MATH0086

1. (a) Forward Euler: $y_{n+1} = y_n + h \times f(y_n, t_n)$

Therefore $T(y_n, t_n) = y_n + h f(y_n, t_n)$

$$g_n = \frac{\partial T(y_n, t_n)}{\partial y_n} = 1 + h \frac{\partial f(y_n, t_n)}{\partial y_n}$$

Depending on the form of $f(y_n, t_n)$ we have several cases:

i) Decay-type problems with $\frac{\partial f}{\partial y} < 0$

$$g_n = 1 - h\left|\frac{\partial f}{\partial y_n}\right|,$$

and for stability

$$|g_n| < 1 \Rightarrow -1 < 1 - h\left|\frac{\partial f(y_n, t_n)}{\partial y_n}\right| < 1$$

$$\Rightarrow \quad h < \frac{2}{\left|\frac{\partial f(y_n, t_n)}{\partial y_n}\right|}$$

which means Forward Euler is conditionally stable.

ii) Growth-problems, $\frac{\partial f}{\partial y} > 0$

$g_n = 1 + h\frac{\partial f}{\partial y} > 1$, therefore it's unconditionally unstable.

iii) The function f is complex valued. $\frac{dy}{dt} = -i w y$

$$g_n = 1 - iwh \Rightarrow |g_n|^2 = 1 + h^2 w^2 > 1$$

hence instability for oscillatory case.

For given equation $f(y, t) = -y^2 - t^4$ $\quad \frac{\partial f}{\partial y} = -2y$ if $y > 0$ then $\frac{\partial f}{\partial y} < 0$

Hence it's decay-type problem. And we find forward Euler is conditional stable with step size $h < \frac{2}{|-2y_n - t_n^4|}$; But if $y<0$, $\frac{\partial f}{\partial y} > 0$ is a growth type problem which implies unstable.

(b) (i) given RK2: $y_{n+1} = y_n + h[a_1 f(y_n, t_n) + a_2 f(y_n + a_3 h f(y_n, t_n), t_n + a_4 h)]$ ①

do Taylor expansion with $f(y_n + a_3 h f(y_n, t_n), t_n + a_4 h)$:

$$f(y_n + a_3 h f(y_n, t_n), t_n + a_4 h)$$

$$= f(y_n, t_n) + a_3 h f(y_n, t_n) \frac{\partial f(y_n, t_n)}{\partial y_n} + a_4 h \times \frac{\partial f(y_n, t_n)}{\partial t_n}$$

$$+ \frac{1}{2}\left[a_3^2 h^2 f^2(y_n, t_n) \times \frac{\partial^2 f}{\partial y^2} + 2 a_3 h f(y_n, t_n) a_4 h \times \frac{\partial^2 f}{\partial t \partial y} + a_4^2 h^2 \frac{\partial^2 f}{\partial t^2}\right] + O(h^3)$$

And we substitude it back to equation ① and do arrangement:

~~y_{n+1}~~

replace y_n with $y(t_n)$

$$y(t_{n+1}) = y(t_n) + h a_1 f(y(t_n), t_n) + h a_2 f(y(t_n), t_n) + h^2 a_2 a_3 f(y(t_n), t_n) \frac{\partial f}{\partial y} + h^2 a_2 a_4 \frac{\partial f}{\partial t}$$

$$+ \frac{h^3}{2} a_2 a_3^2 f^2 \frac{\partial^2 f}{\partial y^2} + h^3 a_2 a_3 a_4 f \frac{\partial^2 f}{\partial t \partial y} + \frac{h^3}{2} a_2 a_4^2 \frac{\partial^2 f}{\partial t^2} + O(h^4)$$

$$= y(t_n) + h(a_1 + a_2) f(y(t_n), t_n) + \frac{1}{2} h^2 \left(2 a_2 a_3 f \cdot \frac{\partial f}{\partial y} + 2 a_2 a_4 \frac{\partial f}{\partial t}\right)$$

$$+ \frac{1}{6} h^3 \left(3 a_2 a_3^2 f^2 \frac{\partial^2 f}{\partial y^2} + 6 a_2 a_3 a_4 f \frac{\partial^2 f}{\partial t \partial y} + 3 a_2 a_4^2 \frac{\partial^2 f}{\partial t^2}\right) + O(h^4)$$

