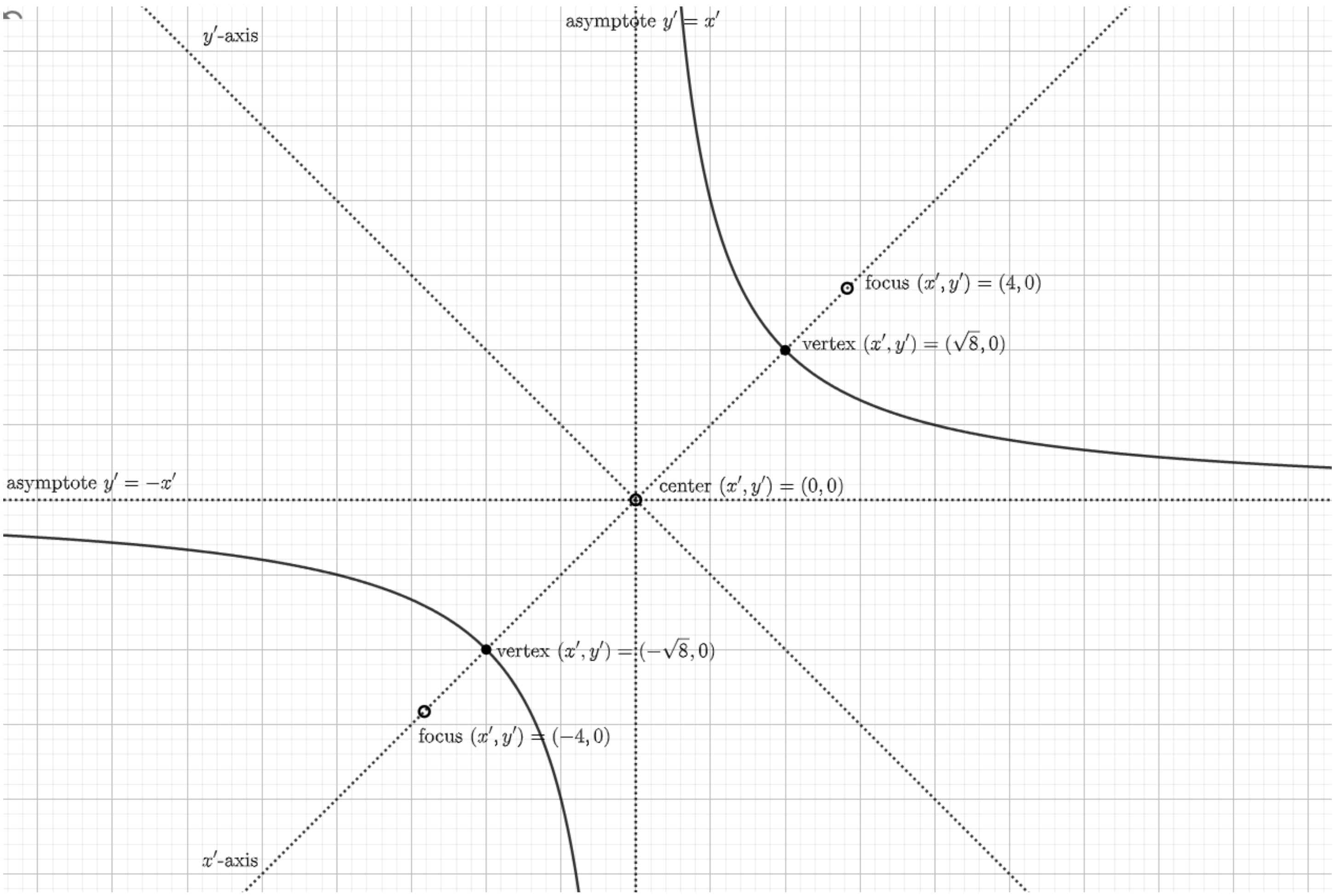
Homework Solutions - Rotation Equations

1a. $xy = 4; \alpha = 45^\circ$

Since $\alpha = 45^\circ$, we get the rotation equations $x = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'$ and $y = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$.

