Math 6014 - Practice Problems 3

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Show that the facial cycles of a 3-connected plane graph are induced and non-separating.
We first show that the facial cycles of a 3-connected plane graph are induced and non-separating. Assume not. Assume that there exists some facial cycle $C = \{v_1, v_1v_2, v_2, \ldots, v_{n-1}, v_{n-1}v_n, v_n, v_nv_1, v_1\}$ which has some chord v_iv_j. By the definition of a face, this means that considering the corresponding planar embedding of G, there are no edges (or vertices) within the region bounded by C. So, recall that the cycle C as embedded in the plane is a simple closed curve, which divides the plane into two regions, one bounded and one unbounded. In particular this means that in such a planar embedding the chord in question must be embedded on the unbounded region (the outer part) and such a chord forms another face bounded by the cycle $\{v_i, v_iv_{i+1}, v_{i+1}, \ldots, v_{j-1}, v_{j-1}v_j, v_j, v_jv_i, v_i\}$ or forms a face bounded by the cycle $\{v_j, v_jv_{j+1}, v_{j+1}, \ldots, v_{i-1}, v_{i-1}v_i, v_i, v_iv_j, v_j\}$ (or both) (where in both cases the addition of indices is done modularly). Without loss of generality say that the chord forms a face bounded by the cycle $\{v_i, v_iv_{i+1}, v_{i+1}, \ldots, v_{j-1}, v_{j-1}v_j, v_j, v_jv_i, v_i\}$. Then, in particular this means that the vertices $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$ have no neighbors other than v_i and v_j (since the path $v_i, v_{i+1}, \ldots, v_{j-1}, v_j$ neighbors exactly two faces neither of which may have an edge embedded in the region they bound). Consequently, deleting the vertices v_i and v_j would isolate the subpath $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$ from the rest of G, resulting in a disconnected graph. We next show that the facial cycles of such a 3-connected graph are non-separating. We do so by induction on the number of edges in G. For our base case we consider the graph K_4. For K_4, each facial cycle is a triangle, whose deletion leaves a single vertex, which is a connected graph. Next, for our inductive step, we start with a 3-connected planar graph with m edges and n vertices. By Tutte's synthesis of 3-connected graphs we know that every 3-connected graph which is not K_4 admits an edge e such that the graph G' obtained by contracting e is also 3-connected. So, contract such an edge to get a multigraph G' which has fewer than m edges and is 3-connected. Now, by induction we know that every facial cycle of G' is non-separating. In particular, this shows that every face in G, which did not contain e was non-separating (because the image of the cycle bounding it in G' is unchanged) unless the cycle bounding it contained one of the endpoints u_1 of the edge $e = u_1v_1$. Then it is possible that removing the cycle C in G isolates v from the rest of G. In that case, the only neighbors of v_1 in G are among those vertices on the cycle C, called $u_1, u_2, \ldots, u_k$. Additionally, v_1 must have at least 3 neighbors in G. Otherwise deleting its neighbors in G isolates v_1 a contradiction. So, in particular, v_1 has u_1 as a neighbor as well as at least 2 other vertices u_r and u_s on C. Additionally, it must hold that u_1, u_r, u_s are consecutive vertices on the cycle. Otherwise, vertices in between these along the cycle have degree 2, a contradiction in a 3 connected graph with more than 3 vertices (well, according to the definition given in class, a 3-connected graph must have at least 4 vertices). Then, we note that there is some middle vertex, w_1, among the 3 consecutive vertices u_1, u_r, u_s along C. The adjacent faces to this middle vertex $w_1 \in \{u_1, u_r, u_s\}$ are exactly the face bounded by

