Case Study

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Introduction

The task of modeling turbulent fluid systems has consistently been one of the most difficult problems to solve in modern mathematics and physics; in fact, despite over two centuries of concerted research into the topic, no analytical method currently exists to accurately predict the evolution of turbulent systems over time.

Nevertheless, the successful simulation of turbulent systems remains uniquely essential to a diverse array of sciences and industries with common use cases including modeling airflow in aerodynamic design, analyzing blood flow in medicine (Sallam and Hwang 1984), and meteorological forecasting (Kim et al. 2015). To perform this simulation, high-resolution direct numerical simulation (DNS) is typically performed using the Navier-Stokes equation, but this is often very complex and computationally expensive to perform on any arbitrary set of inputs. In this case study, we circumvent these time-intensive simulation techniques by interpolating a set of results collected from Navier-Stokes DNS to produce a predictive model given a set of input parameters. Our objectives are therefore as follows:

- Given a set of training and test data describing the inputs of a DNS simulation, namely the Reynolds Number Re, the gravitational acceleration Fr, and the Stokes number St, generate a model or set of models that can accurately predict the particle cluster volume distribution in terms of four moments used as dependent variables. We will call this our *prediction* objective.
- Given the set of input and output parameters, determine how each parameter the distribution of cluster volumes. We will call this our *inference* objective.

Methodology

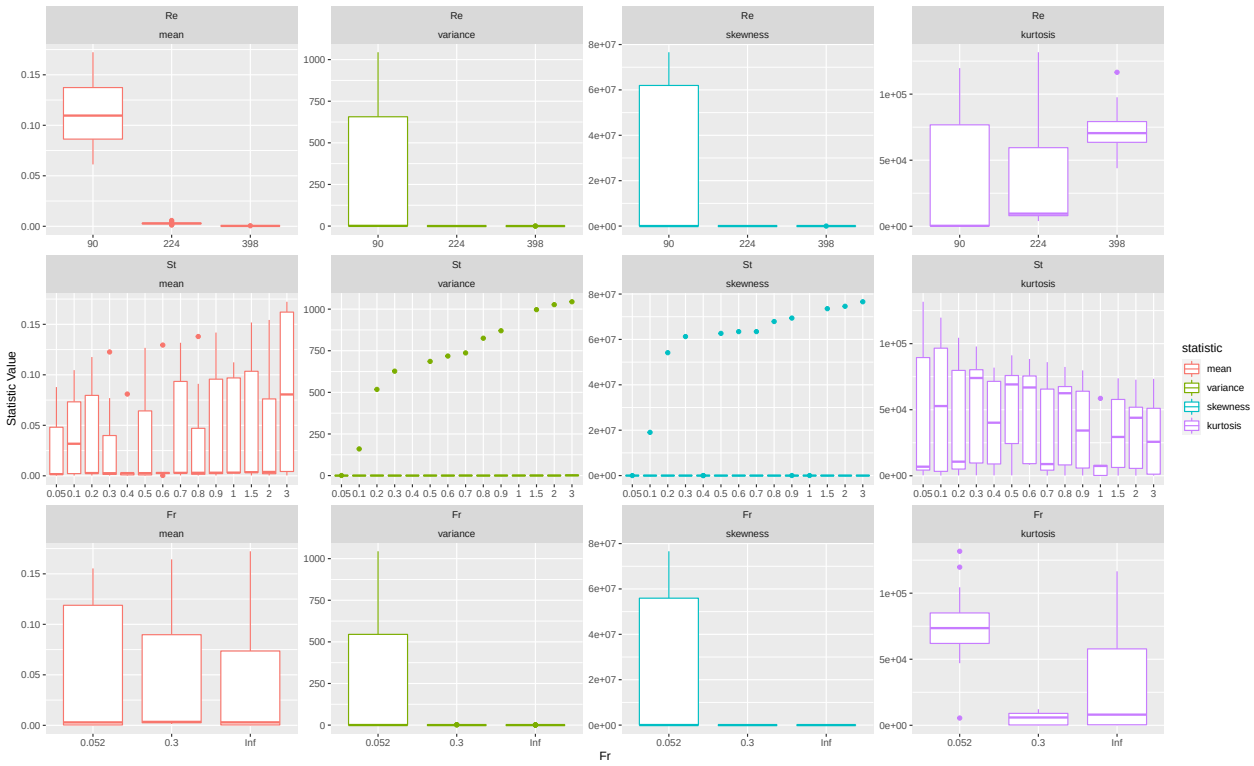
To fulfill the objectives we pose in our introduction, we must collect more information about our modeling objectives. First, we have to define cluster volume in terms of the moments we are given. Since each simulation in our dataset represents a single row of three inputs $< Re, Fr, St >$ and four outputs $< E[X], E[X^2], E[X^3], E[X^4] >$ representing information about a *distribution* of several cluster volumes, we must consider that we are measuring the change in shape of a distribution of volumes rather than the volumes themselves. For this analysis, we defined key characteristics about these models in terms of these moments:

1. Mean: $E[X]$
2. Variance: $E[X^2] - E[X]^2$
3. Skewness: $(\frac{E[X^3]}{E[X^2]})^2$
4. Kurtosis: $\frac{E[X^4]}{E[X^2]^2}$

Our goal here is not only to make interpretations of our model predictions simpler but also to attempt to find some distribution that connects these outputs to our input parameters. To do this, we must first examine how these model characteristics are distributed in our dataset.

Nevertheless, we performed the same analysis on the raw moments in parallel to avoid oversimplifying the problem and so that we could provide correct predicted values (since we are predicting raw moments, *not* these statistics).

Figure 1. Distribution of Statistics across values of Fr, St, and Re



We noted from our EDA (see Figure 1) that the variance and skewness of the dataset appeared to drop dramatically with increasing values of Fr and of Re, and that there seemed to be a possible covariance between the two variables, hinting at an interaction. otherwise, we also noted that the rate at which this interaction variable influenced the data appeared to be either inversely polynomial (rational) or otherwise hyperbolic.

We investigated this in our assessments of the raw moments as well, and found that the distributions were roughly equal, which is unsurprising given that the statistics are derived from the raw moments.

Thus, we hypothesized that the model that would best fit this data would likely look polynomial and have interactions between at least Re and Fr - conceptually, this makes sense, as the turbulence an object faces and the acceleration it undergoes due to gravity might both increase the rate of mixing, leading to uniformly sized clusters.

We created three sets of models across each of the statistics and raw moments (Appendix B); we created a baseline linear model without any interaction variables where each of the raw moments/statistics acted as the response to the set of predictors $< Re, Fr, St >$. We did the same for a linear model with two way interactions which we then performed backwards selection on, optimizing with AIC. Finally, we created a 2-degree polynomial model to measure any squared interactions.

After creating these models, we noted several key lines of evidence to indicate that there were interactions present between Re and St and Re and Fr. We also found that our polynomial model predicting $E[X]$ (the mean) had an excellent fit to the data and showed a low RSE.

We settled on the polynomial model for $E[X]$. For the others, we noted that the residual plots (Appendix B) hinted at a necessary transformation in the response variable, as there was clear deviation from the fitted values as they approached larger values of the moment. Therefore, we log-transformed the predictor of our linear models to see if this would improve model accuracy.

Upon producing log-transformed models for each of the original models, we found much higher accuracy without being overfit. In particular, all of the log-transformed two-degree models showed high adjusted R. squared after backwards selection with AIC and with BIC (used to verify the results in AIC). We settled on

the log-transformed raw moments for prediction, since this will provide the most predictive power regardless of interpretability.

With this evidence settled on the following predictive models for the raw moments:

1. $\log E[X]$
2. $\log E[X^2]$
3. $\log E[X^3]$
4. $\log E[X^4]$

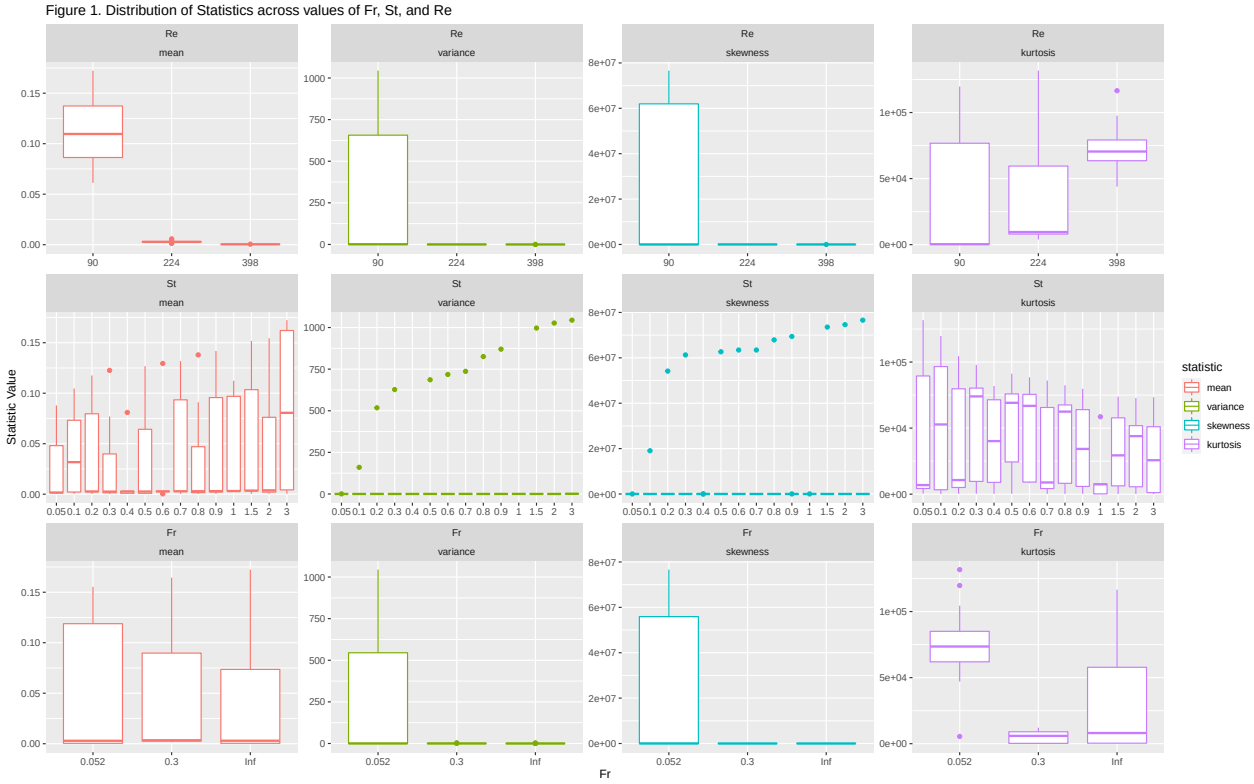
References

Kim, Jung-Hoon, William N Chan, Banavar Sridhar, and Robert D Sharman. 2015. "Combined Winds and Turbulence Prediction System for Automated Air-Traffic Management Applications." *Journal of Applied Meteorology and Climatology* 54 (4): 766–84. https://doi.org/10.1175/JAMC-D-14-0216.1.

Sallam, Ahmed M, and Ned H C Hwang. 1984. "Human red blood cell hemolysis in a turbulent shear flow: Contribution of Reynolds shear stresses." *Biorheology* 21: 783–97. https://doi.org/10.3233/BIR-1984-21605.

Appendix A: EDA

Figure 1



This table of graphs suggests the following key notes:

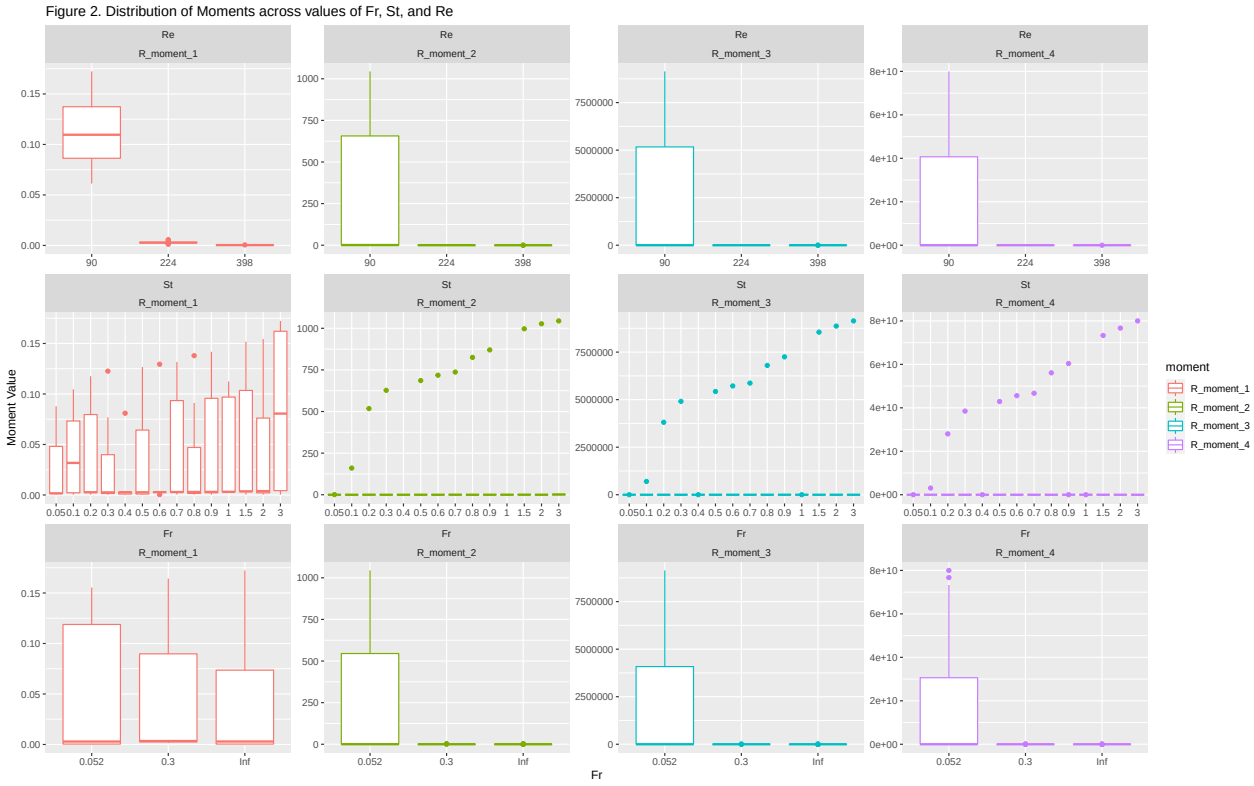
1. The mean, variance, and skewness of the distribution of cluster volumes drops significantly as `Re` increases, while kurtosis appears to increase.
2. There isn't a strong trend for `St` except to note that, as it increases, the maximum value of the mean, variance, and skewness increases fairly dramatically, but the kurtosis falls steadily.

3. The mean of the particle cluster volume remains mostly consistent for values of `Fr`, but the variance and skewness drop quickly. The kurtosis with respect to `Fr` doesn't seem to have a clear trend.

