Perceptron

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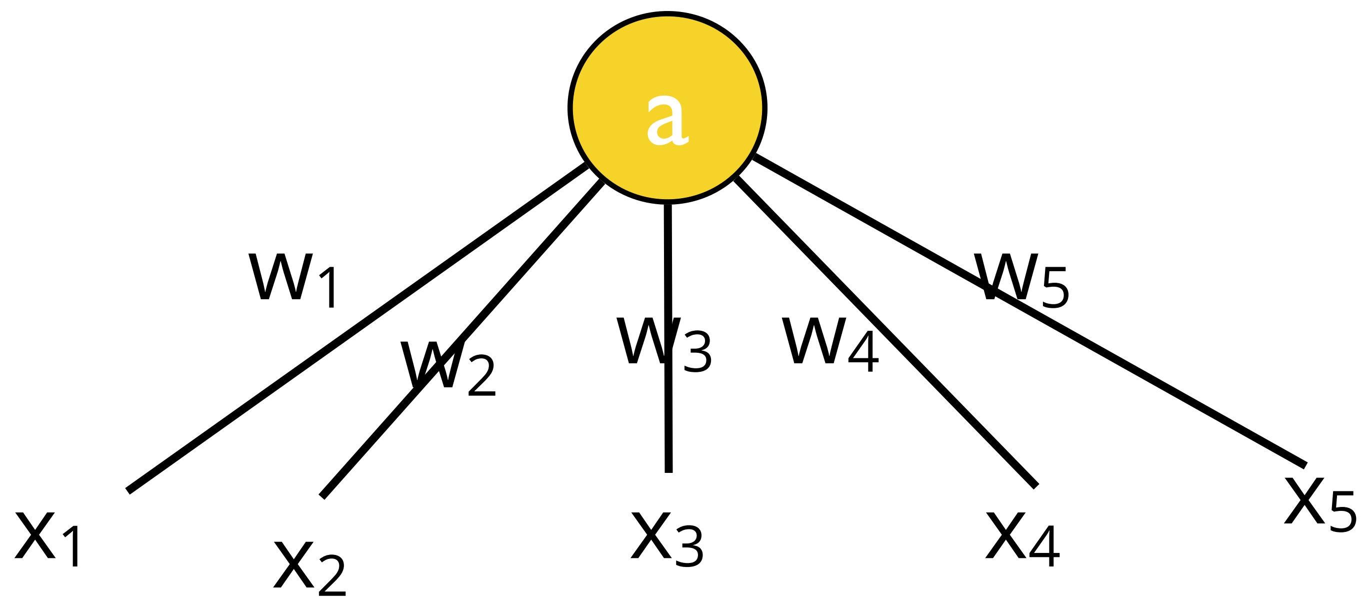
Bio-inspired model

- Perceptron is a bio-inspired algorithm that tries to mimic a single neuron
- We simply multiply each input (feature) by a weight and check whether this weighted sum (activation) is greater than a threshold.
- If so, then we “fire” the neuron (i.e. a decision is made based on the activation)

A single neuron

activation (score) = $a = w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 + w_5x_5$



if $a > \theta$ then
    output = 1
else
    output = 0

If the activation is greater than a predefined threshold, then the neuron fires.

Bias

- Often we need to adjust a fixed *shift* from zero, if the "interesting" region happens to be far from the origin.
- We adjust the previous model by including a bias term b as follows

$$a = b + \sum_{i=1}^{D} w_d x_d$$

Notational trick

- By introducing a feature that is always ON (i.e. $x_0 = 1$ for all instances), we can squeeze the bias term b into the weight vector by setting $w_0 = b$

$$a = \sum_{i=0}^{D} w_d x_d = \boldsymbol{w}^\top \boldsymbol{x}$$

This is more "elegant" as we can write the activation as the inner-product between the weight vector and the feature vector. However, we should keep in mind that bias term still appears in the model.

