# MAST20009 Vector Calculus

## Practice Class 10 Questions

<u>Conservative vector fields</u>

Let $\boldsymbol{F}$ be a $C^1$ vector field defined on $\mathbb{R}^2$ or $\mathbb{R}^3$. The following conditions are all equivalent:

(a) For any oriented simple closed curve $C$, $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{s} = 0$.

(b) For any two oriented simple curves $C_1$ and $C_2$ with the same endpoints, $\displaystyle\int_{C_1} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{s}$

(c) $\boldsymbol{F} = \boldsymbol{\nabla}\phi$ for some scalar function $\phi$.

(d) $\boldsymbol{\nabla}\times\boldsymbol{F} = \boldsymbol{0}$.

A vector field satisfying one (and hence all) of the four conditions is called a conservative vector field.

1. Let
$$\boldsymbol{F}(x,y,z) = \frac{2}{\pi}x\sin(\pi y)\boldsymbol{i} + (x^2\cos(\pi y) - 2ye^{-z})\boldsymbol{j} + y^2e^{-z}\boldsymbol{k}.$$

(a) Show that $\boldsymbol{F}$ is a conservative vector field.

(b) Find a scalar function $\phi(x,y,z)$ such that $\boldsymbol{F} = \boldsymbol{\nabla}\phi$.

(c) Evaluate
$$\int_C \boldsymbol{F}\cdot d\boldsymbol{s}$$
where $C$ is the curve traced out by the path
$$\boldsymbol{c}(t) = (\cos t, \cos t, \sin^2 t), \quad 0 \le t \le \frac{\pi}{2}.$$

(d) Determine the work done by $\boldsymbol{F}$ to move a particle around the parallelogram $P$ with vertices $(-1,1), (0,2), (3,1), (4,2)$.

(a) $\nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{2}{\pi}x\sin(\pi y) & x^2\cos(\pi y) - 2ye^{-z} & y^2e^{-z} \end{vmatrix}$

$= i\,(2ye^{-z} - 2ye^{-z}) - j\,(0-0) + k\,(\cos(\pi y)\cdot 2x - 2x\cos(\pi y))$

$= \underline{0}$.

$F$ is $C^1$ in $R^3$.

(b). $\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$.

$\frac{\partial\phi}{\partial x} = \frac{2}{\pi}x\sin(\pi y) \Rightarrow \phi = \frac{x^2}{\pi}\sin\pi y + C_1(y,z)$

$\frac{\partial\phi}{\partial y} = x^2\cos(\pi y) - 2ye^{-z} \Rightarrow \phi = \frac{x^2}{\pi}\sin\pi y - y^2e^{-z} + C_2(x,z)$

$\frac{d\phi}{dz} = y^2e^{-z} \Rightarrow \phi = -y^2e^{-z} + C_3(x,y)$

combine: $\phi = \frac{x^2}{\pi}\sin(\pi y) - y^2e^{-z} + C$.

(c). since $F$ is conservative field and $F = \nabla\phi$

$\int_C F\cdot ds = \phi(\text{end}) - \phi(\text{start})$

$t=0$

$C(0) = (1,1,0)$

$t=\frac{\pi}{2}$

$C(\frac{\pi}{2}) = (0,0,1)$

$\phi(1,1,0) = \frac{1}{\pi}\cdot 0 - 1 = -1$

$\phi(0,0,1) = 0$.

$\int_C F\cdot ds = 0-(-1) = 1$.

(d) since $E$ is conservative field and parallelogram $P$ is closed curve

then $\int_C E\cdot ds = 0$

> **Gauss' Divergence Theorem**
>
> $$\iiint_{\Omega} \boldsymbol{\nabla}\cdot\boldsymbol{F}\,dV \;=\; \iint_{\partial\Omega} \boldsymbol{F}\cdot d\boldsymbol{S}$$
>
> where
>
> - $\Omega$ is a solid region in space.
> - $\partial\Omega$ is the oriented closed surface that bounds $\Omega$.
> - $\boldsymbol{F}$ is a $C^1$ vector field on $\Omega$.
> - Orientation is defined by the unit outward normal $\hat{\boldsymbol{n}}$ to $\partial\Omega$.

2. Using Gauss' Divergence Theorem, evaluate

$$\iint_{\partial\Omega} \boldsymbol{F}\cdot d\boldsymbol{S}$$

where $\boldsymbol{F}(x,y,z) = 2\boldsymbol{i} + y^2\boldsymbol{j} + z^2\boldsymbol{k}$ and $\Omega$ is the solid sphere

$$x^2 + y^2 + (z-3)^2 \le 9.$$

$\iiint_\Omega \nabla\cdot E\,dV = \iint_{\partial\Omega} E\cdot ds$

$\nabla\cdot F = 0+2y+2z$

$= 2y+2z.$

use spherical coordinate.

