1 Preliminary Result

Theorem 1.1. *For a fixed embedding of a planar graph into the plane, any cycle in such a plane graph (where plane graph is understood to mean planar graph as embedded into the plane in some fixed way) clearly divides the plane into an inner region and outer region by the Jordan Curve Theorem. Then, the claim is that any cycle of odd degree in the plane graph contains a face of odd degree. Furthermore, the number of faces of odd degree contained in such an odd cycle is odd.*

Proof. Note that the facial cycles of a plane graph, G, span the cycle space, and additionally that, (the edge set of) every cycle C in the plane graph is equal to the symmetric difference of the (edge sets of) facial cycles, $F_C := \{C_f : C_f = \delta(f)$ where $f \subset \text{int}(C)$ is a face of $G\}$, that bound faces contained in the interior of C. (The idea of cycle space of a graph is common and standard enough, that we will not dwell on the distinction between a cycle and its edge set, which are in this context assumed to be the same). Now, note that given two cycles C_0 and C_1 whose symmetric difference is a cycle C, we have that the length of C is even if the parities of the lengths of C_0 and C_1 are the same and is odd if the parities of the lengths of C_0 and C_1 differ. As such, for a cycle C in a plane graph, which is the symmetric difference of the facial cycles C_f as defined above, we have that the length of C is odd if the number of facial cycles $f \in F_C$ of odd length is odd and is even otherwise. So, if the length of C is odd, that implies that at least one facial cycle $f \in F_C$ has odd length. Using the terminology of faces, that is to say that at least one such face is of odd degree. ∎

2 Problem-Specific Results

The purpose of this section is to show that $PR(3,3,3) = 13$. This is accomplished in two parts: showing that $PR(3,3,3) \leq 13$ and showing that $PR(3,3,3) > 12$. We show that $PR(3,3,3) \leq 13$ by showing directly that any edge-coloring in three colors of K_{13} such that the subgraph induced by the edges in the first color is planar necessarily contains a monochromatic triangle. We show that $PR(3,3,3) > 12$ by providing a specific edge-coloring in three colors of K_{12} in which the subgraph induced by edges of the first color is planar and so that there are no monochromatic triangles in any of the three colors.

2.1 Relevant Definitions

In the following when we refer to K_n as the complete graph on n vertices, n is assumed to be a positive integer. Also, $E(K_n)$ denotes the edge set of K_n.

Definition 2.1. *A "three-colored K_n", which we can label G in any particular instance, is a complete graph on n vertices such that each edge is assigned one of three colors: green, red and blue. More precisely, it is a graph of the type $G = (V, \hat{E})$ where V denotes the standard collection of vertices $\{v_i : i \in [n]\}$ of the complete graph and $\hat{E} = \{(v_i v_j, c) : i, j \in [n], i \neq j, c \in \{green, red, blue\}\}$ denotes the set of edges each associated with an assigned color.*

Definition 2.2. *The "green subgraph" of $G = (V, \hat{E})$, a three-colored K_n, is the subgraph induced by the green edges. Namely, it is $K_{G,g} = (V_{G,g}, E_{G,g})$ where $E_{G,g} = \{e \in E(K_n) : (e, green) \in \hat{E}\}$ and $V_{G,g} = \{v \in V : (vw, green) \in \hat{E}$ for some $w \in V\}$ "Red subgraphs" and "blue subgraphs" are defined analogously. These three subgraphs of K_n are denoted $K_{G,g}$, $K_{G,r}$, and $K_{G,b}$, respectively.*

Definition 2.3. *The "green neighbors" (resp. "red neighbors", "blue neighbors") of a vertex $v \in K_n$ are the neighbors of that vertex in the green subgraph (resp. red subgraph, blue subgraph). More precisely, for $v \in K_n$ we denote the green neighbors of a by $N_{G,g}(v) = \{w \in K_n : vw \in E_{G,g}\}$ and we denote $N_{G,r}(v) = \{w \in K_n : vw \in E_{G,r}\}$ and $N_{G,b}(v) = \{w \in K_n : vw \in E_{G,b}\}$ likewise for the red and blue neighbors of a vertex $v \in K_n$.*

