EX 1 : CHEESE STORES DATASETS

- p = # stores, each store has n observations. $\Rightarrow i = 1 \dots p$, $j = 1 \dots n$
- for each observation we have:

Q = quantity demanded
P = price
d = dummy variable for displacement in the store

from economic theory we have:

$$Q = aP^{\beta}$$

β: price elasticity.

and we would like to identify β.

$\Rightarrow$ can take:

$$\underbrace{\log Q}_{q} = \log a + \beta \underbrace{\log P}_{p}$$

But consider a more comprehensive model allowing us to identify the different role of covariate:

$$q_{ij} = \beta_{0i} + \beta_{1i} p_{ij} + \beta_{2i} \mathbb{1}(disp = 1) + \beta_{3i} \mathbb{1}(disp = 1) \cdot p_{ij} + \varepsilon_{ij}$$

goal: fit this model using a hierarchical regression.

proposal:

$\log Q_{ij}$

→ Specification of the model:

$$q_{ij} \mid \beta_i \sim N(X_{ij}\beta_i, \sigma^2) \quad , \quad \forall i = 1 \dots p$$
$$\forall j = 1 \dots n$$

$$\beta_i \mid \gamma \sim N(\gamma, \tau^2\sigma^2)$$

with hyperpars: $p(\gamma) \propto c$

$p(\tau^2) \sim \text{Inv-Ga}\left(\frac{1}{2}, \frac{1}{2}\right)$

$p(\sigma^2) \sim \text{Jeffreys}: \ \frac{1}{\sigma^2}$

⇒ posterior inference strategy: Gibbs sampling.

posterior conditionals:

$1 \times p$

- $$p(\beta_i \mid -) \propto \prod_{j=1}^{n} p(y_{ij} \mid X_{ij}'\beta_i, \sigma^2) \cdot p(\beta_i)$$

$$\propto \exp\left\{-\frac{1}{2}(y_i - X_i\beta_i)'(y_i - X_i\beta_i)\right\} \cdot \exp\left\{-\frac{1}{2\sigma^2\tau^2}(\beta_i - \gamma)'(\beta_i - \gamma)\right\}$$

$$\propto \exp\left\{-\frac{1}{2\sigma^2}(-2\beta_i'X_i'y_i + \beta_i'X_iX_i'\beta_i) - \frac{1}{2\sigma^2\tau^2}(-2\beta_i'\gamma + \beta_i'\beta_i)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[-2\beta_i'\left(\frac{X_i'y_i}{\sigma^2} + \frac{\gamma}{\tau^2\sigma^2}\right) + \beta_i'\left(\frac{X_iX_i'}{\sigma^2} + \frac{I}{\sigma^2\tau^2}\right)\beta_i\right]\right\}$$

$$\Rightarrow \beta_i \sim N(m^*, V^*); \quad V^* = \left(\frac{X_iX_i'}{\sigma^2} + \frac{I}{\sigma^2\tau^2}\right)^{-1}; \ m^* = V^*\left(\frac{X_i'y_i}{\sigma^2} + \frac{\gamma}{\tau^2\sigma^2}\right)$$

$$\bullet\ p(\sigma^2|-) \propto \left(\prod_{i=1}^{p}\left(\prod_{j=1}^{n} p(y_{ij}|\beta_i \sigma^2)\right)\cdot p(\beta_i|\sigma^2, \tau^2, \gamma)\right)\cdot p(\sigma^2)$$

$$\propto \left(\prod_{i=1}^{p}\left(\prod_{j=1}^{n}\frac{1}{\sigma}\cdot\exp\left\{-\frac{1}{\sigma^2}(y_{ij}-x_{ji}\beta_i)^2\right)\cdot\frac{1}{\sigma}\exp\left\{-\frac{1}{2\sigma^2\tau^2}(\beta_i-\gamma)'(\beta_i-\gamma)\right\}\right)\cdot\frac{1}{\sigma^2}$$

$$=\left(\prod_{i=1}^{p}\left(\frac{1}{\sigma^n}\cdot\exp\left\{-\frac{1}{2}\frac{1}{\sigma^2}(y_i-x_i\beta_i)'(y_i-x_i\beta_i)\right)\right)$$

