Math 6014 - Homework 3

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1. We proceed by induction. First we consider the base case. If k = 2 and we have a graph on at least 4 vertices, we wish to find a cycle of length at least 4. Pick two vertices in G. By Menger's theorem there exist two internally disjoint paths between them. The union of these two paths is a cycle of length at least 3. If it has length greater than 4, we are done. Otherwise, call the vertices on our cycle u, v, and w. Then, pick a 4th vertex, called y. My Menger's Theorem there exist at least two paths (called p_1 from y to one of u, v, and w and p_2 from y to one of u, v, and w) disjoint except for y. Since there are only 3 vertices on this cycle, the two paths each ending at one of these vertices must end at adjacent vertices, called u and w. Taking the union of the cycle portion $\{u, uv, v, vw, w\}$ with p_1 and p_2 gives a cycle of length at least 4. Now, we consider the inductive step. Say the statement holds for $l \leq k-1$ and we wish to show the statement holds for connectivity k. Say we have a graph with connectivity k and at least 2k vertices. Delete a vertex. The remaining graph G' is k-1-connected. By the inductive hypothesis it has a cycle of length at least 2(k-1). If such a cycle has length 2k we are done. Otherwise it has length 2k-2 or 2k-1. Say it has length 2k-2. Consider this cycle sitting inside the original graph G. Pick another vertex z in G. Since G is k-connected Menger's theorem says that there exist k paths disjoint except for z from z to this cycle C. Since $k > \frac{2k-2}{2}$, some two of these paths p_1 and p_2 from z to C will end at adjacent vertices, called u and w. Consider C' = $C[u, w] \cup p_1 \cup p_2$ where C[u,w] is the part of C from u to w including other vertices. C' is a cycle of length at least 2k-1. If it has length 2k or greater we are done. Otherwise, we pick another vertex r in G. Since G is k-connected, there exist k paths from r to C' disjoint except at r. Since k > $\frac{2k-1}{2}$, some two of these paths q_1 and q_2 will end at two adjacent vertices, say s and t on C'. Taking the union C" = $C'[s, t] \cup q_1 \cup q_2$ gives a cycle of length at least 2k and we are done.

2. We wish to show that every edge in a 3-connected graph is contained in two induced, non-separating cycles. We prove this claim by induction on the number of vertices. As shown in class, every 3-connected graph has a contractible edge, and in particular, if one keeps contracting contractible edges of a 3-connected graph, one will be left with K_4. So, our base case is K_4. By symmetry, every edge of K_4 is the same. So, we notice that, for a given edge of K_4, it is contained in exactly 2 triangles, each of which is induced and non-separating.
   Now, we assume that the statement holds for 3-connected graphs with n-1 vertices and we wish to show that the statement holds for 3-connected graphs with n vertices (where $n \geq 5$). We know that our 3-connected graph with n vertices has a contractible edge, xy. Contract that edge of G to get $G' = (V(G) \setminus \{x, y\} \cup \{e^*\}, E(G) \setminus \{vx \in E(G)\} \setminus \{vy \in E(G)\} \cup \{ve^* | vx \in E(G)\} \cup \{ve^* | vy \in E(G)\})$. G' is also 3-connected and by our inductive hypothesis, each edge of G' is contained in 2 induced, non-separating cycles.
   These induced, non-separating cycles of G' which do not include e^* are also induced, non-separating cycles of G. However, other cycles, C', are not induced in G when uncontracted. Such cycles must contain e*. When such C' is uncontracted, there is some chord, and such a chord must intersect $\{x, y\}$ since otherwise it was not a chord in C'. This chord must, therefore, be the edge xy itself or an edge of the form xs or yr where s is adjacent to y on the cycle C, which is the result of uncontracting C', and r is the vertex adjacent to x on C. It turns out that the chord can't be xy, since that would imply that our cycle C in G contains 2 edges adjacent to y other than xy and also 2 edges adjacent to x other than xy. However, such a graph is not a cycle when xy is contracted to get C' since e^* would have degree 4 in C', which implies that C' wasn't a cycle to begin with in G'. Without loss of generality, say that the chord is of the form xs. (The case where the chord is yr is the same by symmetry as the following

