Advanced numerical analysis

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Contents

Chapter 1

Inverse linear problems

The main objective of the course is dealing with the following objects:

$$\int_{\Omega} (input)(system) = output \tag{1.1}$$

In inverse linear problems we usuall

Chapter 2

Numerical methods for PDE

PDE stands for Partial Differential Equations, and we are going to study in particular Finite Difference Methods, not Finite Element Methods.

PDEs can be divided in three categories: elliptic, parabolic and high-parabolic. We will see a test problem on each of these categories.

2.1 Poisson 2D problem with Dirichlet boundary conditions

$$\text{Poisson equation:} \quad \Delta u(x,y) = f(x,y), \quad (x,y) \in \Omega \tag{2.1}$$

$$\text{Dirichlet Boundary condition:} \quad u(x,y) = g(x,y), \quad (x,y) \in \Gamma \tag{2.2}$$

where $\Omega = (0,1)^2$

Example: the electric potential inside a domain with a defined potential inside, f, and a fixed potential, g, at the border.

The Poisson 2D problem can be solved analytically using *separation of variables* by searching for solutions of this type:

$$u(x,y) = X(x)Y(y) \tag{2.3}$$

Discretization: fix N positive integer and put $h = \frac{1}{N}$ as the stepsize.

Now we discretize the plane $\mathbb{R}^2$ by the mesh:

$$\mathbb{R}^2_h = \{(mh, nh) : m, n \in \mathbb{Z}\} \tag{2.4}$$

For a mesh point (mh, nh), with $m, n \in \mathbb{Z}$ we can define four neighbors:

The discrete version of Ω is

$$\Omega_h := \Omega \cap \mathbb{R}^2_h \tag{2.5}$$

while the discrete version of Γ is the set of points Γ_h in $\mathbb{R}^2_h$ which are not in Ω_h, but have a neighbor in Ω_h.

Notice how $\Gamma_h \neq \Gamma \cap \mathbb{R}^2_h$ because the four corners $\{(0,0),(0,1),(1,0),(1,1)\}$ are not included.

We also define $\overline{\Omega_h} = \Omega_h \cup \Gamma_h$: this is the discrete version of $\overline{\Omega_h} = \Omega \cup \Gamma$.

The discrete version of the Poisson problem with Dirichlet Boundary conditions is:

$$\text{discrete Poisson equation: } \Delta_h u_h(x,y) = f(x,y), (x,y) \in \Omega_h \tag{2.6}$$

$$\text{discrete Dirichlet Boundary condition: } u_h(x,y) = g(x,y), (x,y) \in \Gamma_h \tag{2.7}$$

An operator is a map/function, whose domains and codomains are set of functions.

Δ_h is a discrete Laplacian, i.e. and operator such that:

- it associates (as a map) to a mesh function $\overline{\Omega_h} \to \mathbb{R}$ an interior mesh function $\Omega_h \to \mathbb{R}$;
- it approximates the Laplacian, in the sense that, for function $v : \overline{\Omega_h} \to \mathbb{R}$ sufficiently smooth, which means $C^2(\overline{\Omega_h})$:

$$\Delta_h v|_{\overline{\Omega_h}}(x,y) \approx \Delta v(x,y) \quad (x,y) \in \Omega_h \tag{2.8}$$

where $|_{\overline{\Omega_h}}$ is the restriction of v to $\overline{\Omega_h} \subseteq \overline{\Omega}$.

We are going to show how to construct a discrete Laplacian.

Consider a function v of only one variable t. We discretize $v''(t)$ by:

$$\frac{v(t-h) - 2v(t) + v(t+h)}{h^2} \tag{2.9}$$

.

$$\frac{\frac{v(t+h)-v(t)}{h} - \frac{v(t)-v(t-h)}{h}}{h} \approx \frac{v'(t) - v'(t-h)}{h} \approx v''(t-h) \approx v''(t) \tag{2.10}$$

By Taylor approximations under the assumptions that v is C^4:

$$v(t-h) = v(t) - v'(t)h + \frac{1}{2}v''(t)h^2 - \frac{1}{6}v'''(t)h^3 + \frac{1}{24}v^{(4)}(\alpha_h)h^4 \tag{2.11}$$

where $\alpha_h \in [t-h, t]$;

$$v(t+h) = v(t) + v'(t)h + \frac{1}{2}v''(t)h^2 + \frac{1}{6}v'''(t)h^3 + \frac{1}{24}v^{(4)}(\beta_h)h^4 \tag{2.12}$$

where $\beta_h \in [t, t+h]$.
By summing 2.11 and 2.12 we obtain:

$$v(t-h) + v(t+h) = 2v(t) + v''(t)h^2 + \frac{1}{24}(v^{(4)}(\alpha_h) + v^{(4)}(\beta_h))h^4 \tag{2.13}$$

$$v(t-h) - 2v(t) + v(t+h) = v''(t)h^2 + \frac{1}{24}(v^{(4)}(\alpha_h) + v^{(4)}(\beta_h))h^4 \tag{2.14}$$

$$\frac{v(t-h) - 2v(t) + v(t+h)}{h^2} = v''(t) + \frac{1}{24}(v^{(4)}(\alpha_h) + v^{(4)}(\beta_h))h^2 \tag{2.15}$$

So we got an approximation of the second derivative of the function v.

2.1.1 Discretization of the Laplacian

For a smooth function $v : \overline{\Omega_h} \to \mathbb{R}$ we approximate the Laplacian:

by:

$$\Delta_h v(x,y) = \frac{v(x-h,y) - 2v(x,y) + v(x+h,y)}{h^2} + \frac{v(x,y-h) - 2v(x,y) + v(x,y+h)}{h^2} \tag{2.16}$$

at $(x, y) \in \Omega_h$.

The computation of $\Delta_h v(x, y)$ requires $v(x-h, y)$ and $v(x, y+h)$ with $(x-h, y), (x, y+h) \in \Gamma_h$.

This discretization of the Laplacian is called the *Five-points discretization*:

2.1.2 Assess approximation of the discrete Laplacian

Consider $v : \overline{\Omega} \to \mathbb{R}$ of class C^4. For $(x, y) \in \Omega_h$:

$$|\Delta_h v|_{\overline{\Omega_h}}(x, y)\Delta v(x, y)| \tag{2.17}$$

$$|\frac{v(x-h,y) - 2v(x,y) + v(x+h,y)}{h} + \frac{v(x,y-h) - 2v(x,y) + v(x,y+h)}{h} - \frac{\partial^2 v}{\partial x^2}(x,y) - \frac{\partial^2 v}{\partial y^2}(x, \tag{2.18}$$

which is $\leq$ than:

$$|\frac{v(x-h,y) - 2v(x,y) + v(x+h,y)}{h^2} - \frac{\partial^2 v}{\partial x^2}(x,y)| + |\frac{v(x,y-h) - 2v(x,y) + v(x,y+h)}{h^2} - \frac{\partial^2 v}{\partial y^2}(\tag{2.19}$$

.

2.19 is equal to:

$$|\frac{1}{24}(\frac{\partial^4 v}{\partial x^4}(\alpha_h, y) + \frac{\partial^4 v}{\partial x^4}(\beta_h, y)\ h^2)| + |\frac{1}{24}(\frac{\partial^4 v}{\partial y^4}(x, \gamma_h) + \frac{\partial^4 v}{\partial y^4}(x, \delta_h)\ h^2)| \tag{2.20}$$

where $\alpha_h \in [x-h, x], \beta_h \in [x, x+h], \gamma_h \in [y-h, y], \delta_h \in [y, y+h]$.
2.20 (the error) can be bounded ($\leq$) by:

$$\frac{1}{6}\max\{\max_{(x,y)\in\overline{\Omega}} |\frac{\partial^4 v}{\partial x^4}(x,y)|, \max_{(x,y)\in\overline{\Omega}} |\frac{\partial^4 v}{\partial y^4}(x,y)|\}\ h^2 \tag{2.21}$$

So we can conclude that:

$$\max_{(x,y)\in\Omega_h} |\Delta_h v|_{\overline{\Omega_h}}(x, y)\Delta v(x, y)| \leq \frac{1}{6}\max\{\max_{(x,y)\in\overline{\Omega}} |\frac{\partial^4 v}{\partial x^4}(x,y)|, \max_{(x,y)\in\overline{\Omega}} |\frac{\partial^4 v}{\partial y^4}(x,y)|\}\ h^2 \tag{2.22}$$

This is called the *consistency error* of the five point approximation of the discrete Laplacian.

