Cambridge University Mathematics Ph.D. Program Research Statement

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My primary goal for myself at this point in time is to become as adept and efficient a researcher as possible and to use the next few years in the absolute most optimal way to further my career objective of becoming a respected mathematics researcher at the top of my chosen field. It then seems obvious that the best way to further these ambitions would be to work under the tutelage of an eminent researcher in a respected graduate program under the important necessary condition that I am prepared to thrive in such a program. I believe that the Cambridge University has such a program and that I am prepared to thrive in the program.

Namely, after two years of Ph.D. study at the program ranked #2 in the U.S. for my former subfield of mathematics, I believe I have mastered the necessary introductory topics at a level such that I would be prepared to undertake those same courses at the Ph.D. level at the Cambridge University. I, of course, recognize that there is more to success in a graduate program than coursework, namely research. Luckily, I have two years of research experience with two different faculty members on two separate projects, one completed in a one-on-one setting with Dr. Craig Tovey and the other as part of a research team led by Dr. Xingxing Yu. As a result, I believe that I am equipped to enter the program and undertake research in a timely fashion with an appropriate faculty member or larger research group.

In order to explain what has shaped my research ambitions I will try to be as concrete as possible. Namely, upon deciding to change my research field to some mix of algebraic number theory and representation theory, I consulted with faculty who lent me a copy of the introductory text on modular forms by Schur and Diamond as well as the slightly more niche works in "Modular Forms and Fermat's Last Theorem". My interests developed upon close study of these books, and as so, I outline them as systematically as possible using these books as a guide.

What led me to apply to this specific program is a the presence of faculty members whose research interests align almost perfectly with some subset of mine, combined in interesting and novel ways, to the point that I can clearly forsee a good working relationship for several years. Namely, among others, I would be particularly interested in working with Dr. Wiles or anyone else interested in supervising me. I could belabor the point of which research interests of mine overlap with which of theirs, but these faculty obviously already know what their own research involves and as such only need to see where my interests lie. So, I refer all these faculty to a description of my specific interests below and I further urge them faculty members to view my online GitHub portfolio at www.github.com/beechamcaitlin .

- Modular Curves and Theory of Riemann Surfaces:
  - Modular forms give rise to fundamental domains, by which we can quotient the complex plane and compactify to get a Riemann surface. At the end of the day, all of these Riemann surfaces are all complex tori of some genus. The fact that these are complex tori is important since the fact that there are intuitively several bands along which to integrate gives an intuitive, visual understanding of the Jacobian. I obscured some details there, but the idea is that the Jacobian is formed as a quotient involving the dual space of a space of differential forms and a homology group. The dual space to a space of differential forms can crudely be understood as a space of integration operators, since one can integrate differential forms to get a constant, roughly speaking. Also, any algebraic topology student will recognize the idea of traversing (or integrating) an integral number of times around bands of an n-holed torus as looking at the homology of such a surface. Here the denominator in our useful isomorphic idea of the Jacobian is exactly this idea of integrating an integral number of times around each band and all that is left in the numerator, so to speak, is integration around each band a fractional amount. So one sees a clear bijection between the Riemann surface and its Jacobian since specifying a fractional traversal along each band specifies a location on the surface.
- Representation Theoretic Material:
  - Of course, the actual proof Weil gave for the Modularity Theorem was done via representation theory. Namely, he proved that $a_p(E) = a_p(f)$, for all but finitely many primes, by showing that the formulas for trace and determinant of two operators, one relating to the elliptic curve E and the other to a newform f, agree on a dense subset of the relevant Galois group, and thus are equal as functions since they are continuous. So, the importance is clear, but I will add that the reason the topic stands out to me is my strong interest in and decent knack for the field.
  - It is very interesting here to explore the representation theoretic ideas brought into play here. Namely, a representation theorist would take great care to consider the inherent structure of a group and why it might lend itself to certain almost canonical representations. In such a vein, there might be canonical properties associated with representations of a given group. Namely, when considering representations of the group of symmetries of the square, it seems that to an intelligent human capable of nuanced understanding (not a machine for detecting whether representations are reducible and putting them into blocks) that a canonical choice of dimension for such a representation would be two, and the associated matrices would simply be the collection of the appropriate rotation and reflection matrices. The canonical choice of field in this case also seems clear: namely the smallest field containing all the entries in the intuitively obvious rotation and reflection matrices. So, in this case, I suppose it would be the rationals. However,

there are some choices made in the development of representation theoretic parts of this field that perplex me. Namely, I can accept that the group of torsion points of an elliptic curve in $(\mathbb{Z}/N\mathbb{Z})^2$ and I certainly understand why there is a natural p-adic structure in the set of p^k torsion points for k in the integers, which is what the Tate module captures. What I don't understand is why the relevant two-dimensional Galois representations lie over the contrived field noted in the literature, and why they are a priori two dimensional. (I will add that it seems to me that use of such a contrived field is what guarantees an irreducible two-dimensional representation, but what I think I am missing is what such a choice captures about the structure of the curve and its additive group of torsion points and whether the structure is translated in meaningful ways by these Galois representations).

