The Vicsek Model

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1 Introduction

This research concerns a mathematical model describing active matter: the Vicsek Model of flocking in birds, as initially proposed by Tamas Vicsek in 1995[1].

The collective motion as described in the Vicsek Model is the simplest form of spontaneous phenomenon where birds interact and respond to the behaviour of other individuals local to itself.

In this, we model each bird as a weightless point particle moving with an initial constant velocity of v_0 along a polar direction $\hat{n} = (cos\theta, sin\theta)$. Then, in order to create a swarming behaviour, the birds check in at each discrete time step with all of its neighbours within a radius R and reorientate themselves along the mean direction of all birds within the region. Since birds are natural, they are imperfect creatures and so we take account for this by adding a random normally distributed noise term, η_i, with mean 0 and standard deviation σ into its new direction.

2 The Basic Model

2.1 Mathematical Description

Consider a system of N birds in 2D. The state of an individual i at discrete time t is described by its **position** $\vec{r}_i(t) = (x_i(t), y_i(t))$ and its **angle** $\theta_i(t) \in [0, 2\pi)$. These are used to define the direction of the individual's velocity $\hat{n}_i(t) = (\cos\theta_i(t), \sin\theta_i(t))$. The initial states of the birds are decided using uniformly random positions and orientations.

The discrete time evolution of one particle is defined by two equations. The equations of motion of bird i at time $t+1$ are defined by:

$$\vec{r}_i(t+1) = \vec{r}_i(t) + v_0\vec{n}_i(t) \ (mod\ L) \tag{1}$$

$$\begin{aligned} \theta_i(t+1) &= angle\left[\sum_{j=1}^{N} \mathbf{I}_{\{\|\vec{r}_i(t)-\vec{r}_j(t)\|<R\}}\hat{n}_j(t)\right] + \eta_i(t+1) \\ &= \arctan\left(\frac{\sum_{j=1}^{N} \mathbf{I}_{\{\|\vec{r}_i(t)-\vec{r}_j(t)\|<R\}}\sin\theta_j(t)}{\sum_{j=1}^{N} \mathbf{I}_{\{\|\vec{r}_i(t)-\vec{r}_j(t)\|<R\}}\cos\theta_j(t)}\right) + \eta_i(t+1) \end{aligned} \tag{2}$$

where $\|\vec{r}_i(t) - \vec{r}_j(t)\|$ is the norm under the periodic boundary conditions(see Section 2.2). v_0 is the initial and constant velocity of all the birds at all time steps and $\eta_i(t)$ is a random noise term generated at each time step t for bird i from a $N(0, \sigma^2)$ random variable.

In order to investigate the relationship between the noise term and the collective motion of birds, we calculate the Vicsek order parameter, which is a measurement of alignment of birds between 0 and 1. This is calculated after updating all the birds' states for each time step t and investigated using different values of σ in a single simulation test. The **Vicsek order parameter** is defined as:

$$\begin{aligned}
n(\sigma, t) &= \frac{1}{N}\left|\sum_{i=1}^{N} \hat{n}_i(t)\right| \\
&= \frac{1}{N}\sqrt{\left(\sum_{i=1}^{N} \cos\theta_i(t)\right)^2 + \left(\sum_{i=1}^{N} \sin\theta_i(t)\right)^2} \qquad (3)
\end{aligned}$$

2.2 Algorithm to Calculate the Distance between Two particles Under Periodic Boundary Conditions

In order to produce visual plots of the birds swarming, we introduce periodic boundary conditions to represent the birds' motion on a contained space. This ensures that birds which disappear out of the right re-enter on the left, and those disappearing off the top re-appear in the bottom.

Figure 1 demonstrates the periodic boundary conditions. The central square between 0 and L is copied to its neighbouring squares. Moving over the boundary is equivalent to the particle re-entering the central square. This can be done by taking the modulo operation over the coordinates of the particle(i.e. modulo the periodicity L).

Each particle only interacts with its nearest neighbours, as defined by a search radius R. So when calculating the distance between any two particles $p_1 : \vec{r}_1 = (x_1, y_1)$ and $p_2 : \vec{r}_2 = (x_2, y_2)$ within the central square, we only need to copy p_2 to all the nearest 8 squares and then take the minimum of the 9 distances. Naturally, we would think of constructing 9 position matrices but this is computationally costly. So we want to simplify this problem analytically.

