Exercise 1.

(a) matrix $A = D - E - F$

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} = \underset{D}{\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}} - \underset{E}{\begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}} - \underset{F}{\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}}$$

$P\underline{x}^{k+1} = N\underline{x}^k + \underline{b}$ for $k = 0, 1, 2 \dots$

From stationary method $\underline{x}^{k+1} = B\underline{x}^k + C$, where $N = P - A$, $B = I - P^{-1}A$, $C = P^{-1}\underline{b}$

For Jacobian Method, $P = D$, $N = E + F$

then $B_J = I - D^{-1}A = D^{-1}(E+F)$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}^{-1} \left[\begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{1}{3} \\ -\frac{2}{5} & 0 \end{pmatrix}$$

For G-S method, $P = D - E$, $N = F$

then $B_{GS} = I - (D-E)^{-1}A = (D-E)^{-1}F$

$$= \left[\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} \right]^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 2 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{1}{3} \\ 0 & \frac{2}{15} \end{pmatrix}$$

(b) given initial vector $x^0 = (\frac{1}{2}, \frac{1}{2})^T$

Jacobian Method: $\underline{x}^1 = B_J \underline{x}^0 + D^{-1}\underline{b}$

$$= \begin{pmatrix} 0 & -\frac{1}{3} \\ -\frac{2}{5} & 0 \end{pmatrix} \times \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 1/3 & 0 \\ 0 & 1/5 \end{pmatrix} \times \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{6} \\ -\frac{1}{5} \end{pmatrix} + \begin{pmatrix} \frac{4}{3} \\ \frac{7}{5} \end{pmatrix}$$

$$\underline{x}^1 = \left(\frac{7}{6}, \frac{6}{5}\right)^T$$

(c) By Thm 4.7.2.

If $A \in \mathbb{R}^{n \times n}$ is SDD (strictly diagnolly dominant) then both Jacobi and Gauss–Seidel converge for arbitrary initial guesses.

Since $A = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$, by SDD if $|a_{ii}| > \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}|$. By checking this criteria,

$|3|>1$ and $|5|>2$.

Hence, both method converge.

(ii) using B_J and B_{GS} to calculate eigenvalues

$$\det(B_J - \lambda I) = \begin{vmatrix} -\lambda & -\frac{1}{3} \\ -\frac{2}{5} & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \frac{2}{15} = 0 \quad \text{then } \lambda_{1,2} = \pm\sqrt{\frac{2}{15}}$$

$$\rho(B_J) = \max_{\lambda_i \in \sigma(B_J)} |\lambda_i| = \sqrt{2/15}$$

$$\det(B_{GS} - \lambda I) = \begin{vmatrix} -\lambda & -\frac{1}{3} \\ 0 & \frac{2}{15}-\lambda \end{vmatrix} = -\lambda\frac{2}{15} + \lambda^2 = 0$$

$\lambda_1 = 0 \quad \lambda_2 = \frac{2}{15}$

$\rho(B_{GS}) = \frac{2}{15}$

Since $\rho(B_{GS}) > \rho(B_J)$, Jacobi gives the iteration with the smallest special radius.

(d) The Jacobi iteration matrix with entry $b_{ij} = -\frac{a_{ij}}{a_{ii}}$ and $b_{ii} = 0$ for $i = 1 \ldots n$

$$\|B_J\|_\infty = \max_{i\in\{1\ldots n\}} \sum_{j=1}^{n} |b_{ij}| = \max_{i\in\{1\ldots n\}} \sum_{j=1}^{n} \frac{|a_{ij}|}{|a_{ii}|}$$

$$= \max_{i\in\{1\ldots n\}} \frac{1}{|a_{ii}|} \cdot \sum_{j=1}^{n} |a_{ij}|$$

$$= \max\left\{\frac{a_{12}}{a_{11}}, \frac{a_{21}}{a_{22}}\right\}$$

$$= \max\left\{\frac{1}{3}, \frac{2}{5}\right\}$$

$$= \frac{2}{5}$$

the convergence constant $\|B_J\| = \frac{2}{5} < 1$, then its linear convergence

The G-S iteration matrix $B_{GS} = I - (D-E)^{-1}A$

$$\|B_{GS}\|_\infty = \max_{i\in\{1\ldots n\}} \sum_{j=1}^{n} |b_{ij}| = \max\left\{\left|\frac{a_{12}}{a_{11}}\right|, \left|\frac{a_{21}}{a_{22}}\right|\right\} = \max\left(\frac{1}{3}, \frac{2}{15}\right) = \frac{1}{3} < 1$$

