Appendix A

Derivation of the (Model Implied) Correlation of Item Response Times

In the following, λ_k and ϕ_k are the time intensity parameter and the speed sensitivity parameter of item k, respectively, $\sigma^2_{\epsilon_k}$ is the residual variance parameter of item k and ζ_i is the speed parameter of person i. We assume that over persons, speed is normally distributed $\zeta \sim \mathcal{N}(0, \sigma^2_\zeta)$. Further, if X is a log normally distributed variable with parameters μ and σ, then $X = \exp(Y)$ where $Y \sim \mathcal{N}(\mu, \sigma^2)$, which is the same as $ln(X) \sim \mathcal{N}(\mu, \sigma^2)$. Then, from Johnson Norman, Kotz, and Balakrishnan (1994), the expectation of X is:

$$\begin{aligned} \mathrm{E}(X) &= \mathrm{E}[\exp(Y)] \\ &= \exp\left(\mu + \frac{\sigma^2}{2}\right). \end{aligned} \tag{1}$$

In addition, the expectation of X^2 is:

$$\begin{aligned} \mathrm{E}(X^2) &= \mathrm{E}[(\exp(Y))^2] \\ &= \mathrm{E}[\exp(2Y)] \\ &= \mathrm{E}[\exp(Y^*)] \\ &= \exp\left(2\mu + 2\sigma^2\right), \end{aligned} \tag{2}$$

where $Y^* = 2Y$ and hence $Y^* \sim \mathcal{N}(2\mu, 4\sigma^2)$.

Based on the 3PLN lognormal measurement model for response times RT_{ik}, the expectation of the response time variable $\mathrm{E}(RT_k)$ of an item k can be written as

$$\begin{aligned} \mathrm{E}(RT_k) &= \mathrm{E}\left[\mathrm{E}(RT_{ik}|\zeta_i)\right] \\ &= \mathrm{E}\left[\exp\left(\lambda_k - \phi_k\zeta_i + \frac{\sigma^2_{\epsilon_k}}{2}\right)\right] \\ &= \mathrm{E}\left[\exp(Z)\right] \\ &= \exp\left(\lambda_k + \frac{\sigma^2_{\epsilon_k}}{2} + \frac{\phi_k^2\sigma^2_\zeta}{2}\right), \end{aligned} \tag{3}$$

where Z is a normally distributed: $Z \sim \mathcal{N}(\lambda_k + \frac{\sigma^2_{\epsilon_k}}{2}, \phi^2_k\sigma^2_\zeta)$.

Using similar steps in the derivation, the expectation of the squared response time variable $\mathrm{E}(RT^2_k)$ can be written as

$$\begin{aligned}
\mathrm{E}[RT^2_k] &= \mathrm{E}\left[\mathrm{E}(RT^2_{ik}|\zeta_i)\right] \\
&= \mathrm{E}\left[\exp\left(2\lambda_k - 2\phi_k\zeta_i + 2\sigma^2_{\epsilon_k}\right)\right] \\
&= \exp\left(2\lambda_k + 2\sigma^2_{\epsilon_k} + 2\phi^2_k\sigma^2_\zeta\right)
\end{aligned} \tag{4}$$

Therefore the variance of the response time variable of item k, $\mathrm{Var}(RT_k)$, can be denoted as

$$\begin{aligned}
\mathrm{Var}(RT_k) &= \mathrm{E}(RT^2_k) - \mathrm{E}(RT_k)^2 \\
&= \exp\left(2\lambda_k + 2\sigma^2_{\epsilon_k} + 2\phi^2_k\sigma^2_\zeta\right) - \exp\left(2\lambda_k + \sigma^2_{\epsilon_k} + \phi^2_k\sigma^2_\zeta\right) \\
&= \exp\left(2\lambda_k\right)\exp\left(\sigma^2_{\epsilon_k} + \phi^2_k\right)\exp\left(\sigma^2_{\epsilon_k} + \phi^2_k\right) - \exp\left(2\lambda_k\right)\exp\left(\sigma^2_{\epsilon_k} + \phi^2_k\right) \\
&= \exp\left(2\lambda_k + \sigma^2_{\epsilon_k} + \phi^2_k\sigma^2_\zeta\right)\left(\exp(\sigma^2_{\epsilon_k} + \phi^2_k\sigma^2_\zeta) - 1\right)
\end{aligned} \tag{5}$$

