OPERATOR THEORY HOMEWORK 5

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of problems will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together. Hand in your solutions in class on the due date.

1. Problem 16.3.18. Fix $k \in L^2(\mathbb{R}^2)$. By Theorem 15.8.7, the integral operator L_k with kernel k is a bounded linear map of $L^2(\mathbb{R})$ onto itself. Prove that the adjoint operator $(L_k)^*$ is the integral operator L_{k^*} whose kernel is $k^*(x,y) = \overline{k(y,x)}$.

- It suffices to show that $\langle L_k f, g\rangle = \langle f, L_{k^*} g\rangle$ for all $f, g \in L^2(\mathbb{R})$. Then, by uniqueness of the adjoint, we know that $L_{k^*} = L_k^*$.
- Note that $\langle L_k f, g\rangle = \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} k(x,y)f(y)dy)\overline{g(x)}dx = \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} k(x,y)f(y)\overline{g(x)})dydx$.
- Likewise note that

$$\begin{aligned}
\langle f, L_{k^*} g\rangle &= \int_{-\infty}^{\infty} f(x)\overline{\int_{-\infty}^{\infty} \overline{k(y,x)}g(y)dy}dx \\
&= \int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} k(y,x)\overline{g(y)dy}dx = \int_{-\infty}^{\infty} f(x)(\int_{-\infty}^{\infty} k(y,x)\overline{g(y)}dy)dx \\
&= \int_{-\infty}^{\infty} f(y)(\int_{-\infty}^{\infty} k(x,y)\overline{g(x)}dx)dy = \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} k(x,y)f(y)\overline{g(x)}dx)dy
\end{aligned}$$

Now, we would like to use Fubini's Theorem meaning we would need to show that $\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} |k(x,y)f(y)\overline{g(x)}|dA < \infty$. So, note that $(|k(x,y)| - |f(y)\overline{g(x)}|)^2 \geq 0$ meaning that $|k(x,y)|^2 + |f(y)\overline{g(x)}|^2 - 2|k(x,y)f(y)\overline{g(x)}| \geq 0$ or that $|k(x,y)f(y)\overline{g(x)}| \leq \frac{1}{2}(|k(x,y)|^2 + |f(y)\overline{g(x)}|^2)$. Now further interating our previous process we know that $0 \leq (|f(y)| - |\overline{g(x)}|)^2 = |f(y)|^2 + |\overline{g(x)}|^2 - 2|f(y)\overline{g(x)}|$ or that $|f(y)\overline{g(x)}| \leq \frac{1}{2}(|f(y)|^2 + |\overline{g(x)}|^2)$. So, we know that $|k(x,y)f(y)\overline{g(x)}| \leq \frac{1}{2}(|k(x,y)|^2) + \frac{1}{2}|f(y)|^2|\overline{g(x)}|^2$ and thus

$$\begin{aligned}
\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} |k(x,y)f(y)\overline{g(x)}|dA &\leq \frac{1}{2}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}(|k(x,y)|^2 + |f(y)|^2|\overline{g(x)}|^2)dA \\
&= \frac{1}{2}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} |k(x,y)|^2 dA + \frac{1}{2}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}(|f(y)|^2|\overline{g(x)}|^2)dA.
\end{aligned}$$

Now, by Tonelli we know that

$$\frac{1}{2}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}(|f(y)|^2|\overline{g(x)}|^2)dA = \frac{1}{2}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}(|f(y)|^2|\overline{g(x)}|^2)dxdy = \frac{1}{2}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}(|f(y)|^2|\overline{g(x)}|^2)dydx$$

and of course

$$\frac{1}{2}\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty}(|f(y)|^2|\overline{g(x)}|^2)dydx = \frac{1}{2}(\int_{-\infty}^{\infty}|\overline{g(x)}|^2dx)(\int_{-\infty}^{\infty}|f(y)|^2dy)$$

since $\int_{-\infty}^{\infty} |f(y)|^2 dy =: A < \infty$ and $\int_{-\infty}^{\infty} |\overline{g(x)})|^2 dx = \int_{-\infty}^{\infty} |g(x))|^2 dx =: B < \infty$ that means

$$\frac{1}{2}(\int_{-\infty}^{\infty} |\overline{g(x)}|^2 dx)(\int_{-\infty}^{\infty} |f(y)|^2 dy) \leq \frac{1}{2}AB < \infty.$$

