Exercise 2

$$\begin{cases} y'(t) = -C\arctan(ky), & t>0 \\ y(0) = y_0 \end{cases}$$
C and k are real positive constant

(a) backward Euler: $u_{n+1} = u_n + hf(t_{n+1}, u_{n+1})$

$f(t, y(t)) = -C\arctan(ky)$

applying BE method: $u_{n+1} = u_n - hC \cdot \arctan(ku_{n+1})$

therefore $g(u_n, u_{n+1}, h) = u_n + hf(t_{n+1}, u_{n+1}) = u_n - hC \cdot \arctan(ky_{n+1})$

(b) $g(u_n, u_{n+1}, h) = u_n - hC \cdot \arctan(ku_{n+1}) = u_{n+1}$

the equation satisfy the form $x^{k+1} = g(x^k)$ and u_n is fixed point iteration.

Considering it's a fixed point problem, in order to find global convergence, Contraction Mapping Theorem can be applied.

First, $g(u_n, u_{n+1}, h)$ be a strict contraction on $\mathbb{R}$.

Second, $g(u_n, u_{n+1}, h) \in \mathbb{R}$ for $u_n \in \mathbb{R}$. this condition is clear.

In order to prove g is a strict contraction, we need to prove

$|g'(u_n, u_{n+1}, h)| \leq 1$ for $u_n \in \mathbb{R}$.

$$\begin{aligned} g'(u_n, u_{n+1}, h) &= [u_n + hf(t_{n+1}, u_{n+1})]' \text{ wrt } u_{n+1} \\ &= h\frac{\partial f(t_{n+1}, u_{n+1})}{\partial u_{n+1}} \\ &= -h \times C\frac{k}{1+(ku_{n+1})^2} \end{aligned}$$

Hence $\left|-h \cdot \frac{Ck}{1+(ku_{n+1})^2}\right| \leq 1$

$$-1 \leq \frac{hCk}{1+(ku_{n+1})^2} \leq 1$$

The first inequality can't hold since C, k are real positive

The second inequality gives us $h \leq \frac{1+(ku_{n+1})^2}{Ck}$

(c) $u_{n+1} = u_n - hC\arctan(k u_{n+1})$

This is equivalent to the following nonlinear equation for u_{n+1}

$f(u_{n+1}, u_n, h) = u_{n+1} - u_n + hC\arctan(k u_{n+1}) = 0$

Newton's method for the solution of $f(x)=0$ reads, for some given initial data $x^{(0)}$,

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$

Using this for the solution of the nonlinear equation we obtain,

$$u_{n+1}^{(k+1)} = u_{n+1}^{(k)} - \frac{f(u_{n+1}^{(k)})}{f'(u_{n+1}^{(k)})}, \quad u_{n+1}^{(0)} = u_n$$

where we define

$$f(u_{n+1}^{(k)}) = u_{n+1}^{(k)} + hC\arctan(k u_{n+1}^{(k)}) - u_n$$

$$f'(u_{n+1}^{(k)}) = 1 + hC\cdot\frac{k}{1+(k u_{n+1}^{(k)})^2}$$

Exercise 3

$$\begin{cases} y'(t) = f(t, y(t)), & t\in[0,T],\ 0<T<+\infty \\ y(0) = y_0 \end{cases}$$

$$h = \frac{T}{N}, \quad \begin{cases} u_{n+\frac{1}{2}} = u_n + \frac{h}{2} f(t_n, u_n) \\ P_n = f(t_n + \frac{h}{2}, u_{n+\frac{1}{2}}) \\ u_{n+1} = u_n + hP_n \end{cases}$$

(a) The method can be written as:

$$u_{n+1} = u_n + h f(t_n + \frac{h}{2}, u_n + \frac{h}{2} f(t_n, u_n))$$

The right hand side contain previous steps not u_{n+1}, so it's explicit method.

(b) as defined in (a)

$$\Phi = f(t_n + \frac{h}{2}, u_n + \frac{h}{2} f(t_n, u_n))$$

(c) prove zero-stable.

Since f is assumed to be uniformly Lipschitz continuous wrt its second argument and RK2 is one-step method, then

Thm 5.3.7 can be applied.

Thm states: Suppose $\exists h_0 > 0, L_1 > 0$ and $L_2 > 0$ s.t.
$|\Phi(t, u, v, h) - \Phi(t, u', v', h)| \le L_1 |u - u'| + L_2 |v - v'|$, then the one-step method is zero-stable.

Thus, by Thm.

$$|\Phi(t, u, v, h) - \Phi(t, u', v', h)|$$
$$= |f(t_{n+1/2}, u_{n+1/2}) - f(t_{n+1/2}, u'_{n+1/2})|$$
$$\le L_1 |u_{n+1/2} - u'_{n+1/2}| \quad \blacksquare$$

proved zero-stable.

