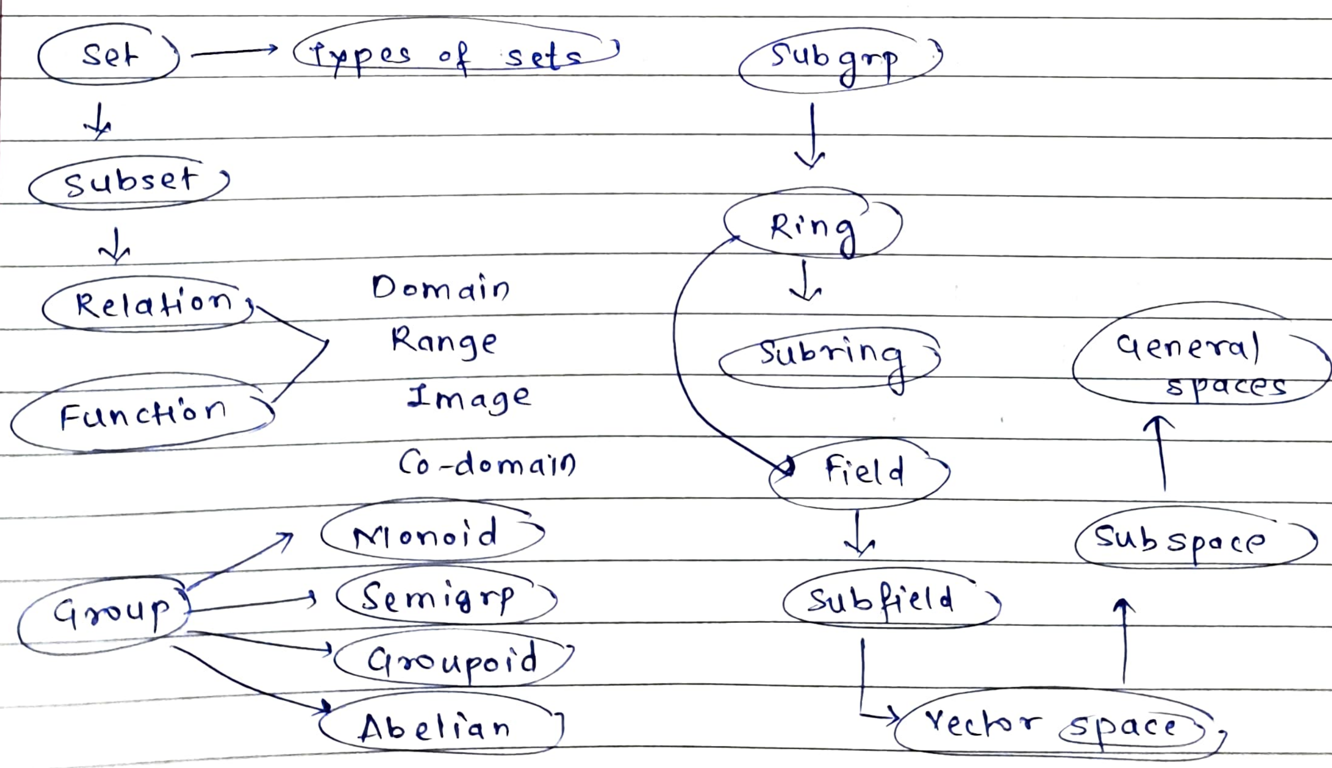
Real Analysis RoadMap

- Point set Topology on $\mathbb{R}$
- Countability of sets
- Sequence and Series of Reals
- Functions
- Limits, Continuity, uniform continuity
- Differentiability
- Riemann Integral
- Improper Integral
- Functions of bounded variation
- Sequence & series of Function (Uniform convergence)
- Several Variable Calculus
- Measure theory

Pure Mathematics



Set: A well-defined & distinct collection of objects

Types of sets:

1. Finite set → Months in a Year
2. Infinite Sets → $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$
3. Singleton set → $\{a\}$
4. Null set (Empty) $\{\}$, ϕ, Nothing
5. Equivalent set $n(A) = n(B)$
(sets having same Cardinal no. (same no. of elements))

$A = \{1, 2, 3\}$

$B = \{2, 3, 4\}$

$n(A) = n(B)$

Cardinal No: No. of elements in a set

$A = \{1, 2, 3, 4\}$

$B = \{2, 3, 9\}$

$C = \{0, 1, 3, 5\}$

no. of elements in $A = 4$ $n(A) = 4$

—— " —— $B = 3$ $n(B) = 3$

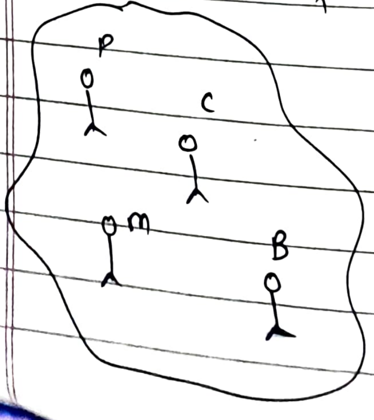
—— " —— $C = 4$ $n(C) = 4$

Cardinality ↓ Size

Equal sets:

1. Equivalent sets
2. Precisely same elements.

Subset: Part of set



$\{\}$ $\{P, C\}$ $\{B, M\}$ $\{P, C, M, B\}$

$\{P\}$ $\{P, M\}$ $\{P, C, M\}$

$\{C\}$ $\{P, B\}$ $\{P, C, B\}$ $= 16 = 2^4$

$\{M\}$ $\{C, M\}$ $\{B, M, C\}$

$\{B\}$ $\{C, B\}$ $\{B, M, P\}$

Date
Page

$A = \{1, 2\}$ $B \subseteq A$
$B = \{2\}$ $A \subseteq A$

Note: Every set is a subset of itself.

$\boxed{\text{No. of subsets of a set} = 2^n}$

Proper Subset:
For $\{P, C, M, B\} \rightarrow n(\text{subset}) < 4$
$A = \{1, 2\} \rightarrow \{\}, \{1\}, \{2\}$
$B = \{2\}$
$B \subset A$
↓
atleast one element less than that of A.

superset: $A \supset B$
↳ A is superset of B

Trivial set:
A → set
↳ subsets → $\{\}$ $\{A\}$
trivial set

$P \subseteq Q$
$P \subset Q$
$Q \supset P$

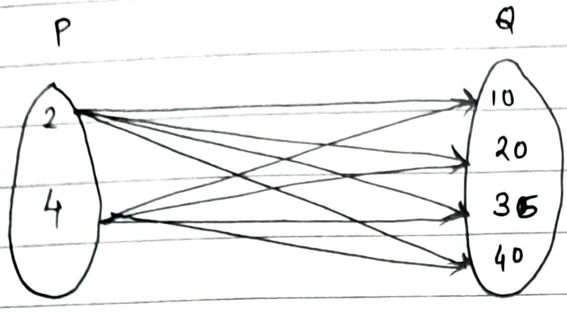
No. of subsets of a set A $= 2^n$
where $n \Rightarrow$ Cardinality of A
No. of proper subsets of a set $= 2^n - 1$

Note: (1) $\phi \subseteq \phi$
(2) $\phi \subseteq A$

Equal $\xrightarrow{\text{Always}}$ Equivalent

Equivalent $\xrightarrow[\text{May not}]{\text{May or}}$ Equal

Relation: set of ordered pair that satisfy a relationship.



