Ex 3.

Since u is sufficient smooth on interval $[x-h_0, x+h_0]$ which means 3 times differentiable on interval. Then we can use Taylor Thm in the neighborhood of x given $0<h<h_0$.

$$u(x+h) = u(x) + h u'(x) + \frac{h^2}{2} u''(\xi), \text{ where } x<\xi<x+h$$
$$= u(x) + h u'(x) + O(h^2)$$

$$u(x+h) - u(x) = h u'(x) + O(h^2)$$
$$|u'(x) - Du(x)| = \left| u'(x) - \frac{u(x+h)-u(x)}{h} \right|$$
$$= \left| u'(x) - u'(x) - \frac{O(h^2)}{h} \right|$$
$$= \left| \frac{O(h^2)}{h} \right|$$
$$\overset{\text{Taylor}}{=} h \times \left| \frac{u''(\xi)}{2} \right| \leq h\, C_1(u, h_0)$$

where $C(u,h_0) = \sup \frac{u''(y)}{2}$ for $h \leq h_0$, $y \in [x, x+h_0]$

So C_1 is dependent on u and h_0

Let's assume u is 4 times differentiable to find second result

① $u(x+h) = u(x) + h u'(x) + \frac{h^2}{2} u''(x) + \frac{h^3}{3!} u'''(x) + \frac{h^4}{4!} u^{(4)}(\xi^+)$

② $u(x-h) = u(x) - h u'(x) + \frac{h^2}{2} u''(x) - \frac{h^3}{3!} u'''(x) + \frac{h^4}{4!} u^{(4)}(\xi^-)$

where $\xi^+ \in [x, x+h]$, $\xi^- \in [x-h, x]$

①+②: $u(x+h) + u(x-h) = 2u(x) + h^2 u''(x) + \frac{h^4}{12} u^{(4)}(\xi)$, where $\xi \in [x-h, x+h]$

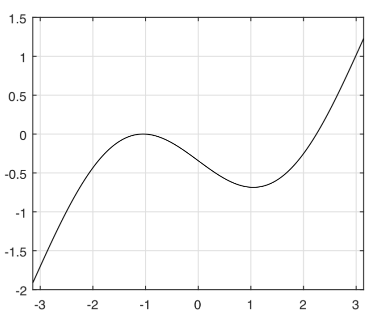
rearrange the formula:

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = u''(x) + \frac{h^2}{12} u^{(4)}(\xi),$$

$$|u''(x) - D^2 u(x)| = \left| \frac{h^2}{12} u^{(4)}(\xi) \right| = \left| \frac{1}{12} u^{(4)}(\xi) \right| \times h^2 \leq C_2(u,h_0) h^2$$ where $C_2(u,h_0) = \sup \frac{1}{12} u^{(4)}(y)$, $y \in [x-h_0, x+h_0]$

So that C_2 is dependent on u and h_0

Ex 3. $f(x) = \frac{x}{2} - \sin x + \frac{\pi}{6} - \frac{\sqrt{3}}{2}$



(a) by using bisection method can't find both roots for interval $[-3,3]$ set $a=-3$ $b=3$, Apply Bisection method

$x_0 = \frac{a+b}{2} = 0$ $f(x_0)<0$, $f(a)<0$, $f(b)>0$;
$f(x_0) \times f(b) < 0$, set $a=0, b=3$

$x_1 = \frac{a+b}{2} = 1.5$ $f(x_1)<0$, $f(a)<0$, $f(b)>0$;
$f(x_1)\cdot f(b) > 0$, set $a = x_1, b=3$

$x_2 = \frac{a+b}{2} = 2.25$ $f(x_2)>0$, $f(a)<0$, $f(b)>0$;
$f(x_2)\cdot f(a) < 0$, set $a=1.5$, $b=2.25$

