Algebraic Topology - Homework 5

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1. Show that if $\tilde{X}$ is an n fold covering space of the CW complex X, then $\chi(\tilde{X}) = n\chi(X)$. (You need to say something about the CW structure on $\tilde{X}$. For this problem you just need to describe the structure, but don't need to prove the structure is correct (though you should think through this).

   Note that if $\tilde{X}$ is an n-fold covering space of CW-complex X, then if l_k is the number of k cells in X then nl_k is the number of k cells in $\tilde{X}$. Thus, we have that

$$\chi(\tilde{X}) = \sum_{k=0}^{\infty}(-1)^k n l_k = n(\sum_{k=0}^{\infty}(-1)^k l_k) = n\chi(X).$$

2. Show that a surface of genus g can be a covering space of a surface of genus h only if $g = n(h-1)+1$ for some integer n. (Here all surfaces are assumed to be oriented).

   Example 2.36 in Hatcher proves that

$$H_n(M_r) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^{2r} & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 2 \\ 0 & \text{if } n \geq 3 \end{cases}$$

   Thus, we see that

$$rank(H_n(M_r)) = \begin{cases} 1 & \text{if } n = 0 \\ 2r & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ 0 & \text{if } n \geq 3 \end{cases}$$

   So, we have that

$$\chi(M_g) = \sum_{i=0}^{2}(-1)^i rank(H_i(X)) = 2 - 2g$$

   and that

$$\chi(M_h) = \sum_{i=0}^{2}(-1)^i rank(H_i(X)) = 2 - 2h.$$

   Now, if M_g is a covering space of M_h, then it is an n-fold covering space for some $n \geq 1$. By Exercise 5, we know that $\chi(M_g) = n\chi(M_h)$ or that $2 - 2g = n(2-2h)$ or that $1 - g = n(1-h)$ or that $g = 1 - n + nh = n(h-1)+1$ as claimed.

3. Given a sequence of abelian groups $G_0 = \mathbb{Z}, G_2, \ldots, G_n$ show that there is a CW complex X such that $H_i(X) \cong G_i$.

   - We let $X = \vee_{k=0}^{n} Y_k$ where Y_k is constructed so that $H_k(Y_k) \cong G_k$ and $H_j(Y_k) = 0$ for all $j \in \mathbb{N}$ with $j \neq k$. The wedge can be performed at any points $\{y_k \in Y_k : k \in [0:n]\}$.
   - Then, since for any $y_k \in Y_k$ we will show that (y_k, Y_k) is a good pair we may invoke Corollary 2.25 which says that $\tilde{H}_j(\vee_{k=0}^{n} Y_k) \cong \bigoplus_{k=0}^{n} \tilde{H}_j(Y_k)$ for all $j \in \mathbb{N}$. Then, since for any space Z and for $j \geq 1$ one has that $\tilde{H}_j(Z) \cong H_j(Z)$ we have that $H_j(\vee_{k=0}^{n} Y_k) \cong \bigoplus_{k=0}^{n} H_j(Y_k)$ for all $j \geq 1$.
   - By our future construction of Y_k as in bullet point 1, one has for each $j \geq 1$ that the direct sum $\bigoplus_{k=0}^{n} H_j(Y_k)$ has only one non-zero term (the jth term) and in particular is exactly $\bigoplus_{k=0}^{n} H_j(Y_k) \cong H_j(Y_j)$. Thus, for $j \geq 1$ one will have that $H_j(\vee_{k=0}^{n} Y_k) \cong G_j$, again by our chosen construction as in bullet point 1. We will postpone the proof that $H_0(\vee_{k=0}^{n} Y_k) \cong \mathbb{Z}$ until the end of our construction of Y_k.
   - We now explain how to construct Y_k. In particular, there are two cases to consider. Either G_k is finitely generated or it is not.
   - If G_k is finite generated, then $G_k \cong \mathbb{Z}^{a_k} \oplus \left(\bigoplus_{i=0}^{m_k} \mathbb{Z}/b_i^k\mathbb{Z}\right)$ where $a_k \in \mathbb{N}_{\geq 0}, m_k \in \mathbb{N}_{\geq -1}$ (with the convention that if $m_k = -1$ then the sum $\bigoplus_{i=0}^{m_k} \mathbb{Z}/b_i^k\mathbb{Z}$ is empty) and $b_i^k \in \mathbb{N}_{\geq 1}$. In this case, we let $Y_k = (\vee_{i=0}^{a_k} S^k) \vee (\vee_{i=0}^{m_k} D^{k+1} \cup_{f_i^k} S^k)$ where $f_i^k : \partial D^{k+1} \to S^k$ is an attaching map of degree b_i^k. To construct this map explicitly we note that ∂D^{k+1} is cannonically isomorphic to S^k via $\psi_k : \partial D^{k+1} \to S^k$.

