Finite éléments on structured meshed

Roland Becker

May 11, 2020

1 The Q^1-element

We denote by $\alpha = (\alpha_1, \ldots \alpha_d) \in \mathbb{N}_0^d$ et multi-index with the usual conventions, such as $|\alpha|_\infty = \max_{1\le i\le d} \alpha_i$, and by $x_\alpha in \mathbb{R}^d$ de points of tensor-product grids on $\Omega = \prod_{i=1}^d]0, L_i[$.

2 Iteration

Let $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ be the euclidian scalar product with norm $\|\cdot\| = \|\cdot\|_{\mathbb{R}^n}$. Let for a SPD matrix W the scalar product $\langle \cdot, \cdot \rangle_W := \langle W\cdot, \cdot \rangle$ with associated norm $\|\cdot\|_W := \left\| W^{\frac{1}{2}} \cdot \right\|$.

Consider an update

$$x_{n+1} = w_n + \omega_n d_n, \quad \phi(\omega) = \frac{1}{2} \|f - A(x_n + \omega d_n)\|_W^2 = \frac{1}{2} \|r_n - \omega A d_n\|_W^2 \tag{1}$$

Then we have a minimizer for

$$\omega_* = \frac{\langle r_n, Ad_n \rangle_W}{\|Ad_n\|_W^2}, \quad \phi(\omega_*) - \phi(0) = -\frac{\omega_*}{2} \langle r_n, Ad_n \rangle_W = -\frac{\omega_*^2}{2} \|Ad_n\|_W^2 \tag{2}$$

If A is SPD, we can either minimize with respect to the norm $\|\cdot\|$ or $\|\cdot\|_{A^{-1}}$

$$\omega^{(1)} = \frac{{r_n}^\mathsf{T} A d_n}{(Ad_n)^\mathsf{T} A d_n}, \quad \omega^{(2)} = \frac{{r_n}^\mathsf{T} d_n}{(Ad_n)^\mathsf{T} d_n}$$

If A is not symmetric, $\omega^{(2)}$ is still well defined, if A is elliptic ($\xi^\mathsf{T} A \xi \ge \alpha \xi^\mathsf{T} \xi$) and corresponds to a Galerkin solution over the space Vect $\{x_n, d_n\}$.

Suppose that d_n is defined py a preconditioner, $Bd_n = r_n$.

Now let us consider a Gauss-Seidel-type iteration with $A = L + U$ and

$$x_{n+1} = (1-\omega)x_n + \omega L^{-1}(b - Ux^n) = x_n + \omega L^{-1} r^n.$$

3 Grid

$$x_{n+1} = x_n + \omega w_n, \quad w_n = Br_n, \quad r_n := b - Ax_n \tag{3}$$

4 Mixed FEM

5 Grid

We suppose the following numbering

$$ii = \sum_{j=0}^{d-1} \left(\prod_{k=j+1}^{d-1} n_k \right) i_j, \quad i = [i_0, \ldots, i_{d-1}] \tag{4}$$

6 Mixed FEM

The mixed formulation on a d-dimensional brick leads to

$$\begin{bmatrix} A_1 & & & & B_1 \\ & A_2 & & & B_2 \\ & & \ddots & & \vdots \\ & & & A_d & B_d \\ C_1 & C_2 & \cdots & C_d & D \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \\ p \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_d \\ f \end{bmatrix} \tag{5}$$

which leads to the pressure equation

$$Sp = f - \sum_{i=1}^{d} C_i A_i^{-1} g_i, \quad S := D - \sum_{i=1}^{d} C_i A_i^{-1} B_i \tag{6}$$

which allows to recover the fluxes by

$$A_i u_i = g_i - B_i p.$$

6.0.1 Elimination in $d = 1$

We have the following equations

$$\begin{cases} a_{i,i-1}u_{i-1} + a_{i,i}u_i + a_{i,i+1}u_{i+1} + b_{i,i-\frac{1}{2}}p_{i-\frac{1}{2}} + b_{i,i+\frac{1}{2}}p_{i+\frac{1}{2}} &= g_i \\ c_{i-\frac{1}{2},i-1}u_{i-1} + c_{i-\frac{1}{2},i}u_i &= f_{i-\frac{1}{2}} \\ c_{i+\frac{1}{2},i}u_i + c_{i+\frac{1}{2},i+1}u_{i+1} &= f_{i+\frac{1}{2}} \end{cases} \tag{7}$$

