

UNCERTAINTY

CHAPTER 13

Outline

- ◇ Uncertainty
- ◇ Probability
- ◇ Syntax and Semantics
- ◇ Inference
- ◇ Independence and Bayes' Rule

Uncertainty

Let action $A_t =$ leave for airport t minutes before flight
Will A_t get me there on time?

Problems:

- 1) partial observability (road state, other drivers' plans, etc.)
- 2) noisy sensors (radio traffic reports)
- 3) uncertainty in action outcomes (flat tyre, etc.)
- 4) immense complexity of modelling and predicting traffic

Hence a purely logical approach either

- 1) risks falsehood: " A_{25} will get me there on time"
- or 2) leads to conclusions that are too weak for decision making:
" A_{25} will get me there on time if there's no accident on the bridge
and it doesn't rain and my tires remain intact etc etc."

(A_{1440} might reasonably be said to get me there on time
but I'd have to stay overnight in the airport ...)

Methods for handling uncertainty

Nonmonotonic logic:

Assume my car does not have a flat tire

Assume A_{25} works unless contradicted by evidence

Issues: What assumptions are reasonable? How to handle contradiction?

Rules with confidence factors:

$A_{25} \mapsto_{0.3}$ get there on time

However, consider the example:

$Sprinkler \mapsto_{0.99} WetGrass$

$WetGrass \mapsto_{0.7} Rain$

causation ×

Issues: Problems with combination, e.g., *Sprinkler causes Rain??*

Probability

Given the available evidence,

A_{25} will get me there on time with probability 0.04

(Fuzzy logic handles *degree of truth* NOT uncertainty e.g.,
 $WetGrass$ is true to degree 0.2)

Probability

Probabilistic assertions *summarize* effects of

laziness: failure to enumerate exceptions, qualifications, etc.

ignorance: lack of relevant facts, initial conditions, etc.

Subjective or **Bayesian** probability:

Probabilities relate propositions to one's own state of knowledge

e.g., $P(A_{25} | \text{no reported accidents}) = 0.06$

These are *not* claims of some **probabilistic tendency** in the current situation
(but might be learned from past experience of similar situations)

Probabilities of propositions change with new evidence:

e.g., $P(A_{25} | \text{no reported accidents, 5 a.m.}) = 0.15$

Making decisions under uncertainty

Suppose I believe the following:

$$P(A_{25} \text{ gets me there on time} | \dots) = 0.04$$

$$P(A_{90} \text{ gets me there on time} | \dots) = 0.70$$

$$P(A_{120} \text{ gets me there on time} | \dots) = 0.95$$

$$P(A_{1440} \text{ gets me there on time} | \dots) = 0.9999$$

Which action to choose?

Depends on my **preferences** for missing flight vs. airport cuisine, etc.

Utility theory is used to represent and infer preferences

Decision theory = utility theory + probability theory

Probability basics

Begin with a set Ω —the *sample space*

e.g., 6 possible rolls of a die.

$\omega \in \Omega$ is a sample point/possible world/atomic event

A *probability space* or *probability model* is a sample space with an assignment $P(\omega)$ for every $\omega \in \Omega$ s.t.

$$0 \leq P(\omega) \leq 1$$

$$\sum_{\omega} P(\omega) = 1$$

e.g., $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$.

An *event* A is any subset of Ω

$$P(A) = \sum_{\{\omega \in A\}} P(\omega)$$

E.g., $P(\text{die roll} < 4) = 1/6 + 1/6 + 1/6 = 1/2$

Random variables

A *random variable* is a function from sample points to some range, e.g., the reals or Booleans

e.g., $Odd(1) = true$.

