Bootcamp Practice Problems

August 2020

1 Distributions

1. Let $X \sim Exp(\theta)$. Show that $cX \sim Exp(\theta/c)$ where c is some constant.

2 Calculus

1. Find the derivatives of the following functions
   - $f(x) = xe^x$;
   - $f(x) = 1 - \cos^2 x$;
   - $f(x) = \frac{\ln x}{x}$.
2. $f(x) = \frac{1}{\sqrt{\gamma}} \exp\left(-\frac{(x-\mu)^2}{\gamma}\right)$ where constants $\gamma > 0$ and $\mu \in \mathbb{R}$, and $x \in \mathbb{R}$. Calculate $f'(x)$ and find $x_0 \in \mathbb{R}$ such that the tangent line of $f(x)$ at x_0 is horizontal.
3. Find $\lim_{x\to 0}(1+x)^{1/x}$.
4. Which following functions are convex?

   A $f_1(x) = |x|, x \in [-1, 1]$;

   B $f_2(x) = \ln(x^2 + 1), x \in \mathbb{R}$;

   C $f_3(x) = e^{-x}, x \in \mathbb{R}$.
5. Let $f(x) = \frac{1}{x}, x > 0$. For every positive integer n, find $f^{(n)}(x)$.
6. $f(x) = 4xe^{-2x}$ with $x \in (0, \infty)$ which is a Gamma(2,2) density. Find the global maximum of $f(x)$.
7. Calculate $\int xe^x \, dx$.
8. Calculate $\int_1^e \frac{\ln x}{x} dx$.
9. Calculate $\int_0^\pi x \cos x dx$.
10. Find the limit of the sequence a_n where $a_n = \left(1 + \frac{1}{n}\right)^n$, $n \in \mathbb{N}$.
11. Let $a_n = \frac{2^n}{n!}$, $n \in \mathbb{N}$. Is the sequence a_n convergent? If so, what is its limit?
12. For a function such that $f'(x) \le 0$, show that $\sum_{i=k+1}^{\infty} f(i) \le \int_k^\infty f(x) \, dx \le \sum_{i=k}^{\infty} f(i)$
13. Let $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$. If $\alpha > 1$, show that $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.

3 Linear Algebra

1. Show, using the definition of a matrix product, that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.
2. Show that the product of two upper-triangular matrices is also upper triangular. What is $\mathbf{AB}_{ii}$?
3. Give an example of two symmetric matrices whose product is not symmetric. When will the product of two symmetric matrices be symmetric?
4. For a symmetric and positive definite $n \times n$ matrix A, show that $tr(AA) = \sum_{i=1}^{n} a_{i,\cdot}^T a_{i,\cdot}$, where $a_{i,\cdot}$ is the ith row of A.
5. Suppose X^TX is an invertable $p \times p$ matrix. Find $tr(X(X^TX)^{-1}X^T)$.
6. Let X be an $n \times p$ matrix. Find the centering matrix C such that CX subtracts the column means of X and leaves the standard deviation of each column unchanged. Find the standardizing matrix operation such that the resulting matrix is equal to X with the column means of X subtracted and the resulting matrix has column standard deviations of 1.

4 Probability

1. Show that $\Gamma(1/2) = \sqrt{\pi}$, where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$.
   - Hint: Use u-substitution with $x = u^2$.
   - Hint: If we have $f(x)$ is an even function about 0 then $\int_{-\infty}^{\infty} f(x)dx = 2\int_0^\infty f(x)dx$
2. Let X be a random variable with pdf

$$f(x) = \begin{cases} \frac{|x|}{10} & -2 \leq x \leq 4 \\ 0 & o.w. \end{cases}$$

   Calculate $\mathbb{E}[X]$.

3. Show that $cov(X, Y) = E[XY] - E[X]E[Y]$.
4. Show that $Var(X + Y) = Var(X) + Var(Y) + 2cov(X, Y)$.
5. Suppose X and Y have joint density

$$f(x, y) = \begin{cases} 0.13e^{-0.5x-0.2y} - 0.04e^{-x-0.1y} - 0.06e^{-0.5x-0.4y} + 0.16e^{-x-0.4y} & x, y > 0 \\ 0 & o.w. \end{cases}$$

   What is the standard deviation of X?

6. Let X and Y have joint density

$$f(x, y) = \begin{cases} \frac{8}{3}xy & 0 \leq x \leq 1, x \leq y \leq 2x \\ 0 & o.w. \end{cases}$$

   Calculate the covariance of X and Y.

