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# Stability analysis and limit cycle in fractional system with Brusselator nonlinearities

V. Gafiychuk [a,b,*], B. Datsko [b]

[a] Physics Department, New York City College of Technology, CUNY 300 Jay street, Brooklyn, NY 11201, USA
[b] Institute of Applied Problems of Mechanics and Mathematics, National Academy of Sciences of Ukraine, Naukova street 3 B, Lviv 79053, Ukraine

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ABSTRACT

The investigation of limit cycles in the fractional dynamical systems with Brusselator nonlinearities is considered. We present analysis of the stability domains as well as possible solutions realizing at different system parameters.



## 1. Introduction

Fractional oscillators have attracted considerable attention in the last decade. Well-known examples are the Lottka–Volterra model [1], Bonhoeffer–van der Pol oscillator [2–5], fractional-order Chuas, Chen, Lorenz, Lü systems [6–8], Brusselator oscillator [2,9,10], etc. The characteristic features of all these models are that fractional derivatives introduce a new parameter—the order of fractional derivative $\alpha$ changing the properties of the solutions. While applications of fractional derivatives have been considered in famous models, some of the works are devoted to particular properties of these models. These works often give only particular information about the systems not presenting the complete pictures of the realized phenomena. For example, a certain question may arise when analyzing article [10] where the Brusselator model was considered. In order to clarify a little bit an impression that the obtained limit cycle at the total index 0.97 (the sum of orders of two derivatives) is in some way unique we will perform a stability analysis for this model. We restrict ourselves by the model, when each equation has the same derivative indice $\alpha$. In the case where the derivative orders are different, the new instability modes may arise. This is caused by the certain increase of the dimensions of the system under consideration making the model more unstable [4].

The starting point of our consideration is the coupled differential equations [2,10]

$$d^\alpha u/dt^\alpha = A-(B+1)u+u^2v, \tag{1}$$

$$d^\alpha v/dt^\alpha = Bu-u^2v. \tag{2}$$

Here $u(t)$, $v(t)$—are activator and inhibitor variables, and $A$, $B$—are external parameters (the relationship between them determines the system dynamics).

Time derivatives of any function $f(t)$ on the left-hand side of Eqs. (1), (2) are the Caputo fractional derivatives in time of the order $0<\alpha<2$ and are represented as [11,12]

$$\frac{d^\alpha}{dt^\alpha}f(t) := \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}}d\tau,$$

where $m-1<\alpha<m$, $m=1,2$. It should be noted that system (1), (2) at integer $\alpha=1$ corresponds to the system ordinary differential equations.

Analyzing a stability domain we determine characteristic parameters where the system is unstable and possesses a limit cycle. As a result, we obtain that the Brusselator oscillator is unstable in a wide region of $\alpha$ $(0<\alpha<2)$ and results presented in [10] correspond to a one point (denoted by letter $E$ in Fig. 1) in the two-dimensional space of system parameters $A$ and $B$.

## 2. Linear stability analysis

The steady-state solution $(u_0, v_0)$ of the system (1), (2)

$$W(u,v) \equiv A-(B+1)u+u^2v=0,$$
$$Q(u,v) \equiv Bu-u^2v=0 \tag{3}$$

* Corresponding author.
*E-mail address:* vagaf@yahoo.com (V. Gafiychuk).

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can be analyzed by linearization of the system nearby this solution. In this case, the system (1), (2) can be transformed to the system of linear fractional ordinary differential equations with right-hand side matrix $F = \{a_{ij}\}$, $i, j = 1, 2$, diagonal form of which is given by eigenvalues $\lambda_{1,2} = \frac{1}{2}(\operatorname{tr} F \pm \sqrt{\operatorname{tr}^2 F - 4 \det F})$ (coefficients $a_{ij}$ represent Jacobian).

