COMP20007 Design of Algorithms

Binary Search Trees and their Extensions

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Lecture 14

Semester 1, 2020

Dictionaries

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- Abstract Data Structure
- Collection of *(key, value)* pairs
  - Values are usually *records*, such as my videogames or the Unimelb students.
  - Keys are (unique) identifiers, such as the name of a game or the student ID.
- Required operations:
  - *Search* for a value (given a key)
  - *Insert* a new pair
  - *Delete* an existent pair (given a key)

Dictionaries - Implementations

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Unsorted array / Linked list

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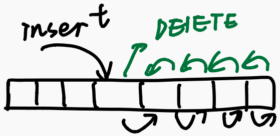

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This lecture: a better data structure

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This lecture: a better data structure

- Search: $\Theta(\log n)$ comparisons;
- Insert/Delete: $\Theta(\log n)$ record swaps;

Binary Search Tree

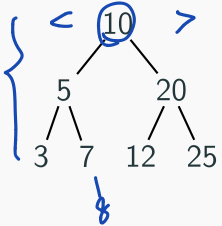
$\boxed{?}$. $\boxed{?}$

Delete

$\theta(\log n) + \theta(\log n)$ bring all subtree up

$\in \theta(\log n)$

$h = \log n$



search

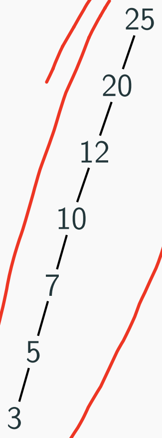
$\theta(\log n)$

Average case

insert $\theta(\log n) + \theta(1) \in \theta(\log n)$

search

Binary Search Tree - Worst Case



Linked List

$\Theta(n)$

for all operations

→ Highly unbalanced

⇓

change to balanced

BST - How to avoid degeneracy?

avoid it to be a linked list

Two options:

BST - How to avoid degeneracy?

Two options:

- Self-balancing
  - **AVL trees**
  - Red-black trees
  - Splay trees

BST - How to avoid degeneracy?

Two options:

- Self-balancing
  - **AVL trees**
  - Red-black trees
  - Splay trees
- Change the representation → NODES to have >1 elements
  - **2-3 trees**
  - 2-3-4 trees
  - B-trees

AVL trees

AVL trees

- Named after Adelson-Velsky and Landis.

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- A BST where each node has a *balance factor*: the difference in height between the left and right subtrees.

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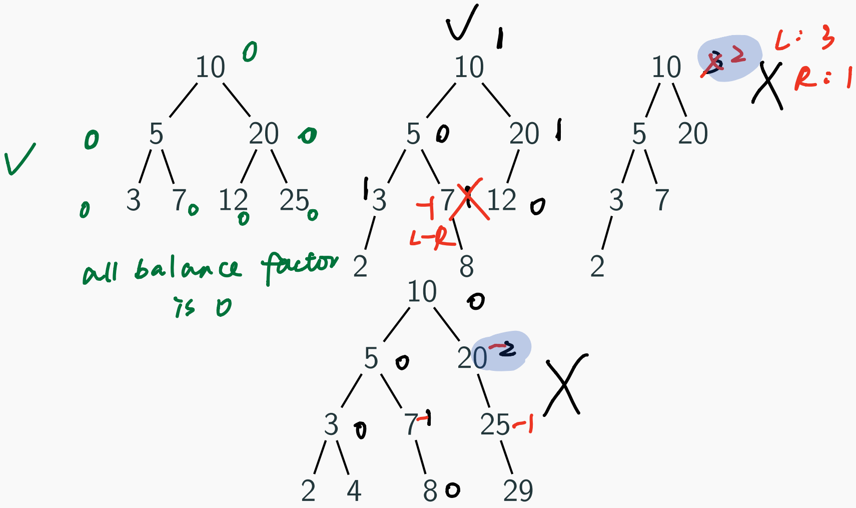
- A BST where each node has a *balance factor*: the difference in height between the left and right subtrees.
- When the balance factor becomes 2 or -2, *rotate* the tree to adjust them.

AVL Trees: Examples and Counter-Examples

Which of these are AVL trees?



AVL Trees - Rotations

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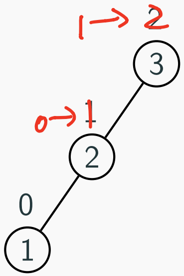
AVL Trees - Rotations

- Search is done as in BSTs.
- Insertion and Deletion also done as in BSTs, with additional steps at the end.
  - Update balance factors.
  - If the tree becomes unbalanced, perform *rotations* to rebalance it.

