Math 6014 - Homework 5

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Let k be a positive integer and let X, Y be a partition of the vertex set of the graph G such that $\chi(G[X]) \leq k$ and $\chi(G[Y]) \leq k$. Suppose $e(X,Y) \leq k-1$. Then $\chi(G) \leq k$.

By assumption, X and Y have k colorings. Color X and Y according to these colorings. Then construct a new bipartite graph as follows. The vertices of this bipartite graph are $X_1, X_2, ..., X_k$ and $Y_1, Y_2, ..., Y_k$ corresponding to the k color classes of each of X and Y. Let there be an edge between X_i and Y_j if and only if $X_i \cup Y_j$ form an independent set in G. The fact that there are at most k-1 edges between X and Y means that in our bipartite graph, there are at least $k^2 - (k-1)$ edges between our partition sets. Now, We check Hall's Condition to show that our bipartite graph has a perfect matching, which can be used to construct a k-coloring of G as follows. Say B has a perfect matching, in which X_i is matched to $Y_{f(i)}$ which means that there was an edge between X_i and $Y_{f(i)}$ in B, which means that $X_i \cup Y_{f(i)}$ form an independent set in G. Thus, for all such pairs, color the independent set $Y_{f(i)}$ with color i. We get a k coloring of G this way. So, assume that Hall's Condition is violated in B. Namely assume that there exists some $A \subseteq X$ such that $|N(A)| < |A|$. We note that in the graph $K_{k,k}$ for all $A \subseteq X$ we have that $|N(A)| = k$. Next, we note that B can be obtained from $K_{k,k}$ by removing at most $k-1$ edges. Now, the set A has at least $k - |A| + 1$ FEWER neighbors in B, than it does in $K_{k,k}$, which means that after the deletion of our set of up to $k-1$ edges, the number of neighbors of A has decreased by at least $k - |A| + 1$. To lose, one neighbor of A (starting from $K_{k,k}$), how many edges must be deleted? At least $|A|$. Thus, to lose $k - |A| + 1$ neighbors, at least $|A|(k - |A| + 1)$ edges must be deleted. We wish to show that $|A|(k - |A| + 1) \geq k$ (which will give us a contradiction). We note that the following equivalences hold for A with $|A| > 1$: $|A|(k - |A| + 1) \geq k$ if and only if $k(|A| - 1) \geq |A|^2 - |A|$ if and only if $k \geq \frac{|A|(|A|-1)}{|A|-1} = |A|$. Now, since $k \geq |A|$ always holds, we know that $|A|(k - |A| + 1) \geq k$ always holds whenever $|A| > 1$. So, in particular, if $|A| > 1$, we must remove at least $|A|(k - |A| + 1) \geq k > k-1$ edges from $K_{k,k}$ to get a B in which Hall's Condition is violated, which is a contradiction as our B was formed from $K_{k,k}$ by removing at most $k-1$ edges. Finally, we must consider the case where $|A| = 1$. Then, we show that $|N(A)| \geq 1$. B is formed from $K_{k,k}$ be removing at most k-1 edges. In $K_{k,k}$, $A = \{X_i\}$ had exactly k neighbors via all edges adjacent to X_i. After the removal of $k-1$ edges there still exists at least one edge from X_i to some vertex in Y. Thus, it has at least one neighbor and we see that in each case Hall's Condition is satisfied and thus we get a perfect matching of B, which as argued earlier, gives us a k coloring of G.

