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YOUVA

~~Real Analysis~~

Partial Differentiation

(~~Strong foundation in limits & continuity~~)

* working rule to find the limit:

1. If $f(x,y) = \frac{x+y}{2x-y}$, show that $\lim_{x\to0}\left[\lim_{y\to0} f(x,y)\right] \neq \lim_{y\to0}\left[\lim_{x\to0} f(x,y)\right]$

→ $f(x,y) = \frac{x+y}{2x-y}$

$$\lim_{x\to0}\left[\lim_{y\to0} f(x,y)\right] = \lim_{x\to0}\left(\frac{x}{2x}\right) = \lim_{x\to0}\left(\frac{1}{2}\right) = \frac{1}{2} \quad \text{— (LHS)}$$

$$\lim_{y\to0}\left[\lim_{x\to0} f(x,y)\right] = \lim_{y\to0}\left[\frac{+y}{-y}\right] = \lim_{y\to0}(-1) = -1 \quad \text{— (RHS)}$$

LHS ≠ RHS

$$\therefore \lim_{x\to0}\left[\lim_{y\to0} f(x,y)\right] \neq \lim_{y\to0}\left[\lim_{x\to0} f(x,y)\right]$$

2. Evaluate $\lim_{\substack{x\to0\\y\to0}} \frac{x^2y}{x^4+y^2}$

→ $f_1 = \lim_{x\to0}\left[\lim_{y\to0}\left(\frac{x^2y}{x^4+y^2}\right)\right] = \lim_{x\to0}\left[\frac{0}{x^4}\right] = 0$

$f_2 = \lim_{y\to0}\left[\lim_{x\to0}\left(\frac{x^2y}{x^4+y^2}\right)\right] = \lim_{y\to0}\left[\frac{0}{y^2}\right] = 0$

$y = mx$

$f_3 = \lim_{x\to0}\left[\frac{mx^3}{x^4+m^2x^2}\right] = \lim_{x\to0}\left[\frac{mx}{x^2+m^2}\right] = \frac{0}{m^2} = 0$

$y = mx^2$

$f_4 = \lim_{x\to0}\left[\frac{mx^4}{x^4+m^2x^4}\right] = \lim_{x\to0}\left[\frac{m}{1+m^2}\right] = \frac{m}{1+m^2}$

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$f_1 = f_2 = f_3 \neq f_4$

Hence, limit does not exist.

3. Evaluate $\lim_{\substack{x \to 0 \\ y \to 0}} (x^3 + y^3)$

$\rightarrow$ $f_1 = \lim_{x \to 0} \left[\lim_{y \to 0} (x^3 + y^3) \right] = \lim_{x \to 0} (x^3) = 0$

$f_2 = \lim_{y \to 0} \left[\lim_{x \to 0} (x^3 + y^3) \right] = \lim_{y \to 0} (y^3) = 0$

Put $y = x$

$f_3 = \lim_{x \to 0} \left[2x^3 \right] = 0$

Put $y = mx$

$f_4 = \lim_{x \to 0} \left(x^3 + m^3 x^3 \right) = \lim_{x \to 0} \left[x^3 (1 + m^3) \right] = 0$

Put $y = mx^2$

$f_5 = \lim_{x \to 0} \left(x^3 + m^3 x^6 \right) = \lim_{x \to 0} \left[x^3 (1 + m^3 x^3) \right] = 0$

$f_1 = f_2 = f_3 = f_4 = f_5$

$\therefore$ limit exists with value 0.

4. Evaluate $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{y^2 - x^2}{y^2 + x^2}$, $x \neq 0, y \neq 0$

$\rightarrow$ $f_1 = \lim_{x \to 0} \left[\lim_{y \to 0} \left(\frac{y^2 - x^2}{y^2 + x^2} \right) \right] = \lim_{x \to 0} \left(\frac{-x^2}{x^2} \right) = \lim_{x \to 0} (-1) = -1$

$f_2 = \lim_{y \to 0} \left[\lim_{x \to 0} \left(\frac{y^2 - x^2}{y^2 + x^2} \right) \right] = \lim_{y \to 0} \left(\frac{y^2}{y^2} \right) = \lim_{y \to 0} (1) = 1$

$f_1 \neq f_2$

$\therefore$ Limit does not exist.

5. Evaluate $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 - y^3}{x^2 + y^2}$; $x \neq 0, y \neq 0$.

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$\rightarrow$ $f_1 = \lim_{x \to 0} \left[\lim_{y \to 0} \left(\frac{x^3 - y^3}{x^2 + y^2} \right) \right] = \lim_{x \to 0} \left(\frac{x^3}{x^2} \right) = \lim_{x \to 0} (x) = 0$

$f_2 = \lim_{y \to 0} \left[\lim_{x \to 0} \left(\frac{x^3 - y^3}{x^2 + y^2} \right) \right] = \lim_{y \to 0} \left(-\frac{y^3}{y^2} \right) = \lim_{y \to 0} (-y) = 0$

Put $y = x$

$f_3 = \lim_{x \to 0} \left[\frac{x^3 - x^3}{x^2 + x^2} \right] = \lim_{x \to 0} \left[\frac{0}{2x^2} \right] = 0$

Put $y = mx$

$f_4 = \lim_{x \to 0} \left[\frac{x^3 - m^3 x^3}{x^2 + m^2 x^2} \right] = \lim_{x \to 0} \left[\frac{x(1 - m^3)}{1 + m^2} \right] = 0$

Put $y = mx^2$

$f_5 = \lim_{x \to 0} \left[\frac{x^3 - m^3 x^6}{x^2 + m^2 x^4} \right] = \lim_{x \to 0} \left[\frac{x - m^3 x^4}{1 + m^2 x^2} \right] = \frac{0}{1} = 0$

$\therefore f_1 = f_2 = f_3 = f_4 = f_5$

$\therefore$ Limit exists with value 0.

6. Evaluate $\lim_{\substack{x \to 1 \\ y \to 2}} \frac{x^2 + 2y}{x + y^2}$, $x \neq 0, y \neq 0$.

$\rightarrow$ $f_1 = \lim_{x \to 1} \left[\lim_{y \to 2} \left(\frac{x^2 + 2y}{x + y^2} \right) \right] = \lim_{x \to 1} \left[\frac{x^2 + 4}{x + 4} \right] = \frac{1 + 4}{1 + 4} = 1$

$f_2 = \lim_{y \to 2} \left[\lim_{x \to 1} \left(\frac{x^2 + 2y}{x + y^2} \right) \right] = \lim_{y \to 2} \left[\frac{1 + 2y}{1 + y^2} \right] = \frac{1 + 4}{1 + 4} = 1$

$\therefore$ limit exists with value 1.