$x_3 = \frac{a+b}{2} = 1.875$ $f(1.875)<0$, $f(2.25)>0$, Set $a=1.875$, $b=2.25$

$x_4 = \frac{a+b}{2} = 2.0625$ $f(x_4)<0$, $f(2.25)>0$, set $a=2.0625$ $b=2.25$
$x_5 = \frac{a+b}{2} = 2.15625$ $f(x_5)<0$, $f(2.25)>0$

$x_6 = 2.2382812$, $x_7 = 2.24414062$, $x_8 = 2.24707031$, $x_9 = 2.245605468$
It's clearly that bisection method converge to one root around 2.246

- given $tol = 10^{-10}$, we have $\frac{|b-a|}{2^{k+1}} \le tol$ and $b=3$ $a=-3$

Hence $\frac{6}{2^{k+1}} \le 10^{-10}$
$6\times10^{10} \le 2^{k+1}$
$\log_2(6\times10^{10}) \le k+1$
$k \geq \log_2(6\times10^{10})-1$

(b) $x^{k+1} = x^k - \frac{f(x^k)}{f'(x^k)}$
$f(x^k) = \frac{x^k}{2} - \sin x^k + \frac{\pi}{6} - \frac{\sqrt{3}}{2}$
$f'(x^k) = \frac{1}{2} - \cos x^k$
$x^{k+1} = x^k - \frac{\frac{x^k}{2} - \sin x^k + \frac{\pi}{6} - \frac{\sqrt{3}}{2}}{\frac{1}{2} - \cos x^k}$

Since $f:[-\pi,\pi] \to \mathbb{R}$ can be twice continuously differentiable,
i.e $f''(x^k) = \sin x^k$ and let one root α_1 be $f(\alpha_1)=0$ which can be find, $\alpha_1 = 0$
and $f'(\alpha_1) = \frac{1}{2} - \cos\alpha_1 = 0$.

Thus it's linear convergence
And for another root α_2 we have calculated by bisection method
From graph it's clear that $f'(\alpha_2) \neq 0$, $f(\alpha_2) = 0$.
Thus, it's quadratic convergence.

(c) $\phi(x) = \sin x + \frac{x}{2} - (\frac{\pi}{6} - \frac{\sqrt{3}}{2})$, $x^0 \in [\frac{\pi}{2}, \pi]$, to prove linear convergence
$\phi: [\frac{\pi}{2}, \pi] \to \mathbb{R}$ is differentiable and let $\alpha \in [\frac{\pi}{2}, \pi]$ be a fixed point of ϕ.

$\phi'(x) = \cos x + \frac{1}{2}$ since $x \in [\frac{\pi}{2}, \pi]$, then we get $|\phi'(x)|<1$. Hence $|\phi'(\alpha)|<1$

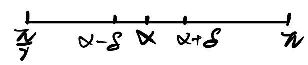
Let $\Lambda = \frac{1+|\phi'(\alpha)|}{2}$ since $|\phi'(\alpha)|<1$, $0<\Lambda<1$
Take $\varepsilon = \Lambda - |\phi'(\alpha)|$ in the def'n of continuity, then $\exists\ \delta$ s.t.
$|x-\alpha| \le \delta \implies |\phi'(x) - \phi'(\alpha)| = \Lambda - |\phi'(\alpha)|$
So that $|\phi'(x)| = |\phi'(x) + \phi'(\alpha) - \phi'(\alpha)| \le |\phi'(\alpha)| + |\phi'(x) - \phi'(\alpha)| \le \Lambda$ for all $x \in [\alpha-\delta, \alpha+\delta] =: I_\delta \subseteq [\frac{\pi}{2}, \pi]$

So ϕ is a strict contraction on $[\frac{\pi}{2}, \pi]$
Next we need to prove $\phi(x) \in I_\delta$ for all $x \in I_\delta$ by mean value theorem.