$P \times Q = \{(p, q) : p \in P,\ q \in Q\}$ → "set-builder form"

$= \{(2,10), (2,20), (2,35), (2,40), (4,10), (4,20), (4,35), (4,40)\}$

↳ ordered pairs

↓

cartesian product
(all possible pairs)

R: q is a multiple of p

R: $\{(p, q) : \boxed{q \text{ is a multiple of } p} : p \in P \ \&\ q \in Q\}$

↓

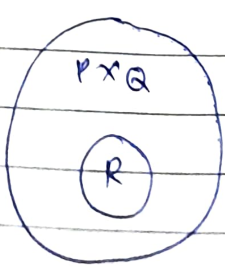
All possible pairs satisfying R

$\boxed{R = \{(2,10), (2,20), (2,40), (4,20), (4,40)\}}$

↳ "Roaster form"

↓

List



$\boxed{R \subseteq P \times Q}$

Domain	Range	Image
p in (p,q)	q in (p,q)	q is image of p
Domain = {2,4}	Range = {10, 20, 35, 40}	

Co-domain

whole q

Domain → all p
Range → all q
Image → q is Image of p
Co-domain → whole Q

$\boxed{\text{Range} \subseteq \text{Co-domain}}$

Function: $\boxed{\text{"Unique Image"}}$

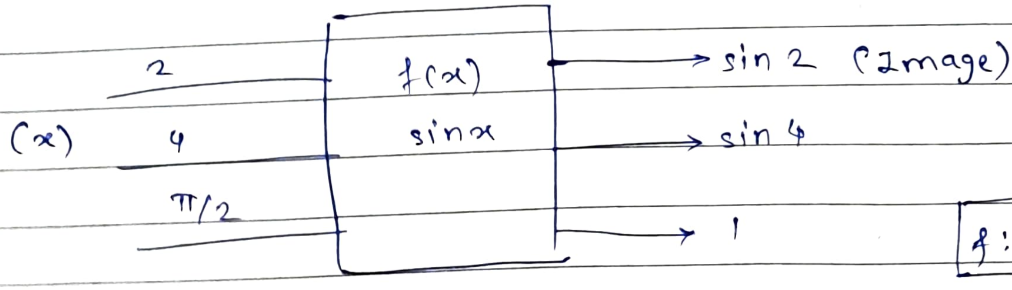
$f(x) = x^2 = 4$

↓

$x = 2, -2$

Relation: set of ordered pair that satisfy a relationship.

Function:



$\boxed{f : X \to Y}$

Input $\xrightarrow{\text{function}}$ Output

Domain → Range

Image

$f(x) = y$

$f: x \to y$

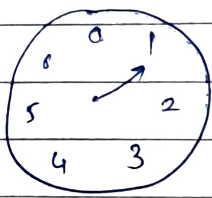
Function

Input
(x)

output
$\sin x$

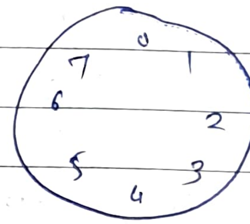
Group Theory:

Clock



$6+6=5$

$4+6=3$



8 hours

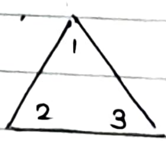
$6+6=4$

$$\boxed{\begin{array}{l} 1+3=4 \\ 5+2=0 \\ 4+3=0 \\ 4+4=1 \end{array}} \to \text{Modular Arithmetic}$$

$$\boxed{7 \text{ hours} = \text{Integer mod } (7)}$$

Symmetry of Δ → doesnot follow commutative law

$1 =$

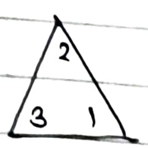


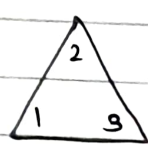
$r \to$ rotation

$f \to$ flip

$$\boxed{\begin{array}{l} 1f = f \\ rf \\ r^2 f \end{array}}$$

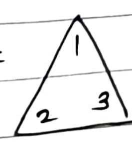
$r =$



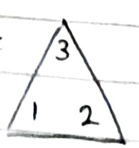


$= rf$

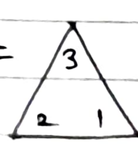
$r^3 = 1 =$



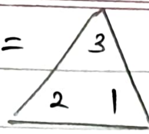
$r^2 =$



$r^2 f =$



$f =$ 

$fr =$ 

$rf \neq fr$

	Clock	Δ	Z
Elements	$\{0, 1, 2, \dots, 6\}$	$\{1, r, r^2, f, rf, r^2f\}$	$\{\dots, -1, 0, 1, \dots\}$
operation	$+$	$\times$	$+$
"Identity"	0	1	0
"Inverse"	$0+0=0$ $0+6=0$ $2+5=0$ $3+4=0$ $4+3=0$ $5+2=0$ $6+1=0$ $6 \leftrightarrow 1$ $5 \leftrightarrow 2$ $3 \leftrightarrow 4$	$r(r^2)=1$ $ff=1$ $r \leftrightarrow r^2$ $f \leftrightarrow f$ $1 \leftrightarrow 1$	$1 \to -1$ $2 \to -2$ $3 \to -3$ $4 \to -4$
Closure	$6+0=6$ closed	$1 \cdot r = r$ $r \cdot r^2 = r^3 = 1$ $r(r^2 f) = r^3 f$ $= 1 \cdot f = f$ Closed	Closed Not closed for $\div$ $\frac{0}{0}, \frac{1}{2}$, etc

Group theory:
set of element G under the operation.

Additive Identity = 0

$\{+, -, \times, \div\}$

$(G, *)$ → operation → $(\mathbb{Z}, +)$

set

$(\mathbb{Z}, \times)$ $(\mathbb{Z}, \div)$

① Closure

$a \in G$

$b \in G$

$a+b \in G$

then closed

"G → groupoid"

$(\mathbb{Z},+)$	$(\mathbb{Z},\times)$	$(\mathbb{Z},\div)$
	$5 \in \mathbb{Z}$	$5 \in \mathbb{Z}$
$35 \in \mathbb{Z}$	$-30 \in \mathbb{Z}$	$-18 \in \mathbb{Z}$
$-168 \in \mathbb{Z}$	$5(-30) \in \mathbb{Z}$	$5 \div (-18)$
$35+(-168) = -133 \in \mathbb{Z}$	$-150 \in \mathbb{Z}$	$= -\frac{5}{18} \notin \mathbb{Z}$
Closed	Closed	Not closed

$(\mathbb{Q}, \div)$

$a \in \mathbb{Q}$

$b \in \mathbb{Q}$

then

$a \div b \in \mathbb{Q}$ given $b \neq 0$

① + ② Associativity

$(a*b)*c = a*(b*c)$ [Not associative for $\div$ in $\mathbb{Z}$]

$(\mathbb{Z}, +)$

$-1, 0, 25$

$(-1+0)+25 = -1+(0+25)$

$-1+25 = -1+25$

$24 = 24$

"Semigroup" $a, b, c \in G$

① + ② + ③ Inverses

$x \in G$

$x \cdot x^{-1} = 1$, $x^{-1} \in G$

then → Monoid

$(\mathbb{Z}, +)$ $1 \to -1$

$1+(-1) = 0$

$2+(-2) = 0$

$(\mathbb{Z}, \times)$ → doesnt satisfy

$(\mathbb{Q}, \times)$ → satisfys except 0

① + ② + ③ + ④ Identity:

$y * e = y = e * y$

where $y \in G$

then $e \in G$

"Group"

(1)(2)(3)(4)(5) Commutativity

of ≠ fr

$\forall\ a, b \in G$ satisfy $ab = ba$, then Group is commutative.

