1. Diffusion equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial^2 t} \quad \text{with } u=u(t,x),\ t\geq 0,\ 0\leq x\leq 1.$$

Given boundary conditions $u|_{t=0,\,0\leq x\leq 1}=u_0(x)$, $u|_{x=1,\,t\geq 0}=g_1(t)$, $\frac{\partial u}{\partial x}\Big|_{x=0,\,t\geq 0}=g_0(t)$

(a) Expand both sides with Taylor expension on each term.

a function $f(x,y)$ depends on 2 variable x and y. the Taylor serious to second order at point (a,b) is

$$f(x,y)=f(a,b)+(x-a)f_x(a,b)+(y-b)f_y(a,b)+\frac{1}{2}\left[(x-a)^2f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2f_{yy}(a,b)\right]$$
$$+\frac{1}{6}\left[(x-a)^3f_{xxx}(a,b)+3(x-a)(y-b)^2f_{xyy}(a,b)+3(x-a)^2(y-b)f_{yxx}(a,b)+(y-b)^3f_{yyy}(a,b)\right]$$

The left hand side can be written as:

$$\frac{u_{m+1,n}-u_{m,n}}{\Delta t}=\frac{1}{\Delta t}\left\{u(t+\Delta t,x)-u(t,x)\right\}$$
$$=\frac{1}{\Delta t}\left\{\cancel{u(t,x)}+\Delta t\,u_t+\frac{1}{2}(\Delta t)^2u_{tt}+O(\Delta t^3)-\cancel{u(t,x)}\right\}$$
$$=u_t+\frac{1}{2}\Delta t\,u_{tt}+O(\Delta t^2)$$

Expand right hand side:

$$\frac{1}{2}\left\{\frac{u_{m+1,n+1}-2u_{m+1,n}+u_{m+1,n-1}}{(\Delta x)^2}+\frac{u_{m,n+1}-2u_{m,n}+u_{m,n-1}}{(\Delta x)^2}\right\}$$
$$=\frac{1}{2(\Delta x)^2}\left\{u(t+\Delta t,x+\Delta x)-2u(t+\Delta t,x)+u(t+\Delta t,x-\Delta x)+u(t,x+\Delta x)-2u(t,x)+u(t,x-\Delta x)\right\}$$
$$=\frac{1}{2(\Delta x)^2}\Big\{\cancel{u}+\cancel{\Delta t\,u_t}+\cancel{\Delta x\,u_x}+\cancel{\tfrac{1}{2}(\Delta t)^2u_{tt}}+\cancel{\Delta x\Delta t\,u_{xt}}+\tfrac{1}{2}(\Delta x)^2u_{xx}+\cancel{\tfrac{1}{6}(\Delta t)^3u_{ttt}}+\tfrac{1}{2}\Delta t(\Delta x)^2u_{txx}+\cancel{\tfrac{1}{2}\Delta x(\Delta t)^2u_{xtt}}$$
$$+\cancel{\tfrac{1}{6}(\Delta x)^3u_{xxx}}$$
$$\cancel{-2u}\cancel{-2\Delta t\,u_t}\cancel{-(\Delta t)^2u_{tt}}\cancel{-\tfrac{1}{3}(\Delta t)^3u_{ttt}}$$
$$\cancel{+u}\cancel{+\Delta t\,u_t}\cancel{-\Delta x\,u_x}\cancel{+\tfrac{1}{2}(\Delta t)^2u_{tt}}\cancel{-\Delta x\Delta t\,u_{xt}}+\tfrac{1}{2}(\Delta x)^2u_{xx}\cancel{+\tfrac{1}{6}(\Delta t)^3u_{ttt}}+\tfrac{1}{2}\Delta t(\Delta x)^2u_{txx}\cancel{-\tfrac{1}{2}(\Delta t)^2\Delta x\,u_{xtt}}$$
$$\cancel{-\tfrac{1}{6}(\Delta x)^3u_{xxx}}$$
$$\cancel{+u}\cancel{+\Delta x\,u_x}+\tfrac{1}{2}(\Delta x)^2u_{xx}\cancel{+\tfrac{1}{6}(\Delta x)^3u_{xxx}}$$
$$\cancel{-2u}$$
$$\cancel{+u}\cancel{-\Delta x\,u_x}+\tfrac{1}{2}(\Delta x)^2u_{xx}\cancel{-\tfrac{1}{6}(\Delta x)^3u_{xxx}}+O\big((\Delta x)^4+(\Delta t)^4\big)\Big\}$$
$$=\frac{1}{2(\Delta x)^2}\times\left\{2(\Delta x)^2u_{xx}+(\Delta x)^2\Delta t\,u_{txx}+O\big((\Delta x)^4+(\Delta t)^4\big)\right\}$$

Left = Right

$$u_t+\frac{1}{2}\Delta t\,u_{tt}+O(\Delta t^2)=u_{xx}+\frac{1}{2}\Delta t\,u_{xxt}+O\left(\frac{1}{2}(\Delta x)^2,\frac{(\Delta t)^4}{(\Delta x)^2}\right)$$

At $O(1)$ there exists $u_t=u_{xx}$, at $O(\Delta x+\Delta t)$ $u_{tt}=u_{txx}$.