Other things to note include:

1. The variance and skewness of the distribution don't seem to be dependent on `St`. The mean and particularly the kurtosis of the distribution does appear to have some relationship with `St`, however, though kurtosis seems more closely related than the variance in what appears to be a roughly linear decline.
2. `Re` evokes a strong change in all of the distribution statistics as it increases, sharply reducing the variance in their distributions. I'd hypothesize that `Re` has a quadratic or otherwise polynomial effect on the distribution of the particle size, since it appears to change the variance and skewness of the set so sharply.
3. `Fr` evokes similar changes in the variance and skewness of the distribution as `Re`, but not in the kurtosis and mean. This could mean there is some interaction effect in the second moment of the data between `Re` and `St`.

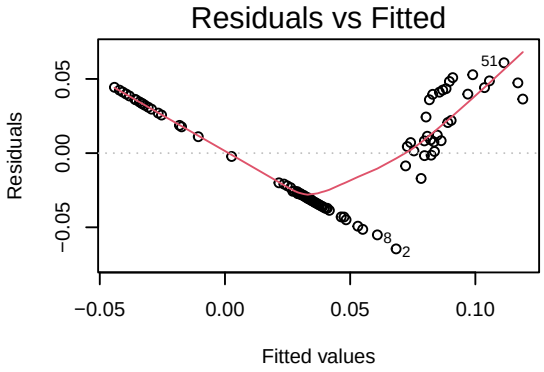
Figure 2

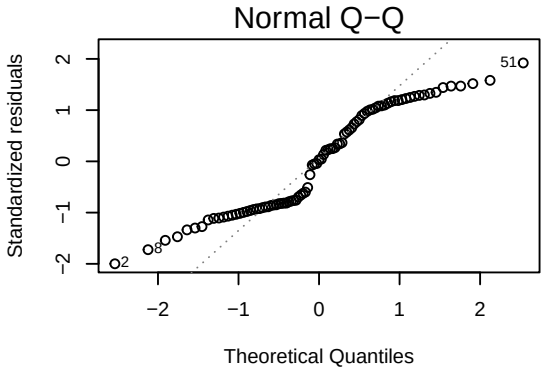


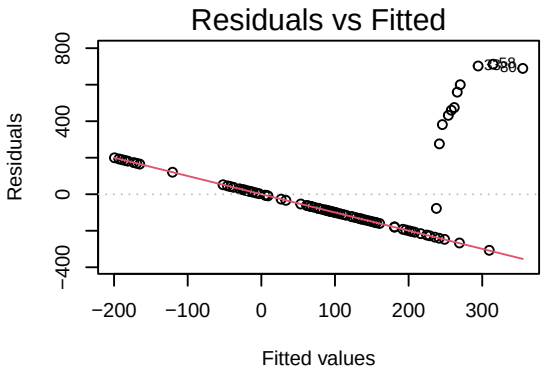
Appendix B: Residual and Q-Q Plots

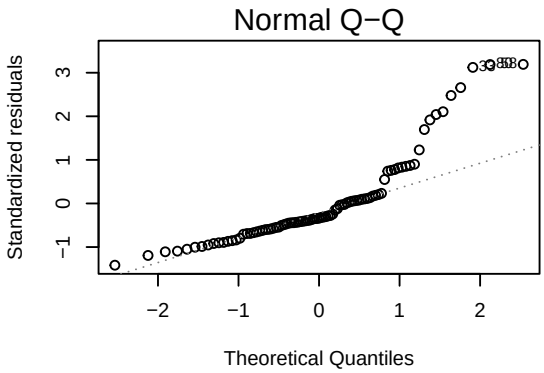
Raw Moments as Response Variables

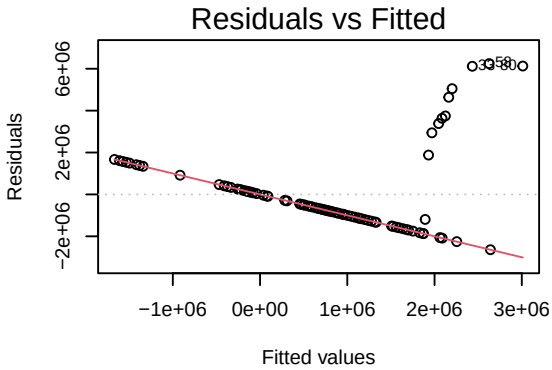
Linear Models, No Interactions

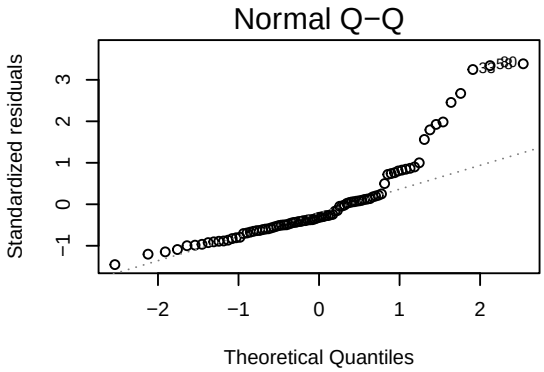


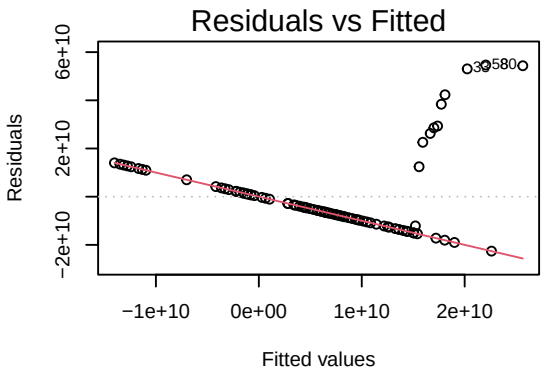


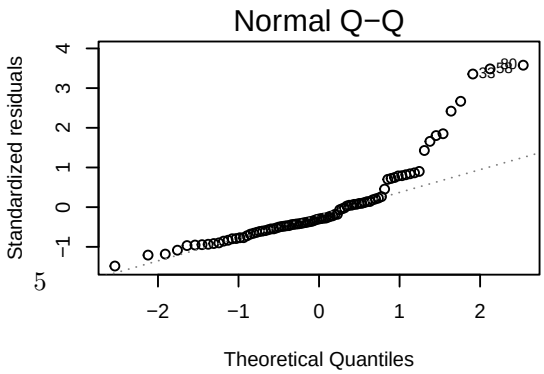












```
## [[1]]
## # A tibble: 4 x 5
##   term         estimate std.error statistic  p.value
##   <chr>           <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)  0.113    0.00950      11.9   7.45e-20
## 2 St           0.0133   0.00463       2.88  5.08e- 3
## 3 Re          -0.000378 0.0000323   -11.7   2.18e-19
## 4 Fr          -0.00790  0.00876      -0.902 3.70e- 1
##
## [[2]]
## # A tibble: 4 x 5
##   term        estimate std.error statistic    p.value
##   <chr>          <dbl>     <dbl>     <dbl>      <dbl>
## 1 (Intercept)  320.       63.9        5.00 0.00000308
## 2 St            40.5      31.2        1.30 0.198
## 3 Re            -0.847     0.217     -3.90 0.000192
## 4 Fr          -184.       59.0       -3.11 0.00251
##
## [[3]]
## # A tibble: 4 x 5
##   term         estimate std.error statistic    p.value
##   <chr>           <dbl>     <dbl>     <dbl>      <dbl>
## 1 (Intercept)  2552522.   535243.      4.77 0.00000760
## 2 St            385763.   260973.      1.48 0.143
## 3 Re             -6896.     1819.     -3.79 0.000279
## 4 Fr          -1495879.   493739.     -3.03 0.00324
##
## [[4]]
## # A tibble: 4 x 5
##   term             estimate   std.error statistic   p.value
##   <chr>               <dbl>       <dbl>     <dbl>     <dbl>
## 1 (Intercept)  20584518108. 4493750683.      4.58 0.0000157
## 2 St            3598811353. 2191050740.      1.64 0.104
## 3 Re             -56618241.   15271463.     -3.71 0.000372
## 4 Fr          -12259674651. 4145287458.     -2.96 0.00402
```

In the graphs above, we've plotted the residuals and Normal Q-Q plots for all of our linear regressions where each of the raw moments represents a response variable to the three predictors $< Re, Fr, St >$. The first row represents information with the first raw moment as the response, the second row represents the second raw moment, and so on. For this first assessment, we have no interaction variables.

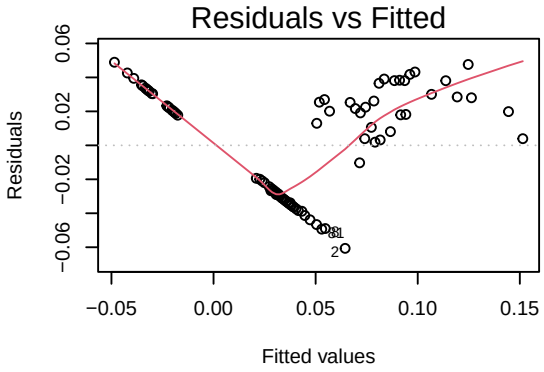
These attempts of linear regression suggests the following key notes:

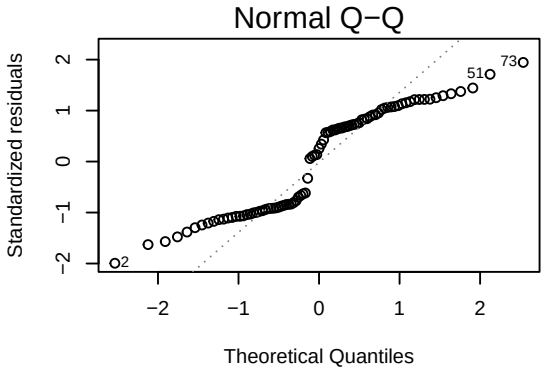
1. Using the Residual Standard Error and Adjusted R^2, they suggest a simple linear regression model does not fit the data well. Raw moments 2, 3, and 4 suffer from a large RSE and very low R^2 values. Raw moment 1 has a higher R^2 value of 0.6271 and a much lower RSE value of 0.0341; however, those values are indicative of weak linearity.

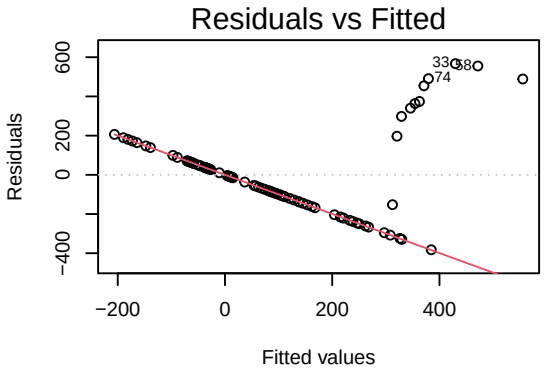
2. Looking at the residual plots, for all 4 raw moments it appears a linear model does fit well up until larger values of the response variable. This could be indicative of a needed transformation for the response values.

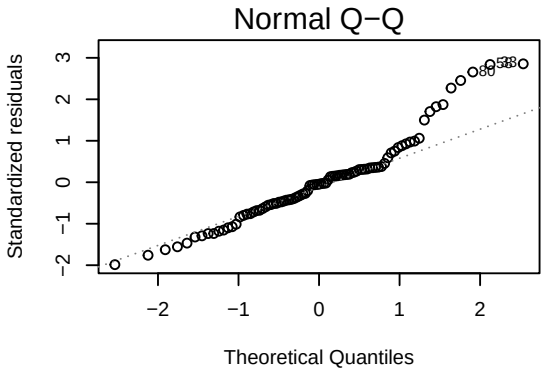
3. The QQ plots for raw moments 2, 3, and 4 also suggest linearity up until the larger values of the response variables. The QQ plot of raw moment 1 follows a "S" pattern that could suggest under-dispersed data.

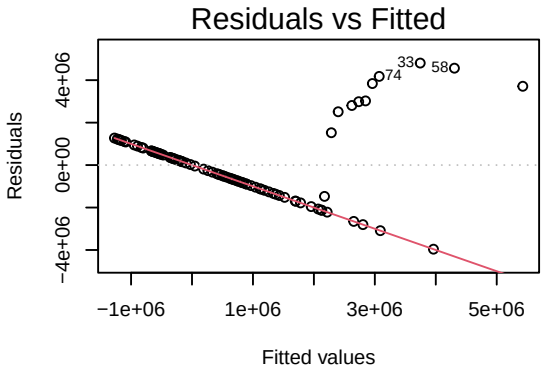
Linear Models, 2-Way interactions with backward stepwise selection optimizing AIC

