5.4 Optimisation in two variables

Consider $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ and let (x_0, y_0) be a point in the interior of D.

We say that *f has a local maximum at (x_0, y_0)* if

We say that *f has a local minimum at (x_0, y_0)* if

Theorem 5.18. If f has a local maximum or local minimum at (x_0, y_0), then either

- $(\nabla f)(x_0, y_0) = (0, 0)$
- or at least one of f_x, f_y does not exist at (x_0, y_0).

The point (x_0, y_0) is called a *critical point of f* if $(\nabla f)(x_0, y_0) = (0, 0)$.

Be careful! Not all critical points are local maxima or minima.

Example 5.19. Find the critical points of $f(x, y) = x^2 + y^2$.

Example 5.20. Find the critical points of $f(x, y) = y^2 - x^2$.

Example 5.21. Find the critical points of $f(x, y) = x^2 + 6xy + 4y^2 + 2x - 4y$.

Suppose f has continuous second order partial derivatives in a neighbourhood of (x_0, y_0).

The symmetric 2×2 matrix

$$\mathbf{H} = \mathbf{H}_f(x_0, y_0) = \begin{bmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{bmatrix}$$

is called *the Hessian of f at (x_0, y_0)*.

Note that we know that $\mathbf{H}$ has real eigenvalues.

Theorem 5.22 (Second Derivative Test)**.** Let (x_0, y_0) be a critical point of f. Then (x_0, y_0) is a

(a) local minimum if $\mathbf{H}$ has positive eigenvalues

(b) local maximum if $\mathbf{H}$ has negative eigenvalues

(c) saddle point if $\mathbf{H}$ has one positive eigenvalue and one negative eigenvalue.

The test is inconclusive if $\mathbf{H}$ is not invertible.

Corollary 5.23 (Second Derivative Test, Alternative Version)**.** Let (x_0, y_0) be a critical point of f. Then (x_0, y_0) is a

(a) local minimum if $\det \mathbf{H} > 0$ and $f_{xx}(x_0, y_0) > 0$

(b) local maximum if $\det \mathbf{H} > 0$ and $f_{xx}(x_0, y_0) < 0$

(c) saddle point if $\det \mathbf{H} < 0$.

The test is inconclusive if $\det \mathbf{H} = 0$.

Example 5.24. Classify the critical points of $f(x, y) = x^2 + 6xy + 4y^2 + 2x - 4y$.

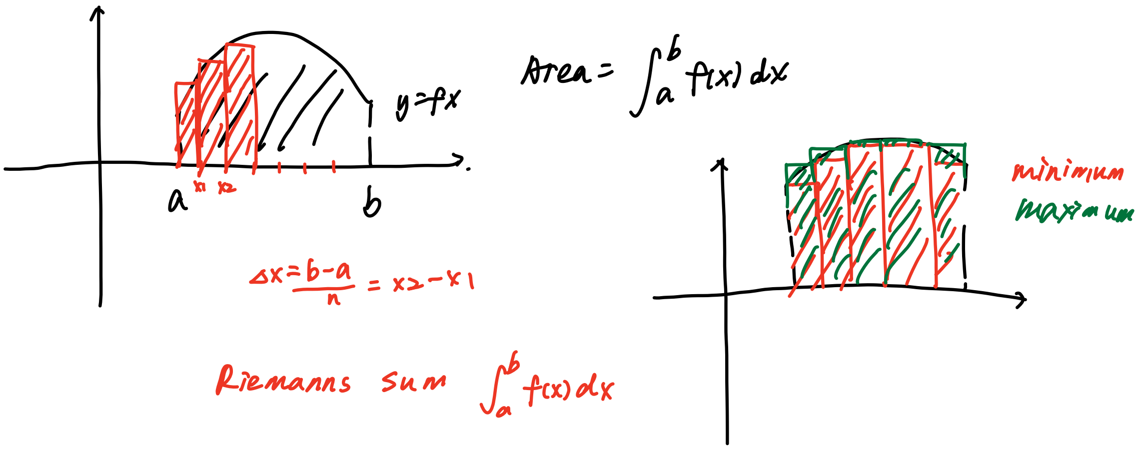
Example 5.25. Classify the critical points of $f(x, y) = x^2y + x^4 - y^3/3$.

