1 Matrices and linear equations

1.1 Systems of linear equations

An equation is *algebraic* if it involves only constants, variables, and algebraic operations (addition, subtraction, multiplication). An algebraic equation is *linear* if each variable occurs to the power one only. A *linear system* is a set of simultaneous linear equations.

The equation

$$2x + y = 5$$

describes a line in the plane $\mathbb{R}^2$. It has solution set

$$x = t, \quad y = 5 - 2t$$

with one parameter $t \in \mathbb{R}$. This can also be written

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 5 - 2t \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix} + t \begin{bmatrix} \\ \end{bmatrix}, \qquad t \in \mathbb{R}.$$

A system of two equations such as

$$\begin{cases} 2x + y = 5 \\ x - y = 1 \end{cases}$$

describes two lines in the plane. These can be

- non-intersecting and parallel:

- intersecting uniquely:

- intersecting in more than one point:

If the two lines don't intersect, we call the system *inconsistent*; otherwise, it is *consistent*.

A system can have many equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

This can be written in matrix form

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The *augmented matrix* of the system puts A and $\mathbf{b}$ together:

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

Example 1.1. Put the following system in augmented form:

$$\begin{cases} 2x + y = 5 \\ x - y = 1 \end{cases}$$

We solve a linear system by replacing it with another linear system that has the same solution set and is easier to solve. This is done via *elementary row operations* on the augmented matrix:

(a) Multiply a row by a nonzero constant: $R \leftarrow \lambda R$, $\lambda \neq 0$.

(b) Swap two rows: $R \leftrightarrow S$.

(c) Add a multiple of one row to another: $R \leftarrow R + \mu S$.

Example 1.2.

$$\begin{cases} x + y + z = 2 \\ 2x + y \quad = 4 \\ x - y - z = 0 \end{cases}$$

We do some elementary row operations on the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 4 \\ 1 & -1 & -1 & 0 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} & & & \\ \\ \\ \end{array}\right] \xrightarrow{R_2 \leftarrow (-R_2)} \left[\begin{array}{ccc|c} & & & \\ \\ \\ \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + 2R_2} \left[\begin{array}{ccc|c} & & & \\ \\ \\ \end{array}\right]$$

This corresponds to a new linear system:

$$\begin{cases} x + y + z = 2 \\ \phantom{x +{}} y + 2z = 0 \\ \phantom{x + y +{}} 2z = -2 \end{cases}$$

This system is very easy to solve because it is *triangular*. We start with the last equation

$$2z = -2 \quad \Rightarrow \quad z =$$

and substitute into the second-to-last equation, and so on

You should check that this is a solution of the original system

$$\begin{cases} x + y + z = 2 \\ 2x + y \quad = 4 \\ x - y - z = 0 \end{cases}$$

1.2 Elimination and echelon forms

We reduced the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 4 \\ 1 & -1 & -1 & 0 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

and we can do one more row operation ($R_3 \leftarrow \frac{1}{2}R_3$) to get

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The final matrix is in *row echelon form*.

A matrix is in *row echelon form (REF)* if

- The leftmost nonzero entry in each nonzero row is 1; we call it a *leading* 1.
- Lower leading 1s appear to the right of higher leading 1s.
- All zero rows are grouped at the bottom of the matrix.

The matrix is in *reduced row echelon form (RREF)* if it also satisfies

- A column that contains a leading 1 has all the entries 0.

Example 1.3 (Some matrices in various states of reduceness)**.**

Gaussian elimination is an algorithm for reducing an (arbitrary) matrix to a matrix in row-echelon form:

Step 1 (If necessary) interchange rows to make the top left entry nonzero.

Step 2 Multiply by the appropriate number to get a leading 1 in the top left entry.

Step 3 Make all entries below this leading 1 into 0s by adding suitable multiples of the first row to the lower rows.

Step 4 Ignore the first row and column of the matrix, and start over from **Step 1**.