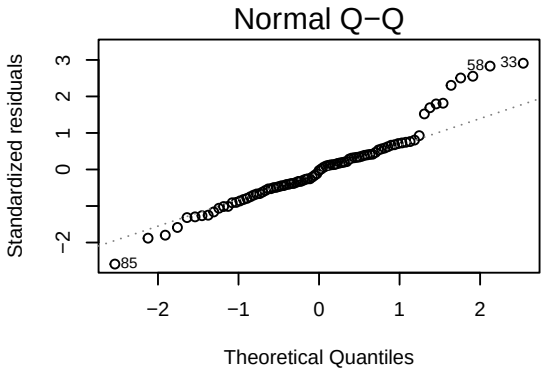


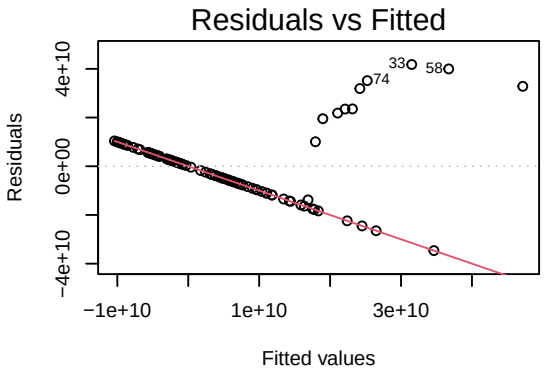


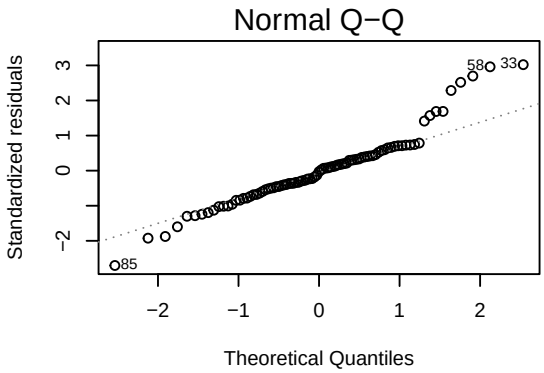










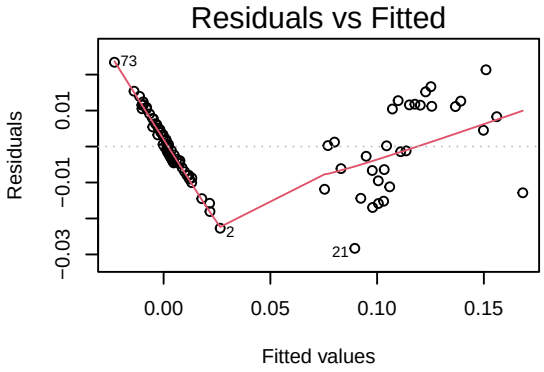


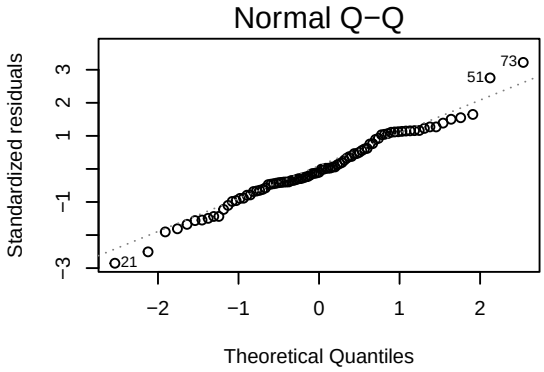
```
## [[1]]
## # A tibble: 6 x 5
##   term         estimate std.error statistic  p.value
##   <chr>           <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)  0.109    0.0137         7.95 8.34e-12
## 2 St           0.0343   0.00895        3.83 2.44e- 4
## 3 Re          -0.000350 0.0000588     -5.95 6.20e- 8
## 4 Fr          -0.0403   0.0181        -2.22 2.91e- 2
## 5 St:Re       -0.000102 0.0000389     -2.62 1.04e- 2
## 6 Re:Fr        0.000134 0.0000704      1.90 6.05e- 2
##
## [[2]]
## # A tibble: 6 x 5
##   term        estimate std.error statistic       p.value
##   <chr>          <dbl>     <dbl>     <dbl>         <dbl>
## 1 (Intercept)   485.      75.3        6.44 0.00000000735
## 2 St             89.1     39.2        2.27 0.0256
## 3 Re             -1.79     0.279     -6.40 0.00000000864
## 4 Fr           -564.     129.        -4.37 0.0000357
## 5 St:Fr        -102.      66.7       -1.53 0.130
## 6 Re:Fr           2.03     0.442      4.59 0.0000157
##
## [[3]]
## # A tibble: 7 x 5
##   term         estimate std.error statistic   p.value
##   <chr>           <dbl>     <dbl>     <dbl>     <dbl>
## 1 (Intercept)  3266842.   761115.      4.29 0.0000481
## 2 St           1434422.   540988.      2.65 0.00962
## 3 Re            -11719.     3124.     -3.75 0.000326
## 4 Fr          -4302932.  1086993.     -3.96 0.000160
## 5 St:Re          -2891.     2071.     -1.40 0.167
## 6 St:Fr       -1006722.   557641.     -1.81 0.0747
## 7 Re:Fr          15708.     3740.      4.20 0.0000673
##
## [[4]]
## # A tibble: 7 x 5
##   term             estimate   std.error statistic  p.value
##   <chr>               <dbl>       <dbl>     <dbl>    <dbl>
## 1 (Intercept)  25353543666. 6374383438.      3.98 0.000149
## 2 St           13419683960. 4530802291.      2.96 0.00400
## 3 Re             -92495227.   26159756.     -3.54 0.000672
## 4 Fr          -34340170960. 9103629018.     -3.77 0.000305
## 5 St:Re          -27759477.   17348818.     -1.60 0.113
## 6 St:Fr        -9237226130. 4670279707.     -1.98 0.0513
## 7 Re:Fr          128102753.   31320533.      4.09 0.000100
```

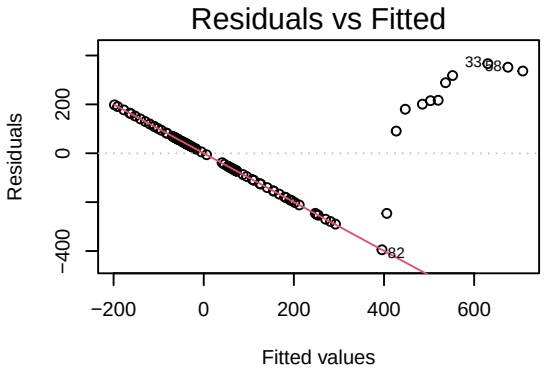
Important takeaways from backward selection:

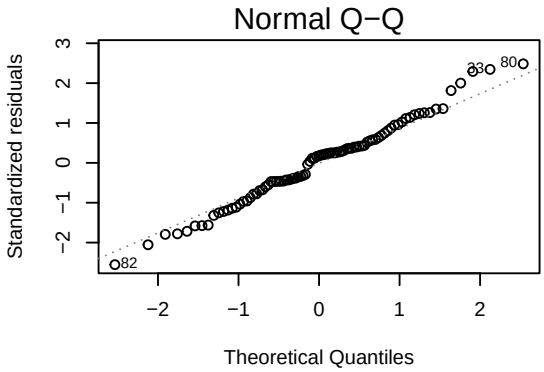
1. In all three raw moments for which backward selection was done, St and Fr resulted in an interaction term as well as Re and Fr.
2. Both raw moment 3 and 4 contain all 3 interaction terms whereas raw moment 2 leaves out the interaction between St and Re.

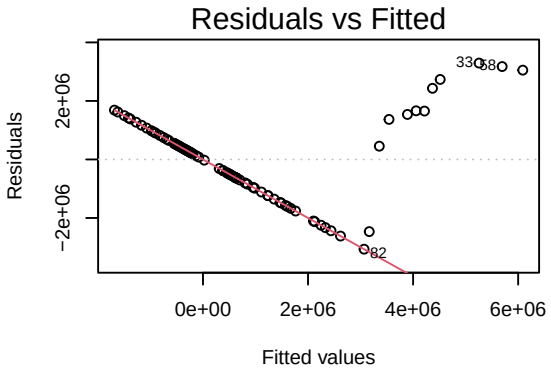
Nonlinear Models, 2-degree polynomials

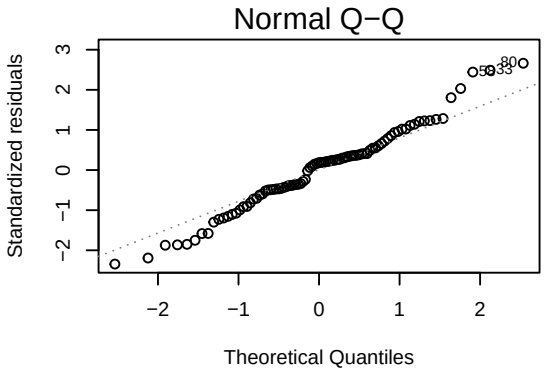


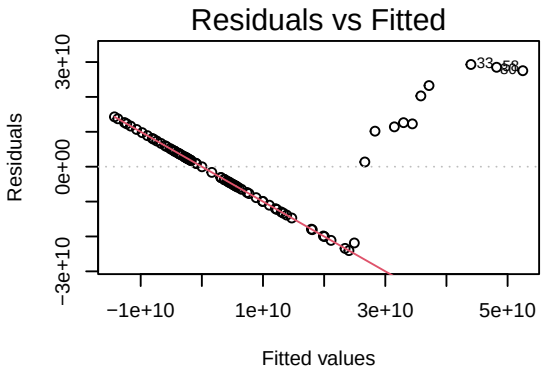


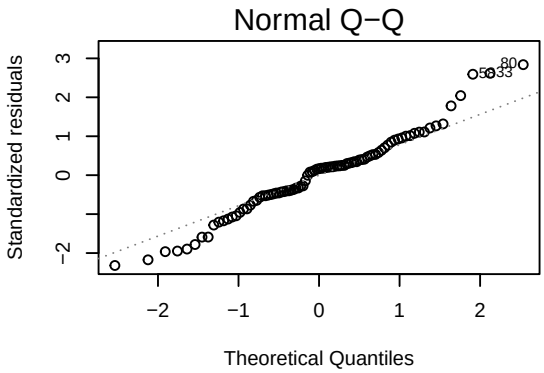












```
## [[1]]
## # A tibble: 10 x 5
##    term                               estimate std.error statistic  p.value
##    <chr>                                 <dbl>     <dbl>     <dbl>    <dbl>
##  1 (Intercept)                          0.0398   0.00112     35.6  2.15e-50
##  2 polym(St, Re, Fr, degree = 2)1.0.0   0.0830   0.0106       7.80 2.16e-11
##  3 polym(St, Re, Fr, degree = 2)2.0.0  -0.0126   0.0106      -1.18 2.41e- 1
##  4 polym(St, Re, Fr, degree = 2)0.1.0  -0.409    0.0110     -37.3  6.80e-52
##  5 polym(St, Re, Fr, degree = 2)1.1.0  -0.795    0.100       -7.93 1.18e-11
##  6 polym(St, Re, Fr, degree = 2)0.2.0   0.273    0.0108      25.3  1.21e-39
##  7 polym(St, Re, Fr, degree = 2)0.0.1  -0.0491   0.0107      -4.61 1.55e- 5
##  8 polym(St, Re, Fr, degree = 2)1.0.1   0.107    0.0995       1.08 2.84e- 1
##  9 polym(St, Re, Fr, degree = 2)0.1.1   0.400    0.0960       4.17 7.87e- 5
## 10 polym(St, Re, Fr, degree = 2)0.0.2   0.0312   0.0111       2.81 6.27e- 3
##
## [[2]]
## # A tibble: 10 x 5
##    term                               estimate std.error statistic  p.value
##    <chr>                                 <dbl>     <dbl>     <dbl>    <dbl>
##  1 (Intercept)                           82.6      17.6      4.69 1.14e- 5
##  2 polym(St, Re, Fr, degree = 2)1.0.0    252.      168.      1.50 1.36e- 1
##  3 polym(St, Re, Fr, degree = 2)2.0.0   -250.      167.     -1.49 1.40e- 1
##  4 polym(St, Re, Fr, degree = 2)0.1.0  -1209.      173.     -7.00 7.40e-10
##  5 polym(St, Re, Fr, degree = 2)1.1.0  -2874.     1579.     -1.82 7.25e- 2
##  6 polym(St, Re, Fr, degree = 2)0.2.0    457.      170.      2.69 8.71e- 3
##  7 polym(St, Re, Fr, degree = 2)0.0.1   -855.      168.     -5.09 2.37e- 6
##  8 polym(St, Re, Fr, degree = 2)1.0.1  -2547.     1567.     -1.63 1.08e- 1
##  9 polym(St, Re, Fr, degree = 2)0.1.1   8206.     1511.      5.43 6.04e- 7
## 10 polym(St, Re, Fr, degree = 2)0.0.2    977.      175.      5.59 3.15e- 7
##
## [[3]]
## # A tibble: 10 x 5
##    term                                 estimate std.error statistic     p.value
##    <chr>                                   <dbl>     <dbl>     <dbl>       <dbl>
##  1 (Intercept)                           669161.   149069.      4.49     2.41e-5
##  2 polym(St, Re, Fr, degree = 2)1.0.0   2393393.  1418774.      1.69     9.56e-2
##  3 polym(St, Re, Fr, degree = 2)2.0.0  -2162325.  1415836.     -1.53     1.31e-1
##  4 polym(St, Re, Fr, degree = 2)0.1.0  -9880668.  1461138.     -6.76     2.12e-9
##  5 polym(St, Re, Fr, degree = 2)1.1.0 -27675056. 13361157.     -2.07     4.16e-2
##  6 polym(St, Re, Fr, degree = 2)0.2.0   3667205.  1438359.      2.55     1.27e-2
##  7 polym(St, Re, Fr, degree = 2)0.0.1  -6985121.  1421149.     -4.92     4.72e-6
##  8 polym(St, Re, Fr, degree = 2)1.0.1 -24637317. 13264480.     -1.86     6.70e-2
##  9 polym(St, Re, Fr, degree = 2)0.1.1  66488589. 12790149.      5.20     1.54e-6
## 10 polym(St, Re, Fr, degree = 2)0.0.2   8012853.  1479412.      5.42     6.38e-7
##
## [[4]]
## # A tibble: 10 x 5
##    term                               estimate     std.error statistic   p.value
##    <chr>                                 <dbl>         <dbl>     <dbl>     <dbl>
##  1 (Intercept)                         5.48e 9   1260858222.      4.34   4.11e-5
##  2 polym(St, Re, Fr, degree = 2)1.0.0  2.23e10  12000294272.      1.86   6.73e-2
##  3 polym(St, Re, Fr, degree = 2)2.0.0 -1.83e10  11975440060.     -1.53   1.30e-1
##  4 polym(St, Re, Fr, degree = 2)0.1.0 -8.14e10  12358620391.     -6.58   4.64e-9
##  5 polym(St, Re, Fr, degree = 2)1.1.0 -2.59e11 113011518584.     -2.30   2.44e-2
```

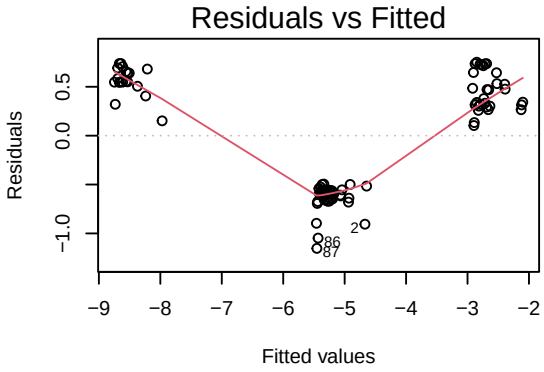
```
##  6 polym(St, Re, Fr, degree = 2)0.2.0  2.97e10  12165944422.      2.44   1.69e-2
##  7 polym(St, Re, Fr, degree = 2)0.0.1 -5.75e10  12020378119.     -4.78   7.97e-6
##  8 polym(St, Re, Fr, degree = 2)1.0.1 -2.31e11 112193808337.     -2.06   4.30e-2
##  9 polym(St, Re, Fr, degree = 2)0.1.1  5.42e11 108181817867.      5.01   3.20e-6
## 10 polym(St, Re, Fr, degree = 2)0.0.2  6.60e10  12513180914.      5.28   1.12e-6
```

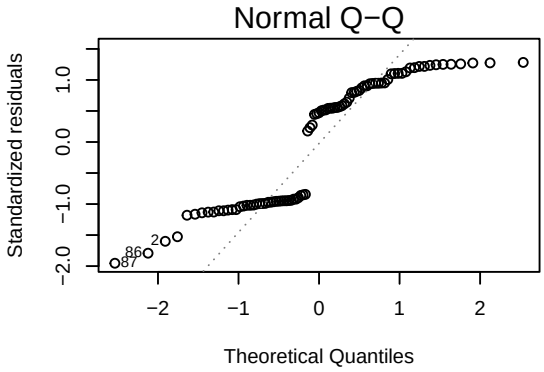
Here again, we now compare a set of 2-degree nonlinear polynomial regression models, again with each of the raw moments as response variables to the predictor set from before.