Perceptron

- Consider only one training instance at a time
  - online learning
  - k-NN considers ALL instances (batch learning)
- Learn only if we make a mistake when we classify using the current weight vector. Otherwise, we do not make adjustments to the weight vector
  - Error-driven learning

Algorithm 5 PerceptronTrain($\mathbf{D}$, *MaxIter*)

$w_d \leftarrow 0$, for all $d = 1 \dots D$ // initialize weights
$b \leftarrow 0$ // initialize bias
for $iter = 1 \dots MaxIter$ **do**
  for all $(\boldsymbol{x}, y) \in \mathbf{D}$ **do**
    $a \leftarrow \sum_{d=1}^{D} w_d \, x_d + b$ // compute activation for this example
    if $ya \leq 0$ **then**
      $w_d \leftarrow w_d + y x_d$, for all $d = 1 \dots D$ // update weights
      $b \leftarrow b + y$ // update bias
    end if
  end for
end for
return $w_0, w_1, \dots, w_D, b$

Algorithm 6 PerceptronTest($w_0, w_1, \dots, w_D, b, \hat{\boldsymbol{x}}$)

$a \leftarrow \sum_{d=1}^{D} w_d \, \hat{x}_d + b$ // compute activation for the test example
return $\textsc{sign}(a)$

slide credit: CIML (Daume III)

Detecting errors

- In Line 6 of PerceptronTrain code we have
  - ya <= 0
  - If the current instance is positive (y = 1), we should have a positive activation (a > 0) in order to have a correct prediction
  - If the current instance is negative (y = -1), we should have a negative activation (a < 0) in order to have a correct prediction
  - In both cases ya > 0.
  - Therefore, if ya <= 0 then we have a misclassification

Update rule — Intuitive Explanation

- Perceptron update rule is
  - $\mathbf{w} = \mathbf{w} + y\mathbf{x}$
- If we incorrectly classify a positive instance as negative
  - We should have a higher (more positive) activation to avoid this
  - We should increase $\mathbf{w}^T\mathbf{x}$
  - Therefore, we should ADD the current instance to the weight vector
- If we incorrectly classify a negative instance as positive
  - We should have a lower (more negative) activation to avoid this
  - We should decrease $\mathbf{w}^T\mathbf{x}$
  - Therefore, we should DEDUCT the current instance from the weight vector

Update rule — Math Explanation

$$
\begin{aligned}
a' &= \sum_{d=1}^{D} w'_d x_d + b' \\
&= \sum_{d=1}^{D} (w_d + x_d) x_d + (b + 1) \\
&= \sum_{d=1}^{D} w_d x_d + b + \sum_{d=1}^{D} x_d x_d + 1 \\
&= a + \sum_{d=1}^{D} x_d^2 + 1 \quad > \quad a
\end{aligned}
$$

If the misclassified instance is a positive one, then after we update using
w = **w** + **x**,
the new activation a’ is greater than the old activation a.

Quiz 1

- Show that the analysis in the previous slide holds when y = -1 (i.e. we misclassified a negative instance)

$$a' = \sum_{d=1}^{D} w'_d x_d + b'$$

The misclassified instance is negative. Therefore, the update will be $w \Leftarrow w - x$ and $b \Leftarrow b - 1$

$$\therefore \ a' = \sum_{d=1}^{D} (w_d - x_d) x_d + (b - 1)$$

$$= \underbrace{\sum_{d=1}^{D} w_d x_d + b}_{a.} - \underbrace{\left(\sum_{d=1}^{D} x_d^2 + 1 \right)}_{>0}$$

$\therefore \ a' < a$ Therefore, the update will decrease the margin.

Things to remember

- There is no guarantee that we will correctly classify a misclassified instance in the next round.
- We have simply increased/decreased the activation but this adjustment might not be sufficient. We might need to do more aggressive adjustments
- There are algorithms that enforce such requirements explicitly such as the Passive Aggressive Classifier (not discussed here)