$$\begin{aligned} xy &= 4 \\ \left(\frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'\right)\left(\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'\right) &= 4 \\ \frac{1}{2}(x')^2 + \frac{1}{2}x'y' - \frac{1}{2}x'y' - \frac{1}{2}(y')^2 &= 4 \\ \frac{1}{2}(x')^2 - \frac{1}{2}(y')^2 &= 4 \\ \frac{(x')^2}{8} - \frac{(y')^2}{8} &= 1. \end{aligned}$$

This is a hyperbola with horizontal transverse axis:

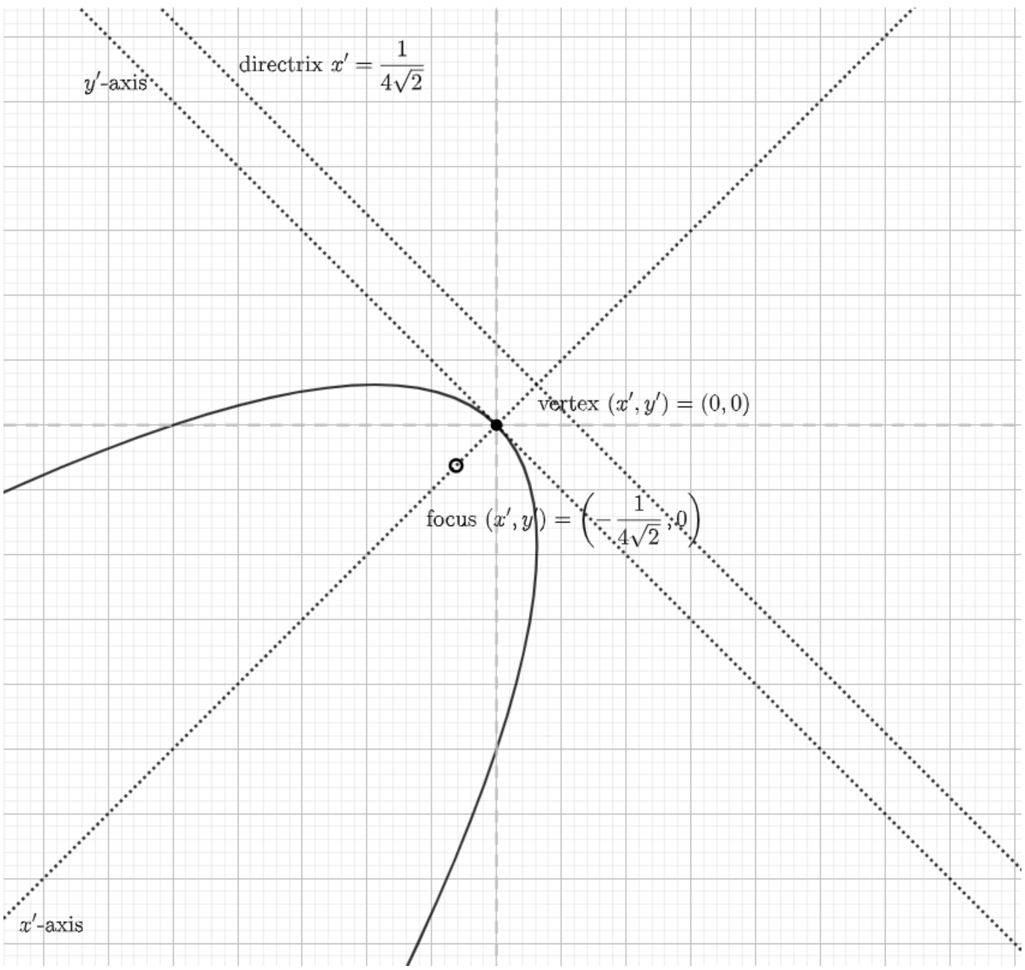


1b. $x^2 - 2xy + y^2 + x + y = 0; \alpha = 45^\circ$

Since $\alpha = 45^\circ$, we get the rotation equations $x = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'$ and $y = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'$.