2.1.3 Exercise

Given $v \in C^6(\Omega)$, find a function $C(x, y)$ of the point $(x, y) \in \overline{\Omega}$ and $M \geq 0$ such that

$$\max_{(x,y)\in\Omega_h} |\Delta_h v|_{\overline{\Omega_h}}(x, y) - \Delta v(x, y) - C(x, y)h^2| \leq Mh^4$$

$$\Delta_h v|_{\overline{\Omega_h}}(x, y) = \Delta v'(x, y) + \text{error}$$

with error $= O(h^2)(*) = C(x, y)h^2 + O(h^4)(**)$.
Remind that:

$$v''(t) \approx \frac{v(t-h) - 2v(t) + v(t+h)}{h^2} \tag{2.23}$$

we can continue with the Taylor expansion:

$$v(t-h) = v(t) - v'(t)h + \frac{v''(t)}{2}h^2 - \frac{v'''(t)}{6}h^3 + \frac{v^4(t)}{24}h^4 - \frac{v^5(t)}{120}h^5 + \frac{v^6(t)}{6!}h^6$$

$$v(t-h) = v(t) + v'(t)h + \frac{v''(t)}{2}h^2 + \frac{v'''(t)}{6}h^3 + \frac{v^4(t)}{24}h^4 + \frac{v^5(t)}{120}h^5 + \frac{v^6(t)}{6!}h^6$$

where $\alpha \in [t-h, t]$ and $\beta \in [t, t+h]$.

$$v(t-h) + v(t+h) = 2v(t) + v''(t)h^2 + \frac{v^4(t)}{12}h^4 + \frac{v^6(\alpha)}{6!}h^6 + \frac{v^6(\beta)}{6!}h^6$$

$$\frac{v(t-h) - 2v(t) + v(t+h)}{h^2} - v''(t)h^2 = \frac{v^4(t)}{12}h^4 + \frac{v^6(\alpha)}{6!}h^6 + \frac{v^6(\beta)}{6!}h^6$$

...
The discrete problem:

$$\text{Poisson equation:} \quad \Delta u_h(x, y) = f(x, y), (x, y) \in \Omega_h$$

$$\text{Dirichlet Boundary condition:} \quad u_h(x, y) = g(x, y), (x, y) \in \Gamma_h$$

is a linear system of $(N-1)^2$ unknowns (where N is the number of subdivisions we do in each dimension of the square) and $(N-1)^2$ equations.

The unknowns are $u_{ij} := u_h(ih, jh)$ with $i \in \{1, \dots, N-1\}$ and $j \in \{1, \dots, N-1\}$.

.. we don't consider the values on the borders as unknowns because they are known, from the second equation of the Dirichlet Boundary condition.

On the border Γ_h:

$$u_{ij} = u_h(ih, jh) = h_{ij} := g(ih, jh)$$

where

$(i,j) \in \{0\}\times\{1,\ldots,N-1\}\cup\{1,\ldots,N-1\}\times\{0\}\cup\{N\}\times\{1,\ldots,N-1\}\cup\{1,\ldots,N-1\}\times\{N\}$

The equations are

$$\Delta_h u_h(x,y) = f(x,y), \quad \in \Omega_h$$

each equation corresponds to a point in Ω_h with $(N-1)^2$ equations.

The equation corresponding to the point (ih, jh), with $i \in \{1,\ldots,N-1\}, j \in \{1,\ldots,N-1\}$ is:

$$\frac{u_{i-1\,j} - 2u_{ij} + u_{i+1\,j}}{h^2} + \frac{u_{i\,j-1} - 2u_{ij} + u_{i\,j+1}}{h^2} = f(ih, jh) = f_{ij}$$

$$u_{i\,j-1} + u_{i-1\,j} - 4u_{ij} + u_{i+1\,j} + u_{i\,j+1} = h^2 f_{ij}$$

In order to apply a linear system of the form $Au = b$, we need to re-arrange the equations of the points of the square (which are 2D) into a 1D vector.

Consider $N = 4$:

...

We want to rewrite our matrix of unknowns u:

$$u = \begin{bmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{22} & u_{32} \\ u_{13} & u_{23} & u_{33} \end{bmatrix}$$

as a vector:

$$u = \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{13} \\ u_{23} \\ u_{33} \end{bmatrix}$$

Then we can rewrite our equation $u_{i\,j-1} + u_{i-1\,j} - 4u_{ij} + u_{i+1\,j} + u_{i\,j+1} = h^2 f_{ij}$ as (for the point $(1,1)$):

$$u_{10} + u_{01} - 4u_{11} + u_{21} + u_{12} = h^2 f_{11}$$

where u_{10} and u_{01} correspond to g_{10} and g_{01} respectively.

$$-4u_{11} + u_{21} + u_{12} = h^2 f_{11} - u_{10} - u_{01}$$

For the point $(2, 1)$:

$$u_{20} + u_{11} - 4u_{21} + u_{31} + u_{22} = h^2 f_{22}$$

rewritten as:

$$u_{11} - 4u_{21} + u_{31} + u_{22} = h^2 f_{22} - g_{20}$$

We can write a matrix $\mathcal{A}$:

$$\mathcal{A} = \begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix}$$

and the unknown vector u:

$$u = \begin{bmatrix} h^2 f_{11} - g_{01} - g_{10} \\ h^2 f_{21} - g_{20} \\ h^2 f_{31} - g_{14} \\ h^2 f_{12} - g_{02} \\ h^2 f_{22} \\ h^2 f_{32} - g_{24} \\ h^2 f_{13} - g_{03} - g_{14} \\ h^2 f_{23} - g_{24} \\ h^2 f_{33} - g_{34} - g_{43} \end{bmatrix}$$

$\mathcal{A}$ can be rewritten as a $3x3$ blocks matrix with blocks of size $3x3$:

$$\mathcal{A} = \begin{bmatrix} A & I_3 & 0 \\ I_3 & A & I_3 \\ 0 & I_3 & A \end{bmatrix}$$

For a general N, $\mathcal{A}$ is a $(N-1)\times(N-1)$ block matrix:

$$\mathcal{A} = \begin{bmatrix} A & I_{N-1} & & & \\ I_{N-1} & A & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & A & I_{N-1} \\ & & & I_{N-1} & A \end{bmatrix}$$

A generic matrix with dimension $n \times n$ is called “sparse” if the number of non-zero elements is $O(n)$.

The matrix $\mathcal{A}$ is sparse, block *tridiagonal* (it has elements on the diagonal, on the diagonal of the upper matrix and on the diagonal of the lower matrix), symmetric ($\mathcal{A}^T = \mathcal{A}$) and negative definite (all the eigenvalues are negative).

Consider a symmetric (real eigen-values) matrix B $n \times n$, B is negative definite when:

$$< x, Bx >= x^T Bx = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} x_i x_j$$

is $\leq 0 \ \ \forall x : n \times 1$ vectors (negative semi-definite) and
< 0 for $x \neq 0$.

2.1.4 $\mathcal{A}$ is negative definite

We are going to prove that $\mathcal{A}$ is negative definite, which then means that it is a non-singular matrix, which means that it can be inverted, which means that our system can be solved.

First we prove that:

$$< v, \mathcal{A}v >\leq -2 \, ||v||^2$$

for all $v \in \mathbb{R}^{N-1}$.

$$< v, \mathcal{A}v >= \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} a_{ij} v_i v_j =$$

$$-4v_1^2+v_1v_2+v_2v_1-4v_2^2+v_2v_3+v_3v_2-4v_3^2+\cdots-4v_{N-2}^2+v_{N-2}v_{N-1}+v_{N-1}v_{N-2}-4v_{N-1}^2 =$$

$$-4v_1^2 - 4v_2^2 - 4v_3^2 + \cdots - 4v_{N-2}^2 - 4v_{N-1}^2 + 2v_1v_2 + 2v_2v_3 + \cdots + 2v_{N-2}v_{N-1} =$$

$$-3v_1^2-2v_2^2-2v_3^2+\cdots-2v_{N-2}^2-3v_{N-1}^2-(v_1^2-2v_1v_2+v_2^2)-(v_2^2-2v_2v_3+v_3^2)+\cdots+(v_{N-2}^2-2v_{N-2}v_{N-}$$

$$-3v_1^2-2v_2^2-2v_3^2+\cdots-2v_{N-2}^2-3v_{N-1}^2-(v_1-v_2)^2-(v_2-v_3)^2+\cdots-(v_{N-2}-v_{N-1})^2 \leq$$

$$-2v_1^2 - 2v_2^2 - 2v_3^2 + \cdots - 2v_{N-2}^2 - 2v_{N-1}^2 = -2\underbrace{\left(v_1^2 + v_2^2 + v_3^2 + \cdots + v_{N-2}^2 + v_{N-1}^2\right)}$$

Consider $v = (v_1, \ldots, v_{N-1}) \in \mathbb{R}^{(N-1)^2}$, where $v_1, \ldots, v_{N-1} \in \mathbb{R}^{N-1}$.