(d) $T_n = \dfrac{y(t_{n+1}) - y(t_n)}{h} - f(t_n + \frac{h}{2}, u_n + \frac{h}{2} f(t_n, y(t_n)))$

$\Phi = f(t_n + \frac{h}{2}, u_n + \frac{h}{2} f(t_n, u_n)) = f(t_{n+1/2}, u_{n+1/2})$

In Taylor Expasion:

$y(t_{n+1}) = y(t_n + h) = y(t_n) + h y'(t_n) + \frac{h^2}{2} y''(t_n) + o(h^3)$

$y''(t_n) = f_t(t_n, y(t_n)) + f_y(t_n, y(t_n)) \times f(t_n, y(t_n))$ by chain rule

Therefore

$$y(t_n + h) = y(t_n) + h f(t_n, y(t_n)) + \frac{h^2}{2}[f_t(t_n, y(t_n)) + f_y(t_n, y(t_n)) \times f(t_n, y(t_n))] + o(h^3)$$

$$= y(t_n) + \frac{h}{2} f(t_n, y(t_n)) + \frac{h}{2}[f(t_n, y(t_n)) + h f_t(t_n, y(t_n)) + h \times f_y(t_n, y(t_n)) \times f(t_n, y(t_n))] + o(h^3) \quad ①$$

Recall multivariate Taylor Expansion:

$f(t+h, y+k) = f(t, y) + h f_t(t, y) + f_y(t, y) k + \cdots$

Hence, $f(t+\frac{h}{2}, y+\frac{h}{2}f(t_n, y(t_n))$ ②

$$= f(t_n, y(t_n)) + \frac{h}{2} f_t(t_n, y(t_n)) + \frac{h}{2} f_y(t_n, y(t_n)) f(t_n, y(t_n)) + O(h^2)$$

Then we can combine ① and ② together:

$$T_n = \frac{1}{2} f(t_n, y(t_n)) + \frac{1}{2}\Big[f(t_n, y(t_n)) + h f_t(t_n, y(t_n)) + h f_y(t_n, y(t_n)) f(t_n, y(t_n)) \Big] + O(h^2) - f(t_n, y(t_n)) - \frac{h}{2} f_t(t_n, y(t_n)) - \frac{h}{2} f_y(t_n, y(t_n)) f(t_n, y(t_n))$$

$$= O(h^2)$$

Hence, RK2 method is consistent with $T_n = O(h^2)$. my assumption is that $y(t_n)$ is three times differentiable, or $f(t, y(t))$

is twice differentiable. Either explanations assumes y and f sufficiently smooth.

(e) Use Theorem 5.5.8 which need 3 assumptions:

1. f satisfied assumption of Picard's Thm. This is verified by f is assumed to be uniformly Lipschitz continuous wrt its second argument

2. method is zero-stable which is proved in (c)

3. uniformly consistent which is proved in (d).

As all condition are satisfied, we can conclude that $E(h) \leq C\,T(h)$. Furthermore, $T(h) = O(h^2)$ as $h \to 0$, then also $E(h) = O(h^2)$. This proves RK2 is second order convergent

Exercise 5

$$\begin{cases} \vec{w}'(t) = A\vec{w}(t) \\ \vec{w}(0) = \vec{w}_0 \end{cases} \qquad \vec{w}(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

(a) Forward Euler:

$$\begin{cases} \underline{w}(0) = \underline{w_0} \\ \underline{w}^{n+1} = \underline{w}^n + h \times A\underline{w}^n \end{cases}, \text{ where } \underline{w}^n = \begin{bmatrix} w_{1,n} \\ w_{2,n} \end{bmatrix} \text{ is estimation of } \vec{w}(t)$$

Backward Euler:

$$\begin{cases} \underline{w}(0) = \underline{w_0} \\ \underline{w}^{n+1} = \underline{w}^n + h \times A\underline{w}^{n+1} \end{cases}$$

Crank-Nicolson:

$$\begin{cases} \underline{w}(0) = \underline{w_0} \\ \underline{w}^{n+1} = \underline{w}^n + \frac{h}{2}[A\underline{w}^n + A\underline{w}^{n+1}] \end{cases}$$

(b) $A = VDV^{-1}$

Define $\underline{x}^n = V^{-1}\underline{w}^n$, where $\underline{x}^n = \begin{bmatrix} x_{1,n} \\ x_{2,n} \end{bmatrix}$ $\underline{w}^n = \begin{bmatrix} w_{1,n} \\ w_{2,n} \end{bmatrix}$