$$\times\ \frac{1}{\sigma}\cdot\exp\left\{-\frac{1}{2\sigma^2}(\beta_i-\gamma)(\tau^2 I)^{-1}(\beta_i-\gamma)\right\}\right)\cdot\ \frac{1}{\sigma^2}$$

$$=\ (\sigma^2)^{-\frac{N+P}{2}-1}\cdot\exp\left\{-\frac{1}{2\sigma^2}\left[\sum^{p}(y_i-x_i\beta_i)'(y_i-x_i'\beta_i)\right\}+(\beta_i-\gamma)(\tau^2 I)^{-1}(\beta_i-\gamma)\right]\right\}$$

$$\Rightarrow \sigma^2|-\sim \text{In-Ga}\left(\frac{N+P}{2};\ \frac{\sum^{p}\left((y_i-x_i'\beta_i)'(y_i-x_i'\beta_i)+(\beta_i-\gamma)'(I\tau^2)^{-1}(\beta_i-\gamma)\right)}{2}\right)$$

$$\bullet\ p(\tau^2|-)\propto p(\tau^2)\cdot\left(\prod_{i=1}^{p}p(\beta_i|\gamma, \tau^2\sigma^2)\right)$$

$$\propto\left(\frac{1}{\tau^2}\right)^{p}\cdot\exp\left\{-\frac{1}{2\tau^2}\left(\sum^{p}(\beta_i-\gamma)(\sigma^2 I)^{-1}(\beta_i-\gamma)\right)\right\}\cdot\tau^{2^{-\frac{1}{2}-1}}\cdot\exp\left\{-\frac{1}{2}\tau^2\right\}$$

$$=(\tau^2)^{-p-\frac{1}{2}-1}\cdot\exp\left\{-\frac{1}{\tau^2}\left[\frac{\sum^{p}(\beta_i-\gamma)'(\sigma^2 I)^{-1}(\beta_i-\gamma)+1}{2}\right]\right\}$$

$$\Rightarrow \tau^2\sim\text{Inv-Ga}\left(p+\frac{1}{2};\ \frac{\sum^{p}(\beta_i-\gamma)'(\sigma^2 I)^{-1}(\beta_i-\gamma)+1}{2}\right)$$

- $p(\gamma|-) \propto \left(\prod^{p} p(\beta_i|\gamma, \delta^2, \tau^2)\right) p(\gamma)$

$$\propto \exp\left\{-\frac{1}{2\delta^2\tau^2} \sum^{p} (\beta_i-\gamma)'(\beta_i-\gamma)\right\}$$

$$\propto \exp\left\{-\frac{1}{2\delta^2\tau^2}\left[p\gamma' I\gamma - 2\gamma'\sum^{p}\beta_i\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[\frac{p\gamma'\gamma}{\delta^2\tau^2} - 2\,\frac{\gamma'\sum\beta_i}{\delta^2\tau^2}\right]\right\}$$

$$\Rightarrow \gamma|- \sim N(M, K)$$

$$K = \left(\frac{p\cdot I}{\delta^2\tau^2}\right)^{-1} \quad ; \quad M = K\cdot\left(\frac{\sum^{p}\beta_i}{\delta^2\tau^2}\right)$$

EX 2 : PRESIDENTIAL SURVEY

Dataset : y : binary → 1: voting for Bush
↘ 0: not voting for Bush

$S = \#$ states

X: predictors

μ_i, β_i

Model : Probit model :

$$P(y_{ij} = 1) = \phi(z_{ij}) \quad i = 1 \dots S, \; j = 1 \dots n_i$$

$$z_{ij} = \mu_i + x_{ij}^T \beta_i = \underbrace{x_{ij}}_{\text{design matrix with intercept}}{}^T \gamma_i$$

problem: posterior $p(\gamma_i | y)$ is intractable:

$$p(\gamma_i | y) \propto \underbrace{p(\gamma_i)}_{\text{not prior such that the posterior is tractable.}} \cdot \prod_{j=1}^{n_i} [\phi(x_{ij}' z_i)]^{y_{ij}} \cdot [1 - \Phi(x_{ij}' z_i)]^{1 - y_{ij}}$$

$\Downarrow$

solution: <u>Augmented Model</u> :

$S =$ # states
$p =$ # parameters

$$y_{ij} = \mathbb{1}(z_{ij} \geq 0)$$

$$z_{ij} \sim N(x_{ij}\gamma_i, 1) \longleftarrow \text{auxiliary variable}$$

$$\gamma_i \sim N(m, \tau^2)$$

hyperpriors:
$p(m) \propto c$
$p(\tau^2) \propto \text{Inv-ga}\left(\frac{1}{2}, \frac{1}{2}\right)$

this augmentation works, indeed:

1) the implied marginal model for y_{ij} is the original one:

$$p(y_{ij} = 1) = p(z_{ij} > 0) = \Phi(x_{ij}\gamma_i) \quad \checkmark$$

2) We obtain tractable conditional posteriors which we can use to construct a Gibbs sampler to conduct posterior inference.