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Prove that if G is a k-color critical graph for some positive integer k then the graph obtained from G by applying Mycielski's construction is $(k+1)$-color critical.
Lemma: A k-color critical graph can always be colored in such a way that there is only one vertex of some specified color.
Proof: Pick a vertex. Remove it. Now what remains can be colored using k-1 colors, those other than the specified color.
We first show that the graph obtained my Mycielski's construction is k+1 colorable. We know that G is k-colorable. Now, color the vertices of the copy of G, $\{u_1, ..., u_n\}$, as follows. The coloring is $C(u_i) = C(v_i)$ where u_i is the vertex in the copy of G corresponding to v_i. Then, color the vertex, w, which is adjacent to all u_i with color $k+1$. Why is this a proper coloring? Assume not. Then some vertex v_i is colored the same as some adjacent vertex u_j. Why are v_i and u_j adjacent? Because the edge v_iv_j is present in G. This provides a contradiction as this would imply that v_i and v_j obtained the same color in the original k-coloring of G.
Next, we need to show that G' is color critical. It suffices to show that the removal of any edge of G' results in a graph that is k-colorable, because the removal of any vertex will result in the removal of some edge. We have three cases: (1) the edge removed is of the form u_iw for some $u_i \in G'$, (2) the edge removed is of the form v_iv_j, or (3) the edge removed is of the form v_iu_j. In case (1), we use the previous lemma. Namely, say that the edge to be removed is u_iw. Then, by the lemma, we can obtain a k-coloring of G, such that the only vertices of G' which receive color k are v_i and u_i (we apply the lemma to G, and then extend the coloring to G' as usual). Then, after removing the edge u_iw, we wish to recolor w with color k. We may do so, as no neighbors (current neighbors) of w have color k. Thus, we obtain a k-coloring of $G' \setminus u_iw$. In case (2), since G is k-color critical, we now know that $G \setminus v_iv_j$ has a k-1 coloring, C'. We can extend this to a k-1 coloring of $G' \setminus (\{w\} \cup v_iv_j)$ (though we may need to further modify it later if this does not work). We color $C'(u_l) = C'(v_l)$ and $C'(w) = k$. Now, since v_ru_s an edge in $G' \setminus (\{w\} \cup v_iv_j)$ implies that either v_rv_s is an edge in $G' \setminus (\{w\} \cup v_iv_j)$ or $v_rv_s = v_iv_j$. We wish to show that the endpoints of any edge in $G' \setminus (\{w\} \cup v_iv_j)$ receive different colors via C' or modify our coloring if this does not hold. Obviously, the endpoints of any edge v_rv_s receive different colors as C' was extended from a proper coloring of $G \setminus v_iv_j$. We wish to show the endpoints of an edge of the form v_ru_s receive different colors. Assume not, then $C'(v_r) = C'(u_s) = C'(v_s)$ which implies that v_r and v_s are not adjacent in $G' \setminus (\{w\} \cup v_iv_j)$, which is a contradiction (by the way we've constructed G') unless $v_rv_s = v_iv_j$. So, either $r = i$ and $s = j$ or $r = j$ and $s = i$. The fact that $C'(v_r) = C'(u_s) = C'(v_s)$ implies that $C'(v_s) = C'(v_s) = C'(v_i) = C'(v_j)$. We modify the coloring C' of G' to get a coloring C'' of G' as follows. The coloring $C''(v) = C'(v)$ for all $v \in \{v_1, ..., v_n\}$. Then, $C''(u) = C'(u)$ for all $u \in \{u_1, ..., u_n\} \setminus C'^{-1}(C'(v_i))$. Then, $C''(x) = k$ for all $x \in C'^{-1}(C'(v_i)) \cap \{u_1, ..., u_n\}$. Then, $C''(w) = C'(v_i)$. This is a proper k-coloring of $G' \setminus v_iv_j$. In case (3), say the edge to be removed is of the form u_iv_j. Then, we relabel the vertices of G' as follows. We set $w' = w$, $v'_l = v_l$ and $u'_l = u_l$ for all $l \neq i$. Then, set $v'_i = u_i$ and $u'_i = v_i$. This relabling is valid as $N(v_i) = N(u_1) \setminus w$. Then, we have basically reduced this new case to case 2, in which an edge of the form v_iv_j has been removed. However, the only difference is that now there is no edge between u'_i and w' in this new labeling, there is an edge v'_iw', and there is an edge $u'_iu'_j$. Regardless, the graph induced by the v'_i's is isomorphic to $G \setminus v_iv_j$, and thus has a k-1 coloring, K. Extend this coloring to color the u'_i's by setting $K(u'_i) = K(v'_i)$. This may not be a proper coloring but as in case (2) we will adjust it. As argued in case two, either all endpoints of edges of the form v_ru_s receive different colors under K' or $K'(v'_r) = K'(u'_s) = K'(v'_s)$, which is a contradiction to the fact that we had a *proper* k-1 coloring of the graph induced by the v'_i's unless $\{r, s\} = \{i, j\}$, in which case we