AVL Trees: R-Rotation



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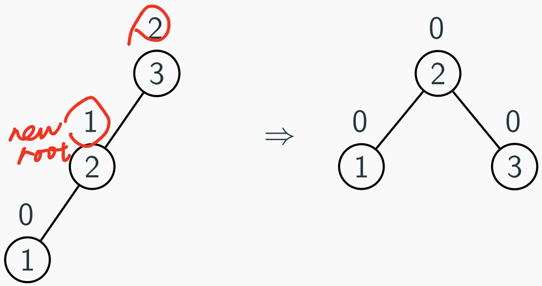
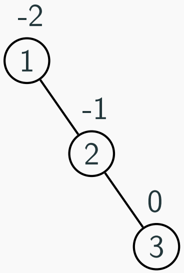


AVL Trees: L-Rotation



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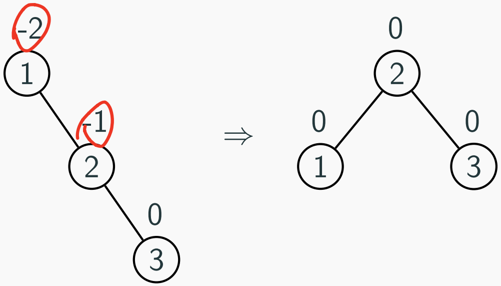
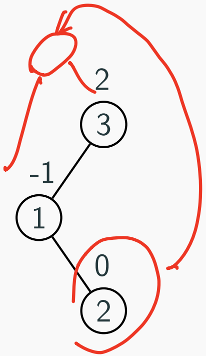


AVL Trees: LR-Rotation



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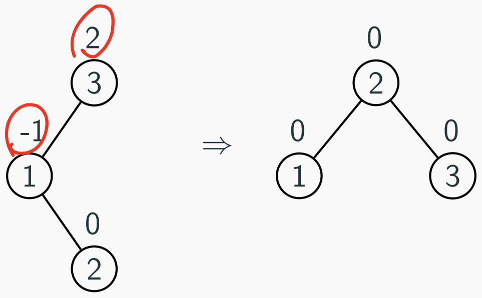

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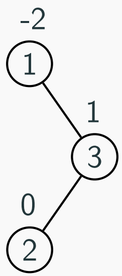
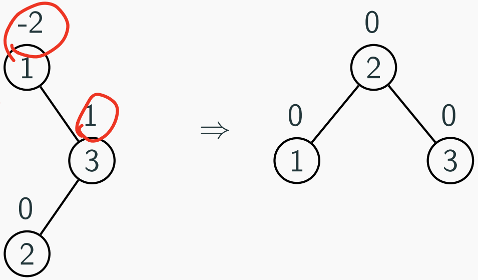


AVL Trees: RL-Rotation

the main while choosing the rotation is to look at balance factor

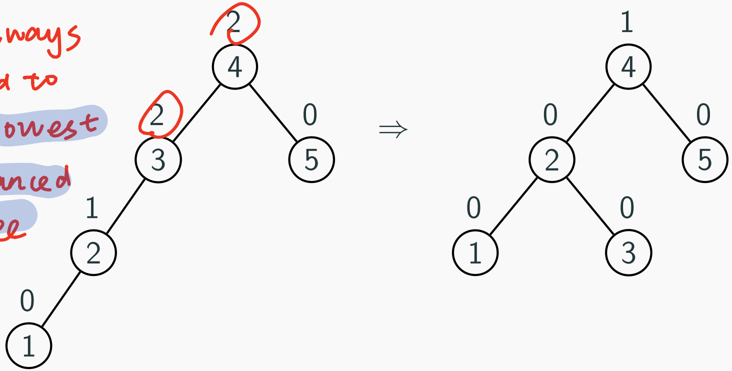


AVL Trees: Where to Perform the Rotation

Along an unbalanced path, we may have several

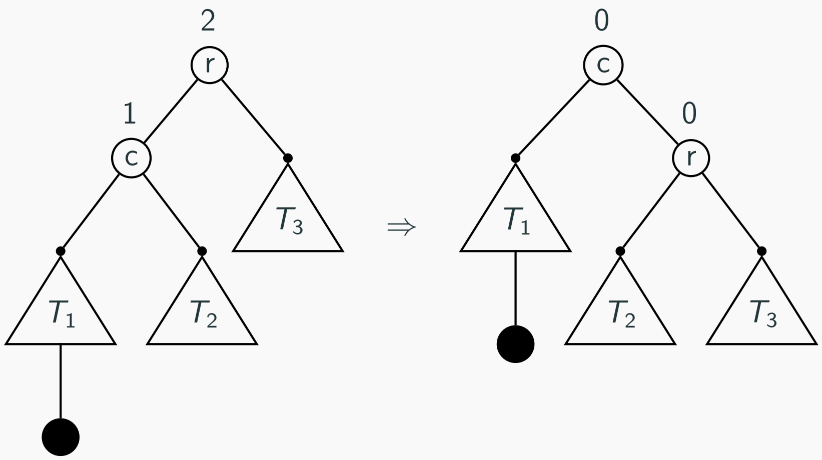
nodes with balance factor 2 (or -2):

