Advanced numerical analysis

Alberto Chiusole's notes

professors:
Paolo Novati (novati@units.it) and
Stefano Maset (maset@univ.trieste.it)

Academic Year 2017-2018

Contents

Chapter 1

Inverse linear problems

The main objective of the course is dealing with the following objects:

$$\int_{\Omega} (input)(system) = output \tag{1.1}$$

In inverse linear problems we usuall

Chapter 2

Numerical methods for PDE

PDE stands for Partial Differential Equations, and we are going to study in particular Finite Difference Methods, not Finite Element Methods.

PDEs can be divided in three categories: elliptic, parabolic and high-parabolic. We will see a test problem on each of these categories.

2.1 Poisson 2D problem with Dirichlet boundary conditions

$$\text{Poisson equation:} \quad \Delta u(x,y) = f(x,y), (x,y) \in \Omega \tag{2.1}$$

$$\text{Dirichlet Boundary condition:} \quad u(x,y) = g(x,y), (x,y) \in \Gamma \tag{2.2}$$

where $\Omega = (0,1)^2$

Example: the electric potential inside a domain with a defined potential inside, f, and a fixed potential, g, at the border.

The Poisson 2D problem can be solved analytically using *separation of variables* by searching for solutions of this type:

$$u(x,y) = X(x)Y(y) \tag{2.3}$$

Discretization: fix N positive integer and put $h = \frac{1}{N}$ as the stepsize.

Now we discretize the plane $\mathbb{R}^2$ by the mesh:

$$\mathbb{R}^2_h = \{(mh, nh) : m, n \in \mathbb{Z}\} \tag{2.4}$$

For a mesh point (mh, nh), with $m, n \in \mathbb{Z}$ we can define four neighbors:

The discrete version of Ω is

$$\Omega_h := \Omega \cap \mathbb{R}_h^2 \tag{2.5}$$

while the discrete version of Γ is the set of points Γ_h in $\mathbb{R}_h^2$ which are not in Ω_h, but have a neighbor in Ω_h.

Notice how $\Gamma_h \neq \Gamma \cap \mathbb{R}_h^2$ because the four corners $\{(0,0),(0,1),(1,0),(1,1)\}$ are not included.

We also define $\overline{\Omega_h} = \Omega_h \cup \Gamma_h$: this is the discrete version of $\overline{\Omega_h} = \Omega \cup \Gamma$.

The discrete version of the Poisson problem with Dirichlet Boundary conditions is:

$$\text{discrete Poisson equation: } \Delta_h u_h(x,y) = f(x,y), (x,y) \in \Omega_h \tag{2.6}$$

$$\text{discrete Dirichlet Boundary condition: } u_h(x,y) = g(x,y), (x,y) \in \Gamma_h \tag{2.7}$$

An operator is a map/function, whose domains and codomains are set of functions.

Δ_h is a discrete Laplacian, i.e. and operator such that:

- it associates (as a map) to a mesh function $\overline{\Omega_h} \to \mathbb{R}$ an interior mesh function $\Omega_h \to \mathbb{R}$;
- it approximates the Laplacian, in the sense that, for function $v : \overline{\Omega_h} \to \mathbb{R}$ sufficiently smooth, which means $C^2(\overline{\Omega_h})$:

$$\Delta_h v|_{\overline{\Omega_h}}(x,y) \approx \Delta v(x,y) \quad (x,y) \in \Omega_h \tag{2.8}$$

where $|_{\overline{\Omega_h}}$ is the restriction of v to $\overline{\Omega_h} \subseteq \overline{\Omega}$.

We are going to show how to construct a discrete Laplacian.