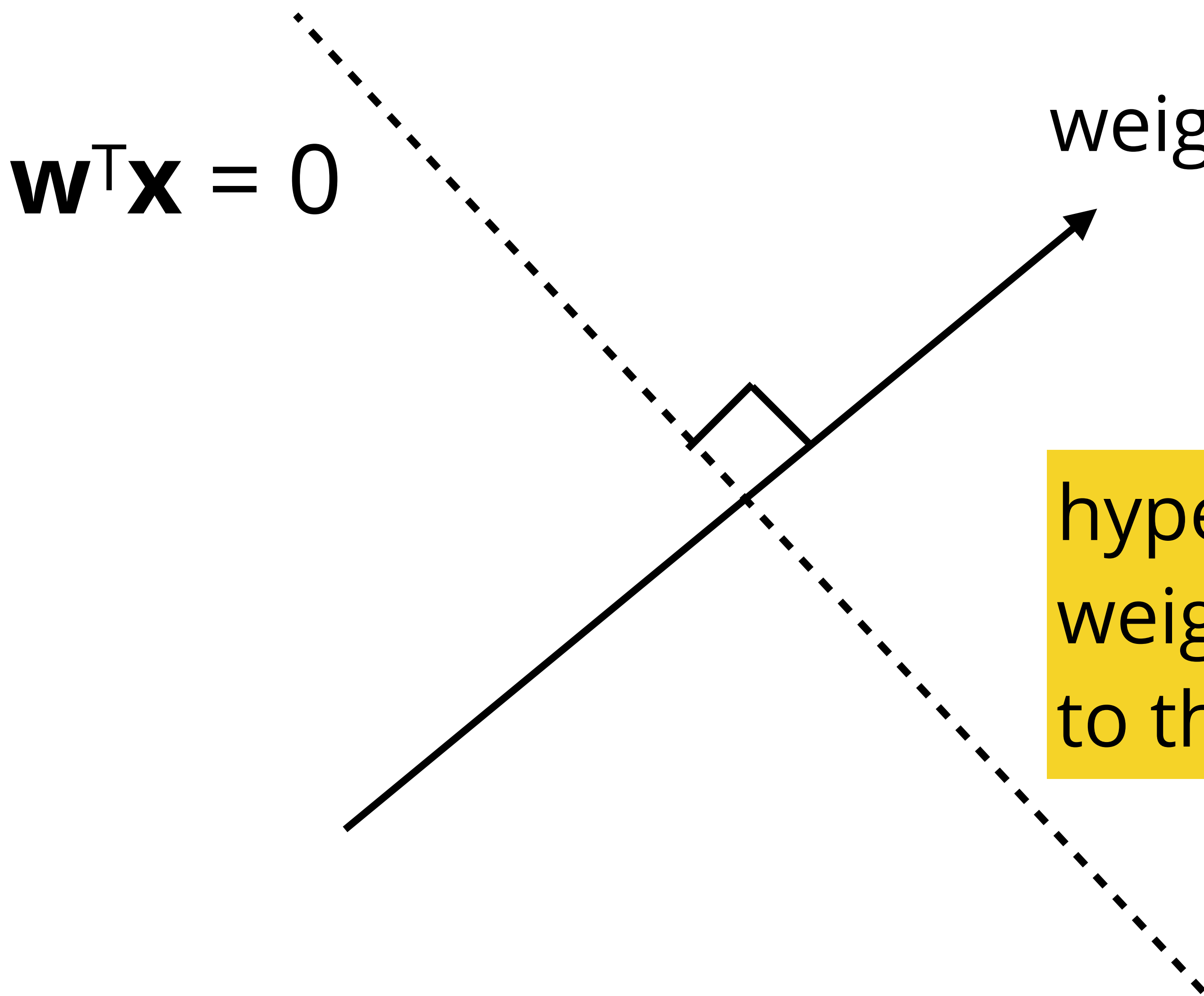
Ordering of instances

- Ordering training instances randomly within each iteration produces good results in practice
- Showing only all the positives first and all the negatives next is a bad idea

