MAST30001 Stochastic Modelling

$\boxed{\textbf{Assignment 1}}$

Please complete the Plagiarism Declaration Form on the LMS before submitting this assignment.

Don't forget to staple your solutions (note that there are no publicly available staplers in Peter Hall Building), and to put your name, student ID, tutorial time and day, and the subject name and code on the first page (not doing so will forfeit marks).

The submission deadline is **4:15pm on Friday, 13 September, 2019**, in the appropriate assignment box in Peter Hall Building (near Wilson Lab).

There are 3 questions, all of which will be marked. No marks will be given for answers without clear and concise explanations. Clarity, neatness, and style count.

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1. A discrete time Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ has the following transition matrix.

$$P = \begin{pmatrix} 1/2 & 1/3 & 0 & 1/12 & 1/12 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 2/3 & 1/3 & 0 & 0 \\ 0 & 3/4 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (a) Write down the communication classes of the chain.

   (b) Find the period of each communicating class.

   (c) Determine which classes are essential.

   (d) Classify each essential communicating class as transient or positive recurrent or null recurrent.

   (e) Describe the long run behaviour of the chain (including deriving long run probabilities where appropriate).

   (f) Find the expected number of steps taken for the chain to first reach state 3, given the chain starts at state 4.

Ans.

(a) The communicating classes of the chain are $S_1 = \{1\}$, $S_2 = \{2, 3, 4\}$ and $S_3 = \{5\}$.

(b) All communicating classes are aperiodic (loop).

(c) S_2 and S_3 are essential.

(d) S_2 and S_3 are positive recurrent because they are finite and essential.

(e) The chain will eventually end up in S_2 or S_3. Because these classes are positive recurrent and aperiodic, they are ergodic, with long run probabilities given by their unique stationary distributions $\pi^{(2)} = (\pi_2, \pi_3, \pi_4)$ and $\pi^{(3)} = (\pi_5)$. Since 5 is an absorbing state, $\pi_5 = 1$, and $\pi^{(2)}$ satisfies

$$\pi^{(2)} \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/3 & 1/3 & 0 \\ 3/4 & 0 & 1/4 \end{pmatrix} = \pi^{(2)},$$

and $\pi_2 + \pi_3 + \pi_4 = 1$. Solving these equations yields

$$\pi = \left(\frac{12}{29}, \frac{9}{29}, \frac{8}{29} \right).$$

(f) For $j \in S, j \neq 3$, let e_j be the expected amount of time to reach state 3 given the chain starts in state j. We want to find e_4. First step analysis implies

$$\begin{aligned} e_4 &= 1 + \frac{1}{4}e_4 + \frac{3}{4}e_2, \\ e_2 &= 1 + \frac{1}{2}e_4, \end{aligned}$$

and solving gives $e_4 = 14/3$.

2. Fix an integer $\alpha \geq 1$ and define the Markov chain $(X_n)_{n\geq 0}$ on $\{\alpha, \alpha+1, \ldots\}$ by the transition probabilities,

$$p_{i,i+1} = 1 - p_{i,i-1} = \frac{\alpha}{i}, \quad i = \alpha, \alpha+1, \alpha+2, \ldots$$

Determine the values of α for which the chain is positive recurrent, and for those values, find a stationary distribution of the chain.

Ans.

We can determine positive recurrence by whether or not $\pi P = \pi$ has a probability vector solution π, where P is the transition matrix of the chain. These equations are, for $i > 0$,

$$\pi_i = \frac{\alpha}{i-1}\pi_{i-1} + \frac{i+1-\alpha}{i+1}\pi_{i+1},$$

and

$$\pi_\alpha = \frac{1}{\alpha+1}\pi_{\alpha+1}.$$

Using the recursion to solve the first few π_j, $j = \alpha+1, \alpha+2, \ldots$ in terms of π_α, we guess the general formula

$$\pi_i = \pi_\alpha \frac{i\alpha^{i-\alpha-1}}{(i-\alpha)!},$$

which is easily seen to satisfy the recursion above. The chain is positive recurrent if the π sequence is summable, so that π_α can be chose to make it a probability vector.

$$\begin{aligned} \sum_{i\geq\alpha} \pi_i &= \pi_\alpha \sum_{i\geq\alpha} \frac{i\alpha^{i-\alpha-1}}{(i-\alpha)!} \\ &= \pi_\alpha \sum_{j\geq 0} \frac{(j+\alpha)\alpha^{j-1}}{j!} \\ &= \pi_\alpha \sum_{j\geq 1} \frac{\alpha^{j-1}}{(j-1)!} + \pi_\alpha \sum_{j\geq 0} \frac{\alpha^j}{j!} \\ &= \pi_\alpha 2e^\alpha. \end{aligned}$$

Thus the chain is positive recurrent for any α, and the stationary distribution is

$$\pi_i = \frac{1}{2e^\alpha} \frac{i\alpha^{i-\alpha-1}}{(i-\alpha)!}.$$

3. Assume that the lifetimes (measured from the beginning of use) of lightbulbs are i.i.d. random variables with distribution

$$\mathbb{P}(T \geq k) = (k+1)^{-\beta}, \quad k = 0, 1, 2, \ldots,$$

for some $\beta > 0$. (Note that time is measured in discrete units.) In a lightbulb socket in a factory, a bulb is used until it fails, and then it is replaced at the next time unit. Let $(X_n)_{n\geq 0}$ be the irreducible Markov chain which records the age of the bulb currently in use in the socket ($X_n = 0$ at times when a bulb is replaced, corresponding to a new bulb).

(a) Derive the transition probabilities of the chain.

(b) For each value of β, determine if the chain is positive recurrent, null recurrent, or transient.

Ans.

(a) The event $X_n = j$ is the same as the current component has lasted at least j time units. Also, given $X_n = i$, $X_{n+1} \in \{0, i+1\}$, and therefore, for $i \geq 0$,

$$p_{i,i+1} = \mathbb{P}(X_n = i+1 | X_n = i) = \frac{\mathbb{P}(T \geq i+1)}{\mathbb{P}(T \geq i)} = \left(\frac{i+1}{i+2}\right)^{\beta},$$

$$p_{i,0} = 1 - p_{i,i+1} = 1 - \left(\frac{i+1}{i+2}\right)^{\beta}.$$

(b) The times between returns to zero are distributed as $T+1$. Thus, for all $\beta > 0$,

$$\mathbb{P}(T(0) < \infty | X_0 = 0) = \mathbb{P}(T+1 < \infty) = 1,$$

since $\lim_{k\to\infty} \mathbb{P}(T \geq k) = 0$, so the chain is always recurrent. Similarly,

$$\mathbb{E}[T(0)|X_0 = 0] = \mathbb{E}[T] + 1 = 1 + \sum_{k\geq 1} \mathbb{P}(T \geq k) = \sum_{k\geq 0} (k+1)^{-\beta},$$

which is finite if $\beta > 1$, and infinite if $\beta \leq 1$. Thus the chain is positive recurrent if $\beta > 1$ and null recurrent if $\beta \leq 1$.