Definition 2.4. *The "green degree" (resp. "red degree", "blue degree") of a vertex $v \in G$ in a three-colored K_n called $G = (V, \hat{E})$ is the degree of that vertex in the green subgraph (resp. red subgraph, blue subgraph). More precisely, for $v \in K_n$ we denote the "green degree' of v by $d_{G,g}(v) = |N_{G,g}(v)|$ and we denote $N_{G,r}(v) = |N_{G,r}(v)|$ and $N_{G,b}(v) = |N_{G,b}(v)|$ likewise for the red and blue degree of a vertex $v \in K_n$.*

Definition 2.5. *A "red-green pentagram" is a three-colored K_5 whose red and green subgraphs are each 5-cycles. A "blue-green pentagram" is likewise a three-colored K_5 whose blue and green subgraphs are each 5-cycles.*

Definition 2.6. *A "monochromatic triangle" in $G = (V, \hat{E})$, a three-colored K_n, is an isomorphic copy of K_3 contained in exactly one of the green, red or blue subgraphs, meaning $K_{G,g}$, $K_{G,r}$ or $K_{G,b}$ respectively.*

Definition 2.7. *A "planar Ramsey counterexample", G, on n vertices is three-colored K_n such that the green subgraph of G is a planar graph and so that no monochromatic triangles exist in G.*

Definition 2.8. *We now define isomorphic subgraphs for three-colored graphs. Given $G = (V, \hat{E})$, a three-colored K_n, and $H = (W, \hat{F})$, a three colored K_m where $1 \leq m \leq n$, we say that G "contains an isomorphic copy of H" if there exists an injective mapping $f : W \to V$ such that $w_i w_j \in N_{n,c}$ if and only if $f(w_i)f(w_j) \in N_{n,c}$ for all colors $c \in \{\text{green}, \text{red}, \text{blue}\}$, namely an injective mapping on vertex sets that respects edge colors.*

Definition 2.9. *We define $v_{G,g} = |V_{G,g}|$, $e_{G,g} = |E_{G,g}|$, and $f_{G,g} = |F_{G,g}|$, to be the number of vertices, edges and faces of $K_{G,g}$ respectively where $F_{G,g}$ denotes the set of faces of the plane graph $K_{G,g}$. (Recall "plane graph" means $K_{G,g}$ has a fixed embedding into the plane).*

2.2 Necessary Lemmas

In the rest of the paper, $G = (V, \hat{E})$ is assumed to be a specific three-colored K_{13}. Furthermore, it is assumed that the green subgraph $K_{G,g}$ is planar and that G has no monochromatic triangles.

Lemma 2.10. *No vertex $v \in G$ has degree 6 or more in any single color, planar or not. More precisely stated, $\max\,(d_{G,g}(v), d_{G,r}(v), d_{G,b}(v)) \leq 5$ for all $v \in V$.*

Proof. Stated intuitively, this follows from the fact that if some vertex $v \in V$ had degree 6 or more in one color, then the induced complete graph on $N(v)$, the neighborhood of v, would need to be colored in the other two colors and cannot be done so without creating monochromatic triangles due to the fact that $R(3,3) = 6$. More precisely stated, if $d_{G,c}(v) \geq 6$ for some $v \in V$ and some $c \in \{\text{green}, \text{red}, \text{blue}\}$, then to avoid triangles in the color c, one has that all edges $(uw, k) \in \hat{E}$ such that $u, w \in N_{G,g}(v), u \neq w$ satisfy $k \neq c$ meaning that $k \in \{\text{green}, \text{red}, \text{blue}\} \setminus \{c\}$. Then, the fact that $R(3,3) = 6$, provides a contradiction. ∎

Lemma 2.11. *All vertices $v \in V$ have green degree at least 2. More precisely stated, $d_{G,g}(v) \geq 2$ for all $v \in V$.*

Proof. If $d_{G,g}(v) \leq 1$ for some $v \in V$, then $d_{G,g}(v) + d_{G,r}(v) + d_{G,b}(v) = 12$ implies that $d_{G,r}(v) + d_{G,b}(v) \geq 11$ meaning that $\max\,(d_{G,r}(v), d_{G,b}(v)) \geq 6$, a contradiction to Lemma 2.10 since $\max\,(d_{G,r}(v), d_{G,b}(v)) \leq \max\,(d_{G,g}(v), d_{G,r}(v), d_{G,b}(v))$ and thus $6 \leq \max\,(d_{G,r}(v), d_{G,b}(v)) \leq \max\,(d_{G,g}(v), d_{G,r}(v), d_{G,b}(v)) \leq 5$. ∎