Hyperplane

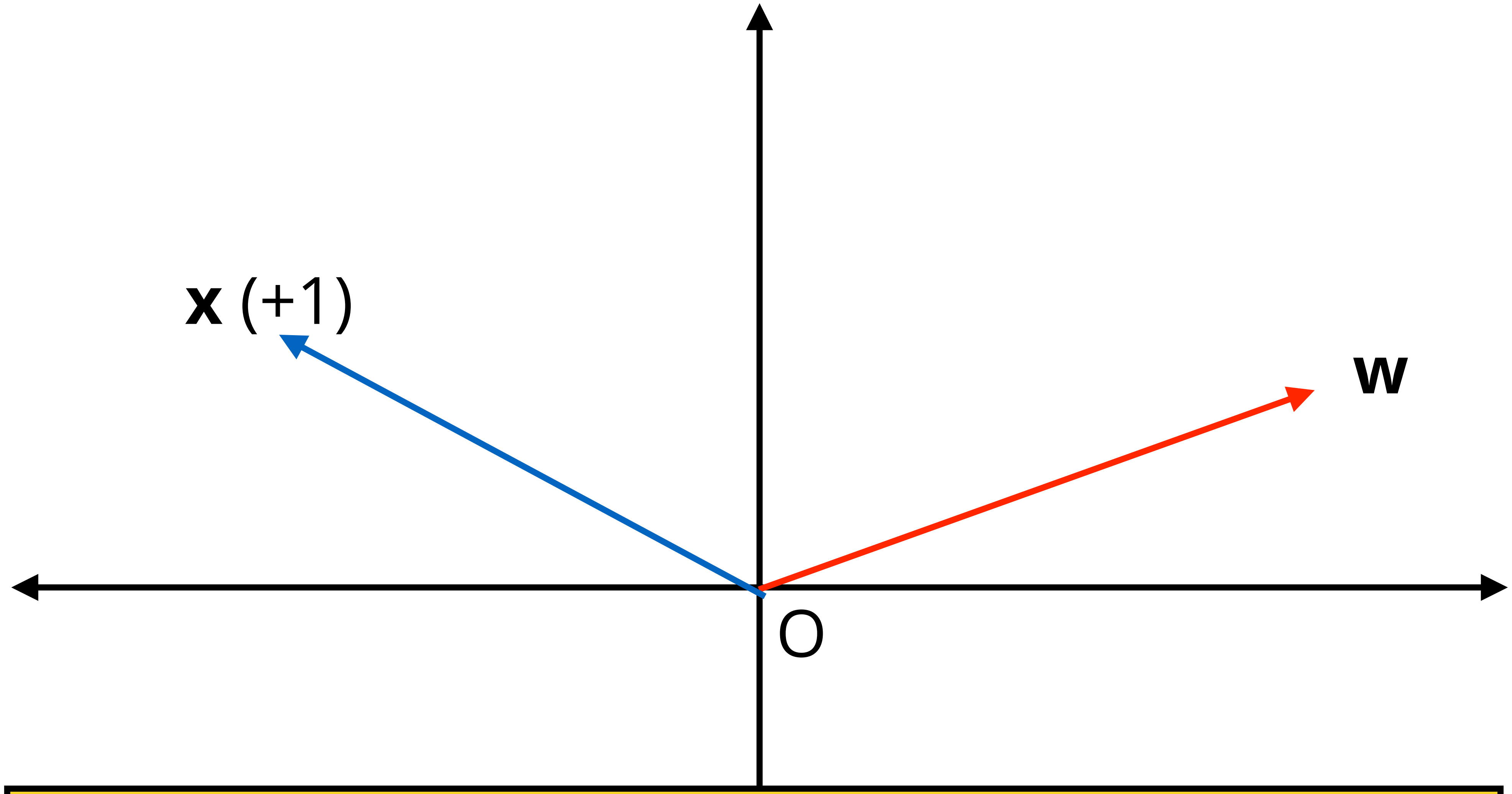
- The decision in perceptron is made depending on $\mathbf{w}^T\mathbf{x} > 0$ or $\mathbf{w}^T\mathbf{x} <= 0$
- Therefore, $\mathbf{w}^T\mathbf{x} = 0$ is the critical region (decision boundary)
- $\mathbf{w}^T\mathbf{x} = 0$ defines a hyperplane
- Example:
    - In 2D space we have $w_1x_1 + w_2x_2 = 0$ (ignoring the bias term), which is a straight line through the origin.
    - In N dimensional space this is an (N-1) dimensional hyperplane