P induces a *probability distribution* for any r.v. X :

$$P(X = x_i) = \sum_{\{\omega: X(\omega) = x_i\}} P(\omega)$$

e.g., $P(Odd = true) = 1/6 + 1/6 + 1/6 = 1/2$

Propositions

Think of a proposition as the event (set of sample points) where the proposition is true

Given Boolean random variables A and B :

event a = set of sample points where $A(\omega) = \text{true}$

event $\neg a$ = set of sample points where $A(\omega) = \text{false}$

event $a \wedge b$ = points where $A(\omega) = \text{true}$ and $B(\omega) = \text{true}$

Often in AI applications, the sample points are *defined* by the values of a set of random variables, i.e., the sample space is the Cartesian product of the ranges of the variables

With Boolean variables, sample point = propositional logic model

e.g., $A = \text{true}$, $B = \text{false}$, or $a \wedge \neg b$.

Proposition = disjunction of atomic events in which it is true

e.g., $(a \vee b) \equiv (\neg a \wedge b) \vee (a \wedge \neg b) \vee (a \wedge b)$

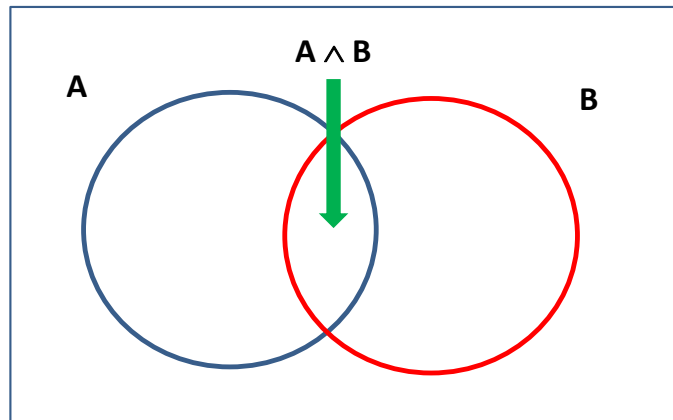
$$\Rightarrow P(a \vee b) = P(\neg a \wedge b) + P(a \wedge \neg b) + P(a \wedge b)$$

Why use probability?

The definitions imply that certain logically related events must have related probabilities

e.g., $P(a \overset{\text{or}}{\vee} b) = P(a) + P(b) - P(a \overset{\text{and}}{\wedge} b)$

True



de Finetti (1931): an agent who bets according to probabilities that violate these axioms can be forced to bet so as to lose money regardless of outcome.

Syntax for propositions

Propositional or Boolean random variables

e.g., *Cavity* (do I have a cavity?)

Discrete random variables (*finite* or *infinite*)

e.g., *Weather* is one of $\langle \textit{sunny}, \textit{rain}, \textit{cloudy}, \textit{snow} \rangle$

Weather = *rain* is a proposition

Values must be exhaustive and mutually exclusive

Continuous random variables (*bounded* or *unbounded*)

e.g., *Temp* = 21.6; also allow, e.g., *Temp* < 22.0.

Arbitrary Boolean combinations of basic propositions

Prior probability

Prior or unconditional probabilities of propositions

e.g., $P(Cavity = true) = 0.1$ and $P(Weather = sunny) = 0.72$
correspond to belief prior to arrival of any (new) evidence

Probability distribution gives values for all possible assignments:

P
represent
a distribution

$P(Weather) = \langle 0.72, 0.1, 0.08, 0.1 \rangle$ (*normalized*, i.e., sums to 1)

Joint probability distribution for a set of r.v.s gives the probability of every atomic event on those r.v.s (i.e., every sample point)

$P(Weather, Cavity) =$ a 4×2 matrix of values:

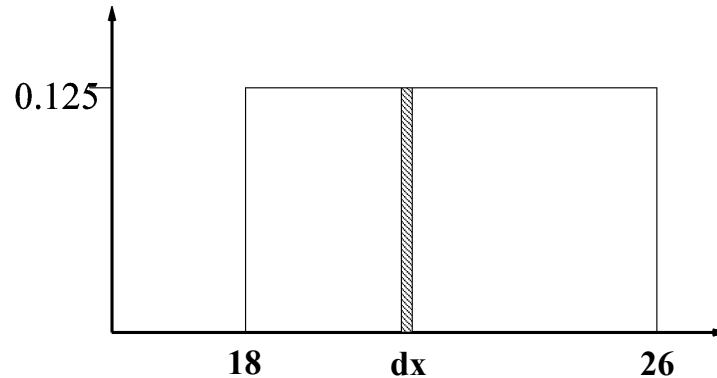
| $Weather =$ | $sunny$ | $rain$ | $cloudy$ | $snow$ |
|------------------|---------|--------|----------|--------|
| $Cavity = true$ | 0.144 | 0.02 | 0.016 | 0.02 |
| $Cavity = false$ | 0.576 | 0.08 | 0.064 | 0.08 |

*Every question about a domain
can be answered by the joint distribution
because every event is a sum of sample points*

Probability for continuous variables

Express distribution as a parameterized function of value:

$$P(X = x) = U[18, 26](x) = \text{uniform density between 18 and 26}$$



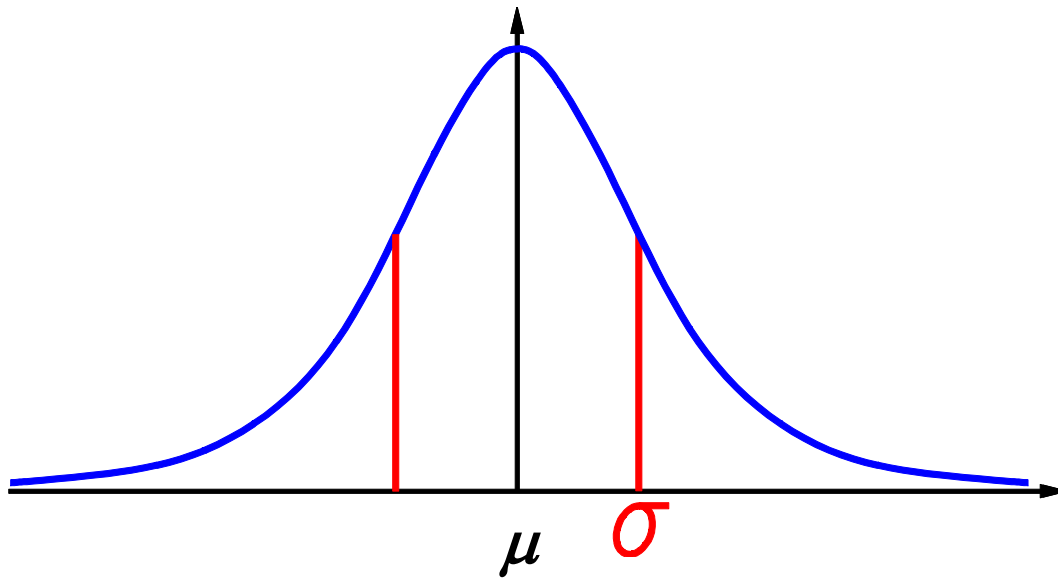
Here P is a *density*; integrates to 1.

$P(X = 20.5) = 0.125$ really means

$$\lim_{dx \rightarrow 0} P(20.5 \leq X \leq 20.5 + dx) / dx = 0.125$$

Gaussian density

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$



Conditional probability

Conditional or posterior probabilities

e.g., $P(\text{cavity}|\text{toothache}) = 0.8$

i.e., given that *toothache* is all I know

NOT “if *toothache* then 80% chance of *cavity*”

(Notation for conditional distributions:

$P(\text{Cavity}|\text{Toothache}) = 2\text{-element vector of } 2\text{-element vectors})$

If we know more, e.g., *cavity* is also given, then we have

$P(\text{cavity}|\text{toothache}, \text{cavity}) = 1$

Note: the less specific belief *remains valid* after more evidence arrives, but is not always *useful*

New evidence may be irrelevant, allowing simplification, e.g.,

$P(\text{cavity}|\text{toothache}, \text{carltonWins}) = P(\text{cavity}|\text{toothache}) = 0.8$