get $\phi'(\xi) = \frac{\phi(\alpha+\delta)-\phi(\alpha-\delta)}{\alpha+\delta-(\alpha-\delta)}$ for $\alpha-\delta<\xi<\alpha+\delta$

since $|\phi'(x)|<1$ for all $x\in[\frac{\pi}{7},\pi] \Longrightarrow |\phi'(\xi)|<1$

$$\frac{|\phi(\alpha+\delta)-\phi(\alpha-\delta)|}{|(\alpha+\delta)-(\alpha-\delta)|}<1 \ , \ \delta,\alpha>0$$



$$|\phi(\alpha+\delta)-\phi(\alpha-\delta)|<|(\alpha+\delta)-(\alpha-\delta)|$$

which means the distance between $\phi(\alpha+\delta)$ and $\phi(\alpha-\delta)$ is smaller than 2δ

proved $\phi\in I_\delta$.

Hence in local area I_δ, we can use contraction mapping Thm to get $|x^{k+1}-\alpha|\le\Lambda<|x^k-\alpha|$

Ex 4.

(a) $f(x)=x^3-2$

$\phi(x)=x(1-\frac{w}{3})+x^3(1-w)+\frac{2w}{3x^2}+2(w-1)$

$f(x)=0 \iff \phi(x)=x$

if $f(x)=0$ then $x^3=2$ $x=(2)^{\frac{1}{3}}$, So we can substitade back to $\phi(x)=x$

$x=x(1-\frac{w}{3})+x^3(1-w)+\frac{2w}{3x^2}+2(w-1)$

$\cancel{x=x}-\frac{w}{3}x+\cancel{2(1-w)}+\frac{2}{3x^2}w+\cancel{2(w-1)}$

$\frac{w}{3}x=\frac{2}{3x^2}w$

$\frac{w}{3}x^3=\frac{2}{3}w \Rightarrow \frac{2}{3}w=\frac{2}{3}w$, which means $w\in\mathbb{R}$

(b) If we want this method convergent, we need for each k

$|x^{k+1}-\alpha|\le\Lambda|x^k-\alpha|$, $k=0,1\ldots$ where $0<\Lambda<1$

and ϕ is a strict construction, i.e $|\phi'(x)|\le\Lambda<1$

Hence, $|\phi'(x)|=|(1-\frac{w}{3})+3(1-w)x^2+(-\frac{4}{3}w\cdot x^{-3})|<1$

As we already know one root $x^3=2$, we can plug it into Eq.

$$\Longrightarrow |1-\frac{w}{3}+3(1-w)\times 2^{\frac{2}{3}}-\frac{4}{6}w|<1$$

$$|1+3\times2^{\frac{2}{3}}-(1+3\cdot2^{\frac{2}{3}})w|<1$$

$$|(1+3\times2^{\frac{2}{3}})(1-w)|<1$$

$$-1<(1+3\times2^{\frac{2}{3}})(1-w)|<1$$

$$1-\frac{1}{1+3\cdot2^{\frac{2}{3}}}<w<1+\frac{1}{1+3\times2^{\frac{2}{3}}}$$

(c) we need to find w such that fixed point iteration is second convergence.

By Thm of quadratic convergence, we need to show ϕ is twice continuously differentiable and $\phi'(\alpha)=0$.

$\phi''(x) = 6(1-w)x + 4w \cdot x^{-4}$, $\phi''(x)$ is quiet smooth in the neighborhood of x

If $\phi'(\alpha)=0$, then $(1+3\cdot 2^{\frac{2}{3}})(1-w)=0$

$w=1$

(d) If we need higher order convergence

$\phi(\alpha)=\alpha$ $\phi'(\alpha)=0$ and $\phi''(\alpha)=0$ where α is the fixed point

As we know that $\phi'(\alpha)=0$ when $w=1$

$$\phi''(x) = 6(1-w) + \frac{4w}{x^4}$$

when $w=1$ and $x^2=2^{\frac{2}{3}}$

$$\phi''(x) = 0 + \frac{4}{2^{4/3}} \neq 0$$

Thus, there is no w such that the method has higher convergence