Assignment 1

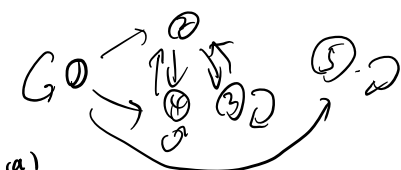Please complete the Plagiarism Declaration Form on the LMS before submitting this assignment.

Don't forget to staple your solutions (note that there are no publicly available staplers in Peter Hall Building), and to put your name, student ID, tutorial time and day, and the subject name and code on the first page (not doing so will forfeit marks).

The submission deadline is **4:15pm on Friday, 13 September, 2019**, in the appropriate assignment box in Peter Hall Building (near Wilson Lab).

There are 3 questions, all of which will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness, and style count.

1. A discrete time Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ has the following transition matrix.

$$P = \begin{pmatrix} 1/2 & 1/3 & 0 & 1/12 & 1/12 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 2/3 & 1/3 & 0 & 0 \\ 0 & 3/4 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(handwritten:)
(a)
$S_1 = \{1\}$
$S_2 = \{2,3,4\}$
$S_3 = \{5\}$

(b) since contains (d) loop, all aperiodic

(c) $\{2,3,4\}$ $\{5\}$ are essential

(d) $\{1\}$ transient
$\{2,3,4\}$ positively pos. recurrent
$\{5\}$ *(recurrent)*

   (a) Write down the communication classes of the chain.
   (b) Find the period of each communicating class.
   (c) Determine which classes are essential.
   (d) Classify each essential communicating class as transient or positive recurrent or null recurrent.
   (e) Describe the long run behaviour of the chain (including deriving long run probabilities where appropriate).
   (f) Find the expected number of steps taken for the chain to first reach state 3, given the chain starts at state 4.

2. Fix an integer $\alpha \geq 1$ and define the Markov chain $(X_n)_{n\geq 0}$ on $\{\alpha, \alpha + 1, \ldots\}$ by the transition probabilities,

$$p_{i,i+1} = 1 - p_{i,i-1} = \frac{\alpha}{i}, \quad i = \alpha, \alpha + 1, \alpha + 2, \ldots$$

   Determine the values of α for which the chain is positive recurrent, and for those values, find a stationary distribution of the chain.

3. Assume that the lifetimes (measured from the beginning of use) of lightbulbs are i.i.d. random variables with distribution

$$\mathbb{P}(T \geq k) = (k + 1)^{-\beta}, \quad k = 0, 1, 2, \ldots,$$

   for some $\beta > 0$. (Note that time is measured in discrete units.) In a lightbulb socket in a factory, a bulb is used until it fails, and then it is replaced at the next time unit. Let $(X_n)_{n\geq 0}$ be the irreducible Markov chain which records the age of the bulb currently in use in the socket ($X_n = 0$ at times when a bulb is replaced, corresponding to a new bulb).

   (a) Derive the transition probabilities of the chain.
   (b) For each value of β, determine if the chain is positive recurrent, null recurrent, or transient.