"Abelian Group"

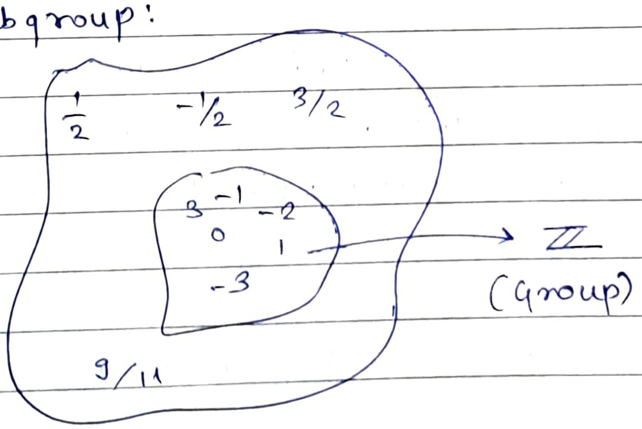
$(G, *)$

1. Closure ⇒ Groupoid
2. ~~\#~~ Closure + Associativity ⇒ semigroup
3. Closure + Associativity + Inverse ⇒ Monoid
4. Closure + Associative + Inverse + Identity ⇒ group
5. Group + Commutativity ⇒ Abelian group

Subgroup:

Q

Group



Set → Subset

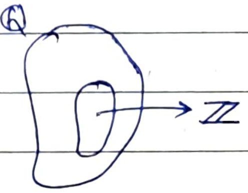
$(G, *) \rightarrow (H, *)$

↓ ↓

Group Group

H is subgroup of G

Q



$(Q, +)$ → Group

C A I e

↓

$(\mathbb{Z}, +)$ → Group

C A I e

Subgroup of Q

Let $(G, *)$ be a group

If subset of $(G, *)$, which is group, $(H, *)$ also form group under the same operation '$*$' then it is

$\boxed{\text{H is a subset of G}}$

called subgroup of G

NOTE:

1. $H \leq G$
2. $G \leq G$
3. if $H \neq G$ then $H < G$, H is called proper subgroup of G

$H \leq G \longrightarrow$ if H ~~is~~ also forms group under same condⁿ (operation)

Subgrp group

Ring

	+	-	×	÷	Form group
$\mathbb{Z}$	✓	✓	×	×	
$\mathbb{R}$	✓	✓	✓	✓	
$\mathbb{R}^{2\times 3}$	✓	✓	×	×	
$\mathbb{C}$	✓	✓	✓	✓	

depending on set we have different operations available

$\mathbb{Z}, \mathbb{R}, \mathbb{R}^{2\times 3}, \mathbb{C} \rightarrow$ Commutative wrt (+)

$\mathbb{Z}, \mathbb{R}, \mathbb{C} \longrightarrow$ have multiplication (Closure)

$\boxed{\mathbb{R}, \mathbb{C} \rightarrow \text{Inverse in multiplication}}$

when a Commutative group under addition also has multiplication (Closure)

$\Downarrow$

"Ring"

$(R, +, \cdot)$

distributive

$a\cdot(b+c) = a\cdot b + a\cdot c$

Ring: A ring is set of R with 2 operation $(R, +, \cdot)$

(+)	(×)	Extra
① Closure	① Closure	① distributive
② Associative	② Associative	$a(b+c) = ab + ac$
③ 0		$a, b, c \in R$
④ $-x$		
⑤ $a+b = b+a$		

$\boxed{\begin{array}{l} (R, +, \cdot) \\ \text{Abelian wrt } + \\ \text{Semigrp with } \cdot \\ \text{Distributive} \end{array}}$

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$

① $(R, +, \cdot)$
② $1 \in R$] "Ring with Unity"

③ Ring with Unity + Commute $ab = ba$] "Commutative Ring"

④ Ring with Unity + Inverse x^{-1}] "Division Ring"

<u>NOTE:</u> Division ring may or may not be commutative.
"Eg: Quatrentons" $x+iy$

(+)	$\cdot$	Extra
① Closure	① Closure	① distributive
② Asso	② Asso	
③ 0	③ 1 (Ring with unity)	
④ $-x$	+ ④ x^{-1} (Division Ring)	may or may not be together
⑤ $a+b = b+a$	⑤ $ab = ba$ (Commutative)	

$+$	$\cdot$	Extra
(1) Closure	(1) Closure	(1) distributive
(2) Ass	(2) Ass	
(3) 0	(3) 1	
(4) $-x$	(4) x^{-1}	
(5) $a+b=b+a$	(5) $ab=ba$	

NOTE: Field = Commutative division ring

$\downarrow$

(Field)

$(F, +, \cdot) \longrightarrow$
- $(F, +)$ abelian
- $(F, \cdot)$ abelian
- distributive

$(\mathbb{Z}, +) \rightarrow$ abelian

$(\mathbb{Z}, \cdot) \longrightarrow$ abelian $\times$ because inverses do not exist

Field $\longrightarrow (\mathbb{Q}, +, \cdot), (\mathbb{R}, +, \cdot), (\mathbb{C}, +, \cdot)$

Noob	Pro
Addition	Addition
Subt.	Additive Inverse
Multiplication	Multiplication
Division	Multiplicative Inverse

No. of ∞ fields $= \infty$

No. of finite fields $= \infty$

(1) $\mathbb{Z}$ is not a field

But $\mathbb{Z} + \{\text{fractions}\} = \text{Field}$

$= \mathbb{Q}$

$$\boxed{\mathbb{Q} = \left\{ \frac{a}{b} : b \neq 0 \ \& \ a, b \in \mathbb{Z} \right\}}$$

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$(\mathbb{Q}, +, \cdot) \to$ Field.

(2) Extending $\mathbb{Q}$:

$\sqrt{2}$ is not a Rational Number.

Q. Is there any field containing $\mathbb{Q}$ & $\sqrt{2}$?

Ans: Yes, $\mathbb{Q}(\sqrt{2})$

$\frac{2\sqrt{2}}{7}$, $2+\sqrt{2}$, --- $\mathbb{Q}(\sqrt{2})$

$\mathbb{Q}(\sqrt{2}) \longrightarrow$
+ abelian
× abelian
distributive

Here, $\mathbb{Q}(\sqrt{2})$ is an extension field of $\mathbb{Q}$.

$\mathbb{Q}(\sqrt{5}) \longrightarrow$ EF of $\mathbb{Q}$

$\mathbb{Q}(\alpha) \longrightarrow$ EF of $\mathbb{Q}$

$\alpha \downarrow$ any irrational no,

NOTE: there are ∞ extension field of $\mathbb{Q}$
NOTE: $\mathbb{Q}(\alpha)$ is an Extension field of $\mathbb{Q}$ "Algebraic Extension"

(3) $a_1, a_2, a_3, \ldots L$

Sequence

$L \to$ limit of sequence

$\boxed{\mathbb{Q} + L} = \mathbb{R}$

$\Downarrow$

$\to$ Field

(4) Extended $\mathbb{R}$

$\mathbb{R}(i)$ $i \to$ imaginary no

$= \sqrt{-1}$

$\to \mathbb{C}$ (Complex number)

$\to$ Field

Rational function

Extension of $\mathbb{C} = \mathbb{C}$

$\mathbb{C}(x) = \left\{ \frac{f(x)}{g(x)} \;\middle|\; f, g \text{ are polynomial with coeff}^n \text{ in } \mathbb{C} \right\}$

$\mathbb{C}(x) = \frac{ix+3}{3-ix^3}$

Rational function $= \frac{f(x)}{g(x)}$ Polynomials

$\mathbb{C}(x) \rightarrow$ Extended $\mathbb{C}$ (Rational function)
$\mathbb{C} \rightarrow$ Complex No.