7. Evaluate $\lim_{\substack{x \to \infty \\ y \to 3}} \frac{2x - 3}{x^3 + 4y^3}$

$\rightarrow$ $f_1 = \lim_{x \to \infty} \left[\lim_{y \to 3} \left(\frac{2x - 3}{x^3 + 4y^3} \right) \right] = \lim_{x \to \infty} \left[\frac{2x - 3}{x^3 + 108} \right]$

$= \lim_{x \to \infty} \left[\frac{\frac{2}{x^2} - \frac{3}{x^3}}{1 + \frac{108}{x^3}} \right] = \frac{0}{1} = 0$

$$f_2 = \lim_{y\to 3}\left[\lim_{x\to\infty}\left(\frac{2x-3}{x^3+4y^3}\right)\right] = \lim_{y\to 3}\left[\lim_{x\to\infty}\left(\frac{\frac{2}{x^2}-\frac{3}{x^3}}{1+\frac{4y^3}{x^3}}\right)\right]$$

$$= \lim_{y\to 3}\left[\frac{0}{1}\right] = 0$$

$f_1 = f_2$

∴ limit exists with value 0.

8. Evaluate $\lim\limits_{\substack{x\to\infty \\ y\to 2}} \dfrac{xy+4}{x^2+2y^2}$

→ $$f_1 = \lim_{x\to\infty}\left[\lim_{y\to 2}\left(\frac{xy+4}{x^2+2y^2}\right)\right] = \lim_{x\to\infty}\left[\frac{2x+4}{x^2+8}\right]$$

$$= \lim_{x\to\infty}\left[\frac{\frac{2}{x}+\frac{4}{x^2}}{1+\frac{8}{x^2}}\right] = 0$$

$$f_2 = \lim_{y\to 2}\left[\lim_{y\to\infty}\left(\frac{xy+4}{x^2+2y^2}\right)\right] = \lim_{y\to 2}\left[\lim_{x\to\infty}\left(\frac{\frac{y}{x}+\frac{4}{x^2}}{1+\frac{2y^2}{x^2}}\right)\right] = 0$$

$f_1 = f_2$

∴ Limit exists with value 0.

• 9. Exercise:

⇒ Evaluate the foll limits:

1. $\lim\limits_{\substack{x\to 1 \\ y\to 2}} \dfrac{2x^2+y^2}{2xy}$

→ $$f_1 = \cancel{\lim_{x\to\infty}\left[\lim_{y\to 2}\left(\frac{xy+4}{x^2+2y^2}\right)\right]} = \cancel{\lim_{x\to\infty}\left[\frac{2x+4}{x^2+8}\right]}$$

$$\cancel{= \lim_{x\to\infty} \left[\frac{\frac{2}{x} + \frac{4}{x^2}}{1+\frac{8}{x^2}} \right] = 0}$$

$$\cancel{f_2 = \lim_{y\to 2} \left[\lim_{x\to\infty} \left(\frac{xy+4}{x^2+2y^2} \right) \right]} = \cancel{\lim_{y\to 2} \left[\lim_{x\to\infty} \left(\frac{\frac{y}{x} + \frac{4}{x^2}}{1+\frac{2y^2}{x^2}} \right) \right] = 0}$$

$\cancel{f_1 = f_2}$

$$f_1 = \lim_{x\to 1} \left[\lim_{y\to 2} \left(\frac{2x^2+y^2}{2xy} \right) \right] = \lim_{x\to 1} \left[\frac{2x^2+4}{4x} \right] = \frac{2+4}{4} = \frac{3}{2}$$

$$f_2 = \lim_{y\to 2} \left[\lim_{x\to 1} \left(\frac{2x^2+y^2}{2xy} \right) \right] = \lim_{y\to 2} \left[\frac{2+y^2}{2y} \right] = \frac{6}{4} = \frac{3}{2}$$

$f_1 = f_2$

$\therefore$ limit exists with value $\frac{3}{2}$.

2. $$\lim_{\substack{x\to 2 \\ y\to 3}} \frac{x^3+y^2}{x^2-y}$$

$\rightarrow$ $$f_1 = \lim_{x\to 2} \left[\lim_{y\to 3} \left(\frac{x^3+y^2}{x^2-y} \right) \right] = \lim_{x\to 2} \left[\frac{x^3+9}{x^2-3} \right] = \frac{8+9}{4-3} = 17$$

$$f_2 = \lim_{y\to 3} \left[\lim_{x\to 2} \left(\frac{x^3+y^2}{x^2-y} \right) \right] = \lim_{y\to 3} \left[\frac{8+y^2}{4-y} \right] = \frac{8+9}{4-3} = 17$$

$\therefore$ limit exists with the value 17.

3. $$\lim_{\substack{x\to \infty \\ y\to 3}} \frac{2xy-3}{x^3+4y^3}$$

$\rightarrow$ $$f_1 = \lim_{x\to\infty} \left[\lim_{y\to 3} \left(\frac{2xy-3}{x^3+4y^3} \right) \right] = \lim_{x\to\infty} \left[\frac{6x-3}{x^3+108} \right]$$

$$= \lim_{x\to\infty} \left[\frac{\frac{6}{x^2} - \frac{3}{x^3}}{1+\frac{108}{x^3}} \right] = 0$$

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$$f_2 = \lim_{y\to 3}\left[\lim_{x\to\infty}\left(\frac{2xy-3}{x^3+4y^3}\right)\right] = \lim_{y\to 3}\left[\lim_{x\to\infty}\left(\frac{\frac{2y}{x^2}-\frac{3}{x^3}}{1+\frac{4y^3}{x^3}}\right)\right] = 0$$

$f_1 = f_2$

$\therefore$ limit exists with value 0.

4. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{xy}{y-x^2}$; $x \neq 0, y \neq 0$

$\rightarrow$ $$f_1 = \lim_{x\to 0}\left[\lim_{y\to 0}\left(\frac{xy}{y-x^2}\right)\right] = \lim_{x\to 0}\left[\frac{0}{-x^2}\right] = 0$$

$$f_2 = \lim_{y\to 0}\left[\lim_{x\to 0}\left(\frac{xy}{y-x^2}\right)\right] = \lim_{y\to 0}\left[\frac{0}{y}\right] = 0$$

Put $y = x$

$$f_3 = \lim_{x\to 0}\left[\frac{x^2}{x-x^2}\right] = \lim_{x\to 0}\left[\frac{x}{1-x}\right] = 0$$

Put $y = mx$

$$f_4 = \lim_{x\to 0}\left[\frac{mx^2}{mx-x^2}\right] = \lim_{x\to 0}\left[\frac{mx}{m-x}\right] = 0$$

Put $y = mx^2$

$$f_5 = \lim_{x\to 0}\left[\frac{mx^3}{mx^2-x^2}\right] = \lim_{x\to 0}\left[\frac{mx}{m-1}\right] = 0$$

Put $y = x^2$

$$f_6 = \lim_{x\to 0}\left[\frac{x^3}{x^2-x^2}\right] = \lim_{x\to 0}\left[\frac{x^3}{0}\right] = \text{undefined}$$

$\therefore$ limit does not exist.

