i, s_j)}{freq(s_i)}$$

$$b_i(o_k) = P(o_k|s_i) = \frac{freq(o_k, s_i)}{freq(s_i)}$$

$$\pi_i = P(q_1 = s_i) = \frac{freq(q_1 = s_i)}{\sum_j freq(q_1 = s_j)}$$

- No state labels? **Unsupervised** case - uses forward-backward algorithm (Baum-Welch algorithm, out of scope).

Reflections

- HMM is a highly efficient approach to structured classification, but with limited representation of context (only observation and state sequences)
- HMM tends to suffer from floating point underflow
  - use scaling coefficients for Forward Algorithm
  - use logs for Viterbi Algorithm

multiple sequences of state can be included

can't use log function in this case since we need have summation in this case.

use maximisation and multiplication

log is useful for multiplication and exponential

Applications

Applications of Sequence Labelling

- **Parts-of-speech Tagging**
  - **INPUT**: Profits soared at Boeing Co., easily topping forecasts on Wall Street, as their CEO Alan Mulally announced first quarter results.
  - **OUTPUT**:

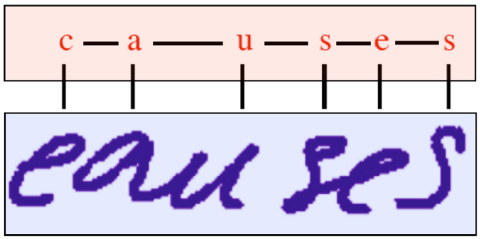
    Profits/N soared/V at/P Boeing/N Co./N,/, easily/ADV topping/V forecasts/N on/P Wall/N Street/N,/, as/P their/POSS CEO/N Alan/N Mulally/N announced/V first/ADJ quarter/N results/N./.

    Labels: N = noun; V = verb; P = preposition; ADV = adverb...

Source: Collin 2013

Applications of Sequence Labelling

- **Optical Character Recognition**



Source: Hofmann 2013

Summary

- What is structured classification? structure between instance, not independently
- How do we evaluate HMMs?
- How do we decode HMMs?
- How do we learn HMMs given labelled training data?
- What are the limitations of HMMs?

References

- Philip Blunsom, *Structured Classification for Multilingual Natural Language Processing*, PhD thesis, University of Melbourne, 2007.
- Michael Collin, *Tagging Problems, and Hidden Markov Models.* 2013. http://www.cs.columbia.edu/~mcollins/hmms-spring2013.pdf.
- Thomas Hofmann, *Structured Prediction (with application in information retrieval)*, 2009. http://mlg.eng.cam.ac.uk/mlss09/mlss_slides/Hoffman_1_2.pdf.
- Lawrence R. Rabiner, *A tutorial on Hidden Markov Models and selected applications in speech recognition*, Proceedings of the IEEE, 77(2), 1989.

Question 4

1 / 1 pts

Given an HMM with N states, for the Forward Algorithm, what is the definition of the forward probability $\alpha_t\ (i)$? How many times a variable $\alpha_t\ (i)$ will be reused at time step $t + 1$?

- $\alpha_t\ (i) = P(q_1, q_2, \ldots, q_t = s_i \mid \mu);\ N^2$
- $\alpha_t\ (i) = P\ (q_1, q_2, \ldots, q_t = s_i \mid \mu);\ N$

Correct!

- $\alpha_t\ (i) = P\ (o_1, o_2, \ldots, o_t, q_t = s_i \mid \mu);\ N$
- $\alpha_t\ (i) = P\ (o_1, o_2, \ldots, o_t, q_t = s_i \mid \mu);\ N^2$

Forward probability is the probability of the partial observation sequence, $o_1,\ o_2,\ \ldots,\ o_t$, and state is s_i at time t. Each term is reused N times for computing N terms of $\alpha_{t+1}\ (i)$. Reusing forward probability can save computation time.