Consider a function v of only one variable t. We discretize $v''(t)$ by:

$$\frac{v(t-h) - 2v(t) + v(t+h)}{h^2} \tag{2.9}$$

.

$$\frac{\frac{v(t+h)-v(t)}{h} - \frac{v(t)-v(t-h)}{h}}{h} \approx \frac{v'(t) - v'(t-h)}{h} \approx v''(t-h) \approx v''(t) \tag{2.10}$$

By Taylor approximations under the assumptions that v is C^4:

$$v(t-h) = v(t) - v'(t)h + \frac{1}{2}v''(t)h^2 - \frac{1}{6}v'''(t)h^3 + \frac{1}{24}v^{(4)}(\alpha_h)h^4 \tag{2.11}$$

where $\alpha_h \in [t-h, t]$;

$$v(t+h) = v(t) + v'(t)h + \frac{1}{2}v''(t)h^2 + \frac{1}{6}v'''(t)h^3 + \frac{1}{24}v^{(4)}(\beta_h)h^4 \tag{2.12}$$

where $\beta_h \in [t, t+h]$.
By summing 2.11 and 2.12 we obtain:

$$v(t-h) + v(t+h) = 2v(t) + v''(t)h^2 + \frac{1}{24}(v^{(4)}(\alpha_h) + v^{(4)}(\beta_h))h^4 \tag{2.13}$$

$$v(t-h) - 2v(t) + v(t+h) = v''(t)h^2 + \frac{1}{24}(v^{(4)}(\alpha_h) + v^{(4)}(\beta_h))h^4 \tag{2.14}$$

$$\frac{v(t-h) - 2v(t) + v(t+h)}{h^2} = v''(t) + \frac{1}{24}(v^{(4)}(\alpha_h) + v^{(4)}(\beta_h))h^2 \tag{2.15}$$

So we got an approximation of the second derivative of the function v.

2.1.1 Discretization of the Laplacian

For a smooth function $v : \overline{\Omega_h} \to \mathbb{R}$ we approximate the Laplacian:
by:

$$\Delta_h v(x,y) = \frac{v(x-h,y) - 2v(x,y) + v(x+h,y)}{h^2} + \frac{v(x,y-h) - 2v(x,y) + v(x,y+h)}{h^2} \tag{2.16}$$

at $(x,y) \in \Omega_h$.

The computation of $\Delta_h v(x,y)$ requires $v(x-h,y)$ and $v(x,y+h)$ with $(x-h,y),(x,y+h) \in \Gamma_h$.

This discretization of the Laplacian is called the *Five-points discretization*:

2.1.2 Assess approximation of the discrete Laplacian

Consider $v : \overline{\Omega} \to \mathbb{R}$ of class C^4. For $(x, y) \in \Omega_h$:

$$|\Delta_h v|_{\overline{\Omega_h}}(x, y)\Delta v(x, y)| \tag{2.17}$$

$$|\frac{v(x-h,y)-2v(x,y)+v(x+h,y)}{h}+\frac{v(x,y-h)-2v(x,y)+v(x,y+h)}{h}-\frac{\partial^2 v}{\partial x^2}(x,y)-\frac{\partial^2 v}{\partial y^2}(x, \tag{2.18}$$

which is $\leq$ than:

$$|\frac{v(x-h,y)-2v(x,y)+v(x+h,y)}{h^2}-\frac{\partial^2 v}{\partial x^2}(x,y)|+|\frac{v(x,y-h)-2v(x,y)+v(x,y+h)}{h^2}-\frac{\partial^2 v}{\partial y^2}(\tag{2.19}$$

.

