Algebraic Topology - Homework 3

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1. Take two copies of the torus $S^1 \times S^1$ and let X be the space obtained by identifying $S^1 \times \{pt\}$ in one torus with $S^1 \times \{pt\}$ in the other torus using the identity map on S^1. Compute $\pi_1(X, x_0)$.

   I claim that it is $F_2 \times \mathbb{Z}$. We use Van Kampen's Theorem to prove this result. Namely, let $A_1 = S^1 \times S^1$ and let $A_2 = S^1 \times S^1$ where $X = A_1 \cup_f A_2$ Now, using the given attaching map f one has that $A_1 \cap A_2 = S^1$. Note that

   $$\pi_1(X) = \pi_1(A_1) *_{\pi_1(A_1 \cap A_2)} \pi_1(A_2)$$

   Note that $\pi_1(A_1) = \mathbb{Z} \times \mathbb{Z} = \langle a, b | aba^{-1}b^{-1} \rangle$ and $\pi_1(A_2) = \langle b', c | b'cb'^{-1}c^{-1} \rangle$.

   Also, note that

   $$\pi_1(X) = \langle a, b, b', c | aba^{-1}b^{-1}, b'cb'^{-1}c^{-1}, bb'^{-1} \rangle$$

   So,

   $$\pi_1(X) = \langle a, b, c | aba^{-1}b^{-1}, bcb^{-1}c^{-1} \rangle = \langle a, b | \rangle \times \langle c | \rangle = F_2 \times \mathbb{Z}$$

   (where the relations $aba^{-1}b^{-1}, bcb^{-1}c^{-1}$ occur as a result of the structure of the direct product $\langle a, b | \rangle \times \langle c | \rangle$.

2. Given a map $f : X \to X$, the mapping torus T_f of f is the space obtained by identifying $S^1 \times \{pt\}$ in one torus with $S^1 \times \{pt\}$ in the other torus using the identity map on S^1. Compute $\pi_1(X, x_0)$.

   We see that $\pi_1(T_f) = (\pi_1(X, x_0) \times \mathbb{Z}) / \sim$, where $\sim$ is defined by $([\gamma], k) = (f_*([\gamma], k)$ for all $k \in \mathbb{Z}$ and all $[\gamma] \in \pi_1(X)$ Thus,

   $$\pi_1(T_f) = \langle a, b, c | aca^{-1}c^{-1}, aba^{-1}b^{-1}, a^{-1}f_*(a), b^{-1}f_*(b) \rangle.$$

3. Let X and Y be two non-empty spaces. If X is path connected and Y has two path components then show that the join $X * Y$ is simply connected. (You may assume the path connected components are open, though that assumption is not necessary).

   Let $[(x, y, t)]_R$ denotes the equivalence class of (x, y, t) via the two relations $(x_1, y, 0) \sim (x_2, y, 0)$ and $(x, y_1, 1) \sim (x, y_2, 1)$ for all $x, x_1, x_2 \in X$ and all $y, y_1, y_2 \in Y$. Also define setwise notation by $[U \times V \times S]_R = \{[(u, v, s)]_R : u \in U, v \in V, s \in S\})$. Let the two path connected components of Y be Y_1, Y_2. Now, let

   $$A_1 = [X \times Y_1 \times [0, 1]]_R$$

   and

   $$A_2 = [X \times Y_2 \times [0, 1]]_R.$$

   Note that $X * Y = A_1 \cup A_2$ and $A_1 \cap A_2 = [X \times \{y\} \times \{1\}]_R$ for arbitrary fixed $y \in Y$. Then, since $A_1 \cap A_2$ is non-empty and path connected, we may invoke the Seifert-Van Kampen theorem to note that $\pi_1(X * Y, [(x', y', 1)]_R) = \pi_1(A_1, [(x', y', 1)]_R) *_{\pi_1(A_1 \cap A_2, [(x', y', 1)]_R)} \pi_1(A_2, [(x', y', 1)]_R)$ where $[(x', y', 1)]_R \in X * Y$ is a fixed basepoint.

   Thus, to show $X * Y$ is simply connected, it suffices to show that $\pi_1(A_1, [(x', y', 1)]_R) = \pi_1(A_2, [(x', y', 1)]_R) = \{e\}$, which is how we proceed.

   For arbitrary loops $\gamma^1 : [0, 1] \to A_1$ and $\gamma^2 : [0, 1] \to A_2$ given by $\gamma^1(t) = [(\gamma_x^1(t), \gamma_y^1(t), q^1(t))]_R$ and $\gamma^2(t) = [(\gamma_x^2(t), \gamma_y^2(t), q^2(t))]_R$ we have the homotopy $h_1' : [0, 1] \times [0, 1] \to A_1$ defined by

   $$h_1'(t, r) = [(\gamma_x^1(t), \gamma_y^1(t), (1 - r)q^1(t))]_R,$$

   meaning that

   $$[(\gamma_x^1(t), \gamma_y^1(t), q^1(t))]_R \text{ is homotopic to } [(\gamma_x^1(t), \gamma_y^1(t), 0)].$$

   However since $[x', y, 0]_R = [x'', y, 0]_R$ for all $x'' \in X$, one obtains that $[(\gamma_x^1(t), \gamma_y^1(t), 0)]_R = [(x, \gamma_y^1(t), 0)]_R$ for any point $x \in X$ which implies that

   $$[(\gamma_x^1(t), \gamma_y^1(t), q^1(t))]_R \text{ is homotopic to } [(x', \gamma_y^1(t), 0)]_R. \tag{1}$$