– As in Example 2.31, one defines $q_i^k : S^k \to \vee_{l=1}^{b_i^k} S^k$ as $\phi_k \circ \pi_k$ where π_k is the quotient map $\pi_k : S^k \to S^k/\sim$ by $z \mapsto [z]$ for all $z \in S^k$ where $\sim$ is defined as follows and $\phi_k : S^k/\sim \to \vee_{i=1}^{k} S^k$ is the cannonical isomorphism. Namely, consider b_i^k pairwise disjoint non-empty open balls $B_1^k, B_2^k \ldots, B_{b_i^k}^k \subseteq S^k$ and define $z_1 \sim z_2$ if and only if $z_1, z_2 \in S^k \setminus (\cup_{j=1}^{b_i^k} B_j^k)$.

– Then, one defines $p_i^k : \vee_{l=1}^{b_i^k} S^k \to S^k$ as in Example 2.31 as the "identification of the summands to a single sphere".

– Finally one defines $f_i^k = p_i^k \circ q_i^k \circ \psi_k$ and one notes that f_i^k has degree b_i^k for all $i \in [0 : m_k]$.

- If G_k is not finitely generated then consider an infinite set of generators $\{g_a\}_{a\in\alpha}$ of G and note that $G_k \cong \langle g_a : a \in \alpha|\rangle^{ab}/N$ where we use additive notation for the abelianization meaning $G_k \cong \langle g_a : a \in \alpha|\rangle_+/N$ where the subscript $+$ just means the operation is addition. $N = \langle y_b : b \in \beta\rangle_+ = \langle \sum_{a\in\alpha} k_a^b x_a : b \in \beta\rangle_+$ (so really $y_b : b \in \beta$ is just the set of relations that hold in G). Now, we let X be the result of the following construction. Namely, we start with
$X^n := \vee_{a\in\alpha} S_a^n$ and construct X by attaching $n+1$ cells e_b^{n+1} using maps $f_b : S^n (\cong \partial e_b^{n+1}) \to X^n$ such that $deg(proj_a \circ f_b) = k_a^b$. Then the value of the cellular boundary map on the $n+1$ cell e_b^{n+1} will be just $\partial_{n+1}^{CW} = \sum_{k \text{ indexes the n cells}} d_{(q_a \circ (X^n \mapsto X^n/X^{n-1}) \circ (inclusion(\partial e_b^{n+1}) \to X^n)} x_a$ (where x_a correspond to the n cells) So, wee see that since $d_{n+1}(y_b) = \sum_a k_a^b x_a$ is just inclusion into F then we get that $H_n(X) \cong G$ and $H_k(X) = 0$ for $k \neq n$.

4. A cochain $\phi \in C^1(X; G)$ can be thought of as a function from paths in X to G. Show that if $\delta\phi = 0$ then
   (a) $\phi(\gamma * \eta) = \phi(\gamma) + \phi(\eta)$
   (b) if γ is homotopic to η rel endpoints, then $\phi(\gamma) = \phi(\eta)$
   (c) ϕ is a coboundary if and only if $\phi(f)$ only depends on the endpoints of f, for all f
   (d) Show that the discussion above gives a homomorphism $H^1(X; G) \to \text{Hom}(\pi_1(X); G)$ (Note: the universal coefficients theorem says that this map is an isomorphism if X is path connected).