Hence the second order of accuracy.

To calculate LTE:

substituting V for u, using Taylor expansion for first 3 terms:

$$E_{m+1,n}=\frac{V_{m+1,n}-V_{m,n}}{\Delta t}-\frac{1}{2}\left\{\frac{V_{m+1,n+1}-2V_{m+1,n}+V_{m+1,n-1}}{(\Delta x)^2}+\frac{V_{m,n+1}-2V_{m,n}+V_{m,n-1}}{(\Delta x)^2}\right\}$$

$$= V_t + \frac{\Delta t}{2} V_{tt} + \frac{(\Delta t)^2}{6} V_{ttt} + O((\Delta t)^3) - \left(V_{xx} + \frac{\Delta t}{2} V_{xxt} + \frac{(\Delta x)^2}{12} V_{xxxx} + O((\Delta x)^3)\right)$$

$$= \underbrace{(V_t - V_{txx})}_{=0} + \frac{\Delta t}{2}\underbrace{(V_{tt} - V_{xxt})}_{=0} + \frac{(\Delta t)^2}{6} V_{ttt} - \frac{(\Delta x)^2}{12} V_{xxxx} + O((\Delta t)^3) + O((\Delta x)^3)$$

Thus, the princial part of the LTE here is

$$\left(\frac{\Delta t}{6}\right)^2 V_{ttt} - \frac{(\Delta x)^2}{12} V_{xxxx}$$

An estimate for the local truncation error in x is:

$$\frac{1}{2\Delta x^2} O(\Delta x^4 + \Delta t^4) = O(\Delta x^2) \approx \frac{\Delta x^2}{12} V_{xxxx}$$

(b) Consider the m-th time step, let

then $u(t_m, x)|_{x=1} = g_1(t_m)$

denote bondary condition $u(t,0) = h(t)$ when $x=0$

then $u(t_m, 0) = h(t_m)$

Solving boundary value problem by re-writing the system as a tri-diagonal matrix

then having form $A\vec{u} = \vec{b} \implies \vec{u} = A^{-1}\vec{b}$

First rearrange Crank-Nicolson algorithm to seperate known and unknown part.

$$-\frac{1}{\Delta t} u_{m+1,n} + \frac{1}{2(\Delta x)^2} u_{m+1,n+1} - \frac{1}{(\Delta x)^2} u_{m+1,n} + \frac{1}{2(\Delta x)^2} u_{m+1,n-1} = \text{left hand side}$$

$$-\frac{1}{\Delta t} u_{m,n} - \frac{u_{m,n+1} - 2u_{m,n} + u_{m,n-1}}{(\Delta x)^2} = \text{right hand side} = b_n, \quad 1 \le n \le max-1$$

Hence the equation can be write as

$$\alpha u_{m+1,n+1} + \beta u_{m+1,n} + \gamma u_{m+1,n-1} = b_n \qquad (1)$$

where $\alpha = \gamma = \frac{1}{2(\Delta x)^2}$, $\beta = -\frac{1}{(\Delta x)^2} - \frac{1}{\Delta t}$,

$$b_n = -\frac{1}{\Delta t} u_{m,n} - \frac{u_{m,n+1} - 2u_{m,n} + u_{m,n-1}}{(\Delta x)^2}$$

then using equation (1) we can form the matrix

$$\begin{array}{c} n=0 \\ \\ \\ \\ \\ \end{array} A=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \gamma & \beta & \alpha & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 \\ & & \gamma & \beta & \alpha \\ 0 & 0 & \cdot & \cdot & 1 \end{pmatrix}, \quad \vec{u}=\begin{pmatrix} u_{m+1,0} \\ u_{m+1,1} \\ u_{m+1,2} \\ \vdots \\ u_{m+1,max-1} \\ u_{m+1,max} \end{pmatrix}, \quad \vec{b}=\begin{pmatrix} h(t_m) \\ b_1 \\ b_2 \\ \vdots \\ g_1(t_m) \end{pmatrix}$$

Thus we can find solution $\vec{u}=A^{-1}\vec{b}$. The procedure continuous to the subsequent time steps.

(c) Stability.

Let $u_{m,n}=\lambda^m e^{ikn}$ then

$$\frac{\lambda^{m+1}e^{ikn}-\lambda^m e^{ikn}}{\Delta t}=\frac{1}{2(\Delta x)^2}\Big\{\lambda^{m+1}e^{ik(n+1)}-2\lambda^{m+1}e^{ikn}+\lambda^{m+1}e^{ik(n-1)}$$
$$+\lambda^m e^{ik(n+1)}-2\lambda^m e^{ikn}+\lambda^m e^{ik(n-1)}\Big\}$$

divide $\lambda^m e^{ikn}$ on both sides:

$$\frac{\lambda-1}{\Delta t}=\left\{\lambda e^{ik}-2\lambda+\lambda e^{-ik}+e^{ik}-2+e^{-ik}\right\}\times\frac{1}{2(\Delta x)^2}$$
$$=\frac{1}{2(\Delta x)^2}\left\{(e^{ik}+e^{-ik}-2)(\lambda+1)\right\}$$

$$\lambda-1=\frac{\Delta t}{(\Delta x)^2}(\lambda+1)(\cos k-1)$$

Since $\cos(2\alpha)=1-2\sin^2(\alpha)$, then $\cos k=1-2\sin^2(\frac{k}{2})$

$$\lambda-1=-\frac{2\Delta t}{\Delta x^2}(\lambda+1)\sin^2(\tfrac{k}{2})$$

Let $\sigma=\frac{2\Delta t}{(\Delta x)^2}\sin^2(\frac{k}{2})$.

