Deriving Beta from independent Gammas

Let $X \sim Ga(a, \theta) \perp\!\!\!\perp Y \sim Ga(b, \theta)$.

Define $U = \frac{X}{X+Y}$ and $V = X + Y$. Note: $U \in (0, 1), V \in (0, \infty)$

Want to show: $U \sim Beta(a, b)$

Joint pdf of X and Y:

$$
\begin{aligned}
f_{X,Y}(x, y) &= \frac{\theta^a}{\Gamma(a)} x^{a-1} e^{-\theta x} \times \frac{\theta^b}{\Gamma(b)} y^{b-1} e^{-\theta y} \\
&= \frac{\theta^{a+b}}{\Gamma(a)\Gamma(b)} x^{a-1} y^{b-1} e^{-\theta(x+y)}
\end{aligned}
$$

From change of variables, we found that $X = UV$ and $Y = V - UV$ Jacobian of transformation is $J = v$. Then joint pdf of U and V is:

$$
\begin{aligned}
g_{U,V}(u, v) = f_{X,Y}(uv, v - uv) \times J \\
&= \frac{\theta^{a+b}}{\Gamma(a)\Gamma(b)} (uv)^{a-1} (v - uv)^{b-1} e^{-\theta(uv+(v-uv))} \times v \\
&= \frac{\theta^{a+b}}{\Gamma(a)\Gamma(b)} u^{a-1} v^{(a+b)-1} (1 - u)^{b-1} e^{-\theta v}
\end{aligned}
$$

To find pdf of U, we need to marginalize out V by integrating over its support:

$$
\begin{aligned}
g_U(u) &= \int_0^\infty g_{U,V}(u, v) dv \\
&= \int_0^\infty \frac{\theta^{a+b}}{\Gamma(a)\Gamma(b)} u^{a-1} v^{(a+b)-1} (1 - u)^{b-1} e^{-\theta v} dv \\
&= \frac{\theta^{a+b}}{\Gamma(a)\Gamma(b)} u^{a-1} (1 - u)^{b-1} \int_0^\infty v^{(a+b)-1} e^{-\theta v} dv \\
&= \frac{\theta^{a+b}}{\Gamma(a)\Gamma(b)} u^{a-1} (1 - u)^{b-1} \int_0^\infty \frac{\Gamma(a + b)}{\Gamma(a + b)} v^{(a+b)-1} e^{-\theta v} dv \\
&= \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} u^{a-1} (1 - u)^{b-1} \int_0^\infty \frac{\theta^{a+b}}{\Gamma(a + b)} v^{(a+b)-1} e^{-\theta v} dv \\
&= \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} u^{a-1} (1 - u)^{b-1} \quad \times 1
\end{aligned}
$$

Since $U \in (0, 1)$ and we recognize $g_U(u)$ as pdf of $Beta(a, b)$, we have that $U \sim Beta(a, b)$!