5. $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x-y}{x^2+y^2}$; $x \neq 0, y \neq 0$

$\rightarrow$ $$f_1 = \lim_{x\to 0}\left[\lim_{y\to 0}\left(\frac{x-y}{x^2+y^2}\right)\right] = \lim_{x\to 0}\left(\frac{x}{x^2}\right) = \lim_{x\to 0}\left(\frac{1}{x}\right) = \text{undefined}$$

$$f_2 = \lim_{y\to 0}\left[\lim_{x\to 0}\left(\frac{x-y}{x^2+y^2}\right)\right] = \lim_{y\to 0}\left(\frac{-y}{y^2}\right) = \lim_{y\to 0}\left(\frac{-1}{y}\right) = \text{undefined}$$

Put $y = x$

$$f_3 = \lim_{x\to 0}\left(\frac{0}{2x^2}\right) = 0$$

$f_1 = f_2 \neq f_3$

$\therefore$ Limit does not exist.

6. $$\lim_{\substack{x\to 1\\ y\to 1}} \frac{xy-2x}{xy-2y}$$

$\rightarrow$ $$f_1 = \lim_{x\to 1}\left[\lim_{y\to 1}\left(\frac{xy-2x}{xy-2y}\right)\right] = \lim_{x\to 1}\left[\frac{x-2x}{x-2}\right] = \frac{1-2}{1-2} = 1$$

$$f_2 = \lim_{y\to 1}\left[\lim_{x\to 1}\left(\frac{xy-2x}{xy-2y}\right)\right] = \lim_{y\to 1}\left(\frac{y-2}{y-2y}\right) = \frac{1-2}{1-2} = 1$$

$f_1 = f_2$

$\therefore$ Limit exists with value 1.

7. $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3+2y^3}{x^2+4y^2} \quad x\neq 0,\ y\neq 0$$

$\rightarrow$ $$f_1 = \lim_{x\to 0}\left[\lim_{y\to 0}\left(\frac{x^3+2y^3}{x^2+4y^2}\right)\right] = \lim_{x\to 0}\left(\frac{x^3}{x^2}\right) = 0$$

$$f_2 = \lim_{y\to 0}\left[\lim_{x\to 0}\left(\frac{x^3+2y^3}{x^2+4y^2}\right)\right] = \lim_{y\to 0}\left[\frac{2y^3}{4y^2}\right] = \frac{0}{2} = 0$$

Put $x = 2y$

$$f_3 = \lim_{y\to 0}\left(\frac{8y^3+2y^3}{4y^2+4y^2}\right) = \lim_{y\to 0}\left(\frac{5y}{4}\right) = 0$$

Put $y = x$

$$f_4 = \lim_{x\to 0}\left(\frac{x^3+2x^3}{x^2+4x^2}\right) = \lim_{x\to 0}\left(\frac{2x}{5}\right) = 0$$

Put $y = mx$

$$f_5 = \lim_{x\to 0}\left(\frac{x^3+2m^3x^3}{x^2+4m^2x^2}\right) = \lim_{x\to 0}\left[\left(\frac{1+2m^3}{1+4m^2}\right)x\right] = 0$$

Put $y = mx^2$

$$f_6 = \lim_{x\to 0}\left(\frac{x^3+2m^3x^6}{x^2+4m^2x^4}\right) = \lim_{x\to 0}\left(\frac{x+2m^3x^4}{1+4m^2x^2}\right) = \frac{0}{1} = 0$$

$\therefore f_1 = f_2 = f_3 = f_4 = f_5 = f_6$

$\therefore$ limit exists with value 0.

8. $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^3}{x^2+y^2}$ $x \neq 0,\ y \neq 0$

$\rightarrow$ $$f_1 = \lim_{x\to 0}\left[\lim_{y\to 0}\left(\frac{x^2y^3}{x^2+y^2}\right)\right] = \lim_{x\to 0}\left[\frac{0}{x^2}\right] = 0$$

$$f_2 = \lim_{y\to 0}\left[\lim_{x\to 0}\left(\frac{x^2y^3}{x^2+y^2}\right)\right] = \lim_{y\to 0}\left[\frac{0}{y^2}\right] = 0$$

Put $y = x$

$$f_3 = \lim_{x\to 0}\left[\frac{x^5}{2x^2}\right] = \lim_{x\to 0}\left[\frac{x^3}{2}\right] = 0$$

Put $y = mx$

$$f_4 = \lim_{x\to 0}\left[\frac{m^3x^5}{x^2+m^2x^2}\right] = \lim_{x\to 0}\left[\frac{m^3x^3}{1+m2}\right] = 0$$

Put $y = mx^2$

$$f_5 = \lim_{x\to 0}\left[\frac{m^3x^8}{x^2+m^2x^4}\right] = \lim_{x\to 0}\left[\frac{m^3x^6}{1+m^2x^2}\right] = \frac{0}{1} = 0$$

$\therefore f_1 = f_2 = f_3 = f_4 = f_5$

$\therefore$ limit ~~does~~ exists with value 0.

9. $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy+2}{x^2+y^2}$, $x \neq 0,\ y \neq 0$

$\rightarrow$ $$f_1 = \lim_{x\to 0}\left[\lim_{y\to 0}\left(\frac{xy+2}{x^2+y^2}\right)\right] = \lim_{x\to 0}\left(\frac{2}{x^2}\right) = \text{undefined}$$

~~Put $y = x$~~

$$f_2 = \lim_{y\to 0}\left[\lim_{x\to 0}\left(\frac{xy+2}{x^2+y^2}\right)\right] = \lim_{y\to 0}\left(\frac{2}{y^2}\right) = \text{undefined}$$

Put $y=x$

$$f_3 = \lim_{x\to 0}\left[\frac{x^2+2}{2x^2}\right] = \frac{2}{0} = \text{undefined}$$

Put $y=mx$

$$f_4 = \lim_{x\to 0}\left[\frac{mx^2+2}{x^2+m^2x^2}\right] = \frac{2}{0} = \text{undefined}$$

Put $y=mx^2$

$$f_5 = \lim_{x\to 0}\left[\frac{mx^3+2}{x^2+m^2x^4}\right] = \frac{2}{0} = \text{undefined}$$

$\therefore$ limit doesnt exist as a finite number.

$\Rightarrow$ Choose the correct alternative:

10. The value of $\lim_{(x,y)\to(0,0)} (x+y)\sin\frac{1}{(x+y)}$, $x\neq 0, y\neq 0$ is

(a) limit does not exist $\quad$ ✓(b) 0

(c) 1 $\quad$ (d) -1

$\rightarrow$ $$f_1 = \lim_{x\to 0}\left[\lim_{y\to 0}\left[(x+y)\sin\frac{1}{(x+y)}\right]\right] = \lim_{x\to 0}\left[x\sin\frac{1}{x}\right] = 0$$

$$f_2 = \lim_{y\to 0}\left[\lim_{x\to 0}\left[(x+y)\sin\frac{1}{(x+y)}\right]\right] = \lim_{y\to 0}\left[y\sin\frac{1}{y}\right] = 0$$

Put $y=x$

$$f_3 = \lim_{x\to 0}\left[2x\sin\frac{1}{2x}\right] = 0$$

Put $y=mx$

$$f_4 = \lim_{x\to 0}\left[(1+m)x\sin\left[\frac{1}{(1+m)x}\right]\right] = 0$$

Put $y=mx^2$

$$f_5 = \lim_{x\to 0}\left[(1+mx)x\sin\left[\frac{1}{(1+mx)x}\right]\right] = 0$$

$\because f_1 = f_2 = f_3 = f_4 = f_5$

$\therefore$ Limit exists with value 0.

