MAST20009 Vector Calculus Assignment 1 Sem 1, 2020

Q1 $f(x,y) = \begin{cases} \dfrac{xy^6}{3x^4+4y^6} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

(a) Approach $(0,0)$ along $y = kx$, $k \in \mathbb{R} \setminus \{0\}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{k^6x^7}{3x^4+4k^6x^6}$$

$$= \lim_{x\to 0} \frac{k^6x^3}{3+4k^6x^2}$$

$$= \frac{0}{3}$$

$$= 0$$

(b) We expect that the limit is zero, so use sandwich rule.

If $(x,y) \neq (0,0)$ then

$$\frac{-|x|y^6}{3x^4+4y^6} \le \frac{xy^6}{3x^4+4y^6} \le \frac{|x|y^6}{3x^4+4y^6}$$

Since $\dfrac{y^6}{3x^4+4y^6} \le 1$ then

$$-|x| \le \frac{xy^6}{3x^4+4y^6} \le |x|$$

Now $\lim_{(x,y)\to(0,0)} (-|x|) = 0$ and $\lim_{(x,y)\to(0,0)} |x| = 0$, so

$$\lim_{(x,y)\to(0,0)} \frac{xy^6}{3x^4+4y^6} = 0 \text{ by sandwich rule.}$$

<u>NOTE:</u> We could also consider sandwich rule bounds for $x \geq 0$ and $y \geq 0$ cases separately

(c) If $(x,y) \neq (0,0)$ then

- $$\frac{\partial f}{\partial x} = \frac{y^6(3x^4+4y^6) - xy^6(12x^3)}{(3x^4+4y^6)^2}$$

$$= \frac{3x^4y^6 + 4y^{12} - 12x^4y^6}{(3x^4+4y^6)^2}$$

$$= \frac{4y^{12} - 9x^4y^6}{(3x^4+4y^6)^2} = \frac{4y^{12} - 9x^4y^6}{9x^8 + 24x^4y^6 + 16y^{12}}$$

- $$\frac{\partial f}{\partial y} = \frac{6xy^5(3x^4+4y^6) - xy^6(24y^5)}{(3x^4+4y^6)^2}$$

$$= \frac{18x^5y^5 + \cancel{24}xy^{11} - \cancel{24}xy^{11}}{(3x^4+4y^6)^2}$$

$$= \frac{18x^5y^5}{(3x^4+4y^6)^2} = \frac{18x^5y^5}{9x^8 + 24x^4y^6 + 16y^{12}}$$

If $(x,y) = (0,0)$ then

- $$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{0}{3h^4} - 0}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

- $$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{0}{4h^6} - 0}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

(d) If $(x,y) \neq (0,0)$

* $4y^{12} - 9x^4y^6$, $18x^5y^5$, $9x^8 + 24x^4y^6 + 16y^2$ are polynomials so are continuous everywhere

* $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are quotients of continuous functions so are continuous everywhere except where denominator is zero.

That is

$$9x^8 + 24x^4y^6 + 16y^{12} = (3x^4 + 4y^6)^2 = 0$$

$$\Rightarrow \quad x = y = 0$$

* Hence $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous for $(x,y) \neq (0,0)$

If $(x,y) = (0,0)$

* If $\frac{\partial f}{\partial x}$ is continuous at $(0,0)$ then $\lim\limits_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}\Big|_{(0,0)}$

• If approach $(0,0)$ along $x=0$ then

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = \lim_{y\to 0} \frac{4y^{12}}{16y^{12}} = \lim_{y\to 0} \frac{1}{4} = \frac{1}{4}$$

• If approach $(0,0)$ along $y=0$ then

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x} = \lim_{x\to 0} \frac{0}{9x^8} = \lim_{x\to 0} 0 = 0$$

Since limiting values are different when approach $(0,0)$ along $x=0$ and $y=0$, $\lim\limits_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}$ does not exist.

Hence $\frac{\partial f}{\partial x}$ is <u>not</u> continuous at $(0,0)$

Summary

Since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous if $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

Then f is C^1 if $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$\boxed{\text{NOTE}}$

We do not need to examine continuity of $\frac{\partial f}{\partial y}$ at $(x,y) = (0,0)$,

Since $\frac{\partial f}{\partial x}$ was not continuous at $(0,0)$, f is not C^1 at $(0,0)$.

$\boxed{Q2}$ $g(x,y) = (3y-x)^{3/2}$, $n=2$, $(a,b)=(2,1)$

(a) $g_x = -\frac{3}{2}(3y-x)^{1/2}$

$g_y = \frac{9}{2}(3y-x)^{1/2}$

$g_{xx} = \left(\frac{-3}{2}\right)\left(\frac{-1}{2}\right)(3y-x)^{-1/2} = \frac{3}{4}(3y-x)^{-1/2}$

$g_{xy} = \left(\frac{-3}{2}\right)\left(\frac{3}{2}\right)(3y-x)^{-1/2} = -\frac{9}{4}(3y-x)^{-1/2}$

$g_{yy} = \left(\frac{9}{2}\right)\left(\frac{3}{2}\right)(3y-x)^{-1/2} = \frac{27}{4}(3y-x)^{-1/2}$

If $(x,y)=(2,1)$ then $3y-x = 3-2=1$. Hence

$g(2,1)=1$, $g_x(2,1) = -\frac{3}{2}$, $g_y(2,1) = \frac{9}{2}$

$g_{xx}(2,1) = \frac{3}{4}$, $g_{xy}(2,1) = -\frac{9}{4}$, $g_{yy}(2,1) = \frac{27}{4}$