$$< v, \mathcal{A}v >= v^T \mathcal{A}v = [v_1^T, v_2^T, \ldots, v_{N-1}^T]$$

$$< x, Bx >= \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} x_i x_j = \sum_{i=1}^{m} \sum_{j=1}^{n} < v_i, \mathcal{A}_i j v_j >=$$

$$< v_1, Av_1 > + < v_1, Iv_2 > + < v_2, Iv_1 > + < v_2, Av_2 > + < v_2, Iv_3 > + < v_3, Iv_2 > + < v_3, Av_3 > +$$

$$+ < v_{N-2}, Av_{N-2} > + < v_{N-2}, v_{N-1} > + < v_{N-1}.v_{N-2} > + < v_{N-1}.Av_{N-1} >=$$

$$\overbrace{< v_1, Av_1 >}^{\leq -2||v_1||_2^2} + \overbrace{< v_1, Iv_2 >}^{\leq -2||v_2||_2^2} + \overbrace{< v_2, Iv_1 >}^{\leq -2||v_3||_2^2} + \cdots + \overbrace{< v_{N-2}, Av_{N-2} >}^{\leq -2||v_{N-2}||_2^2} + \overbrace{< v_{N-1}, Av_{N-1} >}^{\leq -2||v_{N-1}||_2^2} + 2 \ldots \cdots \leq$$

$$-2||v_1||_2^2 - 2||v_2||_2^2 + \cdots - 2||v_{N-1}||_2^2 \ldots \cdots =$$

$$= -||v_1||_2^2 - (||v_1||_2^2 - 2 < v_1, v_2 > + ||v_2||_2^2) - (||v_2||_2^2 - 2 < v_2, v_3 > + ||v_3||_2^2) + \cdots - (||v_{N-2}||_2^2 - 2 < v_{N-2}, v_N$$

$$= -||v_1||_2^2 - ||v_1 - v_2||_2^2 - ||v_2 - v_3||_2^2 + \cdots - ||v_{N-2} - v_{N-1}||_2^2 - ||v_{N-1}||_2^2 \leq 0$$

Now we show that $< v, \mathcal{A}v >< 0$ for $v \neq 0$.

Equivalently, we show that $< v, \mathcal{A}v >= 0 \implies v = 0$. When $< v, \mathcal{A}v >= 0$, also

$$-||v_1||_2^2 - ||v_1 - v_2||_2^2 - ||v_2 - v_3||_2^2 + \cdots - ||v_{N-2} - v_{N-1}||_2^2 = 0$$

$\ldots$

$$||v_1||_2 = ||v_1 - v_2||_2 = \cdots = ||v_{N-2} - v_{N-1}||_2 = ||v_{N-1}||_2 = 0$$

and so:

$$v_1 = 0, v_1 = v_2, \ldots, v_{N-1} = 0$$

and so:

$$v_1 = v_2 = \cdots = v_{N-1} = 0$$

and so:

$$v = 0$$

Exercise

Write the 1D Poisson problem with Dirichlet Boundary condition on $\Omega = (0,1)$.

By definition, the Laplacian is: $\frac{\partial^2}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y)$.

The 1D version of the Poisson problem is: $u''(x) = f(x), \quad x \in \Omega = (0,1)$ $u(x) = g(x), \quad x \in \Gamma : u(0) = g(0), u(1) = g(1)$.

Solve analytically this problem:

$u'(x) = u'(0) + \int_0^x \underbrace{u''(s)}_{f(s)} ds, \quad x \in [0,1]$.

$$u(x) = u(0) + \int_0^x u'(s)ds = u(0) + \int_0^x (u'(0) + (\int_0^s f(\delta)d\delta)\ ds =$$

$$u(0) + u'(0)\int_0^x ds + \int_0^x \int_0^s f(\delta)d\delta ds =$$

$$u(x) = u(0) + u'(0)x + \int_0^x \int_0^s f(\delta)d\delta ds \quad x \in [0,1]$$

$u(0) = g(0)$ $u(1) = u(0) + u'(0) + \int_0^1 \int_0^s f(\delta)d\delta ds \quad x \in [0,1]$

......

Propose a corresponding discrete problem and write the associated linear system:

$\Omega = (0,1)$, $h = 1/N$ with N positive integer. $\Omega_h = h, 2h, \ldots, (N-1)h$

The discrete problem is:

$$\Delta_h u_h(x) = f(x), \quad x \in \Omega_h u_h(0) = g(0) u_h(1) = g(1)$$

Δ_h associates to a function $\overline{\Omega_h} \to \mathbb{R}$ a function $\Omega_h \to \mathbb{R}$ that approximates $\Delta u = u''$.

For a function $v_h : \overline{\Omega_h} \to \mathbb{R}$, we define:

$$\Delta_h v_h(x) = \frac{v_h(x-h) - 2v_h(x) + v_h(x+h)}{h^2}, \quad x \in \Omega_h$$

We can rewrite the discrete problem as:

$$u_i = u_h(ih), \quad i \in 1, \ldots, N-1$$

$$f_i = f(ih)$$

$$g_0 = g(0),\ g_N = g(1)$$

$$\frac{u_{i-1} - 2u_i + u_i + 1}{h^2} = f_i, \quad i \in 1, \dots, N-1$$

with $u_0 = g_0, \quad u_N = g_N,$

$$u_{i-1} - 2u_i + u_i + 1 = h^2 f_i, \quad i \in 1, \dots, N-1$$

with $u_0 = g_0, \quad u_N = g_N.$
The linear system is:
We can write a matrix:

$$\begin{bmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & \ddots & \ddots & & & \\ & & \ddots & & & & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{bmatrix}$$

and the unknown vector is:

$$u = \begin{bmatrix} h^2 f_1 - g_0 \\ h^2 f_2 \\ h^2 f_3 \\ \dots \\ h^2 f_{N-2} \\ h^2 f_{N-1} - g_N \end{bmatrix}$$

For $v \in \mathbb{R}^{N-1}$: $< v, \mathcal{A}v = -2v_1^2 + v_1 v_2 + v_2 v_1 - 2v_2^2 + \dots$
. $+ v_{N-2}v_{N-1} + v_{N-1}v_{N-2} + 2v_N^2 =$

$$-v_1^2 - (v_1 - v_2)^2 - (v_2 - v_3)^2 + \dots - (v_{N-2} - v_{N-1})^2 - v_{N-1}^2 \leq 0$$

If $< v, \mathcal{A}v >= -v_1^2 - (v_1 - v_2)^2 + \dots - (v_{N-2} - v_{N-1})^2 - v_{N-1}^2 = 0$ then $v_1 = 0, v_1 - v_2 = 0, \dots, v_{N-2} - v_{N-1} = 0, v_{N-1} = 0$ and so $v = 0$.

Exercise

Consider the 3D Poisson problem with Dirichlet boundary conditions on $\Omega = (0,1)^3$. Propose a discrete Laplacian and a consequent discrete problem.

$$\Delta u(x, y, z) = f(x, y, z), \quad (x, y, z) \in \Omega = (0,1)^3$$

$$u(x,y,z) = g(x,y,z), \quad (x,y,z) \in \Gamma$$

$$\Delta u(x,y,z) = \frac{\partial^2 u}{\partial x^2}(x,y,z) + \frac{\partial^2 u}{\partial y^2}(x,y,z) + \frac{\partial^2 u}{\partial z^2}(x,y,z)$$

$$\Omega_h = \Omega \cap \mathbb{R}_h^3$$

$$\mathbb{R}_h^3 = (mh, uh, ph) : m, n, p \in \mathbb{Z}$$

Γ_h in 2D was the border without the corners. In 3D is the set of faces, without the edges and the corners. Γ_h = "points of $\mathbb{R}_h^3 \cap \Gamma$ with a neighbor in Ω_h"

The discrete problem then is

$$\Delta_h u(x,y,z) = f(x,y,z), \quad (x,y,z) \in \Omega_h$$

$$u_h(x,y,z) = g(x,y,z), \quad (x,y,z) \in \Gamma_h$$

The discrete Laplacian is given by a mesh function $v : \overline{\Omega_h} = \Omega_h \cup \Gamma_h \to \mathbb{R}$

$$\Delta_h v_h(x,y,z) = \frac{v(x-h,y,z) - 2v(x,y,z) + v(x+h,y,z)}{h^2} + \frac{v(x,y-h,z) - 2v(x,y,z) + v(x,}{h^2}$$

Describe in the matrix of the linear system associated to the discrete problem the row corresponding to a mesh point in Ω_h with nearest neighbors in Ω_h.

In order to write our problem in this way: $\mathcal{A}U = b$ we need to order the unknowns. There are $(N-1)^3$ unknowns, where $h = \frac{1}{N}$; therefore $\mathcal{A}$ is a $(N-1)^3 \times (N-1)^3$ matrix.

What is the equation of a point whose neighbors are part of Ω_h?