For $\alpha$: $0 < \alpha < 2$ for every point inside the parabola $\det F = \operatorname{tr}^2 F/4$, we can introduce a marginal value $\alpha_0 = \frac{2}{\pi}|\operatorname{Arg}(\lambda_i)|$ given by the formula [1–3,6,13,16]

$$\alpha_0 = \frac{2}{\pi}\tan^{-1}\left(\frac{4\det F}{\operatorname{tr}^2 F} - 1\right)^{1/2}_{\operatorname{Re}\lambda>0} \cup 2 - \frac{2}{\pi}\tan^{-1}\left(\frac{4\det F}{\operatorname{tr}^2 F} - 1\right)^{1/2}_{\operatorname{Re}\lambda<0}. \tag{4}$$

The value of $\alpha$ is an additional bifurcation parameter, which switches the stable and unstable states of the system and changes the form of the limit cycle. At lower $\alpha$: $\alpha < \alpha_0 = \frac{2}{\pi}|\operatorname{Arg}(\lambda_i)|$, the system has oscillatory modes, but they are stable. Increasing the value of $\alpha > \alpha_0 = \frac{2}{\pi}|\operatorname{Arg}(\lambda_i)|$ leads to instability and more complicated dynamics of the system.

## 3. Stability domain and nonlinear solutions

It is widely known for the system (1), (2) with integer time derivatives that a steady state solution becomes unstable according to the Hopf bifurcations

$$\operatorname{tr} F = (B - 1 - A^2) > 0, \qquad \det F = A^2 > 0. \tag{5}$$

Nullclines and eigenvalues of the Brusselator model are presented on Fig. 1(a), (b). The intersections of the nullclines at the decreasing part of $W(u, v) = 0$ lead to the limit cycle formation for $\alpha = 1$. In this case, the real part of the eigenvalues passes through zero at $B_0 = 1 + A_0^2$ (Fig. 1(b)). That means that at this condition, the system is unstable according to the Hopf bifurcation because it has two complex conjugate roots with the real part equal to zero. As a result, the real part becomes less than zero at $B < B_0$ and greater than zero at $B > B_0$ (Fig. 1(b)). In the first case, the condition of the Hopf bifurcation (4) realizes at $\alpha > 1$ and in the second case at $\alpha < 1$. The imaginary part of eigenvalues (ellipse like curve) determines the region of values $B$, where the roots are complex. The interval of localization of these roots depends on the intersection point $P_0 = (B_0, A_0)$. The greater the value of $(B_0)$ at the given $(A_0)$ is, than the wider the range with complex conjugate roots. The way how the eigenvalues build up instability domain is shown in Fig. 1(c), (d). On the typical stability domain for Brusselator, we presented marginal curves corresponding to different $\alpha$ (from 0 to 2). On Fig. 1(c) these curves are obtained as a result of solution of Eq. (4)

$$(B - 1 - A^2)^2\gamma^2 = 4A^2 - (B - 1 - A^2)^2,$$

where $\gamma^2 = \tan^2(\pi\alpha_0/2) \cup \tan^2(\pi(2 - \alpha_0)/2)$ for dependance $A$ on $B$ when the value of $\alpha_0$ changes from 0 to 2 by step 0.5 at the condition that the eigenvalues are complex. Explicitly, we have the parametric dependance of $A(B)$ at the given $\gamma^2(\alpha_0)$

$$A^2 = B - 1 + 2/(1 + \gamma^2) \pm \sqrt{(B - 1 + 2/(1 + \gamma^2))^2 - (B - 1)^2}. \tag{6}$$

At $\alpha > \alpha_0$, dynamical system is unstable and at $\alpha < \alpha_0$, is stable. In other words, any value $\alpha_0$ from zero till $\alpha_0 = 2$ divides the region $(B, A)$ into two domains. Inside each particular domain bounded by corresponding curve $\alpha_0$ stationary state is unstable for this value of $\alpha_0$ and outside it is stable (Fig. 1(c)). The matter is that between $\alpha = 0$ and $\alpha = 2$, the roots are complex and there