5.5 Double integrals

If $f\colon (a,b) \subset \mathbb{R} \to \mathbb{R}$ is a function of a single variable, then the definite integral

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

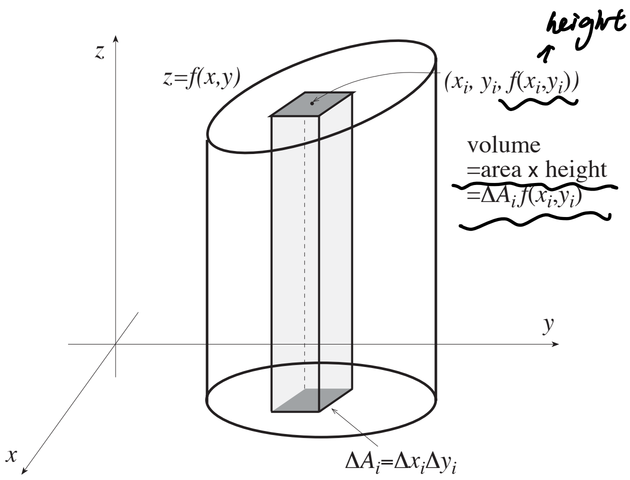
is the (signed) area of the plane region between the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$.



Suppose we are now working with a function $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$. We define the *(double) integral of f over D*, denoted

$$\iint_D f(x,y)\,dA,$$

to be the (signed) volume of the solid region between the graph of f, the xy-plane, and the "cylinder over D":



Of course, almost nobody computes actual definite integrals (whether simple or double) via Riemann sums.

In practice, we proceed in a manner similar to partial differentiation: we integrate with respect to one variable at a time.

Example 5.26.

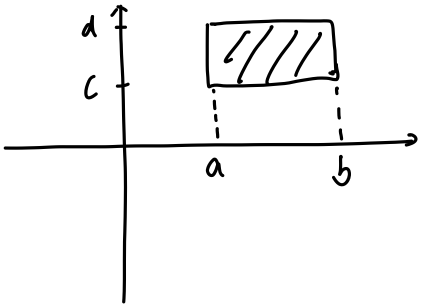
$$\int (3x^2y+12xy)\,dx = 3y\int x^2\,dx + 12y\int x\,dx$$

$$= 3y\cdot\frac{1}{3}x^3 + 12y\cdot\frac{1}{2}x^2$$

$$= x^3y + 6x^2y + C(y)$$

The simplest type of region for integration is a rectangle $D = [a, b] \times [c, d]$.

Then



$$\iint_D f(x,y)\,dA$$

$$= \int_c^d \left(\boxed{\int_a^b f(x,y)\,dx} \right) dy$$

a function related to y

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

Theorem 5.27 (Fubini)**.** If $D = [a, b] \times [c, d]$ then

$$\iint_D f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Example 5.28. Integrate in two ways, changing the order of integration:

$$\int_0^1 \int_0^1 (3x^2y + 12xy)\,dx\,dy = \int_0^1 \left[(x^3y + 6yx^2)\right]_{x=0}^{x=1} dy$$

$$= \int_0^1 (y + 6y)\,dy$$

$$= \int_0^1 7y\,dy$$

$$= \left[\frac{7}{2}y^2\right]_0^1 = \frac{7}{2}$$

$$\int_0^1 \left(\int_0^1 (3x^2y + 12xy)\,dy\right) dx = \int_0^1 \left[\frac{3x^2}{2}y^2 + 6xy^2\right]_0^1 dx$$

$$= \int_0^1 \left(\frac{3}{2}x^2 + 6x\right) dx$$

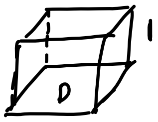
$$= \left(\frac{3}{2} \times \frac{1}{3}x^3 + 3x^2\right)_0^1$$

$$= \frac{1}{2} + 3 = \frac{7}{2}$$

We've already seen that certain volumes can be expressed as double integrals.