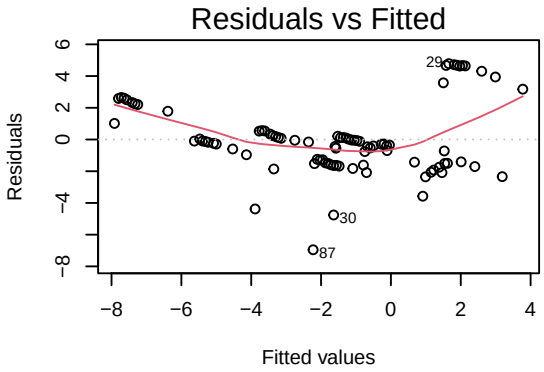
Attempting polynomial regression suggests the following key notes:

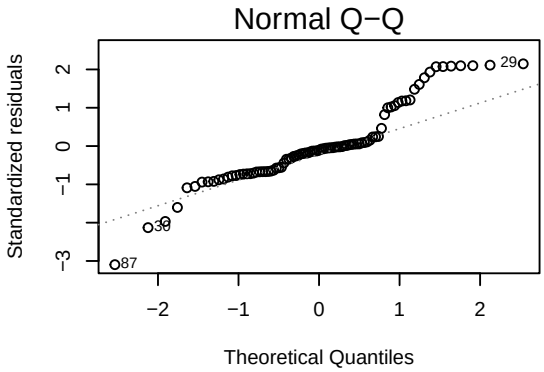
1. Polynomial regression only benefited raw moment 1, increasing its adjusted R^2 to 0.9646 and reducing its RSE to 0.01051. The QQ plot now follows a more linear pattern and the residual vs fitted plot also better follows the line of best fit. Thus, this suggests a polynomial model is better suited for raw moment 1.
2. Raw moments 2, 3, and 4 did not benefit from a polynomial model, each still suffered from a low R^2 and very high RSE.

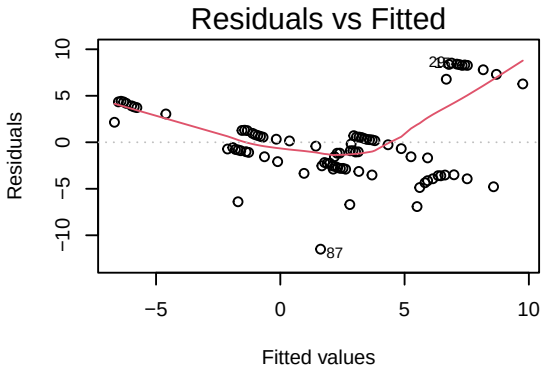
Log-transformed models

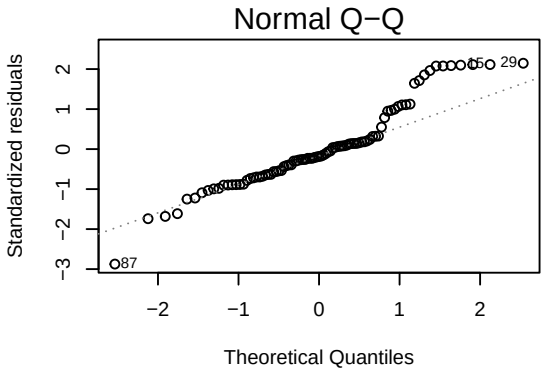


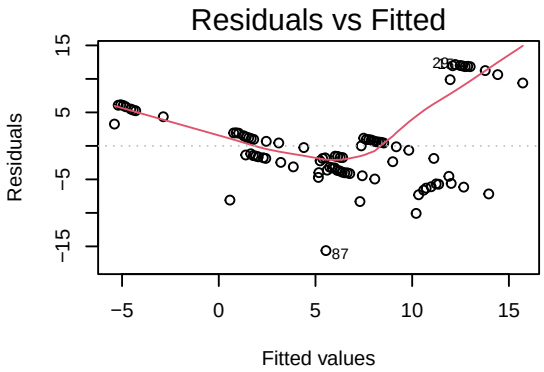


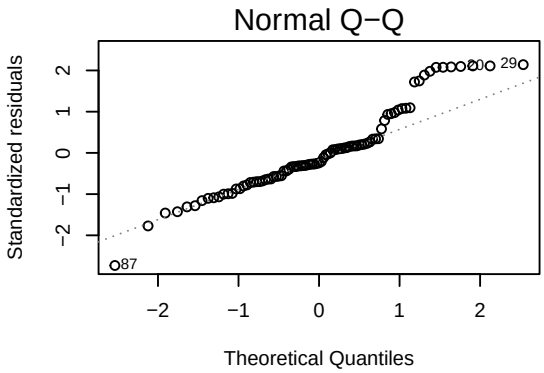




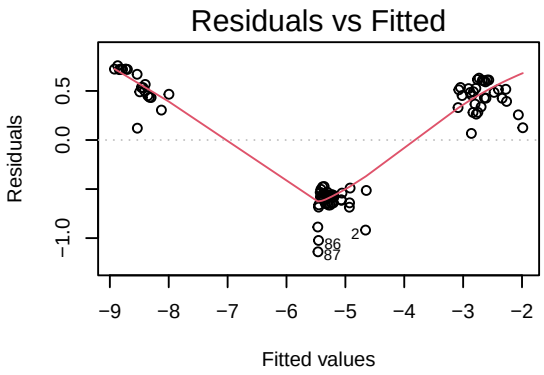


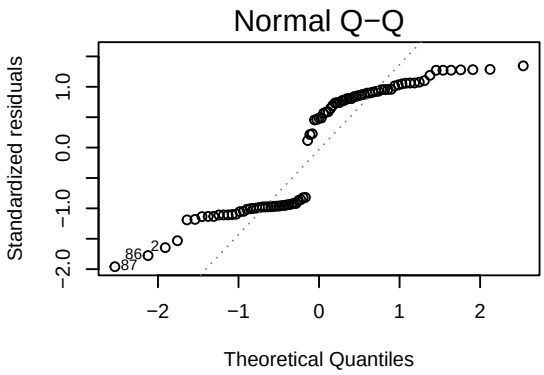


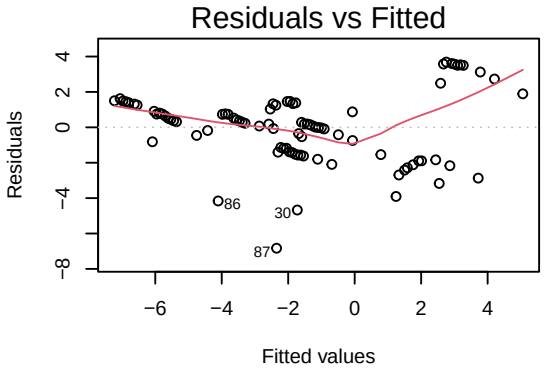


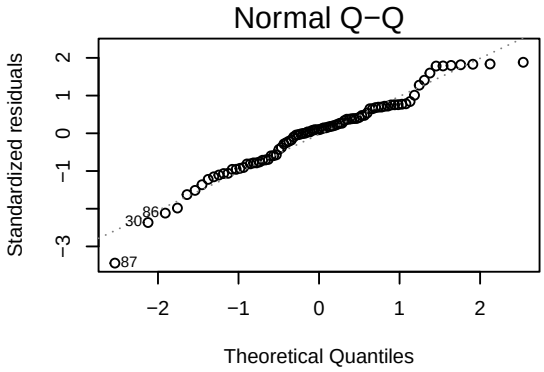


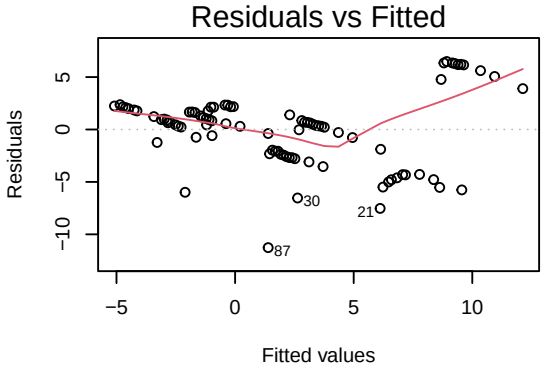
```
## [[1]]
## # A tibble: 4 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)  -1.22    0.166       -7.35  1.11e-10
## 2 St            0.268   0.0812       3.30  1.43e- 3
## 3 Re           -0.0190  0.000566   -33.5   8.56e-51
## 4 Fr            0.0281  0.154        0.183 8.55e- 1
##
## [[2]]
## # A tibble: 4 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)   3.63     0.634        5.73 1.50e- 7
## 2 St            0.784    0.309        2.54 1.29e- 2
## 3 Re           -0.0231   0.00215    -10.8  1.61e-17
## 4 Fr           -2.37     0.584       -4.06 1.09e- 4
##
## [[3]]
## # A tibble: 4 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)   9.39     1.13         8.32 1.27e-12
## 2 St            1.06     0.550        1.93 5.71e- 2
## 3 Re           -0.0286   0.00383     -7.45 7.08e-11
## 4 Fr           -4.74     1.04        -4.56 1.72e- 5
##
## [[4]]
## # A tibble: 4 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)  15.3      1.61         9.47 6.00e-15
## 2 St            1.30     0.787        1.65 1.04e- 1
## 3 Re           -0.0343   0.00549     -6.25 1.54e- 8
## 4 Fr           -7.09     1.49        -4.76 7.79e- 6
```

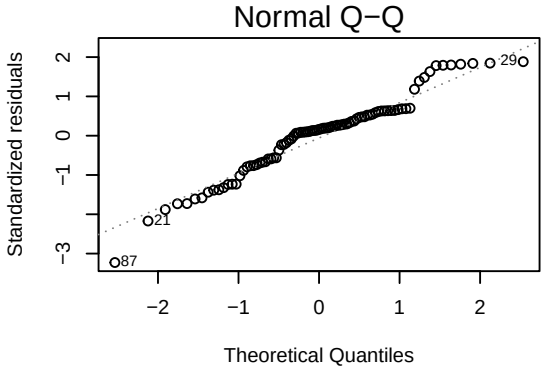

Residuals vs Fitted
Residuals
Fitted values
86
87
2

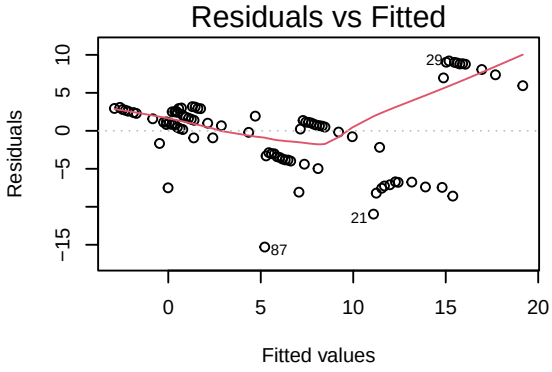

Normal Q−Q
Standardized residuals
Theoretical Quantiles
86
87
2

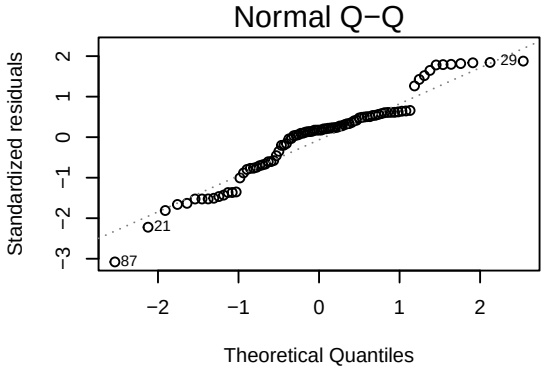

Residuals vs Fitted
Residuals
Fitted values
86
30
87


Normal Q−Q
Standardized residuals
Theoretical Quantiles
30
86
87

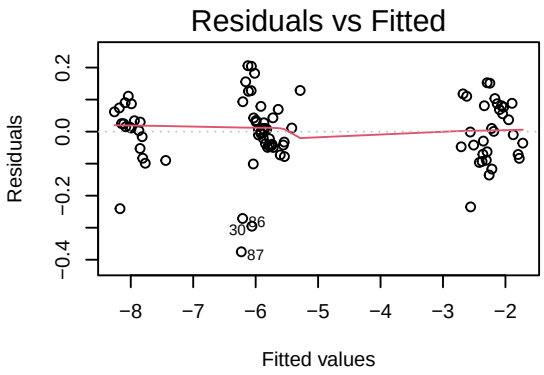

Residuals vs Fitted
Residuals
Fitted values
30
21
87

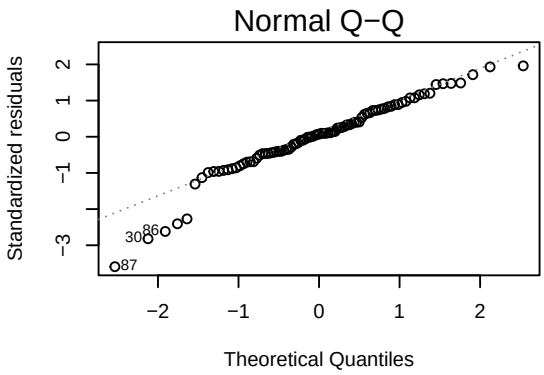

Normal Q−Q
Standardized residuals
Theoretical Quantiles
29
21
87

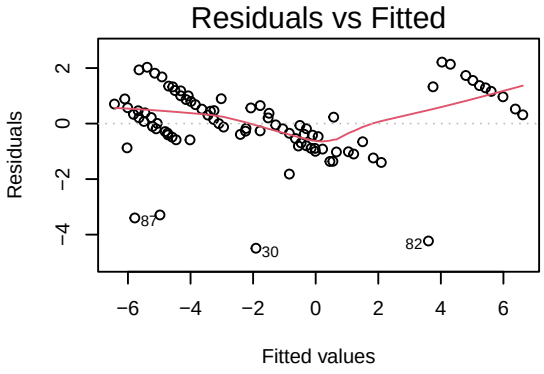

Residuals vs Fitted
Residuals
Fitted values
29
21
87

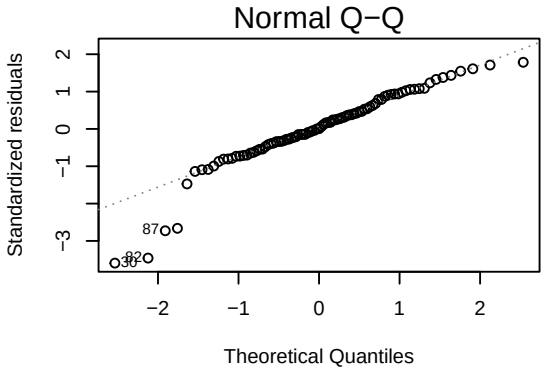

Normal Q−Q
Standardized residuals
Theoretical Quantiles
29
21
87

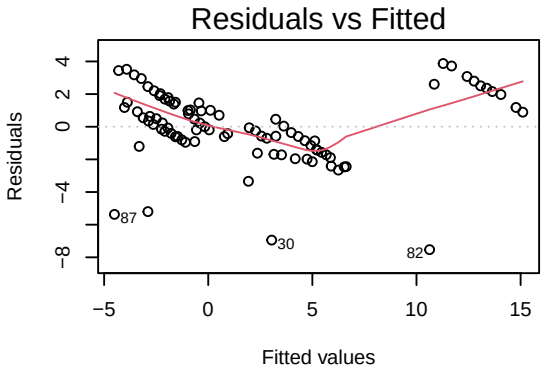
```
## [[1]]
## # A tibble: 5 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept) -0.992    0.206        -4.81 6.52e- 6
## 2 St           0.275    0.0801        3.43 9.28e- 4
## 3 Re          -0.0201   0.000804    -25.0  1.32e-40
## 4 Fr          -0.517    0.330         -1.57 1.20e- 1
## 5 Re:Fr        0.00237  0.00127        1.86 6.61e- 2
##
## [[2]]
## # A tibble: 5 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)   5.75     0.703         8.18 2.68e-12
## 2 St            0.852    0.273         3.12 2.49e- 3
## 3 Re           -0.0330   0.00274     -12.0  5.78e-20
## 4 Fr           -7.35     1.12         -6.54 4.60e- 9
## 5 Re:Fr         0.0216   0.00434       4.98 3.29e- 6
##
## [[3]]
## # A tibble: 5 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)  13.3      1.24         10.8  1.76e-17
## 2 St            1.19     0.481         2.47 1.56e- 2
## 3 Re           -0.0468   0.00482      -9.70 2.33e-15
## 4 Fr          -14.0      1.98         -7.06 4.52e-10
## 5 Re:Fr         0.0400   0.00763       5.24 1.15e- 6
##
## [[4]]
## # A tibble: 5 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)  21.0      1.76         11.9  9.44e-20
## 2 St            1.48     0.685         2.16 3.37e- 2
## 3 Re           -0.0608   0.00687      -8.85 1.18e-13
## 4 Fr          -20.5      2.82         -7.27 1.69e-10
## 5 Re:Fr         0.0581   0.0109        5.35 7.37e- 7
```