We construct the matrix from the equation $\Delta_h u_h(x,y,z) = f(x,y,z)$; which becomes $h^2 \Delta_h u_h(x,y,z) = h^2 f(x,y,z)$

$$-6 \overbrace{u(x,y,z)}^{=u_{ijk}=U_l} + \overbrace{u_h(x-h,y,z)}^{=u_{l-1}} + \overbrace{u_h(x+h,y,z)}^{=u_{l+1}} +$$
$$. \qquad + \overbrace{u_h(x,y-h,z)}^{=u_{l-(N-1)}} + \overbrace{u_h(x,y+h,z)}^{=u_{l+(N-1)}} +$$
$$. \qquad + \overbrace{u_h(x,y,z-h)}^{=u_{l-(N-1)^2}} + \overbrace{u_h(x,y,z+h)}^{=u_{l+(N-1)^2}} = h^2 f(x,y,z)$$

...

2.2 Convergence analysis of the discrete Poisson problem (with D. b.c)

We assume that the continuous problem has a unique solution u.

We want to prove that the discrete problem has a unique solution u_h (already done, but we obtain this by using another technique) and we want to give a bound on the error:

$$\max_{(x,y)\in\overline{\Omega}_h} |u_h(x,y) - u(x,y)|$$

This bound will prove the convergence of μ_h to u as $h \to 0$.

2.2.1 Theorem: Discrete maximum principle

Let $v_h : \overline{\Omega}_h \to \mathbb{R}$ such that:

$$\Delta_h v_h \geq 0 \qquad \forall (x,y) \in \Omega_h$$

Then:

$$\max_{(x,y)\in\Omega_h} v_h(x,y) \leq \max_{(x,y)\in\Gamma_h} v_h(x,y)$$

with equality iff v_h is constant.

Continuous maximum principle

Let $v : \overline{\Omega} \to \mathbb{R}$ of class C^2 such that:

$$\Delta v(x,y) \geq 0 \qquad \forall (x,y) \in \Omega$$

Then:

$$\sup_{(x,y)\in\Omega} v(x,y) \leq \max_{(x,y)\in\Gamma} v(x,y)$$

Proof of the Discrete maximum principle theorem

We have to prove that:

$$\max_{(x,y)\in\Omega_h} v_h(x,y) \leq \max_{(x,y)\in\Gamma_h} v_h(x,y)$$

with $=$ iff v_h is constant.

We begin by observing if v_h is constant, then:

$$\max_{(x,y)\in\Omega_h} v_h(x,y) = \max_{(x,y)\in\Gamma_h} v_h(x,y)$$

Given two logical sentences A and B, $A \implies B$ is equivalent to $\neg B \implies \neg A$

When v_h is not constant (A), then (B):

$$\max_{(x,y)\in\Omega_h} v_h(x,y) < \max_{(x,y)\in\Gamma_h} v_h(x,y)$$

Therefore we prove the counter-opposite ($\neg B$) of the previous equation.

$$\max_{(x,y)\in\Omega_h} v_h(x,y) > \max_{(x,y)\in\Gamma_h} v_h(x,y)$$

which means that v_h is not constant ($\neg A$).

Suppose:

$$\max_{(x,y)\in\Omega_h} v_h(x,y) \geq \max_{(x,y)\in\Gamma_h} v_h(x,y)$$

So:

$$\max_{(x,y)\in\Omega_h} v_h(x,y) = \max_{(x,y)\in\Gamma_h} v_h(x,y)$$

Consider now a point $(x_0, y_0) \in \Omega_h$ where v_h is the maximum.

We prove that all $(x_0 - h, y_0), (x_0 + h, y_0), (x_0, y_0 - h), (x_0, y_0 + h)$ are maximum points for v_h.

We have:

$$h^2 \Delta_h v_h(x_0, y_0) \geq 0$$

$$= v_h(x_0, y_0 - h) + v_h(x_0 - h, y_0) - 4v_h(x_0, y_0) + v_h(x_0 + h, y_0) + v_h(x_0, y_0 + h)$$

and then:

$$4v_h(x_0, y_0) \leq \overbrace{v_h(x_0, y_0 - h)}^{v_h(x_0,y_0)} + \overbrace{v_h(x_0 - h, y_0)}^{v_h(x_0,y_0)} + \overbrace{v_h(x_0 + h, y_0)}^{v_h(x_0,y_0)} + \overbrace{v_h(x_0, y_0 + h)}^{v_h(x_0,y_0)}$$

which means that the second member must be $\leq 4v_h(x_0, y_0)$.

Thus:

$$4v_h(x_0, y_0) = v_h(x_0, y_0 - h) + v_h(x_0 - h, y_0) + v_h(x_0 + h, y_0) + v_h(x_0, y_0 + h)$$

and it follows that:

$$v_h(x_0, y_0 - h) = v_h(x_0 - h, y_0) = v_h(x_0 + h, y_0) = v_h(x_0, y_0 + h) = v_h(x_0, y_0)$$

We have proved that $(x_0 - h, y_0), (x_0 + h, y_0), (x_0, y_0 - h), (x_0, y_0 + h)$ are maximum points for v_h; this argument can be repeated recursively for all the neighbors, which means that they all have value v_{max}; we obtain that all the mesh points obtain the maximum value v_{max}, and v_h is constant. ■

2.2.2 Corollary of the Discrete maximum principle

Let $v_h : \overline{\Omega}_h \to \mathbb{R}$ such that:

$$\Delta_h v_h(x,y) \leq 0 \qquad \forall (x,y) \in \Omega_h$$

$$\min_{(x,y)\in\Omega_h} v_h(x,y) \geq \min_{(x,y)\in\Gamma_h} v_h(x,y)$$

Proof

Consider $w_h = -v_h$. We have:

$$\Delta_h w_h(x,y) = -\Delta_h v_h(x,y) \geq 0 \forall (x,y) \in \Omega_h$$

From the discrete maximum principle, we have:

$$\max_{(x,y)\in\Omega_h} \overbrace{w_h(x,y)}^{=-v_h(x,y)} \leq \max_{(x,y)\in\Gamma_h} \overbrace{w_h(x,y)}^{=-v_h(x,y)}$$

■

2.2.3 Theorem:

The discrete Poisson problem (with D. b.c.) has a unique solution.

Proof:

The discrete problem is written as a linear system: $\mathcal{A}U = b$.

Existence and uniqueness of the solution of the discrete problem follows by the non singularity of the matrix $\mathcal{A}$.

$\mathcal{A}$ is non singular if:

$$\mathcal{A}U = 0 \implies U = 0$$

We are now going to prove this implication.

$\mathcal{A}U = 0$ is obtained for the discrete problem:

$$\Delta_h u_h(x,y) = 0, \quad (x,y) \in \Omega_h$$

$$u_h(x,y) = 0, \quad (x,y) \in \Gamma_h$$

Now we show that $u_h = 0$ (i.e. $U = 0$).

We have:

$$\Delta_h u_h(x,y) \geq 0, \quad (x,y) \in \Omega_h$$

and from the discrete max principle:

$$\max_{(x,y)\in\Omega_h} u_h(x,y) \leq \max_{(x,y)\in\Gamma_h} u_h(x,y) = 0$$

Moreover, we have:

$$\Delta_h u_h(x,y) \leq 0, \quad (x,y) \in \Omega_h$$

and then from the discrete min principle:

$$\min_{(x,y)\in\Omega_h} u_h(x,y) \geq \min_{(x,y)\in\Gamma_h} u_h(x,y) = 0$$

For any $(x,y) \in \Omega_h$ we have: $0 \leq u_h(x,y) \leq 0$, and so: $u_h(x,y) = 0$. ■

2.3 ∞-norm and L^∞ norm

Given a function $f : \mathcal{A} \to \mathbb{R}$, we set:

$$||f||_{L^\infty(\mathcal{A})} := \sup_{x\in\mathcal{A}} |f(x)|$$

When A is finite or $\mathcal{A} sub \mathbb{R}^d$ closed and bounded (i.e. compact) then:

$$||f||_{L^\infty(\mathcal{A})} := \max_{x\in\mathcal{A}} |f(x)|$$

$L^\infty(\mathcal{A})$ is the set of functions $f : \mathcal{A} \to \mathbb{R}$ such that $||f||_{L^\infty(\mathcal{A})} < +\infty$.
We have, for $v : \overline{\Omega} \to \mathbb{R}$ of class C^4:

$$\max_{(x,y)\in\Omega_h} |\Delta_h v|_{\overline{\Omega_h}}(x,y) - \Delta v(x,y)| \leq \frac{h^2}{6} \max\{ \max_{(x,y)\in\overline{\Omega}} |\frac{\partial^4 v}{\partial x^4}(x,y)|, \max_{(x,y)\in\overline{\Omega}} |\frac{\partial^4 v}{\partial y^4}(x,y)|\}$$

$$||\Delta_h v|_{\overline{\Omega_h}}(x,y) - \Delta v(x,y)||_{L^\infty(\Omega_h)} \leq \frac{h^2}{6} \max\{||\frac{\partial^4 v}{\partial x^4}||_{L^\infty(\overline{\Omega_h})}, ||\frac{\partial^4 v}{\partial y^4}||_{L^\infty(\overline{\Omega_h})}\}$$

The stability result says that the map: $(f,g) \rightarrowtail u_h$ (underbraces: data for the discrete problem ... solution of the discrete problem) is uniformly bounded with respect to h, i.e.:

$$||u_h||_{L^\infty(\overline{\Omega_h})} \leq C \cdot \max\{||f||_{L^\infty(\Omega)}, ||g||_{L^\infty(\Gamma)}\}$$

with a constant C independent of f, g and h.