Geometric Interpretation of Hyperplane



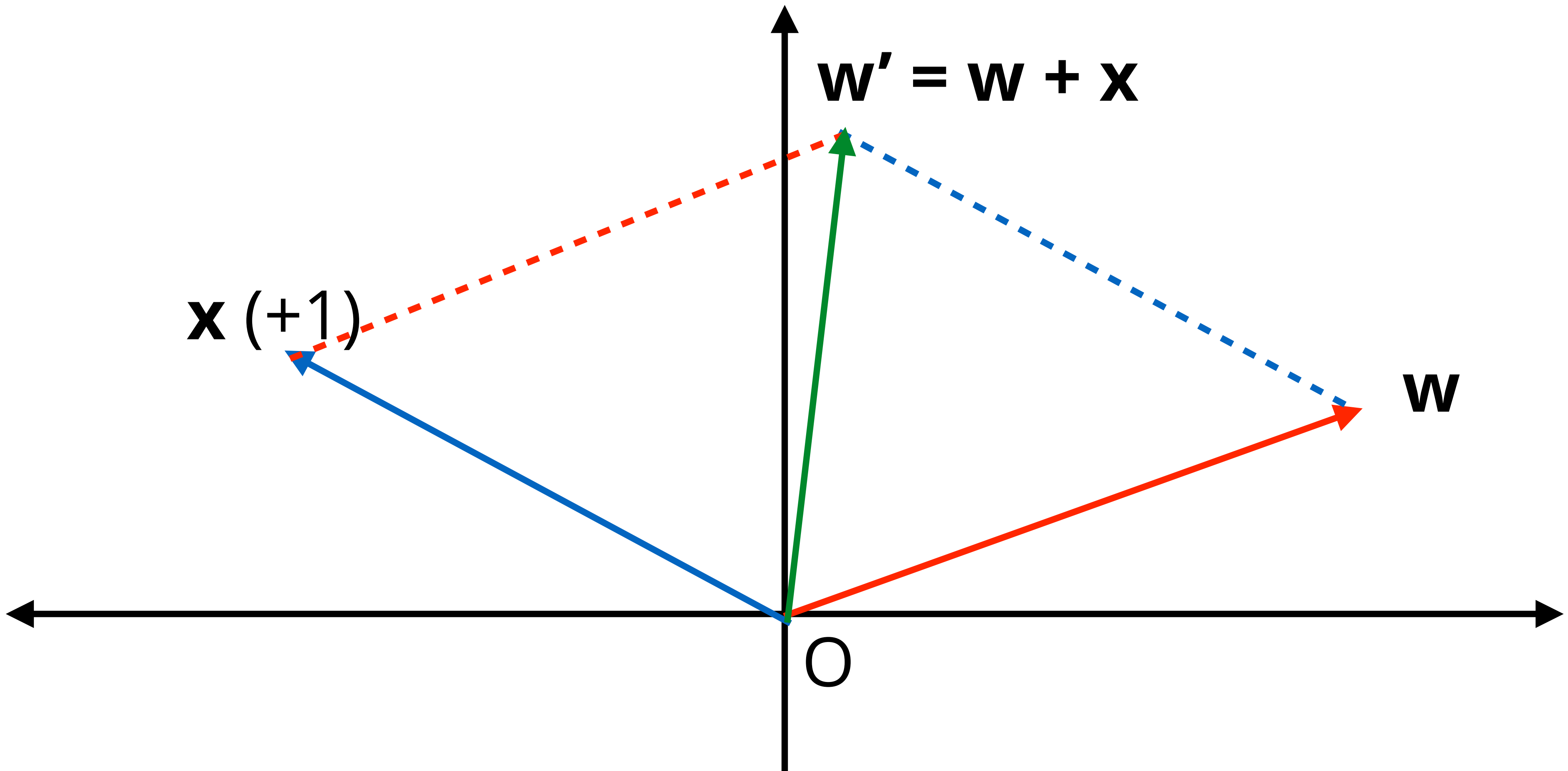
hyperplane defined by the weight vector is perpendicular to the weight vector

Geometric interpretation



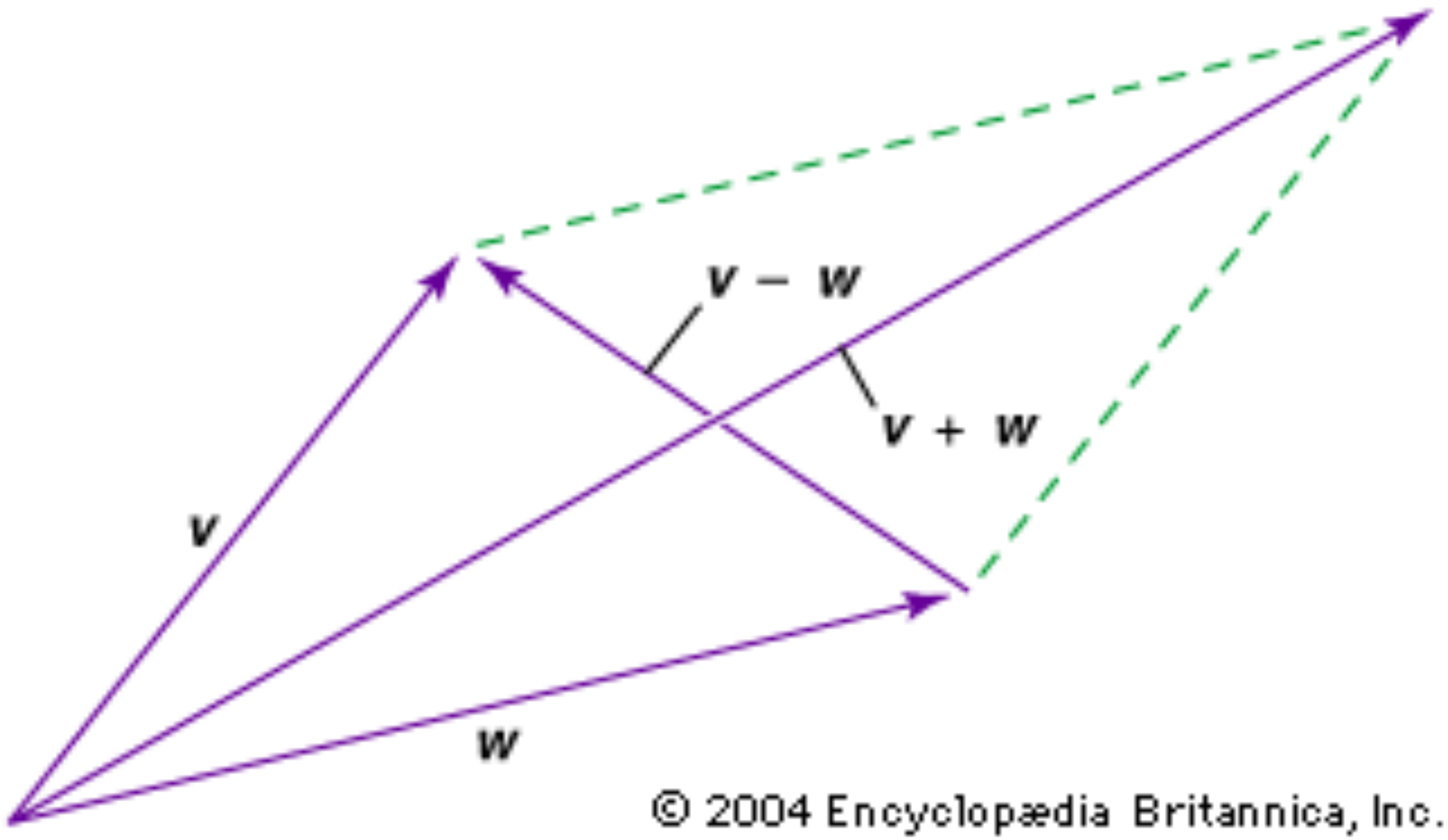
The angle between the current weight vector **w** and the positive instance **x** is greater than 90^o. Therefore, $\mathbf{w}^T\mathbf{x} < 0$, and this instance is going to get misclassified as negative.