$$
\begin{aligned}
x^2 - 2xy + y^2 + x + y &= 0 \\
\left(\frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'\right)^2 - 2\left(\frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'\right)\left(\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'\right)& \\
+\left(\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'\right)^2 + \left(\frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y'\right) + \left(\frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}y'\right) &= 0 \\
\left(\frac{1}{2}(x')^2 - \frac{2}{2}x'y' + \frac{1}{2}(y')^2\right) - 2\left(\frac{1}{2}(x')^2 + \frac{1}{2}x'y' - \frac{1}{2}x'y' - \frac{1}{2}(y')^2\right)& \\
+\left(\frac{1}{2}(x')^2 + \frac{2}{2}x'y' + \frac{1}{2}(y')^2\right) + \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y' + \frac{1}{\sqrt{2}}y' &= 0 \\
\left(\frac{1}{2}(x')^2 - \frac{2}{2}x'y' + \frac{1}{2}(y')^2\right) + \left(-\frac{2}{2}(x')^2 - \frac{2}{2}x'y' + \frac{2}{2}x'y' + \frac{2}{2}(y')^2\right)& \\
+\left(\frac{1}{2}(x')^2 + \frac{2}{2}x'y' + \frac{1}{2}(y')^2\right) + \frac{2}{\sqrt{2}}x' &= 0 \\
\left(\frac{1}{2}(x')^2 - \frac{2}{2}x'y' + \frac{1}{2}(y')^2\right) + \left(-\frac{2}{2}(x')^2 + \frac{2}{2}(y')^2\right) + \left(\frac{1}{2}(x')^2 + \frac{2}{2}x'y' + \frac{1}{2}(y')^2\right) + \frac{2}{\sqrt{2}}x' &= 0 \\
\left(\frac{1}{2}(x')^2 + -\frac{2}{2}(x')^2 + \frac{1}{2}(x')^2\right) + \left(-\frac{2}{2}x'y' + \frac{2}{2}x'y'\right) + \left(\frac{1}{2}(y')^2 + \frac{2}{2}(y')^2 + \frac{1}{2}(y')^2\right) + \frac{2}{\sqrt{2}}x' &= 0 \\
2(y')^2 + \frac{2}{\sqrt{2}}x' &= 0 \\
(y')^2 + \frac{1}{\sqrt{2}}x' &= 0 \\
(y')^2 &= -\frac{1}{\sqrt{2}}x'.
\end{aligned}
$$

This is a leftward opening parabola with vertex $(x', y') = (0, 0)$ and $c = \frac{\sqrt{2}}{4}$ (becuase $4c = \sqrt{2}$):