So it's also linear convergence

Through comparing these 2 constant $\|B_J\|_\infty > \|B_{GS}\|_\infty$

Exercise 4

(a) sufficient condition: A is SDD $\Longrightarrow$ Jacobi and Gauss-Seidel convergence with condition A is strictly dagonally dominant, i.e $|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$

$$A = \begin{pmatrix} \gamma & k & 0 \\ k & \gamma & \beta \\ 0 & \beta & \gamma \end{pmatrix} \Longrightarrow |\gamma| > k, \ |\gamma| > k+\beta, \ |\gamma| > \beta$$

(b) necessary and sufficient condition for convergence is $\rho(B_J) < 1$

$$A = \begin{pmatrix} \gamma & k & 0 \\ k & \gamma & \beta \\ 0 & \beta & \gamma \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma & & 0 \\ & \gamma & \\ 0 & & \gamma \end{pmatrix}}_{D} - \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ -k & 0 & 0 \\ 0 & -\beta & 0 \end{pmatrix}}_{E} - \underbrace{\begin{pmatrix} 0 & -k & 0 \\ 0 & 0 & -\beta \\ 0 & 0 & 0 \end{pmatrix}}_{F}$$

with iteration $\underline{x}^{k+1} = B\underline{x}^k + C \iff P\underline{x}^{k+1} = N\underline{x}^k + \underline{b}$

And $P = D$, $N = E+F$, $B_J = D^{-1}(E+F)$, $\underline{C} = D^{-1}\underline{b}$

So B_J can be $\begin{pmatrix} 0 & -\frac{k}{\gamma} & 0 \\ -\frac{k}{\gamma} & 0 & -\frac{\beta}{\gamma} \\ 0 & -\frac{\beta}{\gamma} & 0 \end{pmatrix}$

$$\det(B_J - \lambda I) = \begin{vmatrix} -\lambda & -\frac{k}{\gamma} & 0 \\ -\frac{k}{\gamma} & -\lambda & -\frac{\beta}{\gamma} \\ 0 & -\frac{\beta}{\gamma} & -\lambda \end{vmatrix} = 0$$

$$\Longrightarrow -\lambda^3 + \frac{k^2}{\gamma^2}\lambda + \frac{\beta^2}{\gamma^2}\lambda = 0$$

$$\lambda\left(\frac{k^2+\beta^2}{\gamma^2} - \lambda^2\right) = 0$$

$$\lambda_1 = 0 \qquad \lambda_{1,2} = \pm\sqrt{\frac{k^2+\beta^2}{\gamma^2}}$$

Then $\lambda_{max} = \lambda_1 = \sqrt{\frac{k^2+\beta^2}{\gamma^2}}$

$$\rho(B_J) = \max_{\lambda\in\sigma(B_J)} |\lambda_1| = \sqrt{\frac{k^2+\beta^2}{\gamma^2}} < 1. \quad \text{with } \gamma^2 \neq 0$$

$$\Longrightarrow \sqrt{k^2+\beta^2} < |\gamma|.$$

Therefore the conditions on Jacobi method to be convergent is $\sqrt{k^2+\beta^2} < |\gamma|$ and $\gamma \neq 0$

(c) $D(\underline{x}^{k+1} - \underline{x}^{k}) = \alpha \underline{r}^{k}$, $k \geq 0$

$$D^{-1}A = \begin{pmatrix} \frac{1}{\gamma} & & 0 \\ & \frac{1}{\gamma} & \\ 0 & & \frac{1}{\gamma} \end{pmatrix} \begin{pmatrix} \gamma & k & 0 \\ k & \gamma & \beta \\ 0 & \beta & \gamma \end{pmatrix} = \begin{pmatrix} 1 & \frac{k}{\gamma} & 0 \\ \frac{k}{\gamma} & 1 & \beta/\gamma \\ 0 & \beta/\gamma & 1 \end{pmatrix}$$

Since $(D^{-1}A)^T = D^{-1}A$ it's symmetric and for symmetric Positive define, we need $\lambda > 0$ for all eigenvalues.

$$\det(D^{-1}A) = \begin{pmatrix} 1-\lambda & \frac{k}{\gamma} & 0 \\ \frac{k}{\gamma} & 1-\lambda & \beta/\gamma \\ 0 & \beta/\gamma & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)^3 - \frac{k^2}{\gamma^2}(1-\lambda) - \frac{\beta^2}{\gamma^2}(1-\lambda) = 0$$

$$(1-\lambda)\left[(1-\lambda)^2 - \frac{k^2+\beta^2}{\gamma^2}\right] = 0$$

$$\lambda_1 = 1 , \lambda_2 = 1 - \frac{\sqrt{k^2+\beta^2}}{|\gamma|} \quad \lambda_3 = 1 + \frac{\sqrt{k^2+\beta^2}}{|\gamma|}$$