Comparing it with Taylor expansion:

$y(t_{n+1}) = y(t_n+h) = y(t_n) + hf(y(t_n), t_n) + \frac{1}{2}h^2 y''(t_n) + \frac{1}{6}h^3 y'''(t_n) + O(h^4)$

Here, $\frac{dy}{dt} = y'$ $\quad y'' = \frac{d}{dt} f(y(t), t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \times \frac{dy}{dt}$

$y''' = \frac{\partial^2 f}{\partial t^2} + 2 \times \frac{\partial^2 f}{\partial t \partial y} \times f + \frac{\partial^2 f}{\partial^2 y} \times f^2 + \frac{\partial f}{\partial t} \cdot \frac{\partial f}{\partial y} + \left(\frac{\partial f}{\partial y}\right)^2 \cdot f$

Hence the Taylor expansion becomes:

$y(t_{n+1}) = y(t_n) + h \times f(y(t_n), t_n) + \frac{1}{2}h^2\left(\frac{\partial f}{\partial t} + f \times \frac{\partial f}{\partial y}\right)$

$+ \frac{1}{6}h^3\left(\frac{\partial^2 f}{\partial t^2} + f \times \frac{2\partial^2 f}{\partial t \partial y} + 2f^2 \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial t \partial y} + \left(\frac{\partial f}{\partial y}\right)^2 f\right) + O(h^4)$

we can conclude that the condition on constants a_i

i.e $a_1 + a_2 = 1$, $2a_2 a_3 = 1$ $\quad 2a_2 a_4 = 1$

And For the form of LTE, it can be given: $LTE = Mh^3 + O(h^4)$

~~Where M =~~ $LTE = \underbrace{\text{Taylor expansion}}_{\cancel{y(t)}\ y_n} - y(t_n)$

$= \frac{1}{6}h^3\Big[(1 - 3a_2 a_3^2) f^2 \frac{\partial^2 f}{\partial y^2} + (2 - 6a_2 a_3 a_4) f \frac{\partial^2 f}{\partial t \partial y} + (1 - 3a_2 a_4^2)\frac{\partial^2 f}{\partial t^2}$

$+ \frac{\partial^2 f}{\partial t \partial y} + \left(\frac{\partial f}{\partial y}\right)^2 f\Big] + O(h^4)$

we substitude conditions found in constant:

$LTE = \frac{1}{6}h^3\left[(1 - \frac{3}{2}a_3) f^2 \frac{\partial^2 f}{\partial y^2} + (2 - 3a_3) f \frac{\partial^2 f}{\partial t \partial y} + (1 - \frac{3}{2}a_3)\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial t \partial y} + \left(\frac{\partial f}{\partial y}\right)^2 f\right] + O(h^4)$

we conclude $M = \left[(1 - \frac{3}{2}a_3) f^2 \frac{\partial^2 f}{\partial y^2} + (2 - 3a_3) f \frac{\partial^2 f}{\partial t \partial y} + (1 - \frac{3}{2}a_3)\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial t \partial y} + \left(\frac{\partial f}{\partial y}\right)^2 f\right] \times \frac{1}{6}$

(ii) when $f = y^P$ we need select value of a_3 to let $M = 0$.

let $a_3 = \frac{2}{3}$ and $\frac{\partial f}{\partial y} = P \times y^{P-1}$ $\quad \frac{\partial^2 f}{\partial y \partial t} = P(P-1)\, y^{P-2} \cdot \frac{dy}{dt}$

then $M = \frac{1}{6}\left[P(P-1) y^{P-2} \frac{dy}{dt} + (P \times y^{P-1})^2 \times y^P\right]$

2. (a) Using a leap-frog approximation for t and central approximation for x:

For equation $u_t + u_{xxx} = 0$ we write:

$$\frac{u_{n,m+1} - u_{n,m-1}}{2\Delta t} + \frac{\frac{u_{n+2,m-2} - 2u_{n+1,m} + u_{n,m}}{(\Delta x)^2} - \frac{u_{n,m} - 2u_{n-1,m} + u_{n+2,m}}{(\Delta x)^2}}{2\Delta x} = 0$$

By re-arrangement:

$$u_{n,m+1} = u_{n,m-1} - \frac{\Delta t}{(\Delta x)^3}\left(u_{n+2,m} - 2u_{n+1,m} + 2u_{n-1,m} - u_{n-2,m}\right)$$ which is consistent with finite-difference scheme.

Let $u_{n+1,m+1} = u(x+\Delta x, t+\Delta t)$

then $u(x, t-\Delta t) - u(x, t+\Delta t) = \frac{\Delta t}{(\Delta x)^3}\left[u(x+2\Delta x, t) - 2u(x+\Delta x, t) + 2u(x-\Delta x, t) - u(x-2\Delta x, t)\right]$

using Taylor expansion:

$$RHS = \frac{\Delta t}{(\Delta x)^3}\Big[u + 2\Delta x\, u_x + 2\Delta x^2 u_{xx} + \frac{8}{6}\Delta x^3 u_{xxx} + \frac{16}{24}(\Delta x)^4 u_{xxxx}$$
$$-2\left(u + \Delta x\, u_x + \frac{(\Delta x)^2}{2} u_{xx} + \frac{(\Delta x)^3}{6} u_{xxx} + \frac{(\Delta x)^4}{24} u_{xxxx}\right)$$
$$+2\left(u - \Delta x\, u_x + \frac{\Delta x^2}{2} u_{xx} + \frac{\Delta x^3}{6} u_{xxx} + \frac{\Delta x^4}{24} u_{xxxx}\right)$$
$$-\left(u - 2\Delta x\, u_x + 2\Delta x^2 u_{xx} - \frac{8}{6}\Delta x^3 u_{xxx} + \frac{16}{24}\Delta x^4 u_{xxxx}\right)$$
$$+ O(\Delta x^5)\Big]$$

So $u(x, t-\Delta t) = -2\Delta t\, u_{xxx} - \Delta t\, O(\Delta x) + u - \Delta t\, u_t + \frac{(\Delta t)^2}{2} u_{tt} - \frac{\Delta t^3}{6} u_{ttt} + O(\Delta t^4)$

To calculate the LTE, at $O(\Delta t)$: $u_t + u_{xxx} = 0$ and $O(\Delta t^2)$ is omitted

hence $LTE = O(\Delta t^3) + O(\Delta t + \Delta x)$.

(b) For von Neumann Stability set $u_{n,m} = \lambda^m e^{ikn}$, $C = \frac{\Delta t}{(\Delta x)^3}$

rewrite formula:

$$\lambda^{m+1} e^{ikn} = \lambda^{m-1} e^{ikn} - C\left(\lambda^m e^{ik(n+2)} - 2\lambda^m e^{ik(n+1)} + 2\lambda^m e^{ik(n-1)} - \lambda^m e^{ik(n-2)}\right)$$

$$\lambda = \frac{1}{\lambda} - C\left[\left(e^{i2k} - e^{i2k}\right) - 2\left(e^{ik} - e^{-ik}\right)\right]$$

$$\lambda - \frac{1}{\lambda} + 2iC(\sin 2k - 2\sin k) = 0$$

$$\lambda^2 + 2iC(\sin 2k - 2\sin k)\cdot\lambda - 1 = 0.$$

$$\lambda = iC(\sin 2k - 2\sin k) \pm \sqrt{-C^2(2\sin k - \sin 2k)^2 + 1}$$

when $C^2(2\sin k - \sin 2k)^2 > 1$, it will be instability since $|\lambda| > 1$

$$\Rightarrow |C| > \frac{1}{2\sin k - \sin k}$$

Set $iC(\sin 2k - 2\sin k) = \Delta$

Therefore, $\lambda^2 = \Delta^2 + \Delta^2 + 1 > 1$ since $\sin 2k - 2\sin k \in \left[-\frac{3}{2}\sqrt{3}, \frac{3}{2}\sqrt{3}\right]$