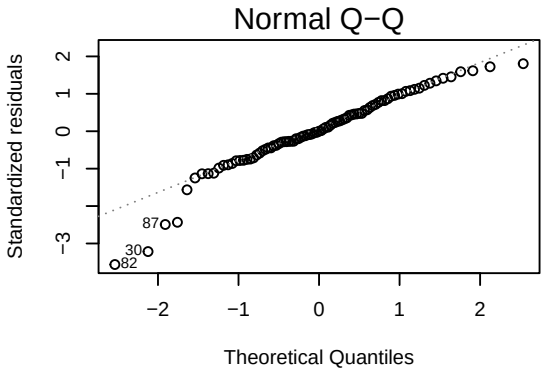

Residuals vs Fitted
Residuals
Fitted values
30
86
87

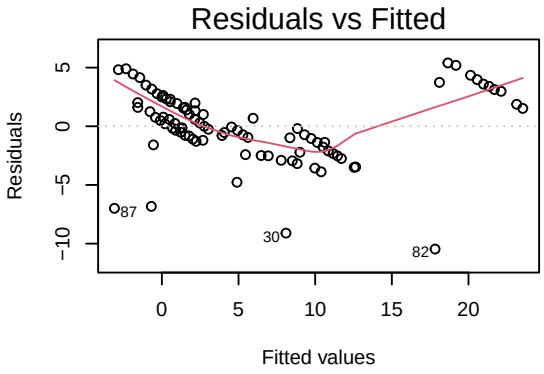

Normal Q−Q
Standardized residuals
Theoretical Quantiles
30
86
87

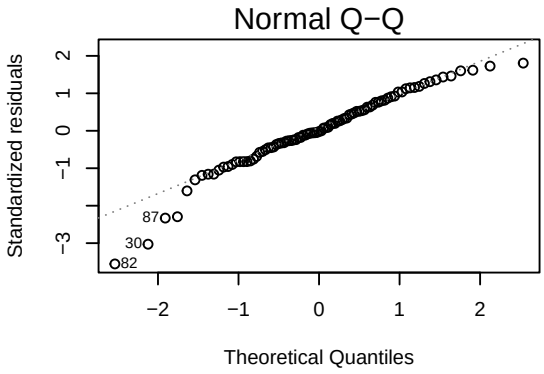

Residuals vs Fitted
Residuals
Fitted values
87
30
82


Normal Q−Q
Standardized residuals
Theoretical Quantiles
87
82
30


Residuals vs Fitted
Residuals
Fitted values
87
30
82


Normal Q−Q
Standardized residuals
Theoretical Quantiles
87
30
82


Residuals vs Fitted
Residuals
Fitted values
87
30
82

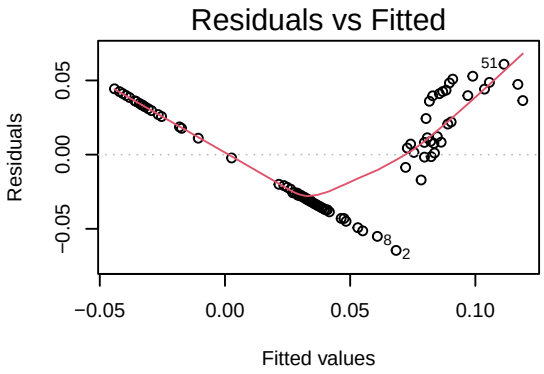

Normal Q−Q
Standardized residuals
Theoretical Quantiles
87
30
82

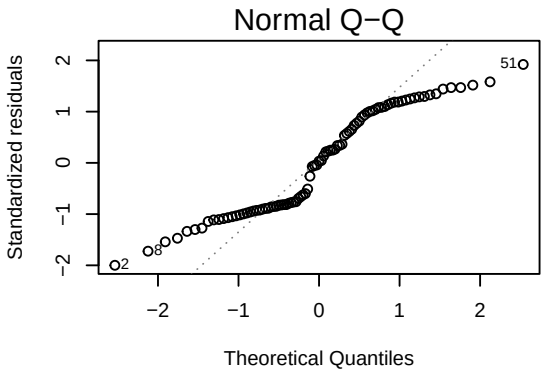
```
## [[1]]
## # A tibble: 10 x 5
##    term                                estimate std.error statistic   p.value
##    <chr>                                  <dbl>     <dbl>     <dbl>     <dbl>
##  1 (Intercept)                           -5.05     0.0117  -433.    4.12e-135
##  2 polym(St, Re, Fr, degree = 2)1.0.0     1.86     0.111     16.7   1.13e- 27
##  3 polym(St, Re, Fr, degree = 2)2.0.0    -0.612    0.111     -5.52  4.14e-  7
##  4 polym(St, Re, Fr, degree = 2)0.1.0   -20.3      0.114   -177.    1.70e-104
##  5 polym(St, Re, Fr, degree = 2)1.1.0    -1.29     1.05      -1.23  2.21e-  1
##  6 polym(St, Re, Fr, degree = 2)0.2.0     5.24     0.113     46.5   3.47e- 59
##  7 polym(St, Re, Fr, degree = 2)0.0.1    -0.110    0.111     -0.991 3.25e-  1
##  8 polym(St, Re, Fr, degree = 2)1.0.1     4.16     1.04       4.01  1.38e-  4
##  9 polym(St, Re, Fr, degree = 2)0.1.1     6.49     1.00       6.48  7.15e-  9
## 10 polym(St, Re, Fr, degree = 2)0.0.2     0.529    0.116      4.57  1.80e-  5
##
## [[2]]
## # A tibble: 10 x 5
##    term                                estimate std.error statistic  p.value
##    <chr>                                  <dbl>     <dbl>     <dbl>    <dbl>
##  1 (Intercept)                            -1.78     0.139   -12.8   6.51e-21
##  2 polym(St, Re, Fr, degree = 2)1.0.0      5.58     1.33      4.21  6.65e- 5
##  3 polym(St, Re, Fr, degree = 2)2.0.0     -4.93     1.32     -3.73  3.59e- 4
##  4 polym(St, Re, Fr, degree = 2)0.1.0    -28.3      1.36    -20.7   1.07e-33
##  5 polym(St, Re, Fr, degree = 2)1.1.0    -16.7     12.5      -1.34  1.86e- 1
##  6 polym(St, Re, Fr, degree = 2)0.2.0      5.12     1.34      3.81  2.75e- 4
##  7 polym(St, Re, Fr, degree = 2)0.0.1    -10.5      1.33     -7.94  1.16e-11
##  8 polym(St, Re, Fr, degree = 2)1.0.1      4.31    12.4       0.348 7.29e- 1
##  9 polym(St, Re, Fr, degree = 2)0.1.1     90.1     11.9       7.54  6.86e-11
## 10 polym(St, Re, Fr, degree = 2)0.0.2     12.4      1.38      9.00  9.79e-14
##
## [[3]]
## # A tibble: 10 x 5
##    term                                estimate std.error statistic  p.value
##    <chr>                                  <dbl>     <dbl>     <dbl>    <dbl>
##  1 (Intercept)                             1.94     0.241     8.08  6.20e-12
##  2 polym(St, Re, Fr, degree = 2)1.0.0      7.58     2.29      3.31  1.41e- 3
##  3 polym(St, Re, Fr, degree = 2)2.0.0     -7.57     2.29     -3.31  1.40e- 3
##  4 polym(St, Re, Fr, degree = 2)0.1.0    -37.5      2.36    -15.9   2.35e-26
##  5 polym(St, Re, Fr, degree = 2)1.1.0    -26.8     21.6      -1.24  2.18e- 1
##  6 polym(St, Re, Fr, degree = 2)0.2.0      5.30     2.32      2.28  2.53e- 2
##  7 polym(St, Re, Fr, degree = 2)0.0.1    -20.8      2.30     -9.07  6.97e-14
##  8 polym(St, Re, Fr, degree = 2)1.0.1      3.94    21.4       0.184 8.54e- 1
##  9 polym(St, Re, Fr, degree = 2)0.1.1    171.      20.7       8.29  2.32e-12
## 10 polym(St, Re, Fr, degree = 2)0.0.2     24.3      2.39     10.2   5.22e-16
##
## [[4]]
## # A tibble: 10 x 5
##    term                                estimate std.error statistic  p.value
##    <chr>                                  <dbl>     <dbl>     <dbl>    <dbl>
##  1 (Intercept)                             5.72     0.335    17.0   3.40e-28
##  2 polym(St, Re, Fr, degree = 2)1.0.0      9.28     3.19      2.91  4.73e- 3
##  3 polym(St, Re, Fr, degree = 2)2.0.0     -9.85     3.18     -3.09  2.75e- 3
##  4 polym(St, Re, Fr, degree = 2)0.1.0    -47.1      3.29    -14.3   1.15e-23
##  5 polym(St, Re, Fr, degree = 2)1.1.0    -35.2     30.0      -1.17  2.45e- 1
```

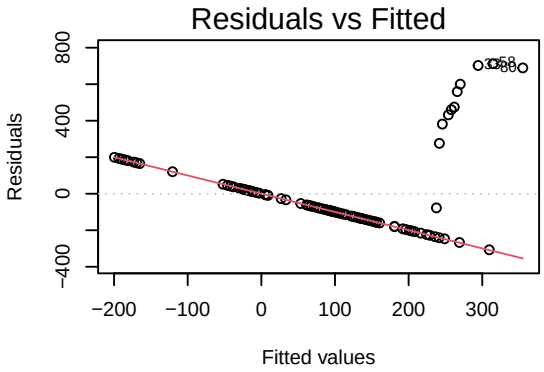
```
##  6 polym(St, Re, Fr, degree = 2)0.2.0     5.57     3.23      1.72  8.91e- 2
##  7 polym(St, Re, Fr, degree = 2)0.0.1   -31.0      3.20     -9.70  4.16e-15
##  8 polym(St, Re, Fr, degree = 2)1.0.1     3.36    29.8       0.113 9.11e- 1
##  9 polym(St, Re, Fr, degree = 2)0.1.1   252.      28.8       8.75  2.99e-13
## 10 polym(St, Re, Fr, degree = 2)0.0.2    36.0      3.33     10.8   2.82e-17
```

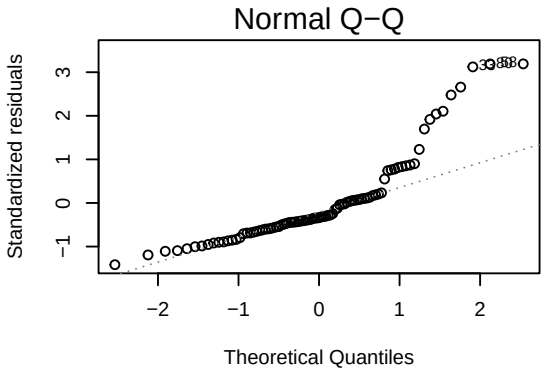
Mean/Variance/Skewness/Kurtosis as Response Variables

