1) Find the inverse z-transform of a) $\frac{8z-19}{(z-2)(z-3)}$ b) $\frac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$

a) $$\frac{8z-19}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

$8z-19 = A(z-3) + B(z-2) \rightarrow$ for $z=2$, $A=3$
$z=3$, $B=5$

$$X(z) = \frac{3}{z-2} + \frac{5}{z-3}$$

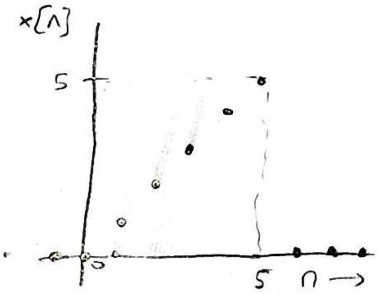
$$\begin{pmatrix} Z\{a^n u[n]\} = \frac{z}{z-a}, \ ROC: |z| > |a| \\ Z\{a^n u[n]\} = \frac{1}{1-az^{-1}}, \ |z| > |a| \end{pmatrix}$$

$$\left(Z^{-1}\left\{\frac{1}{z-a}\right\} = a^n u[n-1] \right.$$

$$\frac{3}{z-2} = 3\frac{z^{-1}}{1-2z^{-1}} \Rightarrow 3\cdot(2^n)u[n]$$

$$\frac{5}{z-3} = 5\frac{z^{-1}}{1-3z^{-1}} \Rightarrow 5\cdot(3^n)u[n]$$

$$\boxed{x[n] = 3(2^n)u[n] + 5(3^n)u[n]}$$



2) Find the z-transform of the signal x[n] depicted below.

$$x[n] = \begin{cases} 0, & n=0 \\ 1, & n=1 \\ 2, & n=2 \\ 3, & n=3 \\ 4, & n=4 \\ 5, & n=5 \\ 0, & n>5 \end{cases}$$

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

since $x[n]=0$ for $n>5$

$$\boxed{X(z) = 1z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + 5z^{-5}}$$

3) Find the response $y[n]$ of an LTID system described by the difference equation:

$$y[n+2] + y[n+1] + 0.22y[n] = x[n+1] + 0.44x[n]$$

for the input $x[n] = (-2)^{-n}u(n)$ with all the initial conditions zero (system in zero state)

i) first take z-transform: z-transform time shifts:

$$Z\{y[n+k]\} = z^k Y(z) - z^k \sum_{m=0}^{k-1} y[m] z^{-m}$$

But since initial conditions zero:

$$Z\{y[n+2]\} = z^2 Y(z)$$
$$Z\{y[n+1]\} = zY(z)$$
$$Z\{y[n]\} = Y(z)$$
$$Z\{x[n+1]\} = zX(z)$$
$$Z\{x[n]\} = X(z)$$

ii) write the z-domain equation =

$$(z^2 + z + 0.22)\,Y(z) = (z + 0.44)\,X(z)$$

iii) find $X(z)$ =

we know:

since $x[n] = (-2)^{-n} u[n] = \left(-\frac{1}{2}\right)^n u[n]$

$$Z\{a^n u[n]\} = \frac{1}{1 - az^{-1}}, \quad |z| > a$$

Here $a = -1/2$: $X(z) = \dfrac{z}{z + \frac{1}{2}} = \dfrac{z}{z + 0.5}$

iiii) solve for $Y(z)$ =

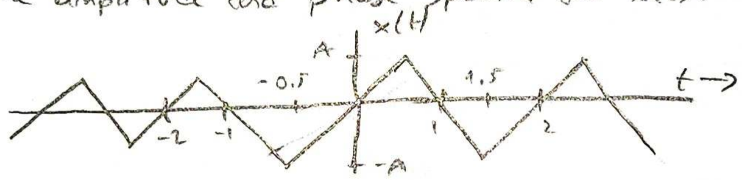
$$Y(z) = \frac{(z + 0.44) \cdot z}{(z^2 + z + 22)(z + 0.5)}$$

iiiii) inverse z-transform =

$$Y(z) = \frac{A}{(z + 0.5)} + \frac{Bz + C}{z^2 + z + 0.22}$$

$$y[n] = A(-0.5)^n u[n] + (Br_1^{\,n} + Cr_2^{\,n})\, u[n] \quad \left(\begin{array}{l} r_1, r_2 \text{ are roots of} \\ z^2 + z + 0.22 = 0 \end{array}\right)$$

v) find the compact fourier series for the triangular periodic signal $x(t)$ shown below, and sketch the amplitude and phase spectra for $x(t)$.



