MATH 421 — Abstract Algebra

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These notes are my own and are not endorsed by the lecturers. I have often made significant modifications to them after the lectures, so they may not accurately reflect the content presented in class. Any errors are almost certainly my own. Additionally, these notes are partially based on *A Book of Abstract Algebra* by *Charles C. Pinter*. Please refer to the original text for a more authoritative source.

Binary Operations

The course begins with an introduction to binary operations, emphasizing their properties on number sets and matrices, including commutativity, associativity, identity, and inverses, alongside a proof-based approach to these concepts.

0 Operations

You often view an operation as an action on any two elements of a set S that produces another element in the same set. Taking a step back, i.e., adopting a more abstract approach, operations in abstract algebra are approached generally rather than being tied to a specific set, denoted by "$*$".

Definition (Binary Operations on set S)**.** An operation $*$ on a set S is a rule that assigns a unique element $a * b \in S$ to every ordered pair (a, b) of elements in S. This is a *binary operation* defined on set S, a rule which takes two elements $a, b \in S$ produces a single element $a * b \in S$. Three important aspects of the definition of an operation are:

1. For every ordered pair $(a, b) \in S$, $a * b$ is defined.
2. The operation $a * b$ must be uniquely defined.
3. If a and b are in S, then it must follow that $a * b \in S$.

Example. The addition of real numbers. Take a pair of any two number $(a, b) \in \mathbb{R} \times \mathbb{R}$. The operation $*$ maps this pair to some $c \in \mathbb{R}$, thus $*$ acts as $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. This is a map. We can also see that f is *bijective*, thus obeying definition two.

Definition (Commutative Binary Operations)**.** A binary operation on a set S is commutative if and only if $a * b = b * a$ if $\forall a, b \in S$.

Definition (Associative Binary Operations)**.** A binary operation is also associative on a set S if $(a * b) * c = a * (b * c)$ if $\forall a, b, c \in S$.

Definition (Identity)**.** An element $e \in S$ is neutral if $a * e = e * a = a$. This is the identity element.

Example. The identity element for the sets $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ is 1 with respect to the multiplication operation.

$$1 \times 4 = 4 = 4 \times 1$$

For the addition operation, 0 is the identity elements for the sets $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$.

$$0 + 4 = 4 = 4 + 0$$

Definition (Inverses)**.** An element $a \in S$ is said to be invertible if $\exists a^{-1} \in S$ and $e \in S$ such that, with respect to an operation "$*$", :

$$a * a^{-1} = e = a^{-1} * a$$

The element a^{-1} is known as the inverse. If the operation $a * a^{-1}$ and $a^{-1} * a$ maps to an identity e of S, then a is said to be invertible.

Example. Let A be the two-element set $A = \{\text{a, b}\}$. Write the tables of all 16 operations on A. Label these operations $0_1 \to 0_{16}$. Given that there is two elements with 4 possible pair combinations. Thus, there are $2^4 = 16$ possible output combinations.

$(x,y)_{01}$	$x*y$	$(x,y)_{02}$	$x*y$	$(x,y)_{03}$	$x*y$	$(x,y)_{04}$	$x*y$
$(a,a)_{01}$	b	$(a,a)_{02}$	b	$(a,a)_{03}$	b	$(a,a)_{04}$	b
$(a,b)_{01}$	b	$(a,b)_{02}$	b	$(a,b)_{03}$	b	$(a,b)_{04}$	b
$(b,a)_{01}$	a	$(b,a)_{02}$	a	$(b,a)_{03}$	b	$(b,a)_{04}$	b
$(b,b)_{01}$	a	$(b,b)_{02}$	b	$(b,b)_{03}$	a	$(b,b)_{04}$	b

$(x,y)_{05}$	$x*y$	$(x,y)_{06}$	$x*y$	$(x,y)_{07}$	$x*y$	$(x,y)_{08}$	$x*y$
$(a,a)_{05}$	a	$(a,a)_{06}$	a	$(a,a)_{07}$	a	$(a,a)_{08}$	a
$(a,b)_{05}$	b	$(a,b)_{06}$	b	$(a,b)_{07}$	b	$(a,b)_{08}$	b
$(b,a)_{05}$	a	$(b,a)_{06}$	a	$(b,a)_{07}$	b	$(b,a)_{08}$	b
$(b,b)_{05}$	a	$(b,b)_{06}$	b	$(b,b)_{07}$	a	$(b,b)_{08}$	b

$(x,y)_{09}$	$x*y$	$(x,y)_{10}$	$x*y$	$(x,y)_{11}$	$x*y$	$(x,y)_{12}$	$x*y$
$(a,a)_{09}$	b	$(a,a)_{10}$	b	$(a,a)_{11}$	b	$(a,a)_{12}$	b
$(a,b)_{09}$	a	$(a,b)_{10}$	a	$(a,b)_{11}$	a	$(a,b)_{12}$	a
$(b,a)_{09}$	a	$(b,a)_{10}$	a	$(b,a)_{11}$	b	$(b,a)_{12}$	b
$(b,b)_{09}$	a	$(b,b)_{10}$	b	$(b,b)_{11}$	a	$(b,b)_{12}$	b

$(x,y)_{13}$	$x*y$	$(x,y)_{14}$	$x*y$	$(x,y)_{15}$	$x*y$	$(x,y)_{16}$	$x*y$
$(a,a)_{13}$	a	$(a,a)_{14}$	a	$(a,a)_{15}$	a	$(a,a)_{16}$	a
$(a,b)_{13}$	a	$(a,b)_{14}$	a	$(a,b)_{15}$	a	$(a,b)_{16}$	a
$(b,a)_{13}$	a	$(b,a)_{14}$	a	$(b,a)_{15}$	b	$(b,a)_{16}$	b
$(b,b)_{13}$	a	$(b,b)_{14}$	b	$(b,b)_{15}$	a	$(b,b)_{16}$	b

Commutativity holds for the following elements:

Operations where $(a,b)=b$:

- 0_{03} since $(a,b)=b=(b,a)$
- 0_{04} since $(a,b)=b=(b,a)$
- 0_{07} since $(a,b)=b=(b,a)$
- 0_{08} since $(a,b)=b=(b,a)$

Operations where $(a,b)=a$:

- 0_{09} since $(a,b)=a=(b,a)$
- 0_{10} since $(a,b)=a=(b,a)$
- 0_{13} since $(a,b)=a=(b,a)$
- 0_{14} since $(a,b)=a=(b,a)$

Identity Elements Present in Set A
The following operations contain identity elements:

$\boxed{\text{Operation } 0_{02}:}$

Identity Definition: $x*e=x \iff e*x=x$

Operation Rules:

$$\begin{cases} a*b=a & \text{(pair operations)} \\ b*a=a \end{cases}$$

$$\begin{cases} a*a=a & \text{(self operations)} \\ b*b=b \end{cases}$$

$\boxed{\text{Operation } 0_{08}\text{:}}$

Identity Definition: $x * e = x \iff e * x = x$

Operation Rules:

$$\begin{cases} a * b = b & \text{(pair operations)} \\ b * a = b \end{cases}$$

$$\begin{cases} a * a = a & \text{(self operations)} \\ b * b = b \end{cases}$$

$\boxed{\text{Operation } 0_{09}\text{:}}$

Identity Definition: $x * e = x \iff e * x = x$

Operation Rules:

$$\begin{cases} a * b = a & \text{(pair operations)} \\ b * a = a \end{cases}$$

$$\begin{cases} a * a = b & \text{(self operations)} \\ b * b = a \end{cases}$$

Theorem. For all operations 0_i where $i \in \{1, \ldots, 16\} \setminus \{2, 8, 9\}$, the set A does not contain an identity element.

Proposition. For inverse elements in set A:

(i) Operations 0_{02}, 0_{08}, and 0_{09} may contain inverse elements since they possess identity elements.

(ii) All other operations 0_i where $i \in \{1, \ldots, 16\} \setminus \{2, 8, 9\}$ cannot have inverse elements as they lack identity elements.

Remark. Even for operations 0_{02}, 0_{08}, and 0_{09}, the existence of an inverse is not guaranteed for all elements in A, despite the presence of identity elements.

1 Groups

$$x * (y * z) = x * (\frac{yz}{5})$$
$$\frac{x(\frac{yz}{5})}{5}$$
$$\frac{xyz}{25}$$

And the reverse

$$(x * y) * z$$
$$(\frac{xy}{5}) * z$$
$$\frac{(\frac{xy}{5}z}{5}$$
$$\frac{xyz}{25}$$

so therefore it is associative.
Problem 3: Solve for $z * e = x$
for e:

$$x * e = x$$
$$\frac{xe}{5} = x$$
$$\frac{xe}{x} = \frac{5x}{x}$$

note that $x \neq 0$

$$e = 5$$

note that 5 is in $\mathbb{R}^*$
For inverse:
Solve $x * x' = 5$ for x'.

$$x * x' = 5$$
$$\frac{x \times x'}{5} = 5$$
$$\frac{x \times x'}{x} = \frac{25}{x}$$
$$x' = \frac{25}{x}$$

Finally step b: check that x and its partner become neutral in both orders.

$$x * \frac{25}{x} = 5$$

and

$$\frac{25}{x} * x = 5$$

now

$$\begin{aligned} x * \frac{25}{x} &= \frac{x \times \frac{25}{x}}{5} \\ &= \frac{25}{5} \\ &= 5\checkmark \end{aligned}$$

Also $\frac{25}{x}$ is in $\mathbb{R}^*$ because x is in $\mathbb{R}^*$.
So, the inverse of x w.r.t. $*$ is $\frac{25}{x}$