homework06

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1 COMPSCI 527 Homework 6

1.0.1 Problem 0 (3 points)

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[171]:
```
### IMPORTS ###
import numpy as np
import matplotlib.pyplot as plt
```

1.2 Part 1: Image Motion Basics

1.2.1 Problem 1.1 (Exam Style)

$$\frac{1}{\sqrt{3^2+4^2}}[3,4]\begin{bmatrix}1\\3\end{bmatrix} = 3$$

1.2.2 Problem 1.2 (Exam Style)

$$(1,1)^T\mathbf{v} - 5 = 0$$
$$(-1,1)^T\mathbf{v} + 1 = 0$$
$$(1,1)^T\mathbf{v} = 5$$
$$(-1,1)^T\mathbf{v} = -1$$

Let $\mathbf{v} = x, y$,

$$x + y = 5$$
$$-x + y = -1$$
$$\mathbf{v} = (3,2)$$

1.3 Part 2: Window Tracking

[132]:
```
g1 = np.array(
    [[6,2,4],
     [8,2,2],
     [2,4,4]])
g2 = np.array(
    [[9,3,6],
     [12,3,3],
```

```
    [3,6,6]])
gradient_g = np.array([g1,g2])

image_diff = np.array(
    [[0, 5, 0],
     [0, 0, 10],
     [15, 0, 0]])
```

1.3.1 Problem 2.1 (Exam Style)

The formula for A is given as:

$$A = \sum_{\mathbf{x}} \nabla g(\mathbf{x}) \left[\nabla g(\mathbf{x})\right]^T w(\mathbf{x} - \mathbf{x}_f)$$

where

$$\nabla g(\mathbf{x}) = \begin{bmatrix} \begin{bmatrix} 6 & 2 & 4 \\ 8 & 2 & 2 \\ 2 & 4 & 4 \end{bmatrix} \\ \begin{bmatrix} 9 & 3 & 6 \\ 12 & 3 & 3 \\ 3 & 6 & 6 \end{bmatrix} \end{bmatrix}$$

[138]:
```
A = np.zeros((2,2))
for i in range(g1.shape[0]):
  for j in range(g1.shape[1]):
    x = np.array([g1[i,j],g2[i,j]])
    x = x.reshape((2,1))
    A = A + np.matmul(x,x.T)
A
```

[138]:
```
array([[164., 246.],
       [246., 369.]])
```

So

$$A = \sum_{\mathbf{x}} \nabla g(\mathbf{x}) \left[\nabla g(\mathbf{x})\right]^T w(\mathbf{x} - \mathbf{x}_f) = \begin{bmatrix} 164 & 246 \\ 246 & 369 \end{bmatrix}$$

Similarly,

$$\mathbf{b} = \sum_{\mathbf{x}} \nabla g(\mathbf{x}) \left[f(\mathbf{x}) - g(\mathbf{x})\right] w(\mathbf{x} - \mathbf{x}_f)$$

and

$$\left[f(\mathbf{x}) - g(\mathbf{x})\right] = \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 10 \\ 15 & 0 & 0 \end{bmatrix}$$

[137]:
```
b = np.sum(
    gradient_g[:, 0:g1.shape[0], 0:g1.shape[1]] *
    image_diff[0:g1.shape[0], 0:g1.shape[1]],
    axis = (1,2)) \
    .reshape((2,1))
b
```

[137]:
```
array([[60],
       [90]])
```

Hence

$$b = \begin{bmatrix} 60 \\ 90 \end{bmatrix}$$

1.3.2 Problem 2.2 (Exam Style)

The given feature x_f does not suffer from the aperture problem because using the Lucas-Kanade tracker eliminates it owing to its assumption of constant motion in the immediate neighborhood.