Geometric interpretation



The new weight vector **w'** is the addition of **w** + **x** according to the perceptron update rule. It lies in between **x** and **w**. Notice that the angle between **w'** and **x** is less than 90^o. Therefore, **x** will be classified as positive by **w'**.

Vector algebra revision



Quiz 2

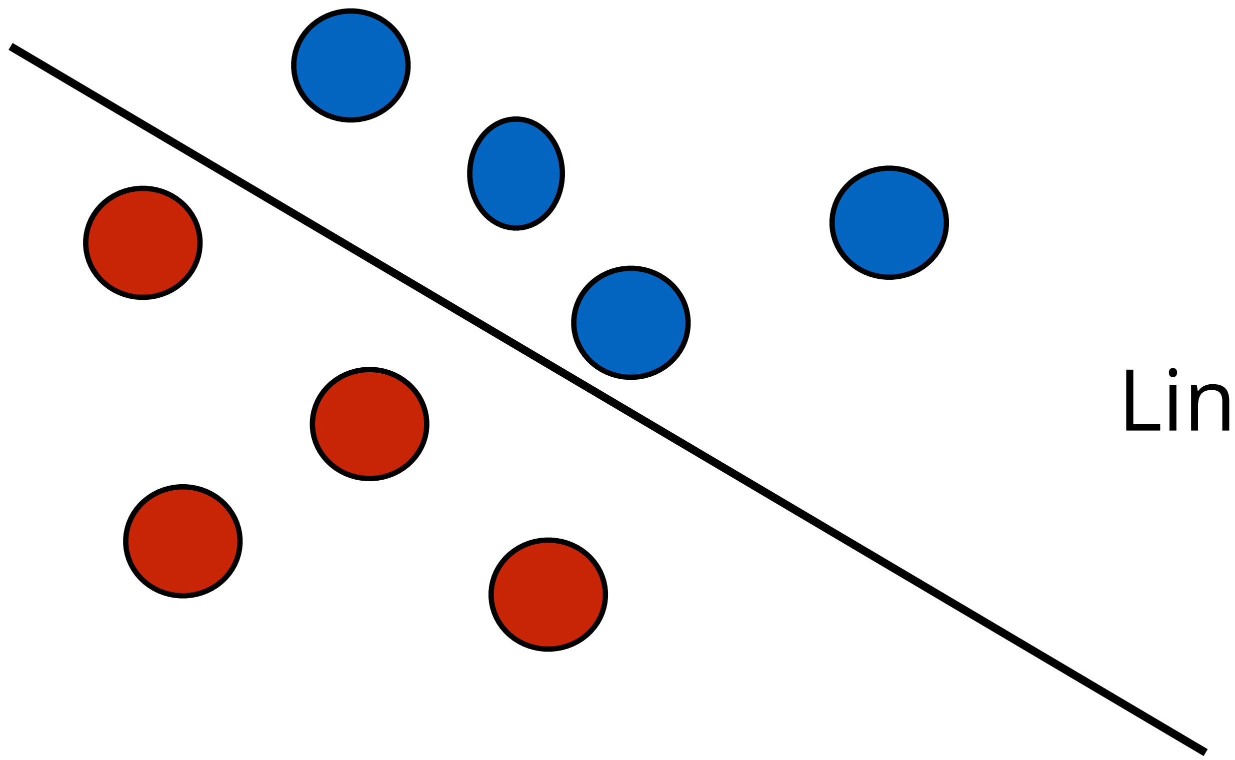
- Let **x** = $(1, 0)^T$ and **y** = $(1, 1)^T$. Compute **x**+**y** and **x**-**y** using the parallelogram approach described in the previous slide.