Space

Set $\longrightarrow$ structure

Space

Vector space
- Internal Composition
- External composition

Internal Comp^n $(*)$
$a \in V$, $b \in V$
then $a * b \in V$
(Closure)
External Comp^n $(\circ)$
$\alpha \in F$, $a \in V$
$\alpha, a \in V$

$V \rightarrow$ Vector Space
$F \rightarrow$ Field
Internal Comp^n
External Comp^n

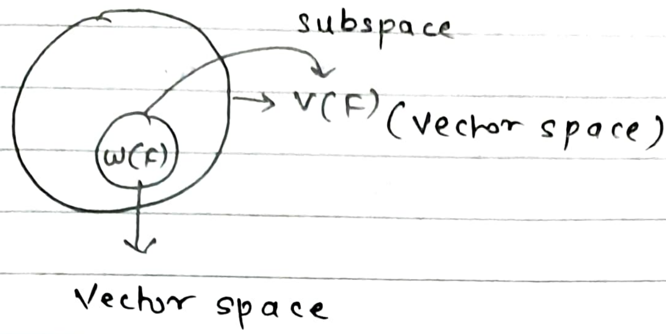
$F \rightarrow \mathbb{Q}$
$\rightarrow \mathbb{R}$
$\rightarrow \mathbb{C}$

$V(F) \rightarrow$ Vector space over a field F

Subspace

V(F)

W(F) → subset of V(F)
→ Vector space



Basic Terminologies in Real Analysis
Point set topology in real analysis.

set] → Topology
↓
Relation
↓
Function (1 variable)
↓ ⟶ sequence & series

$\boxed{\text{Limit, Ctn, diff, UC, Int, R.I, II}}$

↓

2 variables ⟶ sequence & series

$\mathbb{N}$ → set of Natural No.

$\mathbb{N}_0$ → set of Whole No.

$\mathbb{Z}$ → set of Integers

$\mathbb{Q}$ → set of Rational Nos.

$\mathbb{Q}^c$ → set of Irrational Nos.

$\mathbb{R}$ → set of Real Nos.

$\mathbb{C}$ → set of Complex No.

$+\infty$ → arbitrarily large

$-\infty$ → arbitrarily small

A^c → Complement of A
Except A, all elements from Universal set

$A^c \to$ Complement of A
$A^o \to$ Interior of A
$\partial A \to$ boundary of A
$A' \to$ derived set
Iso(A) $\to$ Isolated point of A
st $\to$ such that
stb $\to$ said to be
bdd $\to$ bounded
Unbdd $\to$ Unbounded
nbd $\to$ neighbourhood
Seq $\to$ sequence

cgt $\to$ Convergent
div $\to$ divergent
$\overline{\lim} \to$ limit superior
$\underline{\lim} \to$ limit inferior
Sup(A)/ lub $\to$ least upper bound
Inf (A)/glb $\to$ greatest lower bound
Max^n $\to$ Maximum
Min^n $\to$ Minimum
$\exists \to$ there exists
$\nexists \to$ there DNE

$\exists ! \to$ Exists uniquely
$\forall \to$ for all
$\divideontimes \to$ Contradiction
$\varepsilon \to$ Epsilon
$\in \to$ belongs to
$\notin \to$ doesnot belong to
$\bigcup \quad \bigcap$ $\to$ arbitrary set
($\lambda \in \Lambda \quad \lambda \in \Lambda$) $\to$ both notations are called lambda.

$$AM(a_1, a_2, \dots, a_n) = \frac{a_1 + a_2 + \dots + a_n}{n}$$

$$GM(a_1, a_2, \dots a_n) = (a_1 \cdot a_2 \cdot \dots a_n)^{1/n}$$

$$HM(a_1, \dots, a_n) = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}}$$

$$RMS(a_1, \dots, a_n) = \sqrt{\frac{a_1^2 + a_2^2 + \dots + a_n^2}{n}}$$

Root mean square

$\min(a_1, \dots, a_n) \leq HM \leq AM \leq GM \leq \text{Max}(a_1, \dots, a_n)$

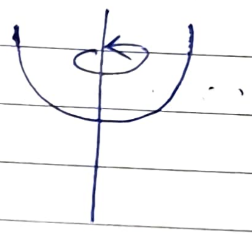
$f(x) = -f(x) \rightarrow$ Reflection about x-axis

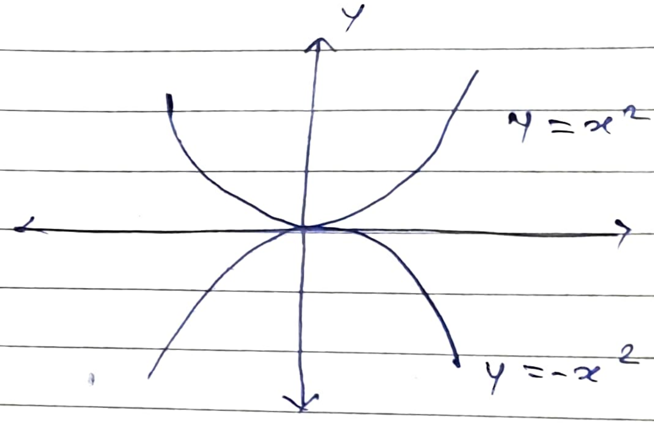
$f(x) = f(-x) \rightarrow$ Reflection about y-axis

$y = x^2$

$\swarrow \quad \searrow$

$y = -x^2 \qquad y = (-x)^2 = x^2$

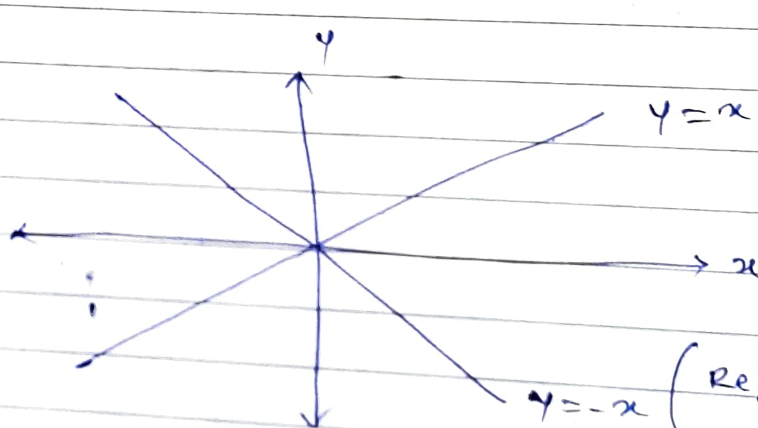




$y = x$

$\swarrow \quad \searrow$

$y = -x \qquad y = -x$



(Reflection abt x & y axis)