2.3.1 Theorem (stability problem)

Let $f \in L^\infty(\Omega)$ and let $g \in L^\infty(\Gamma)$ and let u_h be the solution of the discrete problem:

$$\Delta_h u_h = f(x,y), \qquad (x,y) \in \Omega_f$$

$$u(x,y) = g(x,y) \qquad (x,y) \in \Gamma_h$$

Then we have:

$$||u_h||_{L^\infty(\overline{\Omega})} \leq \frac{1}{8}\overbrace{||f||_{L^\infty(\Omega)}}^{\leq max} + \overbrace{||g|_{L^\infty(\Gamma)}}^{\leq max}$$

Proof:

We introduce the mesh function:

$$\Phi(x,y) = \frac{(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2}{4} = \frac{d^2}{4}, \quad (x,y) \in \overline{\Omega_h}$$

We have:

$$\Delta_h \Phi(x,y) = 1, \quad (x,y) \in \Omega_h$$

$$\Delta_h \Phi(x,y) = \frac{\Phi(x,y-h) + \Phi(x-h,y) - 4\Phi(x,y) + \Phi(x+h,y) + \Phi(x,y+h)}{h^2}$$

$$\Phi(x,y-h) = \frac{(x-\frac{1}{2})^2 + (y-h-\frac{1}{2})^2}{4} = \frac{(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 - 2(y-\frac{1}{2})h + h^2}{4}$$

$$\Phi(x-h,y) = \frac{(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 - 2(x-\frac{1}{2})h + h^2}{4}$$

$$-4\Phi(x,y) = \frac{-4(x-\frac{1}{2})^2 - 4(y-\frac{1}{2})^2}{4}$$

$$\Phi(x,y+h) = \frac{(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 + 2(x-\frac{1}{2})h + h^2}{4}$$

$$\Phi(x+h,y) = \frac{(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 + 2(y-\frac{1}{2})h + h^2}{4}$$

which means

$$\Delta_h \Phi(x,y) = \frac{\frac{4h^2}{4}}{h^2} = 1$$

We have:

$$0 \leq \Phi(x,y) \leq \frac{1}{8}, \quad (x,y) \in \overline{\Omega_h}$$

$$\Phi(x,y) = \frac{d^2}{4} \leq \frac{l^2}{4} = \frac{(\frac{\sqrt{2}}{2})^2}{4} = \frac{(\frac{1}{\sqrt{2}})^2}{4}$$

where l is the diagonal from the center of the square to a corner:
We prove that Δ_h is a linear operator, i.e. for $v_h, w_h : \overline{\Omega_h} \to \mathbb{R}$ we have:

$$\Delta_h(v_h + w_h) = \Delta_h v_h + \Delta_h w_h$$

for $v_h : \overline{\Omega_h} \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ we have:

$$\Delta_h \ldots$$

For $(x,y) \in \Omega_h$:

$$\Delta_h(v_h+w_h)(x,y) = (v_h(x,y-h)+w_h(x,y-h)+v_h(x-h,y)+w_h(x-h,y)-4v_h(x,y)-4w_h(x,y)+$$

$$= (v_h(x,y-h)+v_h(x-h,y)-4v_h(x,y)+v_h(x,y+h)+v_h(x+h,y))\cdot\frac{1}{h^2}+(w_h(x,y-h)+w_h(x-h,y)$$

$$= \Delta_h v_h(x,y) + \Delta_h w_h(x,y)$$

$$\Delta_h(\alpha v_h)(x,y) = (\alpha v_h(x,y-h)+\alpha v_h(x-h,y)-\alpha 4 v_h(x,y)+\alpha v_h(x,y+h)+\alpha v_h(x+h,y))\cdot\frac{1}{h^2} = \alpha$$

Set:

$$A := ||f||_{L^\infty(\Omega)}, B := ||g||_{L^\infty(\Gamma)}$$

For $(x,y) \in \Omega_h$:

$$\Delta_h(u_h+A\Phi)(x,y) = \Delta_h u_h(x,y)+A\underbrace{\Delta_h\Phi(x,y)}_{=1} = \Delta_h u_h(x,y)+A = f(x,y)+A \geq -|f(x,y)|+$$

$$\Delta_h(u_h - A\Phi)(x,y) = \Delta_h u_h(x,y) - A\underbrace{\Delta_h\Phi(x,y)}_{=1} = \Delta_h u_h(x,y) - A = f(x,y) - A \le |f(x,y)| - \underbrace{A}_{=\sup_{(x,y)\in\Omega}|f(}$$

The discrete maximum principle says that:

$$\max_{(x,y)\in\Omega_h}(u_h + A\Phi)(x,y) \le \max_{(x,y)\in\Gamma_h}(u_h + A\Phi)(x,y)$$

$$\min_{(x,y)\in\Omega_h}(u_h - A\Phi)(x,y) \ge \max_{(x,y)\in\Gamma_h}(u_h - A\Phi)(x,y)$$

$$\max_{(x,y)\in\Omega_h} u_h(x,y) \le \max_{(x,y)\in\Omega_h}(\underbrace{u_h + \overbrace{A\Phi(x,y)}^{\ge 0}}_{=(u_h+A\Phi)(x,y)}) \le$$

$$\le \max_{(x,y)\in\Gamma_h}(\underbrace{u_h + A\Phi(x,y)}_{=(u_h+A\Phi)(x,y)}) = \max_{(x,y)\in\Gamma_h}(g(x,y) + A\underbrace{\Phi(x,y)}_{\le\frac{1}{8}}) \le \max_{(x,y)\in\Gamma_h}(g(x,y) - \frac{1}{8}A) = \max_{(x,y)\in\Gamma_h}\underbrace{g(x,y)}_{\ge -|g(x,y)|} + \frac{1}{8}A$$

The opposite with the min.
We have proved, for $(x,y) \in \overline{\Omega_h}$:

$$-\frac{1}{8}A - B \le u_h(x,y) \le \frac{1}{8}A + B$$

This is exactly the same of:

$$|u_h(x,y)| \le \frac{1}{8}A + B$$

In other words:

$$||u_h||_{L^\infty()} \dots$$

$$||\Delta_h v|_{\Omega_h} - (\Delta_h v)|_{\Omega_h}||_{L^\infty(\Omega_h)}$$

$$||u_h - u|_{\overline{\Omega_h}}||$$

2.3.2 Theorem (the convergence theorem)

Let u be the solution of:

$$\Delta u(x,y) = f(x,y) \quad (x,y) \in \Omega$$

$$u(x,y) = g(x,y) \quad (x,y) \in \Gamma$$

and let u_h be the solution of:

$$\Delta u_h(x,y) = f(x,y) \quad (x,y) \in \Omega_h$$

$$u_h(x,y) = g(x,y) \quad (x,y) \in \Gamma_h$$

We have:

$$||u_h - u|_{\Omega_h}||_{L^\infty(\overline{\Omega_h})} \leq \frac{1}{8}||\Delta_h u|_{\Omega_h} - (\Delta u)|_{\Omega_h}||_{L^\infty(\Omega_h)}$$

Proof:

We set:

$$l_h := u_h - u|_{\overline{\Omega_h}}$$

We have:

$$\Delta_h l_h = \Delta_h(u_h - u|_{\overline{\Omega_h}}) = \Delta_h u_h - \Delta_h u|_{\overline{\Omega_h}} = \Delta_h u_h - (\Delta u)|_{\Omega_h} + (\Delta u)|_{\Omega_h} - \Delta_h u|_{\Omega_h}$$

We have, for $(x,y) \in \Omega_h$

$$\Delta_h u_h(x,y) - \Delta u(x,y) = g(x,y) - f(x,y) = 0$$

We have:

$$\Delta_h l_h = -\epsilon_h$$

where

$$\epsilon_h = \Delta_h u|_{\Omega_h} - (\Delta u)|_{\Omega_h}$$

For $(x,y) \in \Gamma_h$:

$$l_h(x,y) = u_h(x,y) - u(x,y) = \ldots$$

By concluding, we can say that the important error l_h satisfies the discrete problem:

$$\Delta_h \epsilon_h(x,y) = -\epsilon_h(x,y) \qquad (x,y) \in \Omega_h$$

$$l_h(x,y) = 0 \qquad (x,y) \in \Gamma_h$$

By means of the previous theorem:

$$||l_h||_{L^\infty(\overline{\Omega_h})} \leq \frac{1}{8}\underbrace{||-\epsilon_h||_{L^\infty(\Omega_h)}}_{=||\epsilon_h||_{L^\infty(\Omega_h)}} + \underbrace{||o||_{L^\infty(\Gamma_h)}}_{=0} \leq \frac{1}{8}||\epsilon_h||_{L^\infty(\Omega_h)} + \dots$$

■

We know that:

$$||\Delta_h u|_{\Omega_h} - (\Delta u)|_{\Omega_h}||_{L^\infty(\Omega_h)} \leq \frac{h^2}{6}\max\{||\frac{\partial^4 u}{\partial x^4}||_{L^\infty(\overline{\Omega_h})}, ||\frac{\partial^4 v}{\partial y^4}||_{L^\infty(\overline{\Omega_h})}\}$$

and then:

$$||u_h - u|_{\overline{\Omega_h}}||_{L^\infty(\overline{\Omega_h})} \leq \frac{h^2}{48}\max\{||\frac{\partial^4 u}{\partial x^4}||_{L^\infty(\overline{\Omega_h})}, ||\frac{\partial^4 v}{\partial y^4}||_{L^\infty(\overline{\Omega_h})}\}$$

when $u \in C^4$.