(indeed, we are back to a normal - normal model):

follows derivation of the full conditionals:

- $p(z_{ij} | -) = p(y_{ij} | z_{ij}) \cdot p(z_{ij})$

truncated normal

$$\Rightarrow \begin{cases} \mathbb{1}(y_{ij} = 1) \cdot N(x_{ij}\gamma_i, 1) \\ \mathbb{1}(y_{ij} = 0) \, N(x_{ij}\gamma_i, 1) \end{cases}$$

$$p(z_{ij} > 0) = p(y_{ij} = 1) = \int_{-\infty}^{x_{ij}^T \beta_i} \phi(z | 0, 1) \, dz = \Phi(x_{ij}' \beta_i)$$

- $p(\gamma_i | -) \propto p(\gamma_i) \cdot \prod_{j=1}^{n_i} p(z_{ij} | x_{ij}\, \gamma_i)$

vector matx

$$\propto \exp\left\{-\frac{1}{2}(\gamma_i - m)'(I\tau^2)^{-1}(\gamma_i - m)\right\} \cdot \exp\left\{-\frac{1}{2}(z_i - x_i\gamma_i)'(z_i - x_i\gamma_i)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(\gamma_i'(I\tau^2)^{-1}\gamma_i - 2\gamma_i'(I\tau^2)m\right) - \frac{1}{2}\left(\gamma_i' x_i' x_i \gamma_i - 2\gamma_i' x_i' z_i\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(\gamma_i'\left(x_i' x_i + (I\tau^2)^{-1}\right)\gamma_i - 2\gamma_i'\left(x_i z_i + \gamma_i' I\tau^2 m\right)\right\}$$

$$\Rightarrow \quad \gamma_i \sim N_p(m^*, V^*)$$

$$V^* = \left(\left(x_i' x_i + (I\tau^2)^{-1}\right)^{-1}\right.$$

$$m^* = V^* \times \left(x_i' z_i \cdot (I\tau^2)^{-1} m\right)$$

- $p(\tau^2 | -) \propto p(\tau^2) \cdot \prod^{S} p(\gamma_i | m, \tau^2)$

$$\propto \tau^{2^{-\frac{1}{2}-1}} \cdot \exp\left\{-\frac{1}{2}\tau^2\right\} \cdot \left(\frac{1}{\tau^2}\right)^S \cdot \exp\left\{-\frac{1}{2\tau^2}\sum^S (\gamma_i - m)'(\gamma_i - m)\right\}$$

$$= (\tau^2)^{-S-\frac{1}{2}-1} \cdot \exp\left\{-\frac{1}{\tau^2}\left(\frac{\sum^S (\gamma_i - m)'(\gamma_i - m) + 1}{2}\right)\right\}$$

$$\Rightarrow \quad \tau^2 \sim \text{Inv-Ga}\left(S + \frac{1}{2},\ \frac{\sum^S (\gamma_i - m)'(\gamma_i - m) + 1}{2}\right)$$

- $p(m|-) \propto p(m) \cdot \prod_{i=1}^{S} p(y_i|m, \tau^2)$

$$\propto \exp\left\{-\frac{1}{2\tau^2}\sum^{S}(y_i-m)'(y_i-m)\right\}$$

$$= \exp\left\{-\frac{1}{2\tau^2}\sum^{S}(y_i'y_i - 2y_i'm + m'm)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(\frac{S}{\tau^2}\cdot m'm - \frac{2}{\tau^2}m'\sum^{S}y_i\right)\right\}$$

$$\Rightarrow \quad m|- \sim N(M, K)$$

$$K = \frac{\tau^2}{S} I \quad ; \quad M = K \times \left(\frac{\sum^{S} y_i}{\tau^2}\right)$$