get that $K'(v_i') = K'(v_j')$. In particular, this means that $r = j$ and $s = i$ as the edge $v_i'u_j'$ does not exist in this graph (that would imply that u_iu_j was an edge in G' which it never was). So the only edge that may have endpoints of the same color is $v_j'u_i'$. We adjust K' to get K'' as follows. We set $K''(v_l') = K'(v_l')$ for all v_l', then set $K''(u_l') = K'(u_l')$ for all $u_l \in \{u_1', ..., u_n'\} \setminus \{u_i'\}$. Then, finally set $K''(u_i') = k$. Also, set $K''(w') = k$. This will not conflict with any vertex u_l' as the only vertex in $\{u_l'\}$ of color k is u_i', which is not adjacent to w'. Additionally, the only neighbor of w' which is not contained in $\{u_l' | l \in [n]\}$ is v_i', which is colored some color $c \in [k-1]$. Thus, we get a proper k-coloring of our entire graph in this way.

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Let G be a graph and k be a positive integer. Show that $\chi(G) \leq k$ if, and only if, G has an acyclic orientation with no directed path of length k.

We first show that if G has an acyclic orientation with no directed path of length k, then G has a proper k coloring. We proceed by first identifying a longest directed path. Let x be the start vertex of this path p. We perform a modified BFS to color the vertices of G. Note that in our modified version of BFS many vertices may be colored some color and then later recolored with a different color. We do as follows: Color x with color 1. Then, look at all directed neighbors of x (namely $N_2 := \{y | xy \text{ is a directed edge of } G\}$). Color these neighbors with color 2. Then look at all directed neighbors of the vertices in N_2, namely the set $N_3 := \{z | yz \text{ is a directed edge of G for some } y \in N_2\}$. Note, that the intersection $N_3 \cap N_2$ need NOT be empty. Color the vertices of N_3 with color 3. Continue in this manner. Once we have colored up to sets $N_2, ...N_i$ in this way, look at all vertices $N_{i+1} := \{u | wu \text{ is a directed edge for some } w \in N_i\}$ and color them with color i+1. Note that, none of the pairwise intersections $N_{i+1} \cap N_k$ for $k \in [i]$ need be empty. Continue until the current set in question $N_l = \emptyset$. We wish to prove that the coloring obtained by this procedure is a proper coloring and uses at most k colors. Assume not. Assume that two adjacent vertices, u and v, are colored with the same color via this procedure. Without loss of generality the edge between them is oriented from u to v. Now, the fact that u and v were colored with the same color, i, means that $u, v \in N_i$. Why? Well, as a lemma, we prove that this algorithm terminates after k iterations, which means that each color j corresponds to exactly one set N_j. Why does this algorithm terminate after k iterations? If not, one would have a sequence of vertices $x_1, x_2, x_3, ..., x_k, x_{k+1}, ..., x_L$ such that these vertices form a directed path. However, such a path would have length strictly greater than k-1, which is a contradiction. This proves our lemma. Now, returning to the main argument, $u, v \in N_i$. However, $uv \in E$ implies that $v \in N_{i+1}$, which implies that v gets recolored $i + 1$ at the step in which those vertices in N_{i+1} are colored. (Note that it may be further recolored later, however never again with color i, by the above lemma). Thus, a contradiction. No two adjacent vertices have the same color, and once again by our lemma, at most k colors are used.

Next, we show that if G has a proper k-coloring, then G has an acyclic orientation with no directed path of length k. Namely, take a specific k coloring of G. Now, for each edge, orient it from the vertex of lower color to the vertex of higher color. Now, the resulting orientation is acyclic as it can never hold that $c_{i_1} < c_{i_2} < ... < c_{i_k} < c_{i_1}$. Also, if there were to be a path of length k or greater, namely $p := x_1, x_2, ..., x_k, x_{k+1}, ...$ this would imply that $c(x_1) < c(x_2) < ... < c(x_k) < c(x_{k+1}) < ...$, which is a contradiction as this implies that $c(x_2) \in \{2, ..., k\}$, $c(x_3) \in \{3, ..., k\}$ and $c(x_i) \in \{i, ..., k\}$, which is a contradiction for $i \geq k + 1$.