`inpaint`: Image in-painting via ℓ-2 norm total variation minimization

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To find a sub-gradient $G \in \partial \mathbf{tv}(U)$ for

$$\mathbf{tv}(U) = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \left\| \begin{bmatrix} U_{i+1,j} - U_{i,j} \\ U_{i,j+1} - U_{i,j} \end{bmatrix} \right\|_2,$$

we note that for (i, j) where all surrounding pixels $(i \pm 1, j \pm 1)$ exist, only the summands involving $(i \pm 1, j \pm 1)$ depend on (i, j). That is,

$$\begin{aligned}
G_{ij} &= \frac{\partial}{\partial U_{i,j}} \mathbf{tv}(U) \\
&= \frac{\partial}{\partial U_{i,j}} \left(\left\| \begin{bmatrix} U_{i+1,j} - U_{i,j} \\ U_{i,j+1} - U_{i,j} \end{bmatrix} \right\|_2 + \left\| \begin{bmatrix} U_{i,j} - U_{i-1,j} \\ U_{i-1,j+1} - U_{i-1,j} \end{bmatrix} \right\|_2 + \left\| \begin{bmatrix} U_{i+1,j-1} - U_{i,j-1} \\ U_{i,j} - U_{i,j-1} \end{bmatrix} \right\|_2 \right) \\
&= \frac{\partial}{\partial U_{i,j}} \Bigg(\sqrt{(U_{i+1,j} - U_{i,j})^2 + (U_{i,j+1} - U_{i,j})^2} + \sqrt{(U_{i,j} - U_{i-1,j})^2 + (U_{i-1,j+1} - U_{i-1,j})^2} + \\
&\quad \sqrt{(U_{i+1,j-1} - U_{i,j-1})^2 + (U_{i,j} - U_{i,j-1})^2} \Bigg) \\
&= (U_{i+1,j} + U_{i,j+1} - 2U_{i,j}) \Big/ \left\| \begin{bmatrix} U_{i+1,j} - U_{i,j} \\ U_{i,j+1} - U_{i,j} \end{bmatrix} \right\|_2 + (U_{i,j} - U_{i-1,j}) \Big/ \left\| \begin{bmatrix} U_{i,j} - U_{i-1,j} \\ U_{i-1,j+1} - U_{i-1,j} \end{bmatrix} \right\|_2 + \\
&\quad (U_{i,j} - U_{i,j-1}) \Big/ \left\| \begin{bmatrix} U_{i+1,j-1} - U_{i,j-1} \\ U_{i,j} - U_{i,j-1} \end{bmatrix} \right\|_2 .
\end{aligned}$$

The total variation is not differentiable with respect to $U_{i,j}$ if and only if any of the norm denominators above is zero, *i.e.*, $U_{i,j} = U_{i-1,j}$ and $U_{i-1,j+1} = U_{i-1,j}$, $U_{i+1,j-1} = U_{i,j-1}$ and $U_{i,j} = U_{i,j-1}$, or $U_{i+1,j} = U_{i,j}$ and $U_{i,j+1} = U_{i,j}$. In these cases, the total variation will be increased if $U_{i,j}$ is changed, so we can set $G_{ij} = 0$.