This kind of inference, sanctioned by domain knowledge, is crucial

Conditional probability

Definition of conditional probability:

$$P(a|b) = \frac{P(a \wedge b)}{P(b)} \text{ if } P(b) \neq 0$$

Product rule gives an alternative formulation:

$$P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$$

A general version holds for whole distributions, e.g.,

$$\mathbf{P}(\textit{Weather}, \textit{Cavity}) = \mathbf{P}(\textit{Weather}|\textit{Cavity})\mathbf{P}(\textit{Cavity})$$

(View as a 4×2 set of equations, *not* matrix multiplication)

Chain rule is derived by successive application of product rule:

$$\begin{aligned} \mathbf{P}(X_1, \dots, X_n) &= \mathbf{P}(X_1, \dots, X_{n-1}) \mathbf{P}(X_n|X_1, \dots, X_{n-1}) \\ &= \mathbf{P}(X_1, \dots, X_{n-2}) \mathbf{P}(X_{n-1}|X_1, \dots, X_{n-2}) \mathbf{P}(X_n|X_1, \dots, X_{n-1}) \\ &= \dots \\ &= \prod_{i=1}^n \mathbf{P}(X_i|X_1, \dots, X_{i-1}) \end{aligned}$$

$$P(x_1, x_2, x_3)$$

$$= P(x_1, x_2) P(x_3|x_1, x_2)$$

$$= P(x_2|x_1) \cdot P(x_1) \cdot P(x_3|x_1, x_2)$$

Inference by enumeration

Start with the joint distribution:

| | <i>toothache</i> | | \neg <i>toothache</i> | |
|----------------------|------------------|---------------------|-------------------------|---------------------|
| | <i>catch</i> | \neg <i>catch</i> | <i>catch</i> | \neg <i>catch</i> |
| <i>cavity</i> | .108 | .012 | .072 | .008 |
| \neg <i>cavity</i> | .016 | .064 | .144 | .576 |

For any proposition ϕ , sum the atomic events where it is true:

$$P(\phi) = \sum_{\omega: \omega \models \phi} P(\omega)$$

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$$P(\text{toothache}) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2$$

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For any proposition ϕ , sum the atomic events where it is true:

$$P(\phi) = \sum_{\omega: \omega \models \phi} P(\omega)$$

$$P(\text{cavity} \vee \text{toothache}) = 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 = 0.28$$

Inference by enumeration

Start with the joint distribution:

| | <i>toothache</i> | | \neg <i>toothache</i> | |
|----------------------|------------------|---------------------|-------------------------|---------------------|
| | <i>catch</i> | \neg <i>catch</i> | <i>catch</i> | \neg <i>catch</i> |
| <i>cavity</i> | .108 | .012 | .072 | .008 |
| \neg <i>cavity</i> | .016 | .064 | .144 | .576 |

Can also compute conditional probabilities:

$$\begin{aligned} P(\neg\textit{cavity}|\textit{toothache}) &= \frac{P(\neg\textit{cavity} \wedge \textit{toothache})}{P(\textit{toothache})} \\ &= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} = 0.4 \end{aligned}$$

Normalization

| | <i>toothache</i> | | \neg <i>toothache</i> | |
|----------------------|------------------|---------------------|-------------------------|---------------------|
| | <i>catch</i> | \neg <i>catch</i> | <i>catch</i> | \neg <i>catch</i> |
| <i>cavity</i> | .108 | .012 | .072 | .008 |
| \neg <i>cavity</i> | .016 | .064 | .144 | .576 |

Denominator can be viewed as a *normalization constant* α

distribution
for P

$$\begin{aligned}
 \rightarrow P(Cavity|toothache) &= \alpha P(Cavity, toothache) && \text{in this case } \alpha = \text{prior probability of } P(toothache) \\
 &= \alpha [P(Cavity, toothache, catch) + P(Cavity, toothache, \neg catch)] \\
 &= \alpha [\langle 0.108, 0.016 \rangle + \langle 0.012, 0.064 \rangle] && \downarrow \langle Cavity, \neg Cavity \rangle \\
 &= \alpha \langle 0.12, 0.08 \rangle = \langle 0.6, 0.4 \rangle
 \end{aligned}$$

query variable: *cavity* or \neg *cavity*

General idea: compute distribution on *query variable*
by fixing *evidence variables* and summing over *hidden variables*

\downarrow
whether we have toothache

\downarrow
variable we don't
have any evidence
catch \neg *catch*

Inference by enumeration, contd.