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$,

$z = r\cos\theta + 3$ $\quad r\in[0,3]$

$\nabla\cdot F = 2r\sin\theta\sin\phi + 2r\cos\theta + 6.$ $\quad \theta\in[0,\pi]$

$\phi\in[0,2\pi]$

$\int_0^{2\pi}\int_0^{\pi}\int_0^{3} (2r\sin\theta\sin\phi + 2r\cos\theta + 6)\,r^2\sin\theta\; dr\,d\theta\,d\phi$

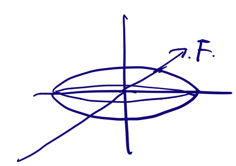
Jacobian

> **Volume via surface integrals**
>
> If $\boldsymbol{F}(x,y,z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ and $\Omega$ is a region to which Gauss' theorem applies, then
>
> $$\text{Volume of }\Omega \;=\; \frac{1}{3}\iint_{\partial\Omega} \boldsymbol{F}\cdot d\boldsymbol{S}.$$

3. Show that the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad a>0,\; b>0,\; c>0$$

is $\dfrac{4\pi abc}{3}$.




$F(x,y,z) = x\,i + y\,j + z\,k.$

ellipsoid is closed solid region

$V = \frac{1}{3}\iint_{\partial\Omega} F\cdot dS$

let $x = a\sin\theta\cos\phi$, $y = b\sin\theta\sin\phi$, $z = c\cos\theta$ $\quad \theta\in[0,\pi]$, $\phi\in[0,2\pi]$

**When you have finished the above questions, continue working on the questions in the Vector Calculus Problem Sheet Booklet.**

$T_\theta = (a\cos\theta\cos\phi,\; b\cos\theta\sin\phi,\; -\sin\theta)$

$T_\phi = (-a\sin\theta\sin\phi,\; b\sin\theta\cos\phi,\; 0).$

$F\cdot dS = F\cdot(T_u\times T_v)$

$T_\theta\times T_\phi = \quad i \quad j \quad k$

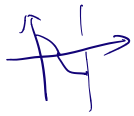
$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} abc\cancel{\sin\theta}\cos^2\theta \, d\theta \, d\phi$$

$$= \frac{abc}{3}\int_0^{2\pi}\int_0^{\pi} \underbrace{-\cancel{\cos^2\theta \, d(\cos\theta)}}_{\downarrow} d\phi$$

$$= \frac{abc}{3}\int_0^{2\pi} \left[\cancel{-\frac{1}{3}\cos^3\theta}\right]_0^{\pi} d\phi.$$

$$= \frac{abc}{3}\int_0^{2\pi} \cancel{\left[\frac{1}{3}+\frac{1}{3}\right]} d\phi$$

$$= \frac{abc}{3}\int_0 \cancel{\frac{4\pi}{3}}$$



$-1$

$$\underline{r_\theta} \times \underline{r_\phi} = \begin{vmatrix} a\cos\theta\cos\phi & b\cos\theta\sin\phi & -c\sin\theta \\ -a\sin\theta\sin\phi & b\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= \underline{i}\,(bc\sin^2\theta\cos\phi) - \underline{j}\left(-ac\sin^2\theta\sin\phi\right)$$

$$+ \underline{k}\left(ab\sin\theta\cos\theta\right).$$

$$= \underline{i}\left(bc\sin^2\theta\cos\phi\right) + \underline{j}\,(ac\sin^2\theta\sin\phi)$$

$$+ \underline{k}\,(ab\sin\theta\cos\theta).$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi} abc\sin^3\theta\underline{\cos^2\phi} + abc\sin^3\theta\underline{\sin^2\phi} + abc\sin\theta\cos^2\theta \, d\theta \, d\phi.$$

$$= \frac{abc}{3}\int_0^{2\pi}\int_0^{\pi} \sin^3\theta + \sin\theta\cos^2\theta \, d\theta \, d\phi$$

$$= \frac{abc}{3}\int_0^{2\pi}\underbrace{\int_0^{\pi}\sin\theta \, d\theta}_{[-\cos\theta]_0^{\pi}} d\phi.$$

$$= \frac{abc}{3}\cdot 2\pi\cdot 2 = \frac{4abc\pi}{3}$$