Math 6014 - Homework 2

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1. Let G be a bipartite graph with partition sets V_1, V_2 and $\delta(G) \geq 1$. Suppose for any edge v_1v_2 with $v_i \in V_i$ for $i = 1, 2$, we have $d(v_1) \geq d(v_2)$. Show that G has a matching such that all vertices in V_1 are matched.

   Assume not. For instance, assume G does not have a matching saturation V_1, which by Hall's Theorem is equivalent to the statement that there exists for $S \subseteq V_1$ such that $|N(S)| < |S|$. Say $|N(S)| = k$ and $|S| = k + r$ for some $r > 0$. Now, we use a counting argument to get a contradiction. Let $S =: \{x_i | i \in [k + r]\}$ and $N(S) =: \{y_i | i \in [k]\}$. The number of edges in the induced subgraph of S union N(S) is $|E(G[S, N(S)])| = \sum_{i=1}^{k+r} deg_G(x_i) = \sum_{i=1}^{k} deg_{G[S,N(S)]}(y_i) \leq \sum_{i=1}^{k} deg_G(y_i)$. We know that $deg_G(x) \geq deg_G(y) \geq deg_{G[S,N(S)]}(y)$ for all $x \in V_1, y \in V_2$. Combining this with the previous equation we get that $|E(G[S, N(S)])| = \sum_{i=1}^{k} deg_G(x_i) + \sum_{i=k+1}^{k+r} deg_G(x_i) \geq \sum_{i=1}^{k} deg_{G[S,N(S)]}(y_i) + \sum_{i=k+1}^{k+r} deg_G(x_i) > \sum_{i=1}^{k} deg_{G[S,N(S)]}(y_i)$ since $\delta(G) \geq 1$. This is contradiction since on one hand $|E(G[S, N(S)])| = \sum_{i=1}^{k} deg_{G[S,N(S)]}(y_i)$ and on the other hand as just shown $|E(G[S, N(S)])| > \sum_{i=1}^{k} deg_{G[S,N(S)]}(y_i)$.

2. Let $r > 0$ and let G be an r-regular graph such that for any $S \subseteq V(G)$ with $|S|$ odd, $e(S, \bar{S}) \geq r$. Show that G has a perfect matching.

   Assume not. Assume there is no perfect matching, which implies that Tutte's condition is violated. We know that there exists a set $X \subseteq V(G)$ with $q(G - X) > |X|$. Now, say there are $q = q(G - X)$ odd components of G-X. The hypothesis of this problem gives us that, for each odd component, the number of edges leaving that component is $\geq r$. Since there are no edges between components of G-X, all these edges must end in X, i.e. be of the form xy with $x \in S, y \in C_i$ where C_i is the ith odd component of $G - X$. Now, the number of edges entering X from $\bar{X}$ is $e(X, \bar{X}) \geq qr$. Now, since $|X| < q(G - X)$ there are k vertices in X with $k < q$. The sum of the degree of the vertices in X is $\sum_{i=1}^{k} r = kr$. Also, the number of edges between X and $\bar{X}$ is $e(X, \bar{X}) \leq \sum_{i=1}^{k} r = kr$ since some edges may be of the form x_1x_2 with $x_1, x_2 \in X$. So, we have $e(X, \bar{X}) \geq qr$ on one hand, and on the other hand $e(X, \bar{X}) \leq kr < qr$, a contradiction.

3. Suppose G is a connected graph and $\delta(G) \geq 3$, and assume that G does not contain two vertex disjoint cycles. Show that $G \cong K_5$, or G is a wheel, or G is a $K_{3,t}^{+}$.

   Consider a maximum path in G. We have 2 cases: (1) the number of vertices on this path is exactly 5, or (2) the number of vertices on this path is greater than 5. Why can't the number of vertices on such a path be strictly less than 5? Say our path has 4 vertices. Then our picture looks like one of the following (Figures (a)) and in each case at least one of the endpoints, p_1 or p_2, by the minimum degree being at least 3, must have some neighbor not on our maximum path p, which implies p was not maximal–a contradiction. If p has 3 vertices as shown in (Figures (b)) then, once again, at least one endpoint has some neighbor not on the path which contradicts maximality. If p has only two vertices, we get a contradiction by the same logic once again. So, a maximal path (a maximum path is always maximal) must have at least 5 vertices. Say our maximum path has exactly 5 vertices. By the cases shown in the following Figures of part 1. Our statement holds.