Typically, we are interested in

the posterior joint distribution of the query variables \mathbf{Y}
given specific values \mathbf{e} for the evidence variables \mathbf{E}

Let the hidden variables be $\mathbf{H} = \mathbf{X} - \mathbf{Y} - \mathbf{E}$ all r.v. query variable evidence variable

Then the required summation of joint entries is done by summing out the hidden variables:

$$\mathbf{P}(\mathbf{Y}|\mathbf{E}=\mathbf{e}) = \alpha \mathbf{P}(\mathbf{Y}, \mathbf{E}=\mathbf{e}) = \alpha \sum_{\mathbf{h}} \mathbf{P}(\mathbf{Y}, \mathbf{E}=\mathbf{e}, \mathbf{H}=\mathbf{h})$$

The terms in the summation are joint entries because \mathbf{Y} , \mathbf{E} , and \mathbf{H} together exhaust the set of random variables

Obvious problems: n.variable, in each variable it has d value

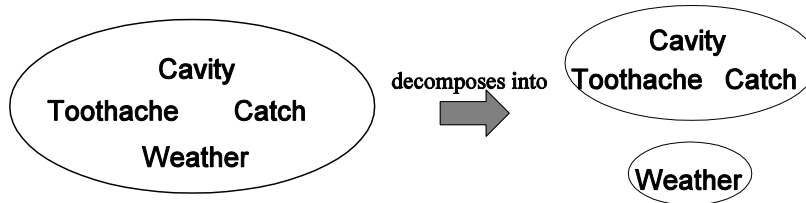
- 1) Worst-case time complexity $O(d^n)$ where d is the largest arity
- 2) Space complexity $O(d^n)$ to store the joint distribution
- 3) How to find the numbers for $O(d^n)$ entries???

in our case
3 variable
boolean
 $2^3 = 8$
 \Rightarrow strictly, only 7
Prob has to add up to 1

Independence

A and B are independent iff

$$\mathbf{P}(A|B) = \mathbf{P}(A) \quad \text{or} \quad \mathbf{P}(B|A) = \mathbf{P}(B) \quad \text{or} \quad \mathbf{P}(A, B) = \mathbf{P}(A)\mathbf{P}(B)$$



$$\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity}, \textit{Weather}) \\ = \mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity}) \mathbf{P}(\textit{Weather})$$

32 entries reduced to 12; for n independent biased coins, $2^n \rightarrow n \rightarrow \mathbf{P}(A) \cdot \mathbf{P}(B) \dots$

Absolute independence powerful but rare

Dentistry is a large field with hundreds of variables, none of which are independent. What to do?

Conditional independence

$P(\text{Toothache}, \text{Cavity}, \text{Catch})$ has $2^3 - 1 = 7$ independent entries

If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:

$$(1) P(\text{catch} | \text{toothache}, \text{cavity}) = P(\text{catch} | \text{cavity})$$

The same independence holds if I haven't got a cavity:

$$(2) P(\text{catch} | \text{toothache}, \neg \text{cavity}) = P(\text{catch} | \neg \text{cavity})$$

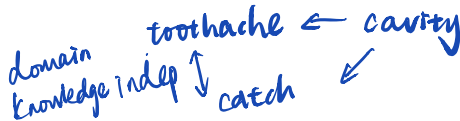
given cavity, then catch is independent of toothache
 Catch is *conditionally independent* of Toothache given Cavity:

$$P(\text{Catch} | \text{Toothache}, \text{Cavity}) = P(\text{Catch} | \text{Cavity})$$

Equivalent statements:

$$P(\text{Toothache} | \text{Catch}, \text{Cavity}) = P(\text{Toothache} | \text{Cavity})$$

$$P(\text{Toothache}, \text{Catch} | \text{Cavity}) = P(\text{Toothache} | \text{Cavity})P(\text{Catch} | \text{Cavity})$$



Conditional independence contd.