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11. The value of the $\lim_{(x,y)\to(0,0)} \frac{x+\sqrt{y}}{\sqrt{(x^2+y)}}$, $x \neq 0, y \neq 0$ is

(a) Limit does not exist (b) 0

(c) 1 (d) -1

→ $f_1 = \lim_{x\to 0}\left[\lim_{y\to 0}\left(\frac{x+\sqrt{y}}{\sqrt{(x^2+y)}}\right)\right] = \lim_{x\to 0}\left[\frac{x}{x}\right] = 1$

$f_2 = \lim_{y\to 0}\left[\lim_{x\to 0}\left(\frac{x+\sqrt{y}}{\sqrt{(x^2+y)}}\right)\right] = \lim_{y\to 0}\left[\frac{\sqrt{y}}{\sqrt{y}}\right] = 1$

Put $y = x^2$

$f_3 = \lim_{x\to 0}\left[\frac{x+x}{\sqrt{x^2+x^2}}\right] = \lim_{x\to 0}\left[\frac{2}{\sqrt{2}}\right] = \sqrt{2}$

$f_1 = f_2 \neq f_3$

∴ Limit does not exist.

★ Working Rule for Continuity at a point (a,b):

9. Test the fun^c $f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2}, & \text{when } x\neq 0, y\neq 0 \\ 0, & \text{when } x=0, y=0 \end{cases}$

→ $f_1 = \lim_{x\to 0}\left[\lim_{y\to 0}\left(\frac{x^3-y^3}{x^2+y^2}\right)\right] = \lim_{x\to 0}\left[\frac{x^3}{x^2}\right] = \lim_{x\to 0}(x) = 0$

$f_2 = \lim_{y\to 0}\left[\lim_{x\to 0}\left(\frac{x^3-y^3}{x^2+y^2}\right)\right] = \lim_{y\to 0}\left(\frac{-y^3}{y^2}\right) = \lim_{y\to 0}(-y) = 0$

Put $y = x$

$f_3 = \lim_{x\to 0}\left[\frac{x^3-x^3}{x^2+x^2}\right] = \lim_{x\to 0}(0) = 0$

Put $y = mx$

$f_4 = \lim_{x\to 0}\left[\frac{x^3-m^3x^3}{x^2+m^2x^2}\right] = \lim_{x\to 0}\left[\frac{x-m^3x}{1+m^2}\right] = 0$

Put $y = mx^2$

$f_5 = \lim_{x\to 0}\left[\frac{x^3-m^3x^6}{x^2+m^2x^4}\right] = \lim_{x\to 0}\left[\frac{x-m^3x^4}{1+m^2}\right] = 0$

∴ $f_1 = f_2 = f_3 = f_4 = f_5$

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$\therefore$ limit exists with value 0.

$\therefore$ ~~limit~~ $f(x,y)$ is continuous at origin $(0,0)$.

10. Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{x}{\sqrt{x^2+y^2}}, & x \neq 0, y \neq 0 \\ 2, & x=0, y=0 \end{cases}$

at the origin.

$\rightarrow$ $f_1 = \lim\limits_{x \to 0} \left[\lim\limits_{y \to 0} \left(\dfrac{x}{\sqrt{x^2+y^2}} \right) \right] = \lim\limits_{x \to 0} \left(\dfrac{x}{x} \right) = 1$

$f_2 = \lim\limits_{y \to 0} \left[\lim\limits_{x \to 0} \left(\dfrac{x}{\sqrt{x^2+y^2}} \right) \right] = \lim\limits_{y \to 0} \left(\dfrac{0}{y} \right) = 0$

$f_1 \neq f_2$

$\therefore$ limit does not exist.

$\therefore$ $f(x,y)$ is not continuous at origin $(0,0)$.

★ Types of Discontinuity:

11. Show that the given func are discontinuous at all the point $(2,-2)$.

$\rightarrow$ $f(x,y) = \begin{cases} \dfrac{x^2+xy+x+y}{x+y}, & (x,y) \neq (2,-2) \\ 4, & (x,y) = (2,-2) \end{cases}$

$\rightarrow$ $f_1 = \lim\limits_{x \to 2} \left[\lim\limits_{y \to -2} \left(\dfrac{x^2+xy+x+y}{x+y} \right) \right] = \lim\limits_{x \to 2} \left[\dfrac{x^2-2x+x-2}{x-2} \right]$

$= \lim\limits_{x \to 2} \left[\dfrac{x^2-x-2}{x-2} \right] = \lim\limits_{x \to 2} \left[\dfrac{(x-2)(x+1)}{x-2} \right] = \lim\limits_{x \to 2} (x+1)$

$f_1 = 3$

$f_1 \neq 4$

$\therefore$ $f(x,y)$ is discontinuous at $(2,-2)$.

• Exercise 1.2:

$\Rightarrow$ Test for continuity:

1. $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{when } x \neq 0, y \neq 0 \\ 0, & \text{when } x=0, y=0 \end{cases}$ at origin.

$\rightarrow$ $f_1 = \lim\limits_{x \to 0}\left[\lim\limits_{y \to 0}\left(\dfrac{xy(x^2-y^2)}{x^2+y^2}\right)\right] = \lim\limits_{x \to 0}\left(\dfrac{0}{x^2}\right) = 0$

$f_2 = \lim\limits_{y \to 0}\left[\lim\limits_{x \to 0}\left(\dfrac{xy(x^2-y^2)}{x^2+y^2}\right)\right] = \lim\limits_{y \to 0}\left(\dfrac{0}{y^2}\right) = 0$

Put $y = x$

$f_3 = \lim\limits_{x \to 0}\left[\dfrac{x^2(0)}{2x^2}\right] = 0$

Put $y = mx$

$f_4 = \lim\limits_{x \to 0}\left[\dfrac{mx^2(x^2-m^2x^2)}{x^2+m^2x^2}\right] = \lim\limits_{x \to 0}\left[\dfrac{mx^2(1-m^2)}{1+m^2}\right] = 0$

Put $y = mx^2$

$f_5 = \lim\limits_{x \to 0}\left[\dfrac{mx^3(x^2-m^2x^4)}{x^2+m^2x^4}\right] = \lim\limits_{x \to 0}\left[\dfrac{mx^3(1-m^2x^2)}{1+m^2x^2}\right] = 0$

$f_1 = f_2 = f_3 = f_4 = f_5 = 0$

$\therefore f(x,y)$ is continuous at origin $(0,0)$.

2. $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & \text{when } x \neq 0, y \neq 0 \\ 0, & \text{when } x=0, y=0 \end{cases}$ at origin.