Also, $k \in L^2(\mathbb{R}^2)$ implies that $\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} |k(x,y)|^2 dA =: C < \infty$
and thus $\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} |k(x,y)f(y)\overline{g(x)}| dA \leq \frac{C}{2} + \frac{1}{2}AB < \infty$ meaning that Fubini's Theorem applies. Thus, $\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} k(x,y)f(y)\overline{g(x)}dx)dy = \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} k(x,y)f(y)\overline{g(x)}dy)dx$ meaning that $\langle f, L_{k^*}g\rangle = \langle L_k f, g\rangle$ for all $f, g \in L^2(\mathbb{R})$ which implies by uniqueness of the adjoint that $L_{k*} = (L_k)^*$.

2. Problem 16.3.20. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence for a Hilbert space H, and let $T\colon H \to \ell^2$ be the *analysis operator* $Tx = \{\langle x, e_n\rangle\}_{n\in\mathbb{N}}$ introduced in Example 16.1.3.

(a) Explicitly identify the *synthesis operator* $T^*\colon \ell^2 \to H$, i.e., find an explicit formula for T^*c for each $c \in \ell^2$.

- We define $T' : \ell^2 \to H$ by $\{c_n\}_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} c_n e_n$.
- Then, we must show that T' is the adjoint of T which amounts to showing that $\langle Tx, c\rangle = \langle x, T'c\rangle$ for all $x \in H$ and all $c \in \ell^2$.
- We do so by direct computation.

$$\langle Tx, c\rangle = \sum_{k=0}^{\infty} \langle x, e_k\rangle \overline{c_k}$$

$$\langle x, T'c\rangle = \langle x, \sum_{k=0}^{\infty} c(k)e_k\rangle = \sum_{k=0}^{\infty} \overline{c_k}\langle x, e_k\rangle$$

meaning $\langle Tx, c\rangle = \langle x, T^*c\rangle$ for all $x \in H$ and all $c \in \ell^2$ and thus $T' = T^*$.

(b) Find explicit formulas for T^*T and TT^*.
Then, $T^*Tx = T^*(\{\langle x, e_n\rangle\}_{n\in\mathbb{N}}) = \sum_{k\in\mathbb{N}} \langle x, e_k\rangle e_k$.
Also, $TT^*c = T(\sum_{k\in\mathbb{N}} c_k e_k) = \{\langle(\sum_{k\in\mathbb{N}} c_k e_k), e_n\rangle\}_{n\in\mathbb{N}} = \{\langle c_n e_n, e_n\rangle\}_{n\in\mathbb{N}} = \{c_n\}_{n\in\mathbb{N}} = c$.

(c) When will T^*T be the identity operator?
T^*T is the identity operator when $span(\{e_n\}_{n\in\mathbb{N}}) = H$.

3. Problem 16.3.22. This problem will establish the Mean Ergodic Theorem. Let $U \in \mathcal{B}(H)$ be a unitary operator on a Hilbert space H. Set $U^0 = I$ (the identity operator), and for each integer $N > 0$ define

$$S_N = \frac{1}{N}\sum_{k=0}^{N-1} U^k.$$

Let $M = \ker(U - I)$, and let P be the orthogonal projection of H onto M. Prove the following statements.

