Assumption:

① incompressible flow: $\operatorname{div} V = 0$, $V = (u, v)$ velocity field.

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

② irrotational flow, no vortices: $\operatorname{rot} V = 0 \Rightarrow \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$

scalar function $\psi(x, y)$:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Then ① is identically satisfied and ② gives:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad (\text{Laplace eq.})$$

ψ is the stream function

$$\left.\begin{aligned} V &= (u, v) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right) \\ \nabla \psi &= \left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right) \end{aligned}\right\} \Rightarrow V \cdot \nabla \psi = 0$$

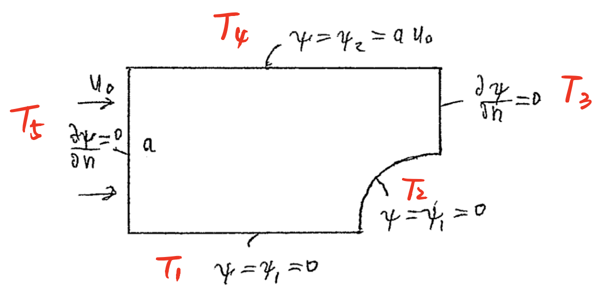
But $\nabla \psi \perp \psi$ contours

V is tangential to contours,

i.e. ψ contours are stream lines.

The boundary conditions are

$$\begin{cases} \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 & ① \\ \psi = \psi_1 = 0 \text{ on } T_1 \text{ and } T_2 & ② \\ \psi = \psi_2 = a u_0 \text{ on } T_4 & ③ \\ \frac{\partial \psi}{\partial n} = 0 \text{ on } T_3 \text{ and } T_5 & ④ \end{cases}$$



Weak formulation:

$$0 = \iint_{\Omega} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) v \, dx\, dy \quad \text{where } v \text{ is arbitrary test function}$$

$$\underline{\underline{\text{Gauss-Green}}} \iint_{\Omega} \left(-\frac{\partial \psi}{\partial x} \cdot \frac{\partial v}{\partial x} - \frac{\partial \psi}{\partial y} \cdot \frac{\partial v}{\partial y}\right) dx\, dy + \int_T \left(\frac{\partial \psi}{\partial x} n_x + \frac{\partial \psi}{\partial y} n_y\right) v \, ds = 0$$

$$= \iint_{\Omega} \left(-\frac{\partial \psi}{\partial x} \cdot \frac{\partial v}{\partial x} - \frac{\partial \psi}{\partial y} \cdot \frac{\partial v}{\partial y}\right) dx\, dy + \int_T \frac{\partial \psi}{\partial n} v \, ds \quad (n \text{ the outward-point unit normal})$$

Apply the boundary conditions to weak formulation:

$$\begin{cases} \text{from ①}: \iint_{\Omega} \left(\frac{\partial \psi}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial v}{\partial y}\right) dx\, dy - \int_{T_3} \frac{\partial \psi}{\partial n} v \, ds - \int_{T_5} \frac{\partial \psi}{\partial n} v \, ds = 0 \\ \text{from ②}: \psi = \psi_1 = 0 \text{ on } T_1 \text{ and } T_2 \text{ for all } v \text{ such that } v = 0 \text{ on } T_1, T_2 \\ \text{from ④}: \psi = \psi_2 = a u_0 \text{ on } T_4 \text{ for all } v \text{ such that } v = 0 \text{ on } T_4 \end{cases}$$

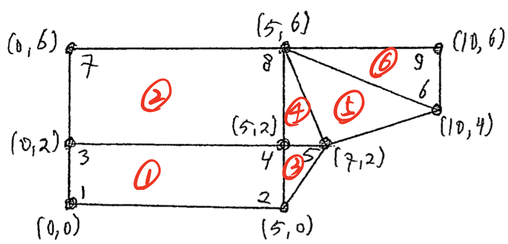
According to the FE eq.: $k_u = f_e + f_b$ where

$$K_{ij} = \iint_{\Omega} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dx\,dy \quad \text{(stiffness Mtx)}$$

$$f_{Li} = 0 \quad \text{(load vector)}$$

$$f_{Bi} = \int_{T_3} \frac{\partial \varphi}{\partial n} N_i \, ds + \int_{T_5} \frac{\partial \varphi}{\partial n} N_i \, ds \quad \text{(boundary vector)}$$