We can use the last two equations to eliminate $u_{i\pm1}$, thus

$$x_i u_i = g_i - \frac{a_{i,i-1}}{c_{i-\frac{1}{2},i-1}} f_{i-\frac{1}{2}} - \frac{a_{i,i+1}}{c_{i+\frac{1}{2},i+1}} f_{i+\frac{1}{2}} - b_{i,i-\frac{1}{2}}p_{i-\frac{1}{2}} - b_{i,i+\frac{1}{2}}p_{i+\frac{1}{2}}, \quad x_i = a_{i,i} - \frac{a_{i,i-1}c_{i-\frac{1}{2},i}}{c_{i-\frac{1}{2},i-1}} - \frac{a_{i,i+1}c_{i+\frac{1}{2},i}}{c_{i+\frac{1}{2},i+1}} \tag{8}$$

For a boundary node we have, say the left, $i = 0$, we have

$$\begin{cases} a_{0,0}u_0 + a_{0,1}u_1 + b_{0,\frac{1}{2}}p_{\frac{1}{2}} &= g_0 \\ c_{\frac{1}{2},0}u_0 + c_{\frac{1}{2},1}u_1 &= f_{\frac{1}{2}} \end{cases} \tag{9}$$

Using the last equation to eliminate u_1, thus

$$x_0 u_0 = g_0 - \frac{a_{0,1}}{c_{\frac{1}{2},1}} f_{\frac{1}{2}} - b_{0,\frac{1}{2}}p_{\frac{1}{2}}, \quad x_0 = a_{0,0} - \frac{a_{0,1}c_{\frac{1}{2},0}}{c_{\frac{1}{2},1}} \tag{10}$$

This gives the following finite difference stencil on the boundary

$$\frac{c_{\frac{1}{2},0}}{x_0} \left(g_0 - \frac{a_{0,1}}{c_{\frac{1}{2},1}} f_{\frac{1}{2}} - b_{0,\frac{1}{2}}p_{\frac{1}{2}} \right) + \frac{c_{\frac{1}{2},1}}{x_1} \left(g_1 - \frac{a_{1,0}}{c_{\frac{1}{2},0}} f_{\frac{1}{2}} - \frac{a_{1,2}}{c_{\frac{3}{2},2}} f_{\frac{3}{2}} - b_{1,\frac{1}{2}}p_{\frac{1}{2}} - b_{1,\frac{3}{2}}p_{\frac{3}{2}} \right) = f_{\frac{1}{2}}$$

and on the interior

$$
\begin{aligned}
&\frac{c_{i+\frac{1}{2},i}}{x_i}\left(g_i - \frac{a_{i,i-1}}{c_{i-\frac{1}{2},i-1}} f_{i-\frac{1}{2}} - \frac{a_{i,i+1}}{c_{i+\frac{1}{2},i+1}} f_{i+\frac{1}{2}} - b_{i,i-\frac{1}{2}} p_{i-\frac{1}{2}} - b_{i,i+\frac{1}{2}} p_{i+\frac{1}{2}}\right) \\
&+\frac{c_{i+\frac{1}{2},i+1}}{x_{i+1}}\left(g_{i+1} - \frac{a_{i+1,i}}{c_{i+\frac{1}{2},i}} f_{i+\frac{1}{2}} - \frac{a_{i+1,i+1+1}}{c_{i+\frac{3}{2},i+1+1}} f_{i+\frac{3}{2}} - b_{i+1,i+\frac{1}{2}} p_{i+\frac{1}{2}} - b_{i+1,i+\frac{3}{2}} p_{i+\frac{3}{2}}\right) = \quad f_{i+\frac{1}{2}}
\end{aligned}
$$

$$
\{ \tag{11}
$$