2.3.3 Some final remarks

Consider:

- u is the solution of the continuous problem;
- u_h is the solution of the discrete problem;
- Δ_h is a discrete Laplacian, not necessarily the five-point discretization.

We define:

$$l_h := u_h - u|_{\overline{\Omega_h}}$$

l_h is called the *convergence error*:

$$\epsilon_h := \Delta_h u|_{\Omega_h} - (\Delta u)|_{\Omega_h}$$

ϵ_h is called the *consistency error.*
We are interested on l_h, since ϵ_h is a "surrogate" of l_h.

We say that, given p a positive integer:

- the discretization is *consistent of order p* if:

$$||\epsilon_h||_{L^\infty(\Omega_h)} = O(h^p) \qquad N \to \infty$$

- the discretization is *convergent of order* p if

$$||l_h||_{L^\infty(\overline{\Omega_h})} = O(h^p) \qquad N \to \infty$$

- the discretization is *stable* if the solution v_h of the discrete problem:

$$\Delta_h v_h(x,y) = l(x,y), \qquad (x,y) \in \Omega_h$$

$$v_h(x,y) = 0$$

we have:

$$||v_h||_{L^\infty(\overline{\Omega_h})} \leq C||l||_{L^\infty(\Omega)}$$

where C is independent of l and h.

We have: consistency of order p and stability $\implies$ convergence of order p.

We have:

$$\Delta_h l_h = \epsilon_h in \Omega_h$$

$$l_h = 0 in \Gamma_h$$

If $||\epsilon_h||_{L^\infty(\Omega_h)} = O(h^p)$ (consistency) and $||l_h||_{L^\infty(\overline{\Omega_h})} \leq C||\epsilon_h||_{L^\infty(\Omega_h)}$ then $||L_h||_{L^\infty(\overline{\Omega_h})} = O(h^p)$ (convergence).

2.4 Fourier Analysis

$L(\Omega_h)$ is the set of functions $\Omega_h \to \mathbb{R}$; $L(\Omega_h)$ is isomorphic, as $\mathbb{R}^{(N-1)^2}$.

The function in $L(\Omega_h)$ are extended to $\overline{\Omega_h}$ by setting value zero there.

So, we can now consider the Laplacian Δ_h as an operator $L(\Omega_h) \to L(\Omega_h)$.

Is this operator invertible? Or, for $f_h \in L(\Omega_h)$, there exists a unique function v_h such that:

$$\Delta_h v_h = f_h$$

$$\Delta_h v_h(x, y) = f_h(x, y) \quad (x, y) in \Omega_h \tag{2.24}$$

$$v_h(x, y) = 0 \quad (x, y) \in \Gamma_h \tag{2.25}$$

This means Δ_h invertible.

We know that the previous problem has a unique solution.

So Δ_h is invertible and Δ_h^{-1} exists. Is Δ_h^{-1} uniformly bounded with respect to h?

Consider $L(\Omega_h)$ with the $L^\infty(\Omega_h)$.

$$||\Delta_h^{-1}|| = \sup_{f_h \in L^\infty(\Omega_h), f_h \neq 0} \frac{||v_h||_{L^\infty(\Omega_h)}}{||f_h||_{L^\infty(\Omega_h)}}$$

Δ_h^{-1} uniformly bounded with respect to h means there exists $C \geq 0$, independent of h, such that:

$$||\Delta_h^{-1}|| \leq C$$

For the five-point discretization, we have:

$$||v_h||_{L^\infty(\Omega_h)} \leq \frac{1}{8} ||f_h||_{L^\infty(\Omega_h)}$$

for any problem with right-hand side f_h.

Then:

$$||\Delta_h^{-1}|| \leq \underbrace{\frac{1}{8}}_{=C}$$

The relation between the convergence error:

$$e_h = u_h - u|_{\Omega_h} \in L(\Omega_h)$$

and the consistency error:

$$\epsilon_h = \Delta_h u|_{\overline{\Omega_h}} - (\Delta u)|_{\Omega_h} \in L(\Omega_h)$$

is:

$$\Delta_h e_h = -\epsilon_h$$

So:

$$e_h = \Delta_h^{-1}(-\epsilon_h)$$

and then:

$$||e_h||_{L^\infty(\Omega_h)} \leq ||\Delta_h^{-1}||||\epsilon_h||_{L^\infty(\Omega_h)}$$

and then:

$$||e_h||_{L^\infty(\Omega_h)} \leq C||\epsilon_h||_{L^\infty(\Omega_h)}$$

L^∞ on $L(\Omega_h)$ is the discrete analogous of the L^∞ on $L(\Omega)$.

We introduce a norm on $L(\Omega_h)$ that is a discrete analogous of the L^2 norm on $L^2(\Omega)$.

For $v : \Omega \to \mathbb{R}$,

$$||v|| = \sqrt{\int_{(x,y)\in\Omega} v(x,y)^2 d(x,y)}$$

is called the L^2 norm of v.

$L^2(\Omega)$ is the set of the function $\Omega \to \mathbb{R}$ such that:

$$\int_{(x,y)\in\Omega} v(x,y)^2 d(x,y) < \infty$$

On $L^2(\Omega)$, we can introduce the scalar product:

$$< v, w >= \int_{(x,y)\in\Omega} v(x,y)w(x,y)d(x,y)$$

with $v, w \in L^2(\Omega)$.

Then, the L^2 norm is the norm derived by this scalar product:

$$||v|| = \sqrt{< v, v >}, \quad L^2(\Omega)$$

Now, we introduce discrete forms of this scalar product and this norm on $L(\Omega_h)$ and then to bound uniformly with respect to h the operator norm:

$$||\Delta_h^{-1}|| = \sup_{f_h \in L^\infty(\Omega_h), f_h \neq 0} \frac{||v_h||_h}{||f_h||_h}$$

where $||.||_h$ is the discrete L^2 norm.

First, we consider the $1D$ version of the Poisson problem.

$$\Omega = (0,1) =: I$$
$$\Omega_h = \{h, 2h, \dots, (N-1)h\} =: I_h$$

$L^2(I)$ is the set of the functions $v : I \to \mathbb{R}$ such that:

$$\int_0^1 v(x)^2 d(x) < +\infty$$

The scalar product on $L^2(I)$ is:

$$< v, w >= \int_0^1 v(x)w(x)d(x), \quad v, w \in L^2(I)$$

and the L^2 norm is:

$$||v|| = \sqrt{\int_0^1 v(x)^2 d(x)}, \quad v \in L^2(I)$$

On $L(I_n)$ we introduce the scalar product:

$$< v_h, w_h >= h \sum_{k=1}^{N-1} v_h(kh) w_h(kh)$$

and the norm:

$$||v_h||_h = \sqrt{< v_h, v_h >} = \sqrt{h \sum_{k=1}^{N-1} v_h(kh)^2}, \quad v_h \in L(I_h)$$

$||.||$ is the L^2 norm on I_h.

Exercise

Explain why the previous scalar product and norm on $L(I_h)$ are discretizations of the scalar product and norm on $L^2(I)$.

$$\int_0^1 g(x)dx \approx h \sum_{k=1}^{N-1} g(kh) = \sum_{k=1}^{N-1} g(kh)h$$

When $g = vw \to$ scalar product, $g = v^2 \to L^2$ norm.