$\rightarrow$ $f_1 = \lim\limits_{x \to 0}\left[\lim\limits_{y \to 0}\left(\dfrac{x^2-y^2}{x^2+y^2}\right)\right] = \lim\limits_{x \to 0}\left(\dfrac{x^2}{x^2}\right) = 1$

$f_1 \neq 0$

$\therefore f(x,y)$ is discontinuous at origin $(0,0)$.

3. $f(x,y) = \begin{cases} \dfrac{x^3y^3}{x^3+y^3}, & \text{when } x \neq 0, y \neq 0 \\ 0, & \text{when } x=0, y=0 \end{cases}$ at origin.

$\rightarrow$ $f_1 = \lim\limits_{x \to 0}\left[\lim\limits_{y \to 0}\left(\dfrac{x^3y^3}{x^3+y^3}\right)\right] = \lim\limits_{x \to 0}\left(\dfrac{0}{x^3}\right) = 0$

$f_2 = \lim\limits_{y \to 0}\left[\lim\limits_{x \to 0}\left(\dfrac{x^3y^3}{x^3+y^3}\right)\right] = \lim\limits_{y \to 0}\left[\dfrac{0}{y^3}\right] = 0$

Put $y=x$

$$f_3 = \lim_{x\to 0}\left[\frac{x^6}{2x^3}\right] = \lim_{x\to 0}\left[\frac{x^3}{2}\right] = 0$$

Put $y=mx$

$$f_4 = \lim_{x\to 0}\left[\frac{m^3x^6}{\cancel{m^3}+x^3+m^3x^3}\right] = \lim_{x\to 0}\left[\frac{m^3x^3}{1+m^3}\right] = 0$$

Put $y=mx^2$

$$f_5 = \lim_{x\to 0}\left[\frac{mx^9}{\cancel{x}^3+m^3x^6}\right] = \lim_{x\to 0}\left[\frac{mx^6}{1+m^3x^3}\right] = 0$$

$f_1 = f_2 = f_3 = f_4 = f_5 = 0$

$\therefore f(x,y)$ is continuous at origin.

=0

4. $f(x,y) = \begin{cases} x^3+y^3, & \text{when } x\neq 0, y\neq 0 \\ 0, & \text{when } x=0, y=0 \end{cases}$ at origin

x^2)] =0 →

$$f_1 = \lim_{x\to 0}\left[\lim_{y\to 0}(x^3+y^3)\right] = \lim_{x\to 0}(x^3) = 0$$

$$f_2 = \lim_{y\to 0}\left[\lim_{y\to 0}(x^3+y^3)\right] = \lim_{y\to 0}(y^3) = 0$$

Put $y=x$

$$f_3 = \lim_{x\to 0}\left[2x^3\right] = 0$$

Put $y=mx$

$$f_4 = \lim_{x\to 0}\left[x^3+m^3x^3\right] = \lim_{x\to 0}\left[x^3(1+m^3)\right] = 0$$

Put $y=mx^2$

$$f_5 = \lim_{x\to 0}\left[x^3+m^3x^6\right] = \lim_{x\to 0}\left[x^3(1+m^3x^3)\right] = 0$$

$f_1 = f_2 = f_3 = f_4 = f_5 = 0$

$\therefore f(x,y)$ is continuous at origin.

5. $f(x,y) = \begin{cases} \dfrac{x^2+2y}{x+y^2} & \\ 1 & \text{when } x=1, y=2 \end{cases}$ at the point $(1,2)$

$\rightarrow$ $f_1 = \lim_{x\to 1}\left[\lim_{y\to 2}\left(\frac{x^2+2y}{x+y^2}\right)\right] = \lim_{x\to 1}\left[\frac{x^2+4}{x+4}\right] = 1$

$f_2 = \lim_{y\to 2}\left[\lim_{x\to 1}\left(\frac{x^2+2y}{x+y^2}\right)\right] = \lim_{y\to 2}\left[\frac{1+2y}{1+y^2}\right] = 1$

$f_1 = f_2 = 1$

$\therefore f(x,y)$ is continuous at $(1,2)$

6. Show that the func $f(x,y) = \begin{cases} 2x^2+y, & (x,y) \neq (1,2) \\ 0, & (x,y) = (1,2) \end{cases}$

is discontinuous at $(1,2)$

$\rightarrow$ $f_1 = \lim_{x\to 1}\left[\lim_{y\to 2}\left(2x^2+y\right)\right] = \lim_{x\to 1}\left[2x^2+2\right] = 4$

$f_1 \neq 0$

$\therefore f(x,y)$ is discontinuous at $(1,2)$

★ Partial Derivatives:

12. If $z(x+y) = x^2+y^2$, show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$\rightarrow$ $z = \frac{x^2+y^2}{x+y}$

$$\frac{\partial z}{\partial x} = \frac{(x+y)(2x) - (x^2+y^2)(1)}{(x+y)^2}$$

$$= \frac{2x^2+2xy-x^2-y^2}{(x+y)^2}$$

$$\therefore \frac{\partial z}{\partial x} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)(2y) - (x^2+y^2)(1)}{(x+y)^2}$$

$$= \frac{2xy+2y^2-x^2-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{y^2+2xy-x^2}{(x+y)^2}$$

$$LHS = \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2$$

$$= \left[\left(\frac{x^2+2xy-y^2}{(x+y)^2}\right) - \left(\frac{y^2+2xy-x^2}{(x+y)^2}\right)\right]^2$$

$$= \frac{(2x^2-2y^2)^2}{(x+y)^4}$$

$$= \frac{4(x+y)^2(x-y)^2}{(x+y)^4}$$

$$LHS = 4\left(\frac{x-y}{x+y}\right)^2$$

$$RHS = 4\left(1-\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$$= 4\left[1-\left(\frac{x^2+2xy-y^2}{(x+y)^2}\right) - \left(\frac{y^2+2xy-x^2}{(x+y)^2}\right)\right]$$

$$= 4\left[\frac{x^2+2xy+y^2-x^2-2xy+y^2-y^2-2xy+x^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2-2xy+y^2}{(x+y)^2}\right]$$

$$RHS = 4\left(\frac{x-y}{x+y}\right)^2$$

$\therefore LHS = RHS$

$$\therefore \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1-\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

13. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

$\rightarrow$ $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)$$

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~~$\therefore \frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right) + \frac{1}{1+}$~~

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{-x}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)$$

$$\therefore \frac{\partial u}{\partial y} = \frac{-x}{y(\sqrt{y^2-x^2})} + \frac{x}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} - \frac{xy}{y\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$$= \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$$= 0$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