we always applied to the lowest unbalanced tree



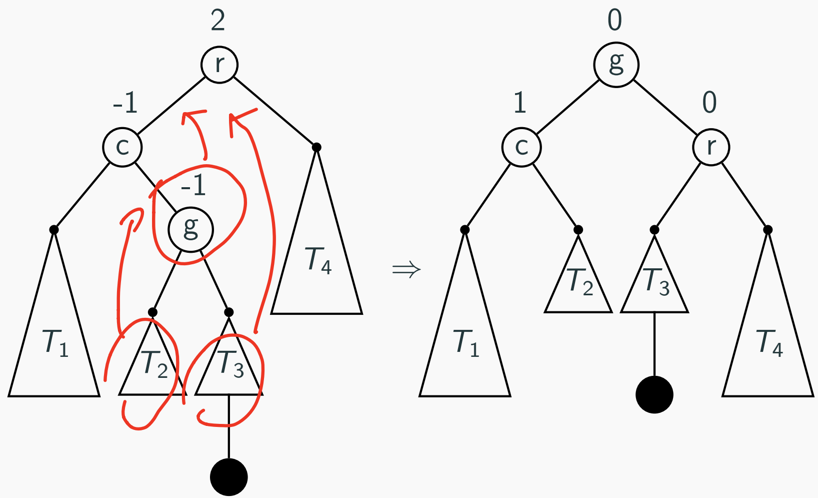
It is always the lowest unbalanced subtree that is re-balanced.

AVL Trees: The Single Rotation, Generally



This shows an R-rotation; an L-rotation is similar.

AVL Trees: The Double Rotation, Generally



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Properties of AVL Trees

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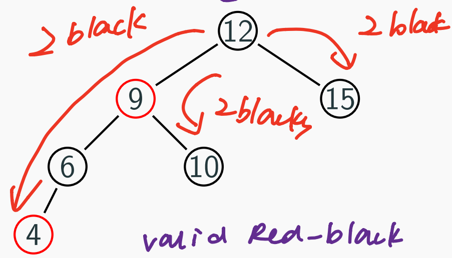
- This ensures all three operations are $\Theta(\log n)$.

Red-black Trees

- A red-black tree is another self-balancing BST.
- Its nodes are coloured red or black so that:

not a valid VAL tree

2 balance factor



1. No red node has a red child.
2. Every path from the root to the leaves has the same number of black nodes.

complexity operation

valid Red-black tree

A worst-case red-black tree (the longest path is twice as long as the shortest path).

Self-balancing trees - In practice

AVL trees vs. red-black trees

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AVL trees vs. red-black trees

- AVL trees are "more balanced" but require more frequent rotations.
  - Better if searches are more frequent than insertions/deletions.

more rotation required

when we require less dynamic

more static structure

Self-balancing trees - In practice

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Key property: rotations keep trees in a shape that guarantees $\Theta(\log n)$ operations.

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  - A 2-node contains one element and at most two children (as in BSTs)
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- Easy way to keep the tree balanced.
- Can be extended in many ways: 2–3–4 trees, B-trees, etc.

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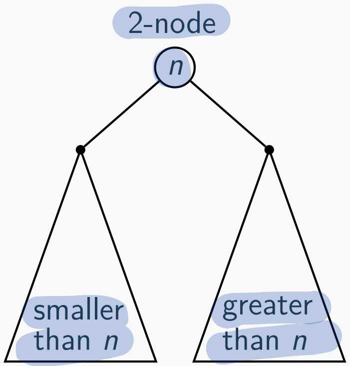

