Tutorial 4: Proofs

Q1. Using the method indicated, prove the following:

(a) The statement "for all $x, y, z \in \mathbb{Z}$ if $y < z$ then $xy < xz$" is not true. (*Counterexample)*

(b) The product of an odd integer with an even integer is even. (*Direct proof)*

(c) Let $n, m \in \mathbb{Z}$. If nm is even then either n or m is even. (*Contrapositive proof)*

(d) The square root of 10 is not rational. (*Contradiction proof)*

(e) $\displaystyle\sum_{k=1}^{n} 4k - 1 = 2n^2 + n$, for all $n \in \mathbb{N}$. (*Induction proof)*

(f) There exists a smallest $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $n! > 2n^2$ (*Induction proof)*

Q2. Prove that the following sets are countably infinite. (Give bijections with $\mathbb{N}$, either by listing the elements without repetition or by giving an explicit formula.)

(a) The natural numbers divisible by 3.

(b) The odd integers.

Q3. Prove the following statements or show that they are false. Indicate the method of proof that you are using.

(a) $\sqrt{2} + \sqrt{5}$ is not rational.

(b) $\displaystyle\sum_{k=1}^{n} \frac{1}{k^2} \geq \frac{3}{2} - \frac{1}{n+1}$, for all $n \in \mathbb{N}$.

(c) If the sum of the digits of a 3-digit number is divisible by 3, then so is the original number.

(d) Every quadrilateral with perpendicular diagonals has equal sides.

(e) $8^{n+1} - 3 \cdot 8^n + 2$ is divisible by 7 for all $n \in \mathbb{N}$.

Q4. Prove that there are infinitely many prime numbers of the form $4n + 3$ where n is an integer.