Find n linearly independent solutions to the n x n constant coefficient homogeneous system

$$x' = Ax$$

Guess solutions have the form

$$x = ve^{\lambda t}$$

where:
λ is a real or complex constant,
v is a nonzero n x 1 vector with constant real components.

$$Ax = x'$$
$$A\,ve^{\lambda t} = \lambda ve^{\lambda t}$$
$$Av = \lambda v$$

λ is an eigenvalue of A,
v is the associated eigenvector.

Let $\lambda_1, \lambda_2, \lambda_k$ be distinct eigenvalues for the matrix A, and v_i an eigenvector associated with λ_i. Then $v_1, v_2, \ldots, v_k$ are linearly independent.

If A has n distinct eigenvalues then

$$x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t} + \cdots + c_n v_n e^{\lambda_n t}$$

is a general solution for the homogeneous equation $x' = Ax$

Let *A* be a matrix with real components.

If λ is a complex eigenvalue and v is a complex eigenvector, then $\bar{v}$ is also an eigenvector with eigenvalue $\bar{\lambda}$.

Let *p+iq* (q≠0) be an eigenvalue of *A* with eigenvector *a+ib*. Then $(a \pm ib)e^{(p \pm iq)t}$ is a complex solution for the homogeneous equation.

The complex solutions from conjugate eigenvalues are linearly independent:

$$x_1 = (a + ib)e^{(p+iq)t} = e^{pt}(a\cos qt - b\sin qt) + ie^{pt}(b\cos qt + a\sin qt)$$
$$x_2 = (a - ib)e^{(p-iq)t} = e^{pt}(a\cos qt - b\sin qt) - ie^{pt}(b\cos qt + a\sin qt)$$

As are also

$$\frac{1}{2}(x_1 + x_2) = e^{pt}(a\cos qt - b\sin qt)$$
$$\frac{1}{2i}(x_1 - x_2) = e^{pt}(b\cos qt + a\sin qt)$$

A general solution for the homogeneous equation $Ax = x'$ is

$$x(t) = c_1 e^{pt}(a\cos qt - b\sin qt) + c_2 e^{pt}(b\cos qt + a\sin qt)$$

Find n linearly independent solutions to the n x n real constant coefficient homogeneous system

$$x'' = Ax$$

Guess solutions have the form

$$x = ve^{\alpha t}$$

$$Ax = x''$$
$$A\,ve^{\alpha t} = \alpha^2 ve^{\alpha t}$$
$$Av = \alpha^2 v$$
$$Av = \lambda v$$

For mass-spring problems, the eigenvalues are negative,

$$\alpha = \sqrt{\lambda}$$
$$\alpha = 0 + i\,\omega$$

and

$$x = ve^{i\omega t} = v\,(\cos \omega t + i\,\sin \omega t)$$

As the real and imaginary parts are linearly independent real-valued solutions,

$$x = a\,(v\cos \omega t) + b\,(v \sin \omega t)$$

is also a solution to $x'' = Ax$.

If *A* has n distinct eigenvalues then

$$x(t) = \sum_{i=1}^{n} a_i\,(v_i \cos \omega_i t) + b_i\,(v_i \sin \omega_i t)$$

$$x(t) = \sum_{i=1}^{n} a_i \cos \omega_i t + b_i \sin \omega_i t$$

is a general solution for the homogeneous equation $Ax = x''$.