It's conditionally stable with condition

$$|C| < \frac{1}{\sin 2k - 2\sin k}$$

3. (a) Deriving the approximation:

$$\frac{u^p_{n,m}-u^{p-1}_{n,m}}{\Delta t} + c\,u^{(p)}_{n,m} = \frac{u^p_{n+1,m}-2u^p_{n,m}+u^p_{n-1,m}}{(\Delta r)^2} + \frac{1}{r}\,\frac{u^p_{n+1,m}-u^p_{n-1,m}}{2\Delta r} + \frac{1}{r^2}\,\frac{u^p_{n,m+1}-2u^p_{n,m}+u^p_{n,m-1}}{(\Delta\theta)^2}$$

re-arrange the equation:

~~(b) set $u^p_{n,m}=\lambda^p e^{i(\alpha n+\beta m)}$~~

~~and re-arrange~~

$$\left(\frac{1}{\Delta t}+c+\frac{2}{(\Delta r)^2}+\frac{2}{r^2(\Delta\theta)^2}\right)\cdot u^p_{n,m} = \frac{1}{\Delta t}u^{p-1}_{n,m} + \left[\frac{1}{(\Delta r)^2}+\frac{1}{2r\Delta r}\right]u^p_{n+1,m} + \left(\frac{1}{(\Delta r)^2}-\frac{1}{2r\Delta r}\right)u^p_{n-1,m} + \frac{1}{r^2(\Delta\theta)^2}u^p_{n,m+1} + \frac{1}{r^2(\Delta\theta)^2}u^p_{n,m-1}$$

(b) set $u^p_{n,m}=\lambda^p e^{i(\alpha n+\beta m)}$

the equation becomes:

$$\left[\frac{1}{\Delta t}+c+\frac{2}{(\Delta r)^2}+\frac{2}{r^2(\Delta\theta)^2}\right]\times\lambda^p e^{i(\alpha n+\beta m)} = \frac{1}{\Delta t}\lambda^{p-1}e^{i(\alpha n+\beta m)} + \left[\frac{1}{(\Delta r)^2}+\frac{1}{2r\Delta r}\right]\lambda^p e^{i(\alpha n+\alpha+\beta m)} + \left[\frac{1}{(\Delta r)^2}-\frac{1}{2r\Delta r}\right]\lambda^p e^{i(\alpha n-\alpha+\beta m)} + \frac{1}{r^2(\Delta\theta)^2}\lambda^p e^{i(\alpha n+\beta m+\beta)} + \frac{1}{r^2(\Delta\theta)^2}\lambda^p e^{i(\alpha n+\beta m-\beta)}$$

$$\frac{1}{\Delta t}+c+\frac{2}{(\Delta r)^2}+\frac{2}{r^2(\Delta\theta)^2} = \frac{1}{\Delta t}\lambda^{-1} + \left[\frac{1}{(\Delta r)^2}+\frac{1}{2r\Delta r}\right]e^{i\alpha} + \left[\frac{1}{(\Delta r)^2}-\frac{1}{2r\Delta r}\right]e^{-i\alpha} + \frac{1}{r^2(\Delta\theta)^2}e^{i\beta} + \frac{1}{r^2(\Delta\theta)^2}e^{-i\beta}$$

$$= \frac{1}{\Delta t}\frac{1}{\lambda} + \frac{1}{(\Delta r)^2}\left[e^{i\alpha}+e^{-i\alpha}\right] + \frac{1}{2r\Delta r}\left[e^{i\alpha}-e^{-i\alpha}\right] + \frac{1}{r^2(\Delta\theta)^2}\left[e^{i\beta}+e^{-i\beta}\right]$$

$$= \frac{1}{\Delta t}\frac{1}{\lambda} + \frac{2}{(\Delta r)^2}\cos\alpha + \frac{i}{r\Delta r}\sin\alpha + \frac{1}{r^2(\Delta\theta)^2}\cos\beta$$

