MAST30001 Stochastic Modelling

Tutorial Sheet 11

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion.

1. Show that $(X_t)_{t>0} = (tB_{1/t})_{t>0}; X_0 = 0$ and $(Y_t)_{t\geq 0} = (cB_{t/c^2})_{t\geq 0}$ are Brownian motions.

   Ans. We need to show that the three axioms of Brownian motion are satisfied.

   - Continuity: Away from zero, both X and Y are continuous functions of B and thus continuous. For X near zero, we argue $\lim_{t\to 0} tB_{1/t} = 0$ by noting that both the mean and variance go to zero (strictly speaking, we need to show that with probability one, this limit is zero, but proving such a statement is outside of the scope of this subject).
   - Independent increments: Disjoint intervals $\{(s_i, t_i)\}_{i=1}^k$ map to disjoint intervals under $t \mapsto t/c^2$, so disjoint increments in Y are functions of the disjoint and hence independent increments of B and thus are themselves independent. For X, increments are functions of the multivariate normal finite dimensional distributions of B, and so are multivariate normal themselves. Thus, for independence, it is enough to check covariances are zero.
   - To show X_t or Y_t is normal mean zero, variance t, note that $B_{1/t}$ and B_{t/c^2} are both normal with respective variances $1/t$ and t/c^2 and then X_t and Y_t are constants times these variables and hence normal (with the correct variances).

2. Show that $W_t = \exp\{cB_t - c^2t/2\}$ has the property $E[W_t|W_s] = W_s$.

   Ans. We first compute

$$\begin{aligned}
E[W_t|B_s] &= \exp\{cB_s - c^2t/2\}E\left[\exp\{c(B_t - B_s)\}|B_s\right] \\
&= \exp\{cB_s - c^2t/2\}\exp\{c^2(t-s)/2\} \\
&= \exp\{cB_s - c^2s/2\} = W_s;
\end{aligned}$$

   where we've used the independent increments property of Brownian motion and then that $B_t - B_s$ is normal with variance $t - s$. Since $x \mapsto \exp\{cx - c^2s/2\}$ is an invertible function, conditioning on B_s is the same as conditioning on W_s and so the result follows.

3. Show that for each $0 < u < 1$, $B_u - uB_1$ is independent of B_1.

   Ans. $(B_u - uB_1, B_1)$ is bivariate normal and so independence is the same as uncorrelated. We check:

$$Cov(B_u - uB_1, B_1) = Cov(B_u, B_1) - uCov(B_1, B_1) = u - u = 0.$$

4. For $0 \leq t_1 < t_2 < t_3$, find $E[B_{t_1}B_{t_2}B_{t_3}]$.

   Ans.

Using the independent increments property and that $E[B_t] = 0$,

$$\begin{aligned}
E[B_{t_1}B_{t_2}B_{t_3}] &= E[(B_{t_3} - B_{t_2})(B_{t_2} - B_{t_1})B_{t_1}] + E[B_{t_2}^2 B_{t_1}] + E[B_{t_3}B_{t_1}^2] - E[B_{t_2}B_{t_1}^2] \\
&= E[(B_{t_3} - B_{t_2})(B_{t_2} - B_{t_1})B_{t_1}] + E[B_{t_2}^2 B_{t_1}] + E[B_{t_3}B_{t_1}^2] - E[B_{t_2}B_{t_1}^2] \\
&= E[(B_{t_2} - B_{t_1})^2 B_{t_1}] + E[B_{t_2}B_{t_1}^2] - E[B_{t_1}^3] + E[B_{t_3}B_{t_1}^2] \\
&= E[(B_{t_2} - B_{t_1})B_{t_1}^2] + E[(B_{t_3} - B_{t_1})B_{t_1}^2] + E[B_{t_1}^3] \\
&= E[B_{t_1}^3] = 0.
\end{aligned}$$

A more clever way is to note that $(B_t)_{t\geq 0} \overset{d}{=} (-B_t)_{t\geq 0}$ and so

$$E[B_{t_1}B_{t_2}B_{t_3}] = E\big[(-B_{t_1})(-B_{t_2})(-B_{t_3})\big] = -E[B_{t_1}B_{t_2}B_{t_3}],$$

which implies $E[B_{t_1}B_{t_2}B_{t_3}] = 0$.