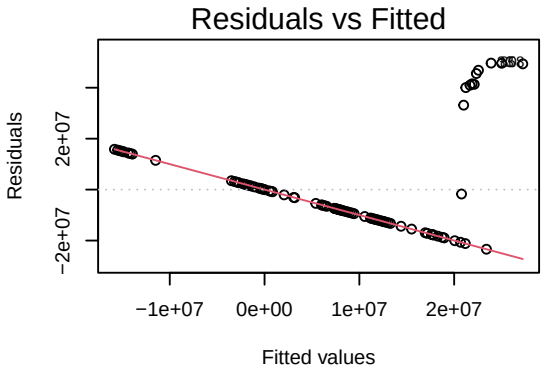
Linear Models, No Interactions

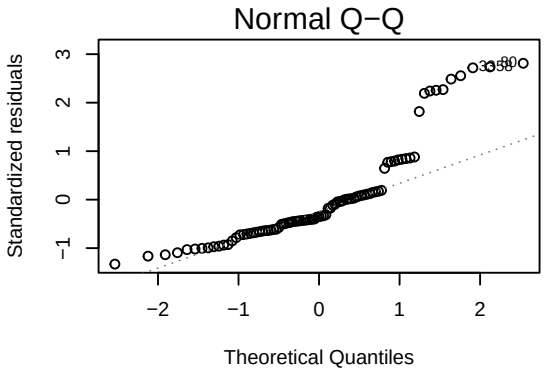


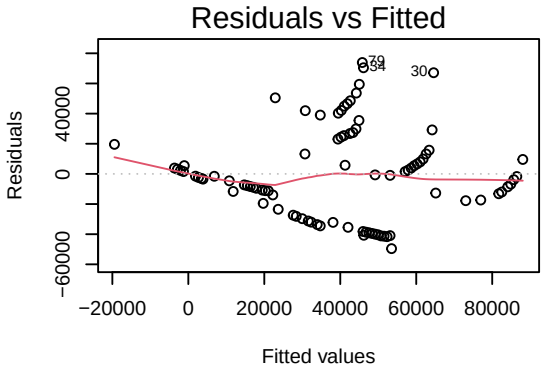


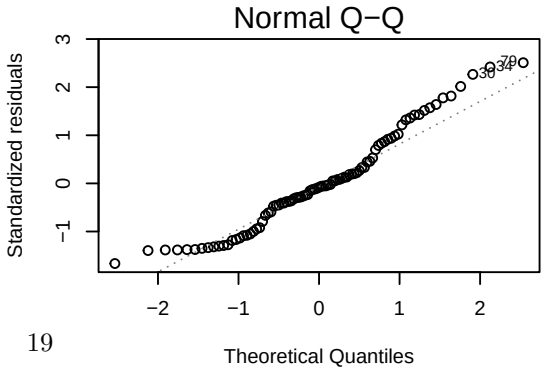






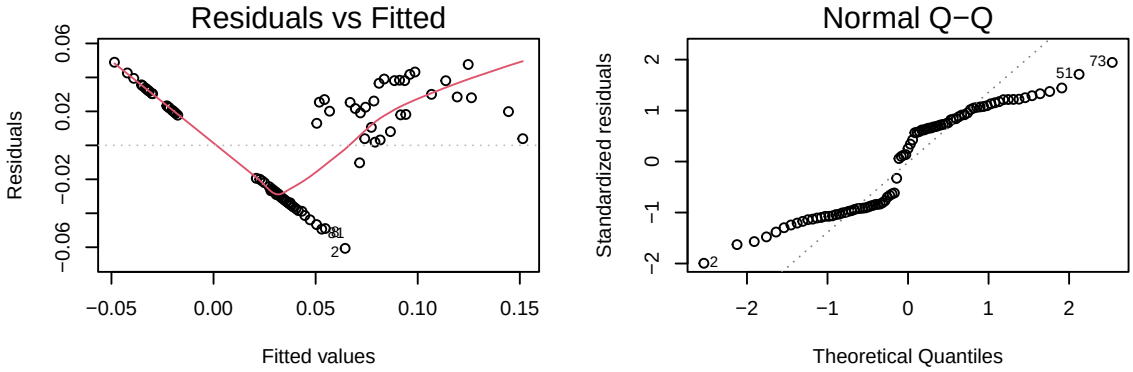


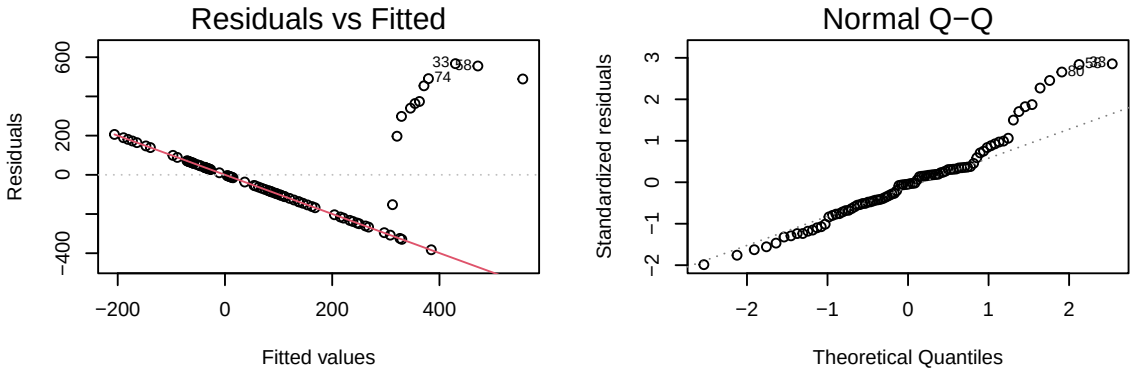


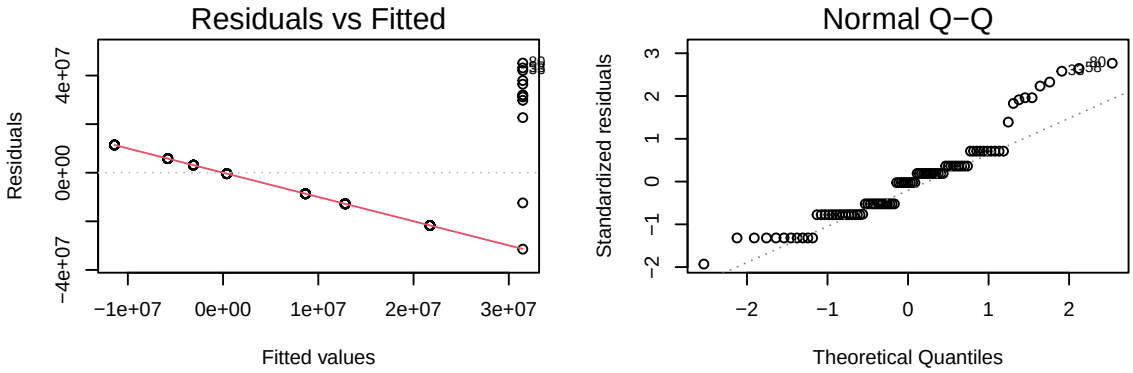


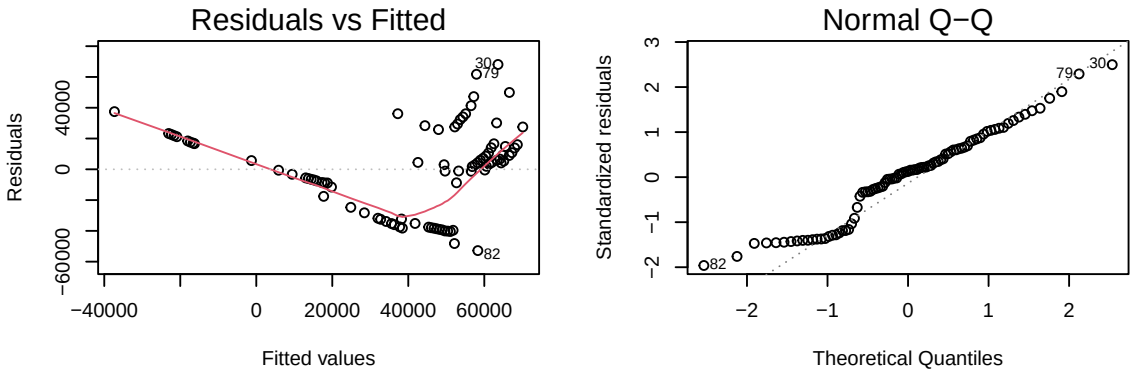
```
## [[1]]
## # A tibble: 4 x 5
##   term         estimate std.error statistic  p.value
##   <chr>           <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)  0.113    0.00950      11.9   7.45e-20
## 2 St           0.0133   0.00463       2.88  5.08e- 3
## 3 Re          -0.000378 0.0000323   -11.7   2.18e-19
## 4 Fr          -0.00790  0.00876      -0.902 3.70e- 1
##
## [[2]]
## # A tibble: 4 x 5
##   term        estimate std.error statistic    p.value
##   <chr>          <dbl>     <dbl>     <dbl>      <dbl>
## 1 (Intercept)  320.       63.9        5.00 0.00000308
## 2 St            40.5      31.2        1.30 0.198
## 3 Re            -0.847     0.217     -3.90 0.000192
## 4 Fr          -184.       59.0       -3.11 0.00251
##
## [[3]]
## # A tibble: 4 x 5
##   term          estimate std.error statistic     p.value
##   <chr>            <dbl>     <dbl>     <dbl>       <dbl>
## 1 (Intercept)  27743016.  5191661.     5.34  0.000000751
## 2 St            2227267.  2531336.     0.880 0.381
## 3 Re             -70896.    17643.    -4.02  0.000126
## 4 Fr          -15471352.  4789079.    -3.23  0.00176
##
## [[4]]
## # A tibble: 4 x 5
##   term        estimate std.error statistic     p.value
##   <chr>          <dbl>     <dbl>     <dbl>       <dbl>
## 1 (Intercept)   36568.    8400.       4.35 0.0000372
## 2 St            -7903.    4096.      -1.93 0.0570
## 3 Re              137.      28.5      4.81 0.00000659
## 4 Fr           -44656.    7749.      -5.76 0.000000129
```

Linear Models, 2-Way interactions with backward stepwise selection optimizing AIC



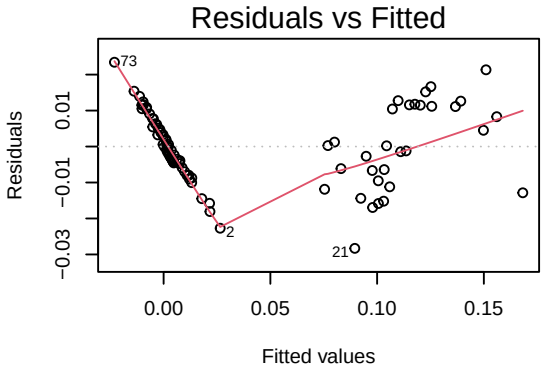


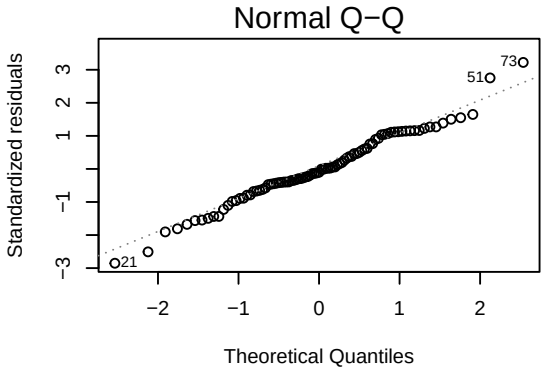


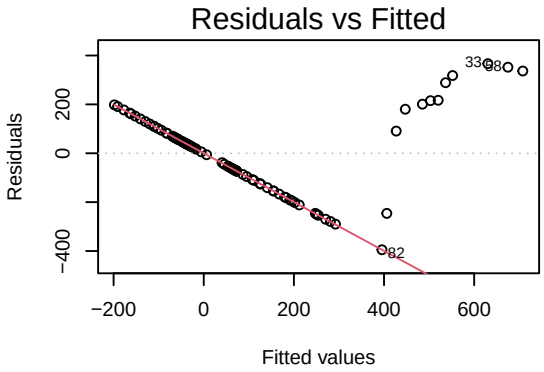


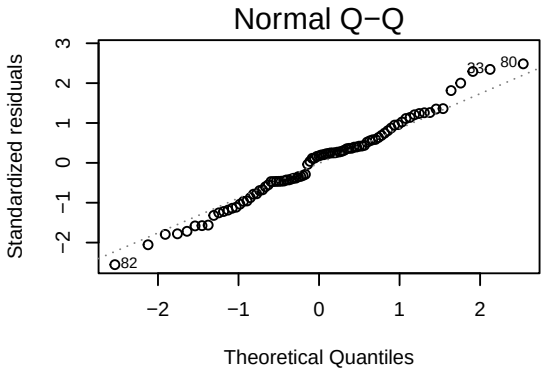
```
## [[1]]
## # A tibble: 6 x 5
##   term         estimate std.error statistic  p.value
##   <chr>           <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)  0.109    0.0137         7.95 8.34e-12
## 2 St           0.0343   0.00895        3.83 2.44e- 4
## 3 Re          -0.000350 0.0000588     -5.95 6.20e- 8
## 4 Fr          -0.0403   0.0181        -2.22 2.91e- 2
## 5 St:Re       -0.000102 0.0000389     -2.62 1.04e- 2
## 6 Re:Fr        0.000134 0.0000704      1.90 6.05e- 2
##
## [[2]]
## # A tibble: 6 x 5
##   term        estimate std.error statistic       p.value
##   <chr>          <dbl>     <dbl>     <dbl>         <dbl>
## 1 (Intercept)   485.      75.3        6.44 0.00000000735
## 2 St             89.1     39.2        2.27 0.0256
## 3 Re             -1.79     0.279     -6.40 0.00000000864
## 4 Fr           -564.     129.        -4.37 0.0000357
## 5 St:Fr        -102.      66.7       -1.53 0.130
## 6 Re:Fr           2.03     0.442      4.59 0.0000157
##
## [[3]]
## # A tibble: 4 x 5
##   term          estimate std.error statistic  p.value
##   <chr>            <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)  46829966.  5506841.      8.50 5.45e-13
## 2 Re            -147947.    22780.     -6.49 5.35e- 9
## 3 Fr          -54452516.  9344209.     -5.83 9.83e- 8
## 4 Re:Fr          168069.    35987.      4.67 1.12e- 5
##
## [[4]]
## # A tibble: 5 x 5
##   term         estimate std.error statistic       p.value
##   <chr>           <dbl>     <dbl>     <dbl>         <dbl>
## 1 (Intercept)   60324.     9711.      6.21  0.0000000192
## 2 St            -7140.     3774.     -1.89  0.0620
## 3 Re               27.0      37.9     0.712 0.478
## 4 Fr          -100438.    15532.     -6.47  0.00000000630
## 5 Re:Fr           242.       59.9     4.04  0.000117
```