Lemma 2.12. *The graph G has at most 22 green edges. More precisely put $|E_{G,g}| \leq 22$.*

Proof. Planarity of the green subgraph $K_{G,g}$ implies that $13 - e_{G,g} + f_{G,g} = 2$ or $f_{G,g} = e_{G,g} - 11$. Additionally, since all faces $f \in F_{G,g}$ have degree at least 4 to avoid triangles, the formula $2e_{G,g} = \sum_{f \in F_{G,g}} \deg(f) \geq 4f_{G,g}$ along with the above gives $2e_{G,g} \geq 4(e_{G,g} - 11)$ or $2e_{G,g} \geq 4e_{G,g} - 44$ which implies that $e_{G,g} \leq 22$. Also note that equality is achieved when all faces $f \in F_{G,g}$ in the green subgraph have degree $\deg(f) = 4$ equal to four. ∎

Lemma 2.13. *At most 9 vertices $v \in V$ have green degree at least 4. More precisely stated, $|\{v \in V : d_{G,g}(v) \geq 4\}| \leq 9$.*

Proof. Otherwise, if $|\{v \in V : d_{G,g}(v) \geq 4\}| \geq 10$, then $44 \geq 2e_{G,g} = \sum_{v \in V} d_{G,g}(v) \geq 40 + a + b + c$ where $a, b, c \in \mathbb{Z}_{\geq 2}$ (by Lemma 2.11), a contradiction since $a, b, c \geq 2$ implies that $a + b + c \geq 6$ and thus $40 + a + b + c \geq 46$ giving $44 \geq 46$. ∎

Lemma 2.14. *At least 4 vertices $v \in V$ have green degree less than or equal to three. More precisely stated, $|\{v \in V : d_{G,g}(v) \leq 3\}| \geq 4$.*

Proof. Otherwise, if $|\{v \in V : d_{G,g}(v) \leq 3\}| \leq 3$ then $V = \{v \in V : d_{G,g}(v) \leq 3\} \sqcup \{v \in V : d_{G,g}(v) \geq 4\}$ and $|V| = 13$ implies that $13 = |\{v \in V : d_{G,g}(v) \leq 3\}| + |\{v \in V : d_{G,g}(v) \geq 4\}| \leq 3 + |\{v \in V : d_{G,g}(v) \geq 4\}|$ proving that $|\{v \in V : d_{G,g}(v) \geq 4\}| \geq 10$, a contradiction to Lemma 2.13. ∎

Lemma 2.15. *If any vertex $v \in V$ has green degree $d_{G,g}(v) = 2$ equal to two, then G contains an isomorphic copy of a red-green pentagram and an isomorphic copy of a blue-green pentagram. The red-green pentagram, blue-green pentagram, and the "closed" green neighborhood $v \cup N_{G,g}(v)$ are pairwise vertex disjoint, meaning that any two of the three mentioned subgraphs have no common vertices.*

Proof. Such a vertex, $v \in V$ must have red degree and blue degree each equal to five by Lemma 2.10, more precisely meaning that $d_{G,r}(v) = 5$ meaning that $d_{G,b}(v) = 5$. Then, to avoid red triangles the edges with both endpoints among $N_{G,r}(v)$ must be colored green or blue. Likewise, to avoid blue triangles the edges with both endpoints among $N_{G,r}(v)$ must be colored green or red. More precisely put, to avoid red triangles, for all $(uw, c) \in \hat{E}$ with $u, w \in N_{G,r}(v)$ and $u \neq w$ one has that $c \in \{\text{green}, \text{blue}\}$. Likewise, to avoid blue triangles, for all $(uw, k) \in \hat{E}$ with $u, w \in N_{G,b}(v)$ and $u \neq w$ one has that $k \in \{\text{green}, \text{red}\}$.

Now, we show that the only edge coloring in two colors of K_5 free of monochromatic triangles is exactly the two-color pentagram described in the definition section. Namely, to show this statement, we assume without loss of generality that the two colors are green and blue and show that the only edge coloring of K_5 in green and blue is a blue-green pentagram.