$$\Rightarrow \frac{1}{\lambda} = \Delta t\left[\frac{4\sin^2(\frac{\alpha}{2})}{(\Delta r)^2} - \frac{i\sin\alpha}{r\Delta r} + \frac{4\sin^2\frac{\beta}{2}}{r^2(\Delta\theta)^2} + \frac{1}{\Delta t} + c\right]$$

let $m = \frac{4\sin^2(\frac{\alpha}{2})}{(\Delta r)^2} + \frac{4\sin^2\frac{\beta}{2}}{r^2(\Delta\theta)^2} + \frac{1}{\Delta t} + c$, $n = -\frac{i\sin\alpha}{r\Delta r}$

then $\frac{1}{\lambda} = m + ni$

the magnitude of $|\lambda|^2 = \frac{1}{m^2+n^2}$

The system is conditionally stable with condition $\frac{1}{m^2+n^2} < 1$

(c)

4. (a) $\frac{d}{dx}(AE + \frac{du}{dx}) + q = 0$ where $q = \rho g A$.

$\int_0^L (\frac{du}{dx} \cdot \frac{du}{dx} \cdot AE + \rho g A) \cdot v \, dx = 0$ where v is the test function.

That means $\int_0^L u_{xx} AE v \, dx + \int_0^L \rho g A \cdot v \, dx = 0$

$\int_0^L AEv \, du' + \int_0^L \rho g A v \, dx = 0$

$AEv \cdot \frac{du}{dx}\Big|_0^L - \int_0^L AE \frac{dv}{dx} \cdot \frac{du}{dx} \cdot dx + \int_0^L \rho g A v \, dx = 0$

And $AEv \cdot \frac{du}{dx}\Big|_0^L = 0$ is given by boundary condition $u(0) = 0$ $F(L) = 0$, where $F = AE \cdot \frac{du}{dx}$

Hence $\cancel{Eq} \Rightarrow \int_0^L \rho g A v \, dx - \int_0^L AE \frac{dv}{dx} \frac{du}{dx} dx = 0$

(b) FE equation for a single 2-node element

$N_1 = 1 - \frac{x}{L}$, $N_2 = \frac{x}{L}$

And the stiffness matrix $k_{ij} = \int_0^L \frac{dN_i}{dx} AE \frac{dN_j}{dx} dx$

$k_{11} = \int_0^L \frac{dN_1}{dx} AE \frac{dN_1}{dx} dx = \int_0^L (-\frac{1}{L}) AE (-\frac{1}{L}) dx = \frac{1}{L^2} AE x\Big|_0^L = \frac{1}{L} AE$

$k_{12} = \int_0^L \frac{dN_1}{dx} AE \frac{dN_2}{dx} dx = \int_0^L (-\frac{1}{L}) AE (\frac{1}{L}) dx = -\frac{1}{L} AE$

$\Rightarrow K = \frac{1}{L} AE \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

And the load vector $f_{li} = \int_0^L q N_i \, dx$

$f_{l_1} = \int_0^L q(1 - \frac{x}{L}) dx = \frac{q}{2} L$, $f_{l_2} = \int_0^L q \frac{x}{L} dx = \frac{q}{2} L$

$f_l = \frac{q}{2} L \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$f_{bi} = -N_i F\Big|_0^L \Rightarrow f_{b_1} = -F(0)$ $f_{b_2} = 0$ $\therefore f_b = \begin{pmatrix} -F(0) \\ 0 \end{pmatrix}$

Hence the FE equation becomes

$\frac{1}{L} AE \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{q}{2} L \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -F(0) \\ 0 \end{pmatrix}$

with the condition $u(0) = 0$, $F(L) = 0$

$\cancel{\frac{1}{L}A}$ $\frac{1}{L} AE u_2 = \frac{q}{2} L \Rightarrow u_2 = \frac{qL^2}{2AE}$

