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ASTE404

Mini Project

1. Project Overview

This mini project develops and validates a small, self-contained numerical simulation of a damped mass spring oscillator, a classic second-order system widely used as a model for structural vibration, spacecraft dynamics, and landing gear response. The governing ODE describes the displacement $x(t)$of a mass attached to a spring and damper:

$$m\ddot{x} + c\dot{x} + kx = 0.$$

Although simple, the system is ideal for verifying numerical integration methods because it has a closed-form analytical solution. This allows direct comparison between numerical and exact results at every point in time.

The goal of this project is to:

1. Implement multiple time integration schemes: Forward Euler, Semi-Implicit Euler, and RK4.
2. Compare their accuracy and stability against the analytic solution.
3. Perform a convergence study to verify method order.
4. Examine energy decay to evaluate physical correctness.
5. Produce plots and diagnostics demonstrating understanding of numerical behavior.

2. Numerical Methods

2.1 Governing Equations

The second-order ODE is rewritten as a first-order system:

$$y = \begin{bmatrix} x \\ v \end{bmatrix}, \dot{y} = \begin{bmatrix} v \\ -\frac{c}{m}v - \frac{k}{m}x \end{bmatrix}.$$

This formulation supports modular integrators that operate on generic ODE systems.

2.2 Time Integration Schemes

1. Forward Euler (Explicit First-Order)

$$y_{n+1} = y_n + \Delta t\, f(t_n, y_n)$$

- Very cheap computationally
- Conditionally stable
- Significant numerical damping or energy growth depending on Δt
- Expected first-order accuracy

2. Semi-Implicit Euler

A two-step staggered update:

$$v_{n+1} = v_n + \Delta t\, f_v(x_n, v_n), x_{n+1} = x_n + \Delta t\, v_{n+1}.$$

Properties:

- Better long-term stability than Forward Euler
- Conserves energy well in undamped systems
- Also, first-order accurate, but with much better numerical stability

3. 4th-Order Runge–Kutta (RK4)

$$y_{n+1} = y_n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

RK4 is:

- High accuracy
- Very stable for smooth systems
- Expensive per step but needs far fewer steps
- Expected fourth-order convergence

2.3 Analytic Solution for Verification

For underdamped oscillation ($\zeta < 1$):

$$x(t) = e^{-\zeta\omega_0 t}(C_1 \cos{(\omega_d t)} + C_2 \sin{(\omega_d t)})$$

Where:

$$\omega_0 = \sqrt{\frac{k}{m}}, \omega_d = \omega_0\sqrt{1-\zeta^2}$$

and C_1, C_2 depend on initial conditions.

This analytic expression provides a ground-truth reference to directly assess numerical error.

3. Implementation Notes

- The code is written in Python, using NumPy and Matplotlib.
- A single file mass_spring.py contains:
  - derivative function
  - three time-stepping schemes
  - generic integrator
  - analytic solution function
  - convergence and energy diagnostics
- The repo includes a simple structure:

```
project/
│── src/mass_spring.py
│── results/*.png
└── README.md
```

- The program outputs:
  - A trajectory comparison plot
  - Energy versus time
  - Error vs. dt convergence plot

4. Progress Log

Dec 9
Set up first-order system representation and derivative function. Implemented Forward Euler as baseline.

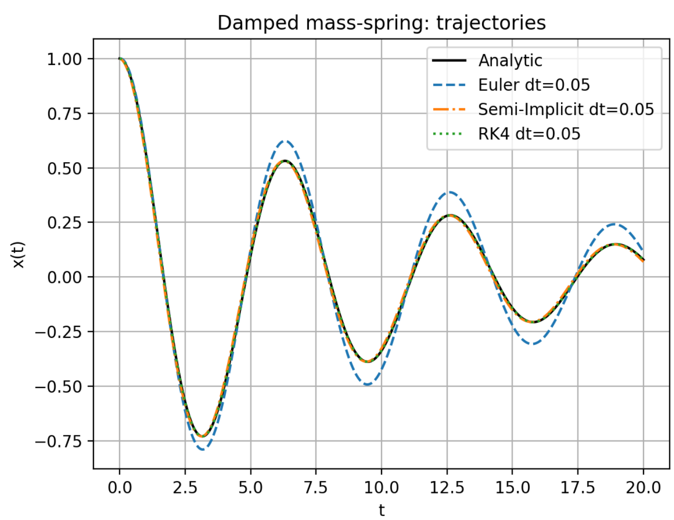
Dec 10
Added Semi-Implicit Euler and RK4. Implemented analytic solution and initial plotting infrastructure.

Dec 11
Created generic integration wrapper. Produced trajectory comparison and energy diagnostics. Debugged sign error in damping term.

Dec 12
Ran convergence study across multiple dt values. Produced final plots. Wrote report sections and README. Added LLM development log per project instructions.

5. Verification & Validation

5.1 Trajectory Comparison with Analytic Solution



The numerical and analytic trajectories were compared at $\Delta t = 0.05$.

