Math 6014 - Homework 4

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1. Let H be a graph with $\Delta(H) \leq 3$. Show that if G contains MH then G also contains TH.
   If G contains MH, that means that after some series of edge/vertex deletions and edge contractions we get H from G. Let ϕ be the contraction map taking G to G'. Note that in the contracted graph, every vertex corresponds to a connected component of the original graph. We consider 3 cases: either $w \in V(G')$ (where G' is our contracted graph isomorphic to H) has degree 1, degree 2 or degree 3. If w has degree 1 in G' (adjacent to edge e in G'), uncontract G' to get G and pick w' to be the endpoint of $\phi^{-1}(e)$ in G which belongs to $\phi^{-1}(w)$. Add $\phi^{-1}(e)$ and w' to T. If w has degree two, adjacent to edges $e_1 = vw$ and $e_2 = wu$, we know that when we uncontract G' to G, we get a connected component in place of w (namely $\phi^{-1}(w)$). Let w_1 be the endpoint of $\phi^{-1}(e_1)$ in G which lies $\phi^{-1}(w)$. Similarly let w_2 be the endpoint of $\phi^{-1}(e_2)$ in G, which lies in the connected component replacing w (namely $\phi^{-1}(w)$). The fact that this subgraph is connected implies that there exists a path between w_1 and w_2 within this connected subgraph. Add $\phi^{-1}(e_1)$, $\phi^{-1}(e_2)$ and this path (edges and vertices) to T. Otherwise if $deg_{G'}(w) = 3$, then there exist edges f_1, f_2, f_3 adjacent to w in G'. Uncontract G' to get G, and now we have some connected subgraph in place of w. Look at the endpoints v_1 of $\phi^{-1}(f_1)$ in this connected subgraph, v_2 of $\phi^{-1}(f_2)$ in this connected subgraph, and v_3 of $\phi^{-1}(f_3)$ in this connected subgraph. The fact that this subgraph is connected means that there exists a path within it from v_1 to v_2. Call this p. Then, again by connectedness there exists a path from v_3 to v_1 in this connected subgraph. Call it q. Let x^* be the first vertex along q which lies on p. Add the edges $\phi^{-1}(f_1)$, $\phi^{-1}(f_2)$, and $\phi^{-1}(f_3)$ to T. Also add p and $q[v_3, x^*]$ (vertices and edges) to T. Now, T is a topological H in G.

2. Let G be a 2-connected plane graph and C a cycle in G. Consider the set $S \subseteq V(G^*)$ consisting of all vertices of G^* inside C. Show that

   (a) $G^*[S]$ is connected, and
   We use the fact that any open region is "polygonally connected" (Professor Yu said we can use as a fact that any two points in a connected open set in $\mathbb{R}^2$ can be joined by a path consisting of finitely many line segments contained in that open set. He also explicitly said by email when I asked him that we don't need to be terribly precise with this question). Take $\pi(s_1), \pi(s_2)$ such that $s_1, s_2 \in S$ (where $\pi : V(G^*) \to \mathbb{R}^2$ is the embedding map). The set $\pi(C)$ is a simple closed curve in the plane which means it divides the plane into 2 connected open sets (by the Jordan Curve Theorem). In particular, the bounded region, A, contained within $\pi(C)$ is one of them. A is polygonally connected. So, there exists a polygonal arc, p, joining $\pi(s_1)$ and $\pi(s_2)$ contained in A. Now, it could happen that $\pi(V(G)) \cap p \neq \emptyset$. In this case, we modify p to avoid the vertices of G. Look at the embedding of a specific vertex, $v_i = ((v_i)_x, (v_i)_y)$, that intersects p. There exists some radius $\epsilon > 0$ such that $|C(v_i, \epsilon) \cap p| = 2$ and $B(v_i, \epsilon) \subseteq A$. Namely there exists some radius such that the circle of that radius around v_i only intersects p at two points. Call those two intersection points $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Modify p to get p_1 by removing the portion of p from a to b and instead joining a and b as follows. Let $s = (s_1, s_2) = a - b$. Now, let $r = (\frac{\sqrt{s_1^2+s_2^2}}{4\sqrt{1+\frac{s_1^2}{s_2^2}}} + (v_i)_x, \frac{-s_1\sqrt{s_1^2+s_2^2}}{4s_2\sqrt{1+\frac{s_1^2}{s_2^2}}} + (v_i)_y)$ if $s_2 \neq 0$. Otherwise let $r = ((v_i)_x, \frac{s_1}{4} + (v_i)_y)$. Then, to get p_1. Let $p_1 = p[s_1, a] \cup [a, r] \cup [r, b] \cup p[b, s_2]$ (where [a,r] is the line segment between a and r in the plane and not WLOG let a be the closest point along p to s_1 out of a and b). Repeatedly use the same construction iteratively to get p_i from p_{i-1} until $p_i \cap \pi(V(G)) = \emptyset$. Then, we have p_i a path consisting of polygonal arcs joining $\pi(s_1)$ and $\pi(s_2)$. Such a path will intersect the embeddings

of edges of G as well as faces of G. Construct a walk in G^* from our path p_i as follows. Let the path q start at s_1. Then trace along p_i. Whenever p_i intersects the embedding of an edge e of G, include the edge $\tau(e)$ in q (where $\tau : E(G) \to E(G^*)$ is our bijection between edges and dual edges). Also, whenever p_i enters a face, f, of G include $\phi(f)$ in q (where $\phi : F(G) \to V(G^*)$ is our bijection between faces of G and vertices of the dual). This process will terminate with q ending at s_2 which will gives us a walk from s_1 to s_2 in G^*. Any ab walk in a graph contains an ab path so we are done.