$f(x) \rightarrow f(x+a)$

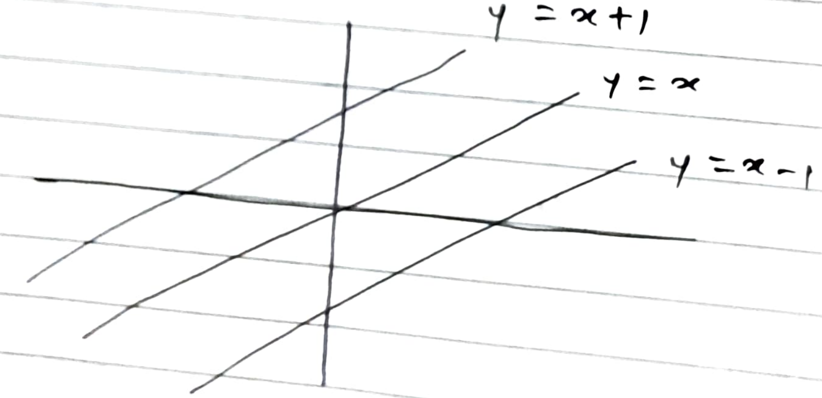
$f(x) \rightarrow f(x) + a$

$\rightarrow$ shift left $f \rightarrow f(x+a)$

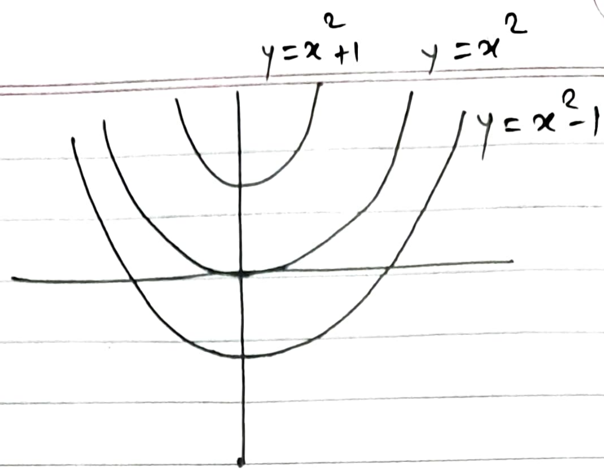
shift right $\rightarrow f(x-a)$

$\rightarrow$ shift up / down

$y = x$



$y = x^2$



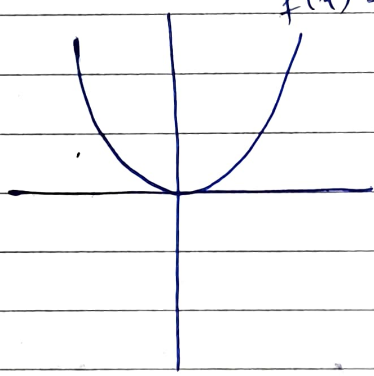
$f(x) \longrightarrow f^{-1}(x) \longrightarrow$ Inverse function

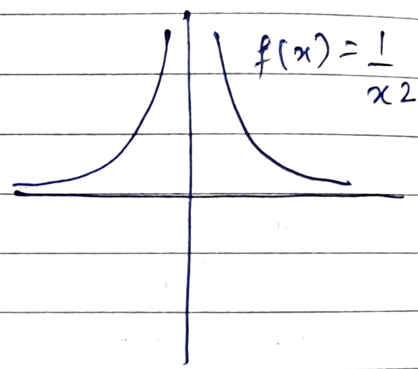
$= \frac{1}{f(x)} \longrightarrow$ Reverse