argument). The end vertex s must lie on the cycle C' in order for xs to be a chord in C and furthermore s must be the vertex adjacent to y which is not x. Otherwise, it was a chord to begin with, or it was the edge adjacent to x on C' other than xy, which is still not a chord in C. So, our picture looks like Figure (a). This cycle was to be a cycle containing some edge on C'. However, the uncontraction, C, may not be induced as described earlier. In the case that the edge to be contained was xr or some edge contained on C' other than xy or ys, we reroute the uncontraction C to an induced cycle C* which still contains that edge. In particular, we let $C* = C - \{y\} - \{xy, ys\} \cup \{xs\}$. Likewise, if the original cycle C' was to be one that included ys, we let the triangle $\{x, y, s\}$ be our cycle C**. Now, as removing C' from G' did not result in a disconnected graph, the removal of C* or C** in only possibly disconnects G if C* or C** only contains one of x or y and all neighbors of the one not contained in C* or C** lie on C in G. Why? Well, G' was 3 connected which implies that removing e^* and s from G' does not disconnect G'. That means that removing x,y, and s (the vertices of C**) does not disconnect G. In the other case, if removing C' did not disconnect G', then removing C* does not disconnect G except for the case just mentioned, as a path between any two vertices in G'-C' which does not pass through e^* is still a valid path in G - C*. There are no paths between vertices in G' - C' which pass through e^* as e^* is not in G'-C'. However, one needs to consider whether there exist paths from y to other vertices in G-C*. If there are not such paths, then y must lie in some component of G-C* separate from other vertices. As there are paths between any 2 vertices in G-C* both of which are not y, this means that y must be an isolated vertex of G-C*. This implies that all neighbors of y in G belong to C*. Since G is 3-connected, the minimum degree of G is at least 3, which implies that y has some neighbor on C* other than x or s. That neighbor must be r. Otherwise, if it is some other vertex t, then e^*t was a chord in C', which is impossible. So, in this case, where C* turns out to be separating, instead of using the cycle C*, if this cycle was to include xr, we use the cycle C*** = {xr, ry, yx}. Otherwise if it was to include some edge of C other than xy or xr, we use C**** $= C - \{x\} - \{xy, xr\} \cup \{yr\}$. Why is C**** non-separating? If it is separating then $N(x) \subseteq C$ and in particular, by similar logic as before, $N(x) = \{y, r, s\}$. If C**** and C* are both separating then $\{s, r\}$ is a 2-cut in G, a contradiction. (We can do the analogous procedure if we have a cycle D' containing ys which when uncontracted gives a separating cycle D which isolates x in G. In this case, similarly to the case just outlined we actually have both chords, yr and xs in G and we let D*** $= \{yx, xs, sy\}$ if ys was the edge to be contained or if it was to contain some other edge on C except xy or ys, we let D**** $= C - \{y\} - \{xy, ys\} \cup \{xs\}$). This (C***, C****, D*** or D****) is an induced cycle and it is non-separating. The process just outlined gives us a way of constructing one induced, non-separating cycle C containing a given edge e of G from a induced, non-separating cycle C' containing e of G'. We repeat the process on the second cycle E' of G' to get E of G. Can the cycles C' and E' be distinct but result in the same cycle C = E in G? No. To get two induced, non-separating cycles for the edge xy, instead of contracting xy, contract another contractible edge, st. By the same argument given in class, one can get that there are at least two contractible edges in a 3-connected graph with at least 5 vertices. By the arguments outlined above, applied instead to the contraction of st, we get two induced, non-separating cycles of G which contain xy.

3. Assume all edges are not contractible. Pick an edge e = e_1e_2. Since it is not contractible, when contracted the resulting graph has a k-1 cut, which means that e was contained in a k-cut of the original graph G. Let E be that k-cut. Pick that edge e such that the size of the smallest component of $G - E$ is minimized. Call that smallest component C_e. Now, consider some edge f = e_1f_1 for which $f_1 \notin E$. (Such an edge exists since $deg(e_1) \geq k$). By assumption, f is also contained in some k-cut. Call that cut F. Now, consider the attached diagram.