1.3.3 Problem 2.3

[157]:
```
np.round(
    np.linalg.pinv(A) @ b,
    decimals = 2)
```

[157]:
```
array([[0.11],
       [0.17]])
```

The approximate minimum-norm solution x_0 to the linear system is (to the nearest hundredth):

$$\begin{bmatrix} 0.11 \\ 0.17 \end{bmatrix}$$

1.3.4 Problem 2.4

The given equation is a linear equation and can be solved by taking the product of the pseudo-inverse of A and b.

$$Ax = b$$

Let the Singular Value Decomposition of A be given as the following:

$$A = USV^T$$

Then,

$$A_{pseudo} = (USV^T)^{-1} = VS^{-1}U^T$$

Hence, the general solution for x can be determined by:

$$x = A_{pseudo} * b = VS^{-1}U^Tb$$

1.4 Part 3: The Lucas-Kanade Tracker

1.4.1 Problem 3.1

[170]:
```
def x_range(sigma):
    trunc_limit = np.ceil(3. * sigma)
    x_range = np.arange(2 * trunc_limit + 1) - trunc_limit
    return x_range
```

[175]:
```
def averaging_kernel(sigma):

    def gauss_in_x(x, sigma):
        return np.exp(-x**2/(2*sigma**2))

    x = x_range(sigma)
    averaging_kernel = gauss_in_x(x, sigma)
    normalized_ave_kernel = 1/(np.linalg.
 ↪norm(averaging_kernel))*averaging_kernel
    return normalized_ave_kernel, x
```

[182]:
```
def differentiation_kernel(sigma):
    x = x_range(sigma)
    def differentiated_gaussian(x, sigma):
        # literally just the derivative of the gaussian function (I hope that's␣
 ↪what it's looking for!)
        return (-x/sigma**2)*np.exp(-x**2/(2*sigma**2))

    differentiation_kernel = differentiated_gaussian(x, sigma)
    normalized_diff_kernel = 1/(np.linalg.
 ↪norm(differentiation_kernel))*differentiation_kernel
    return normalized_diff_kernel, x
```

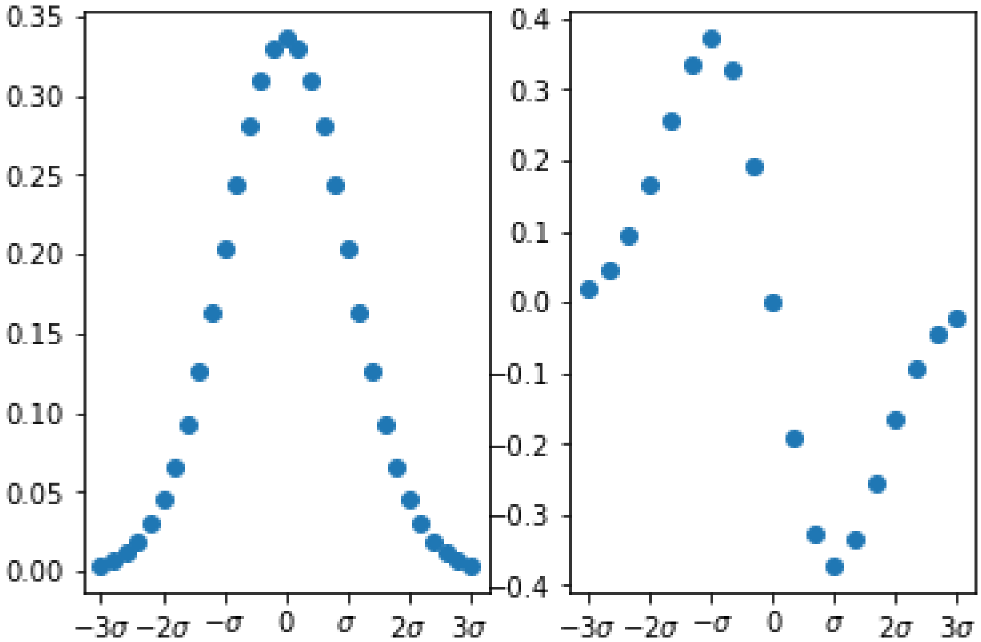
[194]:
```
## PLOTS ##
sigma_list = ['$-3\sigma$', '$-2\sigma$', '$-\sigma$', '0', '$\sigma$',␣
 ↪'$2\sigma$', '$3\sigma$']
ticks = np.linspace(-3, 3, 7)

sigma_ave = 5
plt.subplot(1, 2, 1)
kernel, x = averaging_kernel(sigma_ave)
plt.xticks(sigma_ave*ticks, sigma_list)
plt.scatter(x, kernel)

sigma_diff = 3
plt.subplot(1,2,2)
kernel, x = differentiation_kernel(sigma_diff)
plt.xticks(sigma_diff*ticks, sigma_list)
```

```
plt.scatter(x, kernel)
```

[194]: <matplotlib.collections.PathCollection at 0x159b74db250>



1.4.2 Problem 3.2

Note: I was unable to reach the solution to this problem - I couldn't understand the pseudocode and I was confused by the actions of the kernels, gradients, given parameters and loss function in this problem.

