Advanced Graph Theory: Chromatic Binding Functions and Vizing Pairs

Lecture notes scribed by Caitlin Beecham

April 12, 2019

1 Motivating Examples

Recall that a line graph $L(G)$ of a graph G has vertex set $V(L(G)) = E(G)$ with edges defined by

$$(ab \in E(L(G))) \text{ if and only if } (\exists v \in V(G) \text{ such that } a, b \in \delta(v))$$

where $a, b \in V(L(G))$ correspond to edges $a, b \in E(G)$.

Now, clearly $\chi(L(G)) = \chi'(G)$ since the vertices in $L(G)$ correspond to the edge set of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent.

Now, one recalls that for any simple graph G one has that $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$. Now, note that

$$\omega(L(G)) = \Delta(G)$$

since every clique of $L(G)$ corresponds to the set of edges incident to a vertex in G.

Now, combining this with our earlier observation one has that

$$\chi(L(G)) = \chi'(G) \leq \Delta(G) + 1 = \omega(L(G)) + 1.$$

So, in particular one has that any line graph $L(G)$ satisfies $\chi(L(G)) \leq \omega(L(G)) + 1$ which is known as the Vizing Bound. Alternately stated, the class of line graphs is $\chi-$bounded with binding function $f(x) = x + 1$.

This class of graphs motivates the question, what other classes of graphs satisfy the Vizing Bound?

1.1 Characterization of Line Graphs

Line graphs, which are the set of graphs which arise as the line graph of some other graph, have been completely characterized as the set of graphs not containing an induced copy of any graph in a list of nine small graphs, which are listed at the following link. `https://www.math.wvu.edu/~hjlai/Pdf/Hong-Jian_Lai_Pdf/062-JCTB_2001.pdf`

Now, denote this set of nine graphs B. Then, the class of line graphs is exactly

$$\mathcal{L} = Forb(B),$$

which means that $Forb(B)$ satisfies the Vizing Bound. However, one asks does some superclass of $\mathcal{L}$ also satisfy the Vizing Bound? In particular, one could construct a super class $\mathcal{L}' \supsetneq \mathcal{L}$ by forbidding only some subset $B' \subsetneq B$. In particular, Kirstead showed that this is possible.

Theorem 1.1 (Kirstead 1984) *If $G \in Forb(K_{1,3}, K_5^-)$ then $X(G) \leq \omega(G) + 1$.*
(Note that $\{K_{1,3}, K_5^-\} \subsetneq B$).

1.2 Definitions

Definition 1.2 *A "Vizing Pair" is a pair of graphs (A, B) such that the family $Forb(A, B)$ satisfies the Vizing Bound.*

Definition 1.3 *A "Good Vizing Pair" is a Vizing pair (A, B) such that neither $Forb(A)$ nor $Forb(B)$ satisfy the Vizing Bound.*

Definition 1.4 *A Good Vizing Pair (A, B) is called "saturated" if for every good Vizing Pair (A', B') such that $A \subseteq A'$ and $B \subseteq B'$, one has that $A \cong A'$ and $B \cong B'$.*

Note: Essentially by considering only saturated pairs one is considering pairs which are maximal relative to subset containment, which is useful since $Forb(A, B) \subseteq Forb(A', B')$ whenever $A \subseteq A$ and $B \subseteq B'$. This containment means that showing $Forb(A', B')$ is $\chi-$bounded with binding function f also implies that $Forb(A, B)$ is $\chi-$bounded with the same binding function.

1.3 Further Research

One open question is, for which graphs A, B is the class of graphs $Forb(A, B)$ $\chi-$bounded with binding function $f(x) = 3$? Clearly, for such a class to be χ-bounded at all, one must have that one of A, B is a tree. Otherwise, constructions by Mycielski and Erdos prove that there are graphs with arbitrarily high chromatic number for fixed girth, which means that if both A, B contain cycles, then one can ensure that $G \not\supseteq A, B$ by taking G of girth strictly greater than $M := max(\text{circumference}(A), \text{circumference}(B))$. Then, by Mycielski's construction, one knows that there exists an infinite sequence of graphs $\{G_0, G_1, G_2, \dots\} \subseteq Forb(A, B)$ each of girth $M + 1$ with $\chi(G_{i+1}) = \chi(G_i)$ for all $i \in \mathbb{N}$. Now, assuming $M + 1 \geq 4$, such a graph would also satisfy $\omega(G) = 2$ (the edge set of our graph is non-empty) since girth at least 4 implies triangle-free. So, one cannot have that $Forb(A, B)$ is $\chi-$bounded as that would imply that for all G_i one has that $\chi(G_i) \leq f(\omega(G_i)) = f(2) =: K$. However, as stated earlier $\chi(G_{i+1}) = \chi(G_i) + 1$ for all $i \in \mathbb{N}$, which implies that there exists $j \in \mathbb{N}$ such that $\chi(G_j) > K$, a contradiction. Thus, one of A, B must be a forest. (In particular, it suffices to consider only trees).

Conjecture 1.5 *Every graph $G \in Forb(K_3, Fork)$ satisfies $\chi(G) \leq 3$, where Fork is the graph obtained from $K_{1,4}$ by subdividing two edges exactly once.*

1.4 The Regularity Lemma

First, we prove a helper lemma that demonstrates any well-formed partition with an exceptional set can be refined to increase its parameter q value. This will then be applied a bounded number of times on a given partition, resulting in finding an ϵ-regular partition for any large-enough graph.