   Lemma: At least one of S_1 or S_9 is empty.
   Proof: Assume S_1 is non-empty. Then, $|S_2| + |S_5| + |S_4| \geq k + 1$ since otherwise $S_2 \cup S_3 \cup S_4$ would either be an l-cut with $l \leq k$ or it would be a k-cut resulting in a component S_1 strictly smaller than $C_e := S_1 \cup S_4 \cup S_7$ which was chosen (we chose the cut set E) such that C_e is of minimum size. Now, assume S_9 is non-empty. Then, $|S_6| + |S_5| + |S_8| \geq k$ since otherwise $S_6 \cup S_5 \cup S_8$ is an l-cut with $l < k$. (We know that such a set is a cut since there are no edges between S_9 and S_i for $i \in [9] \setminus \{4, 5, 6, 9\}$). Now, combining $|S_2| + |S_5| + |S_4| \geq k + 1$ and $|S_6| + |S_5| + |S_8| \geq k$, we get that

$|S_2|+|S_5|+|S_4|+|S_6|+|S_5|+|S_8| \geq 2k+1$ which is a contradiction as $|S_2|+|S_5|+|S_4|+|S_6|+|S_5|+|S_8| = 2k$ since $S_2 \cup S_5 \cup S_8$ and $S_4 \cup S_5 \cup S_6$ are each of size k. So, at least one of S_1 or S_9 is empty.

Lemma: At least on of S_3 or S_7 is empty.
Proof: Same proof by symmetry as proof of previous lemma.

Lemma: The set S_7 is empty.
Proof: We first note that $|S_2| + |S_5| + |S_8| = k$ implies that $|S_5| + |S_8| = k - |S_2|$. Now, assume S_7 is non-empty. Then, $|S_4|+|S_5|+|S_8| \geq k+1$ since otherwise $S_4 \cup S_5 \cup S_8$ is an l-cut with $l < k$ or a k-cut resulting in a component, namely S_7, of size strictly smaller than $C_e = S_1 \cup S_4 \cup S_7$, a contradiction in either case. So, we have that $|S_4|+|S_5|+|S_8| = |S_4|+k-|S_2| \geq k+1$, which implies that $|S_4| \geq |S_2|+1$. Now, since C_e is of minimum size (over all possible components resulting from some k-cut), we know that $|S_1| + |S_4| + |S_7| \leq |S_1| + |S_2| + |S_3|$, which implies $|S_4| + |S_7| = |S_2| + r + |S_7| \leq |S_2| + |S_3|$ for some $r > 0$. So, that means that $|S_7| < |S_3|$. Now, as stated before, at least one of S_3 or S_7 is empty. So, if S_7 is non-empty, then S_3 is empty. However, we just showed that if S_7 is non-empty, then $|S_7| < |S_3| = 0$, which gives us a contradiction. Thus, S_7 is empty.

Lemma: The component C_e has at least k vertices.
Proof: Clearly, C_e must have at least one vertex. Otherwise, E was not a cut. Now, C_e must have at least 2 vertices. Why? Because otherwise, if C_e consists of a single vertex, x, we know that $deg(x) \geq k$ since G is k-connected. There are no edges between S_4 and any of S_3, S_6,or S_9, so all edges originating at x must end in vertices of $S_2 \cup S_5 \cup S_8$, of which there are exactly k. However, since by construction there was an edge contained in $E = S_2 \cup S_5 \sup S_8$, such a graph has a triangle, a contradiction. So, $|C_e| \geq 2$ which means that it has at least two vertices, called a and b. They must be adjacent, since otherwise, by the previous argument, each of a or b is adjacent to all of E, creating a triangle. So, a and b are adjacent, and since there are no triangles, a and b have no common neighbor. More precisely, $|N(a) \cap N(b)| = 0$. Now, since $|N(a)| \geq k$ and $|N(b)| \geq k$, that means that $|N(a) \cup N(B)| = |N(a)| + |N(b)| - |N(a) \cap N(b)| \geq 2k$. Then, since $N(a) \cup N(b) \subseteq C_e \cup E$, $|C_e \cup E| = |C_e| + k \geq 2k$, which implies that $|C_e| \geq k$.