14. Find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$ if $u = e^{r\cos\theta}.\cos(r\sin\theta)$

→ $u = e^{r\cos\theta}.\cos(r\sin\theta)$

$$\frac{\partial u}{\partial r} = -e^{r\cos\theta}\sin(r\sin\theta)\cdot\sin\theta + \cos(r\sin\theta)\,e^{r\cos\theta}(\cos\theta)$$

$$= e^{r\cos\theta}[\cos(r\sin\theta)\cos\theta - \sin(r\sin\theta)\sin\theta]$$

$$\therefore \partial u/\partial r = e^{r\cos\theta}[\cos(r\sin\theta+\theta)]$$

$$\frac{\partial u}{\partial \theta} = -e^{r\cos\theta}.\sin(r\sin\theta)\,r\cos\theta + \cos(r\sin\theta)\,e^{r\cos\theta}(-r\sin\theta)$$

$$= -re^{r\cos\theta}[\sin(r\sin\theta)\cos\theta + \cos(r\sin\theta)\cdot\sin\theta]$$

$$\therefore \frac{\partial u}{\partial \theta} = -re^{r\cos\theta}[\sin(r\sin\theta+\theta)]$$

15. If $u = (1-2xy+y^2)^{-1/2}$ prove that, $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3$

$\rightarrow$ $u = (1-2xy+y^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2y)$$

$$\therefore \frac{\partial u}{\partial x} = y(1-2xy+y^2)^{-3/2}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2x+2y)$$

$$\therefore \frac{\partial u}{\partial y} = (x-y)(1-2xy+y^2)^{-3/2}$$

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xy(1-2xy+y^2)^{-3/2} + (y^2-xy)(1-2xy+y^2)^{-3/2}$$

$$= u^3[xy+y^2-xy]$$

$$\therefore x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = u^3y^2$$

16. If $z = e^{ax+by} \cdot f(ax-by)$. prove that

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$$

$\rightarrow$ $z = e^{ax+by} f(ax-by)$

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax-by)(a) + f(ax-by)e^{ax+by}(a)$$

$$\therefore \frac{\partial z}{\partial x} = ae^{ax+by}[f'(ax-by) + f(ax-by)]$$

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax-by)(-b) + f(ax-by)e^{ax+by}(b)$$

$$\therefore \frac{\partial z}{\partial y} = be^{ax+by}[f(ax-by) - f'(ax-by)]$$

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = ab\,e^{ax+by}[f'(ax-by) - f(ax-by)]$$

$$+ abe^{ax+by}[f(ax-by) - f'(ax-by)]$$

$$= abe^{ax+by}[2f(ax-by)]$$

$$= 2abz$$

$$\therefore b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$$

★ Partial Derivatives of Higher Orders:

17. Prove that $y = f(x+at) + g(x-at)$ satisfies
$$\frac{\partial^2 y}{\partial t^2} = a^2\left(\frac{\partial^2 y}{\partial x^2}\right)$$

$\rightarrow$ $y = f(x+at) + g(x-at)$

$$\frac{\partial y}{\partial t} = f'(x+at)(a) + g'(x-at)(-a)$$

$$\frac{\partial y}{\partial t} = a\left[f'(x+at) - g'(x-at)\right]$$

$$\frac{\partial^2 y}{\partial t^2} = a\left[f''(x+at)(a) - g''(x-at)(-a)\right]$$

$$\therefore \frac{\partial^2 y}{\partial t^2} = a^2\left[f''(x+at) + g''(x-at)\right]$$

$$\frac{\partial y}{\partial x} = f'(x+at) + g'(x-at)$$

$$\frac{\partial^2 y}{\partial x^2} = f''(x+at) + g''(x-at)$$

$$a^2\frac{\partial^2 y}{\partial x^2} = a^2\left[f''(x+at) + g''(x-at)\right]$$

$$\therefore \frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 x}{\partial t^2}$$

~~z = tan(y+a~~

18. If $z = \tan(y+ax) + (y-ax)^{3/2}$, then show that
$$\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}.$$

$\rightarrow$ $z = \tan(y+ax) + (y-ax)^{3/2}$

$$\frac{\partial z}{\partial x} = \sec^2(y+ax)(a) + \frac{3}{2}(y-ax)^{1/2}(-a)$$

$$= a\left[\sec^2(y+ax) - \frac{3}{2}(y-ax)^{1/2}\right]$$

$$\frac{\partial^2 z}{\partial x^2} = a\left[2\sec^2(y+ax)\tan(y+ax)(a) - \frac{3}{2}\left(\frac{1}{2}\right)(y-ax)^{-1/2}(-a)\right]$$

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$$= a^2\left[2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2}\right]$$

$$\frac{\partial z}{\partial y} = \sec^2(y+ax) + \frac{3}{2}(y-ax)^{1/2}$$

$$\frac{\partial^2 z}{\partial y^2} = 2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2}$$

$$a^2\frac{\partial^2 z}{\partial y^2} = a^2\left[2\sec^2(y+ax)\tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2}\right]$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$$

19. If $u = e^{xyz}$, find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$.

→ $u = e^{xyz}$

$$\frac{\partial u}{\partial z} = e^{xyz}(xy)$$

$$\frac{\partial^2 u}{\partial y \partial z} = e^{xyz}(x) + (xy)e^{xyz}(xz)$$

$$= xe^{xyz} + x^2yze^{xyz}$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = xe^{xyz}(yz) + e^{xyz} + x^2yze^{xyz}(yz) + e^{xyz}(2xyz)$$

$$= xyze^{xyz} + e^{xyz} + x^2y^2z^2e^{xyz} + 2xyze^{xyz}$$

$$= e^{xyz}(xyz + 1 + x^2y^2z^2 + 2xyz)$$

$$= e^{xyz}(1 + 3xyz + x^2y^2z^2)$$

20. If $z = x\log(x+r) - r$, where $r^2 = x^2+y^2$

Prove that

(i) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{x+r}$ (ii) $\frac{\partial^3 z}{\partial x^3} = -\left(\frac{x}{r^3}\right)$

→ (i) $z = x\log(x+r) - r$

$r^2 = x^2 + y^2$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial z}{\partial x} = \frac{x}{x+r}\left(1+\frac{\partial r}{\partial x}\right) + \log(x+r) - \frac{\partial r}{\partial x}$$

$$= \frac{x}{x+r}\frac{\partial r}{\partial x} + \frac{x}{x+r} + \log(x+r) - \frac{\partial r}{\partial x}$$

$$= \frac{-r}{x+r}\frac{\partial r}{\partial x} + \frac{x}{x+r} + \log(x+r)$$

$$= \frac{-r}{x+r}\left(\frac{x}{r}\right) + \frac{x}{x+r} + \log(x+r)$$

$$= \frac{-x}{x+r} + \frac{x}{x+r} + \log(x+r)$$

$$= \log(x+r)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{x+r}\left[1+\frac{\partial r}{\partial x}\right]$$

$$= \frac{1}{x+r}\left[1+\frac{x}{r}\right]$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{r}$$

$$\frac{\partial z}{\partial y} = \frac{x}{x+r}\left(\frac{\partial r}{\partial y}\right) - \frac{\partial r}{\partial y}$$

