Math 6014 - Practice Problems 4

Caitlin Beecham

December 5, 2018

1

Prove, without using the Four Color Theorem, that every triangle-free planar graph is 4-colorable.

We recall that an edge set $E' \in E(G)$ is a cycle if and only if the corresponding edge set $\phi_E(E') \in E(G)$ is a minimal cut. Next, we note that G triangle free implies that the girth of G is at least 4, which implies that the minimum size of a cut set in G^* is at least 4. Thus, G^* is 4-edge-connected, and by a theorem proved in class, this implies that G^* has a 4 flow, so that in particular, the flow number of G^*, $\phi(G^*) \leq 4$. Correspondingly for G, $\chi(G) \leq 4$ and thus G is 4-colorable and we are done.

2

Prove, without using the Four Color Theorem, that every plane graph with a Hamiltonian cycle is 4-face-colorable.

We proceed by induction on the number of chords on such a Hamiltonian cycle with a slightly stronger statement. We show that every plane graph with a Hamiltonian cycle has a 4 face coloring in which the inner region bounded by the Hamiltonian cycle uses only colors 1 and 2 and the outer region uses only colors 3 and 4. For our base case, say there are 0 chords on such a Hamiltonian cycle. Then, the cycle splits the plane into 2 faces. Color the bounded one with color 1 and the unbounded one with color 3. Now, for the inductive step assume that the statement holds for plane graphs with Hamiltonian cycles with k-1 chords, we wish to show that the statement holds for plane graphs with Hamiltonian cycles with k chords. So, say we have such a plane graph with a Hamiltonian cycle along which there are k chords, some embedded within the region bounded by the Hamiltonian cycle, some embedded within the outer region of the Hamiltonian cycle. Say our Hamiltonian cycle is labeled $v_1, v_2, \ldots, v_n$. We take a chord v_iv_j and delete it. Now, by the inductive hypothesis, this new graph G' has a 4 face coloring in which the inner region uses only colors 1 and/or 2 and the outer region uses only colors 3 and/or 4. Now, we have two cases, either the chord v_iv_j was embedded with the Hamiltonian cycle in G or it was embedded outside. Say that v_iv_j was embedded within the cycle. Now, take the graph G' with its 4 face coloring and add the chord v_iv_j back in. What we have is no longer a proper face coloring so we modify it. We note that the cycle $K := v_i, v_iv_j, v_j, v_jv_{j+1}, v_{j+1}, v_{j+1}v_{j+2}, v_{j+2}, \ldots, v_{i-1}, v_{i-1}v_i, v_i$ (addition of indices is done modularly) is embedded as a closed curve in the plane and thus divides the plane into 2 regions. The inner region contains some subset of the faces contained within our Hamiltonian cycle in G, all colored with colors 1 and/or 2. We modify the colorings of these faces as follows let $c'(f) = 2$

if $c(f) = 1$ and $c'(f) = 1$ if $c(f) = 2$ for all faces f contained within K. Then, let $c'(f) = c(f)$ for all faces f of G not contained within K. I claim that this is a proper coloring of G. Why? Note that for all chords $e \neq v_i v_j$ of G, they bound exactly two faces (both contained within the Hamiltonian cycle). If such a chord e is not contained in K then the coloring has not been modified from what it was in G' and is therefore still "locally proper" (meaning the 2 adjacent faces get different colors). If such a chord e is contained in K, the coloring c' assigns different colors to its neighboring faces as well, since it did so via the coloring c and then c' simply swapped those 2 colors. If $e = v_i v_j$ then we note that one neighboring face still has color $x \in [2]$ where x was the color assigned to the face of G' in which $v_i v_j$ is embedded and the other has color $y \in \{[2] \setminus \{x\}\}$. Thus, the neighboring faces receive different colors. Additionally, all faces contained within the Hamiltonian cycle receive different colors from their neighboring faces embedded outside the cycle as the inner region uses colors 1 and 2 and the outer region uses colors 3 and 4. The argument for the chord $v_i v_j$ being contained outside the Hamiltonian cycle is analogous and I will not repeat it. Thus concludes the proof.

3

Prove that for every bipartite graph H there exists a $\Delta(H)$-regular bipartite graph G such that $H \subseteq G$. Use this to show that $\chi'(H) = \Delta(H)$.