Theoretical exercise sheet 2
Linear systems

Exercises 1, 4 and 5 (marked *) to be submitted via Moodle in pdf format (either handwritten and scanned, or typeset using LaTeX).
Some subset of these questions will be assessed.
Deadline: 10pm UK time Sunday 15th November.

EXERCISE 1(*) Let

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}.$$

(a) The Jacobi and Gauss-Seidel methods for the approximation of the solution to the linear system $A\mathbf{x} = \mathbf{b}$ can both be written in the form

$$P\mathbf{x}^{k+1} = N\mathbf{x}^k + \mathbf{b}.$$

Write down the matrices P and N for each of the two methods. What are the associated iteration matrices B_{J} and B_{GS}?

(b) Compute the vector $\mathbf{x}^1$ obtained after one iteration with the Jacobi method starting from the initial vector $\mathbf{x}^0 = \left(\frac{1}{2}, \frac{1}{2}\right)^T$.

(c) Do both methods converge? Which gives the iteration matrix with the smallest spectral radius?

(d) Prove that both methods converge linearly with respect to the norm $\|\cdot\|_\infty$, and compare the convergence constants.

EXERCISE 2 Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

(a) Write the Gauss-Seidel method for the solution of the linear system $A\mathbf{x} = \mathbf{b}$. What is the associated iteration matrix B_{GS}?

(b) What can be said about the convergence of the Gauss-Seidel method for this system?

(c) Now consider the iteration

$$D(\mathbf{x}^{k+1} - \mathbf{x}^k) = \alpha\mathbf{r}^k \qquad k \geq 0,$$

where D is the diagonal of A and $\mathbf{r}^k = \mathbf{b} - A\mathbf{x}^k$ is the residual. This method is sometimes called the Jacobi over-relaxation (JOR) method; note that for $\alpha = 1$ the method is equivalent to the Jacobi method.

Write down the associated iteration matrix B_{JOR}, and find the optimal choice of α minimising the spectral radius $\rho(B_{\mathrm{JOR}})$.

(d) Starting from the initial vector $\mathbf{x}^{(0)} = (1,1)^T$, compute the first iteration $\mathbf{x}^1$ of the JOR method using the optimal choice of α you derived in part (c).

EXERCISE 3 Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(a) Write down the Jacobi method for this system and establish if the method is convergent.

(b) Now consider the iterative method defined by

$$L\mathbf{x}^{k+1} = L\mathbf{x}^k + \delta \mathbf{r}^k, \qquad k \geq 0, \tag{1}$$

where $\delta > 0$ is a parameter, $\mathbf{r}^k = \mathbf{b} - A\mathbf{x}^k$ is the residual, and

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}.$$

Rewrite (1) in the form $\mathbf{x}^{k+1} = B_\delta \mathbf{x}^k + \mathbf{c}_\delta$, $k \geq 0$, giving explicit expressions for the matrix B_δ and the vector $\mathbf{c}_\delta$.

(c) For what values of the parameter $\delta > 0$ does the method (1) converge? Find the optimal value of δ which minimises the spectral radius $\rho(B_\delta)$.

(d) Take $\delta = \frac{4}{3}$. By using the results from part (c), establish if (1) converges. If yes, which method do you expect to have the faster convergence, (1) or the Jacobi method? Why?

EXERCISE 4(*) Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} \gamma & \kappa & 0 \\ \kappa & \gamma & \beta \\ 0 & \beta & \gamma \end{pmatrix},$$

and κ, β, $\gamma \in \mathbb{R}$ are three real parameters.
(Warning: Before answering this question, make sure you know the difference between "necessary" and "sufficient" conditions.)

(a) Without constructing the iteration matrices, give a *sufficient condition* on the coefficients κ, β and γ for the Jacobi and Gauss-Seidel methods to be convergent.

(b) Write down the iteration matrix B_J for the Jacobi method and give a *necessary and sufficient* condition on κ, β, γ for the method to be convergent.

(c) Now suppose we want to solve the system $A\mathbf{x} = \mathbf{b}$ using the JOR method (cf. Exercise 2)

$$D(\mathbf{x}^{k+1} - \mathbf{x}^k) = \alpha \mathbf{r}^k, \quad k \geq 0, \tag{2}$$

where D is the diagonal of A. This method corresponds to applying the stationary Richardson method (from lectures) to the modified system $D^{-1}A\mathbf{x} = D^{-1}\mathbf{b}$.

Determine for which γ, β, κ the matrix $D^{-1}A$ is symmetric positive definite. Assuming this holds, apply an appropriate theorem from lectures to determine for which α the method (2) converges, and compute the optimal value $\alpha = \alpha_*$ giving the fastest convergence, along with the associated error reduction factor $C > 0$ such that

$$\|\mathbf{x}^{k+1} - \mathbf{x}\|_2 \leq C\|\mathbf{x}^k - \mathbf{x}\|_2\,, \quad k \geq 0.$$

(d) Given that $\|\mathbf{x}^0 - \mathbf{x}\|_2 = 1$, estimate the smallest integer for which $\|\mathbf{x}^k - \mathbf{x}\|_2 \leq \left(\frac{1}{2}\right)^9$, in the case when $\gamma = 1$, $\beta = 2^{-4}$, $\kappa = \sqrt{3}\beta$.

EXERCISE 5(*) Let A be a n-by-n symmetric positive definite matrix, and define the function $\|\mathbf{x}\|_A : \mathbb{R}^n \to \mathbb{R}$ by $\|\mathbf{x}\|_A = \sqrt{\mathbf{x}^T A\mathbf{x}}$.

(a) Let $A^{1/2}$ denote the unique symmetric positive definite square root of A, which satisfies $(A^{1/2})^2 = A$. Show that $\|\mathbf{x}\|_A = \|A^{1/2}\mathbf{x}\|_2$, and use this fact to check that the function $\|\mathbf{x}\|_A$ defines a norm on $\mathbb{R}^n$. Determine constants $0 < c \leq C$ such that

$$c\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_A \leq C\|\mathbf{x}\|_2, \qquad \forall \mathbf{x} \in \mathbb{R}^n.$$

(b) Now consider the stationary Richardson method for the solution of the linear system $A\mathbf{x} = \mathbf{b}$, with iteration matrix $B_\alpha = I - \alpha A$. Show that $A^{1/2}B_\alpha = B_\alpha A^{1/2}$. Use this to prove that

$$\|\mathbf{e}^{k+1}\|_A \leq \rho(B_\alpha)\|\mathbf{e}^k\|_A,$$

where $\mathbf{e}^k = \mathbf{x} - \mathbf{x}^k$ denotes the solution error after the kth iteration.

EXERCISE 6 Consider the (unpreconditioned) gradient method for the solution of a linear system $A\mathbf{x} = \mathbf{b}$.

(a) Show that the acceleration parameter α_k in the gradient method is the unique solution to the minimisation problem

$$\alpha_k = \arg\min \Phi(\mathbf{x}^k + \alpha\mathbf{r}^k),$$

where $\Phi(\mathbf{y}) = \frac{1}{2}\mathbf{y}^T A\mathbf{y} - \mathbf{y}^T\mathbf{b}$ denotes the energy of the system $A\mathbf{x} = \mathbf{b}$.

(b) Show that the residuals in the gradient method satisfy $(\mathbf{r}^{k+1}, \mathbf{r}^k) = 0$ for each k.

(c) Give a geometric interpretation of the gradient method.