So, we have shown that G contains an isomorphic copy of a red-green pentagram and an isomorphic copy of a blue-green pentagram which are vertex disjoint since $N_{G,b}(v) \cap N_{G,r}(v) = \emptyset$. Additionally, since $N_{G,g}(v) \cap N_{G,b}(v) = \emptyset$ and $N_{G,g}(v) \cap N_{G,r}(v) = \emptyset$ the proof is complete. ∎

Lemma 2.16. *If any vertex $v \in V$ has green degree $d_{G,g}(v) = 3$, then G contains an isomorphic copy of a red-green pentagram or contains an isomorphic copy of a blue-green pentagram.*

Proof. If $d_{G,g}(v) = 3$ for some $v \in V$, then $d_{G,r}(v) \geq 5$ or $d_{G,b} \geq 5$ since $12 = d_{G,g}(v) + d_{G,r}(v) + d_{G,b}(v) = 3 + d_{G,r}(v) + d_{G,b}(v)$ implies $9 = d_{G,r}(v) + d_{G,b}(v)$. Consequently, if $d_{G,b}(v) \geq 5$, then by the argument given in the proof of Lemma 2.15, G contains an isomorphic copy of a red-green pentagram. Likewise, if $d_{G,r}(v) \geq 5$, then by the same argument, G contains an isomorphic copy of a blue-green pentagram. ∎

2.3 Main Theorem

We are now ready to prove that $PR(3, 3, 3) = 13$.

Theorem 2.17. *One has that $PR(3, 3, 3) \leq 13$.*

Proof. The result is shown by considering, G, a three-colored K_{13}, and then showing that if the green subgraph $K_{13,g}$ of G is a planar graph, then there necessarily exists a monochromatic triangle in G. We do so by contradiction, namely by assuming G is a three-colored K_{12} such that $K_{G,g}$ is planar and G contains no monochromatic triangles and obtaining a contradiction in each of several cases.

We accomplish this by considering two cases. Either some vertex in $v \in G$ has "green degree", $d_{13,g}(v) = 2$, equal to two or all vertices $v \in G$ have green degree, $d_{13,g}(v) \geq 3$, at least 3.

Case 1: There exists a vertex, $v \in G$, or green degree $d_{13,g}(v) = 2$, equal to two.

Consider the vertex of green degree 2. Either at least one of the vertices adjacent to it has degree 2, or both of the adjacent vertices have degree at least 3. The former leads to a contradiction. Likewise the latter leads to a contradiction since both green neighbors having degree at least 3 forces a green triangle.

Case 2: All vertices $v \in G$ have green degree, $d_{G,g}(v) \geq 3$, at least 3.

Claim 2.18. *The green subgraph $K_{G,g}$ of G has at least 20 edges.*

Proof: This result follows easily from an easy application of Euler's formula. Namely, we see that $2E_{G,g} = \sum_{v \in V_{G,g}} d_{G,g}(v) \geq 3|V_{G,g}| = 39$ which in fact implies that $2E_{G,g} \geq 40$ or $E_{G,g} \geq 20$. □

Claim 2.19. *The green subgraph $K_{G,g}$ of G at most 21 edges.*

Proof: Otherwise, that means the green subgraph $K_{G,g}$ has 22 edges, due to Lemma 2.12.

First we show that such a graph $K_{G,g}$, when embedded into the plane, must have faces exclusively of degree 4. Namely, note that $V_{G,g} - E_{G,g} + F_{G,g} = 2$ implies that $13 - 22 + F_{G,g} = 2$ and thus $F_{G,g} = 11$. However, now note that $44 = 2E = \sum_{v \in V_{G,g}} d_{G,g}(v) = \sum_{f \in \text{faces of} K_{G,g}} d_{G,g}(f)$ and since absence of triangles in $K_{G,g}$ implies $d_{G,g}(f) \geq 4$ for all faces f of $K_{G,g}$ we see that $44 = \sum_{f \in \text{faces of} K_{G,g}} d_{G,g}(f) \geq 11 * 4$ and indeed, any face of degree 5 or more, would provide a contradiction.