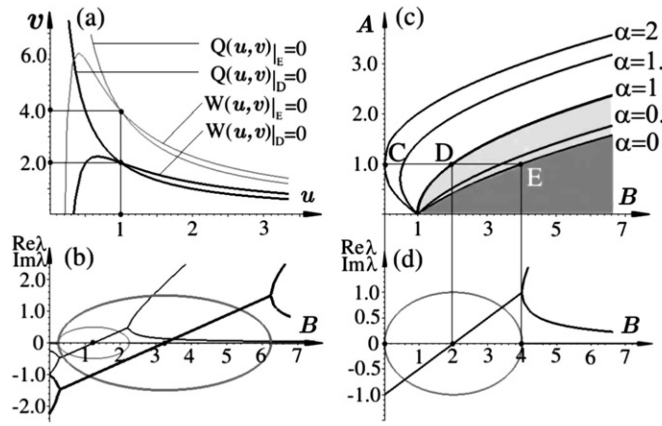


**Fig. 1.** Nullclines in the coordinates $(u, v)$ for $A_0 = 1$, $B_0 = 2$ (black lines) and $A_0 = 1$, $B_0 = 4$ (grey lines)—(a); the eigenvalues ($\operatorname{Re}\lambda$—black lines, $\operatorname{Im}\lambda$—grey lines) of the Brusselator model for $A_0 = 1/2$ and $A_0 = 3/2$—(b); stability domain in the coordinates $(B, A)$—(c); the eigenvalues for the case $A_0 = 1$—(d).

is a minimal value $\alpha_0$ where the system is unstable according to eigenvalues with the complex conjugate roots. It is expected that in the domain inside this particular curves $\alpha_0$ limit cycle arises according to the any value of $\alpha > \alpha_0$. Outside this parabola-alike curves the system is stable according to this value of $\alpha_0$. Of course it can be unstable according to $\alpha$ greater than $\alpha_0$. One of the lines as it was mentioned above corresponds to $\alpha = 1$ ($B = 1 + A^2$) separating the instability domain by two sub-domains where instability takes place at $\alpha < 1$ (grey domain of Fig. 1(c)) and $\alpha > 1$. Because the roots are real and positive in the dark grey domain the system can be unstable even for $\alpha < 0$.

The results of numerical simulation of the initial value problem for the system (1), (2) are presented on Figs. 2–3. The system with the corresponding initial conditions was integrated numerically using explicit and implicit schemes. The fractional derivatives were approximated using a difference schemes on the basis of the Grünwald–Letnikov definition [12,14,15]. In order to obtain limit cycles in the systems we are supposed to take parameters $A$ and $B$ inside instability domain (between the curves $\alpha = 0$ and $\alpha = 2$). Having the values of parameters $A$ and $B$ we calculate $\alpha_0$. In this the system with $\alpha > \alpha_0$ is unstable and possesses limit cycle and with $\alpha < \alpha_0$ is stable.

In Fig. 2, as an example, we present scenario of possible oscillations at different values of $\alpha$. In order to present complete pattern of the possible solutions, oscillations were obtained at different relationships between parameters $A$ and $B$ and the positions they have inside instability region. We start to model the system with the parameters corresponding to point $D$ on Fig. 1(c) ($\alpha = 1$). In this point the standard system is neutrally stable and increase of $\alpha$ leads to limit cycle Fig. 3(a), (b) and decrease of $\alpha$ stabilizes the system. The similar oscillations we obtained in whole domain of the parameters $A$ and $B$ between the curves $\alpha = 1$ and $\alpha = 2$. In the domain above the curve $\alpha = 2$ the oscillations of huge amplitude can be obtained with higher order of the fractional derivative ($\alpha > 2$).