Now, we consider 4 cases: (1) $|S_1| = 0$ and $|S_3| \geq 0$, (2) $|S_1| = 0$ and $|S_3| = 0$, (3) $|S_1| \geq 0$ and $|S_3| = 0$, (4) $|S_1| \geq 0$ and $|S_3| \geq 0$.

Consider Case (1). We first prove the following lemma.
Lemma: The set S_3 non-empty implies that $|S_2| \geq |S_4|$.
Proof: If we have that $|S_3| \geq 0$, then $|S_2|+|S_5|+|S_6| \geq k$. Now, $k = |S_4|+|S_5|+|S_6|$ since $S_4 \cup S_5 \cup S_6$ is a k-cut. So, $|S_2| + |S_5| + |S_6| \geq |S_4| + |S_5| + |S_6|$, which implies the desired result.
Now, $|S_1| = 0$ implies that $|S_1| + |S_4| + |S_7| = |S_4| = k \leq |S_2|$ which means that $|S_2| = k$. Since $|S_2| = |S_4| = k$, we know that $S_5, S_6, S_8 = \emptyset$. This implies that $|S_9| = 0$, since otherwise S_9 is not connected to S_1 in the original graph as there are no edges between S_9 and S_3 or between S_9 and S_2 or between S_9 and S_4. So, S_7, S_8 and S_9 are empty, which is a contradiction since that would imply that $F = S_4 \cup S_5 \cup S_6$ is not a cut.

Next, we consider Case (2). If S_3, S_1 both empty, then $k \leq |S_1| + |S_4| + |S_7| = |S_4|$ implies $|S_4| = k$ and since $k = |S_4| = |S_1| + |S_4| + |S_7| \leq |S_1| + |S_2| + |S_3| = |S_2|$, we get that $|S_2| \geq k$ and since $|S_2| + |S_5| + |S_8| = k$, we get that exactly $|S_2| = k$. This means that S_5, S_6, S_8 all empty. By the same argument given in Case (1), S_9 is also empty. So, S_7, S_8, S_9 are all empty, which contradicts the fact that F was a cut.

Now, we consider Case (3). If S_1 is non-empty, then S_9 is empty by the first lemma. Also, $|C_e| = |S_1| + |S_4| + |S_7| = |S_1| + |S_4| \geq k = |S_4| + |S_5| + |S_6|$, which implies that $|S_1| \geq |S_5| + |S_6|$. Since S_1 is non-empty we know that $|S_2| + |S_4| + |S_5| \geq k + 1$ since otherwise C_e was not of minimum size. So, $|S_2| + |S_4| + |S_5| \geq k + 1 \geq k = |S_2| + |S_5| + |S_8|$, which implies that $|S_4| \geq |S_8|$. However, C_e of minimum size means that $|S_1| + |S_4| + |S_7| = |S_1| + |S_4| \leq |S_7| + |S_8| + |S_9| = |S_8|$, which implies that

$|S_4| \leq |S_8|$ since $|S_1| \geq 0$. This is a contradiction.

Now, we consider Case (4). If both S_1 and S_3 are non-empty, we get by the first two lemmas that $|S_7| = |S_9| = 0$. So, $k \leq |S_1| + |S_4| + |S_7| = |S_1| + |S_4| \leq |S_7| + |S_8| + |S_9| = |S_8|$, which implies that $|S_8| \geq k$. In particular, $|S_8| = k$ as it is totally contained in a k-cut, which has size k. This implies that $|S_2| = |S_5| = 0$. Now, we have two subcases. Either $|S_6| \leq k$ or $|S_6| = k$. If $l := |S_6| \leq k$, then we get a contradiction as S_6 is an l-cut since it separates S_3 from the rest of the graph. If $|S_6| = k$, then $|S_4| = 0$. This provides a contradiction as $|S_4| = 0$ and $|S_2| = |S_5| = 0$ implies that S_1 was not in the same connected component as the rest of G to begin with.
So, in each case, we get a contradiction, and there is some contractible edge.