Unit -3

Distribution functions

observed — theoretical

↳ Assumptions.

Discrete Distributions

① Binomial Distribution

$X \rightarrow$ no. of successes in n trials.

$P[X = x] \equiv x$ no. of successes in n-trials.

$$\boxed{p_X(x) = P[X = x]}$$

Prob. of sucess $\rightarrow p$

Prob. of failure $\rightarrow 1-p$
$= q$

$$p + q = 1$$

$n \rightarrow$ no. of independent trials

$X \rightarrow$ no. of success in n trials

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$p \rightarrow$ prob. of success

$q \rightarrow$ prob. of failure

eg. coin toss 5 trials , $x = 0, 1, 2, 3, 4, 5$

$$P[X=0] = \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0$$

$P[X=1] =$ HTTTT, THTTT, TTHTT, TTTHT, TTTTH

$$= 5 \cdot \frac{1}{2} \left(\frac{1}{2}\right)^4$$

$P[X=2] =$ HTTTH, HTTHT, HTHTT, HHTTT ... $\rightarrow$ like this 10 cases

$$= 10 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3$$

$$\boxed{P[X=x] = {}^nC_x \, p^x \, q^{n-x}}$$

$X \rightarrow$ binomial variate

$\Rightarrow$ Successive prob. are the successive terms of the binomial expansion of $(q+p)^n$.

Recurrence formula

$$r \rightarrow r+1$$

$$P[X=r+1] = {}^nC_{r+1} \, p^{r+1} \, q^{n-r-1}$$

$$= \frac{n!}{(r+1)!\,(n-r-1)!}\, p^{r+1}\, q^{n-r-1}.$$

$$P[X=r] = \frac{n!}{r!\,(n-r)!}\, p^{r} q^{n-r}$$

$$\frac{n!\,(n-r)}{(r+1)!\,(n-r-1)!\,(n-r)}\, p^{r}\cdot p \cdot \frac{q^{n-r}}{q}$$

$$\boxed{P[X=r+1] = P[X=r]\cdot\frac{n-r}{r+1}\left(\frac{p}{q}\right)}$$

Mean & Variance of B.D.

$$P[X=x] = {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$\text{Mean} \cdot E(X) = \sum_{x=0}^{n} x \cdot P[X=x]$$

$$= \sum_{x=0}^{n} x \cdot {}^{n}C_{x}\cdot p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{(n-x)!\,x!}\, p^{x}\cdot q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{n!}{(x-1)!\,(n-x)!}\, p^{x} q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!\,(n-x)!}\, p^{x} q^{n-x}$$

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$$= \sum_{x=1}^{n} \frac{n\,(n-1)!}{(x-1)!\,(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} n \cdot {}^{n-1}C_{x-1}\, p^x q^{n-x}$$

$$= \sum_{x=1}^{n} n \cdot {}^{n-1}C_{x-1}\, \frac{p^{x-1}}{p^{-1}}\, q^{n-x}$$

$$= \sum_{x=1}^{n} np\; {}^{n-1}C_{x-1}\, p^{x-1} q^{n-x}$$

$$\downarrow$$

$$q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= n.p.\,\underbrace{(q+p)}_{1}{}^{n-1}$$

$$\boxed{\text{Mean} = E(X) = n.p.}$$

$\operatorname{var}(X) = E(X^2) - (E(X))^2$

$$\cancel{E(X^2) = \sum_{x=0}^{n} x^2 . P(X=x) = \sum_{x=0}^{n} x^2 . {}^nC_x\, p^x . q^{n-x}}$$

not correct way.

$$E(X^2) = E(X^2 - X + X)$$

$$= E(X^2 - X) + E(X)$$

$$= E[X(X-1)] + E(X)$$

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$$E[x(x-1)] = \sum_{x=0}^{n} x(x-1) \cdot {}^nC_x \, p^x \cdot q^{n-x}$$

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$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x(x-1)} \frac{n!}{\cancel{x(x-1)}(x-2)!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n(n-1)(n-2)!}{(x-2)!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=2}^{n} n(n-1) \, {}^{n-2}C_{x-2} \, p^x q^{n-x}$$

$$= n(n-1) \cdot p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2} \, p^{x-2} q^{(n-2)-(x-2)}$$

$$= p^2 n(n-1) (\cancel{q+p})^{n-2}$$

$$E(x^2) = E(x(x-1)) + E(x)$$

$$= p^2 n(n-1) + np$$

$$= \cancel{np(p(n-1)+1)} = n^2p^2 - np^2 + np$$

$$\text{var}(x) = \cancel{n^2p^2} - np^2 + np - \cancel{(np)^2}$$

$$= np(1-p)$$

$$= npq.$$

$$\boxed{\text{Variance }(x) = npq.}$$

$M_x(t) = E(e^{tx})$

$$= \sum_{x=0}^{n} e^{tx} \cdot {}^nC_x\, p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x}$$

$$= (q + pe^t)^n$$

moment generating fcn.

$$\boxed{M_x(t) = (q+pe^t)^n}$$

$$M'_x(t) = n(q+pe^t)^{n-1}\, pe^t$$

$t=0$, $M_x'(0) = n(q+p)^{n-1}\, p \cdot e^0$

$$\boxed{M_x'(0) = np = E(x)}$$

$$\boxed{M_x''(t)\Big|_{t=0} = E(x^2)}$$

Mgf abt the mean

$$M_x(t) = E(e^{t(x-np)})$$

$$= E(e^{tx} \cdot e^{-npt})$$

$$= e^{-tnp} \cdot E(e^{tx})$$

$$= e^{-tnp} \cdot M_x(t)$$

$$\boxed{\underset{\text{(abt mean)}}{M_x(t)} = e^{-tnp} \cdot (q+pe^t)^n}$$

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$$= \left[e^{-tp}(q+pe^{t})\right]^n$$

$$= \left[qe^{-tp} + pe^{t(1-p)}\right]^n$$

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$M_p(t)$ abt mean $= \left[qe^{-tp} + pe^{tq}\right]^n$

$$\left[e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots\right]$$

$$= \left\{ q\left[1 - tp + \frac{t^2p^2}{2} - \frac{t^3p^3}{3!} + \frac{t^4p^4}{4!} + \dots\right] + p\left[1 + tq + \frac{t^2q^2}{2!} + \frac{t^3q^3}{3!} + \frac{t^4q^4}{4!} \dots\right] \right\}^n$$