There are many other applications:

- If $D \subset \mathbb{R}^2$ is a plane region, then its area is given by



$$\iint_D 1 \, dA$$

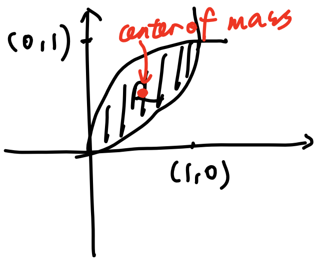
- If a flat object occupies a plane region $D \subset \mathbb{R}^2$ and $\rho \colon D \to \mathbb{R}$ is the function describing the density of the object at various points, then the mass of the object is given by

$$m = \iint_D \rho(x,y) \, dA$$

- The coordinates of the centre of mass of a flat object as above are given by

$$\bar{x} = \frac{\iint_D x\,\rho(x,y)\,dA}{m} \qquad \bar{y} = \frac{\iint_D y\,\rho(x,y)\,dA}{m}$$

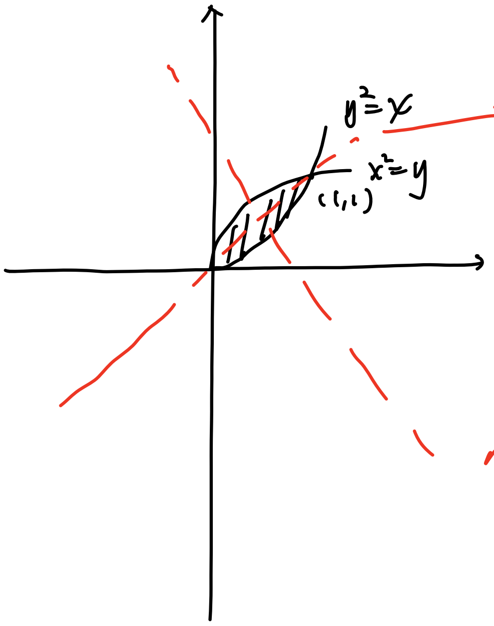
Example: Region as below



symmetric

com $(\frac{1}{2}, \frac{1}{2})$

Example 5.29. Find the centre of mass of a uniformly-dense flat object occupying the plane region R between the parabolas $y^2 = x$ and $x^2 = y$.



symmetric about this line.

not symmetric about the line

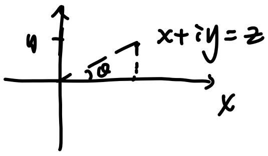
$$m = \int_0^1 \int_{x^2}^{\sqrt{x}} (dx\,dy) = \frac{1}{3}$$

$$\iint x\rho(x,y)\,dA = \int_0^1$$

5.6 Complex numbers

Recall that a complex number $z \in \mathbb{C}$ can be given in Cartesian form

$$z = \underbrace{x}_{\text{real part}} + \underbrace{iy}_{\text{imaginary part}} \qquad x, y \in \mathbb{R}, i^2 = -1,$$



or in polar form

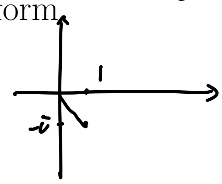
$$z = r\underbrace{\boxed{e^{i\theta}}}_{\text{cis}\,\theta} = r\cos\theta + ir\sin\theta \qquad r \in \mathbb{R}_{\geq 0}, \theta \in \mathbb{R}.$$

$$e^{i\theta} \cdot e^{i\alpha} = e^{i(\theta+\alpha)}$$

$$e^{i\cdot 0} = 1$$

Example 5.30. Write $1 - i$ in polar form.

$$z = \sqrt{2}\, e^{i(-\frac{\pi}{4})}$$



Write $2e^{i\pi/6}$ in Cartesian form.

$$2e^{i(\frac{\pi}{6})} = 2\cdot\left(\cos(\tfrac{\pi}{6}) + i\cdot\sin(\tfrac{\pi}{6})\right)$$

$$= 2\cdot\frac{\sqrt{3}}{2} + 2\cdot i\cdot\frac{1}{2}$$

$$= \sqrt{3} + i$$

The polar form is particularly well-suited for multiplications.