Further, note that there must exist a vertex $v \in K_{G,g}$ of degree $d_{G,g}(v) \leq 3$ since otherwise if all vertices $v \in K_{G,g}$ have degree at least 4, then $44 = 2E_{G,g} = \sum_{v \in V_{G,g}} d_{G,g}(v) \geq 4|V_{G,g}| = 52$, a contradiction. Indeed, by Lemma 2.11 and the fact that the current case prohibits v from having green degree 2, such a vertex v must have green degree $d_{G,g} = 3$.

Then, by presence of a degree 3 vertex, Lemma 2.16 ensures existence of a 5-cycle in the green subgraph $K_{G,g}$. Indeed, then Theorem 1.1 guarantees existence of an odd number of faces of odd degree, which provides a contradiction, since as shown above, all faces have degree 4. □

Case 2a:
The green subgraph $K_{G,g}$ has 20 edges.

Claim 2.20. *Indeed, $K_{G,g}$ has 9 faces.*

Proof: We have that $v_{G,g} - e_{G,g} + f_{G,g} = 2$ and thus $13 - 20 + f_{G,g} = 2$ meaning $f_{G,g} = 9$. □

Claim 2.21. *The list of green degrees of the 13 vertices of G listed with multiplicity is* $3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 4$.

Proof: Specifically, the formula $40 = 2e_{G,g} = \sum_{v \in V_{G,g}} d_{G,g}(v)$ along with the fact that $d_{G,g}(v) \geq 3$ for all $v \in V_{G,g}$ (by virtue of working within Case 2) implies that $40 = \sum_{v \in V_{G,g}} d_{G,g}(v) \geq 39$. Then, since, once again, $d_{G,g}(v) \geq 3$ for all $v \in V_{G,g}$ and since $40 = \sum_{v \in V_{G,g}} d_{G,g}(v)$ we obtain the result. □

Claim 2.22. *The list of degrees of faces of $K_{G,g}$ listed with multiplicity is one of* $4, 4, 4, 4, 4, 5, 5, 5, 5$, $4, 4, 4, 4, 4, 4, 5, 5, 6$, *or* $4, 4, 4, 4, 4, 4, 4, 6, 6$.

Proof: We note that $40 = 2e_{G,g} = \sum_{f \in F_{G,g}} \deg(f) \geq 4f_{G,g} = 36$. Now, since $40 = \sum_{f \in F_{G,g}} \deg(f) \geq 36$ and since $\deg(f) \geq 4$ for all $f \in F_{G,g}$ (by virtue of being triangle-free) we see that the only possibilities for terms in the summand $\sum_{f \in F_{G,g}} \deg(f)$ listed with multiplicity are exactly the three given in the statement. □

Claim 2.23. *The list of degrees of faces of $K_{G,g}$ listed with multiplicity is not* $4, 4, 4, 4, 4, 4, 4, 6, 6$ *or* $4, 4, 4, 4, 4, 5, 5, 5, 5$.

Proof: The fact that there exists a vertex of degree 3, shown in Claim 2.21, guarantees existence of a 5-cycle in $K_{G,g}$, which then by Theorem 1.1 guarantees existence of an odd facial cycle, thus ruling out the possibility of faces all of even degree as above. □

Corollary 2.24. *Thus, the list of degrees of faces of $K_{G,g}$ listed with multiplicity is* $4, 4, 4, 4, 4, 4, 5, 5, 6$.

Case 2b:
The green subgraph $K_{G,g}$ has 21 edges.

Claim 2.25. *Indeed, $K_{G,g}$ has 10 faces.*

Proof: We have that $v_{G,g} - e_{G,g} + f_{G,g} = 2$ and thus $13 - 21 + f_{G,g} = 2$ meaning $f_{G,g} = 10$. □

Claim 2.26. *The list of green degrees of the 13 vertices of G listed with multiplicity is either* $3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 4, 4, 4$ *or* $3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 4, 5$.