One matrix can give rise to many different row echelon forms (depending on choices made during the Gaussian elimination algorithm).

Example 1.4. Use Gaussian elimination to reduce to row echelon form:

$$\begin{bmatrix} -1 & 2 & 1 & -3 & 1 \\ 1 & 0 & 1 & -1 & 3 \\ 3 & 1 & 0 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 2 & 2 & -4 & 4 \\ 0 & 7 & 3 & 6 & 2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 0 & 1 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2 & 3 \end{bmatrix} \text{REF}$$

Gauss–Jordan elimination is an extension that adds a further step to Gaussian elimination, so that the end result is in reduced row echelon form:

Step 5 For each leading 1, make all the entries above it 0 by adding a suitable multiple of this row to the rows above.

The most efficient way to apply **Step 5** is to start at the bottom of the matrix and work toward the top.

There is a unique reduced row echelon form for any given matrix. So the result of Gauss–Jordan elimination is uniquely defined.

arameter in R.
$$\begin{bmatrix} 1 & \boxed{-2} & \boxed{-1} & 3 & -1 \\ 0 & 1 & \boxed{1} & -2 & 2 \\ 0 & 0 & 1 & -2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -1 & 3 \\ 0 & 1 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2 & 3 \end{bmatrix}$$

turn 0

$$\xrightarrow{xxx} \begin{bmatrix} 1 & 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & 3 \end{bmatrix}$$

$$\xrightarrow{xxx} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & 3 \end{bmatrix}$$ RREF.

Example 1.5. Use Gauss–Jordan elimination to reduce to row echelon form:

$$\begin{bmatrix} -1 & 2 & 1 & -3 & 1 \\ 1 & 0 & 1 & -1 & 3 \\ 3 & 1 & 0 & 3 & -1 \end{bmatrix}$$

Example 1.6. Solve the linear system

$$\begin{cases} -x_1 + 2x_2 + x_3 - 3x_4 = 1 \\ x_1 \qquad\quad + x_3 - \ x_4 = 3 \\ 3x_1 + \ x_2 \qquad\quad + 3x_4 = -1 \end{cases}$$

augmented matrix

$$\left[\begin{array}{cccc|c} -1 & 2 & 1 & -3 & 1 \\ 1 & 0 & 1 & -1 & 3 \\ 3 & 1 & 0 & 3 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & 3 \end{array}\right]$$

$$\begin{cases} x_1 + x_4 = 0 \\ x_2 = -1 \\ x_3 - 2x_4 = 3 \end{cases} \Rightarrow \begin{cases} x_1 = -t \quad \text{free p} \\ x_2 = -1 \\ x_3 = 2t+3 \\ x_4 = t \end{cases}$$

? why does this work ?

① solutions from RREF

- ignore zero rows
- for the last row

either $[0 \ \cdots \ | 1]$ $\Rightarrow$ $0=1$ no solution in $\mathbb{R}$

or $[0 \cdots 1 \cdots 0 | ?]$ $\Rightarrow$ if column j has a leading 1, say in row k, then $x_j = $ ____

otherwise x_j is a free parameter

$$\left[\begin{array}{cccc|c} 1 & 5 & 0 & 3 & -1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 + 5x_2 + 3x_4 = -1 \\ x_3 + 2x_4 = 3 \end{cases}$$

② elementary row ops preserve solution

ⓐ $R \leftarrow \lambda R$ $\lambda \neq 0$.

swap. ⓑ $R \leftarrow S$ ⓒ $R \leftarrow R + \mu S$

1.3 Arithmetic with matrices

A *matrix* is a two-dimensional rectangular array of (to start with, real) numbers, called *entries*:

$$\begin{bmatrix} 1 & 2 & 3 \\ \frac{1}{2} & 1.3 & 0 \end{bmatrix}$$

This is a 2×3 matrix, which indicates that it has 2 rows and 3 columns.

A general matrix is often denoted

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

If $m = n$, we say that the matrix is *square*, and we call the entries $a_{11}, a_{22}, \dots, a_{nn}$ the *(main) diagonal*.

