MAST20009 Vector Calculus

Practice Class 5 Questions

> The domain determines the terminals and order of integration.
>
> - Vertical Strips
>   Integrate with respect to y first
> $$\phi_1(x) \le y \le \phi_2(x), \quad a \le x \le b$$
> - Horizontal Strips
>   Integrate with respect to x first
> $$\psi_1(y) \le x \le \psi_2(y), \quad c \le y \le d$$

$\int_1^e 1-\log x\,dx$

$[x]_1^e -$

$\int_1^e \log x\,dx$

$u=\log x$

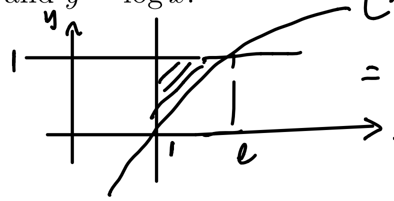
$\frac{dv}{dx}=1$

$\frac{dv}{dx}=\frac{1}{x}$ $v=x$

$[x\log x - 1]_1^e$

$= e\log e - 1 - 0$

1. Let R be the region bounded by the curves $x = 1$, $y = 1$ and $y = \log x$.
   (a) Sketch R.
   (b) Determine the area of R using double integrals.

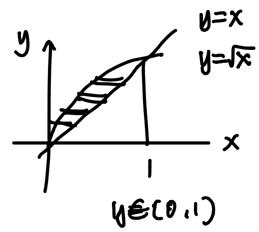


$1 \le x \le e$

$\log x \le y \le 1$

$[y]_{\log x}^{1}$

2. Consider the double integral
$$\int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx$$
   (a) Sketch the region of integration in the x-y plane.
   (b) Evaluate the integral in the order given. $\frac{1}{24}$
   (c) Change the order of integration and re-evaluate the integral.



$\int_0^1 (\frac12 x^2 - \frac12 x^3)\,dx$

$= \frac12 \int_0^1 (x^2 - x^3)\,dx$

$= \frac12 [\frac13 x^3 -$

$\int_0^1 [\frac12 x^2 y]_{y^2}^{y} dy$

$= \int_0^1 \frac12 (y^3 - y^5)\,dy = \frac12 \int_0^1 (\frac14 y^4 - \frac16 y^6) = \frac{1}{24}$

> Triple Integrals
>
> If $f(x, y, z)$ is a continuous function over a domain D in $\mathbb{R}^3$, we can evaluate the triple integral
> $$\iiint_D f(x,y,z)dV = \iiint_D f(x,y,z)dxdydz$$

3. Evaluate
$$\int_1^2 \int_{-1}^1 \int_0^3 \left(3x + 2y^2 + z^3\right) dx\,dy\,dz.$$

$= \int_1^2 \int_{-1}^1 [\frac32 x^2 + 2y^2 x + z^3 x]_{x=0}^{x=3} dy\,dz$

$= \int_1^2 (31 + 6z^3)\,dz$

$= [31z + \frac32 z^4]_1^2$

3

$= \int_1^2 \int_{-1}^{1} [\frac{27}{2} + 6y^2 + 3z^3]\, dy\, dz$

$= \int_1^2 [\frac{27}{2}y + 2y^3 + 3z^3 y]_{y=-1}^{y=1}\, dz$

$= 6z + 24 - 31 - \frac{3}{2}$

31

$= 55 - \frac{3}{2} = \frac{110-3}{2}$

$= \frac{107}{2}$

The domain D is an *elementary region* if one variable is bounded by functions of the other 2 variables, the domains of these functions being described using horizontal or vertical strips.

If z is bounded by two functions of x and y. The projection of D onto the xy-plane gives a region R that can be described by either vertical or horizontal strips. Then the domain D can be described as

$$f_1(x,y) \le z \le f_2(x,y)$$

and either

$$\phi_1(x) \le y \le \phi_2(x), \quad a \le x \le b \quad \underline{\text{OR}}$$

$$\psi_1(y) \le x \le \psi_2(y), \quad c \le y \le d$$

4. Let V be the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, $(0,0,3)$.

A B C

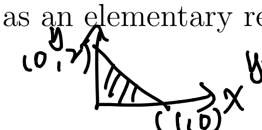
(a) Sketch the region V.

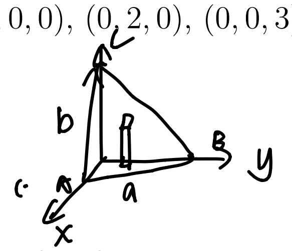
(b) Describe V as an elementary region.

(a) $0 \le z \le \frac{6-6x-3y}{2}$

$0 \le x \le 1$

$0 \le y \le -2x+2$





$\vec{a} = \vec{OB} - \vec{OA}$

$= (-1, 2, 0)$

$\vec{b} = (-1, 0, 3)$

5. Let D be the region bounded by the paraboloids $z = x^2 + y^2 - 7$ and $z = 9 - 3x^2 - 3y^2$.

(a) Sketch the region D.

(b) Determine where the paraboloids intersect.

(c) Using cartesian coordinates, evaluate the triple integral

$$\iiint_D x\, dV.$$

$\vec{a} \times \vec{b}$

$\begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix}$

$= (6, 3, 2)$.

$\boxed{6x + 3y + 2z = 6}$

When you have finished the above questions, continue working on the questions in the Vector Calculus Problem Sheet Booklet.

5.

(c).

$x^2 + y^2 - 7 \le z \le 9 - 3x^2 - 3y^2$

$x^2 + y^2 - 7 = 9 - 3x^2 - 3y^2$

$4x^2 + 4y^2 = 16$

$x^2 + y^2 \le 4$

$x^2 + y^2 = 4 \qquad z = -3$

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

$-2 \le x \le 2$

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2-7}^{9-3x^2-3y^2} x \, dz \, dy \, dx$$

$$9x - 3x^3 - 3y^2x$$

$$-x^3 - y^2x + 7x$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[zx \right]_{z=x^2+y^2-7}^{z=9-3x^2-3y^2} dy \, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(-4x^3 - 4y^2x + 16x \right) dy \, dx$$

$$= \int_{-2}^{2} \left[-4x^3y - \frac{4}{3}y^3x + 16xy \right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left(-8x^3\sqrt{4-x^2} - \frac{8}{3}x(4-x^2)^{\frac{3}{2}} + 32x\sqrt{4-x^2} \right) dx$$

$$= \int_{-2}^{2}$$