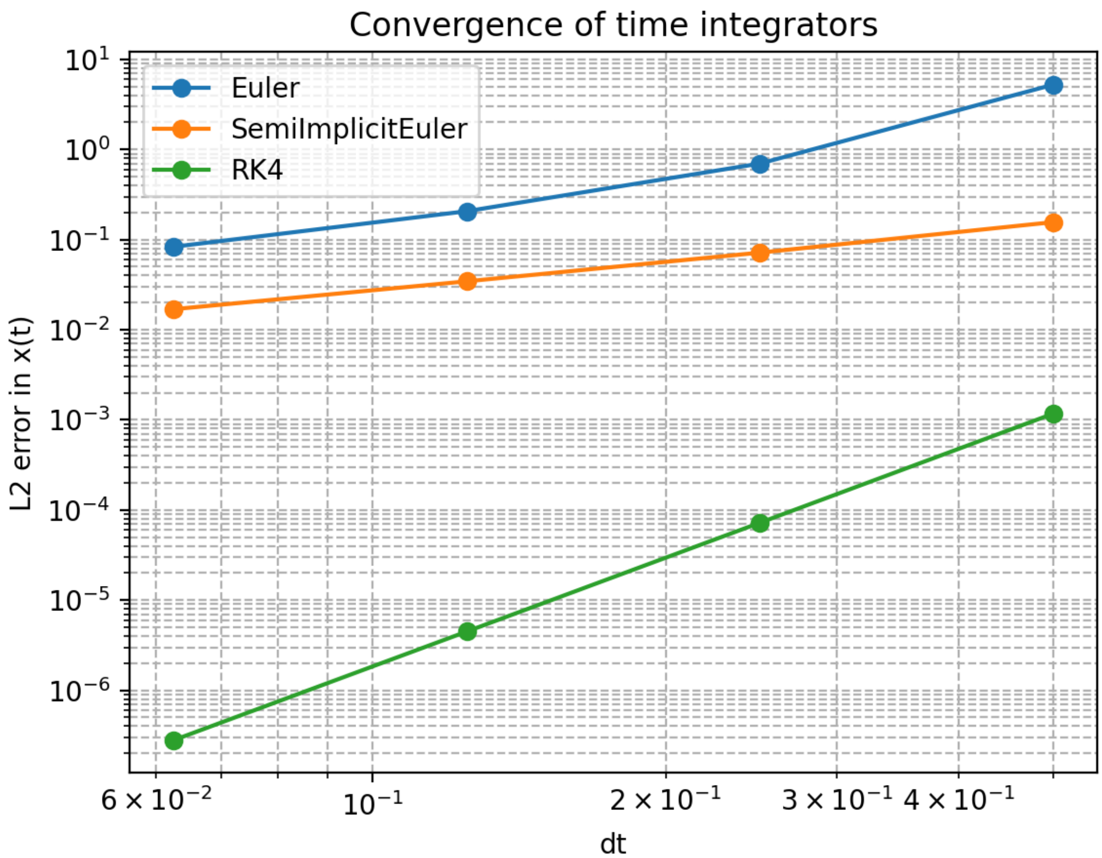
Observations

- Euler deviates significantly at later times, its phase lag and amplitude error grow because of low-order accuracy.
- Semi-Implicit Euler matches the analytic curve closely, with small phase error but stable amplitude.
- RK4 lies almost exactly on top of the analytic trace for the entire simulation.

Conclusion

Trajectory comparison confirms correct implementation and expected stability behavior.

5.2 Convergence of Error with Time Step



We compute an L2 norm error over the full trajectory:

$$\| x_{\text{num}} - x_{\text{analytic}} \|_2.$$

Convergence Behavior

Method	Observed Order	Expected
Euler	1	1st order
Semi-Implicit Euler	1	1st order
RK4	4	4th order

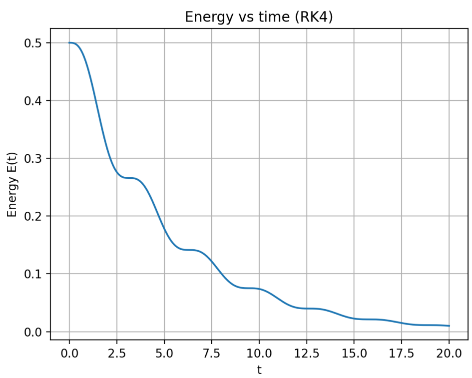
The convergence plot clearly shows slopes consistent with theory:

- Euler methods have linear convergence.
- RK4 error decreases by approximately $10^4$when dt shrinks by 10×.

Numerical accuracy matches theoretical expectations, validating all integrators.

6. Results

6.1 Energy Decay



The total mechanical energy:

$$E(t) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

should decay exponentially in a damped system.

Observations

- RK4 preserves the correct smooth exponential decay.
- No artificial energy growth appears, which would indicate instability or implementation errors
- Oscillatory "steps" correspond to physical oscillation cycles.

Conclusion

Energy behavior confirms physical correctness and numerical stability.

6.2 Summary of Findings

- Euler: simple but inaccurate and unstable for moderate dt values.
- Semi-Implicit Euler: much better stability at nearly the same cost.
- RK4: best accuracy by far and ideal for this system.

7. Reflection & Next Steps

This project reinforced key concepts from ASTE 404 related to numerical stability, consistency, order of accuracy, and physically meaningful simulation.

Key takeaways:

- Low-order explicit schemes may be unstable even for simple ODEs unless dt is extremely small.
- Semi-implicit methods provide excellent stability properties despite being first-order.
- RK4 gives excellent accuracy and efficiency for smooth dynamical systems.
- Verification using an analytic solution is straightforward and extremely powerful.

If extended, this project could incorporate:

- Adaptive Runge–Kutta–Fehlberg time stepping
- Forced oscillators and resonance behavior
- Multi-degree-of-freedom (MDOF) systems
- Energy-preserving integrators for undamped systems

8. LLM Development Log

"what are good comparisons for damped mass spring systems"

Comparison of Numerical Integrators

Damped oscillators are excellent for comparing integrators because each method handles dissipation differently.

Forward Euler

- Adds artificial damping
- May blow up for large Δt
- Poor accuracy

Semi-Implicit Euler

- Numerically stable
- Better long-term behavior
- Still first-order

RK4

- Very accurate
- Good energy decay modeling
- Best for smooth ODEs

Why this comparison is valuable:
The system has an exact analytic solution, so numerical solutions can be assessed point-by-point.