$$= \left(\frac{-r}{x+r}\right)\frac{\partial r}{\partial y}$$

$$= \frac{-r}{x+r}\left(\frac{y}{r}\right)$$

$$= \frac{-y}{x+r}$$

$$\frac{\partial^2 z}{\partial y^2} = -\left[\frac{(x+r)(1) - (y)\left(\frac{\partial r}{\partial y}\right)}{(x+r)^2}\right]$$

$$= -\left[\frac{(x+r) - y^2/r}{(x+r)^2}\right]$$

$$= - \left[\frac{rx + r^2 - y^2}{r(x+r)^2} \right]$$

$$= - \left[\frac{rx + x^2}{r(x+r)^2} \right]$$

$$= \frac{-x}{r(x+r)}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{r} - \frac{x}{r(x+r)}$$

$$= \frac{1}{r} \left[1 - \frac{x}{x+r} \right]$$

$$= \frac{1}{r} \left[\frac{r}{x+r} \right]$$

$$= \frac{1}{x+r}$$

(iii) $$\frac{\partial^3 z}{\partial x^3} = \frac{-1}{r^2} \frac{\partial r}{\partial x}$$

$$= \frac{-x}{r^3}$$

21. Find the value of n so that the equation $v = r^n(3\cos^2\theta - 1)$ satisfies the relation.

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v}{\partial \theta} \right) = 0$$

$\longrightarrow$ $v = r^n(3\cos^2\theta - 1)$

$$\frac{\partial v}{\partial r} = (3\cos^2\theta - 1)\, n r^{n-1}$$

$$= n r^{n-1}(3\cos^2\theta - 1)$$

$$\frac{\partial v}{\partial \theta} = r^n[-6\cos\theta \sin\theta]$$

$$= -6r^n \cos\theta \sin\theta$$

$$\frac{\partial}{\partial r}\left[r^2 \frac{\partial v}{\partial r} \right] + \frac{1}{\sin\theta} \cdot \frac{\partial}{\partial \theta}\left[\sin\theta \cdot \frac{\partial v}{\partial \theta} \right] = 0$$

$$\frac{\partial}{\partial r}\left[r^n n r^{n-1}(3\cos^2\theta-1)\right] + \frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left[\sin\theta \cdot r^n \cdot (-6)\sin\theta\cos\theta\right] = 0$$

$$\therefore (3n\cos^2\theta - n)\frac{\partial}{\partial r}\left[r^{n+1}\right] - \frac{6r^n}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left[\sin^2\theta\cos\theta\right] = 0$$

$$\therefore (3\cos^2\theta-1)n(n+1)r^n - \frac{6r^n}{\sin\theta}\left[-\sin^3\theta + 2\sin\theta\cos^2\theta\right] = 0$$

$$(n^2+n)(3\cos^2\theta-1) = \frac{6}{\sin\theta}\left[2\sin\theta\cos^2\theta - \sin^3\theta\right]$$

$$3n^2\cos^2\theta - n^2 + 3n\cos^2\theta - n = 12\cos^2\theta - 6\sin^2\theta$$

$$= 12\cos^2\theta - 6 + 6\cos^2\theta$$

$$\therefore \cos^2\theta(3n^2+3n) - (n^2+n) = 18\cos^2\theta - 6$$

$$3n^2+3n = 18 \longrightarrow n^2+n = 6$$

$$n^2+n = 6 \longrightarrow n^2+n=6$$

$$n^2+n-6=0$$

$$(n-2)(n+3)=0$$

$$\therefore n = 2 / n = -3$$

22. If $u = (1-2xy+y^2)^{-1/2}$, prove that $\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^2\frac{\partial u}{\partial y}\right] = 0$

$\longrightarrow$ $u = (1-2xy+y^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2y)$$

$$= y(1-2xy+y^2)^{-3/2}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2x+2y)$$

$$= (x-y)(1-2xy+y^2)^{-3/2}$$

$$(1-x^2)\frac{\partial u}{\partial x} = (1-x^2)(y)(1-2xy+y^2)^{-3/2}$$

$$= (y-x^2y)(1-2xy+y^2)^{-3/2}$$

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right] = (y-x^2y)\left(\frac{-3}{2}\right)(1-2xy+y^2)^{-5/2}(-2y)$$

$$+(1-2xy+y^3)^{-3/2}(-2xy)$$

$$= \frac{3y(y-x^2y)}{(1-2xy+y^2)^{5/2}} + \frac{(-2xy)}{(1-2xy+y^2)^{3/2}}$$

$$= \frac{3y^2+3x^2y^2+(-2xy)(1-2xy+y^2)}{(1-2xy+y^2)^{5/2}}$$

$$= \frac{3y^2-3x^2y^2-2xy+4x^2y^2-2xy^3}{(1-2xy+y^2)^{5/2}}$$

$$= \frac{x^2y^2-2xy-2xy^3+3y^2}{(1-2xy+y^2)^{5/2}}$$

$$y^2\frac{\partial u}{\partial y} = y^2(x-y)(1-2xy+y^2)^{-3/2}$$

$$= (xy^2-y^3)(1-2xy+y^2)^{-3/2}$$

$$\frac{\partial}{\partial y}\left[y^2\frac{\partial u}{\partial y}\right] = (xy^2-y^3)\left(\frac{-3}{2}\right)(1-2xy+y^2)^{-5/2}(-2x+2y)$$

$$+(1-2xy+y^2)^{-3/2}(2xy-3y^2)$$

$$=(xy^2-y^3)(-3)(-x+y)(1-2xy+y^2)^{-5/2}+ (2xy-3y^2)(1-2xy+y^2)^{-3/2}$$

$$=(xy^2-y^3)(3x-3y)(1-2xy+y^2)^{-5/2}+ (2xy-3y^2)(1-2xy+y^2)^{-3/2}$$

$$= \frac{(3x^2y^2-3xy^3-3xy^3+3y^4)}{(1-2xy+y^2)^{5/2}} + \frac{2xy-3y^2}{(1-2xy+y^2)^{3/2}}$$

$$= \frac{3x^2y^2-3xy^3-3xy^3+3y^4+(2xy-3y^2)(1-2xy+y^2)}{(1-2xy+y^2)^{5/2}}$$

$$= \frac{3x^2y^2-6xy^3+3y^4+2xy-4x^2y^2+2xy^3-3y^2+6xy^3-3y^4}{(1-2xy+y^2)^{5/2}}$$

$$= \frac{-x^2y^2+2xy^3+2xy-3y^2}{(1-2xy+y^2)^{5/2}}$$

$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^2\frac{\partial u}{\partial y}\right]$$

$$= \frac{x^2y^2-2xy-2xy^3+3y^2-x^2y^2+2xy^3+2xy-3y^2}{(1-2xy+y^2)^{5/2}}$$

$$=0$$

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23. If $x^x y^y z^z = c$, show that at $x=y=z$

$$\frac{\partial^2 z}{\partial x \partial y} = -(x\log ex)^{-1}$$

$\rightarrow$ $x\log x + y\log y + z\log z = \log c$

$x = y = z$

$$1+\log y + (1+\log z)\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\left(\frac{1+\log y}{1+\log z}\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{\left[(1+\log z)(0) - (1+\log y)\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)\right]}{(1+\log z)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(1+\log y)}{z(1+\log z)^2}\frac{\partial z}{\partial x}$$

$$1+\log x + (1+\log z)\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\left(\frac{1+\log x}{1+\log z}\right) \quad \text{——— (2)}$$