Quiz 3

- Provide a geometric interpretation for the update rule in Perceptron when a negative instance is mistaken to be positive.

Linear separability

- If a given set of positive and negative training instances can be separated into those two groups using a straight line (hyperplane), then we say that the dataset is *linearly separable*.



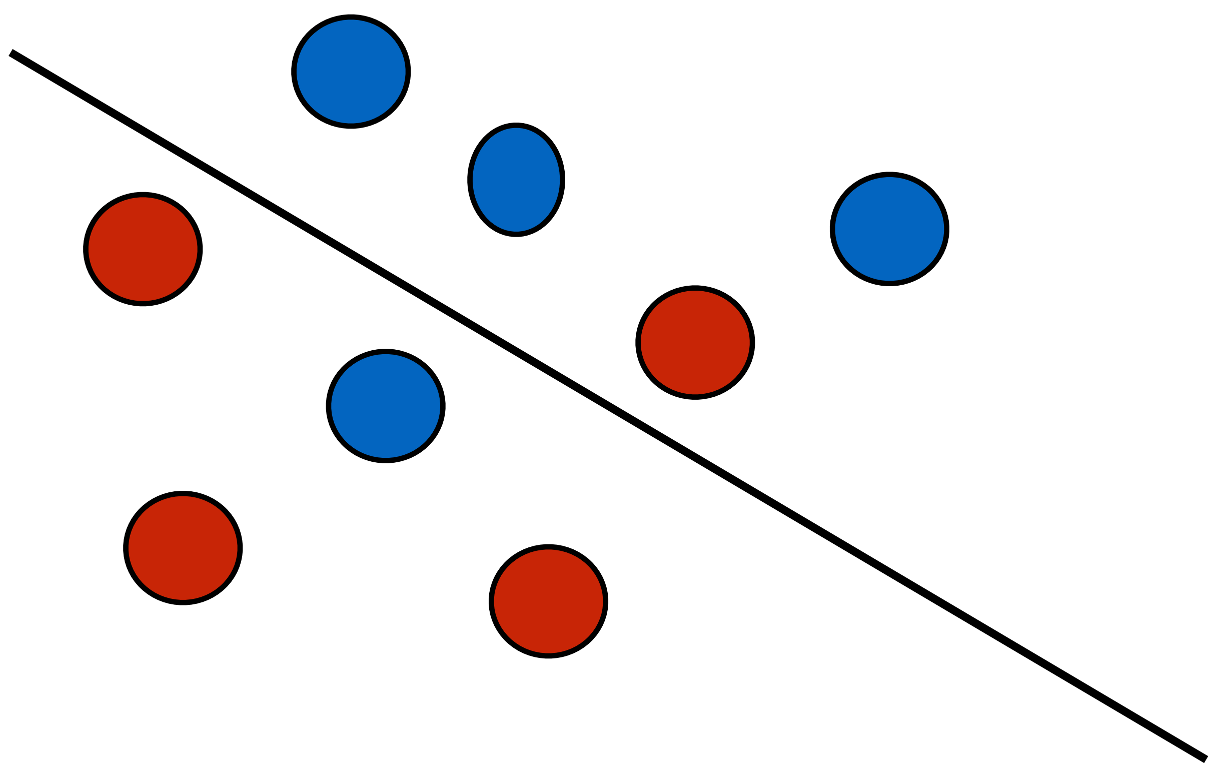
Linearly separable

Remarks

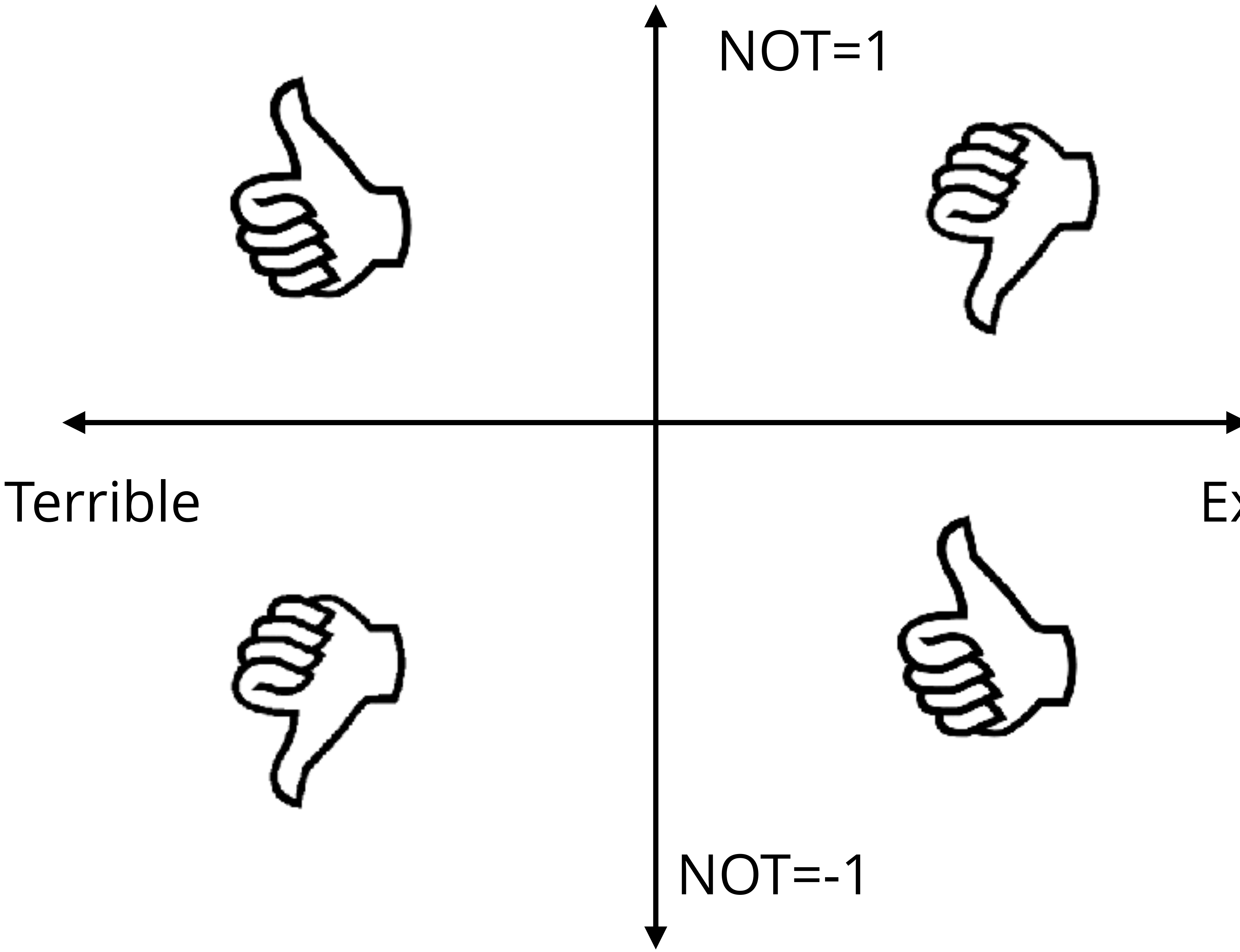
- When a dataset is linearly separable, there can exist more than one hyperplanes that separates the dataset into positive/negative groups.
- In other words, the hyperplane that linearly separates a linearly separable dataset might not be unique.
- However, (by definition) if a dataset is non-linearly separable, then there exist NO hyperplane that separates the dataset into positive/negative groups.

A non-linearly separable case

No matter how we draw straight lines, we cannot separate the red instances from the blue instances



Negation handling in Sentiment Classification



Mutually exclusive OR (XOR): XOR(A,B) = 1 only when one of the two inputs is 1.

Further Remarks

- When a dataset is linearly separable it can be proved that the perceptron will always find a separating hyperplane!
- The final weight vector returned by the Perceptron is more influenced by the final training instances it sees.
  - Take the average over all weight vectors during the training (averaged perceptron algorithm)