Math 6014 - Homework 6

Caitlin Beecham

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We wish to show that every graph with a Hamiltonian cycle admits a 4 flow. We first recall that a graph has a k flow if and only if it has a $\mathbb{Z}_k$ flow. Additionally, for any abelian group, H, of size k, a graph has a $\mathbb{Z}_k$ flow if and only if it has an H flow. So, in particular, a graph has a 4 flow if and only if it has a $\mathbb{Z}_4$ flow. Also, a graph has a $\mathbb{Z}_4$ flow if and only if it has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ flow. We show that any graph with a Hamiltonian cycle has a $\mathbb{Z}_2 \times \mathbb{Z}_2$ flow. To start, note that the fact that G has a Hamiltonian cycle implies that any edge not included in the cycle is a chord of that cycle. Formally, say G has a Hamiltonian cycle given by $C := v_1,\ v_1v_2,\ v_2, \dots, v_{n-1},\ v_{n-1}v_n,\ v_n,\ v_nv_1,\ v_1$. We first build a $\mathbb{Z}_2 \times \mathbb{Z}_2$ circulation of G. Note, that the flow on some edges of this circulation may be 0. We do so as follows. For each directed edge v_iv_{i+1} (and also v_nv_1) let $f_0(v_iv_{i+1}) = (1,0)$ and let $f_0(e) = (0,0)$ for all other $e \in E(G) \setminus E(C)$. This is clearly a circulation as the flow into each vertex is $(1,0)$ and the flow out of each vertex is $(1,0)$. Now, in order to build a nowhere zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ flow on G, we first build the following circulations for each chord $v_jv_{\phi(j)}$. Without loss of generality, let each chord $v_jv_{\phi(j)}$ be oriented such that the initial vertex v_j is the one of lower index and its end vertex $v_{\phi(j)}$ is defined to be the vertex of higher index (this is how we define ϕ). Now, let the circulation f_j be defined as follows. We set $f_j(v_jv_{\phi(j)}) = (0,1)$ and for all $k \in \{0, 1, ..., n - \phi(j) + j - 1\}$ set $f_j(v_{\phi(j)+k}v_{\phi(j)+k+1}) = (0,1)$. Then, finally set $f_j(e) = 0$ for all other e. We recall that the sum of circulations is a circulation. Thus, $f := \sum_{i=0}^{J} f_j$ is a circulation (where J is the number of chords). I also claim that f is a flow, namely that f is a nowhere zero circulation. Why? Note that for all edges, r, along C, we can project f to the first coordinate. I claim such a first coordinate is always non zero. Why? Because $f_0(r) = (1,0)$ for all such r. Then, note that $f_j(r) \in \{(0,0),(0,1)\}$ for all $j \in [J]$. Thus, $f(r) = (1,0) + \sum_{j=1}^{J} f_j(r) \in \{(1,0),(1,1)\}$ and we see that $f(r)$ is nonzero for all r in C. Next we note that $f(s) \neq (0,0)$ for all chords s as well. Why? Well, there is a unique $j_0 \in [J]$ for which $f_{j_0}(s) \neq (0,0)$ and in particular $f_{j_0}(s) = (0,1)$. Then, $f(s) = \sum_{j=0}^{J} f_j(s) = (0,1) + \sum_{j \in \{0,1,\dots,J\} \setminus \{j_0\}} f_j(s) = (0,1) \neq (0,0)$. Thus, we see, that f is a nowhere zero $\mathbb{Z}_2 \times \mathbb{Z}_2$ flow. Thus, G has a nowhere zero 4 flow as well.