De Moivre's formula:

$$(re^{i\theta})^n = r^n (e^{i\theta})^n = r^n e^{i\theta n}$$

Example 5.31. Compute $(1-i)^{123}$.

$$1-i = \sqrt{2}\, e^{i(-\frac{\pi}{4})}$$

$$\left(\sqrt{2}\, e^{i(-\frac{\pi}{4})}\right)^{123} = (\sqrt{2})^{123} e^{i(123 \times (-\frac{\pi}{4}))}$$

$$= (\sqrt{2})^{123} e^{i(-\frac{123\pi}{4})}$$

$123 = 30\cdot 4 + 3$

$\frac{123\pi}{4} = \frac{30\cdot 4\pi}{4} + \frac{3\pi}{4}$

$$= 2^{61}\cdot\sqrt{2}\, e^{i(-\frac{3\pi}{4})}$$

$$= 2^{61}\cdot\sqrt{2}\cdot e^{i\cdot\frac{5\pi}{4}}$$

$$= 2^{61}\cdot\sqrt{2}\left(\cos\left(\tfrac{5\pi}{4}\right) + i\sin\left(\tfrac{5\pi}{4}\right)\right)$$

$$= 2^{61}\cdot\sqrt{2}\left(-\tfrac{\sqrt{2}}{2} + i\cdot -\tfrac{\sqrt{2}}{2}\right)$$

$$= -2^{61}(1+i)$$

5.6.1 Nasty trigonometric identities

We can exploit the relations

$e^{i\theta} = \cos\theta + i\cdot\sin\theta$

$$\cos\theta = \mathrm{Re}(e^{i\theta}), \qquad \sin\theta = \mathrm{Im}(e^{i\theta})$$

to find identities for sine or cosine of multiple angles.

Example 5.32. Express $\cos(3\theta)$ as a polynomial in $\cos\theta$. → Go to complex number → return to Real number

linear combination to power

$$\cos 3\theta = \mathrm{Re}(e^{i3\theta}) = \mathrm{Re}(e^{i\theta})^3 = \mathrm{Re}(\cos\theta + i\sin\theta)^3$$

$$= \mathrm{Re}(\cos^3\theta + 3\cos^2\theta\cdot i\sin\theta + 3\cos\theta\cdot(i\sin\theta)^2 + (i\sin\theta)^3$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$$

$$= 4\cos^3\theta - 3\cos\theta$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

pascal's triangle

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Here is another way of relating trigonometric functions to complex exponentials:

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$
$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

proof $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

Example 5.33. Prove that

Power to combination

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta).$$

$\sin\theta = \left[\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right]^3$

$= \frac{1}{8i^3}\left(e^{i\theta} - e^{-i\theta}\right)^3$

$= \frac{1}{-8i}\left(e^{i3\theta} - 3e^{i2\theta}e^{-i\theta} + 3e^{i\theta}\cdot e^{-2i\theta} + e^{-i3\theta}\right)$

$= \frac{i}{8}\left(e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} + e^{-i3\theta}\right)$

$= \frac{i}{8}\left(2i\sin 3\theta - 3\left(2i\sin\theta\right)\right).$

$= \frac{-1}{4}\sin 3\theta + \frac{3}{4}\sin\theta = RHS.$

5.6.2 Real calculus of trigonometric functions via complex exponentials

We can extend the complex exponential to a function $\exp\colon \mathbb{C} \to \mathbb{C}$ by

$$e^{x+iy} = e^x\, e^{iy} = e^x\,(\cos y + i \sin y).$$

Let $z = a + ib \in \mathbb{C}$.

We can make sense of

$$\frac{d}{dt} e^{zt} = \frac{d}{dt}(e^{at+ibt}) = \frac{d}{dt}(e^{at} \cdot e^{ibt}) = \frac{d}{dt}\left[e^{at} \cdot (\cos bt + i \cdot \sin bt)\right]$$

$$= a \cdot e^{at}(\cos bt + i \cdot \sin bt) + e^{at}(-b\sin bt + ib\cos bt)$$

$$= e^{at}\left[a \cdot e^{ibt} + ib(\cos bt + i\sin bt)\right]$$

$$= e^{at} \cdot (a \cdot e^{ibt} + ib \cdot e^{ibt}) = e^{at} \cdot e^{ibt}(a+ib) = e^{a+ibt}(a+ib)$$

$$= e^{zt} \cdot z$$

$$= z \cdot e^{zt}$$

Then

$$\frac{d^n}{dt^n} e^{zt} =$$

$$\frac{d^2}{dt^2} e^{zt} = z^2 e^{zt}$$

$$\frac{d^n}{dt^n} = z^n e^{zt}$$

Example 5.34.

$$\frac{d}{dt}e^t\cos(5t) = e^t\cos(5t) - 5e^t\sin(5t)$$
$$= e^t(\cos(5t) - 5\sin(5t))$$