For the expectation of the distribution of the product of the response times of items k and l, $\mathrm{E}(RT_kRT_l)$, this gives

$$\begin{aligned}
\mathrm{E}(RT_kRT_l) &= \mathrm{E}\left[\exp\left(N(\lambda_k + \lambda_l - (\phi_k + \phi_l)\zeta, \sigma^2_{\epsilon_k} + \sigma^2_{\epsilon_l})\right)\right] \\
&= \exp\left(\lambda_k + \lambda_l + \frac{\sigma^2_{\epsilon_k} + \sigma^2_{\epsilon_l}}{2} + \frac{(\phi_k + \phi_l)^2\sigma^2_\zeta}{2}\right)
\end{aligned} \tag{6}$$

Therefore the covariance of the response time variables of two items k and l, $\mathrm{Cov}(RT_k, RT_l)$ and the respective product of the variances of these two items can be denoted as

$$\begin{aligned}
&\mathrm{Cov}(RT_k, RT_l) = \mathrm{E}(RT_kRT_l) - \mathrm{E}(RT_k)\mathrm{E}(RT_k) \\
&= \exp\left(\lambda_k + \lambda_l + \frac{\sigma^2_{\epsilon_k} + \sigma^2_{\epsilon_l}}{2} + \frac{(\phi_k + \phi_l)^2\sigma^2_\zeta}{2}\right) - \exp\left(\lambda_k + \lambda_l + \frac{\sigma^2_{\epsilon_k} + \sigma^2_{\epsilon_l}}{2} + \frac{(\phi^2_k + \phi^2_l)\sigma^2_\zeta}{2}\right) \\
&= \exp\left(\lambda_k + \lambda_l + \frac{\sigma^2_{\epsilon_k} + \sigma^2_{\epsilon_l}}{2} + \frac{(\phi^2_k + \phi^2_l)\sigma^2_\zeta}{2}\right)\left[\exp\left(\phi_k\phi_l\sigma^2_\zeta\right) - 1\right]
\end{aligned} \tag{7}$$

$$
\begin{aligned}
&\mathrm{Var}(RT_k)\mathrm{Var}(RT_l)\\
&\qquad\qquad = \exp\left(2\lambda_k + \sigma^2_{\epsilon_k} + \phi^2_k\sigma^2_\zeta\right)\left(\exp(\sigma^2_{\epsilon_k} + \phi^2_k\sigma^2_\zeta) - 1\right)\\
&\qquad\qquad\quad \exp\left(2\lambda_l + \sigma^2_{\epsilon_l} + \phi^2_l\sigma^2_\zeta\right)\left(\exp(\sigma^2_{\epsilon_l} + \phi^2_l\sigma^2_\zeta) - 1\right)\\
&\qquad\qquad = \exp\left(2\lambda_k + \sigma^2_{\epsilon_k} + \phi^2_k\sigma^2_\zeta\right)\exp\left(2\lambda_l + \sigma^2_{\epsilon_l} + \phi^2_l\sigma^2_\zeta\right)\\
&\qquad\qquad\quad \left(\exp(\sigma^2_{\epsilon_k} + \phi^2_k\sigma^2_\zeta) - 1\right)\left(\exp(\sigma^2_{\epsilon_l} + \phi^2_l\sigma^2_\zeta) - 1\right)\\
&\qquad\qquad = \exp\left(2\lambda_k + 2\lambda_l + \sigma^2_{\epsilon_k} + \sigma^2_{\epsilon_l} + \phi^2_k\sigma^2_\zeta + \phi^2_l\sigma^2_\zeta\right)\\
&\qquad\qquad\quad \left(\exp(\sigma^2_{\epsilon_k} + \phi^2_k\sigma^2_\zeta) - 1\right)\left(\exp(\sigma^2_{\epsilon_l} + \phi^2_l\sigma^2_\zeta) - 1\right)
\end{aligned}
\tag{8}
$$