Similarly the map $h_2' : [0,1] \times [0,1] \to A_2$ defined by

$$h_2'(t,r) = [(\gamma_x^2(t), \gamma_y^2(t), (1-r)q^2(t))]_R$$

is a homotopy, and thus, for any path $[(\gamma_x^2(t), \gamma_y^2(t), q^2(t))]_R \subseteq A_2$ we see that

$$[(\gamma_x^2(t), \gamma_y^2(t), q^2(t))]_R \text{ is homotopic to } [(x', \gamma_y^2(t), 0)]_R. \tag{2}$$

Also, note that we have homotopies $H_1 : [0,1] \times [0,1] \to A_1$ and $H_2 : [0,1] \times [0,1] \to A_2$ defined by

$$H_1([(\gamma_x^1(t), \gamma_y^1(t), q^1(t))]_R, s) = [(\gamma_x^1(t), \gamma_y^1(t), (1-s)q^1(t) + s)]_R$$

and also

$$H_2([(\gamma_x^2(t), \gamma_y^2(t), q^2(t))]_R, s) = [(\gamma_x^2(t), \gamma_y^2(t), (1-s)q^2(t) + s)]_R$$

meaning that in particular one has that $[x', \gamma_y^1(t), 0]_R$ is homotopic to $[x', \gamma_y^1(t), 1]_R$ but since $[x', y', 1]_R = [x', y'', 1]_R$ for all $y'' \in X$ that means $[x', \gamma_y^1(t), 1]_R = [x', y', 1]_R$. Thus, for any $x \in X$ and any $\gamma_y^1 : [0,1] \to Y_1$

$$[x', \gamma_y^1(t), 0]_R \text{ is homotopic to } [x', y', 1]_R. \tag{3}$$

Likewise we have in particular that for any $x \in X$ and any $\gamma_y^2 : [0,1] \to Y_2$

$$[x', \gamma_y^2(t), 1]_R \text{ is homotopic to } [x', y', 1]_R. \tag{4}$$

Now, note that (1) and (3) imply that all paths $\gamma^1 : [0,1] \to A_1$ are homotopic to the constant path $[(x', y', 1)]_R$. Likewise (2) and (4) imply that all paths $\gamma^2 : [0,1] \to A_2$ are homotopic to the constant path $[(x, y, 1)]_R$. Thus, each of A_1 and A_2 are simply connected. Now, recall that for any space U one has that U is simply connected if and only if U is path connected and $\pi_1(U, u) = \{e\}$ for all points $u \in U$. Thus, A_1, A_2 simply connected implies that $\pi_1(A_1, [(x', y', 1)]_R) = \pi_1(A_2, [(x', y', 1)]_R) = \{e\}$ and thus $\pi_1(X * Y, [(x', y', 1)]_R) = \{e\}$ which implies that $X * Y$ is simply connected since $X * Y$ is path connected.

4. If X is a space with a contractible universal cover then show that any map $S^n \to X$, $n \geq 2$ can be extended to a map $D^{n+1} \to X.$

Call our contractible universal cover $\tilde{X}$. First, note that $D^{n+1} \cong (S^n \times (0,1]) \cup \{0\}$ via the homeomorphism $\phi : D^{n+1} \cong (S^n \times (0,1])$.
Say $f : S^n \to X$ is our given map.
Note that S^n is path connected and locally path connected. Also, since for any $y_0 \in Y$ we have that $\pi_1(S^n, y_0) = \{e\}$ is trivial for any $y_0 \in S^n$ we certainly have that $f_*(\pi_1(S^n, y_0)) \subseteq p_*(\pi_1(\tilde{X}, f(\tilde{y_0}))$ since $f_*(\pi_1(S^n, y_0)) = \{e\}$ is the trivial group in $\pi_1(X, f(y_0))$. That means Prop 1.33 says we have a lift $\tilde{f} : S^n \to \tilde{X}$.
Say that our contraction map is $\tau : \tilde{X} \times [0,1] \to \{\tilde{x}\}$.
Define, $y_0 \in f^{-1}(p(\tilde{x}))$.
Now, define $\hat{F} : S^n \times [0,1] \to X$ by

$$\hat{F}(x,t) = \tau(\tilde{f}(x), 1-t).$$

If we define

$$F = \hat{F} \circ \phi$$

we are done.

5. Let X be a path connected, locally path connected space with $\pi_1(X, x_0)$ finite, show that any map $X \to S^1$ is nullhomotopic.

$\pi_1(X, x_0)$ finite means that $f_*(\pi_1(X, x_0)) \subseteq \pi_1(S^1, f(x_0))$ a finite subgroup of Z meaning just the trivial group $f_*(\pi_1(X, x_0)) = \{e\}$.

So, as in Prop 1.33, we have that

$$f_*(X, x_0) \subseteq p_*(\pi_1(Z, 0)) = \{f(x_0)\}$$

the constant loop. So, Prop 1.33 says we can lift f to $\tilde{f} : X \to \mathbb{R}$.
Now, we define the homotopy

$$H_t = g^{-1} \circ h_t \circ g : \mathbb{R} \to \mathbb{R}$$

by

$$gx \mapsto x - f(\tilde{x_0})$$

and

$$g_t : x \mapsto (1-t)x.$$

We note that this defines a homotopy

$$\tilde{H}_t = H_t \circ \tilde{f} : X \to \mathbb{R}.$$

Note that indeed we have that

$$\tilde{H}_1(X) = \{f(\tilde{x_0})\}.$$

Finally, we project this homotopy down to S^1 to get

$$\hat{H}_t : X \to S^1$$

defined by

$$\hat{H}_t = p \circ \tilde{H}_t.$$

Note that indeed

$$\hat{H}_1(X) = p(\{f(\tilde{x_0})\}) = \{f(x_0)\}$$

and thus any such arbitrary $f : X \to S^1$ is homotopic.