Putting (2) in (1)

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log y)}{z(1+\log z)^2}\left(\frac{1+\log x}{1+\log z}\right)$$

$$= -\frac{(1+\log y)(1+\log x)}{z(1+\log z)^3}$$

As $x = y = z$,

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log x)(1+\log x)}{x(1+\log x)^3}$$

$$= \frac{-1}{x(1+\log x)}$$

$$= \frac{-1}{x(\log e + \log x)}$$

$$= \frac{-1}{x\log ex}$$

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = -(x\log ex)^{-1}$$

24. Prove that if $f(x,y) = \frac{1}{\sqrt{y}} e^{-\frac{(x-a)^2}{4y}}$ then $f_{xy}(x,y) = f_{yx}(x,y)$

→ $f(x,y) = \dfrac{e^{-(x-a)^2/4y}}{\sqrt{y}}$

$$f_y = \frac{\sqrt{y}\cdot e^{-(x-a)^2/4y}\dfrac{(x-a)^2}{4y^2} - e^{-(x-a)^2/4y}\cdot\frac{1}{2}\cdot y^{-1/2}}{y}$$

$$= \frac{e^{-(x-a)^2/4y}}{y}\left[\frac{(x-a)^2}{4y^{3/2}} - \frac{1}{2y^{1/2}}\right]$$

$$f_y = \frac{e^{-(x-a)^2/4y}}{4y^{5/2}}\left[(x-a)^2 - 2y\right]$$

$$f_{xy} = \frac{e^{-(x-a)^2/4y}}{4y^{5/2}}\left[2(x-a)\right] + \left[\frac{(x-a)^2 - 2y}{4y^{5/2}}\right] e^{-(x-a)^2/4y}\left(\frac{-2(x-a)}{4y}\right)$$

$$f_{xy} = e^{-(x-a)^2/4y}\left[\frac{2(x-a)}{4y^{5/2}} + \left(\frac{-2(x-a)^3 + 2y\cdot 2\cdot(x-a)}{16y^{5/2}\cdot y}\right)\right]$$

$$= e^{-(x-a)^2/4y}\left[\frac{x-a}{2y^{5/2}} + \frac{4y(x-a) - 2(x-a)^3}{16y^{7/2}}\right]$$

$$= e^{-(x-a)^2/4y}\left[\frac{x-a}{2y^{5/2}} + \frac{2y(x-a) - (x-a)^3}{8y^{7/2}}\right]$$

$$= e^{-(x-a)^2/4y}\left[\frac{4y(x-a) + 2y(x-a) - (x-a)^3}{8y^{7/2}}\right]$$

$$= e^{-(x-a)^2/4y}\left[\frac{6y(x-a) - (x-a)^3}{8y^{7/2}}\right]$$

$$\therefore f_{xy} = \frac{e^{-(x-a)^2/4y}(x-a)}{8y^{7/2}}\left[6y - (x-a)^2\right] \quad \text{———} (1)$$

$$f_x = \frac{\sqrt{y}\left(e^{-(x-a)^2/4y}\right)\dfrac{(-2)(x-a)}{4y} - e^{-(x-a)^2/4y}(0)}{y}$$

$$= -\frac{e^{-(x-a)^2/4y}(x-a)(\sqrt{y})}{2y^2}$$

$$= -\frac{e^{-(x-a)^2/4y}(x-a)}{2y^{3/2}}$$

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$$f_{yx} = -\left\{ \frac{2y^{3/2} e^{-(x-a)^2/4y} (x-a) \frac{(x-a)^2}{4y^2} - (x-a) e^{-(x-a)^2/4y} \cdot (3) \cdot y^{1/2}}{4y^3} \right\}$$

$$= -\frac{e^{-(x-a)^2/4y}}{4y^3} \left[\frac{(x-a)^3}{2y^{1/2}} - \frac{3(x-a)}{y^{-1/2}} \right]$$

$$= +\frac{e^{-(x-a)^2/4y}}{4y^3} \left[\frac{3(x-a)}{y^{-1/2}} - \frac{(x-a)^3}{2y^{1/2}} \right]$$

$$= \frac{e^{-(x-a)^2/4y} (x-a)}{8y^{7/2}} \left[\frac{6y^{1/2}}{y^{-1/2}} - \frac{2(x-a)^2}{2} \right]$$

$$\therefore f_{yx} = \frac{e^{-(x-a)^2/4y} (x-a)}{8y^{7/2}} \left[6y - (x-a)^2 \right] \quad \text{——— (2)}$$

From (1) & (2)

$f_{xy} = f_{yx}$

25. If $u = x^y$, show that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

$\rightarrow$ $u = x^y \rightarrow \log y = y \log x$

$$\frac{1}{u} \frac{\partial u}{\partial y} = \log x$$

$$\frac{\partial u}{\partial y} = x^y \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^y}{x} + \log x . y x^{y-1}$$

$$= x^{y-1} [1 + y \log x]$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = x^{y-1} \left[\frac{y}{x} \right] + (1 + y \log x)(y-1) x^{y-2}$$

$$= y x^{y-2} + (1 + y \log x)(y-1) x^{y-2}$$

$$\therefore \frac{\partial^3 u}{\partial x^2 \partial y} = x^{y-2} [y + (y-1)(1 + y \log x)] \quad \text{——— (1)}$$

$$\frac{\partial u}{\partial x} = y x^{y-1} \longrightarrow \log\left(\frac{\partial u}{\partial x}\right) = \log y + (y-1) \log x$$

$$\frac{1}{\partial u / \partial x} \frac{\partial^2 u}{\partial y \partial x} = \frac{1}{y} + \log x$$

$$\frac{\partial^2 u}{\partial y \partial x} = x^{y-1}[1+y\log x]$$

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = x^{y-1}\left[\frac{y}{x}\right] + \ldots$$

$$\therefore \frac{\partial^3 u}{\partial x \partial y \partial x} = x^{y-2}[y+(y-1)(1+y\log x)] \quad \text{--- (2)}$$

From (1) & (2)

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$$

26. If $u = \log(x^3+y^3+z^3-3xyz)$, show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

$\rightarrow$ $u = \log(x^3+y^3+z^3-3xyz)$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right) u$$

$$= \frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2-3yz)$$

$$= \frac{3}{x^3+y^3+z^3-3xyz}(x^2-yz)$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{3(y^2-xz)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{x^3+y^3+z^3-3xyz}[x^2+y^2+z^2-yz-xz-xy]$$

$$= \frac{3(x^2+y^2+z^2-yz-xz-xy)}{(x^2+y^2+z^2-xy-yz-xz)(x+y+z)}$$

$$= \frac{3}{x+y+z} \quad (\text{say } S)$$