# Calibration of Instrument Terms - Error Derivation

$$I = \frac{I^+ - I^-}{I^+ + I^-} = Q\eta_Q + U_i \chi_{u\to Q}$$

$$S_{in} = M_{HWP}(\Theta) R_{alt} M_{M3} \underbrace{R_{PA}}_{\text{parallactic angle}} \underbrace{S_{sky}}_{\text{from star}}$$

Choosing $\Theta$ s.t. $Q$ & $U$ are modulated b/w...

$$\left.\begin{aligned}
&① \; I^{\theta_1} = \begin{pmatrix} Q_1^{\theta_1} & U_1^{\theta_1} \end{pmatrix} \begin{pmatrix} \eta_Q \\ \chi_{u\to Q} \end{pmatrix} \\
&② \; I^{\theta_2} = \begin{pmatrix} Q_2^{\theta_2} & U_2^{\theta_2} \end{pmatrix} \begin{pmatrix} \eta_Q \\ \chi_{u\to Q} \end{pmatrix}
\end{aligned}\right\} \text{Case for one standard}$$

For multiple measurements...

$$\overbrace{\begin{pmatrix} I_1 \\ \vdots \\ I_n \end{pmatrix}}^{I_{data}} = \overbrace{\begin{pmatrix} Q_1 & U_1 \\ \vdots & \vdots \\ Q_n & U_n \end{pmatrix}}^{QU_{data}} \begin{pmatrix} \eta_Q \\ \chi_{u\to Q} \end{pmatrix}$$

$$I_{data} = QU_{data} \begin{pmatrix} \eta_Q \\ \chi_{u\to Q} \end{pmatrix}$$

For an overdetermined sys ($n > 2$)...

$$\to QU_{data}^T QU_{data} \begin{pmatrix} \eta_Q \\ \chi_{u\to Q} \end{pmatrix} = QU_{data}^T I_{data}$$

$$\to \underbrace{\begin{pmatrix} \hat{\eta}_Q \\ \hat{\chi}_{u\to Q} \end{pmatrix}}_{\text{LSQ estimates of inst. params}} = \left(QU_{data}^T QU_{data}\right)^{-1} QU_{data}^T I_{data}$$

Finding the noise in $I_{data}$ ...

$\rightarrow$ sum $= I^+ + I^-$

$\rightarrow$ diff $= I^+ - I^-$

error in $I^+$

$\rightarrow \sigma_{sum}^2 = (\sigma_+)^2 + (\sigma_-)^2$ error in $I^- = \sigma_+$

$\sigma_{sum} = \sqrt{2}\,\sigma_{+/-}$

$\rightarrow \sigma_{diff}^2 = \sigma_+^2 + \sigma_-^2$

$\sigma_{diff} = \sqrt{2}\,\sigma_{+/-}$

$$\rightarrow \overbrace{\text{n. diff}}^{\text{normalized diff}} = I = \frac{I^+ - I^-}{I^+ + I^-} = \frac{diff}{sum}$$

$$\rightarrow \sigma_I^2 = I^2\left(\frac{\sigma_{diff}^2}{diff^2} + \frac{\sigma_{sum}^2}{sum^2}\right)$$

$$= \frac{diff^2}{sum^2}\left(\frac{\sigma_{diff}^2}{diff^2} + \frac{\sigma_{sum}^2}{sum^2}\right)$$

$$= \frac{\sigma_{diff}^2}{sum^2} + \frac{diff^2\,\sigma_{sum}^2}{sum^4}$$

$$= \frac{\sigma_{diff}^2}{sum^2}\left(1 + \cancel{I^2}\right) \quad \cancel{I^2} \rightarrow 0$$ (assuming cov. of

$$\sim \frac{\sigma_{diff}^2}{sum^2}$$

$$= \frac{2\sigma_{I^{+/-}}^2}{sum^2}$$

As signal in one beam is sum/2 for $\overbrace{\text{low-polarization}}^{\text{\% pol} \sim \text{a couple \%}}$ sources

$$\rightarrow \sigma_I^2 = \left(\sigma_{I^{+/-}}^2\right)\left(\frac{1}{2}\left(I^{+/-}\right)^{-2}\right)$$

$$= \frac{1}{2\,SNR_{I^{+/-}}^2}$$

As SNR of the source before wollaston $SNR_{I_{tot}} = 2SNR_{I^{+/-}}$ ...

$$\rightarrow \boxed{\sigma_I^2 = \frac{1}{SNR_{I_{tot}}^2}}$$ this value can be found w/ NIRC2 expTime calculator

For the measured normalized intensities $I_{data}$, we have the covariance matrix $\Sigma_{I_{data}}$. The covariance matrix of the inst. params is then...

$$\rightarrow \boxed{\Sigma_{\eta,x} = \left(QU_{data}^T \Sigma_{I_{data}}^{-1} QU_{data}\right)^{-1}}$$

↳ in this exp, $\Sigma_{I_{data}}$ has diagonal elements $= \sigma_{I_n}^2 = \frac{1}{SNR_{I_{tot}}}$

↳ all off-diagonal elements assumed to be zero (noise of different sources & Q & U are completely uncorrelated)

In case of SNR being equivalent for all standards...

$$\rightarrow \Sigma_{\eta,x} = \sigma_I^2 \, QU_{data}^+ (QU_{data}^+)^T$$

$$= \boxed{\left(\frac{1}{SNR}\right)^2 QU_{data}^+ (QU_{data}^+)^T}$$