So $V\underline{x}^n = \underline{w}^n$

In Forward Eular:

$$\underline{w_{n+1}} = \underline{w_n} + h \times A\underline{w_n} \Rightarrow V\underline{x}^{n+1} = V\underline{x}^n + h \times VDV^{-1} \times V\underline{x}^n$$

$$\Rightarrow \underline{x}^{n+1} = \underline{x}^n + h \times D \cdot \underline{x}^n$$

Backward Eular:

$$\underline{w}^{n+1} = \underline{w}^n + h \times A\underline{w}^{n+1} \Rightarrow \underline{x}^{n+1} = \underline{x}^n + h \times D \cdot \underline{x}^{n+1}$$

Crank-Nicolson:

$$\underline{w}^{n+1} = \underline{w}^n + \frac{h}{2}[A\underline{w}^n + A\underline{w}^{n+1}]$$

$$\Rightarrow \underline{x}^{n+1} = \underline{x}^n + \frac{h}{2}\left[D\,\underline{x}^n + D\,\underline{x}^{n+1}\right]$$

(C) $d_1, d_2 < 0$, $D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$

Forward Eular:

$\underline{x}^{n+1} = \underline{x}^n + hD\underline{x}^n = B\underline{x}^n$, where $B = (I + hD)$ is a diagonal matrix

for stability, $\rho(B) < 1$. i.e $\max\limits_{i=1,2}\{|1+hd_i|\} < 1$

$$\Rightarrow -1 < 1 + hd_i < 1 \quad , i=1,2$$

$$\Rightarrow 0 < h < \min_{i=1,2}\left\{-\frac{2}{d_i}\right\}$$

Backword Eular:

$\underline{x}^{n+1} = \underline{x}^n + hD\underline{x}^{n+1} \Rightarrow \underline{x}^{n+1} = (I - hD)^{-1}\underline{x}^n$

For stability. $\rho((I-hD)^{-1}) < 1$,

$$\text{i.e } \max_{i=1,2}\left\{\left|\frac{1}{1-hd_i}\right|\right\} < 1$$

$$\Rightarrow \max_{i=1,2}\{|1-hd_i|\} > 1$$

$$1 - hd_i > 1 \quad \text{or} \quad 1 - hd_i < -1$$

$(-hd_i > 0$ for sure$)$ or $hd_i > 0$ is not possible

Hence as long as $h > 0$, the stability holds. i.e unconditional stability.

For Crank-Nicolson:

$$\underline{x}^{n+1} = \underline{x}^n + \frac{h}{2}\left[D\,\underline{x}^n + D\,\underline{x}^{n+1}\right]$$

$\underline{x}^{n+1} = B\,\underline{x}^n$, where $B = [I + \frac{h}{2}D][I - \frac{h}{2}D]^{-1}$

For stability: $\rho(B) < 1$,

$$\text{i.e } \max_{i=1,2}\left\{\left|\frac{1+\frac{h}{2}d_i}{1-\frac{h}{2}d_i}\right|\right\} < 1$$

which is for sure since $d_i < 0$, $1 + \frac{h}{2}d_i < 1 - \frac{h}{2}d_i$ holds forever

Hence as long as $h>0$, the stability holds. i.e unconditional stability.

(d) $\underline{w}'(t) = \begin{bmatrix} w_1'(t) \\ w_2'(t) \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -\lambda & -\mu \end{bmatrix}$, $w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$

$\underline{w}(0) = \begin{bmatrix} w_1(0) \\ w_2(0) \end{bmatrix} = \begin{bmatrix} w_{1,0} \\ w_{2,0} \end{bmatrix}$

So in matrix form:

$$\begin{bmatrix} w_1'(t) \\ w_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\lambda & -\mu \end{bmatrix} \times \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \quad \text{and} \quad \underline{w}(0) = \begin{bmatrix} w_{1,0} \\ w_{2,0} \end{bmatrix}$$

To calculate eigenvalue of A

$$\det(A - dI) = 0 \Rightarrow \begin{vmatrix} -d & 1 \\ -\lambda & -\mu - d \end{vmatrix} = 0$$

$$d(\mu + d) + \lambda = 0$$

$$d_{1,2} = \frac{-\mu \pm \sqrt{\mu^2 - 4\lambda}}{2}$$

From (c), it's clear that backward Euler and Crank-Nicolson are independent of h, i.e unconditional stable.

For Forward Euler method has condition on h, which means $0 < h < \min\left\{ -\frac{4}{-\mu \pm \sqrt{\mu^2 - 4\lambda}} \right\}$ and $d_{1,2} < 0$

If $\lambda = 6$ and $\mu = 5$, then $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$

I can calculate eigenvalue value and eigenvector,

$d_1 = -2$, $d_2 = -3$ so $D = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$

As discussed in (c), $h_1 = -\frac{2}{-2} = 1$, $h_2 = -\frac{2}{-3} = \frac{2}{3}$

So condition for h is $0 < h < \frac{2}{3}$.