[211]:
```
import urllib.request
import pickle
from os import path as osp

pickle_file_name = 'inputs.pkl'
if not osp.exists(pickle_file_name):
    fmt = 'https://www2.cs.duke.edu/courses/spring21/compsci527/homework/6/{}'
    url = fmt.format(pickle_file_name)
    urllib.request.urlretrieve(url, pickle_file_name)
with open(pickle_file_name, 'rb') as file:
    inputs = pickle.load(file)
f, g = inputs['f'], inputs['g']
```

[212]:
```
params = {
    "window" : 1.5,
    "differentiation" : 1,
```

```
    "termination_delta" : 10**(-3),
    "termination_epsilon" : 10**(-6),
    "termination_rho" : 10,
    "max_iterations" : 30
}
```

[221]:
```
def lk_iterator(f, g, params):

        params['X'] = params['X'] + params['shift_g']
        params['j'] = g[params['X']]

        gradient_kernel, gradient_x =␣
↪differentiation_kernel(params['differentiation'])
        A = gradient_x.T * (gradient_x * np.array([params['w'], params['w']]))
        b = gradient_x.T * ((params['i'] - params['j']) * params['w'])
        s = np.linalg.lstsq(A, b)
        displacement_old = params['displacement']
        params['displacement'] = params['displacement'] + s

        def L(d):
            return np.sum((g[params['X'] + d] - f[params['X']])**2*params['w'])

        if (np.linalg.norm(s) <= params['termination_delta']) or \
           (L(displacement_old) - L(params['displacement']) <=␣
↪params['displacement_epsilon']):
           done = True

        if (np.linalg.norm(params['displacement']- displacement_old) >␣
↪params['termination_rho'] or params['iteration'] > params['max_iterations']):
            lost = True

        return done, lost, params
```

[256]:
```
def lucas_kanade(f, g, params):


    # construct support of W, which is just [0, ..., 100] = omega where␣
↪w(omega) > 0
    half_width = 5          # Piazza @223
    Omega = np.linspace(0, 100, 101, dtype = int)
    X_support_of_w = Omega[np.where(np.absolute(Omega) <= half_width)]
    # use W indicator to build vector of nonzero values of w(x)
    ave_kernel, w_x = averaging_kernel(params['window'])

    # window coords in f added to space
    window_f = np.linspace(0, 10, 11, dtype = int) # the window is centered at␣
↪position 2 with indices [0,1,2,3,4]
```

```python
    X_support = X_support_of_w + window_f   # add these coordinates to the set X
    f_space = f[X_support]                  # find all coordinates of X in f

    # initialize variables for iteration in the lucas kanade iterator

    initialization_variables = {
        'displacement' : np.zeros(f.shape),
        'shift_g' : np.zeros(f.shape),
        'X' : X_support,
        'w' : w_x,
        'i' : f_space,
        'j' : None,
        'iteration' : 0
    }

    # add these parameters to the parameters passed to lk iterator
    params.update(initialization_variables)

    done, lost = False, False

    while (not (done or lost)):
        params['iteration'] += 1
        done, lost, params = lk_iterator(f, g, params)

    if lost:                        # no need to consider L_max (Piazza @236)
        return None
    else:
        return displacement
```

[257]:
```python
lucas_kanade(f, g, params)
```

```
       ␣
↪---------------------------------------------------------------------------

       ValueError                                Traceback (most recent call␣
↪last)

       <ipython-input-257-fa8c73175557> in <module>
    ----> 1 lucas_kanade(f, g, params)


       <ipython-input-256-3bb8d561741d> in lucas_kanade(f, g, params)
        11     # window coords in f added to space
        12     window_f = np.linspace(0, 10, 11, dtype = int) # the window is␣
↪centered at position 2 with indices [0,1,2,3,4]
    ---> 13     X_support = X_support_of_w + window_f   # add these coordinates␣
↪to the set X
```

```
     14     f_space = f[X_support]                  # find all coordinates of X in f
     15


ValueError: operands could not be broadcast together with shapes (6,) (11,)
```