5. The price of a stock at the start of the day is 100 dollars. Suppose that the logarithm of the price of the stock at time t hours after the start of the day is $B_t + 2t + \log(100)$.

   (a) Find the chance that the price of the stock one hour after the start of the day is higher than the starting price.

   (b) If you buy one share of the stock and sell it at after one hour, how much money would you expect to make?

   (c) Given the price of the stock two hours after the start of the day is 100 dollars, what is the chance the price of the stock one hour after the start of the day is higher than the starting price.

   Ans. Let $S_t = 100\exp\{B_t + 2t\}$ be the price of the stock t hours into the day.

   (a) $P(S_1 > 100) = P(B_1 > -2) \approx .98$, since B_1 is standard normal.

   (b) On average you would make

   $$E[S_1] - 100 = 100(E[\exp\{B_1 + 2\}] - 1) = 100(e^{5/2} - 1).$$

   (c) $P(S_1 > 100 | S_2 = 100) = P(B_1 > -2 | B_2 = -4)$. Since (B_1, B_2) are bivariate normal with correlation $1/\sqrt{2}$, we can set

   $$B_1 = B_2/2 + Z/\sqrt{2},$$

   where Z is standard normal independent of B_2. Thus

   $$P(B_1 > -2 | B_2 = -4 = P(Z/\sqrt{2} - 2 > -2) = P(Z > 0) = 1/2.$$

6. Show that for $x > 0$, $T_x = \inf\{t : B_t = x\}$ has the same distribution as x^2/Z^2, where Z is standard normal.

   Ans. From lecture we saw that T_x satisfies, for $x, t > 0$

   $$P(T_x \leq t) = \sqrt{\frac{2}{\pi}} \int_{x/\sqrt{t}}^{\infty} e^{-u^2/2} du = P(|Z| \geq x/\sqrt{t}) = P(Z^2 \geq x^2/t) = P(x^2/Z^2 \leq t),$$

   which implies x^2/Z^2 and T_x have the same distribution.

7. Let $M_t = \min_{0 \le s \le t} B_s$.

(a) Use the reflection principle to show that for $z \le 0$ and $x \ge z$,

$$P(M_t \le z, B_t \ge x) = P(B_t \le 2z - x).$$

(b) Find the joint density of (M_t, B_t).

(c) Show that $B_t - M_t$ has the same distribution as $|B_t|$. [Interesting fact: this distributional identity holds at the process level: $(B_t - M_t)_{t \ge 0}$ has the same distribution as $(|B_t|)_{t \ge 0}$.]

Ans.

(a) If $M_t \le z$, then there is a first moment T in $(0, t)$ such that $B_T = z$. Any Brownian path from this point (T, z) with $B_t \ge x$ can be reflected about the line $y = z$ to create a path such that $\{M_t \le z, B_t \le 2z - x\}$ and all such paths can be reflected back to create a path in the event that we're computing the probability. Thus the reflection principle says that

$$P(M_t \le z, B_t \ge x) = P(M_t \le z, B_t \le 2z - x) = P(B_t \le 2z - x),$$

where the second equality is because $B_t \le 2z - x$ implies $M_t \le z$ (since $x \ge z$).

(b) From (a), we have

$$P(M_t \le z, B_t \ge x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{2z-x} e^{-u^2/(2t)} du.$$

Differentiating with respect to z and then x and taking the negative (since one of the signs is the wrong way), we find the density of (M_t, B_t) to be

$$t^{-3/2} \sqrt{\frac{2}{\pi}} (x - 2z) e^{-(2z-x)^2/(2t)}, \quad z < 0, z < x.$$

(c) From (b), for $y > 0$,

$$P(B_t - M_t \le y) = t^{-3/2} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{0} \int_{z}^{y+z} (x - 2z) e^{-(2z-x)^2/(2t)} dx dz.$$

Differentiating with respect to y we have that $B_t - M_t$ has density

$$t^{-3/2} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{0} (y - z) e^{-(z-y)^2/(2t)} dz = \sqrt{\frac{2}{\pi t}} e^{-y^2/t},$$

for $y > 0$ and this is the density of the absolute value of a normal distribution with mean zero and variance t.