CSCI 115

Lab 2 Solutions to Textbook Exercises

2.1 **(a)** Not reflexive (for any female). Not symmetric (consider a brother and sister). Not antisymmetric (consider two brothers). Transitive (for any 3 brothers).

(b) Not reflexive. Not symmetric, and is antisymmetric. Not transitive (only goes one level).

(c) Not reflexive (for nearly all numbers). Symmetric since $a + b = b + a$, so not antisymmetric. Transitive, but vacuously so (there can be no distinct a, b, and c where aRb and bRc).

(d) Reflexive. Symmetric, so not antisymmetric. Transitive (but sort of vacuous).

(e) This relation consists of $\langle 1, 1\rangle$, $\langle 2, 1\rangle$, $\langle 2, 2\rangle$, $\langle 3, 1\rangle$, $\langle 3, 3\rangle$, $\langle 4, 1\rangle$, $\langle 4, 2\rangle$, and $\langle 4, 4\rangle$. Reflexive since any a, $a \bmod a = 0$. It is not symmetric ($\langle 4, 2\rangle$ is in, but $\langle 2, 4\rangle$ is not). It is antisymmetric. It is transitive (the only non-vacuous case is $\langle 4, 2\rangle$ and $\langle 2, 1\rangle$).

(f) Not reflexive. It is symmetric, antisymmetric, and transitive, since no element violates any of the rules (these are all if-then rules, and there is nothing in the "if" part).

(g) Trivially reflexive, since there are no elements to contradict the requirement. It is symmetric, antisymmetric, and transitive, since no element violates any of the rules (these are all if-then rules, and there is nothing in the "if" part).

2.2 In general, prove that something is an equivalence relation by proving that it is reflexive, symmetric, and transitive.

(a) This is an equivalence that effectively splits the integers into odd and even sets. It is reflexive ($x + x$ is even for any integer x), symmetric (since $x + y = y + x$) and transitive (since you are always adding two odd or even numbers for any satisfactory a, b, and c).

(b) This is not an equivalence. To begin with, it is not reflexive for any integer.

(c) This is an equivalence that divides the non-zero rational numbers into positive and negative. It is reflexive since $x\dot{x} > 0$. It is symmetric since $x\dot{y} = y\dot{x}$. It is transitive since *any* two members of a given equivalence class (that is, the positives and negatives) satisfy the relationship.

(d) This is not an equivalance relation since it is not symmetric. For example, $a = 1$ and $b = 2$.

(e) This is an eqivalance relation that divides the rationals based on their fractional values. It is reflexive since for all a, $a - a = 0$. It is symmetric since if $a - b = x$ then $b - a = -x$. It is transitive since any two rationals with the same fractional value will yield an integer.

(f) This is not an equivalance relation since it is not transitive. For example, $4 - 2 = 2$ and $2 - 0 = 2$, but $4 - 0 = 4$.

2.3 A relation is a partial ordering if it is antisymmetric and transitive.

(a) Not a partial ordering because it is not transitive.

(b) Is a partial ordering bacause it is antisymmetric (if a is an ancestor of b, then b cannot be an ancestor of a) and transitive (since the ancestor of an ancestor is an ancestor).

(c) Is a partial ordering bacause it is antisymmetric (if a is older than b, then b cannot be older than a) and transitive (since if a is older than b and b is older than c, a is older than c).

(d) Not a partial ordering, since it is not antisymmetric for any pair of sisters.

(e) Not a partial ordering because it is not antisymmetric ($\langle a, b\rangle$ and $\langle b, a\rangle$ are in the relation).

(f) This is a partial ordering. It is antisymmetric (no violations exist) and transitive (no violations exist).

2.4 A total ordering can be viewed as a permuation of the elements. Since there are $n!$ permuations of n elements, there must be $n!$ total orderings.

2.29 **(a)** **Proof**: By contradiction. Assume that the theorem is false. Then, each pigeonhole contains at most 1 pigeon. Since there are n holes, there is room for only n pigeons. This contradicts the fact that a total of $n+1$ pigeons are within the n holes. Thus, the theorem must be correct. $\square$

(b) **Proof**:

i. **Base case.** For one pigeon hole and two pigeons, there must be two pigeons in the hole.

ii. **Induction Hypothesis.** For n pigeons in $n-1$ holes, some hole must contain at least two pigeons.

iii. **Induction Step.** Consider the case where $n+1$ pigeons are in n holes. Eliminate one hole at random. If it contains one pigeon, eliminate it as well, and by the induction hypothesis some other hole must contain at least two pigeons. If it contains no pigeons, then again by the induction hypothesis some other hole must contain at least two pigeons (with an extra pigeon yet to be placed). If it contains more than one pigeon, then it fits the requirements of the theorem directly.