Divide the domain of the flow into 6 parts:



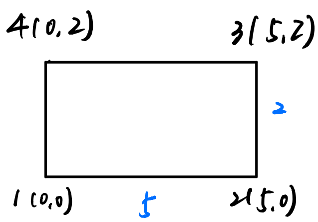
From the graph, we know that ① ② are 4-note rectangular ③ ④ ⑤ ⑥ are 3-note triangular

To calculate the 4-node rectangular element

Interpolation: $\varphi = \sum_{i=1}^{4} N \varphi_i$ FE equation: $Kg = f_b$ $g = (u_1, u_2, u_3)^T$

Galerkin: $v_i = N_i$ ($i=1 \ldots 4$)

Element ①



$2a=2$, $2b=5$ $4ab=10$

shape functions:

$$N_1^e(x,y) = \frac{1}{4ab}(x-x_2)(y-y_4) = \frac{1}{10}(x-5)(y-2) = \frac{xy}{10} - \frac{x}{5} - \frac{y}{2} + 1$$

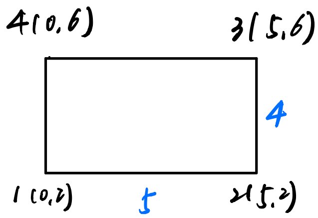
$$N_2^e(x,y) = -\frac{1}{4ab}(x-x_1)(y-y_3) = -\frac{1}{10}x(y-2) = -\frac{xy}{10} + \frac{x}{5}$$

$$N_3^e(x,y) = -\frac{1}{4ab}(x-x_4)(y-y_2) = \frac{1}{10}xy$$

$$N_4^e(x,y) = \frac{1}{4ab}(x-x_3)(y-y_1) = -\frac{1}{10}(x-5)(y-0) = -\frac{xy}{10} + \frac{y}{2}$$

$$K' = \begin{pmatrix} \frac{29}{30} & \frac{17}{60} & -\frac{23}{30} & -\frac{29}{60} \\ \frac{17}{60} & \frac{29}{30} & -\frac{29}{60} & -\frac{23}{30} \\ -\frac{29}{30} & -\frac{23}{60} & \frac{29}{30} & \frac{17}{60} \\ -\frac{23}{60} & -\frac{29}{30} & \frac{17}{60} & \frac{29}{30} \end{pmatrix} \qquad u' = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_4 \\ \varphi_3 \end{pmatrix}$$

Element ②:



$2a=4 \quad 2b=5 \quad 4ab=20$

Shape functions:

$$N_1^e(x,y)=\frac{1}{4ab}(x-x_2)(y-y_4)=\frac{1}{20}(x-5)(y-6)=\frac{1}{20}xy-\frac{3}{10}x-\frac{1}{4}y+\frac{3}{2}$$

$$N_2^e(x,y)=-\frac{1}{4ab}(x-x_1)(y-y_3)=-\frac{1}{20}x(y-6)=-\frac{1}{20}xy+\frac{3}{10}x$$

$$N_3^e(x,y)=-\frac{1}{4ab}(x-x_3)(y-y_1)=\frac{1}{20}x(y-2)=\frac{1}{20}xy-\frac{1}{10}x$$

$$N_4^e(x,y)=\frac{1}{4ab}(x-x_4)(y-y_2)=-\frac{1}{20}(x-5)(y-2)=-\frac{1}{20}xy+\frac{1}{10}x+\frac{1}{4}y-\frac{1}{2}$$