Let us consider solutions in the domain between the curves $\alpha = 0$ and $\alpha = 1$. If we increase $B$ by moving in the direction of the point $E$ (Fig. 1(c)) the system becomes unstable according to the less values of $\alpha$. The characteristic solutions are presented on Fig. 2(c), (d). Successive decrease of $\alpha$ shows that the oscillations are realized in the wide range of system parameters. For a sufficient small $\alpha$ (Fig. 3) such oscillations become the oscillations of the relaxation type. It should be noted that these oscillations are realized exactly at the same parameters as solutions obtained by authors of [10]. Because the point $E$ is located on the curve $\alpha = 0$ instability conditions are satisfied for $\alpha > 0$ and, consequently, the

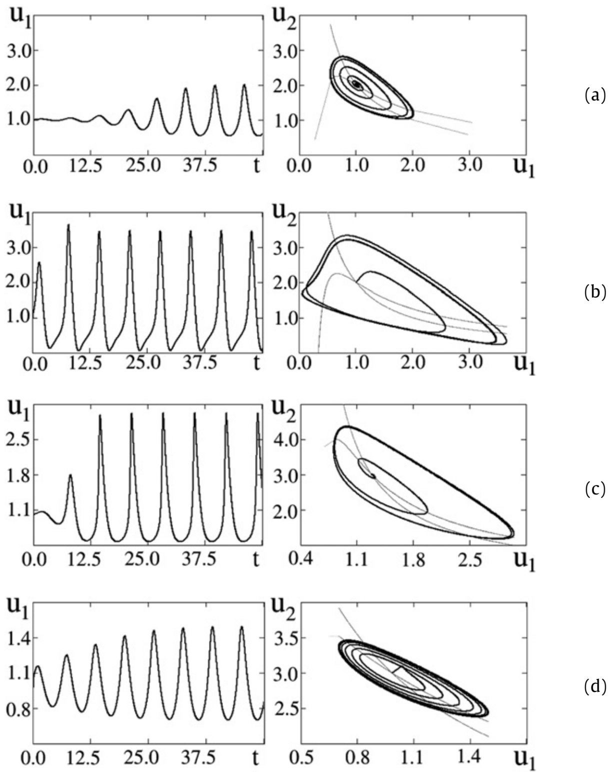


**Fig. 2.** Dynamics of the Brusselator model for $A = 1$, $B = 2$, $\alpha = 1.05$–(a); $A = 1$, $B = 2$, $\alpha = 1.4$–(b); $A = 1$, $B = 3$, $\alpha = 0.9$–(c); $A = 1$, $B = 3$, $\alpha = 0.7$–(d).

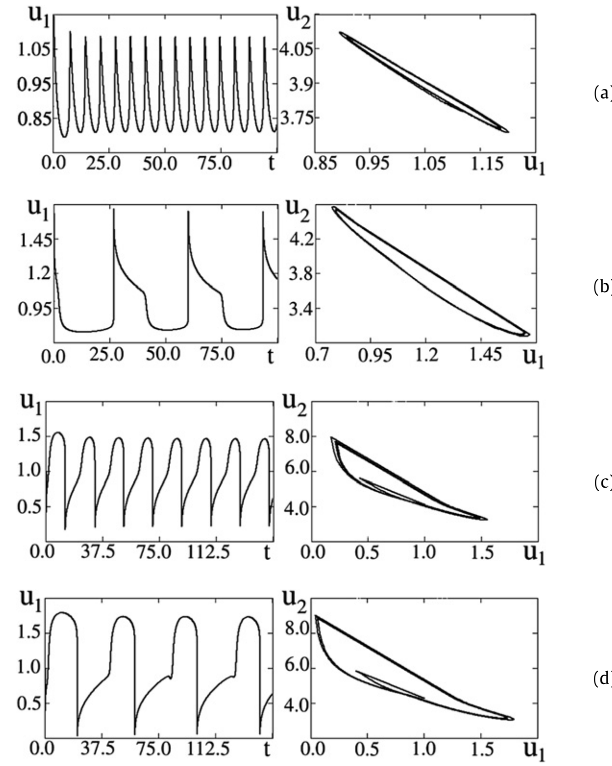


**Fig. 3.** Dynamics of the Brusselator model for $A = 1$, $B = 3.98$, $\alpha = 0.1$–(a); $A = 1$, $B = 4.05$, $\alpha = 0.1$–(b); $A = 1$, $B = 4.1$, $\alpha = -0.25$–(c); $A = 1$, $B = 4.25$, $\alpha = -0.25$–(d).