7. Let $X = \{-1, 0, 1\}$ each with probability 1/3, and let $Y = X^2$.
   (a) What is $cov(X, Y)$?
   (b) Are X and Y independent?
8. Let the joint pdf of X, Y, Z be defined as

$$f(x, y, z) = \begin{cases} \frac{1-\sin(x)\sin(y)\sin(z)}{8\pi^3} & 0 \leq x, y, z \leq 2\pi \\ 0 & o.w. \end{cases}$$

   (a) Find the pdf for (X, Y).
   (b) Find the pdf for X.
   (c) Are X and Y independent?
   (d) Are X, Y, Z independent?

9. Show that if $X \sim Pois(\lambda)$ is independent of $Y \sim Pois(\theta)$, then $X + Y \sim Pois(\lambda + \theta)$.
   (a) Brute force approach
   (b) MGF approach

10. A random variable X has pdf

$$f(x) = \begin{cases} 0.005(20 - x) & 0 < x < 20 \\ 0 & o.w. \end{cases}$$

Given that $X > 8$, calculate the probability that $X > 16$.

11. Suppose we model X_1 and X_2 as independent $Exp(\lambda_1)$ and $Exp(\lambda_2)$ random variables. What is $\mathbb{E}[min(X_1, X_2)]$?

12. **Challenge?** Let $X_1, \ldots, X_n$ be i.i,d, with pdf

$$f(x|\theta) = \begin{cases} \frac{2}{\pi\theta} \exp\left\{-\frac{x^2}{\pi\theta^2}\right\} & x \geq 0 \\ 0 & o.w. \end{cases}$$

(a) (Optional) Find the MLE for θ.

(b) Define the random variable $Y = \frac{2}{\pi\theta^2} \sum X_i^2$. What distribution does Y follow? *Hint:* First find the distribution of $W_1 = \frac{2}{\pi\theta^2} X_i^2$.

5 Inference

1. For $X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$, for known variance σ^2, show that $\overline{X}$, the sample mean, is sufficient for μ.

2. Let $X_1, \ldots, X_n$ be i.i.d $Poisson(\theta)$. Find the MLE for θ.

3. Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$. Find the MLEs of the mean μ and variance σ^2.

4. Let $X_1, \ldots, X_n$ be i.i.d. $Unif(0, \theta)$, $\theta > 0$ unknown with pdf

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} & 0 \leq x \leq \theta \\ 0 & o.w. \end{cases}$$

   Find the MLE for θ.

5. Let $X_1, \ldots, X_n$ be i.i.d. $Unif(0, \theta)$, $\theta > 0$ unknown with pdf

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} & 0 \leq x < \theta \\ 0 & o.w. \end{cases}$$

   Show the MLE for θ does not exist.

6. Let $X_1, \ldots, X_n$ be i.i.d. $Unif(\theta, \theta + 1)$, θ unknown with pdf

$$f(x|\theta) = \begin{cases} 1 & \theta \leq x \leq \theta + 1 \\ 0 & o.w. \end{cases}$$

   Show that the MLE for θ is not unique.

7. Let $X_1, \ldots, X_n$ be i.i.d $Poisson(\theta)$. Find the Fisher Information $I_n(\theta)$ of the n observations.

8. Show that the MSE can be decomposed into the sum of the bias squared plus the variance of the estimator. Specifically, suppose δ is some estimate of the parameter of interest, θ. Show that $E[(\delta - \theta)^2] = \text{Bias}(\delta)^2 + \text{Var}(\delta)$.

9. Consider an experiment where the sample size is random. First, draw $N \sim \text{Pois}(\lambda)$ and then collect the data from the sampling distribution: $X|N \sim \text{Binom}(N + 1, p)$. We desire inference on p.

   (a) Is $X/(N + 1)$ an unbiased estimate of p?

   (b) Compare this experiment to one where N is fixed at N_f and $X_f \sim \text{Binom}(N_f, p)$. Is the variance of X_f/N_f always lower than the variance of $X/(N + 1)$ when N is a random variable?

   (c) What does the fixed sample size have to be to ensure equal variances? Will the expected value of N be bigger or smaller than this fixed sample size?

10. Show that if a two-sided, equal tailed hypothesis test rejects the null, the confidence interval of the same significance level will not include 0.