According to the stiffness matrix: $K_{ij}=\iint_{\Omega}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)dx\,dy$

$$K^2=\begin{pmatrix} \frac{41}{60} & -\frac{7}{120} & -\frac{41}{120} & -\frac{17}{60} \\ -\frac{7}{120} & \frac{41}{60} & -\frac{17}{60} & -\frac{41}{120} \\ -\frac{41}{120} & -\frac{17}{60} & \frac{41}{60} & -\frac{7}{120} \\ -\frac{17}{60} & -\frac{41}{120} & -\frac{7}{120} & \frac{41}{60} \end{pmatrix} \qquad u^2=\begin{pmatrix} \varphi_3 \\ \varphi_4 \\ \varphi_8 \\ \varphi_7 \end{pmatrix}$$

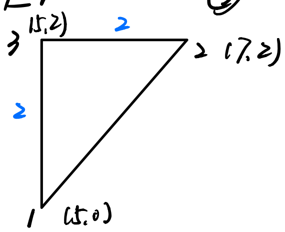
To calculate the 3-Node rectangular elements

The shape functions for a 3-Node triangular:

$$N_i^e(x,y)=\frac{1}{2A}\left[x_j y_k - x_k y_j + (y_j-y_k)x + (x_k-x_j)y\right] \quad (i,j,k \text{ cyclic})$$

where (x_i, y_i) are the coordinates of the i-th node and A is the area of the element.

Element ③



$$A = \frac{1}{2} \times 2 \times 2 = 2$$

Shape functions: $N_1^e(x,y) = \frac{1}{2A}[x_2 y_3 - x_3 y_2 + (y_2 - y_3)x + (x_3 - x_2)y]$

$$= \frac{1}{4}[2\times 7 - 2\times 5 + (-2y)] = 1 - \frac{1}{2}y$$

$$N_2^e(x,y) = \frac{1}{2A}[x_3 y_1 - x_1 y_3 + (y_3 - y_1)x + (x_1 - x_3)y]$$

$$= \frac{1}{4}[5\times 0 - 5\times 2 + 2x] = \frac{1}{2}x - \frac{5}{2}$$

$$N_3^e(x,y) = \frac{1}{2A}[x_1 y_2 - x_2 y_1 + (y_1 - y_2)x + (x_2 - x_1)y]$$

$$= \frac{1}{4}[5\times 2 - 7\times 0 - 2x + 2y] = \frac{5}{2} - \frac{x}{2} + \frac{y}{2}$$

According to the stiffness matrix: $k_{ij} = \iint \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right) dx\,dy$

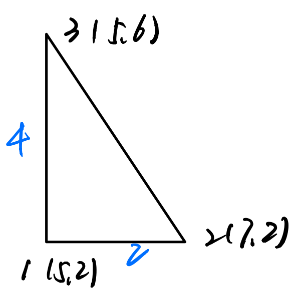
$$k_{11} = \iint \frac{1}{4}\,dx\,dy = \frac{1}{2} = k_{22}$$

$$k_{12} = k_{21} = 0$$

$$k_{13} = k_{31} = \iint \left(0 - \frac{1}{4}\right) dx\,dy = -\frac{1}{2} = k_{23} = k_{32}$$

$$k_{33} = \iint \left(\frac{1}{4} + \frac{1}{4}\right) dx\,dy = 1$$

Element ④:



Shape functions:

$$N_1^e(x,y) = \frac{1}{2A}[x_2 y_3 - x_3 y_2 + (y_2 - y_3)x + (x_3 - x_2)y]$$

$$= \frac{1}{8}[7\times 6 - 5\times 2 + (-4x) + (-2y)] = 4 - \frac{1}{2}x - \frac{1}{4}y$$