$\frac{1}{L} AE u_2 = F(0) - \frac{q}{2} L \Rightarrow F(0) = qL$

$\Rightarrow u(x) = \sum N_i(x) u_i = \frac{qL}{2AE} x$

(c) 2-node elements:

$$K = \frac{AE}{L} \times \begin{bmatrix} 1 & -1 & \cdots & & 0 \\ -1 & 2 & -1 & \cdots & \vdots \\ 0 & -1 & 2 & -1 & \vdots \\ \vdots & & -1 & 2 & \vdots \\ 0 & \cdots & & & 1 \end{bmatrix}, \quad P = [u_1 + \frac{qL}{n}, \frac{qL}{n}, \ldots, \frac{qL}{n}, \frac{qL}{2n}]^T$$ with length $(n+1)$.

In equation $K \times \underline{\delta} = P$ where $\underline{\delta} = [u_1, u_2, \ldots, u_{n+1}]^T$

then write in equation form:

$$\frac{AE}{L}(2u_2 - u_3) = \frac{qL}{n}$$

$$\vdots$$

$$\frac{AE}{L}(-u_{n-1} + 2u_n - u_{n+1}) = \frac{qL}{n}$$

$$\vdots$$

$$\frac{AE}{L}(-u_n + u_{n+1}) = \frac{qL}{2n}$$

$$\frac{2AE}{L}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{qL}{4} \\ \frac{qL}{2} \\ \frac{qL}{4} \end{pmatrix} + \begin{pmatrix} F(0) \\ 0 \\ 0 \end{pmatrix}$$

when $n = L$, we could get

$$\begin{cases} \frac{2AE}{L}(-u_2 + u_3) = \frac{qL}{4} + F(0) \\ \frac{AE}{L}(+2u_2 - u_3) = \frac{qL}{2} \\ \frac{AE}{L}(-u_2 + u_3) = \frac{qL}{4} \end{cases}$$

Solving for u_2 and u_3

$$\begin{cases} 2u_2 - u_3 = \frac{qL^2}{2AE} \\ -u_2 + u_3 = \frac{qL^2}{4AE} \end{cases} \Rightarrow \begin{cases} u_2 = \frac{3qL^2}{\cancel{4}8AE} \\ u_3 = \frac{qL^2}{2AE} \end{cases}$$

$$F(0) = -qL$$

Thus $u = [0, \frac{3qL^2}{\cancel{4}8AE}, \frac{qL^2}{2AE}]$

(d) The last term would be $\frac{qL}{2n} + M$.

5. (a) $\iint (-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}) - g(x,y)\, v\, dx\, dy = 0$, v is the test function.

$$\Rightarrow \iint (-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial y})\,dx\,dy + \int_\Gamma (\frac{\partial u}{\partial x} n_x + \frac{\partial u}{\partial y} n_y)\, v\, ds - \iint g(x,y)\,dx\,dy = 0$$

then $\iint (\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + g(x,y)\,v)\,dx\,dy - \int_0^4 \frac{\partial u}{\partial n}\,dy = 0$

(b) FE equation: $Ku = f_i + f_b$ where

Stiffness matrix: $K_{ij} = \iint (\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\cdot\frac{\partial N_j}{\partial y})\,dx\,dy$

Load vector: $f_{Ii} = \iint g N_i\,dx\,dy$ where N_i are shape functions.

$N_1 = \frac{1}{4}(\xi - 1)(\eta - 1)$, $N_2 = -\frac{1}{4}(\xi + 1)(\eta - 1)$

$N_3 = \frac{1}{4}(\xi + 1)(\eta + 1)$, $N_4 = -\frac{1}{4}(\xi - 1)(\eta + 1)$

Since $(x_1, y_1) = (0,0)$, $(x_2, y_2) = (2,0)$, $(x_3, y_3) = (4,4)$, $(x_4, y_4) = (2,4)$

then $x = N_1 x_1 + N_2 x_2 + N_3 x_3 + \cancel{N_4 y_4}\, N_4 x_4$

$= 2N_2 + 4N_3 + 2N_4$

$= -\frac{1}{2}(\xi+1)(\eta-1) + (\xi+1)(\eta+1) - \frac{1}{2}(\xi-1)(\eta+1) = \xi + \eta + 2$