Indeed an element $\phi \in C^1(X; G)$ is just $\phi \in \text{Hom}(C_1(X), G)$. Indeed, since $C_1(X) = \{$ paths in $X\}$. Note that $\delta = \partial_2^*$ meaning that $\delta(\phi)(D) = \phi(\partial(D))$ and thus, we must have that $\phi(\partial(D)) = 0$ for all $D \in C_2(X)$.
Throughout this problem we will need the following functions:

$$
\begin{aligned}
&p : \mathbb{R}^3 \to \mathbb{R}^2 && p(x, y, z) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&q : \mathbb{R}^2 \to \mathbb{R}^2 && q(x, y, z) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
&r : \mathbb{R}^2 \to \mathbb{R}^2 && r(x, y) = (x, 2y - |x|) \\
&s : \mathbb{R}^2 \to \mathbb{R}^2 && s(x, y) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
&f : \mathbb{R}^2 \to \mathbb{R}^2 && f(x, y) = (y - x)^+(0, 1) + (y - x)^-(1, 0)
\end{aligned}
$$

Note that all of the above are continuous, either because they are linear or because their component functions are continuous functions of x and y. (In particular note that $f(a) = a^+ := max(a, 0)$ and $f(a) = a^- := -min(a, 0)$ are continuous functions of $a \in \mathbb{R}$).

(a) Let
$$g : \{0 \times [0, 1]\} \cup \{[1, 0] \times 0\} \to X$$
be defined by
$$g(0, a) = \gamma(1 - a)$$
$$g(b, 0) = \eta(b).$$
Then, we have that
$$g(f(p(\Delta^2)))$$
is a 2-simplex in X.
Now, note that since $\delta = \partial^*$ we have that $\delta(\phi D) = \phi(\partial D)$ for all 2-simplices $D \subseteq X$. So, by hypothesis $\phi(\partial D) = 0$ for all 2-simplices meaning that in particular the statement holds for the 2-simplex $g(f(p(\Delta^2)))$.
So, finally we obtain $\partial(g(f(p(\Delta^2)))) = g(f(p([\hat{e_0}, e_1, e_2]))) - g(f(p([e_0, \hat{e_1}, e_2]))) + g(f(p([e_0, e_1, \hat{e_2}]))) = \eta - (\gamma * \eta) + \gamma$. Now, since $\phi(\partial D) = \phi(\eta) - \phi(\gamma * \eta) + \phi(\gamma) = 0$ we see that $\phi(\eta) + \phi(\gamma) = \phi(\gamma * \eta)$.

(b) • Now, note that $s(r(q(p(\Delta^n))) = [0,1] \times [0,1]$ and note that $\gamma \sim \eta$ relative to endpoints means that there exists a continuous function
$$F : \mathbb{R}^2 \to X$$
such that $F(0,t) = \gamma(t)$ and $F(1,t) = \eta(t)$ and so that $F(s,0)$ and $F(s,1)$ are constant for $s \in [0,1]$.

• Now, note that if one partitions γ into $\gamma = \alpha * \beta$ (in the obvious cannonical way) then we see that by our above construction

$$s(r(q(p([\hat{e_0}, e_1, e_2])))) = \alpha$$
$$s(r(q(p([e_0, \hat{e_1}, e_2])))) = \eta$$
$$s(r(q(p([e_0, e_1, \hat{e_2}])))) = \beta$$

and since
$$D := s(r(q(p(\Delta^n))))$$
is a 2-simplex in X satisfying $\partial D = \alpha - (-\eta) + \beta = 0$ and since by hypothesis $\phi(\partial D) = \phi(\alpha) - \phi(\eta) + \phi(\beta) = 0$ we see that $\phi(\eta) = \phi(\alpha) + \phi(\beta)$ and by part (a) we get that $\phi(\eta) = \phi(\alpha * \beta) = \phi(\gamma)$.