$\square$

2.31 **Base case**: $\mathbf{T}(n-1) = 1 = 1(1+1)/2$.
Induction hypothesis: $\mathbf{T}(n-1) = (n-1)(n)/2$.
Induction step:

$$\begin{aligned} \mathbf{T}(n) &= \mathbf{T}(n-1) + n \\ &= (n-1)(n)/2 + n \\ &= n(n+1)/2. \end{aligned}$$

Thus, the theorem is proved by mathematical induction.

2.32 If we expand the recurrence, we get

$$\mathbf{T}(n) = 2\mathbf{T}(n-1) + 1 = 2(2\mathbf{T}(n-2) + 1) + 1) = 4\mathbf{T}(n - 2 + 2 + 1.$$

Expanding again yields

$$\mathbf{T}(n) = 8\mathbf{T}(n-3) + 4 + 2 + 1.$$

From this, we can deduce a pattern and hypothesize that the recurrence is equivalent to

$$\mathbf{T}(n) = \sum_{i=0}^{n} -12^i = 2^n - 1.$$

To prove this formula is in fact the proper closed form solution, we use mathematical induction.

Base case: $\mathbf{T}(0) = 2^0 - 1 = 0$. $\mathbf{T}(1) = 2^1 - 1 = 1$.

Induction hypothesis: $\mathbf{T}(n-1) = 2^{n-1} - 1$.

Induction step:

$$\begin{aligned}
\mathbf{T}(n) &= 2\mathbf{T}(n-1) + 1 \\
&= 2(2^{n-1} - 1) + 1 \\
&= 2^n - 1.
\end{aligned}$$

Thus, as proved by mathematical induction, this formula is indeed the correct closed form solution for the recurrence.

2.33 Expanding the recurrence, we get

$$\begin{aligned}
\mathbf{T}(n) &= \mathbf{T}(n-1) + 3n + 1 \\
&= \mathbf{T}(n-2) + 3(n-1) + 1 + 3n + 1 \\
&= \mathbf{T}(n-3) + 3(n-2) + 1 + 3(n-1) + 1 + 3n + 1 \\
&= \mathbf{T}(n-i) + \sum_{j=0}^{i-1}(3(n-j)+1)
\end{aligned}$$

For $i = n$, we get

$$= T(0) + \sum_{j=0}^{n-1}(3(n-j)+1).$$

Changing the variable j and reversing the order gives

$$
\begin{aligned}
&= 1 + \sum_{i=1}^{n}(3i+1) \\
&= n + 1 + 3\sum_{i=1}^{n} i \\
&= \frac{3n^2+3n+2n+2}{2} = \frac{3n^2+5n+2}{2}
\end{aligned}
$$

for all $n \geq 0$.

Now, to prove this is correct.
Base case: $\mathbf{T}(1) = \frac{3(1)^2+5(1)+2}{2} = 1$.
Induction hypothesis: $\mathbf{T}(n-1) = \frac{3(n-1)^2+5(n-1)+2}{2}$.
Induction step:

$$
\begin{aligned}
T(n) &= T(n-1) + 3n + 1 \\
&= \frac{3(n-1)^2+5(n-1)+2}{2} + 3n + 1 \\
&= \frac{3n^2-6n+3+5n-5+2+6n+2}{2} \\
&= \frac{3n^2+5n+2}{2}.
\end{aligned}
$$

Thus, as proved by mathematical induction, this formula is indeed the correct closed form solution for the recurrence.

2.34 **(a)** The probability is 0.5 for each choice.

(b) The average number of "1" bits is $n/2$, since each position has 0.5 probability of being "1."

(c) The leftmost "1" will be the leftmost bit (call it position 0) with probability 0.5; in position 1 with probability 0.25, and so on. The number of positions we must examine is 1 in the case where the leftmost "1" is in position 0; 2 when it is in position 1, and so on. Thus, the expected cost is the value of the summation

$$\sum_{i=1}^{n} \frac{i}{2^i}.$$

The closed form for this summation is $2 - \frac{n+2}{2^n}$, or just less than two. Thus, we expect to visit on average just less than two positions. (Students at this point will probably not be able to solve this summation, and it is not given in the book.)