In case (2), we decompose the graph into blocks, which we know is possible from class. As noted, this decomposition consists of 2-connected blocks and blocks which are a single edge. Now, since there are not two disjoint cycles we know that any two blocks which are not single edges must overlap. Also, as we all learned in an undergraduate level Graph Theory class, every pair of blocks which overlap overlap in a single vertex. Furthermore, as stated in class, all non-trivial blocks have an ear decomposition, which consists of a cycle and some ears. In addition, if we consider the tree-like structure comprised of the blocks in our block decomposition, we cannot have single edges as blocks as leaves because the minimum degree is at least 3 and if such blocks were leaves there would be a vertex of degree 1 because there are no cycles in this tree-like structure as stated in class. (This means that there are no cycles not contained in a single block, and as a result the only edge adjacent to the end vertex in such a trivial block is to its neighbor in that block). Also, if there is more than one non-trivial block, there is some vertex v, such that for every pair of non-trivial blocks B_1 and B_2, $B_1 \cap B_2 = \{v\}$. (Otherwise, we have two disjoint cycles). We can also not have single edge blocks between non-trivial blocks since that would result in 2 disjoint cycles. So, our block decomposition consists only of non-trivial blocks which all overlap in one vertex. Now, we have several cases.

In Case (a), we suppose that every ear of a given block is a chord, consisting of a single edge, that there are at least two non-trivial blocks, and that each block has some ear (it is not just a cycle). In this case, all chords (ears) of both blocks must have one end at the "overlapping vertex". Otherwise, if we have a chord on each, which does not end at the overlapping vertex, we get two disjoint cycles. Now, we also know that each vertex in our graph G has degree at least 3. In addition, the only neighbors to vertices in a given block which do not overlap with other blocks are exclusively neighbors within that block. This means that every vertex on the cycle in our ear decomposition must have another neighbor on the cycle (via a chord) other than the two on the cycle. These chords must all end at the overlapping vertex. In addition, the two vertices adjacent to the overlapping vertex must be adjacent to each other. Otherwise, they either have degree less than 3, or they are each adjacent to some other vertex on the cycle which gives a cycle that does not overlap with the cycle in the other blocks ear decomposition meaning we have two disjoint cycles. So, our image looks like figure (Figure (2a)). However, we also note that this image has two disjoint cycles, which means that this case cannot happen.

Consider Case (b), in which we have exactly one block (2-connected) and every ear of its Ear Decomposition is a chord. We have several cases. If the number of vertices on the cycle in this ear decomposition is 4, we get K_4, which is isomorphic to $K_{3,t}^+$. If there are 5 vertices on this cycle, we get something isomorphic to $K_{3,t}^+$, K_5, or a wheel (as shown in the Figures: (1)(a)(i), (1)(b)(i), (1)(b)(ii), (1)(d)(i), (1)(e)(i), (1)(e)(ii), and (1)(e)(iii)). If there are 6 or more, we must have that all chords except one are incident to the same vertex to avoid creating 2 disjoint cycles and we also have that the two vertices adjacent to that vertex are each adjacent to each other. So, we get a wheel.

Now, consider Case (c), in which there are at least two blocks, each of which must be non-trivial by our earlier argument, whose ears aren't necessarily chords but rather paths of length possibly greater than 1. Once again, all ears on each of these ear decompositions must have one end at the overlapping vertex. Also, the two vertices adjacent to this vertex in each of the blocks must be adjacent to each other, to avoid creating two disjoint cycles within each block and keeping the degree of these neighboring vertices at least 3. However, such an edge creates two disjoint cycles, one in each block, (similar to what is shown in Figure (2a)), so this case cannot happen.

Consider Case (d), in which there is one block, whose ears are not necessarily chords. If there are at least 6 vertices on the cycle, we know it is a subdivision of a wheel, and in addition, the new vertices which result from subdividing the chords we had earlier now have degree 2 and need neighbors, which as we said before, must lie within this block. For each chord, which previously ended at the overlapping vertex, we now possibly have a new vertex resulting from subdivision which is adjacent to the overlapping vertex and has one other neighbor on the subdivided path. It needs a third neighbor. We see that, except for in the case where there are 4 vertices on the cycle of this block, for any neighbor we assign it, we get 2 disjoint cycles. In the case where there are 4 vertices on our cycle, we can assign a neighbor to one vertex resulting from a subdivision without creating 2 disjoint cycles in which case (if only that edge was subdivided once) we get a wheel. In the case that more than one chord was subdivided, we see that we cannot assign neighbors to each of the new vertices without creating two disjoint cycles. In the case where there are 5 vertices on the cycle, and some ears are not chords, please refer to figures on the following sheet. (Note: those figures are the only ones we need to consider because of the following fact. The fact that there is one block means that a maximum path of G is contained completely within that block. In the case that the maximum path is not of length 5, we revert to the previous case in which there are more than 6 vertices on the cycle). We see that, in all viable cases, we get a wheel, K_5, or $K_{3,t}^{+}$.