(c) By definition a coboundary ϕ is an element $\phi \in im(\delta_0)$. Then that implies that $\phi = \delta_0(h)$ for some $h \in \text{Hom}(C_0(X), G)$. Indeed, then $\phi(v) = (\delta_0 \circ h)(v) = h(\partial_1(v))$. Now, note that $\partial_1(v)$ depends only on the endpoints of v for all paths v proving necessity. Now, to prove sufficiency suppose that $\phi(v)$ depends only on the endpoints of v,

(d) We simply define $\psi : H^1(X, G) \to \text{Hom}(\pi_1(X), G)$ by
$$\psi([\phi])([\gamma]) = \phi(\gamma)$$
for all loops $\gamma \subseteq X$ and all $[\phi] \in H^1(X, G)$. Note that this map is well defined for two reasons. Firstly because part (b) insures that $\phi(a) = \phi(b)$ for all loops $a, b \in X$ with $[a]_{\pi_1(X)} = [b]_{\pi_1(X)}$ and secondly because $\psi([\phi]) = 0$ for all $\phi \in im(\delta_0)$. Namely note that by part (c) for such ϕ we have that $\phi(\gamma)$ depends only on the endpoints of γ and thus for a loop γ we have that $\phi(\gamma) = \phi(\{\gamma(0)\}) = \phi(\{\gamma(0)\} * (-\{\gamma(0)\})) = \phi(\{\gamma(0)\}) - \phi(\{\gamma(0)\}) = 0 \in G$. Finally, note that $\psi([\phi])$ is a homomorphism for all $[\phi] \in H^1(X, G)$ since $\psi([\phi])([a][b]) = \psi([\phi])([a * b]) = \phi(a * b) = \phi(a) + \phi(b)$.

5. In Hatcher's book, the cohomology of $\mathbb{R}P^n$ is computed. Use this computation (you do not have to reproduce it) to show that there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m; \mathbb{Z}/2\mathbb{Z}) \to H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ if $n > m$.

Note that as Hatcher correctly states (as a total ring) $H^*(RP^n, \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ meaning that each individual coordinate (corresponding to the kth powers of α for $0 \leq k \leq m$) are each $\mathbb{Z}_2$.

So, if there were such a non-trivial map, it would send $f : H^*(RP^m, \mathbb{Z}_2)(\cong \mathbb{Z}_2[\alpha]/(\alpha^{m+1})) \to H^*(RP^n, \mathbb{Z}_2)(\cong \mathbb{Z}_2[\beta]/(\beta^{n+1}))$ it would send $\alpha \mapsto^f \beta$. Now, since f is a homomorphism and $\alpha^{m+1} = 0$ we would need that $\beta^{m+1} = 0$ in $\mathbb{Z}_2[\beta]/(\beta^{n+1})$. However, that does not hold since $m + 1 < n + 1 = min(k \in \mathbb{Z} : \beta^k = 0)$.

6. Show that any map $S^4 \to S^2 \times S^2$ must induce the zero map on H^4.

First note that $H^4(S^2 \times S^2) \cong H_4(S^2 \times S^2) \cong H^2(S^2) \otimes_{\mathbb{Z}} H^2(S^2) \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}$. Take a generator $\alpha \in H^2(S^2)$ and then note that $\alpha \cup \alpha$ is a generator of $H^4(S^2 \times S^2)$ and Prop 3.10 in Hatcher says that $f^*(\alpha \cup \alpha) = f^*(\alpha) \cup f^*(\alpha) \in H^4(S^4)$. However, if $f^*(\alpha) \cup f^*(\alpha) \in H^4(S^4)$ that implies that $f^*(\alpha) \in H^2(S^4)$ but $H_2(S^4) = 0$ which implies that $f^*(\alpha) \cup f^*(\alpha) = 0$ and thus $f^* = 0$.