Write out full joint distribution using chain rule:

$$\begin{aligned} & \mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity}) \\ &= \mathbf{P}(\textit{Toothache} | \textit{Catch}, \textit{Cavity}) \mathbf{P}(\textit{Catch}, \textit{Cavity}) \\ &= \mathbf{P}(\textit{Toothache} | \textit{Catch}, \textit{Cavity}) \mathbf{P}(\textit{Catch} | \textit{Cavity}) \mathbf{P}(\textit{Cavity}) \\ \text{conditional indep} &= \mathbf{P}(\textit{Toothache} | \textit{Cavity}) \mathbf{P}(\textit{Catch} | \textit{Cavity}) \mathbf{P}(\textit{Cavity}) \end{aligned}$$

I.e., 2 + 2 + 1 = 5 independent numbers (equations 1 and 2 remove 2)

In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in n to linear in n .

Conditional independence is our most basic and robust form of knowledge about uncertain environments.

Bayes' Rule

Product rule $P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$

$$\Rightarrow \text{Bayes' rule } P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

or in distribution form

$$\mathbf{P}(Y|X) = \frac{\mathbf{P}(X|Y)\mathbf{P}(Y)}{\mathbf{P}(X)} = \alpha \mathbf{P}(X|Y)\mathbf{P}(Y)$$

Useful for assessing diagnostic probability from causal probability:

$$P(\overset{\text{disease}}{\downarrow} \text{Cause} | \overset{\text{symptom}}{\downarrow} \text{Effect}) = \frac{P(\text{Effect}|\text{Cause})P(\text{Cause})}{P(\text{Effect})}$$

E.g., let M be meningitis, S be stiff neck:

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.8 \times 0.0001}{0.1} = 0.0008$$

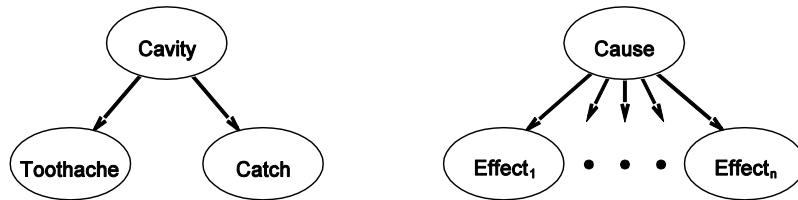
Note: posterior probability of meningitis still very small!

Bayes' Rule and conditional independence

$$\begin{aligned} & \mathbf{P}(Cavity|toothache \wedge catch) \\ &= \alpha \mathbf{P}(toothache \wedge catch|Cavity)\mathbf{P}(Cavity) \\ &= \alpha \mathbf{P}(toothache|Cavity)\mathbf{P}(catch|Cavity)\mathbf{P}(Cavity) \end{aligned}$$

This is an example of a *naive Bayes* model:

$$\mathbf{P}(Cause, Effect_1, \dots, Effect_n) = \mathbf{P}(Cause) \prod_i \mathbf{P}(Effect_i|Cause)$$



Total number of parameters is *linear* in n

Summary

Probability is a rigorous formalism for uncertain knowledge

Joint probability distribution specifies probability of every **atomic event**

Queries can be answered by summing over atomic events

For nontrivial domains, we must find a way to reduce the joint size

Independence and **conditional independence** provide the tools

Examples of skills expected:

- ◇ Calculate conditional probabilities using inference by enumeration
- ◇ Use conditional independence to simplify probability calculations
- ◇ Use Bayes' rule for solving diagnostic problems