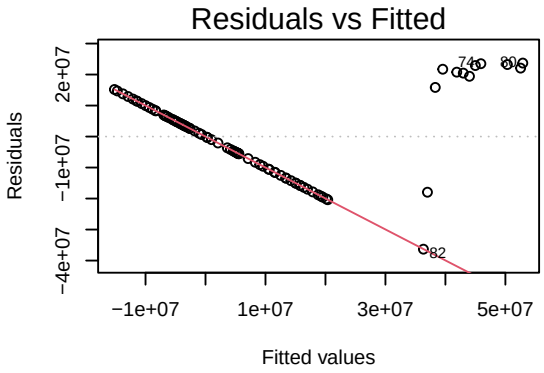
Nonlinear Models, 2-degree polynomials

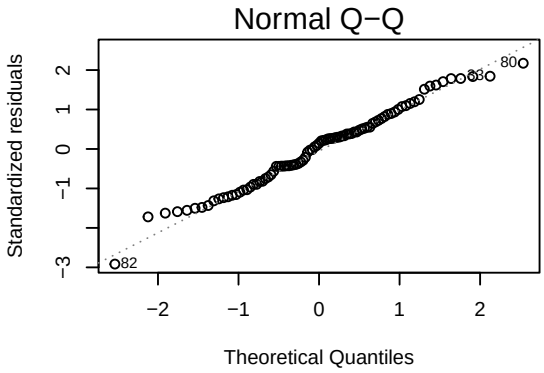


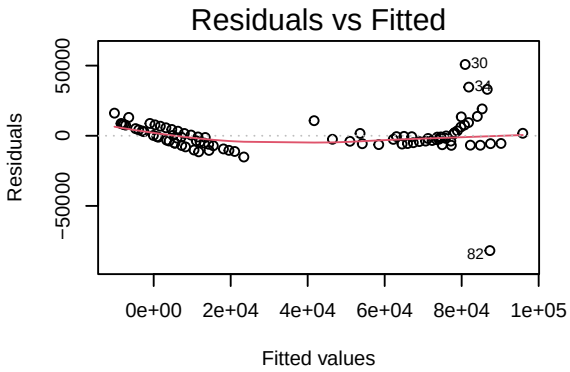


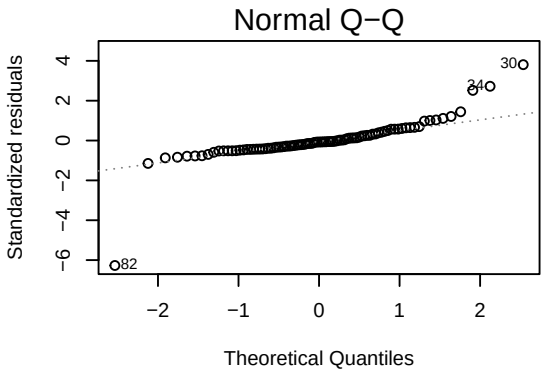








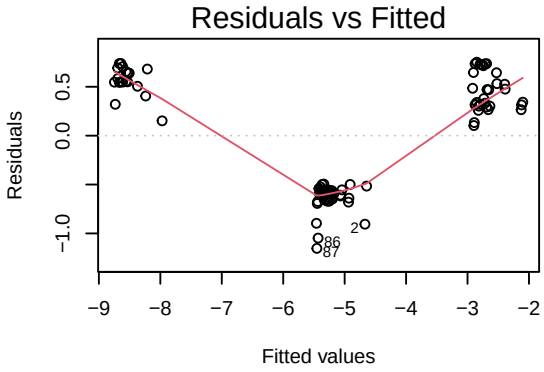


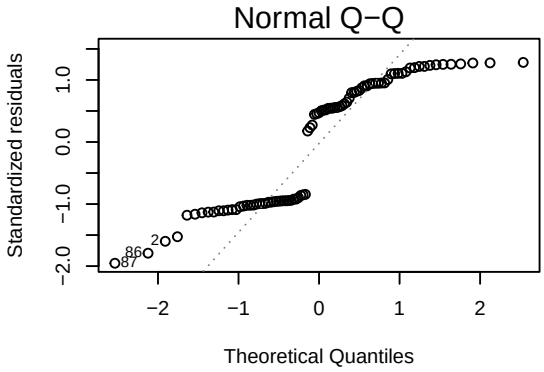


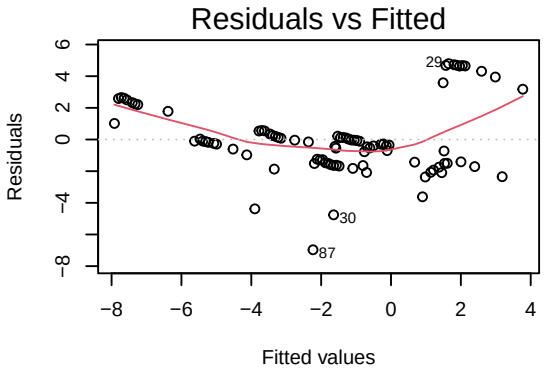
```
## [[1]]
## # A tibble: 10 x 5
##    term                                estimate std.error statistic  p.value
##    <chr>                                  <dbl>     <dbl>     <dbl>    <dbl>
##  1 (Intercept)                           0.0398   0.00112     35.6  2.15e-50
##  2 polym(St, Re, Fr, degree = 2)1.0.0    0.0830   0.0106       7.80 2.16e-11
##  3 polym(St, Re, Fr, degree = 2)2.0.0   -0.0126   0.0106      -1.18 2.41e- 1
##  4 polym(St, Re, Fr, degree = 2)0.1.0   -0.409    0.0110     -37.3  6.80e-52
##  5 polym(St, Re, Fr, degree = 2)1.1.0   -0.795    0.100       -7.93 1.18e-11
##  6 polym(St, Re, Fr, degree = 2)0.2.0    0.273    0.0108      25.3  1.21e-39
##  7 polym(St, Re, Fr, degree = 2)0.0.1   -0.0491   0.0107      -4.61 1.55e- 5
##  8 polym(St, Re, Fr, degree = 2)1.0.1    0.107    0.0995       1.08 2.84e- 1
##  9 polym(St, Re, Fr, degree = 2)0.1.1    0.400    0.0960       4.17 7.87e- 5
## 10 polym(St, Re, Fr, degree = 2)0.0.2    0.0312   0.0111       2.81 6.27e- 3
##
## [[2]]
## # A tibble: 10 x 5
##    term                                estimate std.error statistic  p.value
##    <chr>                                  <dbl>     <dbl>     <dbl>    <dbl>
##  1 (Intercept)                             82.6      17.6      4.69 1.14e- 5
##  2 polym(St, Re, Fr, degree = 2)1.0.0     252.      168.       1.50 1.36e- 1
##  3 polym(St, Re, Fr, degree = 2)2.0.0    -250.      167.      -1.49 1.40e- 1
##  4 polym(St, Re, Fr, degree = 2)0.1.0   -1209.      173.      -7.00 7.41e-10
##  5 polym(St, Re, Fr, degree = 2)1.1.0   -2874.     1579.      -1.82 7.25e- 2
##  6 polym(St, Re, Fr, degree = 2)0.2.0     457.      170.       2.69 8.72e- 3
##  7 polym(St, Re, Fr, degree = 2)0.0.1    -855.      168.      -5.09 2.37e- 6
##  8 polym(St, Re, Fr, degree = 2)1.0.1   -2547.     1567.      -1.63 1.08e- 1
##  9 polym(St, Re, Fr, degree = 2)0.1.1    8206.     1511.       5.43 6.04e- 7
## 10 polym(St, Re, Fr, degree = 2)0.0.2     977.      175.       5.59 3.15e- 7
##
## [[3]]
## # A tibble: 10 x 5
##    term                                   estimate  std.error statistic  p.value
##    <chr>                                     <dbl>      <dbl>     <dbl>    <dbl>
##  1 (Intercept)                            6945617.   1419066.      4.89 5.11e- 6
##  2 polym(St, Re, Fr, degree = 2)1.0.0    14106783.  13506044.      1.04 2.99e- 1
##  3 polym(St, Re, Fr, degree = 2)2.0.0   -17980770.  13478071.     -1.33 1.86e- 1
##  4 polym(St, Re, Fr, degree = 2)0.1.0  -100510268.  13909331.     -7.23 2.75e-10
##  5 polym(St, Re, Fr, degree = 2)1.1.0  -153220839. 127191756.     -1.20 2.32e- 1
##  6 polym(St, Re, Fr, degree = 2)0.2.0    39243409.  13692479.      2.87 5.32e- 3
##  7 polym(St, Re, Fr, degree = 2)0.0.1   -71445362.  13528648.     -5.28 1.10e- 6
##  8 polym(St, Re, Fr, degree = 2)1.0.1  -137986773. 126271443.     -1.09 2.78e- 1
##  9 polym(St, Re, Fr, degree = 2)0.1.1   698478835. 121756044.      5.74 1.70e- 7
## 10 polym(St, Re, Fr, degree = 2)0.0.2    81234318.  14083285.      5.77 1.49e- 7
##
## [[4]]
## # A tibble: 10 x 5
##    term                                estimate std.error statistic  p.value
##    <chr>                                  <dbl>     <dbl>     <dbl>    <dbl>
##  1 (Intercept)                           38719.     1487.    26.0   1.59e-40
##  2 polym(St, Re, Fr, degree = 2)1.0.0   -64206.    14150.    -4.54  2.01e- 5
##  3 polym(St, Re, Fr, degree = 2)2.0.0    23760.    14121.     1.68  9.64e- 2
##  4 polym(St, Re, Fr, degree = 2)0.1.0    91237.    14572.     6.26  1.86e- 8
##  5 polym(St, Re, Fr, degree = 2)1.1.0  -391634.   133254.    -2.94  4.32e- 3
```

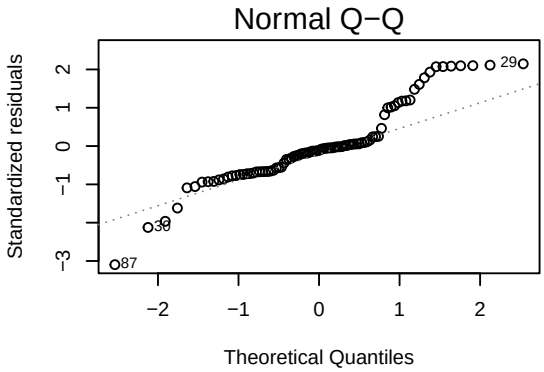
```
##  6 polym(St, Re, Fr, degree = 2)0.2.0   50261.    14345.     3.50  7.59e- 4
##  7 polym(St, Re, Fr, degree = 2)0.0.1 -191182.    14173.   -13.5   3.39e-22
##  8 polym(St, Re, Fr, degree = 2)1.0.1  129677.   132290.     0.980 3.30e- 1
##  9 polym(St, Re, Fr, degree = 2)0.1.1 1055166.   127560.     8.27  2.57e-12
## 10 polym(St, Re, Fr, degree = 2)0.0.2  208720.    14755.    14.1   2.34e-23
```

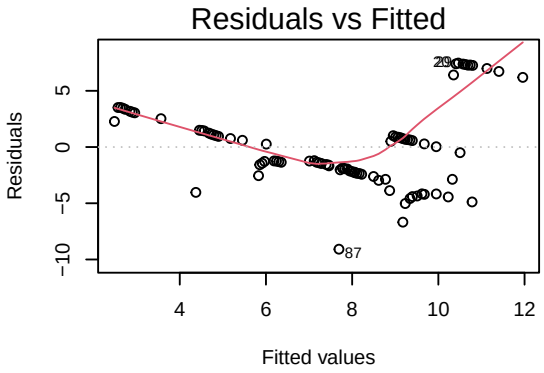
Log-transformed Models

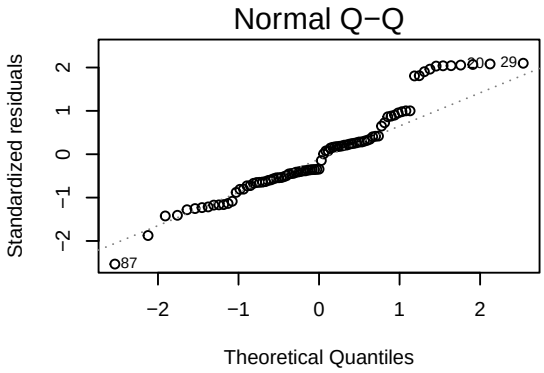


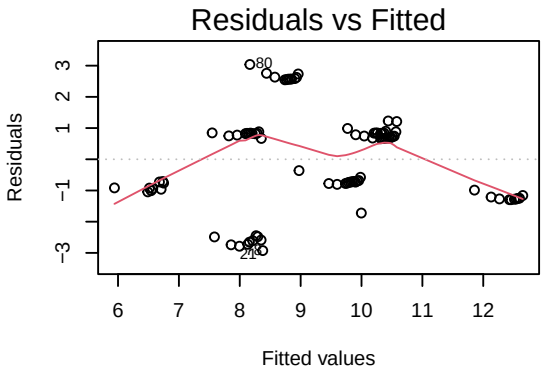


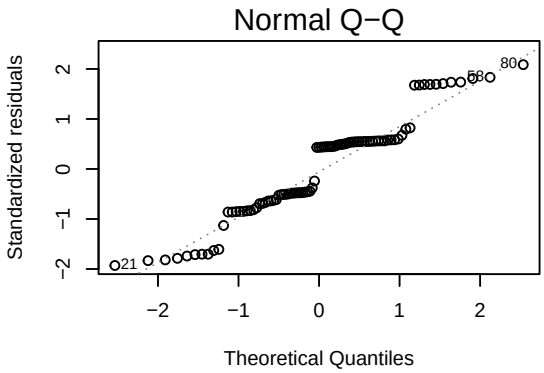




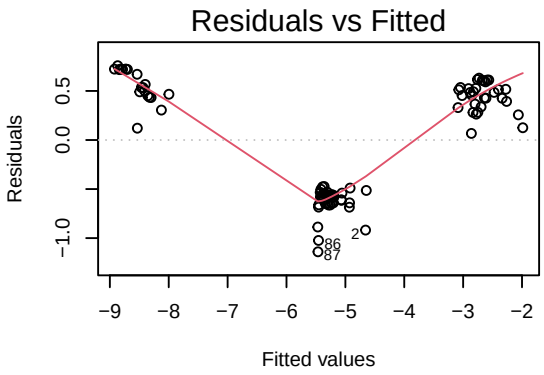


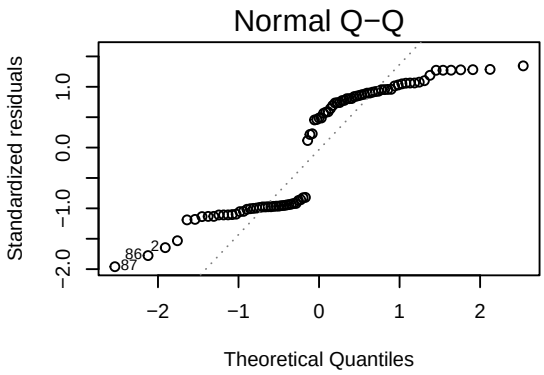


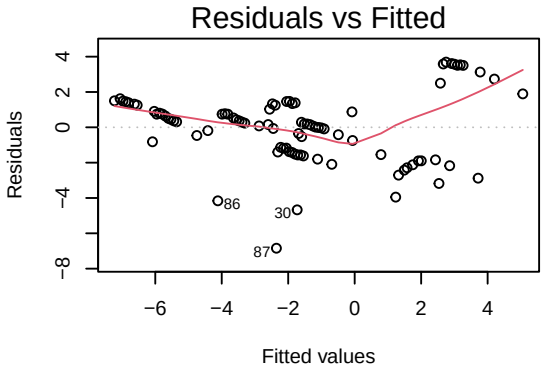


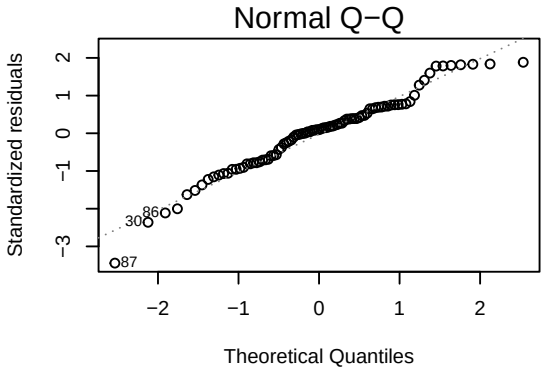


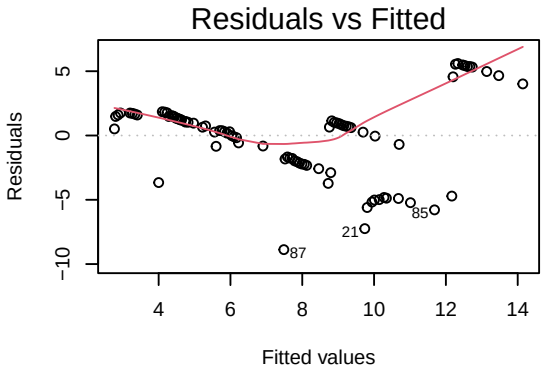
```
## [[1]]
## # A tibble: 4 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)  -1.22    0.166       -7.35  1.11e-10
## 2 St            0.268   0.0812       3.30  1.43e- 3
## 3 Re           -0.0190  0.000566   -33.5   8.56e-51
## 4 Fr            0.0281  0.154        0.183 8.55e- 1
##
## [[2]]
## # A tibble: 4 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)   3.62     0.635        5.70 1.71e- 7
## 2 St            0.787    0.309        2.54 1.29e- 2
## 3 Re           -0.0231   0.00216    -10.7  1.93e-17
## 4 Fr           -2.38     0.586       -4.06 1.08e- 4
##
## [[3]]
## # A tibble: 4 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)  11.5      1.01        11.4  8.82e-19
## 2 St            0.554    0.493        1.12 2.65e- 1
## 3 Re           -0.0109   0.00344     -3.16 2.21e- 3
## 4 Fr           -4.74     0.933       -5.08 2.19e- 6
##
## [[4]]
## # A tibble: 4 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)   8.03     0.431       18.6  8.74e-32
## 2 St           -0.274    0.210       -1.30 1.96e- 1
## 3 Re            0.0120   0.00146      8.19 2.39e-12
## 4 Fr           -2.35     0.397       -5.91 6.98e- 8
```