2.19 is equal to:

$$|\frac{1}{24}(\frac{\partial^4 v}{\partial x^4}(\alpha_h, y) + \frac{\partial^4 v}{\partial x^4}(\beta_h, y)\ h^2)| + |\frac{1}{24}(\frac{\partial^4 v}{\partial y^4}(x, \gamma_h) + \frac{\partial^4 v}{\partial y^4}(x, \delta_h)\ h^2)| \tag{2.20}$$

where $\alpha_h \in [x-h, x], \beta_h \in [x, x+h], \gamma_h \in [y-h, y], \delta_h \in [y, y+h]$.
2.20 (the error) can be bounded ($\leq$) by:

$$\frac{1}{6}\max\{\max_{(x,y)\in\overline{\Omega}}|\frac{\partial^4 v}{\partial x^4}(x,y)|, \max_{(x,y)\in\overline{\Omega}}|\frac{\partial^4 v}{\partial y^4}(x,y)|\}\ h^2 \tag{2.21}$$

So we can conclude that:

$$\max_{(x,y)\in\Omega_h}|\Delta_h v|_{\overline{\Omega_h}}(x, y)\Delta v(x, y)| \leq \frac{1}{6}\max\{\max_{(x,y)\in\overline{\Omega}}|\frac{\partial^4 v}{\partial x^4}(x,y)|, \max_{(x,y)\in\overline{\Omega}}|\frac{\partial^4 v}{\partial y^4}(x,y)|\}\ h^2 \tag{2.22}$$

This is called the *consistency error* of the five point approximation of the discrete Laplacian.

2.1.3 Exercise

Given $v \in C^6(\Omega)$, find a function $C(x, y)$ of the point $(x, y) \in \overline{\Omega}$ and $M \geq 0$ such that

$$\max_{(x,y)\in\Omega_h}|\Delta_h v|_{\overline{\Omega_h}}(x, y) - \Delta v(x, y) - C(x, y)h^2| \leq Mh^4$$

$$\Delta_h v|_{\overline{\Omega_h}}(x,y) = \Delta v'(x,y) + \text{error}$$

with error $= O(h^2)(*) = C(x,y)h^2 + O(h^4)(**)$.
Remind that:

$$v''(t) \approx \frac{v(t-h) - 2v(t) + v(t+h)}{h^2} \tag{2.23}$$

we can continue with the Taylor expansion:

$$v(t-h) = v(t) - v'(t)h + \frac{v''(t)}{2}h^2 - \frac{v'''(t)}{6}h^3 + \frac{v^4(t)}{24}h^4 - \frac{v^5(t)}{120}h^5 + \frac{v^6(t)}{6!}h^6$$

$$v(t-h) = v(t) + v'(t)h + \frac{v''(t)}{2}h^2 + \frac{v'''(t)}{6}h^3 + \frac{v^4(t)}{24}h^4 + \frac{v^5(t)}{120}h^5 + \frac{v^6(t)}{6!}h^6$$

where $\alpha \in [t-h, t]$ and $\beta \in [t, t+h]$.

$$v(t-h) + v(t+h) = 2v(t) + v''(t)h^2 + \frac{v^4(t)}{12}h^4 + \frac{v^6(\alpha)}{6!}h^6 + \frac{v^6(\beta)}{6!}h^6$$

$$\frac{v(t-h) - 2v(t) + v(t+h)}{h^2} - v''(t)h^2 = \frac{v^4(t)}{12}h^4 + \frac{v^6(\alpha)}{6!}h^6 + \frac{v^6(\beta)}{6!}h^6$$

...
The discrete problem:

$$\text{Poisson equation:}\quad \Delta u_h(x,y) = f(x,y), (x,y) \in \Omega_h$$

$$\text{Dirichlet Boundary condition:}\quad u_h(x,y) = g(x,y), (x,y) \in \Gamma_h$$

is a linear system of $(N-1)^2$ unknowns (where N is the number of subdivisions we do in each dimension of the square) and $(N-1)^2$ equations.

The unknowns are $u_{ij} := u_h(ih, jh)$ with $i \in \{1, \ldots, N-1\}$ and $j \in \{1, \ldots, N-1\}$.

.. we don't consider the values on the borders as unknowns because they are known, from the second equation of the Dirichlet Boundary condition.