This gives

$$
\begin{aligned}
\rho_{RT_k,RT_l} &= \frac{\mathrm{Cov}(RT_k, RT_l)}{\sqrt{\mathrm{Var}(RT_k)\mathrm{Var}(RT_l)}}\\
&= \frac{\left[\exp\left(\phi_k\phi_l\sigma^2_\zeta\right) - 1\right]}{\sqrt{\left(\exp(\sigma^2_{\epsilon_k} + \phi^2_k\sigma^2_\zeta) - 1\right)\left(\exp(\sigma^2_{\epsilon_l} + \phi^2_l\sigma^2_\zeta) - 1\right)}}
\end{aligned}
\tag{9}
$$

This is the model implied correlation of the two response time distributions of items k and i under the 3PLN model. Under the 2PLN model $phi_k = phi_l = 1$, therefore the respective correlation is

$$
\frac{\left[\exp\left(\sigma^2_\zeta\right) - 1\right]}{\sqrt{\left(\exp(\sigma^2_{\epsilon_k} + \sigma^2_\zeta) - 1\right)\left(\exp(\sigma^2_{\epsilon_l} + \sigma^2_\zeta) - 1\right)}}
\tag{10}
$$

Appendix B

Item Log Response TIme Distributions

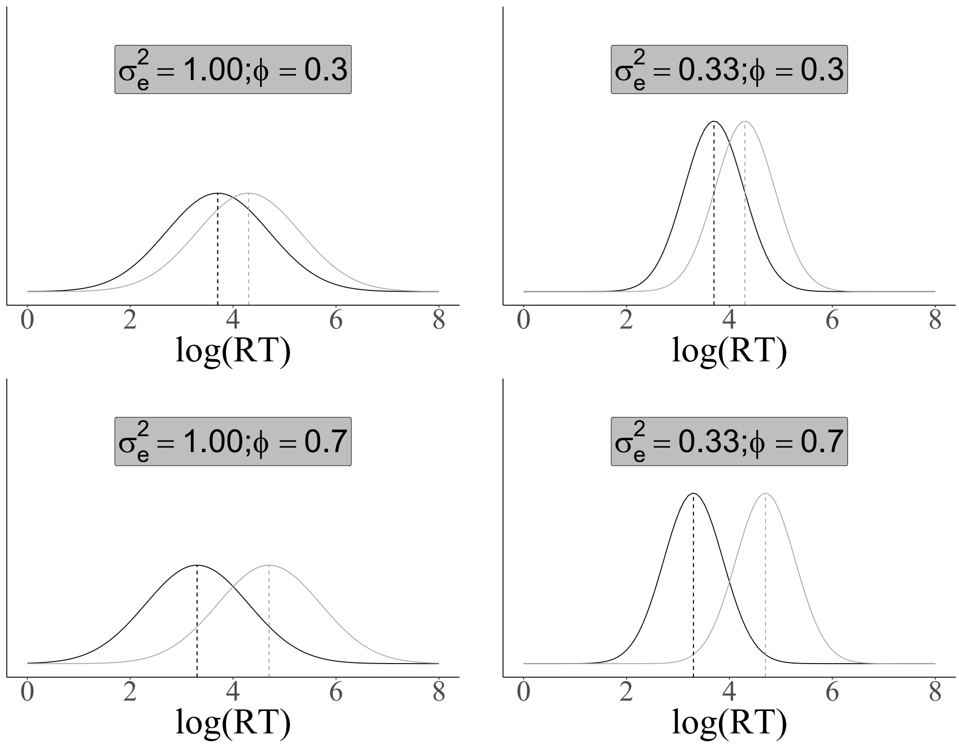

Figure B1. Expected log response time distributions of a fast person with $\zeta_1 = 1$ (black line) and a slow person with $\zeta_2 = -1$ (grey line) on four different items, all with $\lambda_k = 4$. Dotted lines indicate the medians of the corresponding distributions.

