COMP20007 Design of Algorithms

Hashing

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Lecture 15

Semester 1, 2020

Dictionaries - Recap

- Abstract Data Structure: collection of *(key, value)* pairs.

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- This lecture: Hash Tables.

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like array

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key

in example
output: color

in real hash table,
the output is a position

Hash Tables

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- Average case performance for Search, Insert and Delete: $\Theta(1)$
- Requires a *hash function*: $h(K) \to i \in [0, m-1]$.
- A hash function should:
  - Be efficient ($\Theta(1)$).
  - Distribute keys evenly (uniformly) along the table.

want to avoid collison

Identity Hash Function

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- Note that $K \in [0, m-1]$. In other words *we need to know the maximum number of keys in advance*.
- Sometimes this is possible: postcodes, for example.
- Many times it is not:
  - *m is too large* (need to preallocate) → eg. for postcode, you need to preallocate 10000 element
  - Unbounded integers (student IDs)
  - Non-integer keys (games)

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- Allow us to set the size m.
- Small m results in lots of collisions, large m takes excessive memory. Best m will vary.

Hashing Strings

- Assume A $\mapsto 0$, B $\mapsto 1$, etc.
- Assume 26 characters and $m = 101$.
- Each character can be mapped to a *binary* string of length 5 ($2^5 = 32$).

map each character to
0-25

A — 00000
B — 00001
C — 00010

✓ $2^4 = 16 < 26$
$2^5 = 32$

Hashing Strings

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- Each character can be mapped to a *binary* string of length 5 ($2^5 = 32$).

We can think of a string as a long binary number:

$$\text{M Y K E Y} \mapsto 0110011000010100010011000 \ (= 13379736)$$

$$13379736 \bmod 101 = 64$$

So 64 is the position of string M Y K E Y in the hash table.

Hashing Strings

We deliberately chose m to be prime.

$$13379736 = \overbrace{\boxed{12 \times 32^4 + 24 \times 32^3 + 10 \times 32^2 + 4 \times 32}}^{\text{will be cancelled by 32}} + \overset{\text{only this matter}}{24}$$

With $m = 32$, the hash value of any key is the last character's value! if last character is the same, many collisions

another problem for hashing strings:

① very long string, the number calculated is very large → overflow

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Then we have

$$h(s) = (\sum_{i=0}^{|s|-1} chr(s_i) \times 32^{|s|-i-1}) \bmod m,$$

where m is a prime number. For example,

$$h(\mathrm{V\ E\ R\ Y\ L\ O\ N\ G\ K\ E\ Y}) = (21 \times 32^{10} + 4 \times 32^{9} + \cdots)\ mod\ 101$$

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The term between parenthesis can become quite large and result in overflow.

Horner's Rule

Instead of

$$21 \times 32^{10} + 4 \times 32^9 + 17 \times 32^8 + 24 \times 32^7 \cdots$$

factor out repeatedly:

$$(\cdots ((21 \times 32 + 4) \times 32 + 17) \times 32 + \cdots) + 24$$

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Instead of

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factor out repeatedly:

instead of calculating big number

$$(\cdots ((21 \times 32 + 4) \times 32 + 17) \times 32 + \cdots) + 24$$

Now utilize these properties of modular arithmetic:

$$\begin{aligned} (x + y) \bmod m &= ((x \bmod m) + (y \bmod m)) \bmod m \\ (x \times y) \bmod m &= ((x \bmod m) \times (y \bmod m)) \bmod m \end{aligned}$$

So for each sub-expression it suffices to take values modulo m.

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Practical efficiency will depend on the table **load factor**: $\alpha = n/m$ n = # total of records

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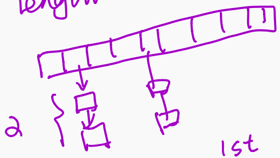
ave length $= \alpha$



1st 1 op

2st 2 op

$\vdots$

α ops

$$\frac{1}{\alpha}(1+\cdots+\alpha) = \frac{1}{\cancel{\alpha}} \cdot \frac{\cancel{\alpha}(\alpha+1)}{2} = \frac{\alpha+1}{2}$$

if $0<\alpha<1$, $1+\alpha/2 \sim 1$ operation

if $\alpha \geq 1$

eg. $\alpha=2$, each cell has average 2 elements

$\rightarrow$ 2 operations

$\Leftrightarrow$ each cell has linked list of 2 elements, $\rightarrow$ search for two operation

Separate Chaining

Assign multiple records per cell (usually through a linked list)

- Assuming even distribution of the *n* keys.
- A sucessful search requires $1 + \alpha/2$ operations on average.
- An unsucessful search requires α operations on average.

try to look up but fails

$\boxed{\text{why ?}}$

look through whole linked list

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- Worst case $\Theta(n)$ only with a bad hash function (load factor is more of an issue).

mapping all elements into same cell

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- Almost same numbers for Insert and Delete.
- Worst case $\Theta(n)$ only with a bad hash function (load factor is more of an issue).
- Requires extra memory.

Linear Probing

Populate successive empty cells.

move until find an empty cell

Linear Probing

we will only have $0<\alpha<1$ in this case

Populate successive empty cells.

- Much harder analysis, simplified results show:
- A sucessful search requires $(1/2) \times (1 + 1/(1-\alpha))$ operations on average.
- An unsucessful search requires $(1/2) \times (1 + 1/(1-\alpha)^2)$ operations on average.

only make sense for $0<\alpha<1$

* we don't allocate extra space, if $\alpha>1$, means all spaces are fully allocated, when $\alpha=1$, worst case $O(n)$

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- Similar numbers for Insert. Delete virtually impossible.

we don't actually delete element, just assign a flag show it's deleted

(如果真delete, 我们在搜索下一个会loss track.

不删除的缺点, 占用内存.

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- Similar numbers for Insert. Delete virtually impossible.
- Does not require extra memory.
- Worst case $\Theta(n)$ with a bad hash function *and/or* clusters.

eg ?

Double Hashing

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- First try: $h(K)$ $(h(k)+0\cdot s(k)) \bmod m = h(k)$
- Second try: $(h(K)+s(K)) \mod m$
- Third try: $(h(K)+2s(K)) \mod m$
- ...

if $s(k)=1$

$(h(k)+0\cdot s(k)) \bmod m = h(k)$ since xx mod m $= h(k)$

$(h(k)+1\cdot 1) \bmod m$ $h(k) \bmod m = h(k)$

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Both Linear Probing and Double Hashing are sometimes referred as *Open Addressing* methods.

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- High load factors deteriorate the performance of a hash table (for linear probing, ideally we should have $\alpha < 0.9$).
- *Rehashing* allocates a new table (usually around double the size) and move every item from the previous table to the new one.
- *Very expensive operation*, but happens infrequently.

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- Allow $\Theta(1)$ Search, Insert and Delete in the average case.
- Preallocates memory (size m).
- Requires good *hash functions*.
- Requires good collision handling.

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- Hash Tables ignore *key ordering*, unlike BSTs.
- Queries like "give me all records with keys between 100 and 200" are easy within a BST but much less efficient in a hash table.

↑

if require ordering of keys

BST better

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- Also: memory requirements of a hash table are much higher.

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- Also: memory requirements of a hash table are much higher.

That being said, if hashing is applicable, a well-tuned hash table will typically outperform BSTs.

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Next lecture: what happens if records/data is too large?