since it's symmetric about $t = 0$, the signal is even, so only cosine terms will appear in its fourier series.

ii) general form of trigonometric fourier series.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) \quad \text{where} \quad \omega_0 = \frac{2\pi}{T_{=2}} = \pi$$

iii) compute coefficients

a_0 (DC component) $\quad a_0 = \dfrac{1}{T} \displaystyle\int_{-T/2}^{T/2} x(t)\,dt$ $\quad$ since the positive and negative areas cancel $\underline{a_0 = 0}$

a_n (cosine coefficients) $\quad a_n = \dfrac{2}{T} \displaystyle\int_0^{T/2} x(t) \cos(n\omega_0 t)\,dt$

since $T = 2$ $\quad a_n \displaystyle\int_0^1 x(t) \cos(n\pi t)\,dt$

on $0 \leq t \leq 1$, $x(t) = A(1 - t)$

$$a_n = A \int_0^1 (1 - t) \cos(n\pi t)\,dt$$

$$A\left[\int_0^1 \cos(n\pi t)\,dt - \int_0^1 t\cos(n\pi t)\,dt\right]$$

$$a_n = -A \frac{1}{(n\pi)^2}\left[\cos(n\pi) - 1\right]$$

$$a_n = -A \frac{1}{(n\pi)^2}\left[(-1)^n - 1\right]$$

- $(-1)^n - 1 = 0$ when n even $\rightarrow a_n = 0$
- $(-1)^n - 1 = -2$ when n odd

$\rightarrow a_n = \dfrac{2A}{(n\pi)^2}$, n odd

$$\boxed{x(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2A}{(n\pi)^2} \cos(n\pi t)}$$

5. Determine the fundamental frequency and period of the following signals:

a) $x(t) = 2 + 7\cos(\frac{1}{2}t + \theta_1) + 3\cos(\frac{2}{3}t + \theta_2) + 5\cos(\frac{7}{6}t + \theta_3)$

b) $x(t) = 2\cos(2t + \theta_1) + 5\sin(\pi t + \theta_2)$

c) $x(t) = 3\sin(3\sqrt{2}t + \theta) + 7\cos(6\sqrt{2}t + \phi)$

a) angular frequencies: $w_1 = \frac{1}{2}$, $w_2 = \frac{2}{3}$, $w_3 = \frac{7}{6}$

find fundamental frequency: we need the gcd of these:

$w_0 = gcd(\frac{1}{2}, \frac{2}{3}, \frac{7}{6}) \approx gcd(\frac{3}{6}, \frac{4}{6}, \frac{7}{6})$

their gcd is $\frac{1}{6}$, $\boxed{w_0 = \frac{1}{6}}$

period: $T_0 = \frac{2\pi}{w_0} = 2\pi \cdot 6 = \boxed{12\pi}$

b) angular frequencies: $w_1 = 2$, $w_2 = \pi$

fundamental frequencies: $gcd(2, \pi)$, since π is irrational, no common multiple, non-periodic signal

$\boxed{\text{No fundamental period}}$

c) angular frequencies: $w_1 = 3\sqrt{2}$, $w_2 = 6\sqrt{2}$

fundamental frequencies: $gcd(3\sqrt{2}, 6\sqrt{2}) = 3\sqrt{2}$

since $\sqrt{2}$ is irrational, the signal is non-periodic.

$\boxed{\text{No fundamental period}}$

6. Find the Fourier transform of $e^{-a|t|}$

$X(jw) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-jwt} dt$ ← write the fourier transform definition

• split the integral at $t=0$ $\Rightarrow \int_{-\infty}^{0} e^{at} e^{-jwt} dt + \int_{0}^{\infty} e^{-at} e^{-jwt} dt$

$\Rightarrow \int_{-\infty}^{0} e^{(a-jw)t} dt + \int_{0}^{\infty} e^{-(a+jw)t} dt$

• solve both integrals

first integral $= \frac{1}{a-jw}$, second integral $= \frac{1}{a+jw}$

• combine

$X(jw) = \frac{1}{a-jw} + \frac{1}{a+jw} \Rightarrow \frac{2a}{a^2+w^2}$

$$\boxed{X(jw) = \frac{2a}{a^2+w^2}}$$

7. Using the time-shifting property, find the fourier transform of $e^{-a|t-t_0|}$.

- from previous problem: $\mathcal{F}\{e^{-a|t|}\} = X(jw) = \frac{2a}{a^2+w^2}$
- apply time-shifting property: $x(t-t_0) \Rightarrow X(jw)e^{-jwt_0}$
- final result: $\mathcal{F}\{e^{-a|t-t_0|}\} = \frac{2a}{a^2+w^2} \cdot e^{-jwt_0}$

$$X(jw) = \frac{2ae^{-jwt}}{a^2+w^2}$$

8. Consider a signal $x(t) = \text{sinc}^2(5\pi t)$ whose spectrum is $X(w) = 0.2\Delta\left(\frac{w}{20\pi}\right)$. Plot the frequency spectrum when $f_s = 5, 10, 20$ Hz.

- $X(w)$ is a triangle function centered at 0 and has bandwidth:

Base form -20π to $20\pi \Rightarrow B = 20\pi$

- sampling frequency:

$f_s = 5, 10, 20$ Hz, since $f_s = \frac{w_s}{2\pi} \Rightarrow w_s = 2\pi f_s$

thus: $w_s = \begin{cases} 10\pi & f_s = 5 \\ 20\pi & f_s = 10 \\ 40\pi & f_s = 20 \end{cases}$

no graph found

9. A continuous-time sinusoid $\cos(2\pi f t + \theta)$ is sampled at a rate $f_s = 1200$ Hz. Determine the apparent (aliased) sinusoid of the resulting samples if the input signal frequency f is (a) 200 Hz, (b) 600 Hz, (c) 1000 Hz, (d) 2400 Hz.

Aliased frequency formula: when sampled, apparent frequency:

$f_a = |f - mf_s|$ where m is an integer chosen so that:

$f_a \in \left[0, \frac{f_s}{2}\right]$

a) $f = 200$ Hz, since $200 < 600$: $\boxed{f_a = 200\text{ Hz}}$, $\boxed{\text{No aliasing}}$

b) $f = 600$ Hz : Exactly at Nyquist frequency

$\boxed{f_a = 600\text{ Hz}}$: $\boxed{\text{No aliasing, but highest freq representable}}$

c) $f = 100$ Hz : $1000 - 1 \cdot 1200 = -200$

$\boxed{f_a = |-200| = 200\text{ Hz}}$ $\boxed{\text{Aliased to 200 Hz}}$

d) $f = 2400$ Hz : $2400 - 2 \cdot 1200 = 0$: $\boxed{f_a = 0 \text{ Hz}}$

$\boxed{\text{Aliased to } 0 \text{ Hz}}$

10) A signal $x[n] = \text{sinc}(\pi n/4)$ modulates a carrier $\cos \Omega n$. find and sketch the spectrum of the modulated signal $x[n] \cos \Omega_c n$ for $\Omega_c = \frac{\pi}{2}$.

The fourier transform of a sinc is a rectangular function:

$$X(e^{i\Omega}) = \begin{cases} 4, & |\Omega| \leq \frac{\pi}{4} \\ 0, & \text{otherwise} \end{cases}$$

so:

$$X(e^{i\Omega}) = 4 \,\text{rect}\left(\frac{\Omega}{\pi/2}\right)$$

modulation in frequency domain:

multiplying by $\cos(\frac{\pi}{2} n) \rightarrow$ frequency shift:

$$\frac{1}{2}\left[X(e^{j(\Omega - \pi/2)}) + X(e^{j(\Omega + \pi/2)})\right]$$

The spectrum is two rectangles:

- One centered at $+\pi/2$
- One centered at $-\pi/2$
- Each has hight 2 (because of $1/2 \cdot 4$)
- width = $\pi/2$ (from $\pi/4$ left + right of center)