C as well as the faces bounded by w_0, w_1, v_1 and w_2, w_1, v_1 (where $\{w_0, w_2\} = \{u_1, u_r, u_s\} \setminus \{w_1\}$). Since all facial cycles in G are induced, this means that the neighbors of w_1 in G are exactly w_0, w_2 and v_1. In particular, (under the continuing assumption that we are in the problem case where deleting C isolates v_1), deleting w_0 and w_2 results in a graph G'' where the only neighbor of w_1 in G'' is v_1 and vice versa, so that the path $\{w_1, v_1\}$ is an isolated component of G'', a contradiction to the fact that G is 3 connected (as long as there are vertices other than $\{w_0, w_1, w_2, v_1\}$, which there are since by assumption we are looking at a 3 connected graph with strictly more edges than K_4 which in particular forces at least 5 vertices). Thus, we see that deleting C in G cannot isolate v_1 and so now we have proven that for any cycle C in G, which does not contain this edge, deleting this cycle results in a connected graph. What if the facial cycle in question contains e? Well, firstly note that this can be the case for at most 2 facial cycles. Well we have two cases: either the length of such a cycle C in G which contains e is 3 or it is strictly greater than 3. If it is strictly greater than 3, then the image of this face still exists in the contracted graph G' and its image in G' is non-separating. The neighbors of the cycle C in G are exactly the neighbors of the image C' in G'. We note that C' non-separating in G' means that for all vertices $w, v \in G' \setminus C'$, there exists a path from w to v in $G' \setminus C'$, which is in particular a path using only vertices and edges in $G' \setminus C' = G \setminus C$. So, since $G' \setminus C' = G \setminus C$, such a path between any two vertices in $V(G') \setminus V(C') = V(G) \setminus V(C)$ still exists in G. Thus, C is non-separating in G. Now, what if the length of the facial cycle $C = \{z_1, z_2, z_3\}$ in G is 3 and the contracted edge $e = z_2z_3 \in C$ (endpoints z_2, z_3 chosen without loss of generality)? We note that in this case, once e is contracted to obtain G' the face bounded by C no longer exists in G'. Namely, the image of such a face is a single edge z_1z_*. However, since G' is 3 connected, we note that deleting the endpoints of this edge results in a graph that is still connected. It remains to show that deleting the three vertices z_1, z_2, z_3, G is still connected. Well, note that the vertices $V(G) \setminus C$ of G are exactly the vertices $V(G') \setminus \{z_1z_*\}$ of G'. The fact that $G' \setminus \{z_1, z_*\}$ is connected means that for any $r, s \in V(G') \setminus \{z_1, z_*\}$, there exists a path from r to s in $G' \setminus \{z_1z_*\}$. Such a path still exists in the uncontracted graph $G \setminus \{z_1, z_2, z_3\}$. Thus, $G \setminus \{z_1, z_2, z_3\}$ is connected and we are done.

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Let G be a 3-connected plane graph and let $x \in V(G)$. Show that the vertices and edges of G cofacial with x form a 'generalized' wheel.

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Let G be a connected plane graph and let g denote the girth of G. We wish to show $e(G) \leq g|V(G)|/(g-2)$. (I am assuming that we are only proving this statement for graphs with finite girth, namely those with cycles).

We proceed by induction on the number of edges. First, we handle the base case in which G has 3 edges and is in particular K_3. In this case, $g = 3$, $|V(G)| = e(G) = 3$, which implies that $\frac{g*|V(G)|}{g-2} = 9$ and clearly $e(G) = 3 \leq 9$. Now, we handle the inductive step. Assume that the statement holds for graphs with $k < m$ edges and now we wish to show that the statement holds for graphs with m edges. Take a graph G with m edges, n vertices and girth g. Now, there are two cases. Either (1) there exists a cycle, C, in G of length g and an edge $e \notin C$ such that e is not contained in any triangle or (2) there does not exist such a pair C,e. In case (1), we contract edge, e, to get the graph G' which has $n-1$ vertices and now has $m-1$ edges (since e was not contained in any triangles) and still has girth g. We know by the inductive hypothesis that $e(G') = e(G) - 1 \leq \frac{g(n-1)}{g-2}$ which implies that $e(G) \leq \frac{g(n-1)}{g-2} + 1 = \frac{g(n-1)}{g-2} + \frac{g-2}{g-2} = \frac{gn-g+g-2}{g-2} = \frac{gn-2}{g-2} \leq \frac{gn}{g-2}$, which is what we wanted to prove. Next, we handle case (2). In particular, we assume that for every cycle C of length G, any edge $e \notin C$ is contained in some triangle. In particular, this means that the girth of G is actually $g = 3$. Thus, this reduces to the case where every edge is contained in a triangle and in particular, it suffices to prove the statement for plane triangulations as these have the maximal number of edges for a fixed g and n. We recall that Eulers formula says that $n - m + \phi = 2$ (where f is the number of faces of our planar graph). So, in particular $\phi = 2 - n + m$. Also, $2m = \sum_{f \in F} deg(f) = 3\phi$ which implies that $\phi = \frac{2}{3}m = 2 - n + m$. So, in particular $m = 3n - 6$. Now, is it true that $e(G) = m = 3n - 6 \leq \frac{gn}{g-2} = \frac{3n}{3-2} = 3n$. Yes, thus, the inductive step is complete and we are done.

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Exercise 39 of Chapter 4