Appendix C

Response Time Characteristic Curve

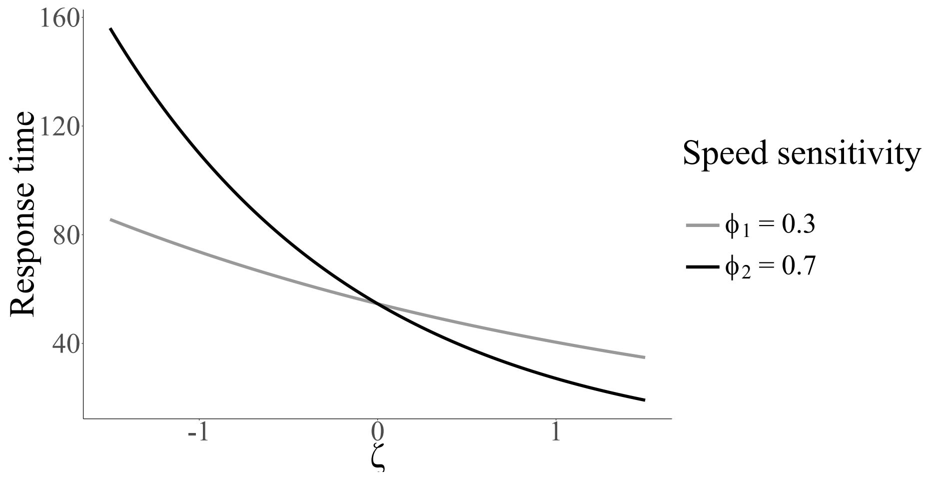

Figure C1. Response Time Characteristic Curve of two items with identical time intensity ($\lambda_k = 4$) and differing item speed sensitivity parameters ϕ_k.

Appendix D

Priors for Empirical Data Analysis

The identity matrix is notated as I_n with the size of n. $\sigma_{\theta_i,\zeta_i}$ is truncated to stay in range of $-\sqrt{\sigma_\theta^2\sigma_\zeta^2}$ and $\sqrt{\sigma_\theta^2\sigma_\zeta^2}$ (with $\sigma_\theta^2 = 1$ for model identification) to keep the person parameter covariance matrix positive definite. Priors for the HRT with a 2PL model for ability and a 2PLN model for speed:

$$\begin{gathered}
\Sigma_P \sim InverseWishart(I_3, 4) \\
\sigma_{\theta_i,\zeta_i} \sim N(0, 10000) \text{ truncated at } [-\sigma_\zeta, \sigma_\zeta] \\
\frac{1}{\sigma_\zeta^2} \sim \Gamma(0.01, 0.01) \\
\frac{1}{\sigma_{\sigma_\epsilon}^2} \sim \Gamma(0.01, 0.001) \\
\mu_{\sigma_\epsilon} \sim N(0, 1000000) \\
\mu_b \sim N(0, 1000000) \\
\mu_a \sim N(1, 1000000) \\
\mu_\lambda \sim N(1, 1000000)
\end{gathered} \tag{11}$$

Priors for the HRT with a 2PL model for ability and a 3PLN model for speed:

$$\begin{gathered}
\Sigma_P \sim InverseWishart(I_4, 5) \\
\sigma_{\theta_i,\zeta_i} \sim N(0, 10000) \\
\frac{1}{\sigma_{\sigma_\epsilon}^2} \sim \Gamma(0.01, 0.001) \\
\mu_{\sigma_\epsilon} \sim N(0, 1000000) \\
\mu_b \sim N(0, 1000000) \\
\mu_a \sim N(1, 1000000) \\
\mu_\lambda \sim N(1, 1000000) \\
\mu_\phi \sim N(1, 1000000)
\end{gathered} \tag{12}$$

Appendix E

Empirical Model Fit

Table E1

DIC for the HRT with the 2PLN and the 3PLN and the corresponding difference for all Math booklets.

Booklet	$DIC(3PLN)$	$DIC(2PLN)$	Δ_{DIC}
M01	251955	253042	1087
M02	213884	215179	1295
M03	231336	231690	354
M04	256032	256617	585
M05	257370	257703	333
M06ab	267682	268551	869

Appendix F

Multivariate Normal Distributions for Data Generation

Means of the multivariate normal distribution:

$$\mu_I = (\mu_a = 1.12, \mu_b = 0.54, \mu_\phi = 0.3, \mu_\lambda = 4.26) \tag{13}$$

Covariances of the multivariate normal distribution:

$$\Sigma_I = \begin{pmatrix} \sigma_a^2 = 0.45 & & & \\ \sigma_{b,a} = 0.05 & \sigma_b^2 = 1.00 & & \\ \sigma_{\phi,a} = 0.01 & \sigma_{\phi,b} = 0.03 & \sigma_\phi^2 = 0.01 & \\ \sigma_{\lambda,a} = -0.02 & \sigma_{\lambda,b} = 0.13 & \sigma_{\lambda,\phi} = 0.01 & \sigma_\lambda^2 = 0.25 \end{pmatrix}. \tag{14}$$