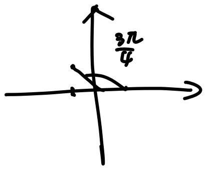
Alternative

$$\frac{d}{dt}(e^t\cos(5t)) = \frac{d}{dt}(e^t\,\mathrm{Re}(e^{5it})) = \frac{d}{dt}(\mathrm{Re}(e^{(1+5i)t}))$$
$$= \mathrm{Re}\left(\frac{d}{dt}(e^{(1+5i)t})\right)$$
$$= \mathrm{Re}\left((1+5i)\,e^{(1+5i)t}\right)$$
$$= \mathrm{Re}\left((1+5i)(e^t\cdot e^{5it})\right)$$
$$= e^t\,\mathrm{Re}\left[(1+5i)\cdot(\cos(5t)+i\cdot\sin(5t))\right]$$
$$= e^t(\cos(5t) - 5\sin(5t))$$

$$= e^t(\cos(5t) - 5\sin(5t))$$

Example 5.35.

$$\frac{d^{56}}{dt^{56}}e^{-t}\sin t = \frac{d^{56}}{dt^{56}}(e^{-t}\,\mathrm{Im}(e^{it}))$$
$$= \mathrm{Im}\left(\frac{d^{56}}{dt^{56}}e^{(i-1)t}\right)$$
$$= \mathrm{Im}\left[\frac{d^{56}}{dt^{56}}(i-1)\,e^{(i-1)t}\right]$$
$$= \mathrm{Im}\left[\frac{d^{56}}{dt^{56}}e^{i\cdot\frac{3\pi}{4}}\cdot e^{(i-1)t}\right]$$



$$= \mathrm{Im}\left[(\sqrt{2}e^{i\frac{3\pi}{4}})^{56}\,e^{(i-1)t}\right]$$
$$= \mathrm{Im}(2^{28}\cdot \underbrace{e^{42\pi i}}_{=e^0=1}\,e^{(i-1)t})$$
$$= \mathrm{Im}(2^{28}e^{(i-1)t})$$

$$= \mathrm{Im}\left[2^{28}(e^{-1t}\cdot e^{it})\right]$$
$$= \mathrm{Im}\left[2^{28}e^{-t}\cdot(\cos t + i\sin t)\right]$$
$$= 2^{28}e^{-t}\sin(t)$$

Example 5.36.

$$\frac{d}{dt}(e^{zt}) = ze^{zt}$$

$$\int e^{zt}\,dt = \frac{1}{z}\int e^{zt} + C$$

$$\begin{aligned}
\int e^t\cos(2t)\,dt &= \int e^t \,\mathrm{Re}(e^{2it})\,dt \\
&= \mathrm{Re}\int e^t e^{2it}\,dt \\
&= \mathrm{Re}\int e^{(1+2i)t}\,dt \\
&= \mathrm{Re}\left(\frac{1}{1+2i}e^{(1+2i)t}\right) + C \\
&= \mathrm{Re}\left[\left(\frac{1-2i}{(1+2i)(1-2i)}e^{(1+2i)t}\right)\right] + C \\
&= \frac{1}{5}\mathrm{Re}\left[1-2i\, e^{(1+2i)t}\right] + C \\
&= \frac{1}{5}\mathrm{Re}\left[(1-2i)e^t(\cos 2t + i\sin 2t)\right] + C, \\
&= \frac{1}{5}e^t(\cos 2t + 2\sin 2t) + C.
\end{aligned}$$

Example 5.37. Show that

Recall $\cos t = \mathrm{Re}(e^{it})$

$\sin t = \mathrm{Im}(e^{it})$

$$\int_0^\pi e^t\sin^2(t)\,dt = \frac{2}{5}(e^\pi - 1).$$

$e^{it} = \cos t + i\sin t$

$e^{-it} = \cos t - i\sin t$

$\sin t = \frac{1}{2i}(e^{it} - e^{-it})$

$$= \int_0^\pi e^t\left(\frac{1}{2i}(e^{it}-e^{-it})\right)^2 dt$$

$$= \int_0^\pi e^t\left(\frac{1}{-4}(e^{2it}+e^{-2it}-2)\right)dt$$

$$= -\frac{1}{4}\int_0^\pi (e^t\cdot e^{2it} + e^t\cdot e^{-2it} - 2e^t)\,dt$$

$$= -\frac{1}{4}\left(\int_0^\pi e^{(2i+1)t} + e^{(1-2i)t} - 2e^t\right)dt$$

$$= -\frac{1}{4}\left(\left[\frac{1}{2i+1}e^{(2i+1)t}\right]_0^\pi + \left[\frac{1}{1-2i}e^{(1-2i)t}\right]_0^\pi - 2\left[e^t\right]_0^\pi\right)$$