The 2nd order Taylor polynomial about $(2,1)$ is

$$P_2(x,y) = g(2,1) + (x-2)g_x(2,1) + (y-1)g_y(2,1)$$
$$+\frac{1}{2}\left[(x-2)^2 g_{xx}(2,1) + 2(x-2)(y-1)g_{xy}(2,1) + (y-1)^2 g_{yy}(2,1)\right]$$
$$= 1 - \frac{3}{2}(x-2) + \frac{9}{2}(y-1) + \frac{1}{2}\left[\frac{3}{4}(x-2)^2 - \frac{9}{2}(x-2)(y-1) + \frac{27}{4}(y-1)^2\right]$$
$$= 1 - \frac{3}{2}(x-2) + \frac{9}{2}(y-1) + \frac{3}{8}(x-2)^2 - \frac{9}{4}(x-2)(y-1) + \frac{27}{8}(y-1)^2$$

(b) $g(2.1, 0.9) \approx P_2(2.1, 0.9)$

$$= 1 - \frac{3}{2}(0.1) + \frac{9}{2}(-0.1) + \frac{3}{8}(0.1)^2 - \frac{9}{4}(0.1)(-0.1) + \frac{27}{8}(0.1)^2$$
$$= 1 - \frac{3}{20} - \frac{9}{20} + \frac{3}{800} + \frac{9}{400} + \frac{27}{800}$$
$$= \frac{800 - 120 - 360 + 3 + 18 + 27}{800}$$
$$= \frac{368}{800}$$
$$= \frac{23}{50}$$
$$= 0.46$$

(c) Find error $R_2(x,y)$

$$g_{xxx} = \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)(3y-x)^{-3/2} = \frac{3}{8}(3y-x)^{-3/2}$$

$$g_{xxy} = \left(\frac{3}{4}\right)\left(\frac{-3}{2}\right)(3y-x)^{-3/2} = -\frac{9}{8}(3y-x)^{-3/2}$$

$$g_{xyy} = \left(-\frac{9}{4}\right)\left(\frac{-3}{2}\right)(3y-x)^{-3/2} = \frac{27}{8}(3y-x)^{-3/2}$$

$$g_{yyy} = \left(\frac{27}{4}\right)\left(\frac{-3}{2}\right)(3y-x)^{-3/2} = \frac{-81}{8}(3y-x)^{-3/2}$$

The line $\underline{a} + \xi(\underline{x} - \underline{a}) = (2,1) + \xi\left((2.1, 0.9) - (2,1)\right)$

$$= (2,1) + \xi(0.1, -0.1)$$

$$= (2+0.1\xi, 1-0.1\xi) \quad \text{where } 0 < \xi < 1$$

The error in $P_2(x,y)$ is

$$R_2(x,y) = \frac{1}{6}\Big[(x-2)^3 g_{xxx}\big|_{(2+0.1\xi,\,1-0.1\xi)} + 3(x-2)^2(y-1)g_{xxy}\big|_{(2+0.1\xi,\,1-0.1\xi)}$$
$$+ 3(x-2)(y-1)^2 g_{xyy}\big|_{(2+0.1\xi,\,1-0.1\xi)} + (y-1)^3 g_{yyy}\big|_{(2+0.1\xi,\,1-0.1\xi)}\Big]$$

Now $3y - x = 3(1-0.1\xi) - (2+0.1\xi)$

$$= 3 - 0.3\xi - 2 - 0.1\xi$$

$$= 1 - 0.4\xi$$

$$\Rightarrow R_2(x,y) = \frac{1}{6}\Big[(0.1)^3 \cdot \frac{3}{8}(1-0.4\xi)^{-3/2} + 3(0.1)^2(-0.1)\left(-\frac{9}{8}\right)(1-0.4\xi)^{-3/2}$$
$$+ 3(0.1)(-0.1)^2(1-0.4\xi)^{-3/2}\left(\frac{27}{8}\right) + (-0.1)^3\left(-\frac{81}{8}\right)(1-0.4\xi)^{-3/2}\Big]$$

$$= \frac{(0.1)^3(1-0.4\xi)^{-3/2}}{6}\left[\frac{3}{8} + \frac{27}{8} + \frac{81}{8} + \frac{81}{8}\right]$$

$$= \frac{(0.1)^3(1-0.4\xi)^{-3/2}}{6}\left(\frac{192}{8}\right)$$

$$= 4(0.1)^3(1-0.4\xi)^{-3/2}$$

$$= \frac{1}{250(1-0.4\xi)^{3/2}}$$

Since $(1-0.4\xi)^{-3/2}$ is an increasing function for $0<\xi<1$, an upper bound for the error is

$$|R_2(x,y)| < \frac{1}{250(1-0.4)^{3/2}}$$

$$= \frac{1}{250(0.6)^{3/2}}$$

$$\simeq 0.008606$$

(d) The actual error of approximation is

$$|g(2.1, 0.9) - P_2(2.1, 0.9)|$$

$$= |(3(0.9)-2.1)^{3/2} - 0.46|$$

$$= |(0.6)^{3/2} - 0.46|$$

$$= |0.464758 - 0.46|$$

$$\simeq 0.004758$$

which is smaller than 0.008606 as expected.

It is about 0.003848 smaller than the upper bound using Taylor's formula.