A matrix of size $1 \times n$ is a *row matrix*. A matrix of size $n \times 1$ is a *column matrix*.

A *zero matrix* is a matrix all of whose entries are zero.

Example 1.7. $0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

We say that two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are *equal* if they have the same size and the same entries, that is

$$a_{ij} = b_{ij} \qquad \text{for all } i, j.$$

Example 1.8. Suppose

$$A = \begin{bmatrix} 1 & y & 0 \\ -7 & 2 & 3 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 3 & 0 \\ x & 2 & 3 \end{bmatrix},$$

then $A = B$ if

$y = 3$
$x = -7$

Scalar multiplication

Let $A = [a_{ij}]$ be a matrix and λ be a scalar. We define a new matrix $\lambda A = [\lambda a_{ij}]$.

Example 1.9. Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 2 & -7 \end{bmatrix}, \qquad \text{then} \qquad -2A = \begin{bmatrix} 0 & -2 \\ -2 & 2 \\ -4 & 14 \end{bmatrix} \text{and} \qquad cA = \begin{bmatrix} 0 & c \\ c & -c \\ 2c & -7c \end{bmatrix}$$

Addition of matrices

Given two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size, their *sum* is the matrix $A + B = [a_{ij} + b_{ij}]$.

Example 1.10.

$$\begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 3 \end{bmatrix} + \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix} = \text{not defined}$$

Properties of scalar multiplication

$$
\begin{aligned}
(\lambda+\mu)C &= \lambda C + \mu C \\
\lambda(B+C) &= \lambda B + \lambda C \\
\lambda(\mu C) &= (\lambda\mu)C
\end{aligned}
$$

Properties of matrix addition

$$
\begin{aligned}
A+B &= B+A \\
A+(B+C) &= (A+B)+C \\
A-A &= 0 \\
A+0 &= A
\end{aligned}
$$

Matrix multiplication

Suppose $A=[a_{ij}]$ has size $m\times n$ and $B=[b_{ij}]$ has size $n\times p$. Then we define an $m\times p$ matrix $AB=[c_{ij}]$, where

$$c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}.$$

Example 1.11. $(3\times 2)(2\times 1)$ $(2\times 1)(3\times 1)$

$$\begin{bmatrix}1 & -1\\ 3 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}4\\ -7\end{bmatrix}=\begin{bmatrix}11\\ 12\\ -7\end{bmatrix} \qquad \begin{bmatrix}4\\ -7\end{bmatrix}\begin{bmatrix}1 & -1\\ 3 & 0\\ 0 & 1\end{bmatrix}=\text{not defined}$$

Example 1.12.

$$\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}\begin{bmatrix}4 & 3\\ 2 & 1\end{bmatrix}=\begin{bmatrix}4 & 3\\ 14 & 9\end{bmatrix} \qquad \begin{bmatrix}4 & 3\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & 3\end{bmatrix}=\begin{bmatrix}10 & 9\\ 4 & 3\end{bmatrix}$$

So, in general, $AB\neq BA$. We say that A and B *commute* if $AB=BA$. Of course this requires that A and B be square of the same size (but this is not sufficient).

The *identity matrix* I_n is the $n \times n$ matrix with 1s on the diagonal and 0s everywhere else.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Properties of matrix multiplication

$$\begin{aligned} A(B+C) &= AB + AC \\ (A+B)C &= AC + BC \\ A(BC) &= (AB)C \\ \underbrace{AI = IA}_{} &= A \quad \to \text{for all matrices } A \\ A0 = 0A &= 0 \end{aligned}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

0_{21}

Transpose

The *transpose* of an $m \times n$ matrix $A = [a_{ij}]$ is the $n \times m$ matrix $A^T = [a_{ji}]$ obtained by interchanging the rows and columns of A.

Example 1.13.