$$= -\frac{1}{4}\left[\frac{1}{2i+1}(e^\pi - 1) + \frac{1}{1-2i}(e^\pi - 1) - 2(e^\pi - 1)\right]$$

$$= -\frac{1}{4}\left(\frac{1}{2i+1} + \frac{1}{1-2i} - 2\right)\cdot(e^{\pi}-1)$$

5.6.3 Solving polynomial equations in $\mathbb{C}$

$$= -\frac{1}{4}\left(\frac{1-\cancel{2i}}{5} + \frac{1+\cancel{2i}}{5} - 2\right)(e^{\pi}-1) = \frac{2}{5}(e^{\pi}-1)$$

Theorem 5.38 (Fundamental Theorem of Algebra)**.** Any polynomial of degree n with complex coefficients has n complex roots (counted with multiplicity).

How do we actually find these roots?

Let's start with the special case of solving for z in

$$z^n = w,$$

where $n \in \mathbb{N}$ and $w \in \mathbb{C}$ is a fixed, given complex number.

Example 5.39. Find all complex solutions of $z^3 = 1$.

Method: 1 $\quad (z-1)(z^2+az+b)=0$

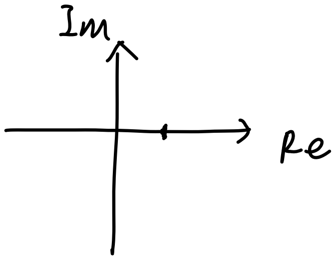
$$z^3+az^2+bz-z^2-az-b=0$$

$$z^3+(a-1)z^2+(b-a)z-b=0$$

$$a=1 \quad b=1$$

$$(z-1)(z^2+z+1)=0$$

Method 2: In polar form

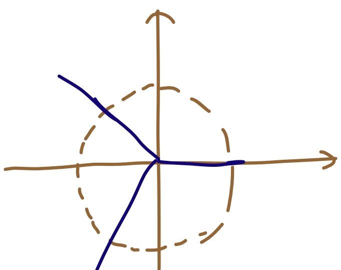


$$1 = 1+i0 = 1\cdot e^0$$

so $z = e^{i0},\ e^{i\cdot\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}}$

$$r^3 e^{i3\theta} = 1\cdot e^0$$

$$r=1$$



$$3\theta = 0+2k\pi,\quad k\in\mathbb{Z}$$

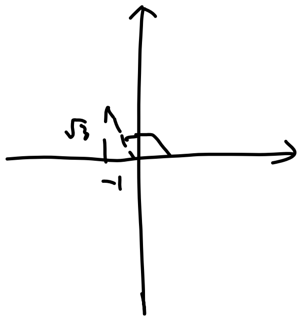
$$\theta = 0+\frac{2}{3}k\pi.\quad k\in\mathbb{Z}$$

$0, \frac{2\pi}{3}, \frac{4\pi}{3}$

Example 5.40. Find the fourth roots of $-1 + i\sqrt{3}$.

$$z^4 = -1 + i\sqrt{3}$$

$$r^4 e^{4\theta i} = 2e^{i\frac{2\pi}{3}}$$



$$r^4 = 2$$
$$r = 2^{\frac{1}{4}}$$

$$4\theta = \frac{2\pi}{3} + 2k\pi$$
$$\theta = \frac{\pi}{6} + \frac{1}{2}k\pi \quad k \in \mathbb{Z}$$

Roots are

$$z_1 = \sqrt[4]{2}\, e^{i\frac{\pi}{6}} \quad k=0$$

$$z_2 = \sqrt[4]{2}\, e^{i\frac{2\pi}{3}} \quad k=1$$

$$z_3 = \sqrt[4]{2}\, e^{i\frac{7\pi}{6}} \quad k=2$$

$$z_4 = \sqrt[4]{2}\, e^{i\frac{13\pi}{6}} \quad k=3$$

The general case $z^n = w$ is treated in the same fashion.

