MAST30001 Stochastic Modelling – 2019

Assignment 2

If you haven't already, please complete the Plagiarism Declaration Form (available through the LMS) before submitting this assignment.

Don't forget to staple your solutions (note that there are no publicly available staplers in Peter Hall Building), and to put your name, student ID, tutorial time and day, and the subject name and code on the first page (not doing so will forfeit marks).

The submission deadline is **4:15pm on Friday, 25 October, 2019**, in the appropriate assignment box in Peter Hall Building (near Wilson Lab).

There are 2 questions, both of which will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness, and style count.

---

1. Let $\big(N(t)\big)_{t\geq 0}$ be a Poisson process with rate 1, and for $\varepsilon > 0$, let $X_1^{(\varepsilon)}, X_2^{(\varepsilon)}, \ldots$ be i.i.d. with density
$$C_\varepsilon^{-1} y^{-1} e^{-y}, \quad y > \varepsilon,$$
where the normalising constant is
$$C_\varepsilon = \int_\varepsilon^\infty x^{-1} e^{-x} dx.$$
Finally, for $t > 0$, define the time-scaled compound Poisson process
$$Z_t^{(\varepsilon)} = \sum_{j=1}^{N(tC_\varepsilon)} X_j^{(\varepsilon)}.$$

   (a) Show that for any $\varepsilon > 0$,
$$C_\varepsilon \leq \varepsilon^{-1} e^{-\varepsilon},$$
and that for any $0 < \varepsilon < 1$,
$$e^{-1} \log(1/\varepsilon) \leq C_\varepsilon.$$

   (b) Show that the number of jumps $N(tC_\varepsilon)$ of the process $(Z_s^{(\varepsilon)})_{s\geq 0}$ up to time t converges to infinity in probability as $\varepsilon \to 0^+$.

   (*That is, show that for all $n \in \mathbb{N}$, $\mathbb{P}(N(sC_\varepsilon) \geq n) \to 1$ as $\varepsilon \to 0^+$.*)

   (c) Show that the Laplace transform of $Z_t^{(\varepsilon)}$:
$$L_\varepsilon(\theta) := \mathbb{E}\big[e^{-\theta Z_t^{(\varepsilon)}}\big], \quad \theta \geq 0,$$
converges pointwise as $\varepsilon \to 0^+$, and identify the limit as the Laplace transform of a well-known distribution.

   (d) Explain in one or two sentences how the number of jumps can go to infinity, but the distribution of $Z_t^{(\varepsilon)}$ can converge.

In fact, the whole process $(Z_t^{(\varepsilon)})_{t\geq 0}$ converges to a process having independent increments and marginals given by part (c). The limit is a non-decreasing pure jump process, with the times of the jumps dense in the positive line.

Ans.

(a) For the upper bound, we have that $x^{-1} \le \varepsilon^{-1}$ for $x \ge \varepsilon$, so

$$C_\varepsilon \le \varepsilon^{-1} \int_\varepsilon^\infty e^{-x} dx \le \varepsilon^{-1} e^{-\varepsilon},$$

as required. For the lower bound, we have that $e^{-1} \le e^{-x}$ for $x \le 1$, so

$$C_\varepsilon \ge \int_\varepsilon^1 x^{-1} e^{-x} dx \ge e^{-1} \int_\varepsilon^1 x^{-1} dx = e^{-1} \log(1/\varepsilon).$$

(b) Letting T_n be the time of the nth jump in the Poisson process $N(t)_{t\ge 0}$, we have

$$\mathbb{P}(N(sC_\varepsilon) \ge n) = \mathbb{P}(T_n \le sC_\varepsilon) \to 1 \text{ as } C_\varepsilon \to \infty.$$

The result now easily follows by part (a), which says $C_\varepsilon \to \infty$ as $\varepsilon \to 0^+$.

(c) Using a formula from lecture (or conditioning on $N(tC_\varepsilon)$),

$$\begin{aligned}\mathbb{E}\big[e^{-\theta Z_t^{(\varepsilon)}}\big] &= \exp\big\{ - tC_\varepsilon(1 - \mathbb{E}[e^{-\theta X_1^{(\varepsilon)}}]\big\} \\ &= \exp\Big\{ - t \int_\varepsilon^\infty (1 - e^{-\theta x}) x^{-1} e^{-x} dx \Big\}.\end{aligned}$$

Now rewriting the integrand as an integral, we have

$$\begin{aligned}\int_\varepsilon^\infty (1 - e^{-\theta x}) x^{-1} e^{-x} dx &= \int_\varepsilon^\infty \int_0^\theta e^{-x(s+1)} ds dx \\ &= \int_0^\theta \int_\varepsilon^\infty e^{-x(s+1)} dx ds \\ &= \int_0^\theta (1+s)^{-1} e^{-\varepsilon(s+1)} ds.\end{aligned}$$

On the bounded interval, we can upper and lower bound the integrand to find

$$e^{-\varepsilon(\theta+1)} \log(1+\theta) \le \int_0^\theta (1+s)^{-1} e^{-\varepsilon(s+1)} ds \le \log(1+\theta).$$

Thus

$$\lim_{\varepsilon\to 0^+} \int_\varepsilon^\infty (1 - e^{-\theta x}) x^{-1} e^{-x} dx = \log(1+\theta),$$

and

$$L_\varepsilon(\theta) \to (1+\theta)^{-t},$$

which is the Laplace transform of the standard gamma with parameter t.

