`inpaint`: Image in-painting via ℓ-2 norm total variation minimization

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1 Problem set-up

Say we've got an image $X \in \mathbf{R}^{m\times n}$ whose pixel values $X_{i,j}$ we know for indices $(i,j) \in \mathcal{K}$. To recover the values of the missing pixels whose indices are not in $\mathcal{K}$, we leverage the assumption that natural images have small *total variation*, which quantifies local deviations between neighboring pixels. We want a reconstruction Y that minimizes the ℓ_2 total variation, under the constraint that we preserve the known pixel values:

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{m-1}\sum_{i=1}^{n-1}\left\|\left[\begin{array}{c} Y_{i+1,j}-Y_{i,j} \\ Y_{i,j+1}-Y_{i,j}\end{array}\right]\right\|_2 \\ \text{subject to} & Y_{i,j}=X_{i,j},\ \forall\,(i,j)\in\mathcal{K} \end{array}$$

2 Subgradient

We'll use the projected subgradient method to solve the problem. To find a sub-gradient $G \in \partial \mathbf{tv}(Y)$ for

$$\mathbf{tv}(Y)=\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\left\|\begin{bmatrix} Y_{i+1,j}-Y_{i,j} \\ Y_{i,j+1}-Y_{i,j}\end{bmatrix}\right\|_2,$$

we note that for (i,j) where all surrounding pixels $(i \pm 1, j \pm 1)$ exist, only the summands involving $(i \pm 1, j \pm 1)$ depend on (i,j). That is,

$$
\begin{aligned}
G_{ij} &= \frac{\partial}{\partial Y_{i,j}} \mathbf{tv}(Y) \\
&= \frac{\partial}{\partial Y_{i,j}} \left(\left\| \begin{bmatrix} Y_{i+1,j} - Y_{i,j} \\ Y_{i,j+1} - Y_{i,j} \end{bmatrix} \right\|_2 + \left\| \begin{bmatrix} Y_{i,j} - Y_{i-1,j} \\ Y_{i-1,j+1} - Y_{i-1,j} \end{bmatrix} \right\|_2 + \left\| \begin{bmatrix} Y_{i+1,j-1} - Y_{i,j-1} \\ Y_{i,j} - Y_{i,j-1} \end{bmatrix} \right\|_2 \right) \\
&= \frac{\partial}{\partial Y_{i,j}} \left(\sqrt{(Y_{i+1,j} - Y_{i,j})^2 + (Y_{i,j+1} - Y_{i,j})^2} + \sqrt{(Y_{i,j} - Y_{i-1,j})^2 + (Y_{i-1,j+1} - Y_{i-1,j})^2} + \right. \\
&\left. \sqrt{(Y_{i+1,j-1} - Y_{i,j-1})^2 + (Y_{i,j} - Y_{i,j-1})^2} \right) \\
&= (Y_{i+1,j} + Y_{i,j+1} - 2Y_{i,j}) \Big/ \left\| \begin{bmatrix} Y_{i+1,j} - Y_{i,j} \\ Y_{i,j+1} - Y_{i,j} \end{bmatrix} \right\|_2 + (Y_{i,j} - Y_{i-1,j}) \Big/ \left\| \begin{bmatrix} Y_{i,j} - Y_{i-1,j} \\ Y_{i-1,j+1} - Y_{i-1,j} \end{bmatrix} \right\|_2 + \\
&(Y_{i,j} - Y_{i,j-1}) \Big/ \left\| \begin{bmatrix} Y_{i+1,j-1} - Y_{i,j-1} \\ Y_{i,j} - Y_{i,j-1} \end{bmatrix} \right\|_2 .
\end{aligned}
$$

The total variation is not differentiable with respect to $Y_{i,j}$ if and only if any of the norm denominators above is zero, *i.e.*, $Y_{i,j} = Y_{i-1,j}$ and $Y_{i-1,j+1} = Y_{i-1,j}$, $Y_{i+1,j-1} = Y_{i,j-1}$ and $Y_{i,j} = Y_{i,j-1}$, or $Y_{i+1,j} = Y_{i,j}$ and $Y_{i,j+1} = Y_{i,j}$. In these cases, the total variation will be increased if $Y_{i,j}$ is changed, so we can set $G_{ij} = 0$.

To project onto the feasible set, we just set all pixels in $\mathcal{K}$ to their known values.