$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$

$= 4N_3 + 4N_4$

$= 2\eta + 2$

Jacobian: $J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

Since $\det(J) = 2 \neq 0$ we could have inverse matrix $J^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$

$(J^T)^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$$K_{11} = \iint \left(\frac{\partial N_1}{\partial x}, \frac{\partial N_1}{\partial y} \right) \begin{pmatrix} \frac{\partial N_1}{\partial x} \\ \frac{\partial N_1}{\partial y} \end{pmatrix} dx\, dy$$

$$= \iint \left(\frac{\partial N_1}{\partial \xi} \quad \frac{\partial N_1}{\partial \eta} \right) J^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (J^T)^{-1} \begin{pmatrix} \frac{\partial N_1}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} \end{pmatrix} \det(J)\, d\xi\, d\eta.$$

$$= \int_{-1}^{1} \int_{-1}^{1} \left[\tfrac{1}{4}(\xi-1) \quad \tfrac{1}{4}(\eta-1) \right] \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{4}(\eta-1) \\ \frac{1}{4}(\xi-1) \end{bmatrix} \times 2\, d\xi\, d\eta.$$

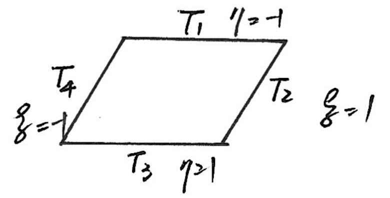
$$= \frac{1}{32} \int_{-1}^{1} \int_{-1}^{1} (5\eta^2 + \xi^2 - 2\eta\xi - 8\eta + 4)\, d\xi\, d\eta$$

$$= \frac{1}{32} \left[\frac{10}{3}\eta^3 + \frac{26}{3}\eta \right]_{-1}^{1} = \frac{3}{4}$$

The Load vector: $f_{1} = \iint_{-1}^{1} g(x, y) N_1 \det(J)\, d\xi\, d\eta.$

$$= \int_{-1}^{1} \int_{-1}^{1} g(\xi + \eta + 2, 2\eta + 2) \tfrac{1}{4}(\xi-1)(\eta-1) \times 2\, d\xi\, d\eta.$$

(c) The boundary vector for this problem is:



$f_b = \int_\Gamma N_i \frac{\partial N}{\partial n}\, ds$

where $ds = \sqrt{(dx)^2 + (dy)^2}$

and $dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta$

$dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta$

$$\therefore \int_{\Gamma_A} y N_i(\xi, \eta)\, ds = \int_{T_2} y(\xi, \eta) N_i(\xi, \eta) \sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2}\, d\eta.$$

$$= \int_{T_2} \left[(\xi+1)(\eta+1) - (\xi-1)(\eta+1) \right] N_i \sqrt{5}\, d\eta.$$

Therefore $f_{b_1} = \int_{-1}^{1} (2\eta + 2) N_1(\xi, \eta) \sqrt{5}\, d\eta = 2\sqrt{5} \int_{-1}^{1} (\eta+1) N_1(\xi, \eta)\, d\eta = 0$

$f_{b_2} = 2\sqrt{5} \int_{-1}^{+1} (\eta+1) N_2\, d\eta = \frac{4}{3}\sqrt{5}$

$f_{b_3} = 2\sqrt{5} \int_{-1}^{1} (\eta+1) N_3\, d\eta = \frac{8}{3}\sqrt{5}$

$f_{b_4} = 2\sqrt{5} \int_{-1}^{1} (\eta+1) N_4\, d\eta = 0$

(d) The mesh is conforming if it guarantees continuity of solutions across element boundaries.

Yes, it is conforming. Because a mesh of square parent elements are conforming and the isoparametric transformation is continuous.

If additional midside nodes were introduced, it wouldn't be changed. Since the new model with 4 midside nodes is contributed by a group of parallelogram,