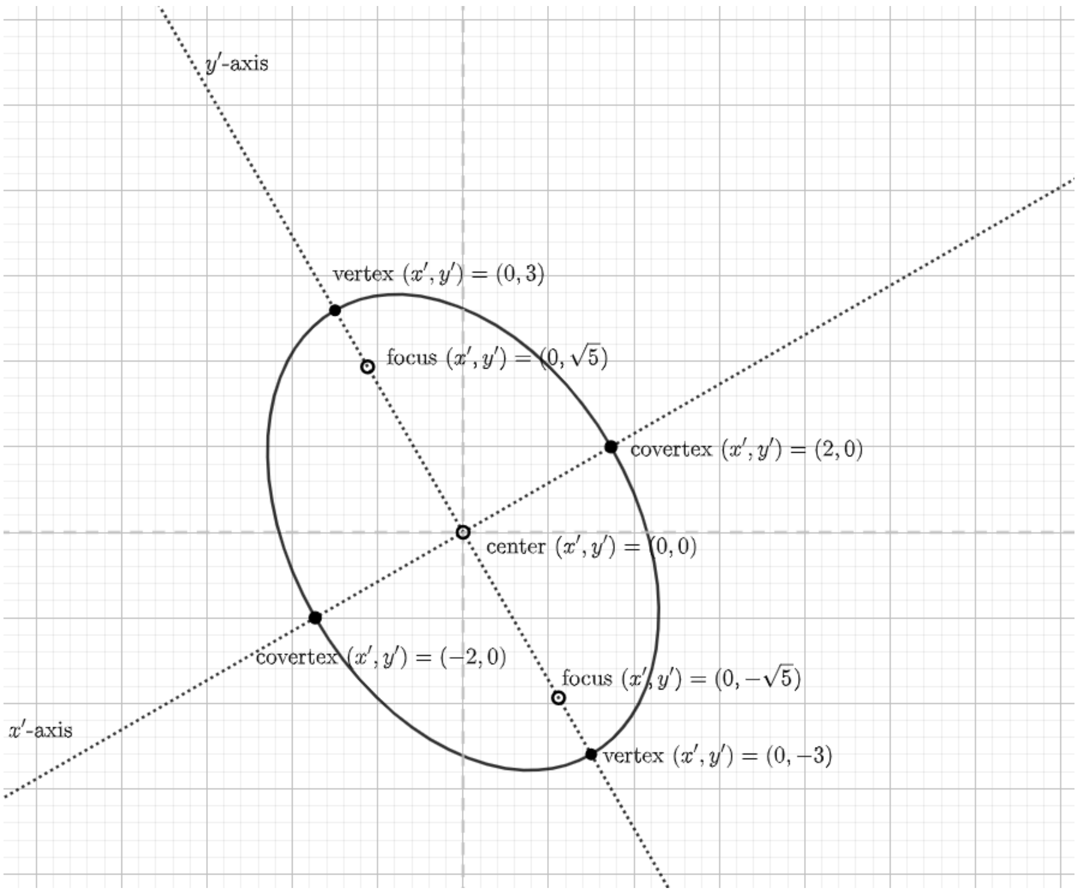


1c. $31x^2 + 10\sqrt{3}xy + 21y^2 - 144 = 0; \alpha = 30^\circ$

Since $\alpha = 30^\circ$, we get the rotation equations $x = \frac{\sqrt{3}}{2}x' - \frac{1}{2}y'$ and $y = \frac{1}{2}x' + \frac{\sqrt{3}}{2}y'$.

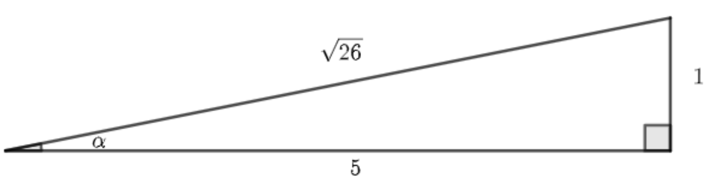
$$
\begin{aligned}
31x^2 + 10\sqrt{3}xy + 21y^2 - 144 &= 0 \\
31\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)^2 + 10\sqrt{3}\left(\frac{\sqrt{3}}{2}x' - \frac{1}{2}y'\right)\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right) + 21\left(\frac{1}{2}x' + \frac{\sqrt{3}}{2}y'\right)^2 - 144 &= 0 \\
31\left(\frac{3}{4}(x')^2 - \frac{2\sqrt{3}}{4}x'y' + \frac{1}{4}(y')^2\right)^2 + 10\sqrt{3}\left(\frac{\sqrt{3}}{4}(x')^2 + \frac{3}{4}x'y' - \frac{1}{4}x'y' - \frac{\sqrt{3}}{4}(y')^2\right) & \\
+21\left(\frac{1}{4}(x')^2 + \frac{2\sqrt{3}}{4}x'y' + \frac{3}{4}(y')^2\right) &= 144 \\
\frac{93}{4}(x')^2 - \frac{62\sqrt{3}}{4}x'y' + \frac{31}{4}(y') + \frac{30}{4}(x')^2 + \frac{30\sqrt{3}}{4}x'y' - \frac{10\sqrt{3}}{4}x'y' - \frac{30}{4}(y')^2 & \\
+\frac{21}{4}(x')^2 + \frac{42\sqrt{3}}{4}x'y' + \frac{63}{4}(y') &= 144 \\
\left(\frac{93}{4}(x')^2 + \frac{30}{4}(x')^2 + \frac{21}{2}(x')^2\right) + \left(-\frac{62\sqrt{3}}{2}x'y' + \frac{20\sqrt{3}}{4}x'y' + \frac{42\sqrt{3}}{4}x'y'\right) + \left(\frac{31}{4}(y')^2 + -\frac{30}{4}(y')^2 + \frac{63}{4}(y')^2\right) &= 144 \\
\frac{144}{4}(x')^2 + \frac{64}{4}(y')^2 &= 144 \\
36(x')^2 + 16(y')^2 &= 144 \\
\frac{(x')^2}{4} + \frac{(y')^2}{9} &= 1.
\end{aligned}
$$