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 7 & -2 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 7 \\ 2 & -2 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}$$

Properties of transpose

$$\begin{aligned} (A^T)^T &= A \\ (\lambda A)^T &= \lambda(A^T) \quad \lambda \text{ number} \\ (A+B)^T &= A^T + B^T \\ (AB)^T &= B^T A^T \end{aligned}$$

?

$$(ABC)^T = ((AB)C)^T = C^T(AB)^T = C^T B^T A^T$$

Trace

The *trace* of a square matrix A, denoted $\mathrm{tr}(A)$, is the sum of the diagonal entries of A.

Example 1.14.

$$\mathrm{tr}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 0 & -1 & 7 \end{bmatrix} = [1-2+7] = 6$$

trace is a similarity invariant
不变的

1.4 Solving linear systems *redux*

We use what we have learnt about matrices to look at linear systems in a more meaningful way.

Elementary matrices

An $n \times n$ matrix is said to be *elementary* if it can be obtained by applying a single elementary row operation (p. 5) to the identity matrix I_n.

Example 1.15. The matrix

$$E = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \end{matrix}$$

is elementary, resulting from

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R_2 \leftarrow R_2 + 2R_1$$

Example 1.16. What matrix do we obtain from

$$M = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

after

are pair do the same thing

(a) multiplying it on the left by E? $EM = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -5 & -2 & -1 \end{bmatrix}$

(b) applying the row operation $R_2 \leftarrow R_2 - 2R_1$? $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & -5 & -2 & -1 \end{bmatrix}$

This holds for all elementary row operations, and all matrix sizes.

real number (except 0) has a inverse in real number

A matrix A is *invertible* if there exists a matrix B such that

$$AB = BA = I_n. \tag{1.17}$$

Proposition 1.18. If a matrix B as in (1.17) exists, then it is unique.

Proof. Suppose B_1 and B_2 are two matrices satisfying (1.17).

so $AB_1 = B_1A = I = AB_2 = B_2A$

$B_2 = B_2I = B_2AB_1 = B_1AB_2 = B_1(AB_2) = B_1I = B_1$

If it exists, the unique matrix B is called the *inverse* of A and is denoted A^{-1}.

If A and B are $n \times n$ matrices, then $AB = I_n$ is equivalent to A and B being inverses of each other. (We will prove this later.)

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A row operation ρ is *invertible* if there exists a row operation π that undoes its effect (no matter what matrix we apply this to).

Example 1.19. A non-example is the row operation ρ: $R \leftarrow 0R$, because this results in the zero row, and there is no way to recover the lost information.

Proposition 1.20. Any elementary row operation is invertible.

Proof. We consider each type of elementary row operation in turn:

(a) If ρ is $R \leftarrow \lambda R$ with $\lambda \neq 0$, we can take π $R \leftarrow \frac{1}{\lambda} R$

(b) If ρ is $R \leftrightarrow S$, we can take π $S \leftrightarrow R$

(c) If ρ is $R \leftarrow R + \lambda S$, we can take π $R \leftarrow R - \lambda S$

?

Theorem 1.21. Any elementary matrix is invertible.

Proof. let E be an elementary matrix

$I \xrightarrow{\rho} E$ write E_ρ for E

since ρ is an elementary row operation. by ...

ρ has an inverse π

$I \xrightarrow{\rho} E \xrightarrow{\pi} I.$

For matrice $(E_\pi E_\rho) I = I.$

$\Downarrow$

$E_\pi E_\rho \Rightarrow$ E_ρ is invertible and $E_\rho^{-1} = E_\pi$

Example 1.22.

$$\begin{bmatrix}1&1&1&2\\0&-1&2&0\\1&-1&-1&0\end{bmatrix}\xrightarrow{R_3\leftarrow R_3-R_1}\begin{bmatrix}1&1&1&2\\0&-1&2&0\\0&-2&-2&-2\end{bmatrix}\xrightarrow{R_2\leftarrow(-R_2)}\begin{bmatrix}1&1&1&2\\0&1&-2&0\\0&-2&-2&-2\end{bmatrix}\xrightarrow{R3\leftarrow R_3+2R_2}\begin{bmatrix}1&1&1&2\\0&-1&-2&0\\0&0&-6&-2\end{bmatrix}$$

E_1M $\quad E_2(E_1M).\quad E_3E_2E_1M$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\rightarrow\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}}_{E_1}\qquad\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\rightarrow\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\longrightarrow\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$$

$= [\, I \mid A^{-1} \,]$

Inverting a matrix

Given an $n \times n$ matrix A, we want to find the inverse of A, if it exists.

