THE UNIVERSITY OF MELBOURNE

Student ID

Semester 1 Assessment, 2016

School of Mathematics and Statistics

MAST20004 Probability

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 4 pages (including this page)

Authorised materials:

- Students may bring one double-sided A4 sheet of handwritten notes into the exam room.
- Hand-held electronic scientific (but not graphing) calculators may be used.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- This paper has 9 questions. Attempt as many questions, or parts of questions, as you can.

  The number of marks allocated to each question is shown in the brackets after the question statement. The total number of marks available for this examination is 100.

  There is a table of normal distribution probabilities at the end of the exam.

  Working and/or reasoning must be given to obtain full credit. Clarity, neatness and style count.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

1. Consider a random experiment with state space Ω.

   (a) Write down the axioms which must be satisfied by a probability mapping P defined on the events of the experiment.

   (b) Using the axioms, prove that for events A and B,

   $$P(A) = P(A \cap B) + P(A \cap B^c).$$

   (c) Is it possible to have a probability space with events A, B, C such that $P(A) = 1/2$, $P(B) = 1/2$, $P(C) = 3/4$, $P(B \cap C) = 1/4$, $P(A \cap B) = 1/4$, $P(A \cap C) = 3/8$, and $P(A \cap B \cap C) = 1/8$? If your answer is no, then prove it's not possible, and if your answer is yes, then define a probability space and events A, B, C that satisfy the constraints.

   [9 marks]

2. The probabilities with which we choose one of three target shooters are 0.2 for Angela, 0.5 for Brunhilde, and 0.3 for Chelsea. The probabilities of each hitting a target from one shot are 0.3 for Angela, 0.4 for Brunhilde, and 0.1 for Chelsea.

   (a) The chosen shooter fires one shot and misses the target. Find the probability that we have chosen Angela.

   (b) Are the events that Angela was chosen and the target was missed positively related, negatively related, or independent? Justify your answer.

   (c) Suppose Angela and Brunhilde each fired one shot at the target. If one bullet is found in the target, find the probability that it came from Brunhilde's gun. Assume that the two shooters fire at the target independently.

   [8 marks]

3. Let T be the number of failures before the first success in a series of independent Bernoulli trials with probability $p \in (0, 1)$, U be the number of failures between the first and the second successes, and V the number of failures between the second and the third successes.

   (a) Find the probability mass functions and identify the distribution by name for

      (i) $X = T + U + V$;

      (ii) $Z = \min\{T, U\}$.

   (b) (i) Find the joint probability mass function of (T, Z).

      (ii) Compute the probability generating function of $T + Z$.

   [10 marks]

4. Let X and Y have joint probability density function given by $ce^{-(x+y)}$, $0 < x < y$.

   (a) Evaluate the constant c.

   (b) Find $f_X(x)$, the marginal density function of X, and identify this distribution by name.

   (c) Evaluate $P(Y > 2 | X = 1)$.

   (d) Evaluate $P(Y > 2 | X > 1)$.

   (e) Are X and Y independent? Justify your answer.

   (f) Compute $E[e^{(X+Y)/2}]$. [14 marks]

5. Let X be uniform on the interval $(0, \pi)$. Let $Y = \cos X$ and $Z = Y^2$.

   (a) Find $F_Y(y)$, the distribution function of Y.

   (b) Find $f_Y(y)$, the density function of Y, state the values of y for which it is defined, and show that it is a density function.

   Recall that $\arccos : [-1, 1] \to [0, \pi]$ and, for $-1 < x < 1$,

$$\frac{d}{dx}(\arccos x) \;=\; -\frac{1}{\sqrt{1-x^2}}.$$

   (c) Evaluate $E(Y)$.

   (d) Find $F_Z(z)$, the distribution function of Z.

   (e) Find $f_Z(z)$, the density function of Z, state the values of z for which it is defined.

   (f) Approximate $E(Z)$ and $Var(Z)$ using suitable Taylor series expansions.

[17 marks]

6. Two casino games are as follows. In Game 1 the chance of winning \$1 is 12/25 and the chance of losing \$1 is 13/25. In Game 2, the chance of winning \$23 is 1/25 and the chance of losing \$1 is 24/25. Let W_1 be the total winnings after playing Game 1 100 times. Let W_2 be the total winnings after playing Game 2 100 times. Losses count as negative in “winnings”.

   (a) Find $E[W_1]$ and $Var(W_1)$.

   (b) Find $E[W_2]$ and $Var(W_2)$.

   (c) Compute $P(W_1 > 0)$ using a normal approximation.

   (d) Compute $P(W_2 > 0)$ using a Poisson approximation.

   (e) Which game would you play and why?

[13 marks]

7. Let Z be a standard normal random variable and let N have distribution given by $P(N = 1) = P(N = 2) = 1/2$ with N and Z independent. Set $X = Z^N$.

   (a) Compute $P(X > 1)$.

   (b) Compute $P(N = 1|X = -1)$.

   (c) Compute $P(N = 1|X > 1)$.

   (d) Compute $P(N = 1|X = 1)$.

   (e) Describe the distribution of $E[X|N]$.

   (f) Compute $E[X]$.

   (g) Describe the distribution of $Var[X|N]$.

   (h) Compute $Var(X)$.

   (i) Find the covariance and correlation of X and N.

[13 marks]

8. A computer can simulate independent standard normal random variables $Z_1, Z_2, \ldots$.

   (a) How can you transform the independent standard normal random variables Z_1, Z_2 to generate a vector (X_1, X_2) having the standard bivariate normal distribution with correlation ρ?

   (b) Given you have a standard bivariate normal vector (X_1, X_2) with correlation ρ, how can you transform (X_1, X_2) to obtain a vector (Y_1, Y_2) having the bivariate normal distribution with means μ_1, μ_2, variances σ_1^2, σ_2^2, and correlation ρ?

   (c) If (X_1, X_2) has the standard bivariate normal distribution with correlation ρ, describe a method to use the computer-generated independent normal random variables $Z_1, Z_2, \ldots, Z_{200}$ to estimate $E[\max\{X_1, X_2\}]$.

   (d) How can you transform the random variables Z_1, Z_2 to obtain a number U that is uniformly distributed between 0 and 1?

[6 marks]

9. Consider the branching process $\{X_n, n = 0, 1, 2, \ldots\}$ where X_n is the population size of the nth generation. Assume $P(X_0 = 1) = 1$ and that the probability generating function of the offspring distribution is $A(z) = (1 - p) + pz^2$ for some $0 < p < 1$.

   (a) What is the probability mass function of the offspring distribution?

   (b) Find a simple expression for $E[X_n]$ in terms of p.

   (c) If $q_n = P(X_n = 0)$ for $n = 0, 1, \ldots$, write down an equation relating q_n and q_{n+1}. Hence or otherwise, find an expression for q_n, $n = 0, 1, \ldots$.

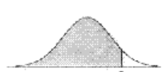
   (d) Find the extinction probability $q = \lim_{n\to\infty} q_n$.

   (e) Find a simple expression for $P(X_3 = 0)$ in terms of p.

   (f) What is $P(X_{100} = 3)$?

[10 marks]

Tables of the Normal Distribution



Probability Content from -oo to Z

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817