Proof: Specifically, the formula $42 = 2e_{G,g} = \sum_{v \in V_{G,g}} d_{G,g}(v)$ along with the fact that $d_{G,g}(v) \geq 3$ for all $v \in V_{G,g}$ (by virtue of working within Case 2) implies that $42 = \sum_{v \in V_{G,g}} d_{G,g}(v) \geq 39$. Then, since, once again, $d_{G,g}(v) \geq 3$ for all $v \in V_{G,g}$ and since $42 = \sum_{v \in V_{G,g}} d_{G,g}(v)$ we see that the only possibilities for terms in the summand $\sum_{v \in V_{G,g}} d_{G,g}(v)$ listed with multiplicity are exactly the two given in the statement. □

Claim 2.27. *The list of degrees of faces of $K_{G,g}$ listed with multiplicity is one of* $4, 4, 4, 4, 4, 4, 4, 4, 4, 6$ *or* $4, 4, 4, 4, 4, 4, 4, 4, 5, 5$.

Proof: We note that $42 = 2e_{G,g} = \sum_{f \in F_{G,g}} \deg(f) \geq 4f_{G,g} = 40$. Now, since $42 = \sum_{f \in F_{G,g}} \deg(f) \geq 40$ and since $\deg(f) \geq 4$ for all $f \in F_{G,g}$ (by virtue of being triangle-free) we see that the only possibilities for terms in the summand $\sum_{f \in F_{G,g}} \deg(f)$ listed with multiplicity are exactly the two given in the statement. □

Claim 2.28. *The list of degrees of faces of $K_{G,g}$ listed with multiplicity is not* $4, 4, 4, 4, 4, 4, 4, 4, 4, 6$.

Proof: The fact that there exists a vertex of degree 3, shown in Claim 2.21, guarantees existence of a 5-cycle in $K_{G,g}$, which then by Theorem 1.1 guarantees existence of an odd facial cycle, thus ruling out the possibility of faces all of even degree as above. □

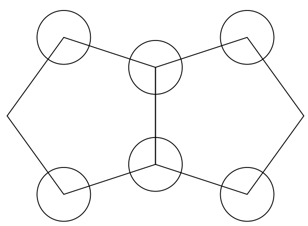
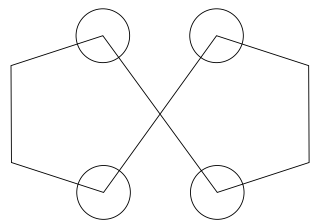
Corollary 2.29. *Thus, the list of degrees of faces of $K_{G,g}$ listed with multiplicity is* $4, 4, 4, 4, 4, 4, 4, 4, 5, 5$.

Now, we merge our cases of consideration to prove a useful result. Namely, notice that in both cases, the following hold.

- At least 10 vertices of $K_{G,g}$ have degree 3.
- All faces of odd degree of $K_{G,g}$ have degree 5.
- Exactly two faces of $K_{G,g}$ have degree 5.

We will use the above to get a contradiction. In the case that the two facial 5-cycles of $K_{G,g}$ share at least one vertex, we can show jointly without regard for number or edges in $K_{G,g}$ that a contradiction arises. Then, in the case that the two facial 5-cycles of $K_{G,g}$ are vertex disjoint, we will need to again split into Cases 2a and 2b.

For now, however, we handle the former situation in which the two facial 5-cycles have non-empty intersection. Namely, suppose that the facial cycles of the two faces of degree 5 in $K_{G,g}$ share a vertex. To avoid vertices of degree 2, which are forbidden by the fact that we are working within Case 2, the facial cycles of these two faces of degree 5 either intersect in exactly one vertex or in exactly one edge. These cases are pictured below.