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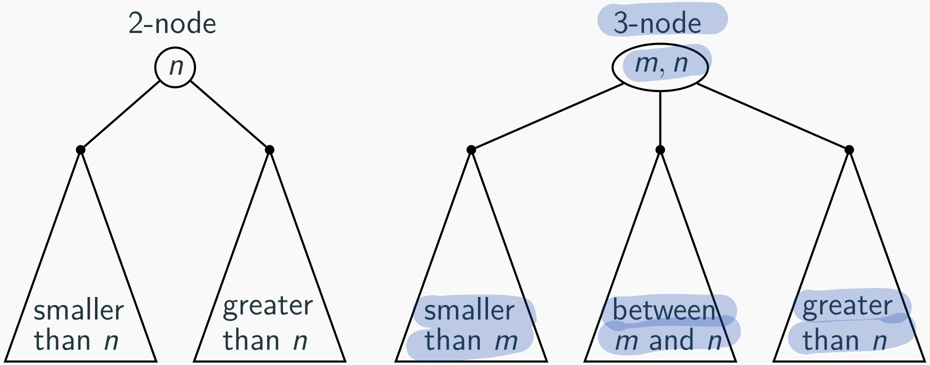

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2 nodes   3 nodes

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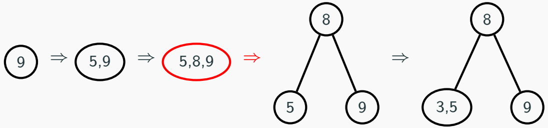

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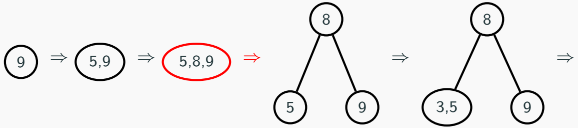

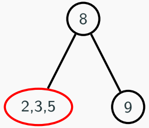

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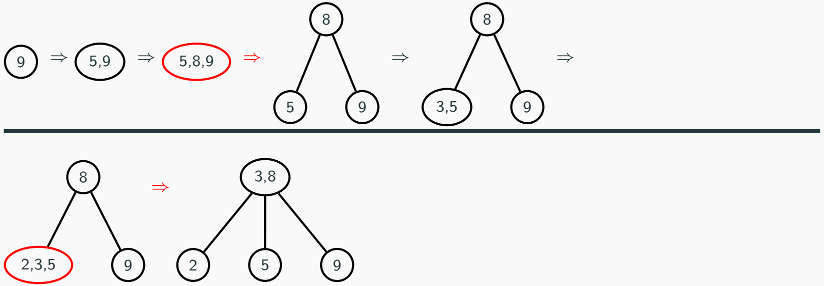

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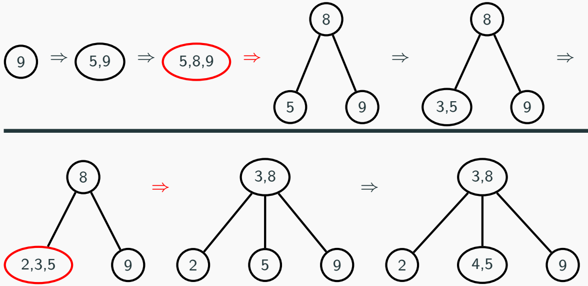

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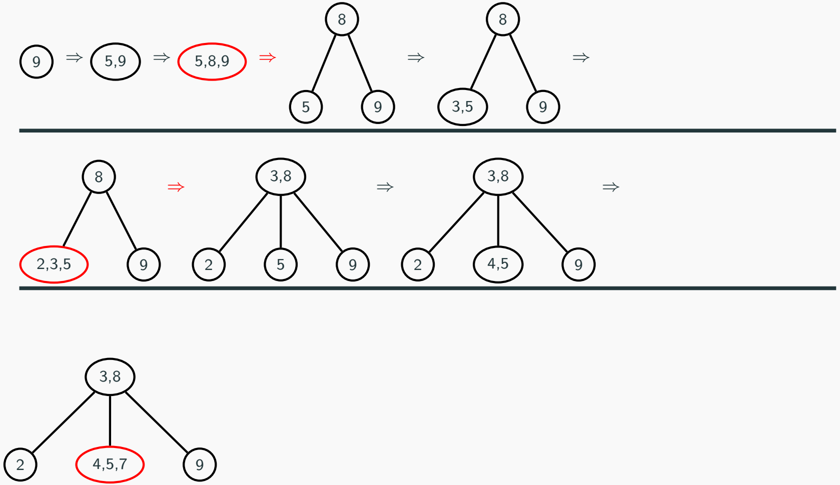