$f(x) = x^2$

$f(x) = \frac{1}{x^2}$

$f(x) = x^2$

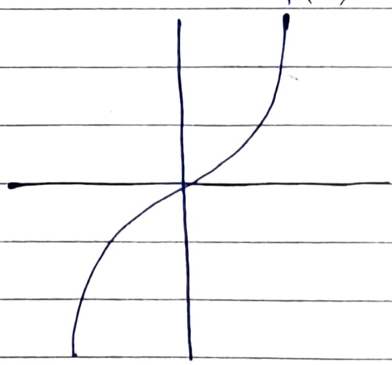


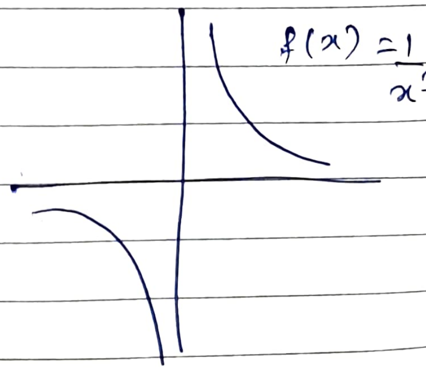


$f(x) = x^3$

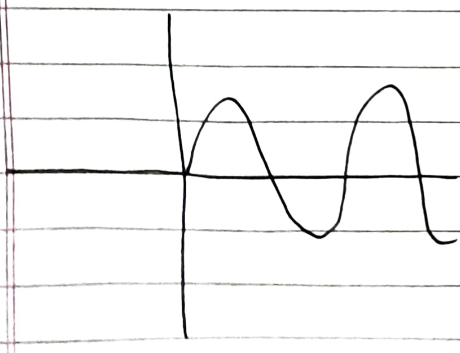
$f(x) = \frac{1}{x^3}$

$f(x) = x^3$

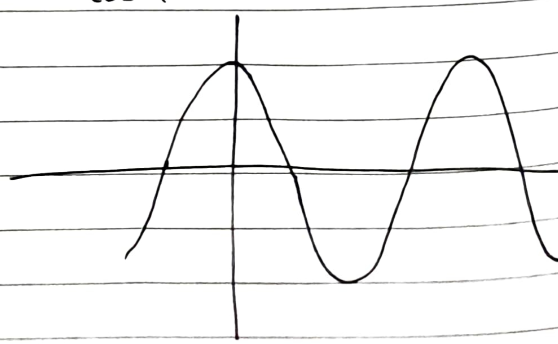




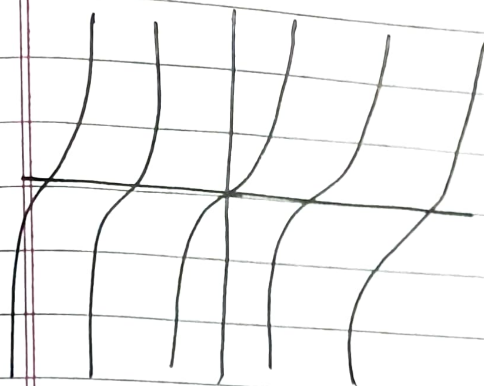
$\sin x$



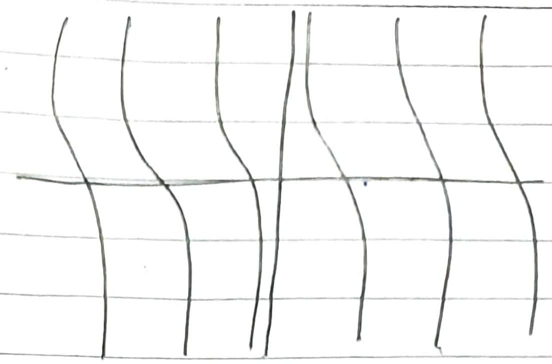
$\cos x$



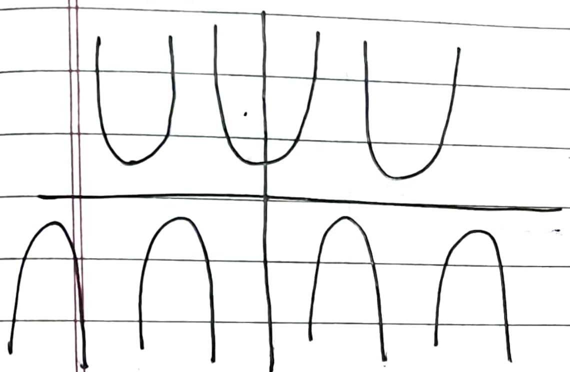
$\tan x$



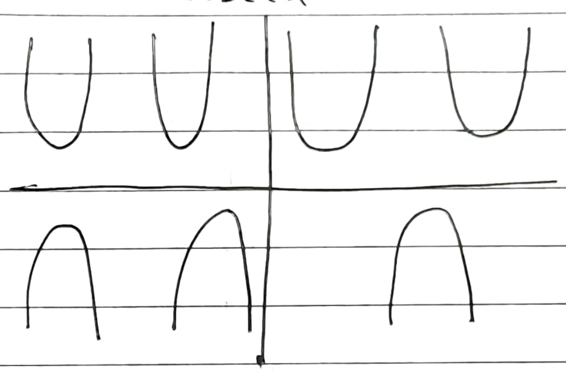
$\cot x$



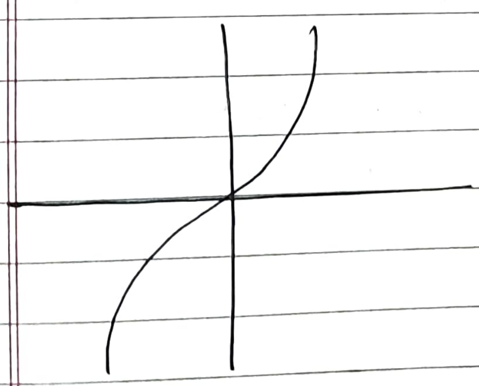
$\sec x$



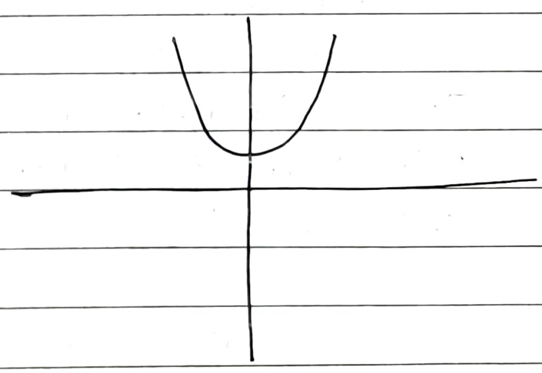
$\operatorname{cosec} x$



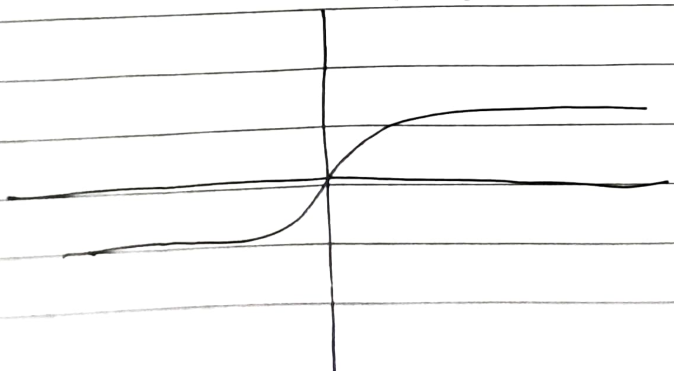
$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$



$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$

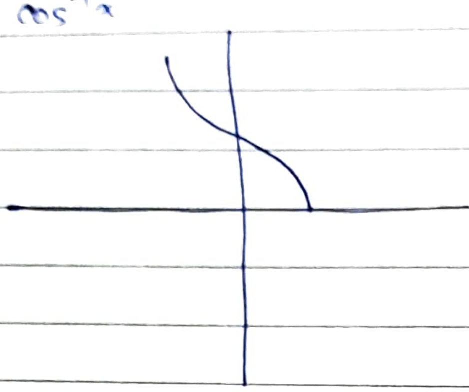


$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

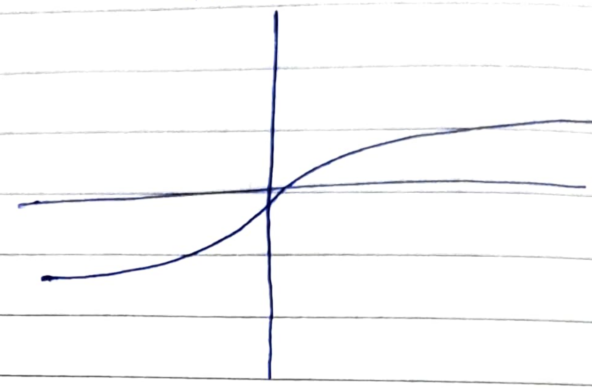


$\sin^{-1}x$

$\cos^{-1}x$



$\tan^{-1}x$



Even functions are symmetric about y-axis.
$f(-x) = f(x)$

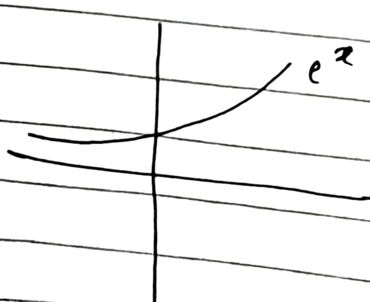
Odd functions are symmetric about opp quadrant
$f(-x) = -f(x)$

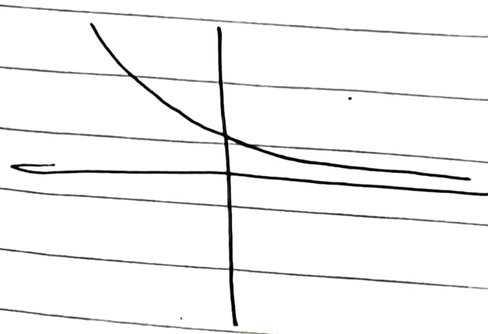
$[x] \rightarrow$ Greatest Integer function GIF

$\{x\} \rightarrow$ fractional part $(0 \leq \{x\} < 1)$

$[x] = x - \{x\}$

$|x| \rightarrow$ modulus of x
$\downarrow$
Absolute value of x





e^{-x}

Set, Bounded above, Bounded Below

Point set topology in real Analysis

Set: A well-defined collection of distinct objects

Collection	Set	Collection	Set
ϕ	✓	$\{x \in \mathbb{R}, x^2 \geq 0\}$	✓
$\{\phi\}$	✓	$\{x \in \mathbb{R}, x^2 > 0\}$	×
$\mathbb{N}$	✓	$\{1, 2, 3\}$	✓
$\mathbb{N}_0$	✓		
$\mathbb{Z}$	✓		
$\mathbb{Q}$	✓		
$\mathbb{Q}^c$	✓		
$\mathbb{R}$	✓		
$\mathbb{C}$	✓		

Set bdd above: A set $A \subseteq \mathbb{R}$ is std bdd above if $\forall\ x \in A\ \exists\ k \in \mathbb{R}$ such that $k \geq x$

$\boxed{\{1, 2, 3, \ldots, 7\}}$ upper bounds

$(0, 1) \rightarrow$ open $\quad (1, 2, 3, \ldots, \infty)$

$[0, 1] \rightarrow$ closed

$\mathbb{R}$ ×

$\mathbb{Q}$ ×

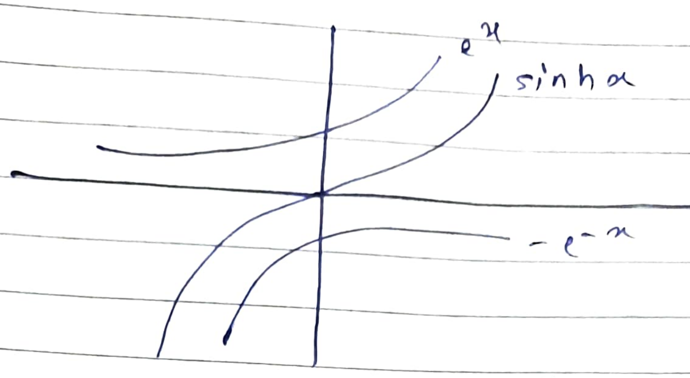
$(0, 1)$ ✓ bdd above

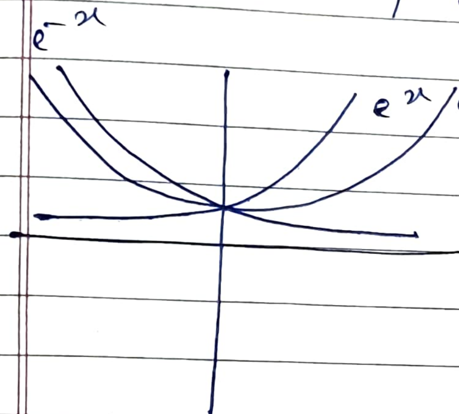
Otherwise, the set is said to be unbounded above