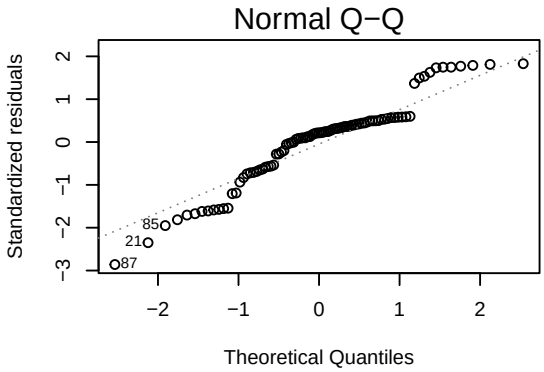

Residuals vs Fitted
Residuals
Fitted values

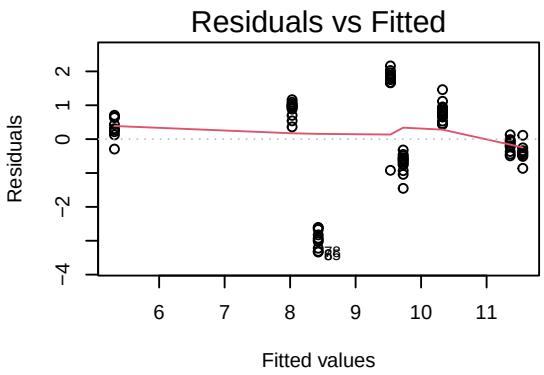

Normal Q−Q
Standardized residuals
Theoretical Quantiles

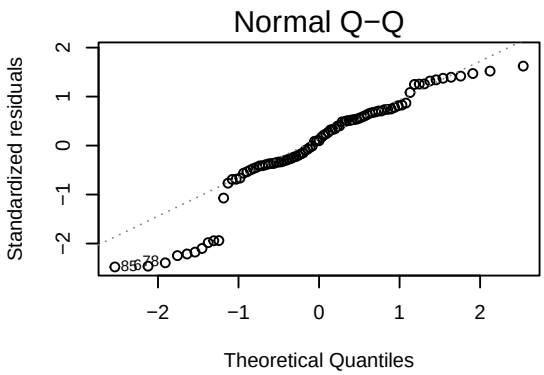

Residuals vs Fitted
Residuals
Fitted values


Normal Q−Q
Standardized residuals
Theoretical Quantiles

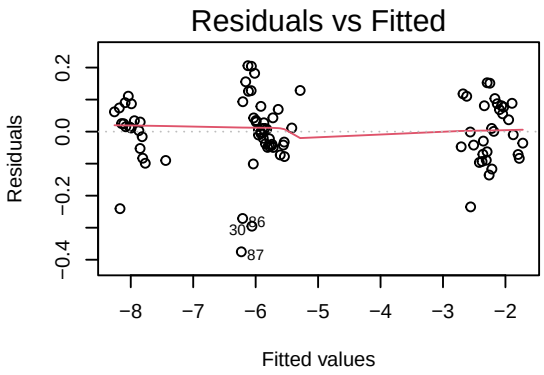

Residuals vs Fitted
Residuals
Fitted values

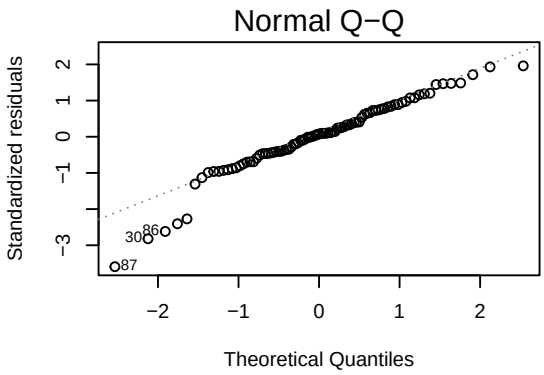

Normal Q−Q
Standardized residuals
Theoretical Quantiles

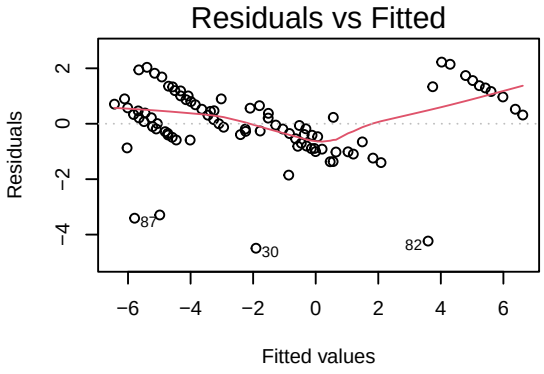

Residuals vs Fitted
Residuals
Fitted values

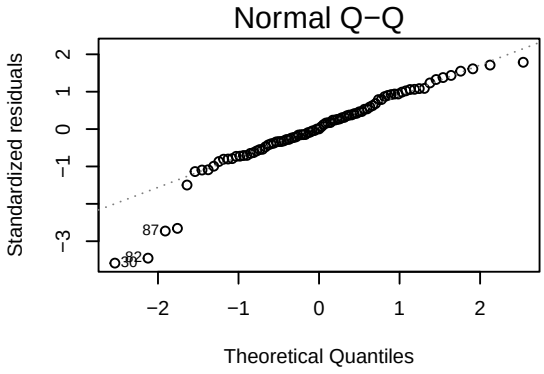

Normal Q−Q
Standardized residuals
Theoretical Quantiles

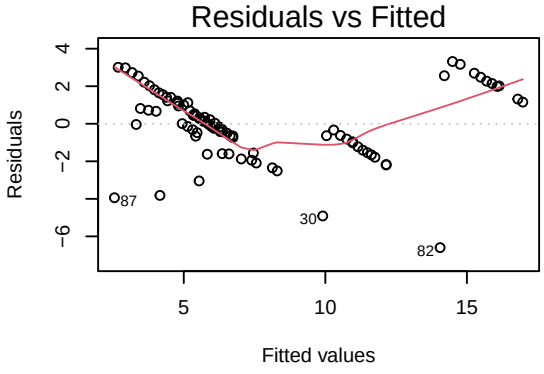
```
## [[1]]
## # A tibble: 5 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept) -0.992    0.206        -4.81 6.52e- 6
## 2 St           0.275    0.0801        3.43 9.28e- 4
## 3 Re          -0.0201   0.000804    -25.0  1.32e-40
## 4 Fr          -0.517    0.330         -1.57 1.20e- 1
## 5 Re:Fr        0.00237  0.00127        1.86 6.61e- 2
##
## [[2]]
## # A tibble: 5 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)   5.74     0.704         8.15 3.00e-12
## 2 St            0.855    0.274         3.12 2.46e- 3
## 3 Re           -0.0330   0.00275     -12.0  6.50e-20
## 4 Fr           -7.37     1.13         -6.54 4.50e- 9
## 5 Re:Fr         0.0217   0.00434       4.99 3.24e- 6
##
## [[3]]
## # A tibble: 5 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)  15.1      1.10         13.8  3.20e-23
## 2 St            0.670    0.428         1.57 1.21e- 1
## 3 Re           -0.0276   0.00429      -6.44 7.16e- 9
## 4 Fr          -13.2      1.76         -7.52 5.62e-11
## 5 Re:Fr         0.0368   0.00678       5.43 5.48e- 7
##
## [[4]]
## # A tibble: 4 x 5
##   term        estimate std.error statistic  p.value
##   <chr>          <dbl>     <dbl>     <dbl>    <dbl>
## 1 (Intercept)  9.30      0.450        20.7  6.53e-35
## 2 Re           0.00515   0.00186       2.77 6.89e- 3
## 3 Fr          -5.80      0.763        -7.60 3.52e-11
## 4 Re:Fr        0.0151    0.00294       5.13 1.77e- 6
```

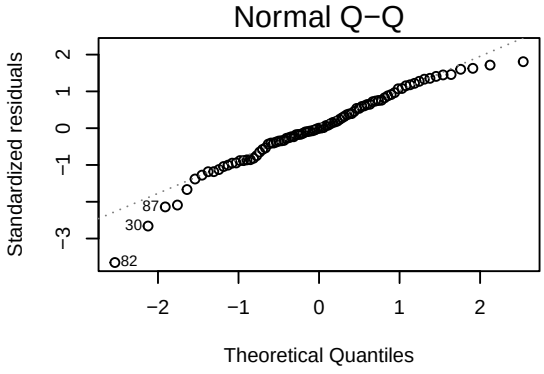

Residuals vs Fitted
Residuals
Fitted values
30
86
87

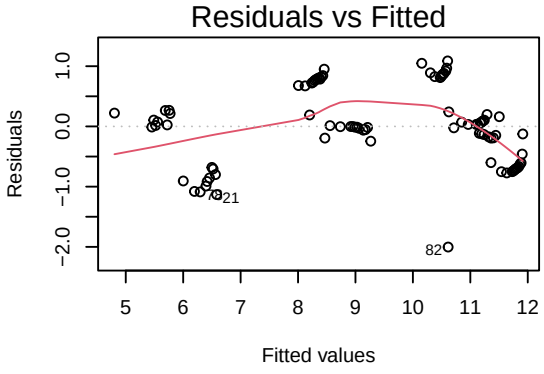

Normal Q−Q
Standardized residuals
Theoretical Quantiles
30
86
87

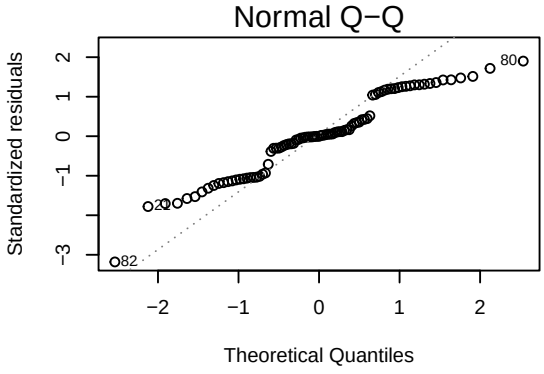

Residuals vs Fitted
Residuals
Fitted values
87
30
82


Normal Q−Q
Standardized residuals
Theoretical Quantiles
87
82
30


Residuals vs Fitted
Residuals
Fitted values
87
30
82


Normal Q−Q
Standardized residuals
Theoretical Quantiles
87
30
82


Residuals vs Fitted
Residuals
Fitted values
21
82


Normal Q−Q
Standardized residuals
Theoretical Quantiles
80
82

```
## [[1]]
## # A tibble: 10 x 5
##    term                                estimate std.error statistic   p.value
##    <chr>                                  <dbl>     <dbl>     <dbl>     <dbl>
##  1 (Intercept)                           -5.05     0.0117  -433.    4.12e-135
##  2 polym(St, Re, Fr, degree = 2)1.0.0     1.86     0.111     16.7   1.13e- 27
##  3 polym(St, Re, Fr, degree = 2)2.0.0    -0.612    0.111     -5.52  4.14e-  7
##  4 polym(St, Re, Fr, degree = 2)0.1.0   -20.3      0.114   -177.    1.70e-104
##  5 polym(St, Re, Fr, degree = 2)1.1.0    -1.29     1.05      -1.23  2.21e-  1
##  6 polym(St, Re, Fr, degree = 2)0.2.0     5.24     0.113     46.5   3.47e- 59
##  7 polym(St, Re, Fr, degree = 2)0.0.1    -0.110    0.111     -0.991 3.25e-  1
##  8 polym(St, Re, Fr, degree = 2)1.0.1     4.16     1.04       4.01  1.38e-  4
##  9 polym(St, Re, Fr, degree = 2)0.1.1     6.49     1.00       6.48  7.15e-  9
## 10 polym(St, Re, Fr, degree = 2)0.0.2     0.529    0.116      4.57  1.80e-  5
##
## [[2]]
## # A tibble: 10 x 5
##    term                                estimate std.error statistic  p.value
##    <chr>                                  <dbl>     <dbl>     <dbl>    <dbl>
##  1 (Intercept)                            -1.78     0.140   -12.8   6.50e-21
##  2 polym(St, Re, Fr, degree = 2)1.0.0      5.60     1.33      4.21  6.63e- 5
##  3 polym(St, Re, Fr, degree = 2)2.0.0     -4.95     1.33     -3.73  3.53e- 4
##  4 polym(St, Re, Fr, degree = 2)0.1.0    -28.3      1.37    -20.7   1.40e-33
##  5 polym(St, Re, Fr, degree = 2)1.1.0    -16.8     12.5      -1.34  1.83e- 1
##  6 polym(St, Re, Fr, degree = 2)0.2.0      5.09     1.35      3.77  3.09e- 4
##  7 polym(St, Re, Fr, degree = 2)0.0.1    -10.6      1.33     -7.93  1.17e-11
##  8 polym(St, Re, Fr, degree = 2)1.0.1      4.36    12.4       0.351 7.27e- 1
##  9 polym(St, Re, Fr, degree = 2)0.1.1     90.3     12.0       7.54  6.90e-11
## 10 polym(St, Re, Fr, degree = 2)0.0.2     12.5      1.39      8.99  1.00e-13
##
## [[3]]
## # A tibble: 10 x 5
##    term                                estimate std.error statistic  p.value
##    <chr>                                  <dbl>     <dbl>     <dbl>    <dbl>
##  1 (Intercept)                            7.45      0.206   36.2    6.35e-51
##  2 polym(St, Re, Fr, degree = 2)1.0.0     4.00      1.96     2.04   4.47e- 2
##  3 polym(St, Re, Fr, degree = 2)2.0.0    -5.28      1.96    -2.70   8.45e- 3
##  4 polym(St, Re, Fr, degree = 2)0.1.0   -18.5       2.02    -9.16   4.70e-14
##  5 polym(St, Re, Fr, degree = 2)1.1.0   -20.3      18.5     -1.10   2.75e- 1
##  6 polym(St, Re, Fr, degree = 2)0.2.0     0.355     1.99     0.179  8.59e- 1
##  7 polym(St, Re, Fr, degree = 2)0.0.1   -20.6       1.96   -10.5    1.28e-16
##  8 polym(St, Re, Fr, degree = 2)1.0.1    -0.738    18.3     -0.0403 9.68e- 1
##  9 polym(St, Re, Fr, degree = 2)0.1.1   163.       17.7      9.20   3.95e-14
## 10 polym(St, Re, Fr, degree = 2)0.0.2    23.7       2.04    11.6    9.43e-19
##
## [[4]]
## # A tibble: 10 x 5
##    term                                estimate std.error statistic  p.value
##    <chr>                                  <dbl>     <dbl>     <dbl>    <dbl>
##  1 (Intercept)                            9.27      0.0717  129.     1.04e-93
##  2 polym(St, Re, Fr, degree = 2)1.0.0    -1.89      0.683    -2.77   7.05e- 3
##  3 polym(St, Re, Fr, degree = 2)2.0.0     0.0204    0.681     0.0300 9.76e- 1
##  4 polym(St, Re, Fr, degree = 2)0.1.0     9.54      0.703    13.6    2.43e-22
##  5 polym(St, Re, Fr, degree = 2)1.1.0    -1.88      6.43     -0.292  7.71e- 1
```

```
##  6 polym(St, Re, Fr, degree = 2)0.2.0  -4.67      0.692    -6.74   2.30e- 9
##  7 polym(St, Re, Fr, degree = 2)0.0.1  -9.94      0.684   -14.5    5.01e-24
##  8 polym(St, Re, Fr, degree = 2)1.0.1  -5.26      6.38     -0.824  4.12e- 1
##  9 polym(St, Re, Fr, degree = 2)0.1.1  71.6       6.15     11.6    8.72e-19
## 10 polym(St, Re, Fr, degree = 2)0.0.2  11.2       0.712    15.7    5.87e-26
```