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We wish to show that a graph has an 8-flow if, and only if, it can be expressed as the union of three even graphs. We first show that if G has an 8 flow, then it can be expressed as the union of 3 even graphs. We first recall that for any abelian group, H, of order 8, a graph G has an 8 flow if and only if it has an H flow. Say $H := \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus, G has a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ flow, f. Now, define G_i for $i \in [3]$ as follows. Let $V(G_i) = V(G)$. Now, let $E(G_i) = \{e \in E(G) | \pi_i(f(e)) = 1\}$ where $\pi_i : \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ is projection of $f(e)$ onto the ith component of $f(e)$. I claim that G_i is an even graph for all i. Why? Assume not. Assume the degree of some $w \in V(G_i)$ is odd for some i. Then, recall that in any flow, the flow into every vertex w equals the flow out of w. In particular, for each ith component ($i \in \{1,2,3\}$) the ith component of the flow in equals the ith component of the flow out. Now, when we restrict ourselves to considering the flow in or out of w on only the ith component, the flow on edges not in G_i do not affect this value. Thus, the ith component of the net flow out of w is $n := \sum_{\{w|wy \in E(G_i)\}} \pi_i(f(wy)) - \sum_{\{u|uw \in E(G_i)\}} \pi_i(f(uw))$. Note, that since $deg_{G_i}(w) = 1 \mathrm{mod} 2$, it cannot hold that $|\{u|uw \in E(G_i)\}| = |\{y|wy \in E(G_i)\}|$ and since $\pi_i(f(uw)) = \pi_i(f(wy)) = 1$ for all such u and y, this implies that $n \neq 0$, a contradiction. Thus, all G_i are even. Now, finally we show that the union of the G_i's is all of G. Why? G has a nowhere zero $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ flow, f, means that $f(e) \neq (0,0,0)$ for all edges e. Thus, at least once component of $f(e)$ is 1, which implies that for all edges e there is at least one G_i such that $e \in G_i$. Finally, we show the reverse implication: namely that if G can be expressed as the union of 3 even graphs, then it has an 8 flow. We first recall that a graph has an Eulerian circuit if and only if it is an even graph with only one nontrivial component. So, in particular, each non trivial component of each G_i has an Eulerian circuit, namely q sequence of directed edges which form a closed walk traversing each edge of the given component of the given G_i exactly. So say, the Eulerian circuit of this component is given by $v_1, v_1v_2, v_2, ..., v_{l-1}, v_{l-1}v_l, v_l, v_lv_1, v_1$, where v_i is not necessarily distinct from v_j for $i \neq j$ but $v_iv_{i+1} \neq v_jv_{j+1}$ for all $i \neq j$. For each edge v_iv_{i+1} assign flow $f_1(v_iv_{i+1}) = 1$. Then, repeat this process for all components of G_1. Do the analogous process for G_2 and G_3. Namely, for each component of G_2, find an Eulerian circuit and assign the flow $f_2(e) = 2$ for each directed edge, e, of that circuit. Then for each component of G_3, find an Eulerian circuit and assign the flow value $f_3(e) = 4$ for each directed edge e of that circuit. Now, extend the flow f_1 of G_1 to a circulation, g_1, of G by setting $g_1(e) = f_1(e)$

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We wish to prove that a plane triangulation is 3-colorable if, and only if, it is an even graph.
We first show that any even plane triangulation is 3-colorable. We consider the dual graph G*. Clearly G* is a cubic multigraph. Now, we recall the theorem presented in class which states that a cubic multigraph has a 3-flow if and only if it is bipartite. Thus, to show that G* has a 3 flow, it suffices to show that G^* is bipartite. Then, we will apply the theorem that the chromatic number of a plane graph equals the flow number of its dual graph. (Clearly, G is not 2 colorable, as it contains a K_3. Thus, to show its chromatic number is 3 it suffices to show that it is 3 colorable). (Similarly, the dual graph G* does not have a 2-flow as it is not an even graph. Thus, to show that the flow number of G* is 3, it suffices to show that it has a 3-flow). So, let's show that G^* is bipartite. This statement is equivalent to G^* having no odd cycles. Well, the fact that G is even implies that all facial cycles in G^* are even. Then, we recall that the facial cycles of a planar graph generate its cycle space which in particular means that the facial cycles generate all cycles. Is it true that the symmetric difference of two non-edge-disjoint cycles is an even cycle? I claim so. Why? Take two cycles C_1 and C_2 both of even lengths a_1 and a_2. Let $x := |E(C_1) \cap E(C_2)|$. This implies that the number of edges in the symmetric difference is $|E(C_1) \triangle E(C_2)| = a_1 + a_2 - 2x$ which is even. Thus, all cycles in G^* are even and we are done with the forward direction. Next, we wish to show that any plane triangulation which is not an even graph is not 3-colorable. We make use of the same theorems employed in the proof of the forward direction. Namely, the chromatic number of G equals the flow number of G^*. Thus, it suffices to show that G^* does not have a 3-flow, which is equivalent to showing that G^* is not bipartite. It is true that G^* is not bipartite, namely because it has an odd cycle. In particular, G not an even graph implies that there is some vertex $w \in G$ of odd

degree. The corresponding face f_w of G^* is bounded by an odd cycle and we are done.