All 9 possible positions of p_2 comprises the set:

$$\{(x_2 + t_1, y_2 + t_2) : t_1, t_2 \in \{0, \pm L\}\}$$

Accordingly, the 9 distances calculated from this set forms the new set:

$$\{\sqrt{(x_2 + t_1 - x_1)^2 + (y_2 + t_2 - y_1)^2} : t_1, t_2 \in \{0, \pm L\}\}$$

We define the distance between the two particles under periodic boundary conditions as the minimum of the distance set, i.e.

$$\|\vec{r}_2 - \vec{r}_1\| = \min\{\sqrt{(x_2 + t_1 - x_1)^2 + (y_2 + t_2 - y_1)^2} : t_1, t_2 \in \{0, \pm L\}\}$$

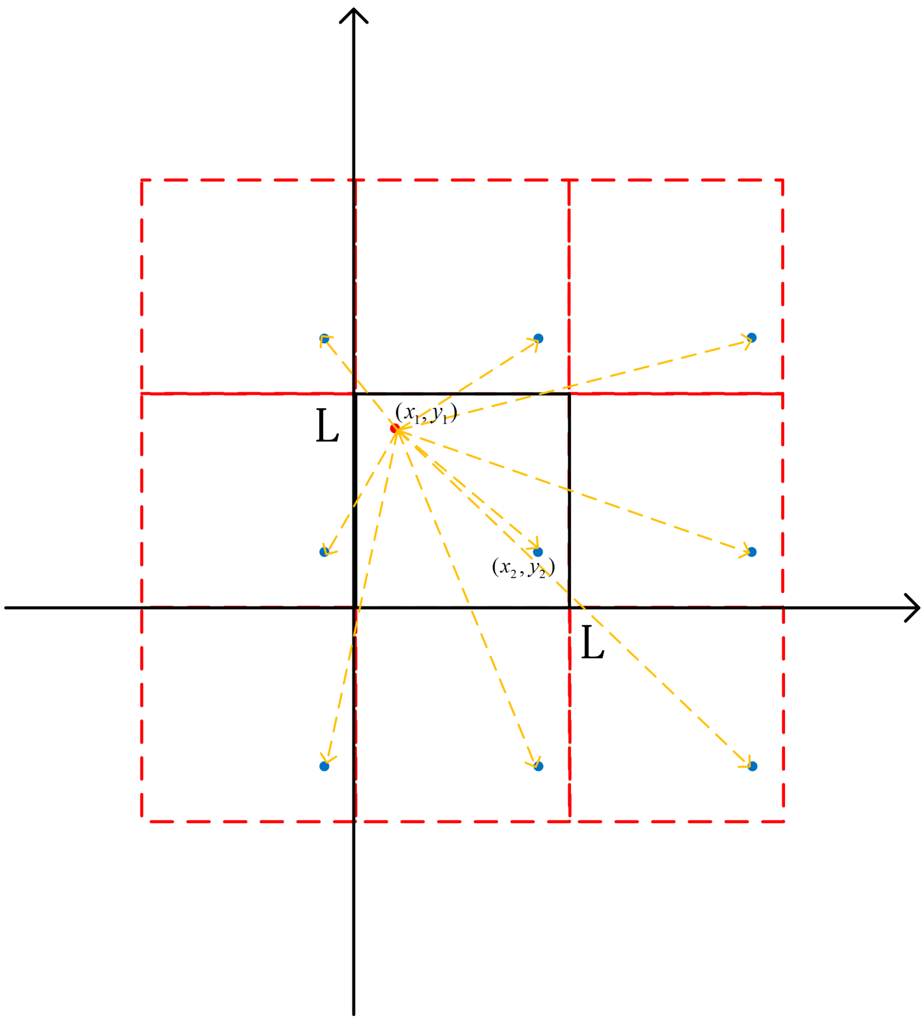

Figure 1: Illustration of periodic boundary conditions

We want to solve the following optimization problem:

$$
\begin{aligned}
& \min_{t_1, t_2 \in \{0, \pm L\}} \sqrt{(x_2 + t_1 - x_1)^2 + (y_2 + t_2 - y_1)^2} \\
\Longleftrightarrow & \min_{t_1, t_2 \in \{0, \pm L\}} (x_2 + t_1 - x_1)^2 + (y_2 + t_2 - y_1)^2 \\
\Longleftrightarrow & \min_{t_1, t_2 \in \{0, \pm L\}} (x_2 - x_1)^2 + (y_2 - y_1)^2 + t_1^2 + 2t_1(x_2 - x_1) + t_2^2 + 2t_2(y_2 - y_1) \\
\Longleftrightarrow & \min_{t_1, t_2 \in \{0, \pm L\}} t_1^2 + 2t_1(x_2 - x_1) + t_2^2 + 2t_2(y_2 - y_1) \\
\Longleftrightarrow & \min_{t_1 \in \{0, \pm L\}} t_1^2 + 2t_1(x_2 - x_1) \quad \text{and} \quad \min_{t_2 \in \{0, \pm L\}} t_2^2 + 2t_2(y_2 - y_1) \\
\Longleftrightarrow & \min_{t_1 \in \{0, \pm L\}} |t_1 - (x_2 - x_1)| \quad \text{and} \quad \min_{t_2 \in \{0, \pm L\}} |t_2 - (y_2 - y_1)|
\end{aligned}
$$

x_1, x_2, y_1, y_2 are fixed constants, so we want to minimize the two quadratic terms including t_1 and t_2 separately.

The solution gives:

$$t_1 = \begin{cases} -L & \text{if } x_2 - x_1 > \frac{L}{2} \\ 0 & \text{if } -\frac{L}{2} \leq x_2 - x_1 \leq \frac{L}{2} \\ L & \text{if } x_2 - x_1 < -\frac{L}{2} \end{cases} \tag{4}$$

and

$$t_2 = \begin{cases} -L & \text{if } y_2 - y_1 > \frac{L}{2} \\ 0 & \text{if } -\frac{L}{2} \leq y_2 - y_1 \leq \frac{L}{2} \\ L & \text{if } y_2 - y_1 < -\frac{L}{2} \end{cases} \tag{5}$$

3 Model Added with Predators

Now modify the model by adding M predators to it. The predators moves with a constant velocity $v_{predator}$. The state of an individual predator i at discrete time t is described by its **position** $\vec{r'}_i(t) = (x'_i(t), y'_i(t))$ and its **angle** $\theta'_i(t) \in [0, 2\pi)$. These are used to define the direction of the individual's velocity $\hat{n'}_i(t) = (\cos\theta'_i(t), \sin\theta'_i(t))$. The initial states of the predators are defined as before using uniformly random positions and orientations.

3.1 Death Rule for Preys

Assume that one predator can only eat at most one prey at each time step. Therefore at each time step, for each predator in the system, identify the nearest prey to it using the same rules as nearing neighbours for the prey. If the distance between them is less than $\frac{1}{2}(v_{predator} - v_0)$, then we assume that the prey is eaten by the predator and thus removed from the system.

3.2 Updating Rules for Preys

The updating rule for the position of the prey at time $t+1$ is the same as Equation (1) shows: position at time t plus a velocity vector in the direction of the angle at time t.

However, the updating rule for the angular component changes. It not only depends on the neighbouring prey, but also depends on the predators. The birds near enough to the predator will gain an extra repulsion term that turns them away from the position of the predator, which takes the form of the angle pointing directly from the predator to the prey. Denote the radius within which for preys to spot the predator and run away to be R_{run_away}. So there is a trade off between rules of alignment and repulsion. We can take into account both of them and define the new direction of velocity by using a linear combination of the weighted normalised direction vectors.

To distinguish, we define the angle of alignment at time $t+1$ as

$$\begin{aligned}\psi_i(t+1) &= angle\left[\sum_{j=1}^{N} \mathbf{I}_{\{\|\vec{r}_i(t)-\vec{r}_j(t)\|<R\}} \hat{n}_j(t)]\right] \\ &= \arctan\left(\frac{\sum_{j=1}^{N} \mathbf{I}_{\{\|\vec{r}_i(t)-\vec{r}_j(t)\|<R\}} \sin\theta_j(t)}{\sum_{j=1}^{N} \mathbf{I}_{\{\|\vec{r}_i(t)-\vec{r}_j(t)\|<R\}} \cos\theta_j(t)}\right)\end{aligned}$$

The angle of repulsion of prey i with respect to predator j at time $t+1$ under periodic boundary condition is defined as

$$\begin{aligned}\phi_{i,j}(t+1) &= angle[\vec{r}_i(t) - \vec{r'}_j(t)] \\ &= \arctan\frac{y'_j(t) - t_2 - y_i(t)}{x'_j(t) - t_1 - x_i(t)}\end{aligned}$$

where t_1 and t_2 are calculated using equations (4) and (5) by taking predator j as p_1 and prey i as p_2 in Section 2.2.