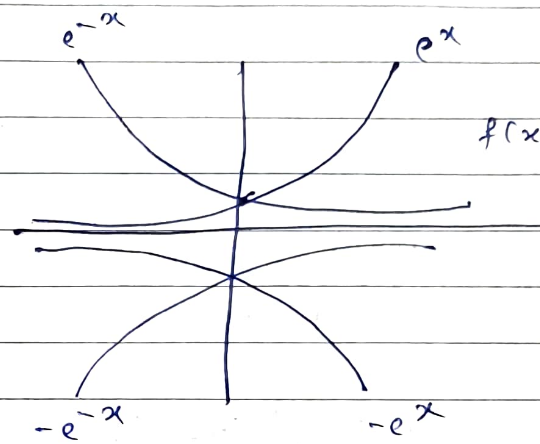
Set	bdd above	upper bound			
ϕ	✓	$\mathbb{R}$	$(0,1)$	✓	$[1,\infty)$
$\{a\}$	✓	$[a,\infty)$	$(0,\infty)$	×	×
$\{a_1,\ldots a_n\}$	✓	$[a_n,\infty)$	$(-\infty,0)$	✓	$[0,\infty)$
$\mathbb{N}$, $\mathbb{Q}^c$, $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{R}$	×	×	$(0,1)\cap\mathbb{Q}^c$	✓	$[1,\infty)$
			$(0,1)\cap\mathbb{Q}$	✓	$[1,\infty)$

$\left\{\frac{1}{n} \middle| n\in\mathbb{N}\right\}$ $(0,1)$	✓	$[1,\infty)$	$\{x^2 \mid x\in\mathbb{R}\}$	×	×
			$\left\{\frac{1}{x} \middle| x\in\mathbb{R}\right\}$	×	×
$\left\{\frac{1}{n+1} \middle| n\in\mathbb{N}\right\}$ $\left(0,\frac{1}{2}\right)$	✓	$\left[\frac{1}{2},\infty\right)$	$\left\{\frac{1}{x-1} \middle| x\in\mathbb{R}-\{1\}\right\}$	×	×
			$\{\sin x \mid x\in\mathbb{R}\}$	✓	$[1,\infty)$
			$\{\cos x \mid x\in\mathbb{R}\}$	✓	$[1,\infty)$
$\left\{\frac{1}{x} \middle| x\in\mathbb{R}\right\}$	×	×			

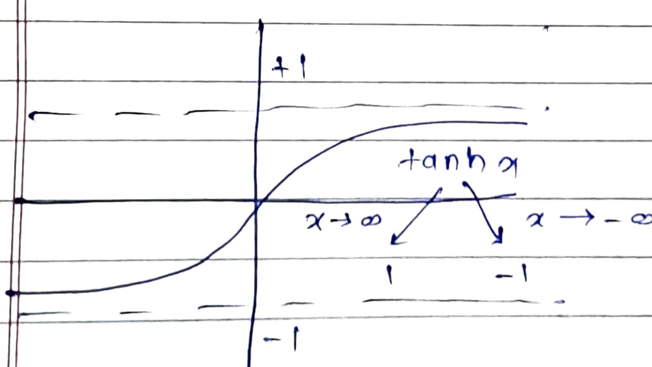
$\left\{\tan x \middle| x\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]\right\}$	×	×	$\left\{\frac{1}{\sin x} \middle| x\in(0,\pi)\right\}$	×	×
$\sinh x = \frac{e^x - e^{-x}}{2}$	×	×			
$\cosh x = \frac{e^x + e^{-x}}{2}$	×	×			
$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$	×	×			

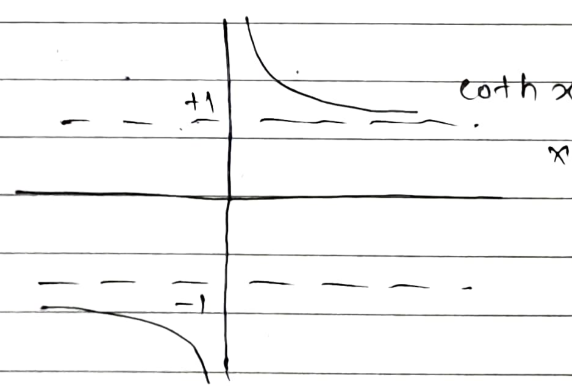






$f(x) \to -f(x)$





$\{\tan^{-1}(x) \mid x \in \mathbb{R}\} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	✓	$\left[\frac{\pi}{2}, \infty\right)$
$\left\{\sin\left(\frac{1}{x}\right) \mid x > 0\right\} = [-1,1]$	✓	$[1, \infty)$
$\left\{\cos\left(\frac{1}{x}\right) \mid x \in (0,1)\right\} = [-1,1]$	✓	$[1, \infty)$
$\mathbb{Z} + \sqrt{2}\,\mathbb{Z}$	✗	✗
$\left\{\frac{m}{2^n},\ m \in \mathbb{Z},\ n \in \mathbb{N}\right\}$	✗	✗

$(0,1) + \mathbb{Q}$ ✗ ✗

$(0,1) + \mathbb{Q}^c$ ✗ ✗

$\left(0, \frac{1}{100}\right) + \mathbb{Q}$ ✗ ✗

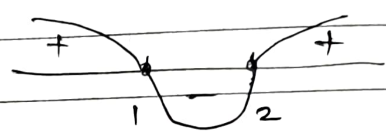
$(0,1) + [2,3]$ ✓ $[3,\infty)$

$\bigcup_{n=1}^{\infty} \left(1-\frac{1}{n}, 2+\frac{1}{n}\right) = (0,3)$ ✓ $[2,\infty)$

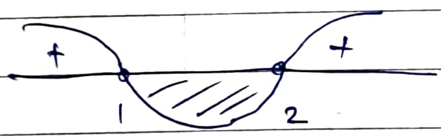
$\mathbb{Q}+\mathbb{Q}, \quad \mathbb{Q}+\mathbb{Q}^c = \mathbb{Q}^c, \quad \mathbb{Q}^c + \mathbb{Q}^c$ ✗ ✗

$\{x \in \mathbb{R} \mid (x-1)(x-2) > 0\}$ $x = 1, 2$

$(\infty, 1) \cup (2, \infty)$



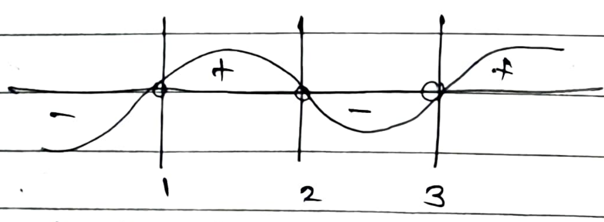
$\{x \in \mathbb{R} \mid (x-1)(x-2) < 0\}$ ✓ $[2,\infty)$



$(1,2)$ ✓

$\{x \in \mathbb{R} \mid (x-1)(x-2) < 0\}$ ✓ $[2,\infty)$

$\{x \in \mathbb{R} \mid (x-1)(x-2)(x-3) > 0\}$ ✗ ✗



$(1,2) \cup (3,\infty)$

$\{x \in \mathbb{R} \mid (x-1)(x-2)(x-3) < 0\} = (-\infty, 1) \cup (2,3)$

✓ $[3,\infty)$

$\left\{ \frac{m}{n} + \frac{4n}{m}, \quad m, n \in \mathbb{N} \right\}$ ✗ ✗

$\left\{ n + \frac{1}{n}, \quad n \in \mathbb{N} \right\}$ ✗

$\left\{ m + \frac{1}{n}, \quad m, n \in \mathbb{N} \right\}$ ✗

$\{ 2^n, \quad n \in \mathbb{N} \}$ ✗

$\left\{ \frac{1}{2^m} + \frac{1}{3^n} + \frac{1}{5^p}, \quad m, n, p \in \mathbb{N} \right\}$ ✓ $\left[\frac{1}{2} + \frac{1}{3} + \frac{1}{5}, \infty \right)$