Lemma 1.6 *Let $G = (V, E)$ be a graph, $0 < \epsilon < \frac{1}{4}$, and $\mathcal{P} = \{C_0, C_1, \dots, C_k\}$ be a partition of V with exceptional set C_0 such that $|C_0| \leq \epsilon |V|$ and $|C_1| = \cdots = |C_k| = c$. If $\mathcal{P}$ is not ϵ-regular then there exists a refinement $\mathcal{P}' = \{C_0', C_1', \dots, C_l'\}$ of $\mathcal{P}$ such that*

- $|C_0'| \leq |C_0| + \frac{|V|}{2^k}$
- $|C_1'| = \cdots = |C_l'|$
- $k \leq l \leq k4^k$
- $q(\mathcal{P}') \geq q(\mathcal{P}) + \frac{\epsilon^5}{2}$

Proof. For every non-ϵ-regular pair (C_i, C_j), we apply Property **??** above. That is, there exists partitions $\mathcal{C}_{ij}, \mathcal{C}_{ji}$ of C_i, C_j respectively, such that $|\mathcal{C}_{ij}| = |\mathcal{C}_{ji}| = 2$, and

$$q(\mathcal{C}_{ij}, \mathcal{C}_{ji}) \geq q(C_i, C_j) + \frac{\epsilon^4 |C_i| |C_j|}{|V|^2}.$$

For every (C_i, C_j) pair that is ϵ-regular, let $\mathcal{C}_{ij} = \{C_i\}$ and $\mathcal{C}_{ji} = \{C_j\}$.

Now, for every $i \in [k]$, we find a minimal partition $\mathcal{C}_i$ of C_i that refines all $\mathcal{C}_{ij}$. Note that $|\mathcal{C}_i| \leq 2^{k-1}$, because each set may be partitioned into two at most $k - 1$ times. Let $\mathcal{C} = \{C_0\} \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k$ be a partition of V. We have

$$\begin{aligned} q(\mathcal{C}) &\geq \sum_{i<j} q(\mathcal{C}_i, \mathcal{C}_j) \\ &\geq \left(\sum_{i<j} q(C_i, C_j) \right) + \frac{\epsilon^4 c^2}{|V|^2} \epsilon k^2 \\ &\geq q(\mathcal{P}) + \frac{\epsilon^5}{2}. && \text{(since } k^2 c^2 > (1-\epsilon)^2 |V|^2 \text{ and } \epsilon < 1/4) \end{aligned} \tag{1.1}$$

At this point we have a partition $\mathcal{C}$ that will satisfy the pairwise ϵ-regular goal, but the cardinality of each partition may not be uniform and the exceptional set may be too large. We next find the partition $\mathcal{P}'$ that retains the pairwise ϵ-regularity of $\mathcal{C}$ but has uniform sized elements with an exceptional set.

Let $\{C'_1, \ldots, C'_l\}$ denote a maximal collection of sets such that $|C'_i| = \lfloor \frac{c}{4^k} \rfloor$, $C'_i \subseteq C_j$ for some $C_j \in \mathcal{C} \setminus \{C_0\}$. Let $C'_0 = V \setminus C'_1 \setminus \cdots \setminus C'_l$. Then, $\mathcal{P}' = \{C'_0\} \cup \{C'_1, \ldots, C'_l\}$ is a partition of V. Now, we need to show that the four desired requirements of $\mathcal{P}'$ hold to complete the proof.

We have

$$|C'_0| \le |C_0| + k2^{k-1}\frac{c}{4^k} \le |C_0| + \frac{|V|}{2^k}, \tag{1.2}$$

satisfying the first requirement. Note that $|C'_i| = \lfloor \frac{c}{4^k} \rfloor$ was chosen in order to satisfy (1.2). Also, by construction, $|C'_1| = \cdots = |C'_l| = \lfloor \frac{c}{4^k} \rfloor$, satisfying the second requirement.

Since $\mathcal{P}'$ is a linear refinement, $k \le l$. Each k-set C_i in $\mathcal{C}$ contains at most $\frac{|C_i|}{\lfloor \frac{c}{4^k} \rfloor} \le 4^k$ sets in $\{C'_1, \ldots, C'_l\}$, so $l \le k4^k$.

By Property **??** of q and equation (1.1), we have

$$q(\mathcal{P}') \ge q(\mathcal{C}) \ge q(\mathcal{P}) + \frac{\epsilon^5}{2}, \tag{1.3}$$

satisfying the final requirement. ∎

Theorem 1.7 (Szemerédi, 1976) *For every $\epsilon > 0$ and $m \in \mathbb{Z}, m \ge 1$, $\exists M \in \mathbb{Z}$ such that every graph $G = (V, E)$ of order at least m admits an ϵ-regular partition $\{V_0, V_1, \ldots, V_l\}$ with exceptional set V_0 and $m \le l \le M$.*

Proof. This follows from repeated applications of Lemma 1.6.

Let $f(x) = x4^x$. Let $s = \lceil 2/\epsilon^5 \rceil$. Choose $k \ge m$ such that $s/2^k < \epsilon/2$. Choose M such that $M = \max\{f^s(k), \lceil 2k/\epsilon \rceil\}$.

If $|V| \le M$, partition G into singleton sets.

If $|V| > M$, then start with the partition $\{C_0, \ldots, C_k\}$ of V where $|C_0| < k$ and $|C_1| = \cdots = |C_k|$. Apply Lemma 1.6 up to s times, and let $\{V_0, V_1, \ldots, V_l\}$ denote the resulting partition with exceptional set V_0, such that

- $|V_0| \le |C_0| + s\frac{|V|}{2^k} \le k + \epsilon\frac{|V|}{2} \le \epsilon\frac{|V|}{2} + \epsilon\frac{|V|}{2} \le \epsilon\,|V|$
- $|V_1| = \cdots = |V_l|$
- $m \le k \le l \le M$
- all but ϵl^2 pairs (V_i, V_j) are ϵ-regular.

∎