Then we have obtained several normalized direction vectors for prey i corresponding to these angles:

1. direction vector with angle of alignment $\psi_i(t+1)$

$$\hat{n}_{align\_i}(t+1) = (\cos\psi_i(t+1), \sin\psi_i(t+1))$$

2. direction vectors with angle of repulsion $\phi_{i,j}(t+1)$ for $j \in \{1,\ldots,M\}$

$$\hat{n}_{i,j}(t+1) = (\cos\phi_{i,j}(t+1), \sin\phi_{i,j}(t+1))$$

Now we want to choose an appropriate function to assign weight to our normalized direction vectors. For simplicity, let the weight for $\hat{n}_{align\_i}(t+1)$ be 1. Note that the closer the prey is to the predator, the higher the priority for it to turn the opposite direction of the predator. An appropriate function is $-ln(x)$, which tends to infinity when x tends to 0 and vanishes when $x = 1$. So we define our weight for $\hat{n}_{i,j}(t+1)$ as(see Figure 2):

$$w_{i,j}(t+1) = \begin{cases} -ln\frac{\left\|\vec{r}_i(t)-\vec{r'}_j(t)\right\|}{R_{run\_away}} & \text{if } \left\|\vec{r}_i(t) - \vec{r'}_j(t)\right\| < R_{run\_away} \\ 0 & \text{if } \left\|\vec{r}_i(t) - \vec{r'}_j(t)\right\| \geq R_{run\_away} \end{cases} \tag{6}$$

Finally, the angle of prey i at time $t+1$ is given by:

$$\begin{aligned}\theta_i(t+1) &= angle\left[\hat{n}_{align\_i}(t+1) + \sum_{j=1}^{M} \mathbf{I}_{\{\left\|\vec{r}_i(t)-\vec{r'}_j(t)\right\|<R_{run\_away}\}} w_{i,j}(t+1)\hat{n}_{i,j}(t+1)\right] + \eta_i(t+1) \\ &= \arctan\left(\frac{\sin\psi_i(t+1) + \sum_{j=1}^{M} \mathbf{I}_{\{\left\|\vec{r}_i(t)-\vec{r'}_j(t)\right\|<R_{run\_away}\}} w_{i,j}(t+1)\sin\phi_{i,j}(t+1)}{\cos\psi_i(t+1) + \sum_{j=1}^{M} \mathbf{I}_{\{\left\|\vec{r}_i(t)-\vec{r'}_j(t)\right\|<R_{run\_away}\}} w_{i,j}(t+1)\cos\phi_{i,j}(t+1)}\right) + \eta_i(t+1)\end{aligned}$$

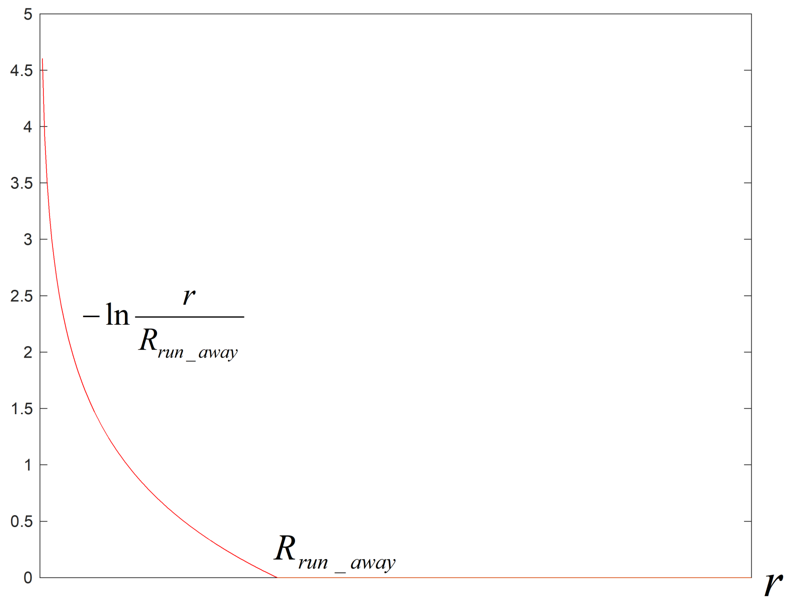

Figure 2: weight for direction vectors with angle of repulsion
r is the distance between prey and predator under periodic boundary conditions

3.3 Updating Rules for Predators

The updating rule for the position of predator j at time $t+1$ is the same as Equation (1) shows, i.e. the position at time t plus a velocity vector in the direction of the angle at time t.