In $L^2(I)$ we have Fourier series. Consider the functions $\Phi_m \in L^2(I)$, $m \in \{1, 2, 3, \dots\}$ given by:

$$\Phi_m(x) = sin(m\pi x), \quad x \in (0,1)$$

These functions are orthogonal:

$$< \Phi_m, \Phi_n >= 0 \quad \forall m \neq n$$

They constitute an orthogonal base for $L^2(I)$; for any $v \in L^2(I)$ we have the *Fourier series of* v:

$$v = \sum_{m=1}^{\infty} c_m \Phi_m$$

with:

$$c_m = \frac{< v, \Phi_m >}{< \Phi_m, \Phi_m >}, \quad m \in \{1, 2, 3, \dots\}$$

The Fourier series converges in L^2:

$$\lim_{M \to \infty} ||v - \sum_{m=1}^{M} c_m \Phi_m|| = 0$$

We have the *Parseval's identity*:

$$||v||^2 = \sum_{m=1}^{\infty} c_m^2 ||\Phi_m||^2, \quad v \in L^2(I)$$

The functions Φ_m, with $m \in \{1, 2, 3, \dots\}$ are eigenvectors of the 1D Laplacian: $\Delta =$ "second derivative".

For $m \in \{1, 2, 3, \dots\}$, we have:

$$\Delta\Phi_m(x) = \frac{d^2}{dx^2}\sin(m\pi x) = \frac{d}{dx}(m\pi\cos(m\pi x)) = m\pi(-m\pi\sin(m\pi x)) = -m^2\pi^2\sin(m\pi x)$$

$$= -m^2\pi^2\Phi_m(x), \quad x \in (0,1)$$

Φ_m is an eigenvector of Δ with relevant eigenvalue $-m^2\pi^2 = \lambda_m$.

We do the same for $L(I_h)$.

We introduce the functions $\Phi_{m,h} \in L(I_h), m \in \{1, 2, \dots\}$, given by:

$$\Phi_{m,h}(x) = \sin(m\pi x), \quad x \in I_h$$

The functions $\Phi_{m,h}, m \in \{1, 2, \dots\}$, are eigenvectors of the discrete laplacian:

$$\Delta_h v_h(x) = \frac{v_h(x-h) - 2v_h(x) + v_h(x+h)}{h^2}, \quad x \in I_h and v_h \in L(I_h)$$

For $m \in \{1, 2, 3, \dots\}$,

$$\Delta_h \Phi_{m,h}(x) = \frac{\sin(m\pi(x-h)) - 2\sin(m\pi x) + \sin(m\pi(x+h))}{h^2}$$

Using Prostapheresis formulas:

$$= \frac{2\sin(m\pi x)\cos(m\pi x) - 2\sin(m\pi x)}{h^2}$$

$$= -\frac{1 - 2\cos(m\pi x)}{h^2} \underbrace{\sin(m\pi x)}_{\Phi_{m,h}(x)}, \quad x \in I_h$$

$\Phi_{m,h}$ is an eigenvector of the discrete 1D laplacian with relevant eigenvalue:

$\lambda_{m,h} = -2\frac{1-\cos(m\pi h)}{h^2} = -2\frac{2\sin^2(\frac{m\pi x}{2})}{h^2} = -4\frac{\sin^2(\frac{m\pi x}{2})}{h^2}$

We have:

$$0 > \lambda_{1,h} > \lambda_{2,h} > \cdots > \lambda_{N-1,h}$$

Exercise

Prove that:

1. $\lim_{h\to 0} \lambda_m = -m^2\pi^2$
2. $\lambda_{N-1,h} > -\frac{4}{h^2}$
3. $-8 \geq \lambda_{1,h}$

Proofs:

1. $$\lim_{h\to 0} \lambda_m = \lim_{h\to 0}(-4\frac{\sin^2(\frac{m\pi x}{2})}{h^2})$$

   Using the limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{x\to 0} \frac{\sin^2 x}{x^2} = 1$

   $$= -m^2h^2 \lim_{h\to 0}(\frac{\sin(\frac{m\pi x}{2})}{\frac{m\pi x}{2}})^2 = -m^2h^2$$

2. $$\lambda_{N-1,h} = -4\frac{\sin^2(\frac{(N-1)\pi x h}{2})}{h^2} > -4\frac{\sin^2(\pi/2)}{h^2} = -\frac{4}{h^2}$$

3.

$$\lambda_{1,h} = -4\frac{\sin^2(\frac{\pi h}{2})}{h^2} = -\pi^2(\frac{\sin(\frac{\pi h}{2})}{\frac{\pi h}{2}})^2$$

As a function of h, $\lambda_{1,h}$ obtains the maximum value when h is the maximum.

The maximum h possible is $h = \frac{1}{2}$.

$$\lambda_{1,h} \leq \lambda_{1,\frac{1}{2}} = -4\frac{\sin^2(\pi\frac{1}{2})}{(\frac{1}{2})^2} = -4\frac{\sin^2(\frac{\pi}{4})}{\frac{1}{4}} = -16(\frac{\sqrt{2}}{2})^2 = -8$$

■

$\Phi_{1,h}, \ldots, \Phi_{N-1,h}$ are a bases for the space $L(I_h) = \mathbb{R}^{N-1}$.
So for any $v_h \in \mathbb{R}^{N-1}$, we have $v_h = \sum_{m=1}^{N-1} c_{m,h}\Phi_{m,h}$.
Moreover, observe that Δ_h is symmetric and so $\Phi_{1,h}, \ldots, \Phi_{N-1,h}$ are orthogonal in the standard scalar product:

$$< v_h, w_h >= \sum_{x \in I_h} v_h(x) w_h(x)$$

of $\mathbb{R}^{N-1}$. The scalar product introduced in $L(I_h)$ is:

$$< v_h, w_h >_h = h < v_h, w_h >$$

So $\Phi_{1,h}, \ldots, \Phi_{N-1,h}$ are orthogonal also in the scalar product $< ., . >_h$

Exercise

For $v_h \in L(I_h)$, prove that:

$$c_{m,h} = \frac{< v_h, \Phi_{m,h} >}{||\Phi_{m,h}||_h^2} =< \Phi_{m,h}, \Phi_{m,h} > \quad , m \in \{1, \ldots, N-1\}$$

Moreover, prove that the discrete Parceval's identity:

$$||v_h||_h^2 = \sum_{m=1}^{N-1} c_{m,h}^2 ||\Phi_{m,h}||_h^2$$

We have:

$$< v_h, \Phi_{m,h} >_h =< \sum_{n=1}^{N-1} c_{n,h}\Phi_{n,h}, \Phi_{m,h} >_h = \sum_{n=1}^{N-1} c_{n,h} \underbrace{< \Phi_{n,h}, \Phi_{m,h} >_h}_{=0 \ \forall n \neq m} = c_{m,h} < \Phi_{m,h}, \Phi_{m,h} >$$

$$||v_h||_h^2 =< v_h, v_h >_h =< \sum_{n=1}^{N-1} c_{n,h}\Phi_{n,h}, \sum_{m=1}^{N-1} c_{m,h}\Phi_{m,h} >_h = \sum_{n=1}^{N-1}\sum_{n=1}^{N-1} c_{n,h}c_{m,h} < \Phi_{n,h}, \Phi_{m,h} >_h = \sum_{m=1}^{N-1} c_{m,h}^2 < \Phi$$

Now we're ready for the band of:

$$||\Delta_h^{-1}|| = \sup_{f_h \in L(I_h)} \frac{||v_h||_h}{||f_h||_h}$$

where v_h is the solution of:

$$\Delta_h v_h = f_h$$

Let:

$$v_h = \sum_{m=1}^{N-1} c_{m,h}\Phi_{m,h}$$

the discrete Fourier series for v_h.

We have:

$$f_h = \Delta_h v_h = \Delta_h \sum_{m=1}^{N-1} c_{m,h}\Phi_{m,h} = \sum_{m=1}^{N-1} c_{m,h} \underbrace{\Delta_h \Phi_{m,h}}_{=\lambda_{m,h}\Phi_{m,h}} = \sum_{m=1}^{N-1} (c_{m,h}\lambda_{m,h})\Phi_{m,h}$$

which is the discrete Fourier series of f_h.

Now we use the discrete Parseval's identity:

$$||f_h||_h^2 = \sum_{m=1} N-1 (c_{m,h}\lambda_{m,h})^2 ||\Phi_{m,h}||_h^2 = \sum_{m=1} N-1 c_{m,h}^2 \underbrace{\lambda_{m,h}^2}_{=|\lambda_{m,h}|^2} ||\Phi_{m,h}||_h^2$$

$$-8 \geq \lambda_{1,h} \geq \lambda_{2,h} \geq \cdots \implies 8 \leq |\lambda_{1,h}| \leq |\lambda_{2,h}| \leq \ldots$$

$$\geq \sum_{m=1}^{N-1} 8^2 c_{m,h}^2 ||\Phi_{m,h}||_h^2 = 8^2 \underbrace{\sum_{m=1}^{N-1} c_{m,h}^2 ||\Phi_{m,h}||_h^2}_{=||v_h||_h^2}$$

So, we have proved:

$$||v_h||^2 \leq \frac{1}{8^2} \cdots$$

Since $f_h \in L(I_h)$ is arbitrary we have:

$$||\Delta_h^{-1}|| \sup_{f_h \in L(I_h)} \frac{||v_h||_h}{||f||_h} \leq \frac{1}{8}$$

Now we pass to the 2D case.

