COMP20007 Design of Algorithms

Complexity Theory

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Lecture 21

Semester 1, 2020

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- Complexity theory asks a different question: "What is the inherent difficulty of the problem?"
- It does however also uses asymptotic notation, although usually more concerned with lower bounds (Ω notation).
- Tigther ("larger") lower bounds give us guarantees on best possible algorithms.

Complexity Theory - Examples

Comparison Sorting (worst case)

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if we gonna to sort array of n numbers,
we need to look at n elements

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n ! possible remutation

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- A less trivial lower bound: $\Omega(\log n!) \approx \Omega(n \log n)$
  - Can be found by using a technique called Decision Trees.

$\boxed{?}$ BINAry tree

with n ! leaves

height = log(# leaves).

minimum compariasion.

$h = \log(n!) = \Omega(n\log n)$

any possiblealgorithm
based on comparison
sorting has to perform
at least nlogn

(cannot go better
than $\Omega(n\log n)$)
for worst case

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Matrix Multiplication square $n \times n$

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STRASSEN's MATRIX ALGORITHMS $O(n^{2.187})$

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This lecture: discussion about “hardness” of problems.

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  - Knapsack: "Is there a set of items of values at least i and weight at most j?"

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True $\rightarrow$ n is not prime

False $\rightarrow$? ?

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  - "Given a graph G and a sequence of nodes $\{v_1, v_2, \ldots, v_n\}$, verify if there is a ~~path~~ circuit that follows that sequence.

start from v1
keep check whether
next node is
connected
$\boxed{\text{linear}}$

Now we can define what is P and what is NP.

P and NP

- A problem is in *P* if its decision version has a solution which is polynomial in the input size.

eg. matrix multiplication

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  will be able generate all candidates in parallel

  - 1) A non-deterministic "machine" generates a candidate.
  - 2) The verification algorithm verifies the solution.
  - 3) Repeat until verified.

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That’s where the *N* in *NP* comes from. =)

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any problem that can be decided in polynomial time

can also be verified in polynomial time

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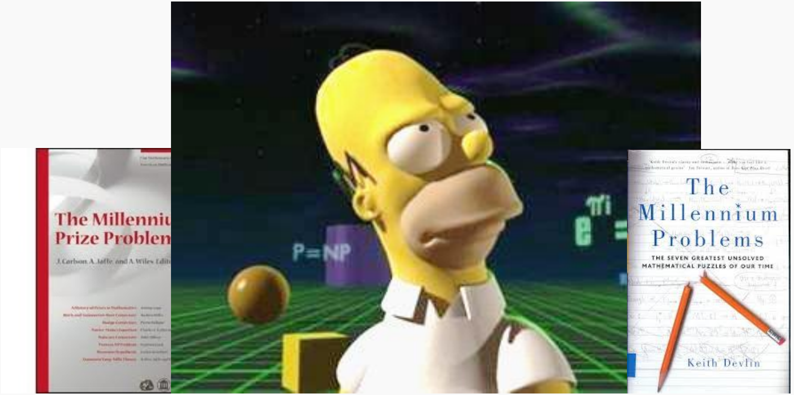
$$\mathbf{P} \stackrel{?}{=} \mathbf{NP}$$

A Million Dollar Question: Is P = NP?

This is one of the seven “millennium problems”: The Clay Institute’s seven most important unsolved mathematical problems.



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- For instance, the Hamiltonian Circuit (HAM) problem can be reduced to the decision version of TSP.

NP complete

Travelling salesman problem

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- It's a daunting task to find and prove bounds for every new problem.
- Reductions allow us to ease this by, roughly, framing a problem as equivalent to another one we know the class.
- For instance, the Hamiltonian Circuit (HAM) problem can be reduced to the decision version of TSP.
- The reduction function is polynomial. Therefore, since HAM is in *NP*, the decision version of TSP is also in *NP*.

From HAM to TSP

- Suppose we have a Hamiltonian Circuit in a graph G with n nodes.

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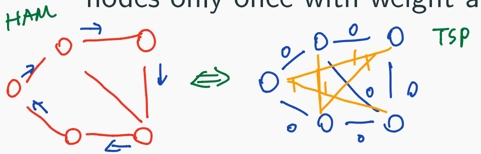
    - This can be done in polynomial time.

iterate all nodes

From HAM to TSP

- Suppose we have a Hamiltonian Circuit in a graph G with n nodes.
- Build a new graph G' where connected nodes in G have an edge of weight 0 and non-connected nodes have weight 1.
  - This can be done in polynomial time.
- Frame Decision-TSP as "Is there a circuit that visit all nodes only once with weight at most 0?"



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- $D \in NP$ **and**
  - Every problem in NP has a polynomial reduction to D **or**
  - A polynomial reduction from a known NP-complete problem to D exists.

Key property: if one finds a polynomial time algorithm to solve an NP-complete problem, then $P = NP$.

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Proving that every problem in *NP* has a polynomial reduction to *D* is hard.

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- "Given a boolean formula with a maximum of three literals, is there an assignment that results in TRUE?"
  - $(x_1 \overset{\text{or}}{\vee} \bar{x}_2 \vee \bar{x}_3) \overset{\text{and}}{\wedge} (\bar{x}_1 \vee x_2) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3) = \text{true}$
  - $\{x_1 = \textit{true}, x_2 = \textit{true}, x_3 = \textit{false}\}$

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- SAT
- Clique → find the maximum clique subgraph which is complete
- Vertex Cover → find the set of vertexes that cover all edges
- Hamiltonian Circuit
- Decision-TSP
- ...

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- Clique
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- Hamiltonian Circuit
- Decision-TSP
- ...

A polynomial time algorithm for any of these would imply that $P = NP$.

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- Decision problems: P contains problems with polynomial time solutions.
- Verification problems: NP contains problems with polynomial time solutions.
- Reductions let us analyse new problems by framing them as existing ones.
- NP-completeness: solving one NP-complete problem implies in $P = NP$ due to reductions.

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- Most scientists believe $P \neq NP$.

since NP complete problem

Last Words

- Some problems are undecidable: COMP30026
- Some scientists tried to prove that $P \stackrel{?}{=} NP$ is undecidable.
- Most scientists believe $P \neq NP$.
- While the problem itself still eludes computer scientists, proposed solutions led to advancements in theory, even though they were wrong.