  – It is also my tentative understanding that these Galois groups act on the torsion points, but if so, I am unclear as to how. I think it may be that torsion points can be embedded as roots of unity via the Weil pairing and thus a member of the Galois group which permutes elements of C, including roots of unity, might permute these torsion points (which are members of $(\mathbb{Z}/N\mathbb{Z})^2$). However, then, it is unclear to me, what, structurally, such a member of the Galois group permuting the roots of unity, might translate to visually and algebraically in terms of permuting torsion points on the relevant elliptic curve. Again, let me emphasize that my strategy so far has been to take a bird's eye view of the subject, so I am allowing myself to develop tentative ideas that may be incorrect with the understanding that they will later be corrected should I choose definitively to specialize in this area.

- Algebraic Geometry material:
  – There are other statements in terms of algebraic curves and varieties, whose difference is exactly that curves have dimension one over the appropriate field while varieties may have dimension two or more. However, Schur and Diamond restrict attention to algebraic curves in the interest of appealing to students with little background in algebraic geometry. If I were admitted to a program with available courses in algebraic geometry, I would hope to deepen my understanding of varieties and also schemes, which unlike varieties account for the fact that some roots of polynomials are multiple roots, a fact which is unfortunately obscured in the theory of varieties since the associated vanishing ideals are always radical ideals.
  – I am also learning briefly about the relation between varieties and function fields. Of course, the fact that Riemann surfaces are isomorphic if and only if they have the same function field was instrumental in at least one proof given in the textbook. I believe that fact was used in the handling of the finitely many primes at which an elliptic curve may have bad reduction.
  – I was also happy to see concrete motivation in the field of modular forms for abstract practices in algebraic geometry. In particular, the practices of homogenizing and dehomogenizing ideals came up, due to the importance of working with non-singular curves, namely those for which the associated gradients are nowhere zero. I noticed that adding a homogenizing variable could in theory turn a singular curve in two variables into a non-singular variety in three variables.

- Algebraic Number Theoretic material:
  – The fields whose inherent machinery I enjoy the most are representation theory and algebraic number theory. Thus, I was quite happy to find that considering solutions over the algebraic closure of the rationals brings into serious play the theory of rings of integers and prime factorization of ideals in Dedekind domains, and in the section on Galois representations draws more serious connections by introducing the notions of inertia groups, decomposition groups, and Frobenius elements which seem to extend the notion of working modulo a prime to modulo a prime ideal. Of course, it is then due to the fact that such Frobenius elements are dense in relevant Galois groups, that one obtains the Modularity Theorem in the way that Weil actually proved it.

- Arithmetic Geometry Material:
  – The Eichler-Shimura relation could be viewed as the culmination of Schur and Diamond. However, a deeper understanding of pullbacks and pushforwards of Jacobians (whose composition may or may not multiply a stated quantity by a given integer of important consequence) seems needed to make sense of such a result.
  – I will also mention that I have a decently deep understanding about some topics, like Jacobians and the Picard group, but that there are still other holes in my knowledge that need to be filled. Namely, it would be nice to study results such as the Riemann-Roch and Mordell-Weil Theorems in depth in a formal class setting, complete with assignments and tests.

- Operator Theoretic material related to Hecke Operators:
  – The space of modular forms decomposes into the space of cusp forms and the quotient space of the space of modular forms by cusp forms, known as the Eisenstein space.
  – The space of cusp forms is broken into the mutually orthogonal (with respect to the Petersson inner product) spaces of oldforms and newforms, which are each stable under the action of the various Hecke operators. Newforms are of the most interest since the requisite modular forms f, found to satisfy the representation theoretic version of the Modularity Conjecture, end up being newforms that are able to be systematically constructed due to the fact that the space of newforms has a meaningful "canonical basis" of simultaneous eigenfunctions.