In the 2D case:

$$\Omega = (0,1)^2 = I^2$$

$$\Omega_h = I^2 = \{(mh, nh) : m, n \in \{1, \dots, N-1\}\}$$

$L(\Omega_h)$ is isomorphic to $\mathbb{R}^{(N-1)^2}$:

$$L(\Omega_h) = \mathbb{R}^{(N-1)^2}$$

We introduce, for $m, n \in \{1, \dots, N-1\}$, the functions $\Phi_{m,n,h} \in L(\Omega_h)$ given by:

$$\Phi_{m,n,h}(x,y) = \Phi_{m,h}(x)\Phi_{n,h}(y) \quad (x,y) \in \Omega_h$$

where $\Phi_{m,h}$ and $\Phi_{n,h}$ were defined in the 1D case.

The scalar product on $L(\Omega_h)$ is:

$$< v_h, w_h >= h^2 \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} v_h(m_h, n_h) w_h(m_h, n_h)$$

and the L^2 norm is:

$$||v_h||_h = \sqrt{< v_h, v_h >} = \sqrt{h^2 \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} v_h(m_h, n_h)^2}$$

for $v_h, w_h \in L(\Omega_h)$.

For $m, n \in \{1, \dots, N-1\}$, $\Phi_{m,n,h}$ is an eigenvector of the 2D discrete laplacian Δ_h.

$$\Delta_h \Phi_{m,n,h}(x,y) = \frac{\Phi_{m,n,h}(x-h,y) - 2\Phi_{m,n,h}(x,y) + \Phi_{m,n,h}(x+h,y)}{h^2} + \frac{\Phi_{m,n,h}(x,y-h) - 2\Phi_{m,}}{}$$

$$= \frac{\Phi_{m,h}(x-h)\Phi_{n,h}(y) - 2\Phi_{m,h}(x)\Phi_{n,h}(y) + \Phi_{m,h}(x+h)\Phi_{n,h}(y)}{h^2} + \frac{\Phi_{m,h}(x)\Phi_{n,h}(y-h) - 2\Phi_{m,h}}{}$$

$$= \frac{\Phi_{m,h}(x-h) - 2\Phi_{m,h}(x) + \Phi_{m,h}(x+h)}{h^2}\Phi_{m,h}(y) + \Phi_{m,h}(x)\frac{\Phi_{n,h}(y-h) - 2\Phi_{n,h}(y) + \Phi_{n,h}(y+}{h^2}$$

$$= \lambda_{m,h}\Phi_{m,h}(x)\Phi_{n,h}(y) + \Phi_{m,h}(x)\lambda_{n,h}\Phi_{n,y}(y) = (\lambda_{m,h} + \lambda_{n,h}) \underbrace{\Phi_{m,h}(x)\Phi_{n,h}(y)}_{=\Phi_{m,n,h}(x,y)}$$

We have proved that $\Phi_{m,n,h}$ is an eigenvector of 2D Δ_h with relevant eigenvalue $\lambda_{m,h} + \lambda_{n,h}$.

Now, we prove that $\Phi_{m,n,h}, m, n \in \{1, \ldots, N-1\}$ are orthogonal in the scalar product of $L(\Omega_h)$ we have introduced.

For $m, n, p, q \in \{1, \ldots, N-1\}$ with $(m, n) \neq (p, q)$:

$$< \Phi_{m,n,h}, \Phi_{p,q,h} >_h = h^2 \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} \Phi_{m,n,h}(kh, lh) \Phi_{p,q,h}(kh, lh)$$

$$= h^2 \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} \Phi_{m,h}(kh) \Phi_{n,h}(lh) \Phi_{p,h}(kh) \Phi_{q,h}(lh)$$

$$= h^2 (\sum_{k=1}^{N-1} \Phi_{m,h}(kh) \Phi_{p,h}(kh)) (\sum_{l=1}^{N-1} \Phi_{n,h}(lh) \Phi_{q,h}(lh))$$

$$= \underbrace{< \Phi_{m,h}, \Phi_{p,h} >_h}_{1D scalar product} \cdot \underbrace{< \Phi_{n,h}, \Phi_{q,h} >_h}_{1D scalar product}$$

Remember $(m, n) \neq (p, q)$:

$$m \neq p \text{ or } mn \neq q$$

so the above scalar product is zero.

Since the $(N-1)^2$ functions $\Phi_{m,n,h}$, $m, n \in \{1, \ldots, N-1\}$ are nonzero orthogonal functions, they $(N-1)^2$ linearly independent functions of $\mathbb{R}^{(N-1)^2}$ and so they constitute a bases for $\mathbb{R}^{(N-1)^2}$.

So any $v_h \in L(\Omega_h)$ can be written as:

$$v_h = \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} \ldots$$

Exactly as in the 1D case, we have:

$$c_{m,n,h} = \frac{< v_h, \Phi_{m,n,h} >_h}{||\Phi_{m,n,h}||} \ldots$$

...

Now we are ready to give a bond for

$$||\Delta_h^{-1}|| = \sup_{f_h \in L(\Omega_h)} \frac{||v_h||_h}{||f_h||_h}$$

where v_h is the solution of: $\Delta_h v_h = f_h$

...

Let $f_h \in L(\Omega_h)$ and let v_h be the solution of $\Delta_h v_h = f_h$. Let:

$$v_h = \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} c_{m,n,h} \Phi_{m,n,h}$$

be the Fourier series of v_h.

We have:

$$f_h = \Delta_h v_h = \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} c_{m,n,h} \underbrace{\Delta_h \Phi_{m,n,h}}_{(\lambda_{m,h}+\lambda_{n,h})\Phi_{m,n,h}} = \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} c_{m,n,h}(\lambda_{m,h}+\lambda_{n,h}) c_{m,n,h} \Phi_{m,n,h}$$

which is the discrete Fourier series of f_h.

So:

$$||f_h||_h^2 = \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} \underbrace{(\lambda_{m,h} + \lambda_{n,h})^2}_{||\Phi_{m,n,h}||_h^2} c_{m,n,h}^2$$

$$= \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} (\underbrace{|\lambda_{m,h}|}_{\geq 8} + \underbrace{|\lambda_{n,h}|}_{\geq 8})^2 c_{m,n,h}^2 \geq \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} 16^2 c_{m,n,h}^2$$

Then we have:

$$||v_h||_h^2 \leq \frac{1}{16^2} ||f_h||_h^2$$

$$||v_h||_h \leq \frac{1}{16} ||f_h||_h$$

Since $f_h \in L(\Omega_h)$ is arbitrary:

$$||\Delta_h^{-1}|| = \sup_{f_h \in L(\Omega_h)} \frac{||v_h||_h}{||f_h||_h} \leq \frac{1}{16}$$

Remember we have:

$$\Delta_h \underbrace{e_h}_{convergence error} = - \underbrace{\epsilon_h}_{consistency error}$$

So:

$$||e_h||_h \leq ||\Delta_h^{-1}|| ||\epsilon_h||_h \leq \frac{1}{16} ||\epsilon_h||_h$$

In case of the L^∞ norm, we had:

$$||e_h||_{L^\infty(\Omega_h)} \leq \frac{1}{8} ||\epsilon_h||_{L^\infty(\Omega_h)}$$

Exercise

Prove that, for $v_h \in L(\Omega_h)$, we have:

$$||v_h|| \leq ||v_h||_{L^\infty(\Omega_h)}$$

and then conclude with the estimate:

$$||e_h||_h = O(h^2), h \to 0$$

$$||v_h||_h = \sqrt{h^2 \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} \underbrace{v_h(kh, nh)^2}_{\leq ||v_h||^2_{L^\infty(\Omega_h)}}}$$

$$\leq \sqrt{h^2 \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} ||v_h||_{L^\infty(\Omega_h}} = ||v_h||_{L^\infty(\Omega_h} \sqrt{h^2 \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} 1}$$

$$= ||v_h||_{L^\infty(\Omega_h} \underbrace{\sqrt{h^2(N-1)}}_{=h(N-1)}$$

$$\leq ||v_h||_{L^\infty(\Omega_h} \underbrace{hN}_{=1} = ||v_h||_{L^\infty(\Omega_h}$$

Then:

$$||e_h||_h \leq \frac{1}{16} ||\epsilon_h||_h \leq \frac{1}{16} \overbrace{||\epsilon_h||_{L^\infty(\Omega_h)}}^{=O(h^2),\, h \to 0}$$

and then:

$$||e_h||_h = O(h^2),\ h \to 0 \qquad \blacksquare$$