$$N_2^e(x,y) = \frac{1}{2A}[x_3 y_1 - x_1 y_3 + (y_3 - y_1)x + (x_1 - x_3)y]$$

$$= -\frac{5}{2} + \frac{1}{2}x$$

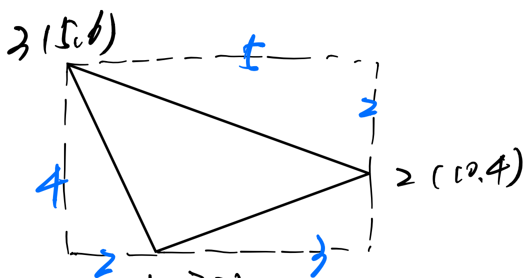
$$N_3^e(x,y) = \frac{1}{2A}[x_1y_2 - x_2y_1 + (y_1-y_2)x + (x_2-x_1)y]$$

$$= -\frac{1}{2} + \frac{1}{4}y$$

Similarly, we can get K_{ij}

$$K^4 = \begin{pmatrix} \frac{5}{4} & -1 & -\frac{1}{4} \\ -1 & -1 & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix} \qquad u^4 = \begin{pmatrix} \varphi_4 \\ \varphi_5 \\ \varphi_8 \end{pmatrix}$$

Element ⑤



$$A = (2+3)\times 4 - \frac{1}{2}\times 2\times 4 - \frac{1}{2}\times 2\times 3 - \frac{1}{2}\times 2\times 5$$

$$= 20 - 12 = 8$$

shape functions:

$$N_1^e(x,y) = \frac{5}{2} - \frac{1}{8}x - \frac{5}{16}$$

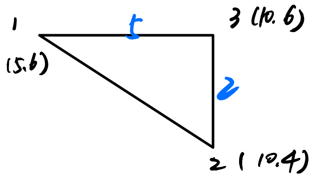
$$N_2^e(x,y) = -2 + \frac{1}{4}x + \frac{1}{8}y$$

$$N_3^e(x,y) = \frac{1}{2} - \frac{1}{8}x + \frac{3}{16}y$$

get $K_{ij} = \iint \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} + \frac{\partial N_j}{\partial y}\right) dx\, dy$

$$K^5 = \begin{pmatrix} \frac{29}{32} & -\frac{9}{16} & -\frac{11}{32} \\ -\frac{9}{16} & \frac{5}{8} & -\frac{1}{16} \\ -\frac{11}{32} & -\frac{1}{16} & \frac{13}{32} \end{pmatrix} \qquad u^5 = \begin{pmatrix} \varphi_5 \\ \varphi_6 \\ \varphi_8 \end{pmatrix}$$

Element ⑥



$A = 2\times 5\times\frac{1}{2} = 5$

shape functions:

$N_1^e(x,y) = \frac{1}{10}(10\times 6 - 10\times 4 + (-2x)] = 2 - \frac{1}{5}x$

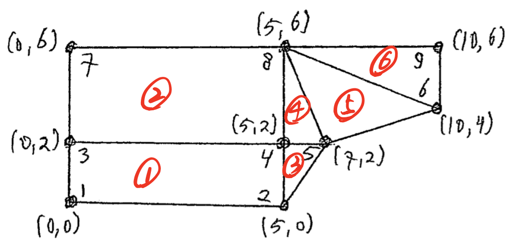
$N_2^e(x,y) = \frac{1}{10}(10\times 6 - 5\times 6 - 5y) = 3 - \frac{1}{2}y$

$N_3^e(x,y) = \frac{1}{10}(5\times 4 - 10\times 6 + 2x + 5y) = -4 + \frac{1}{5}x + \frac{1}{2}y$

Similarly, get stiffness matrix K_{ij}.

$$K^6 = \begin{pmatrix} \frac{1}{5} & 0 & -\frac{1}{5} \\ 0 & \frac{5}{4} & -\frac{5}{4} \\ -\frac{1}{5} & -\frac{5}{4} & \frac{29}{20} \end{pmatrix} \qquad u^6 = \begin{pmatrix} \varphi_8 \\ \varphi_6 \\ \varphi_9 \end{pmatrix}$$

To assemble the element data into global data.