$$\vec{r'}_j(t+1) = \vec{r'}_j(t) + v_0\vec{n'}_j(t)\ (mod\ L)$$

The updating rule for angles of predators at time $t+1$ is analogous to that of preys. The only difference is that we do not need to consider the rule of alignment and also that the rule of repulsion turns into a rule of attraction. If there is no prey within $R_{predator}$ (the radius within which to catch the preys) then the predators keep in the same direction as last step plus a noise term. Here we use the same weight factor for each direction vector $\hat{n}_{i,j}(t+1)$(see Figure 3):

$$w'_{i,j}(t+1) = \begin{cases} -ln\frac{\left\|\vec{r}_i(t)-\vec{r'}_j(t)\right\|}{R_{predator}} & \text{if } \left\|\vec{r}_i(t)-\vec{r'}_j(t)\right\| < R_{predator} \\ 0 & \text{if } \left\|\vec{r}_i(t)-\vec{r'}_j(t)\right\| \geq R_{predator} \end{cases} \tag{7}$$

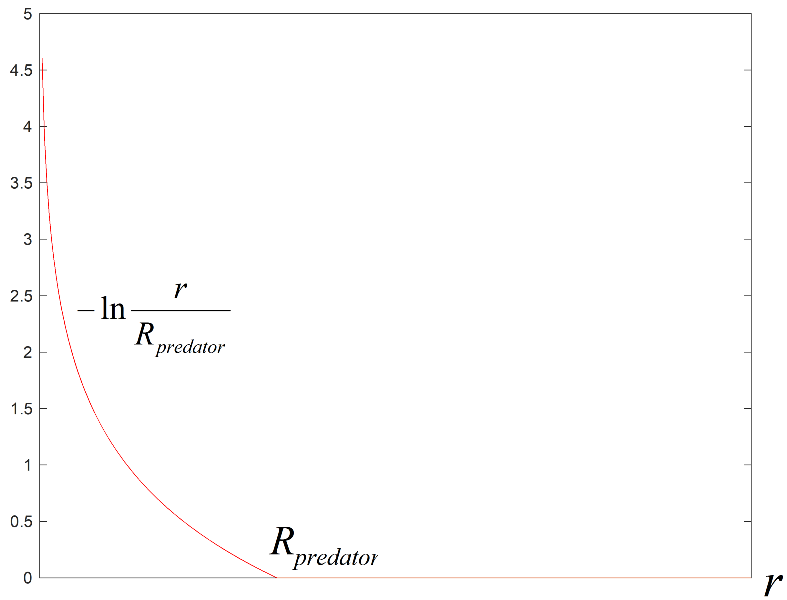

Figure 3: weight for direction vectors with angle of attraction
r is the distance between prey and predator under periodic boundary conditions

Similarly, the angle of predator j at time $t+1$ is given by:

$$\begin{aligned}\theta'_j(t+1) &= angle\left[\sum_{i=1}^{N} \mathbf{I}_{\{\|\vec{r}_i(t)-\vec{r'}_j(t)\|<R_{predator}\}} w'_{i,j}(t+1)\hat{n}_{i,j}(t+1)\right] + \eta'_j(t+1) \\ &= \arctan\left(\frac{\sum_{i=1}^{N} \mathbf{I}_{\{\|\vec{r}_i(t)-\vec{r'}_j(t)\|<R_{predator}\}} w'_{i,j}(t+1)\sin\phi_{i,j}(t+1)}{\sum_{i=1}^{N} \mathbf{I}_{\{\|\vec{r}_i(t)-\vec{r'}_j(t)\|<R_{predator}\}} w'_{i,j}(t+1)\cos\phi_{i,j}(t+1)}\right) + \eta'_j(t+1)\end{aligned}$$

if $\sum_{i=1}^{N} \mathbf{I}_{\{\|\vec{r}_i(t)-\vec{r'}_j(t)\|<R_{predator}\}} \geq 1$. Alternatively,

$$\theta'_j(t+1) = \theta'_j(t) + \eta'_j(t+1)$$

if $\sum_{i=1}^{N} \mathbf{I}_{\{\|\vec{r}_i(t)-\vec{r'}_j(t)\|<R_{predator}\}} = 0$.

References

[1] Vicsek, T., Czirok, A., Ben-Jacob, E., Cohen, I. and Shochet, O. (1995) *Novel Type of Phase Transition in a System of Self-Driven Particles* Phys. Rev. Lett.