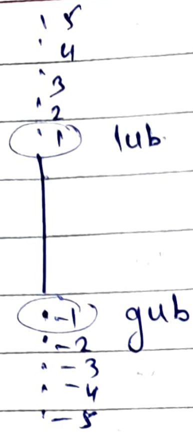
$\left(0, \frac{1}{2} + \frac{1}{3} + \frac{1}{5} \right)$

Set Bdd below: A set $A \subseteq \mathbb{R}$ is stb bdd below if $\exists \, K \in \mathbb{R}$ st $K \leq x \quad \forall \, x \in A$

otherwise, a set is said to be unbdd below

lower bound

UB	LB
• $K' > K$ is an upper bound of A.	• $K' < K$ is a lower bound of A
• bdd above $\Rightarrow \infty$ no. of upper bound.	• bdd below $\Rightarrow \infty$ no. of lower bounds
• Not bdd above $\Rightarrow$ No upper bound	• Not bdd below $\Rightarrow$ no lower bound
• bdd above $\Rightarrow$ No largest upper bound (not known)	• bdd below $\Rightarrow$ Smallest LB (not known)
• Every non-empty bdd set has lub	• Every non-empty bdd set has glb



$(0,1)$

glb lub

(completeness property of $\mathbb{R}$)

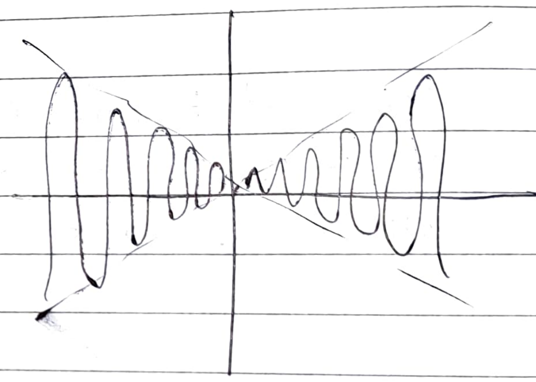
$\left\{ x\sin\left(\frac{1}{x}\right) : x > 0 \right\}$

$\hookrightarrow -1 \leq \sin\left(\frac{1}{x}\right) \leq 1$

$-x \leq x\sin\left(\frac{1}{x}\right) \leq 1$

$-\infty \leq x\sin\left(\frac{1}{x}\right) \leq \infty$

X X



$\left\{ x\sin\left(\frac{1}{x-5}\right) : x \in (0,5) \right\}$

$\hookrightarrow -x \leq x\sin\left(\frac{1}{x-5}\right) \leq x$

$\hookrightarrow -5 < x\sin\left(\frac{1}{x-5}\right) < 5$

glb

lower bounds

$(-\infty, -5]$

$\hookrightarrow$ glb

Upper bounds

$[5, \infty)$

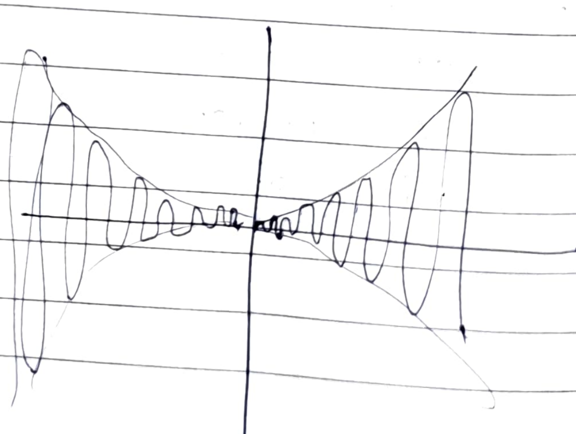
lub

$\left\{ x^2\sin\left(\frac{1}{x}\right) : x > 0 \right\}$

$\hookrightarrow -x^2 \leq x^2\sin\left(\frac{1}{x}\right) \leq x^2$

$-\infty \leq x^2\sin\left(\frac{1}{x}\right) \leq \infty$

X X



Date
Page

Bounded set, Intervals, Supremum, Infimum
Point set topology

Bounded set: A set $A \subseteq \mathbb{R}$ is stb bdd set if it is bdd above & bdd below.

Bounded function: ~~If~~ $f: A \subseteq \mathbb{R} \to \mathbb{R}$ is stb bdd func if its range is bdd

Set	Bdd		
$\phi \subseteq [0,1]$	✓	$\mathbb{Q}^c$	✗
$\mathbb{N}$	✗	$(0,1) \subseteq [0,1]$	✓
$\mathbb{N}_0$	✗		
$\mathbb{Q}$	✗		
$\mathbb{R}$	✗		

$(0,1)$: 0, 1 will not be considered; $[0,1]$: 0, 1 will be considered

$\left\{ \frac{m}{n}, m, n \in \mathbb{N} \right\}$ ✗

Interval:

$x \in$

$[a,b]$	$a \le x \le b$
$[a,b)$	$a \le x < b$
$(a,b]$	$a < x \le b$
(a,b)	$a < x < b$

Set	Interval
ϕ	✓
$\{a\}$	✓
$\{a_1, a_2, \ldots, a_n, \ldots\}$	✗
$\mathbb{N}, \mathbb{N}_0, \mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{Q}^c$	✗
$(0,\infty)$	✓
$(-\infty, 0)$	✓
$\left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$	✗
$\mathbb{Z} + \sqrt{2}\,\mathbb{Z}$	✗
$\left\{ \frac{m}{2^n}, m \in \mathbb{Z}, n \in \mathbb{N} \right\}$	✗
$(0,1)$	✓
$[0,1]$	✓
$\{ x \in \mathbb{R} \mid x^3 < 8 \}$ $(-\infty, 2)$	✓
$(0,1) \cap \mathbb{Q}$	✗
$(0,1) \cap \mathbb{Q}^c$	✗
$(0,\infty) \cap \mathbb{Q}$	✗

Supremum of a set: Let A be a non-empty bdd above set, then A has ∞ no. of upper bounds, smallest of all is called supremum of A.

$(0,1)$
$\swarrow \quad \downarrow$
glb $\quad$ lub

$$lub(A) = sup(A)$$

If not bdd above, $sup(A) = +\infty$

Infimum of a set; let A be a non-empty bdd below set, then A has ∞ no. of lower bounds, largest of all is called Infimum.

$$glb(A) = Inf(A)$$

If not bdd below, $Inf(A) = -\infty$

Maximum of a set A:
$Max(A) = Sup(A)$ | $Min(A) = Inf(A)$

NOTE:

- Sup / Inf may not Exist
- Non-Empty bdd set has Inf & Sup.
- $Sup(A) \notin A$ $[Inf(A) \notin A]$

  $A = (0,1)$ $\quad sup(A) = 1$

  $Inf(A) = 0$

- Bdd set may not have max or min value.
- A, B $\rightarrow$ bdd

  $Sup(A \pm B) = Sup(A) \pm Sup(B)$

  ~~Sup~~ $Inf(A \pm B) = Inf(A) \pm Inf(B)$
- $Sup(A)$, $Inf(A)$, $max(A)$, $min(A)$ If exist then they are unique