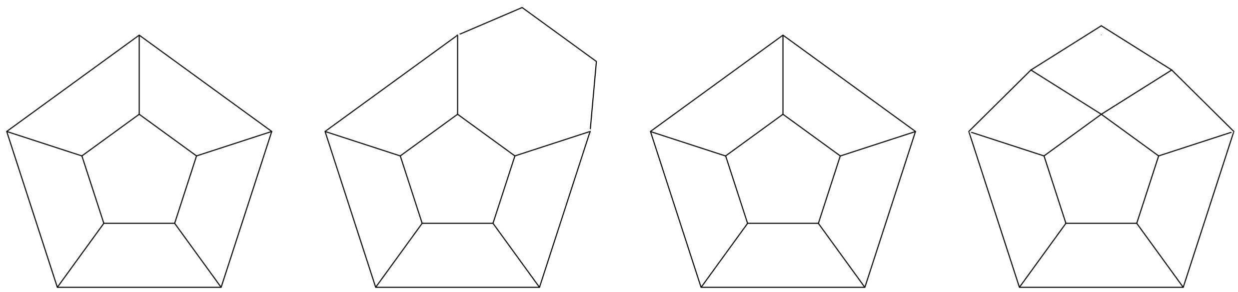
 

Consider the figure on the left, in which the two facial 5-cycles in $K_{G,g}$ have a common edge. Recall, that any degree 3 vertex w in $K_{G,g}$ guarantees existence of a 5-cycle that is vertex disjoint from $w \cup N_{G,g}(w)$. Such a 5-cycle necessarily contains an odd number of faces of odd degree in its interior, by Theorem 1.1, which must all have degree 5 by Corollaries 2.24 and 2.29. Since there are exactly two faces of degree 5 by Corollaries 2.24 and 2.29, we have that any 5-cycle must contain exactly one of the two facial 5-cycles shown above.

However, not now that since there are at least 10 vertices of degree 3 in $K_{G,g}$ and 13 vertices total, at least one of the circled vertices in the figure on the left must have degree 3. Then, the 5-cycle disjoint from $w \cup N_{G,g}(w)$ guaranteed by Lemma 2.16 necessarily cannot contain the edge e common to both facial 5-cycles. So, we obtain a contradiction since any 5-cycle that contains exactly one of the two faces of degree 5 pictured above must contain the edge e.

Now, consider the figure on the right. Again, since there are at least 10 vertices of degree 3 out of 13 total vertices in $K_{G,g}$ we see that at least one of the four circled vertices must have degree 3. Then, for such a circled vertex $w \in K_{G,g}$ we are guaranteed again by Lemma 2.16 a 5-cycle, C, disjoint from $w \cup N_{G,g}(w)$. Similarly, by Theorem 1.1, this 5-cycle C must contain exactly one of the pictured faces of degree 5 in its interior. However, that is impossible since, if that were the case, such a 5-cycle would need to contain the vertex v common to both the pictured facial 5-cycles, which it cannot since $v \in N_{G,g}(w)$.

As mentioned before, in the event that the two facial 5-cycles guaranteed by Corollaries 2.24 and 2.29 are vertex disjoint, we will need to split into cases meriting separate consideration. We do so now, and picture the ultimate determinations of those cases below.



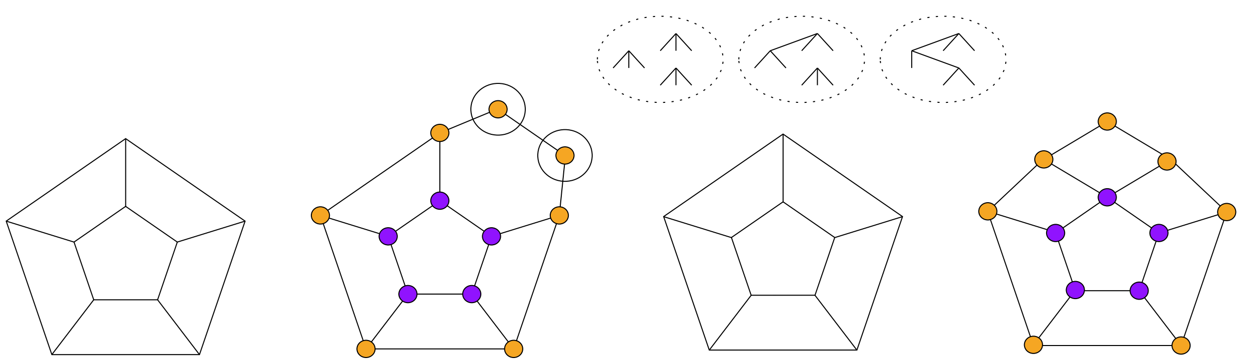
Case 2a : $K_{G,g}$ *has* 20 *edges.* *Case 2b :* $K_{G,g}$ *has* 21 *edges.*

Indeed, the pictured cases above are exhaustive, without loss of generality.

Namely, note that in Case 2a, at least one of the two facial 5-cycles of $K_{G,g}$ has vertices all of degree 3 by Claim 2.21. Then, Corollary 2.24 along with vertex disjointness of the facial 5-cycles ensures that at least one of the faces of degree 5 has neighboring faces as depicted in the two leftmost figures.

