Computational homework 1
Nonlinear equations

Exercises 1 and 2 (marked *) to be submitted. A subset of these will be assessed.
Deadline: 10pm Sunday 8th November.

Please submit your solutions using the link on the course Moodle page. You should submit a single pdf file, created using the Matlab `publish` command, formatted as per the template provided on the Moodle page.

EXERCISE 1(*) (see theoretical sheet 1, exercise 3)
The function $f(x) = \frac{x}{2} - \sin(x) + \frac{\pi}{6} - \frac{\sqrt{3}}{2}$, has two zeros in the interval $[-\pi/2, \pi]$.

a) The bisection method can be used only to approximate one of the two zeros. Apply the bisection method (command `bisection`) to compute an approximation of this root with a tolerance $tol = 10^{-10}$ on the error, that is, $|\alpha - x^k| \leq 10^{-10}$. Choose a suitable interval for the intial data by inspecting the graph of the function f.

b) Compute both roots α and β of the function f using Newton's method (command `newton`). Use the tolerance $tol = 10^{-10}$ on the increment between successive iterates $(x^{k+1} - x^k)$ as stopping criterion, and choose $x_\alpha = \pi$, $x_\beta = -\pi/2$ as initial data for the method.

   Compare the number of iterations used for each of α and β, and explain the difference in the number of iterations (if there is one).

c) For the negative root β we can reduce the number of iterations required by applying the *modified* Newton method

$$x^{k+1} = x^k - 2\frac{f(x^k)}{f'(x^k)}$$

   which is second order if $f'(\beta) = 0$. Use the command `fixpoint` to implement the fixed point iteration that corresponds to the modified Newton method. Report the number of iterations needed to find β using this method, with the same initial guess and increment tolerance as in part (b).

EXERCISE 2(*)
We wish to find the root of $f(x) = x + \exp(-20x^2)\cos(x)$ using Newton's method.

a) Plot a graph of f between $(-1, 1)$.

   Apply Newton's method with $x^0 = 0$ as starting point and $tol = 10^{-10}$, and report the first 10 iterates $x^0, \ldots, x^9$.

   What happens to the iterates x^k as k tends to infinity?

   Why does this behaviour not contradict our theoretical analysis of Newton's method?

b) Use 5 iterations of the bisection method (command `bisection`) applied to the function $f(x)$ on the interval $[-1, 1]$ to find a better starting point for Newton's method.

Use this starting point to compute the root using Newton's method with the tolerance $tol = 10^{-10}$ on the increment. What happens now?

EXERCISE 3

Let's investigate the computation of $\alpha = 2^{1/4}$, which is a zero of the function $f(x) = x^4 - 2$.

a) Using the command `fixpoint`, compute the iterates x^k of the method

$$x^{k+1} = x^k - \frac{(x^k)^4 - 2}{4(x^k)^3},$$

starting from $x^{(0)} = 4$ and using $tol = 10^{-10}$ as tolerance on the increment $(x^{k+1} - x^k)$ as stopping criterion.

Store all the iterates in the vector `x`.

b) Define the error `e = x - 2^(1/4)`, that is the vector for which the kth component is $e^k = x^k - \alpha$. The method is said to be order $p > 0$ if there exists a constant $C > 0$ such that

$$a_p^k = \frac{|e^{k+1}|}{|e^k|^p} \leq C \quad \forall k > 0.$$

Compute the values of a_p^k, for $p = 1, 2, 3$. Using the command `semilogy` visualise the values of a_p^k in the three cases. What is the convergence order p of the given method? Estimate the constant C. How does your estimate compare to the asymptotic value predicted by the theory? (You will need to quote an appropriate theorem from lectures.)