(d) The number of jumps in the process increases as ε decreases, but the sizes of the jumps decrease (in distribution) with ε in such a way that their sum in the limit is finite.

2. A certain queuing system has two types of customers and two types of servers. Type A customers arrive according to a Poisson process with rate 3, and, independently, Type B customers arrive according to a Poisson process with rate 2. If Server A is free, then an arriving Type A customer begins service with Server A. If Server A

is busy but Server B is free, then an arriving Type A customer will begin service with Server B. If an arriving Type A customer finds both servers busy, they will leave the system. If Server B is free, then an arriving Type B customer will be served by Server B, and otherwise will leave the system. Server A takes an exponential rate 2 time to finish a service, Server B takes an exponential rate 1 time to finish a service, and all service times are independent and independent of arrivals.

(a) Model the system as a four state Markov chain and write down its generator.

(b) Find the stationary distribution of the Markov chain.

(c) What is the stationary average number of customers in the system?

(d) Given a customer is not immediately rejected from the system, what is the average time they spend in the system?

(e) What is the long-run proportion of time there is a Type A customer being served by Server B?

Ans.

(a) We view the system as a CTMC with states $\{(0,0),(1,0),(0,1),(1,1)\}$, where $(0,0)$ means the system is empty, $(1,0)$ means the first server is busy and the second is not, $(0,1)$ means the second is busy and the first is not, and $(1,1)$ means both servers are busy. The generator is

$$A = \begin{pmatrix} -5 & 3 & 2 & 0 \\ 2 & -7 & 0 & 5 \\ 1 & 0 & -4 & 3 \\ 0 & 1 & 2 & -3 \end{pmatrix},$$

(b) Solving $\pi A = 0$ gives

$$\pi = \frac{1}{105}\left(11, 12, 31, 51\right).$$

(c) The average number of customers in the system is

$$\pi_{(0,1)} + \pi_{(1,0)} + 2\pi_{(1,1)} = 29/21 = 1.381.$$

(d) Using PASTA and thinning, we think of the arrival stream as a rate 5 Poisson process, and an arrival is Type A with probability $3/5$ and Type B with probability $2/5$. Thus, the expected time in the system of a customer given they have entered is

$$\frac{\frac{3}{5}\left((\pi_{(0,0)}+\pi_{(0,1)})(1/2)+\pi_{(1,0)}\right)+\frac{2}{5}\left(\pi_{(0,0)}+\pi_{(1,0)}\right)}{\frac{3}{5}\left((\pi_{(0,0)}+\pi_{(0,1)})+\pi_{(1,0)}\right)+\frac{2}{5}\left(\pi_{(0,0)}+\pi_{(1,0)}\right)} = \frac{145}{208} = 0.697.$$

(e) Since Server B serves both customers at the same rate, the proportion of time it is working on a Type A customer is the proportion of its customers that are of Type A, times the proportion of time the server is working. The argument from the previous part implies the first quantity is

$$\frac{\frac{3}{5}\pi_{(1,0)}}{\frac{3}{5}\pi_{(1,0)}+\frac{2}{5}(\pi_{(0,0)}+\pi_{(1,0)})} = \frac{18}{41},$$

and the second is

$$\pi_{(0,1)} + \pi_{(1,1)} = \frac{84}{105},$$

so the answer is

$$\frac{18}{41} \times \frac{82}{105} = \frac{36}{105} = 0.342857.$$

Alternatively, we can model the system as a six state Markov chain

$$\{(0,0), (A,0), (B,0), (0,A), (A,A), (A,B)\},$$

where the states represent what type of customer is being served by which server, with generator

$$\tilde{A} = \begin{pmatrix} -5 & 3 & 2 & 0 & 0 & 0 \\ 2 & -7 & 0 & 0 & 3 & 2 \\ 1 & 0 & -4 & 0 & 0 & 3 \\ 1 & 0 & 0 & -4 & 3 & 0 \\ 0 & 1 & 0 & 2 & -3 & 0 \\ 0 & 1 & 2 & 0 & 0 & -3 \end{pmatrix},$$

and stationary distribution

$$\tilde{\pi} = \frac{1}{105}\left(11, 12, 19, 12, 24, 27\right).$$

Then the solution is

$$\pi_{(0,A)} + \pi_{(A,A)} = \frac{36}{105} = 0.342857.$$