Likewise, note that in Case 2b, Claim 2.26 dictates that at least one of the two facial 5-cycles, guaranteed by Corollary 2.29, contains five vertices of degree 3 or one vertex of degree 4 and four vertices of degree 3. Then, the fact that all faces other than those of degree 5 have degree 4 and the fact that the two faces of degree 5 are disjoint implies that at least one of the faces of degree 5 in $K_{G,g}$ has neighboring faces as depicted in one of the two rightmost figures.

Now, we see that each case leads to a contradiction. See the following figure, marked with color for ease of reading.



Case 2a : $K_{G,g}$ *has* 20 *edges.* *Case 2b :* $K_{G,g}$ *has* 21 *edges.*

Namely, in the leftmost figure, we note that there exist three vertices not pictured and five edges not pictured. Note that since $K_{G,g}$ is triangle free, there are at most two edges among the three vertices not pictured in $K_{G,g}$. Additionally, of the five edges not pictured, none can lie among the 10 vertices pictured since then two vertices would have degree 4, a contradiction to Claim 2.21. So, at least three of the edges not pictured must have one endpoint among the vertices pictured and one endpoint among those not-pictured. This provides a contradiction, since Claim 2.21 implies that there may be at most one edge between the vertices pictured and those not pictured since otherwise, one obtains more than one vertex of degree 4 or a vertex of degree at least 5.

In the figure second from the left, we note that there is one vertex not pictured and three edges not pictured. By virtue of working within Case 2, all vertices $v \in K_{G,g}$ have degree at least 3. So, note that the two circled vertices each have another neighbor in $K_{G,g}$. At most one of them can have a neighbor among the seven orange vertices since otherwise a triangle is formed. Also, neither can have a neighbor among the five purple vertices due to the fact that the graph in the figure is drawn such that all cycles with empty interior as drawn are indeed facial in the fixed embedding of $K_{G,g}$. Furthermore, at most one of the two circled vertices can have the vertex not pictured as a neighbor since otherwise a triangle is formed. So, indeed we see that one of the two circled vertices has a neighbor among the orange vertices and the other has the vertex not pictured as a neighbor. Thus, we see at this point that one of the 12 pictured vertices has degree 4 in $K_{G,g}$ and as such Claim 2.21 dictates that the vertex not pictured has degree 3 in $K_{G,g}$. So, it must have two neighbors other than the established neighbor among the two circled vertices. This, however provides a contradiction since it would either force a vertex of degree at least 5 or more than one vertex of degree 4.

In the figure second from the right, we note that there are three vertices not pictured and six edges not pictured. We will get a contradiction by showing that the fact that all vertices in $K_{G,g}$ there are actually at least seven edges not pictured. In particular, the subgraph of $K_{G,g}$ is isomorphic to either (1) the graph on 3 vertices with no edges, (2) the graph on 3 vertices with one edge, or (3) the graph on 3 vertices with two edges. Now, the fact that the three unpictured vertices each have degree at least 3 in $K_{G,g}$ by virtue of being in Case 2, implies that the number of edges in $K_{G,g}$ with one endpoint among the three unpictured and the other endpoint among the 10 pictured is (1) at least 9 in the first case, (2) at least 7 in the second case, and (3) at least 6 in the third case. Then, in each case adding up the number of edges with both endpoints among the three unpictured vertices and the number of edges with one endpoint pictured and the other unpictured we see that the number of unpictured edges is (1) at least 9 in the first case, (2) at least 8 in the second case, and (3) at least 7 in the third case. So, all three possibilities provide a contradiction to the fact that there are 6 unpictured edges.

In the rightmost figure, there is one vertex not pictured and three edges not pictured. Since the unpictured vertex must have degree at least 3 by virtue of being in Case 2, all three of the unpictured edges have the unpictured vertex as one endpoint and one of the 12 pictured vertices as the other endpoint. Due to the way the graph is drawn, namely so that all cycles with empty interiors as drawn are facial in the plane graph $K_{G,g}$, we have that all three neighbors of the unpictured vertex lie among the orange vertices, since otherwise if any neighbors lied among the purple vertices, then the unpictured vertex would lie embedded in one of pictured faces f of degree 4 or 5 in the figure, at which point the fact that this vertex has degree 3 would result in the face f being further divided into smaller faces, a contradiction. ∎