Connectivity matrix C.

$$C = \begin{pmatrix} 1 & 2 & 4 & 3 \\ 3 & 4 & 8 & 7 \\ 2 & 5 & 4 & 0 \\ 4 & 5 & 8 & 0 \\ 5 & 6 & 8 & 0 \\ 8 & 6 & 9 & 0 \end{pmatrix}$$

Sum over elements: $Ku = fl$.

global stiffness matrix:

$$K = \begin{bmatrix} \frac{29}{30} & \frac{17}{60} & -\frac{23}{30} & -\frac{29}{60} & 0 & 0 & 0 & 0 & 0 \\ \frac{17}{60} & \frac{29}{30} & -\frac{29}{60} & -\frac{19}{15} & 0 & 0 & 0 & 0 & 0 \\ -\frac{23}{30} & -\frac{29}{60} & \frac{33}{20} & \frac{9}{40} & 0 & 0 & -\frac{17}{60} & -\frac{41}{120} & 0 \\ -\frac{29}{60} & -\frac{19}{15} & \frac{9}{40} & \frac{39}{10} & -\frac{3}{2} & 0 & -\frac{41}{120} & -\frac{8}{15} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} & \frac{77}{32} & -\frac{9}{16} & 0 & -\frac{11}{32} & 0 \\ 0 & 0 & 0 & 0 & -\frac{19}{16} & \frac{15}{8} & 0 & -\frac{1}{16} & -\frac{5}{4} \\ 0 & 0 & -17/60 & -\frac{41}{120} & 0 & 0 & \frac{41}{60} & -7/120 & 0 \\ 0 & 0 & -41/120 & -\frac{8}{5} & -\frac{11}{32} & -\frac{1}{16} & -\frac{7}{120} & 739/480 & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 & -\frac{5}{2} & 0 & -1/5 & \frac{29}{20} \end{bmatrix}$$

by the essential boundary conditions:

$$\begin{cases} \varphi_1 = \varphi_2 = \varphi_5 = \varphi_6 = 0 \\ \varphi_7 = \varphi_8 = \varphi_9 = aU_0 \end{cases}$$

So we need to calculate φ_3 and φ_4

And, the node 4 is an internal node so $R_4 = 0$

the node 3 according to the boundary conditions the line integral on the boundary is $R_3 = 0$

$$K \begin{pmatrix} 0 \\ 0 \\ \varphi_3 \\ \varphi_4 \\ 0 \\ 0 \\ aU_0 \\ aU_0 \\ aU_0 \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ R_7 \\ R_8 \\ R_9 \end{pmatrix}$$

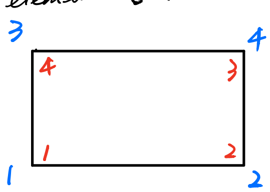
To solve $\varphi_3 . \varphi_4$ we have

$$\begin{cases} \frac{33}{20}\varphi_3 + \frac{9}{40}\varphi_4 - \frac{17}{60} aU_0 - \frac{41}{120} aU_0 = 0 \\ \frac{9}{40}\varphi_3 + \frac{39}{10}\varphi_4 - \frac{41}{120} aU_0 - \frac{8}{15} aU_0 = 0 \end{cases}$$

$$\begin{cases} \varphi_3 = \frac{239}{681} aU_0 \\ \varphi_4 = \frac{139}{681} aU_0 \end{cases}$$

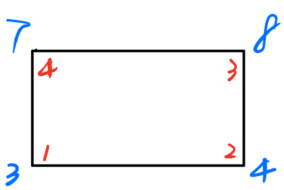
Thus:

element ①:



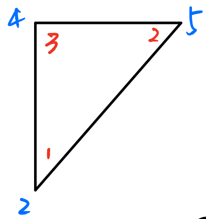
$$\varphi = N_1\varphi_1 + N_2\varphi_2 + N_3\varphi_4 + N_4\varphi_3$$
$$= \frac{1}{10} xy \times \frac{139}{681} aU_0 + \left(-\frac{1}{10} xy + \frac{y}{2}\right) \frac{239}{681} aU_0$$
$$= \left(-\frac{10}{681} xy + \frac{239}{1362} y\right) aU_0$$

element ②:



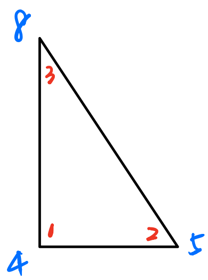
$$\varphi = N_1\varphi_3 + N_2\varphi_4 + N_3\varphi_8 + N_4\varphi_7$$
$$= \left(\frac{1}{20} xy - \frac{3}{10} x - \frac{1}{4} y + \frac{3}{2}\right) \times \frac{239}{681} aU_0 + \left(-\frac{1}{20} xy + \frac{3x}{10}\right) \times \frac{139}{681} aU_0$$
$$+ \left(\frac{1}{20} xy - \frac{1}{10} x\right) aU_0 + \left(-\frac{1}{20} xy + \frac{1}{10} x + \frac{1}{4} y - \frac{1}{2}\right) aU_0$$
$$= aU_0 \left(\frac{6}{227} - \frac{10}{227} x + \frac{221}{1362} y + \frac{5}{681} xy\right)$$

element ③



$$\varphi = N_1\varphi_2 + N_2\varphi_5 + N_3\varphi_4$$
$$= \frac{139}{681}\,\text{allo} \times \left(\frac{5}{2} - \frac{1}{2}x + \frac{1}{2}y\right)$$

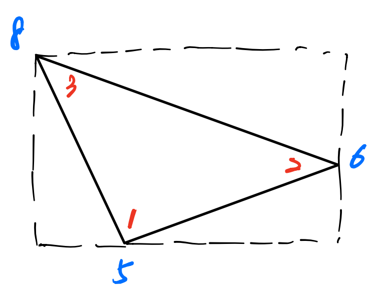
element ④



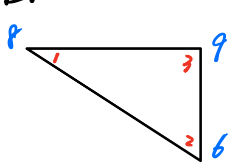
$$\varphi = N_1\varphi_4 + N_2\varphi_5 + N_3\varphi_8$$
$$= \left(4 - \frac{1}{2}x - \frac{1}{4}y\right) \times \frac{139}{681}\,\text{allo} + \left(-\frac{1}{2} + \frac{1}{4}y\right)\text{allo}$$
$$= \text{allo}\left(\frac{431}{1362} - \frac{139}{1362}x + \frac{271}{1362}y\right)$$

element ⑤ :

$$\varphi = N_1\varphi_5 + N_2\varphi_6 + N_3\varphi_8$$
$$= \text{allo}\left(\frac{1}{2} - \frac{1}{8}x + \frac{3}{16}y\right)$$

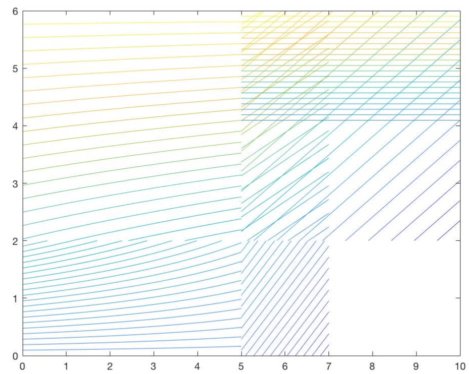


Element ⑥



$$\varphi = N_1\varphi_8 + N_2\varphi_6 + N_3\varphi_9$$
$$= \text{allo}\left(-2 + \frac{1}{2}y\right)$$

To plot contour line:



The physical sense of the solution is as following:
Unlike a real fluid, this solution indicates a net zero drag on the body.