$$w = se^{i\alpha}$$

$$(re^{i\theta})^n = se^{i\alpha}$$

$$r^n e^{in\theta} = se^{i\alpha}$$

$$r^n = s \Rightarrow r = \sqrt[n]{s}$$

$$n\theta = \alpha + 2\pi k \quad k \in \mathbb{Z}$$

$$\theta = \frac{\alpha}{n} + \frac{2\pi}{n}k \quad k \in \mathbb{Z} \qquad k = 0 \dots n-1$$

$$z = \sqrt[n]{s}\, e^{i\left(\frac{\alpha}{n} + \frac{2\pi}{n}k\right)}$$

In the case of a general complex polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

we have

- $c \in \mathbb{C}$ is a root of p (that is $p(c) = 0$) if and only if $(z - c)$ is a factor of $p(z)$
- the Fundamental Theorem of Algebra tells us that $p(z)$ factors into linear terms:

$$p(z) = a_n(z - c_1)(z - c_2)\dots(z - c_n)$$

  with $c_i \in \mathbb{C}$.

- if p has **real coefficients** and $c \in \mathbb{C}$ is a root of p, then the complex conjugate $\overline{c}$ is also a root of p.

$a_i \in \mathbb{R} \rightarrow \overline{a_i} = a_i$

$p(c) = 0 \quad a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0 = 0$

$p(\overline{c}) = 0 \quad a_n(\overline{c})^n + a_{n-1}(\overline{c})^{n-1} + \cdots + a_1 \overline{c} + a_0$

$= \overline{a_n c^n} + \overline{a_{n-1} c^{n-1}} + \cdots + \overline{a_1 c} + a_0$

conjugate multiply rule.

conjugate add rule.

$= \overline{a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0} = \overline{0} = 0$

Example 5.41. Solve

$$z^4 - 2z^2 + 4 = 0.$$

$w = z^2$

$w^2 - 2w + 4 = 0$

$(w-1)^2 = -3$

$w - 1 = -\sqrt{3}i$ or $w - 1 = \sqrt{3}i$

$w = 1 \pm \sqrt{3}i$

$z^2 = 1 + \sqrt{3}i$ or $z^2 = 1 - \sqrt{3}i$

$z^2 = 2e^{i\cdot\frac{\pi}{3}}$ or $z^2 = 2\cdot e^{i\cdot\frac{-\pi}{3}}$

$\Downarrow$

$r^2 e^{i2\theta} = 2e^{i\frac{\pi}{3}}$

$r > 0 \quad r = \sqrt{2}$

$2\theta = \frac{\pi}{3} + 2k\pi$

$\theta = \frac{\pi}{6} + k\pi$

$z_1 = \sqrt{2}e^{i\frac{\pi}{6}}$

$\boxed{z_3 = \sqrt{2}e^{-i\frac{\pi}{6}} = \sqrt{2}e^{i\frac{11\pi}{6}} \quad z_4 = \sqrt{2}\cdot e^{i\cdot\frac{7\pi}{6}} = \sqrt{2}e^{i\frac{5\pi}{6}}}$

