

Finite difference

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1 Grid

$$x_{n+1} = x_n + \omega w_n, \quad w_n = Br_n, \quad r_n := b - Ax_n \quad (1)$$

If A is SPD, we can either minimize with respect to the norm $\|A \cdot\|$ or $\|A^{\frac{1}{2}} \cdot\|$

$$\omega^{(1)} = \frac{r_n^\top Aw_n}{(Aw_n)^\top Aw_n}, \quad \omega^{(2)} = \frac{r_n^\top w_n}{(Aw_n)^\top w_n}$$

If A is not symmetric, $\omega^{(2)}$ is still well defined, if A is elliptic ($\xi^\top A \xi \geq \alpha \xi^\top \xi$) and corresponds to minimization over the space $\text{Vect} \{x_n, w_n\}$.

Now let us consider a Gauss-Seidel-type iteration with $A = L + U$ and

$$x_{n+1} = (1 - \omega)x_n + \omega L^{-1}(b - Ux_n) = x_n + \omega L^{-1}r_n.$$

2 Mixed FEM

3 Grid

We suppose the following numbering

$$ii = \sum_{j=0}^{d-1} \left(\prod_{k=j+1}^{d-1} n_k \right) i_j, \quad i = [i_0, \dots, i_{d-1}] \quad (2)$$

4 Mixed FEM

The mixed formulation on a d -dimensional brick leads to

$$\begin{bmatrix} A_1 & & & B_1 \\ & A_2 & & B_2 \\ & & \ddots & \vdots \\ C_1 & C_2 & \cdots & C_d & D \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \\ p \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_d \\ f \end{bmatrix} \quad (3)$$

which leads to the pressure equation

$$Sp = f - \sum_{i=1}^d C_i A_i^{-1} g_i, \quad S := D - \sum_{i=1}^d C_i A_i^{-1} B_i \quad (4)$$

which allows to recover the fluxes by

$$A_i u_i = g_i - B_i p.$$

4.0.1 Elimination in $d = 1$

We have the following equations

$$\begin{cases} a_{i,i-1}u_{i-1} + a_{i,i}u_i + a_{i,i+1}u_{i+1} + b_{i,i-\frac{1}{2}}p_{i-\frac{1}{2}} + b_{i,i+\frac{1}{2}}p_{i+\frac{1}{2}} = g_i \\ c_{i-\frac{1}{2},i-1}u_{i-1} + c_{i-\frac{1}{2},i}u_i = f_{i-\frac{1}{2}} \\ c_{i+\frac{1}{2},i}u_i + c_{i+\frac{1}{2},i+1}u_{i+1} = f_{i+\frac{1}{2}} \end{cases} \quad (5)$$

We can use the last two equations to eliminate $u_{i\pm 1}$, thus

$$x_i u_i = g_i - \frac{a_{i,i-1}}{c_{i-\frac{1}{2},i-1}} f_{i-\frac{1}{2}} - \frac{a_{i,i+1}}{c_{i+\frac{1}{2},i+1}} f_{i+\frac{1}{2}} - b_{i,i-\frac{1}{2}} p_{i-\frac{1}{2}} - b_{i,i+\frac{1}{2}} p_{i+\frac{1}{2}}, \quad x_i = a_{i,i} - \frac{a_{i,i-1} c_{i-\frac{1}{2},i}}{c_{i-\frac{1}{2},i-1}} - \frac{a_{i,i+1} c_{i+\frac{1}{2},i}}{c_{i+\frac{1}{2},i+1}} \quad (6)$$

For a boundary node we have, say the left, $i = 0$, we have

$$\begin{cases} a_{0,0}u_0 + a_{0,1}u_1 + b_{0,\frac{1}{2}}p_{\frac{1}{2}} = g_0 \\ c_{\frac{1}{2},0}u_0 + c_{\frac{1}{2},1}u_1 = f_{\frac{1}{2}} \end{cases} \quad (7)$$

Using the last equation to eliminate u_1 , thus

$$x_0 u_0 = g_0 - \frac{a_{0,1}}{c_{\frac{1}{2},1}} f_{\frac{1}{2}} - b_{0,\frac{1}{2}} p_{\frac{1}{2}}, \quad x_0 = a_{0,0} - \frac{a_{0,1} c_{\frac{1}{2},0}}{c_{\frac{1}{2},1}} \quad (8)$$

This gives the following finite difference stencil on the boundary

$$\frac{c_{\frac{1}{2},0}}{x_0} \left(g_0 - \frac{a_{0,1}}{c_{\frac{1}{2},1}} f_{\frac{1}{2}} - b_{0,\frac{1}{2}} p_{\frac{1}{2}} \right) + \frac{c_{\frac{1}{2},1}}{x_1} \left(g_1 - \frac{a_{1,0}}{c_{\frac{1}{2},0}} f_{\frac{1}{2}} - \frac{a_{1,2}}{c_{\frac{3}{2},2}} f_{\frac{3}{2}} - b_{1,\frac{1}{2}} p_{\frac{1}{2}} - b_{1,\frac{3}{2}} p_{\frac{3}{2}} \right) = f_{\frac{1}{2}}$$

and on the interior

$$\begin{aligned} & \frac{c_{i+\frac{1}{2},i}}{x_i} \left(g_i - \frac{a_{i,i-1}}{c_{i-\frac{1}{2},i-1}} f_{i-\frac{1}{2}} - \frac{a_{i,i+1}}{c_{i+\frac{1}{2},i+1}} f_{i+\frac{1}{2}} - b_{i,i-\frac{1}{2}} p_{i-\frac{1}{2}} - b_{i,i+\frac{1}{2}} p_{i+\frac{1}{2}} \right) \\ & + \frac{c_{i+\frac{1}{2},i+1}}{x_{i+1}} \left(g_{i+1} - \frac{a_{i+1,i}}{c_{i+\frac{1}{2},i}} f_{i+\frac{1}{2}} - \frac{a_{i+1,i+1+1}}{c_{i+\frac{3}{2},i+1+1}} f_{i+\frac{3}{2}} - b_{i+1,i+\frac{1}{2}} p_{i+\frac{1}{2}} - b_{i+1,i+\frac{3}{2}} p_{i+\frac{3}{2}} \right) = f_{i+\frac{1}{2}} \end{aligned} \quad (9)$$