On the border Γ_h:

$$u_{ij} = u_h(ih, jh) = h_{ij} := g(ih, jh)$$

where

$(i,j) \in \{0\}\times\{1,\ldots,N-1\}\cup\{1,\ldots,N-1\}\times\{0\}\cup\{N\}\times\{1,\ldots,N-1\}\cup\{1,\ldots,N-1\}\times\{N\}$

The equations are

$$\Delta_h u_h(x,y) = f(x,y), \quad \in \Omega_h$$

each equation corresponds to a point in Ω_h with $(N-1)^2$ equations.

The equation corresponding to the point (ih, jh), with $i \in \{1,\ldots,N-1\}, j \in \{1,\ldots,N-1\}$ is:

$$\frac{u_{i-1\,j} - 2u_{ij} + u_{i+1\,j}}{h^2} + \frac{u_{i\,j-1} - 2u_{ij} + u_{i\,j+1}}{h^2} = f(ih, jh) = f_{ij}$$

$$u_{i\,j-1} + u_{i-1\,j} - 4u_{ij} + u_{i+1\,j} + u_{i\,j+1} = h^2 f_{ij}$$

In order to apply a linear system of the form $Au = b$, we need to rearrange the equations of the points of the square (which are 2D) into a 1D vector.

Consider $N = 4$:

...

We want to rewrite our matrix of unknowns u:

$$u = \begin{bmatrix} u_{11} & u_{21} & u_{31} \\ u_{12} & u_{22} & u_{32} \\ u_{13} & u_{23} & u_{33} \end{bmatrix}$$

as a vector:

$$u = \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{12} \\ u_{22} \\ u_{32} \\ u_{13} \\ u_{23} \\ u_{33} \end{bmatrix}$$

Then we can rewrite our equation $u_{i\,j-1} + u_{i-1\,j} - 4u_{ij} + u_{i+1\,j} + u_{i\,j+1} = h^2 f_{ij}$ as (for the point $(1,1)$):

$$u_{10} + u_{01} - 4u_{11} + u_{21} + u_{12} = h^2 f_{11}$$

where u_{10} and u_{01} correspond to g_{10} and g_{01} respectively.

$$-4u_{11} + u_{21} + u_{12} = h^2 f_{11} - u_{10} - u_{01}$$

For the point $(2, 1)$:

$$u_{20} + u_{11} - 4u_{21} + u_{31} + u_{22} = h^2 f_{22}$$

rewritten as:

$$u_{11} - 4u_{21} + u_{31} + u_{22} = h^2 f_{22} - g_{20}$$

We can write a matrix $\mathcal{A}$:

$$\mathcal{A} = \begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix}$$

and the unknown vector u:

$$u = \begin{bmatrix} h^2 f_{11} - g_{01} - g_{10} \\ h^2 f_{21} - g_{20} \\ h^2 f_{31} - g_{14} \\ h^2 f_{12} - g_{02} \\ h^2 f_{22} \\ h^2 f_{32} - g_{24} \\ h^2 f_{13} - g_{03} - g_{14} \\ h^2 f_{23} - g_{24} \\ h^2 f_{33} - g_{34} - g_{43} \end{bmatrix}$$

$\mathcal{A}$ can be rewritten as a $3x3$ blocks matrix with blocks of size $3x3$:

$$\mathcal{A} = \begin{bmatrix} A & I_3 & 0 \\ I_3 & A & I_3 \\ 0 & I_3 & A \end{bmatrix}$$

For a general N, $\mathcal{A}$ is a $(N-1) \times (N-1)$ block matrix:

$$\mathcal{A} = \begin{bmatrix} A & I_{N-1} & & & \\ I_{N-1} & A & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & A & I_{N-1} \\ & & & I_{N-1} & A \end{bmatrix}$$

A generic matrix with dimension $n \times n$ is called “sparse” if the number of non-zero elements is $O(n)$.

The matrix $\mathcal{A}$ is sparse, block tridiagonal (it has elements on the diagonal, on the diagonal of the upper matrix and on the diagonal of the lower matrix), symmetric and $\ldots$.