$$z_2 = \sqrt{2}\, e^{i\frac{7\pi}{6}}$$

Example 5.42. Solve

$$z^4 - z^3 + 27iz - 27i = 0.$$

$$z^3(z-1) + 27i(z-1) = 0$$

$$(z-1)(z^3+27i) = 0$$

$$z = 1 \qquad z^3 + 27i = 0$$

$$z^3 = -27i$$

Example 5.43. Let $k \in \mathbb{N}$. Solve

$$z^k + z^{k-1} + \cdots + z + 1 = 0.$$

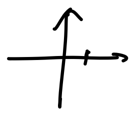
$$(z-1)(z^k + z^{k-1} + \cdots z + 1)$$

$$= (z^{k+1} + \cdots + z) - (z^k + \cdots z^{k-1} + \cdots z + 1)$$

$$= z^{k+1} - 1$$

$$z^{k+1} - 1 = 0 \qquad z \neq 1$$

$$z^{k+1} = 1 \cdot e^{i0}$$



$$r^{k+1} \cdot e^{i(k+1)\theta} = 1 \cdot e^{i0}$$

$$r = 1$$

$$(k+1)\theta = 2\pi w \quad w \in \mathbb{Z}$$

$$\theta = \frac{2\pi w}{k+1} \qquad w = 0 \ldots k$$

$$\theta = 0, \frac{2\pi}{k+1}, \frac{4\pi}{k+1}, \ldots, \frac{2\pi k}{k+1}$$

$$z = e^{i\frac{2\pi w}{k+1}} \quad w \in \{1, \dots, k\}$$

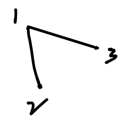
$\{0 \dots k$

$z = 1$ rule out

Adejacency matrix of Γ

$A\,(a_{uv})_{u}\quad v\in V$

$a_{uv} = \begin{cases} 1 & \text{if there are neighbours} \\ 0 & \text{otherwise} \end{cases}$



$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ real symmetric matrix

vector space $\mathcal{F}(V,\mathbb{R}) = \{f : V \to \mathbb{R}\}$

inner product on $\mathcal{F}(V,\mathbb{R})$

$$\langle f, g\rangle = \sum_{v\in V} f(v)g(v)$$

finite dimension ($\dim = \# V$)

Adjacent linear transformation

$$\mathcal{A} : \mathcal{F}(V,\mathbb{R}) \longrightarrow \mathcal{F}(V,\mathbb{R})$$

$$(\mathcal{A}f)(v) = \sum f(v)$$

$$\langle \mathcal{A}f, f\rangle = \sum_{u,v\in V} f(u)f(v) = \sum_{\{u,v\}\in E} f(u)f(v)$$

A real symmetric $\Rightarrow$ there is an orthonormal Spectrum basis of eigenvector for A, with real eigenvalues ~~order~~

$$\lambda_{\dim} = \lambda_n \le \cdots \le \lambda_2 \le \lambda_1 = \lambda_{max}$$

Theorem 1

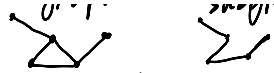
The eigenvalue λ_{max} has multiplicity one and has a positive eigenfuction

Theorem 2

$d_{avg} \le \lambda_{max} \le d$ with equality if and only if f is regular

averge degree

graph subgraph

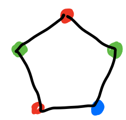


Theorem 3. let Γ' be a subgraph of Γ then

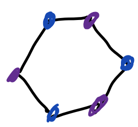
$$d_{max}(\Gamma') \le d_{max}(\Gamma)$$

The chromatic number of Γ is the smallest number of colours necessary for painting the vertices of Γ in such a way that no two neighbouring vertices have the same colour.

easy

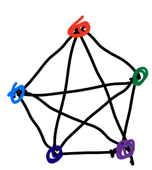


$\chi(C_5) = 3$



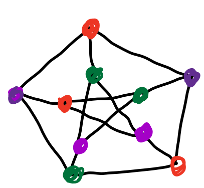
$\chi(C_6) = 2$

complete



$\chi(K_n) = n$

Petersen graph



$\chi = 3$

Bounds on the d chromatic number

$$1 + \frac{d_{max}(\Gamma)}{-d_{min}(\Gamma)} \le \chi(\Gamma) \le 1 + d_{max}(\Gamma)$$

Proof by induction

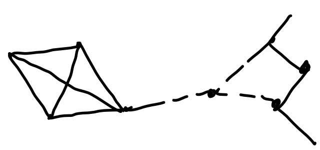
Base case, $n=1$, $\chi=1$ $d_{max}=0$ $A=[0]$

Induction step

Suppose true for all graph on n vertices

Let Γ be a graph on $n+1$ vertices

choose a vertix of Γ of minimal degree



Γ has no vertices

Γ can be coloured by at most $1+d_{max}(\Gamma')$ colours

$\le 1 + d_{max}(\Gamma)$

$deg(v) \le d_{ave} \le d_{max}$

so there is at least one of the $1+d_{max}(\Gamma)$ colours is available

$Tr(A) = 0 = \sum$ eigenvalues

Peterson graph $\chi = 3$

Bounds $1 + \frac{3}{-(-2)} \le 3 \le 1+3$

$2.5 \le 3 \le 4$