(b) if G has no vertex inside C then $G^*[S]$ is a tree.
Assume not. Assume $G^*[S]$ contains a cycle, D. Some embedding of this cycle, $\pi(D)$, will be contained within the region, A, bounded by $\pi(C)$ (since $\pi(S) \subseteq A$ and A "polygonally connected"). Then, by the Jordan Curve Theorem, the embedding of this cycle, $\pi(D)$, divides the plane into two regions. In particular, the inner region bounded by $\pi(D)$ is either itself a face of G^* or contains a face of G^*. There is a bijection $\phi : V(G) \to F(G^*)$ mapping vertices of G to faces of G^* which contain that vertex (as a plane embedding). This means the inverse map ϕ^{-1} is also a bijection with a similar correspondence. So, we get that there is a vertex of G contained within the region B bounded by $\pi(D)$ which is contained in A, the region bounded by $\pi(C)$, a contradiction.

3. Let G be a 2-connected plane graph and let $\sigma : V(G) \cup F(G) \to \mathbb{Z}$ such that $\sigma(x) = d(x) - 4$ for all $x \in V(G) \cup F(G)$. (if x is a face then $d(x)$ is the number of edges in its facial cycle.) Show that

(a) $\sum_{x \in V(G) \cup F(G)} \sigma(x) = -8$ and
First, by the Handshake Lemma we know that $\sum_{v \in V} deg(v) = 2E$ and similarly $\sum_{f \in F} deg(f) = 2E$ since every edge belongs to the boundary of exactly two faces in a 2-connected plane graph. Now, $\sum_{x \in V(G) \cup F(G)} \sigma(x) = \sum_{v \in V(G)} (deg(v) - 4) + \sum_{f \in F(G)} (deg(f) - 4) = \sum_{v \in V(G)} deg(v) + \sum_{f \in F(G)} deg(f) - 4V - 4F = 2E + 2E - 4V - 4F = 4(-V + E - F) = 4 * (-2) = -8$ by Euler's Formula.

(b) if $d(x) \geq 5$ for all $x \in V(G)$ then G contains K_4^- as a subgraph.
First, we note that G 2-connected implies that every face is bounded by a cycle. Now, assume G does not contain K_4^- as a subgraph. Then, we do as follows. (NOTE: When i use the term frontier I mean exclusively the edge sets of the frontiers, not including the vertices). Start with charges assigned according to $\sigma : V(G) \cup F(G) \to \mathbb{Q}$. Then, for every face f_i, consider neighboring triangle faces $\{f_j | \text{frontier}(f_i) \cap \text{frontier}(f_j) \neq \emptyset \text{ and } \deg(f_j) = 3\}$. Send $\frac{1}{3}a_j^i$ charge from f_i to f_j (where $a_j^i = |\text{frontier}(f_i) \cap \text{frontier}(f_j)|$)for all such i,j. Now, the charge on every face, $\tau_1(f_k) \geq 0$ unless $\deg(f_k) \in \{4, 5\}$. Why? The fact that there is no subgraph isomorphic to K_4^- implies that there do not exist f_i, f_j with $deg(f_i) = deg(f_j) = 3$ and $\text{frontier}(f_i) \cap \text{frontier}(f_j) \neq \emptyset$, so for all f_k with $deg(f_k) = 3$, $\tau_1(f_k) = \sigma(f_k) + 3 * \frac{1}{3} = deg(f_k) - 4 + 1 = 0$ (since intuitively all neighboring faces have degree strictly greater than 3). Also, if $\deg(f_k) \geq 6$ then there are at most 6 faces $\{f_l | deg(f_l) = 3 \text{ and } \text{frontier}(f_k) \cap \text{frontier}(f_l) \neq \emptyset\}$. So, we get that $\tau_1(f_k) \geq \sigma(f_k) - 6 * \frac{1}{3} \geq 6 - 4 - 2 = 0$. Now, consider the initial charge σ on the vertices. We note that $\sigma(v) \geq 1$ for all $v \in V(G)$ since $\sigma(v) = deg(v) - 4$ and $deg(v) \geq 5$ for all vertices in G. We then modify the charge on the vertices to get $_1$ as follows. For all vertices with $deg(v) \geq 6$ or vertices with $deg(v) = 5$ who do not fall into cases 1,2, or 3 (described below) give $\frac{1}{3}$ charge to each neighboring face which has degree 4 and give $\frac{2}{15}$ charge to each neighboring face which has degree 5 (and give no charge to other neighboring faces). Case 1: $deg(v) = 5$ and 4 neighboring faces are squares and the fifth is anything. Case 2: $deg(v) = 5$ and 3 neighboring faces of v are squares, 1 is a pentagon and the other is anything. Case 3: deg(v) = 5 and 2 neighboring faces are squares and 3 are pentagons. Consider the attached figures:

We see that in every case, we can get that $\tau_i(f) \geq 0$ for all f and $\tau_i(v) \geq 0$ for all v (where τ_i is the charge map after i iterations of altering charge). This would imply that $\sum_v \tau_i(v) + \sum_f \tau_i(f) \geq 0$. However, $\sum_v \tau_i(v) + \sum_f \tau_i(f) = \sum_v \sigma(v) + \sum_f \sigma(f) = -8$, a contradiction.