solutions with total derivative order $\alpha + \alpha$ equal 0.97 [10] are only particular solutions realized in the system. As you can see from the Fig. 3(b), (c) the limit cycle with $\alpha = 0.1$ can be easily realized in the system. Corresponding attractors in the instability domain between $\alpha = 0$ and $\alpha = 1$ are different. For $\alpha \lesssim 1$ oscillations are like more quasiharmonic and for a sufficiently small values of $\alpha$ they have a form of relaxation oscillations and schematically look similar to those ones presented on Fig. 2(c), (d) and Fig. 3(a), (b), correspondingly.

In should be noted that inside the whole region bounded by $B$-axis and curve $\alpha = 0$ the system can possess limit cycle even for negative values of $\alpha$. The characteristic view of such limit cycle is presented on Fig. 3(c), (d). The matter is that in this case the system has an attractor in the form of relaxation oscillations due to two real positive eigenvalues. These solutions have a longer period, are higher in amplitude and can also be characterized by the form combining fast and slow motions. It should be noted that Grünwald–Letnikov formula can diverge at $\alpha < 0$ [11] and this topic will be investigated somewhere else.

We considered a simple dynamical system without space operators. Originally Brusselator model was introduced as reaction–diffusion system for morphogenic modeling. The diversity of attractors in the model (1), (2) will determine space–time behavior of Brusselator model. In this case except standard Hopf and Turing bifurcations a new type of instability arises from the Hopf bifurcation of the perturbations with a certain wave number not equal to zero [3,16].

## 4. Conclusion

In this Letter we considered solutions of the Brusselator model with fractional time-derivative indices. Special attention is paid to linear theory of instability which is analyzed in detail for different parameters. It was shown that the system is unstable practically for all values of $\alpha$: $0 < \alpha < 2$. The characteristic time domain oscillations and two-dimensional phase portrait obtained by computer simulation are presented for different parameters of Brusselator model.

## References


[1] E. Ahmed, A.M.A. El-Sayed, H.A.A. El-Saka, Phys. Lett. A 358 (2006) 1.
[2] V. Gafiychuk, B. Datsko, V. Meleshko, D. Blackmore, Chaos Solitons Fractals, doi:10.1016/j.chaos.2008.04.039.
[3] V.V. Gafiychuk, B.Y. Datsko, Phys. Rev. E 75 (2007) 055201R.
[4] V. Gafiychuk, B. Datsko, V. Meleshko, Physica A 387 (2008) 418.
[5] V. Gafiychuk, B. Datsko, V. Meleshko, J. Comput. Appl. Math., doi:10.1016/j.cam.2007.08.011.
[6] E. Ahmed, A.M.A. El-Sayed, H.A.A. El-Saka, J. Math. Anal. Appl. 325 (2007) 542.
[7] I. Petras, Chaos Solitons Fractals 38 (2008) 140.
[8] M.S. Tavazoei, M. Haeri, Phys. Lett. A 367 (2007) 102.
[9] T.S. Zhou, C.P. Li, Physica D 212 (2005) 111.
[10] Y. Wang, C. Li, Phys. Lett. A 363 (2007) 414.
[11] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Newark, NJ, 1993.
[12] I. Podlubny, Fractional Differential Equations, Academic Press, 1999.
[13] D. Matignon, Comput. Eng. Sys. Appl. 2 (1996) 963.
[14] R. Gorenflo, E.A. Abdel-Rehim, J. Comput. Appl. Math. 205 (2007) 871.
[15] A.C. Galucio, J.-F. Deu, S. Mengue, F. Dubois, Comput. Methods Appl. Mech. Engrg. 195 (2006) 6073.
[16] V. Gafiychuk, B. Datsko, Phys. Lett. A 372 (2008) 619.