This is an ellipse whose major axis is vertical with vertex $(x', y') = (0, 0)$, $a = 3$, $b = 2$, and $c = \sqrt{5}$:



1d. $8x^2 + 5xy - 4y^2 - 4 = 0; \alpha = \tan^{-1}(\frac{1}{5})$

Since $\alpha = \tan^{-1}(\frac{1}{5})$, we can draw a triangle to find $\cos\alpha$ and $\sin\alpha$:



Thus $\cos\alpha = \frac{5}{\sqrt{26}}$ and $\sin\alpha = \frac{1}{\sqrt{26}}$. We get the rotation equations $x = \frac{5}{\sqrt{26}}x' - \frac{1}{\sqrt{26}}y'$ and $y = \frac{1}{\sqrt{26}}x' + \frac{5}{\sqrt{26}}y'$.

$$8x^2 + 5xy - 4y^2 - 4 = 0$$

$$8\left(\frac{5}{\sqrt{26}}x' - \frac{1}{\sqrt{26}}y'\right)^2 + 5\left(\frac{5}{\sqrt{26}}x' - \frac{1}{\sqrt{26}}y'\right)\left(\frac{1}{\sqrt{26}}x' + \frac{5}{\sqrt{26}}y'\right) - 4\left(\frac{1}{\sqrt{26}}x' + \frac{5}{\sqrt{26}}y'\right)^2 - 4 = 0$$

$$8\left(\frac{25}{26}(x')^2 - \frac{10}{26}x'y' + \frac{1}{26}(y')^2\right)^2 + 5\left(\frac{5}{26}(x')^2 + \frac{25}{26}x'y' - \frac{1}{26}x'y' - \frac{5}{26}(y')^2\right)$$

$$+ -4\left(\frac{1}{26}(x')^2 + \frac{10}{26}x'y' + \frac{25}{26}(y')^2\right) = 4$$

$$\frac{200}{26}(x')^2 - \frac{80}{26}x'y' + \frac{8}{26}(y') + \frac{25}{26}(x')^2 + \frac{125}{26}x'y' - \frac{5}{26}x'y' - \frac{25}{26}(y')^2$$

$$+ - \frac{4}{26}(x')^2 - \frac{40}{26}x'y' - \frac{100}{26}(y') = 4$$

$$\left(\frac{200}{26}(x')^2 + \frac{25}{26}(x')^2 + -\frac{4}{26}(x')^2\right) + \left(-\frac{80}{26}x'y' + \frac{120}{26}x'y' + -\frac{40}{26}x'y'\right) + \left(\frac{8}{26}(y')^2 + -\frac{25}{26}(y')^2 + -\frac{100}{26}(y')^2\right) = 4$$

$$\frac{221}{26}(x')^2 - \frac{117}{26}(y')^2 = 4$$

$$\frac{(x')^2}{8/17} - \frac{(y')^2}{8/9} = 1.$$

This is a hyperbola whose transverse axis is horizontal vertex $(x', y') = (0, 0)$, $a = \sqrt{8/17}$, $b = \sqrt{8/9}$, and $c = \sqrt{208/153}$: