MATH 421 — Abstract Algebra

Based on lectures by Dr. A. Steurer
Notes taken by Pablo C. Bedolla Ortiz

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These notes are my own and are not endorsed by the lecturers. I have often made significant modifications to them after the lectures, so they may not accurately reflect the content presented in class. Any errors are almost certainly my own. Additionally, these notes are partially based on *A Book of Abstract Algebra* by *Charles C. Pinter*. Please refer to the original text for a more authoritative source.

Binary Operations

The course begins with an introduction to binary operations, emphasizing their properties on number sets and matrices, including commutativity, associativity, identity, and inverses, alongside a proof-based approach to these concepts.

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0 Introduction

The course assumes that students have prior knowledge of proof writing, set manipulation, and linear algebra concepts. Students are encouraged to revisit the foundational principles of these topics, including the formal rules of algebra on the set of real numbers, basic proof techniques, and essential operations and properties of sets and matrices, to ensure they are adequately prepared for the course material.

1 Operations

You often view an operation as an action on any two elements of a set S that produces another element in the same set. Taking a step back, i.e., adopting a more abstract approach, operations in abstract algebra are approached generally rather than being tied to a specific set, denoted by "$*$".

Definition (Binary Operations on set S)**.** An operation $*$ on a set S is a rule that assigns a unique element $a * b \in S$ to every ordered pair (a, b) of elements in S. This is a *binary operation* defined on set S, a rule which takes two elements $a, b \in S$ produces a single element $a * b \in S$. Three important aspects of the definition of an operation are:

1. For every ordered pair $(a, b) \in S$, $a * b$ is defined.
2. The operation $a * b$ must be uniquely defined.
3. If a and b are in S, then it must follow that $a * b \in S$.

Example. The addition of real numbers. Take a pair of any two number $(a, b) \in \mathbb{R} \times \mathbb{R}$. The operation $*$ maps this pair to some $c \in \mathbb{R}$, thus $*$ acts as $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. This is a map. We can also see that f is *bijective*, thus obeying definition two.

Definition (Commutative Binary Operations)**.** A binary operation on a set S is commutative if and only if $a * b = b * a$ if $\forall a, b \in S$.

Definition (Associative Binary Operations)**.** A binary operation is also associative on a set S if $(a * b) * c = a * (b * c)$ if $\forall a, b, c \in S$.

Definition (Identity)**.** An element $e \in S$ is neutral if $a * e = e * a = a$. This is the identity element.

Example. The identity element for the sets $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ is 1 with respect to the multiplication operation.

$$1 \times 4 = 4 = 4 \times 1$$

For the addition operation, 0 is the identity elements for the sets $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$.

$$0 + 4 = 4 = 4 + 0$$

Definition (Inverses)**.** An element $a \in S$ is said to be invertible if $\exists a^{-1} \in S$ and $e \in S$ such that, with respect to an operation "$*$", :

$$a * a^{-1} = e = a^{-1} * a$$

The element a^{-1} is known as the inverse. If the operation $a * a^{-1}$ and $a^{-1} * a$ maps to an identity e of S, then a is said to be invertible.