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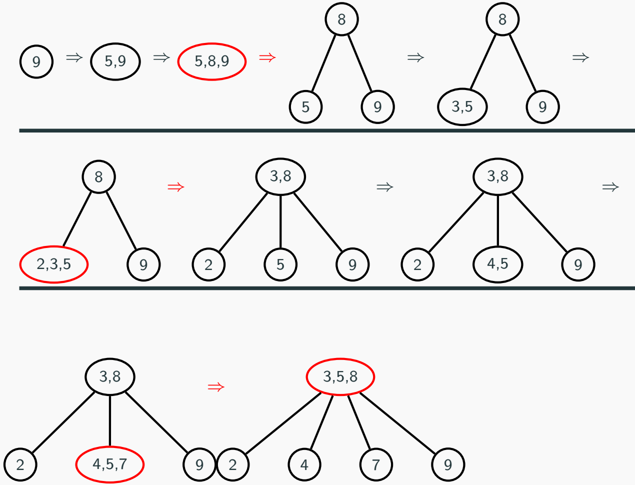

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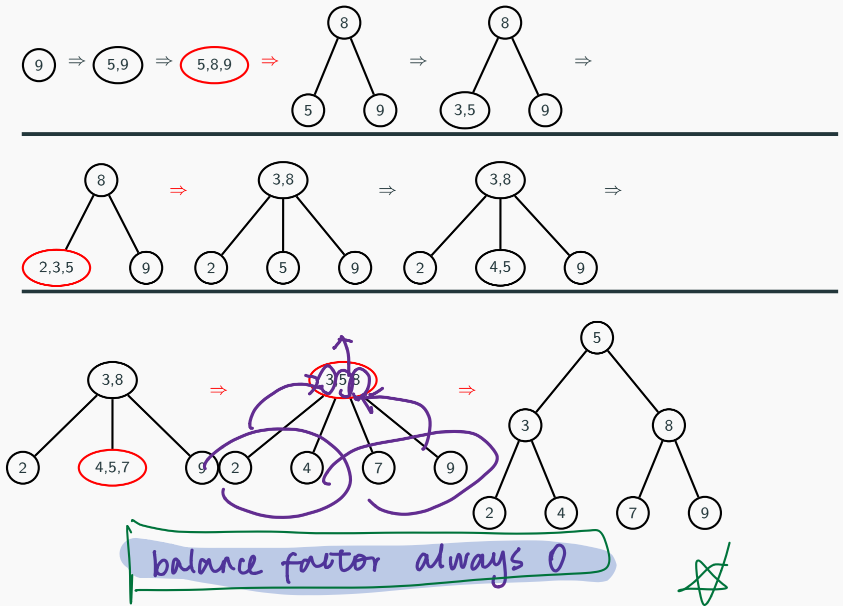


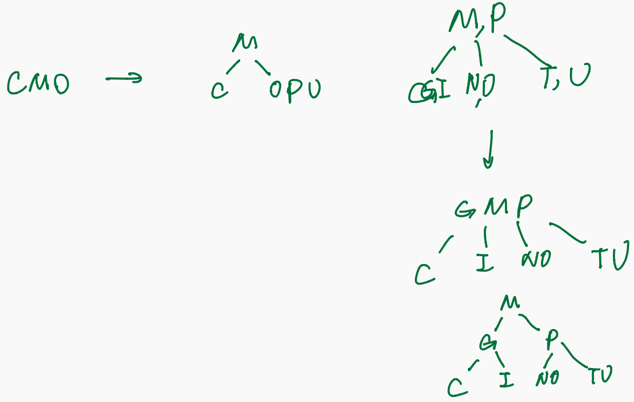
Exercise: 2–3 Tree Construction

Build the 2–3 tree that results from inserting these keys, in the given order, into an initially empty tree:

C, O, M, P, U, T, I, N, G

Extensions

- 2–3–4 trees: includes 4-nodes.



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Key property: balance is achieved by allowing multiple elements per node.

BSTs - Summary

element
record

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- BSTs provide $\Theta(\log n)$ search, insert and delete operations.
- Standard BSTs can degrade to linked lists and $\Theta(n)$ worst case performance.
  - Self-balancing: AVL trees
  - Change of representation: 2–3 trees

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insertion is the
most frequent operation

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hard drive

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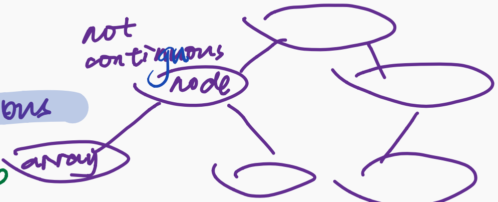
  - C++ *maps*: implemented via Red-black trees.
- Multiple elements per node are a better choice when secondary memory is involved.
  - B-trees (and its variants) are widely used in SQL databases (PostgreSQL, SQLite) and filesystems (Ext4, Reiser4).



contiguous

not much jump

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Next week: C++ maps use BSTs. What about Python *dicts*, do they also use BSTs? (spoiler: no)