Appendix G

Item Numbers Not Reached in Simulation

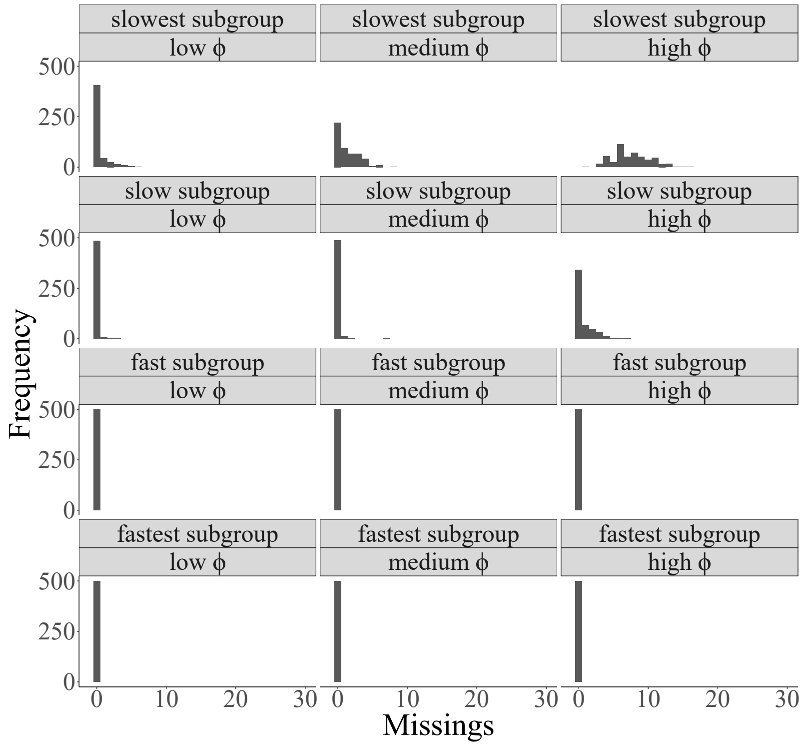

Figure G1. Number of not reached items for the low, medium and high speed sensitivity test form, across the four subgroups. Results shown for a randomly selected single replication.

Appendix H

Standard Deviations for Simulation Results Across Replications

Table H1

Standard Deviations for Test Statistics per Test Form and per Speed Group, Across All Replications.

Test Form	ζ_i	$M(RT)$	$SD(RT)$	$M(mis)$	$SD(mis)$	$cor(\theta, \theta)$	RMSE	$M(\theta_{diff})$
low ϕ	slowest	42.06	31.42	0.01	0.01	0.02	0.05	0.03
low ϕ	slow	33.39	24.36	0.00	0.01	0.01	0.05	0.02
low ϕ	fast	21.51	17.30	0.00	0.00	0.02	0.06	0.02
low ϕ	fastest	18.04	14.39	0.00	0.00	0.02	0.06	0.02
medium ϕ	slowest	46.88	33.10	0.02	0.02	0.02	0.07	0.05
medium ϕ	slow	33.97	25.37	0.00	0.01	0.02	0.06	0.02
medium ϕ	fast	19.86	16.64	0.00	0.00	0.01	0.05	0.02
medium ϕ	fastest	16.25	13.48	0.00	0.00	0.02	0.06	0.02
high ϕ	slowest	63.29	47.33	0.04	0.02	0.04	0.15	0.14
high ϕ	slow	41.06	29.78	0.01	0.01	0.02	0.06	0.03
high ϕ	fast	17.54	13.72	0.00	0.00	0.01	0.05	0.02
high ϕ	fastest	12.45	9.82	0.00	0.00	0.02	0.06	0.02

Note: Standard deviations across replications are depicted for mean cumulative response times M(RT) and the corresponding standard deviation SD(RT), mean proportion of missings M(mis), the corresponding standard deviation SD(mis), correlation between true and estimated ability $cor(\theta, \theta)$, root mean square error (RMSE) and average difference between true and estimated ability $M(\Delta_\theta)$.

References

Johnson Norman, L., Kotz, S., & Balakrishnan, N. (1994). *Lognormal distributions, continuous univariate distributions (vol. 1).* Wiley Series in Probability and Mathematical Statistics: Applied Probability.