Accuracy:

2D approximation: $T = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 y + \dots$

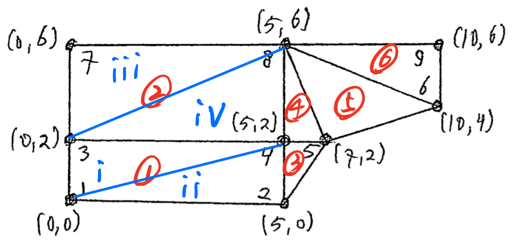
Since Taylor expansion:

$$T = T(x_i, y_i) + \frac{\partial T}{\partial x}\Big|_i (x - x_i) + \frac{\partial T}{\partial y}\Big|_i (y - y_i) + \frac{1}{2}\frac{\partial^2 T}{\partial x^2}\Big|_i (x - x_i)^2 + \frac{1}{2}\frac{\partial^2 T}{\partial y^2}\Big|_i (y - y_i)^2 + \frac{\partial^2 T}{\partial x \partial y}\Big|_i (x - x_i)(y - y_i) + \dots$$

Thus, truncation after a finite number of terms gives an error of $O(h^{p+1})$ and this estimate assumes a 'balanced mesh' in which both x and y dimensions of all elements are of order h. Here element ① and element ② are 4-node rectangular elements with error $O(h_1^2)$ and $O(h_2^2)$ seperetaly.
element ③ ④ ⑤ ⑥ are 3-node triangular elements, with error $O(h_3^2)$. $O(h_4^2)$. $O(h_5^2)$ $O(h_6^2)$ seperately.

the error $O(h_i^2)_{(i=1\ldots6)}$ has 2 ways to improve the accuracy of the solution.

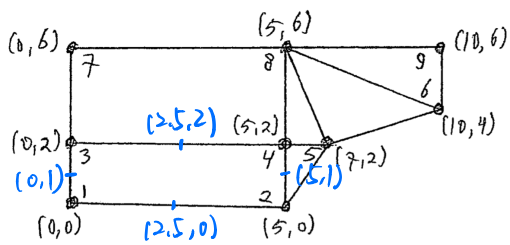
1. reduce h by refining the mesh



For example: change previous element ① into element i and element ii.

change previous element ② into element iii and element iv.

2. increase P by using higher-order shape function:



Turn element ① from 4-node rectangular elements to 8-node triangular elements.

Appendix A MATLAB Code

This is the matlab code using to plot solution.

```
a = 1;
u_0 = 1;
x = linspace(0,5);
y = linspace(0,2);
[X,Y] = meshgrid(x,y);

% figure(1)
z1 = a*u_0.*Y.*(239/1362 - 10*X/681 );
contour(X,Y,z1,20)
hold on

x = linspace(5,7);
y = linspace(0,2);
[X,Y] = meshgrid(x,y);
% figure(2)
z2 = (139/681) *a*u_0*(5/2 - 0.5*X + 0.5*Y );
contour(X,Y,z2,20)
hold on
x = linspace(0,5);
y = linspace(2,6);
[X,Y] = meshgrid(x,y);
% figure(3)
z3 = a*u_0*(6/227 - 10*X/227  + (221/1362)*Y+(5/681)*X.*Y );
contour(X,Y,z3,20)
hold on
x = linspace(5,7);
y = linspace(2,6);
[X,Y] = meshgrid(x,y);
% figure(4)
z4 = a*u_0*(432/1362 - (139/1362)*X+(271/1362)*Y );
contour(X,Y,z4,20)
hold on

x = linspace(5,10);
y = linspace(2,6);
[X,Y] = meshgrid(x,y);
% figure(5)
z5 = a*u_0*(0.5-X/8+3*Y/16 );
contour(X,Y,z5,20)
hold on
x = linspace(5,10);
y = linspace(4,6);
[X,Y] = meshgrid(x,y);
% figure(6)
z6 = a*u_0*(-2+Y/2);
contour(X,Y,z6,20)
hold off
```

Listing 1: Example code from external file.