This is a linear system in disguise:

Example 1.23.

$$A = \begin{bmatrix}1&0&1\\2&-1&3\\2&2&1\end{bmatrix} \qquad B = \begin{bmatrix}x_{11}&x_{12}&x_{13}\\x_{21}&x_{22}&x_{23}\\x_{31}&x_{32}&x_{33}\end{bmatrix}$$

Give A, find B s.t.

$AB = I_3$

$$AB = \begin{bmatrix}1&0&1\\2&-1&3\\2&2&1\end{bmatrix}\begin{bmatrix}x_{11}&x_{12}&x_{13}\\x_{21}&x_{22}&x_{23}\\x_{31}&x_{32}&x_{33}\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}x_{11}+x_{31} & x_{12}+x_{32} & x_{13}+x_{33}\\ 2x_{11}-x_{21}+3x_{31} & 2x_{12}-x_{22}+3x_{32} & 2x_{13}-x_{23}+3x_{33}\\ 2x_{11}+2x_{21}+x_{31} & 2x_{12}+2x_{22}+x_{32} & 2x_{13}+2x_{23}+x_{33}\end{bmatrix}$$

We approach this slightly differently, by considering

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3. \end{array}$$

$$\overset{R_2 \leftarrow R_2 - R_3}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -3 & 2 & 0 & 1 & -1 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3 - 2R_1}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -3 & 2 & 0 & 1 & -1 \\ 0 & 2 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -3 & 2 & 0 & 1 & -1 \\ 0 & 1 & -\frac{1}{2} & -1 & 0 & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -1 & 0 & \frac{1}{2} \\ 0 & -3 & 2 & 0 & 1 & -1 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3 + 3R_2}{\Rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -6 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & - & \\ 0 & 1 & -\frac{1}{2} & -1 & 0 & \\ 0 & 0 & 1 & -6 & 2 & \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

rank: 3 (3 lead

So this is an algorithm for finding the inverse of a matrix A, if the latter is invertible:

1. Apply Gauss–Jordan elimination on $[A \mid I]$ to get a matrix $[C \mid D]$.
2. If $C \neq I$, then A is not invertible.
3. If $C = I$, then A is invertible and $A^{-1} = D$.

eg: $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

if 3 0 is in 2nd or 3rd row not invertible.

Why does this work?

Example 1.24 (A non-invertible matrix).

$[A \mid I] \quad \cdots\cdots \to [C \mid D]$

$\underbrace{E_\ell \cdots E_2 E_1}_{M} [A \mid I] = [MA \mid MI]$

each is elementary

$= [MA \mid M]$

$= [I \mid A^{-1}]$

come 0

invertible

solution x=0

Rank, invertibility, and solvability

The *rank* of a matrix in row echelon form is defined to be the number of leading 1s.

The *rank* of an arbitrary matrix A is the rank of any REF matrix obtained from A via Gaussian elimination.

The rank of the matrix from Example 1.23 is 3, while the rank of the matrix from Example 1.24 is 2.

Theorem 1.25. Let A be an $n \times n$ matrix. The following are equivalent (*TFAE*):

(a) A is invertible.

(b) A has rank n. (We also say it has *full rank*.)

(c) The RREF of A is I_n.

(d) The *homogeneous system* $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$.

