THE UNIVERSITY OF MELBOURNE

Semester 1 Assessment, 2015

School of Mathematics and Statistics

MAST20004 Probability

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam.

This paper consists of 5 pages (including this page)

Authorised materials:

- Students may bring one double-sided A4 sheet of handwritten notes into the exam room.
- Hand-held electronic scientific (but not graphing) calculators may be used.

Instructions to Students

- You may remove this question paper at the conclusion of the examination.
- This paper has 9 questions. Attempt as many questions, or parts of questions, as you can.

  The number of marks allocated to each question is shown in the brackets after the question statement. The total number of marks available for this examination is 100.

  There is a table of normal distribution probabilities at the end of the exam.

  Working and/or reasoning must be given to obtain full credit. Clarity, neatness and style count.

Instructions to Invigilators

- Students may remove this question paper at the conclusion of the examination.

1. Consider a random experiment with state space Ω.

   (a) Write down the axioms which must be satisfied by a probability mapping P defined on the events of the experiment.

   (b) Using the axioms, prove that for an event A,
$$P(A^c) = 1 - P(A).$$

   (c) Using the axioms, prove that for events $A \subseteq B$,
$$P(A) \leq P(B).$$

[9 marks]

2. A bag has three dice in it, one is fair and the other two are weighted so the chances of a six coming up are 7/36 and 8/36, respectively. You choose a die at random from the bag and roll it.

   (a) What is the chance you roll a six?

   (b) Given you roll a six, what is the chance you rolled the fair die?

   (c) Given you roll a six, what is the chance that if you roll the same die again you get a six?

[8 marks]

3. Let X have density
$$f_X(x) = \frac{C}{\sqrt{x}}, \;\; 0 < x < 1.$$

   (a) What is the value of C?

   (b) Compute $P(1/4 < X < 25/16)$.

   (c) Find the cumulative distribution $F_Y(y)$ and probability density function $f_Y(y)$ of $Y = X(1-X)$.

   (d) Find probability density function $f_Z(z)$ of $Z = X^{1/4}$.

   (e) Find the mean and variance of X.

   (f) Calculate the expected value of Z by

      (i) evaluating $\int_{-\infty}^{\infty} \psi(x) f_X(x) dx$ for an appropriate function $\psi(x)$,

      (ii) evaluating $\int_{-\infty}^{\infty} z f_Z(z) dz$,

      (iii) approximation using an appropriate formula based on Taylor series expansion of $x^{1/4}$.

   (g) Calculate the variance of Z and compare it to the approximation using an appropriate formula based on Taylor series expansion of $x^{1/4}$.

[19 marks]

4. The amount of time T (in hours) that a certain electrical component takes to fail has an exponential distribution with parameter $\lambda > 0$. The component is found to be working at midnight on a certain day. Let N be the number of *full* days after this time before the component fails (so if the component fails before midnight the next day, $N = 0$).

   (a) What is the probability that the component lasts at least 24 hours?

   (b) Find the probability mass function of N.

   (c) Identify the distribution of N by name.

   (d) Given that the component is found to have failed on a certain day, what is the distribution function of the number of hours H past midnight that the component failed?

[7 marks]

5. The density of (X, Y) is given by

$$f(x, y) = Ce^{-x},\ 0 < y < x.$$

   (a) What is the constant C?

   (b) What is $P(X < 2 - Y)$?

   (c) What is the density of Y?

   (d) What is the density of X given $Y = y$?

   (e) What is $P(X^2 \geq 1/4 | Y \leq 3/4)$?

   (f) Find $E[X]$ and $E[Y]$.

   (g) What is the covariance of X and Y?

   (h) Are X and Y independent (and why)?

   (i) What is $E\left[\frac{Y}{X}\right]$?

   (j) Find the distribution function of Y/X and name the distribution.

[18 marks]

6. The price of a stock at the start of a trading day is 25 dollars. The price of the stock two hours into the trading day is $25e^X$ and the price of the stock four hours into the trading day is $25e^Y$, where (X, Y) is bivariate normal with $E[X] = -2$, $Var(X) = 2$, $E[Y] = -4$, $Var(Y) = 4$ and $Cov(X, Y) = 2$.

   (a) Find the chance that the price of the stock two hours after the start of the day is at least $25e^{-2\sqrt{2}}$.

   (b) If you buy one share of the stock at the start of the trading day and sell it at after two hours, how much money would you expect to make (losses count negative)?

   (c) What is the correlation of X and Y?

   (d) Given the price of the stock four hours after the start of the day is $25e^2$ dollars, what is the chance the price of the stock two hours after the start of the day was more than 25 dollars?

[10 marks]

7. Roll a fair die and let N be the number of rolls *before* the first six appears. Now toss a fair coin N times and let X be the number of heads in the N tosses.

   (a) Compute $E[X]$.

   (b) Compute $Var(X)$.

   (c) Find the probability generating function of X and identify the distribution of X by name.

   (d) Find the conditional probability mass function of N given $X = x$, for $x = 0, 1, 2, \ldots$.

[11 marks]

8. A gambler is going to make \$100 worth of bets at a roulette table. The gambler is considering two strategies:

   Strategy 1: make a single \$100 bet on black where the chance of winning \$100 is 18/37 and the chance of losing \$100 is 19/37.

   Strategy 2: make 100 sequential \$1 bets on black where for each bet, the chance of winning \$1 is 18/37 and the chance of losing \$1 is 19/37.

   Let W_1 be the winnings (counting losses as negative) of the gambler if Strategy 1 is used and W_2 the winnings if Strategy 2 is used.

   (a) Compute the mean and variance of W_1.

   (b) Compute the mean and variance of W_2.

   (c) Approximate the chance that $W_2 > 0$.

   (d) Which strategy should the gambler use to maximize the chance of ending with more than they started?

[9 marks]

9. Consider the branching process $\{X_n, n = 0, 1, 2, \ldots\}$ where X_n is the population size of the nth generation. Assume $P(X_0 = 1) = 1$ and that the common offspring distribution is geometric with parameter $1/2$.

   (a) Show that the generating function of the offspring distribution is

   $$A_1(z) = \frac{1}{2 - z}, \quad -2 < z < 2.$$

   [Note that part of the question is to justify the range of z where the formula is valid.]

   (b) Find $E[X_n]$.

   (c) If $q_n = P(X_n = 0)$ for $n = 0, 1, \ldots$, write down an equation relating q_n and q_{n+1}. Hence or otherwise, find an expression for q_n, $n = 0, 1, \ldots$.

   (d) Find the extinction probability $q = \lim_{n \to \infty} q_n$.

   (e) Find an expression for the generating function $A_n(z)$ of X_n.

[9 marks]

Tables of the Normal Distribution

Probability Content from -oo to Z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817