$\boxed{\begin{array}{l} qp^2 + pq^2 \\ qp(p+q) \\ = qp \end{array}}$

$$= \left\{ q - \cancel{tqp} + \frac{t^2qp^2}{2!} - \frac{t^3qp^3}{3!} + \frac{t^4p^4q}{4!} + \dots + p + \cancel{tpq} + \frac{t^2pq^2}{2!} + \frac{t^3pq^3}{3!} + \frac{t^4pq^4}{4!} + \dots \right\}^n$$

$$= \left[1 + \left[\frac{t^2}{2}pq + \frac{t^3}{3!}pq(q^2-p^2) + \frac{t^4}{4!}pq(p^3+q^3) + \dots\right]\right]^n$$

$$= 1 + {}^nC_1\left(\frac{t^2}{2}pq + \frac{t^3}{3!}pq(q^2-p^2) + \frac{t^4}{4!}pq(p^3+q^3) + \dots\right)$$
$$+ {}^nC_2\left(\frac{t^2}{2}pq + \frac{t^3}{3!}pq(q^2-p^2) + \frac{t^4}{4!}pq(p^3+q^3) + \dots\right)^2 + \dots$$

$\Rightarrow$ moment abt mean

↳ $\mu_1 = 0$

↳ $\mu_2 \cdot$ coff of $\frac{t^2}{2!} = npq$

↳ $\mu_3 =$ coff of $\frac{t^3}{3!} =$ –

x is binomially distributed with parameters n, p

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Q. $X \sim B(n, p)$

What is distri of $Y = n - X$

Ans. $M_X(t) = (q + pe^t)^n$

$M_Y(t) = E(e^{tY})$

$= E(e^{t(n-X)})$

$= e^{nt} \cdot E(e^{-tX})$

$= e^{nt} M_X(-t)$

$M_Y(t) = [e^{t}(q + pe^{-t})]^n$

$= (qe^t + p)^n = (p + qe^t)^n$

$X \sim B(n, p) \Rightarrow M_X(t) = (q + pe^t)^n$

$Y = n - X \sim B(n, q) \Rightarrow M_X(t) = (p + qe^t)^n$

Q During a war, one ship out of 9 sank on an avg in making a certain voyage. What is the prob. that exactly 3 out of 6 ships would arrive safely?

$X \rightarrow$ that the ship arrives safely.

p (arriving safely) $= \frac{8}{9}$

$q = \frac{1}{9}$

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$n=6$

$P[x=3] = {}^6C_3 \left(\frac{8}{9}\right)^3 \left(\frac{1}{9}\right)^3$

Q. one telephone no out of 15 called b/w 2 pm & 3 pm on weekdays is busy. What is the probability that if 6 randomly selected nos. are called (i) not more than 3 will be busy.
(ii) atleast 3 will be busy.

$X \rightarrow$ telephone line busy

$p = \frac{1}{15}$, $q = \frac{14}{15}$, $n = 6$

(i) $P[x \leq 3] = P[x=0] + P[x=1] + P[x=2] + P[x=3]$

$= {}^6C_0 \left(\frac{1}{15}\right)^0 \left(\frac{14}{15}\right)^6 + {}^6C_1 \left(\frac{1}{15}\right) \left(\frac{14}{15}\right)^5 + {}^6C_2 \left(\frac{1}{15}\right)^2 \left(\frac{14}{15}\right)^4$

$+ {}^6C_3 \left(\frac{1}{15}\right)^3 \left(\frac{14}{15}\right)^3$

(ii) $P[x \geq 3] = P[x=3] + P[x=4] + P[x=5] + P[x=6]$

(or)

$= 1 - P[x < 3]$

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(2) Poisson Distribution

in B.D.,

when $n \to \infty$, $p \to 0$

a new variable $\boxed{\lambda = np}$

this is the limiting case of binomial distri. (or)
Poisson's distribution

$$P[X=r] = {}^nC_r\, p^r q^{n-r} \qquad \left[\begin{array}{l} np = \lambda \\ p = \frac{\lambda}{n} \end{array}\right]$$

$$= \frac{n!}{r!\,(n-r)!} (1-p)^{n-r} p^r$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)\cancel{(n-r)!}}{r!\,\cancel{(n-r)!}}\, p^r (1-p)^{n-r}$$

$$= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \left[1-\frac{\lambda}{n}\right]^{n-r} \left(\frac{\lambda}{n}\right)^r$$

$$= \frac{\lambda^r}{r!} \frac{\overbrace{n(n-1)(n-2)\cdots(n-r+1)}^{r\text{ terms}}}{\underbrace{n^r}_{(n.n.n.\cdots r\text{ times})}} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!} \left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-r+1}{n}\right) \left\{\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}\right.$$

- $n \to \infty$

$$\frac{n-1}{n} = 1 - \frac{1}{n} = 1 - 0 = 1$$

$$\frac{n-2}{n} = 1 - \frac{2}{n} = 1 - 0 = 1$$

$$\frac{n-r+1}{n} = 1$$

Spiral