$\Longrightarrow -1 < \frac{\sqrt{k^2+\beta^2}}{|\gamma|} < 1$, therefore $k^2+\beta^2 < \gamma^2$ matrix is SPD.

$$\lambda_{max} = 1 + \frac{\sqrt{k^2+\beta^2}}{|\gamma|} \qquad \lambda_{min} = 1 - \frac{\sqrt{k^2+\beta^2}}{|\gamma|}$$

Sationary Richardson method is convergence iff $0 < \alpha < \frac{2}{\lambda_{max}}$

The method with linear convergence has $\|\underline{x}^{k+1} - \underline{x}\|_2 \leq \rho(B) \|\underline{x}^{k} - \underline{x}\|_2$

where $\rho(B) = \max\{|1-\alpha\lambda_{max}|, |1-\alpha\lambda_{min}|\}$ is minimized when

$1-\alpha\lambda_{max} = -(1-\alpha\lambda_{min})$

$$\Longrightarrow \rho(B) = \frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} = \frac{2\sqrt{k^2+\beta^2}}{2\gamma}$$

and $\alpha^* = \frac{2}{\lambda_{max}+\lambda_{min}} = \frac{2}{2} = 1$ giving the fastest convergence

$C = \rho(B_{\alpha^*}) = \frac{\sqrt{k^2+\beta^2}}{\gamma}$ along with $\|x^{k+1} - x\|_2 \leq C \| x^k - x\|_2$

(d) from last question we get $C = \frac{\sqrt{k^2+\beta^2}}{\gamma}$

$\gamma = 1$, $\beta = 2^{-4}$, $k = \sqrt{3} \times 2^{-4}$

and then $\|x^k - x\|_2 \leq (\rho(B))^k \| x^0 - x\|_2$

$$\leq \left[\frac{\sqrt{(\sqrt{3}\times 2^{-4})^2+(2^{-4})^2}}{1}\right]^k \times 1$$

$$=\left(\frac{1}{2}\right)^{3k} \leq \left(\frac{1}{2}\right)^9 \quad \text{thus } k=3$$

Excercise 5.

(a) $\|x\|_A = \sqrt{x^T A x} = \sqrt{x^T (A^{1/2})^2 x} = \sqrt{x^T A^{1/2} \cdot A^{1/2} \cdot x} = \sqrt{(A^{1/2} \cdot \vec{x})^T \cdot (A^{1/2} \cdot x)} = \| A^{1/2} \cdot \vec{x} \|_2$

$\|x\|_A = \|A^{\frac{1}{2}} \cdot x\|_2 \leq \|A^{\frac{1}{2}}\|_2 \|\vec{x}\|_2 = C \cdot \|\vec{x}\|_2$ where $C = \|A^{\frac{1}{2}}\|_2$

With $c=1 < \|x^{\frac{1}{2}}\|_2$ satisfying that $\|x\|_A \geq c\|x\|_2$ where $c=1$

Hence for $c=1$ $C = \|A^{\frac{1}{2}}\|$,

$$c\|\vec{x}\|_2 < \|\vec{x}\|_A < C\|\vec{x}\|_2$$

(b) Show $A^{\frac{1}{2}} B_\alpha = B_\alpha A^{\frac{1}{2}}$, where $A^{\frac{1}{2}}$ is unique SPD square root of A.

$(A^{\frac{1}{2}})^2 = A$

$B_\alpha = I - \alpha (A^{\frac{1}{2}})^T (A^{\frac{1}{2}})$

$A^{\frac{1}{2}} B_\alpha = A^{\frac{1}{2}} - \alpha [A^{\frac{1}{2}} \cdot (A^{\frac{1}{2}})^T \cdot A^{\frac{1}{2}}]$

$= (I - \alpha A) A^{\frac{1}{2}}$

$= B_\alpha \cdot A^{\frac{1}{2}}$

$x - x^{k+1} = ((I-\alpha A)\underline{x} + \alpha \underline{b}) - ((I-\alpha A)\underline{x}^k + \alpha \underline{b})$

$= (I-\alpha A)(x - x^k)$

So $e^{k+1} = B_\alpha e^k$, therefore $\|e^{k+1}\|_A = \|B_\alpha e^k\|_A$

$= \|A^{\frac{1}{2}} B_\alpha \cdot e^k\|_2$

$\leq \|A^{\frac{1}{2}} B_\alpha A^{-\frac{1}{2}}\|_2 \cdot \|A^{\frac{1}{2}} e^k\|_2$

$= \|B_\alpha\|_2 \cdot \|e^k\|_A$

Since A is $n \times n$ symmetric matrix, we can conclude $\|B_\alpha\|_2 = \rho(B_\alpha)$, thus $\|e^{k+1}\| \leq \rho(B_\alpha)\|e^k\|$