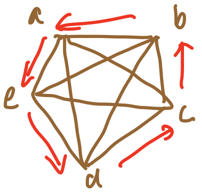
(e) Given any $n \times 1$ matrix $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has a unique solution. (If $\mathbf{b} \neq \mathbf{0}$, we call the system *inhomogeneous*.)

(c) ⇒ (b) definition of rank

(b) ⇒ (a) full rank

(b) ⇒ (c) ⇒ (a) ?

Proof. We show that (a) ⇒ (e) ⇒ (d) ⇒ (c) ⇒ (b) ⇒ (a).



$Ax = b$

(a) ⇒ (e) there exists A^{-1}

$AA^{-1} = I = A^{-1}A$

⇓

$A^{-1}(Ax) = A^{-1}b$

⇓

$(A^{-1}A)x = A^{-1}b$

$Ix = A^{-1}b$

$x = A^{-1}b$

(e) ⇒ (d) Take $b = 0$

(d) ⇒ (c)

Proof by contradiction

$\boxed{\text{If H then C} \Downarrow \text{if not C, then not H}}$

Assume RREF of A is not I_n,

RREF(A) =

after transform

2 leading 1s

column without leading 1

so x_j is a free parameter.

so we couldn't take a unique so

bec

no

Example 1.26. Consider the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

1. Find a row echelon form of $[A \mid \mathbf{b}]$.
2. Find the rank of the matrix A.
3. Find all the solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$.
4. For which values of a, b, and c does the system $A\mathbf{x} = \mathbf{b}$ have infinitely many solutions, a unique solution, or no solutions?

1. $\left[\begin{array}{ccc|c} 0 & -1 & 1 & a \\ 1 & 1 & -1 & b \\ 2 & -1 & 1 & c \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 1 & -1 & b \\ 0 & -1 & 1 & a \\ 2 & -1 & 1 & c \end{bmatrix} \xrightarrow[R_2 \leftarrow (-R_2)]{R_3 \leftarrow R_3 - 2R_1} \begin{bmatrix} 1 & 1 & -1 & b \\ 0 & 1 & -1 & -a \\ 0 & -3 & 3 & c-2b \end{bmatrix}$

$\xrightarrow{R_3 \leftarrow R_3 + 3R_2} \begin{bmatrix} 1 & 1 & -1 & b \\ 0 & 1 & -1 & -a \\ 0 & 0 & 0 & c-2b-3a \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & b+a \\ 0 & 1 & -1 & -a \\ 0 & 0 & 0 & c-2b-3a \end{bmatrix}$

3. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

z is free parameter

$z = t$.

$x = 0$

$y - z = t$

$z = y = t$

$\Rightarrow t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

4.

2.

if $c-2b-3a=0$

rank 2

$\begin{bmatrix} 1 & 0 & 0 & b+a \\ 0 & 1 & -1 & -a \\ 0 & 0 & 0 & 0 \end{bmatrix}$

infinite solution

if $c-2b-3a \neq 0$

rank 3

$\begin{bmatrix} 1 & 0 & 0 & b+a \\ 0 & 1 & -1 & -a \\ 0 & 0 & 0 & \ldots \end{bmatrix}$

no solution

In general, given an inhomogeneous system $A\mathbf{x} = \mathbf{b}$, if

$$\text{rank} A < \text{rank}[A \mid \mathbf{b}],$$

then the system has no solutions (it is inconsistent).

Note also, in part 3, that

?

$$\text{rank} A + \#\text{solution parameters} = \#\text{unknowns} = \#\text{columns}.$$

$2 + 1 = 3$

Example 1.27. Using the result of Example 1.23, solve

$$\begin{cases} x \phantom{{}-y} + z = 0 \\ 2x - y + 3z = 0 \\ 2x + 2y + z = 0 \end{cases}$$

$$A \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \\ 2 & 2 & 1 \end{bmatrix} \qquad A\vec{x} = \vec{0}$$

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A^{-1}(A\vec{x}) = A^{-1}\vec{0}$$

$$\vec{x} = \vec{0}$$

?