then $\lambda-1=-\Delta(\lambda+1) \Rightarrow \lambda=\frac{1-\Delta}{1+\Delta}$,

Since $\Delta\geq 0$, we have $|\lambda|\leq 1$. therefore unconditional stability.

2. In five-point finite-difference approximation with $\Delta x=\Delta y=h$

$(n+1, m)$

$(n, m-1)$ (n, m) $(n, m+1)$

$(n-1, m)$

$$\frac{u_{n+1,m}-2u_{n,m}+u_{n-1,m}}{(\Delta x)^2}+\frac{u_{n,m+1}-2u_{n,m}+u_{n,m-1}}{(\Delta y)^2}+\frac{u_{n+1,m}-u_{n-1,m}}{2(\Delta y)}+\frac{u_{n,m+1}-u_{n,m-1}}{2(\Delta x)}=0$$

$$u_{n+1,m}-4u+u_{n-1,m}+u_{n,m+1}+u_{n,m-1}+\frac{1}{2}h\,u_{n+1,m}-\frac{1}{2}h\,u_{n-1,m}+\frac{1}{2}h\,u_{n,m+1}-\frac{1}{2}h\,u_{n,m-1}=0$$

$$(1+\tfrac{1}{2}h)u_{n+1,m}+u_{n-1,m}(1-\tfrac{1}{2}h)+(1+\tfrac{1}{2}h)u_{n,m+1}+u_{n,m-1}(1-\tfrac{1}{2}h)-4u=0$$

let $A=1+\frac{1}{2}h$ $\quad B=1-\frac{1}{2}h$

then $A u_{n+1,m}+B u_{n-1,m}+A u_{n,m+1}+B u_{n,m-1}+4u_{n,m}=0$

Let $u^{l}_{n,m}$ be the solution at iteration l, write

$$A u^{l}_{n+1,m}+B u^{l}_{n-1,m}+A u^{l}_{n,m+1}+B u^{l}_{n,m-1}+4u^{l+1}_{n,m}=0$$

to compute next iteration. To test for convergence, write

$$u^{l}_{n,m}=\rho^{l}e^{i(k_1 n+k_2 m)}$$

$$A\times\rho^{l}e^{i(k_1 n+k_1+k_2 m)}+B\times\rho^{l}e^{i(k_1 n-k_1+k_2 m)}+A\times\rho^{l}e^{i(k_1 n+k_2 m+k_2)}+B\times\rho^{l}e^{i(k_1 n,k_2 m-k_2)}-4\rho^{l+1}e^{i(k_1 n+k_2 m)}=0$$

Divide $\rho^{l}\cdot e^{ik_1 n}\cdot e^{ik_2 m}$ on both sides,

$$Ae^{ik_1}+Be^{-ik_1}+Ae^{ik_2}+Be^{-ik_2}-4\rho=0$$

substitude $A=1+\frac{1}{2}h$ $\quad B=1-\frac{1}{2}h$ back to equation

$$e^{ik_1}+\tfrac{1}{2}he^{ik_1}+e^{ik_2}+\tfrac{1}{2}he^{ik_2}+e^{-ik_1}-\tfrac{1}{2}he^{-ik_1}+e^{-ik_2}-\tfrac{1}{2}he^{-ik_2}-4\rho=0$$

$$\rho=\tfrac{1}{4}(e^{ik_1}+e^{-ik_1})+\tfrac{1}{8}h(e^{ik_1}+e^{-ik_1})+\tfrac{1}{4}(e^{ik_2}+e^{-ik_2})+\tfrac{1}{8}h(e^{ik_2}-e^{-ik_2})$$

$$=\tfrac{1}{2}\left[\cos k_1+\cos k_2+\tfrac{h}{2}i\sin k_1+\tfrac{h}{2}i\sin k_2\right]$$

$$=\tfrac{1}{2}\left\{\sqrt{1-\tfrac{h^2}{4}}\sin(k_1+\varphi)+\sqrt{1-\tfrac{h^2}{4}}\sin(k_2+\varphi)\right\}$$

as long as $\left|\sqrt{1-\frac{h^2}{4}}\sin(k_i+\varphi)\right|\le 1$, $i=1,2$

i.e. $-1<\sqrt{1-\frac{h^2}{4}}\sin(k_i+\varphi)\le 1$ $\quad i=1,2$

so that $|\rho|\le 1$