(a) $M^\perp = \overline{range}(U-I)$.
First, we show that $range(U-I) \subseteq M^\perp$. Consider arbitrary $h \in range(U-I)$ and also arbtirary $m \in ker(U-I)$. Now, note that we need to show that $\langle (U-I)x, m\rangle = 0$. Also, note that $\langle (U-I)x, m\rangle = \langle Ux, m\rangle - \langle x, m\rangle$, but since $m \in ker(U-I)$ we know that $Um = m$ meaning that $\langle Ux, m\rangle = \langle Ux, Um\rangle$ and since U is unitary we also know that $\langle Ux, Um\rangle = \langle x, m\rangle$. So, indeed $\langle (U-I)x, m\rangle = \langle Ux, m\rangle - \langle x, m\rangle = \langle x, m\rangle - \langle x, m\rangle = 0$ which shows $range(U-I) \subseteq M^\perp$. Now, by Lemma 8.2.6 we know that $M^\perp$ is a closed subspace of H containing $range(U-I)$ and since $\overline{range}(U-I) = \cap\{W : W \subseteq H \text{ closed and } range(U-I) \subseteq W\}$. Thus, one has that $\overline{range}(U-I) \subseteq M^\perp$. Now, we wish to show the reverse inclusion, namely that $M^\perp \subseteq \overline{range}(U-I)$. To show this it suffices to show that $range(U-I)^\perp \subseteq (M^\perp)^\perp = M$. (Note that $\overline{range}(U-I)^\perp \subseteq range(U-I)^\perp$). Well, consider $y \in range(U-I)^\perp$ and then we wish to show that $Uy = y$ which holds if and only if $y = U^{-1}y$ (since U surjective by defintion of a unitary operator). Well, we know that for all $w \in range(U-I)$ we have $\langle y, w\rangle = 0$. Now, note that $\langle y, w\rangle = 0$. Now, since $w \in range(U-I)$ one has $v \in H$ such that $(U-I)v = w$ and thus $\langle y, w\rangle = \langle y, Uv - v\rangle = \langle y, Uv\rangle - \langle y, v\rangle = \langle U^{-1}y, v\rangle - \langle y, v\rangle = \langle U^{-1}y - y, v\rangle = 0$. In fact for all $v \in H$ one considers $\langle y, Uv - v\rangle = 0$ which as above gives $0 = \langle U^{-1}y - y, v\rangle$ for all $v \in H$ meaning that $U^{-1}y - y = 0$ or that $U^{-1}y = y$ which implies $y = Uy$ as desired.

(b) $S_N x \to Px$ for each $x \in M$ and each $x \in range(U-I)$.

First, note that if $x \in M$ then $Ux = x$ meaning that $S_N x = \frac{1}{N}\sum_{n=0}^{N-1} x = x$. Also, since $x \in M$ one has that $Px = x$. Thus, $S_N x = Px$ for any $N \in \mathbb{N}$. Now, suppose that $x \in range(U-I)$ meaning there exists $w \in H$ such that $x = Uw - w$. Then, one notes that $S_N x = \frac{1}{N}\sum_{n=0}^{N-1} U^{n+1}w - \frac{1}{N}\sum_{n=0}^{N-1} U^n w = \frac{1}{N}\sum_{n=1}^{N} U^n w - \frac{1}{N}\sum_{n=0}^{N-1} U^n w = \frac{1}{N}(U^N w - w)$ and note that $||S_N x|| = ||\frac{1}{N}(U^N w - w)|| \leq \frac{1}{N}(||U^N w|| + ||w||) = \frac{1}{N}(2||w||) \to 0$ as $N \to \infty$ meaning that $S_N x \to 0$ as $N \to \infty$. Finally, since $x \in range(U-I) \subseteq M^\perp$ one knows that $Px = 0$. Thus, $S_N x \to Px$ for $x \in range(U-I)$ and we are done.

(c) $S_N x \to Px$ as $N \to \infty$ for each $x \in H$ (and therefore, using the terminology of Section 15.5.1, $S_N \to P$ *in the strong operator topology*).

Note that if $x \in H$ then $x = Px + e$ where $e = x - Px \in M^\perp$. Also, note that $Px \in ker(U-I)$ for all $x \in H$. So, $S_N x = S_N(Px + e) = S_N Px + S_N e$. Also, note that $e \in M^\perp$ implies that $e \in \overline{range}(U-I)$. Thus, for fixed $\epsilon > 0$ there exists a sequence $\{y_n\}_{n \in \mathbb{N}} \subseteq range(U-I)$ such that $||e - y_n|| \to 0$ as $n \to \infty$. Then, note that $S_N e = S_N y_n + S_N(e - y_n)$ and now note that $||S_N(e-y_n)|| = ||\frac{1}{N}\sum_{k=0}^{N-1} U^k(e-y_n)|| \leq \frac{1}{N}\sum_{k=0}^{N-1} ||U^k(e-y_n)|| = \frac{1}{N}\sum_{k=0}^{N-1} ||e - y_n|| = ||e - y_n||$. Then, to show $S_N e = S_N y_n + S_N(e - y_n) \to 0$ as $N \to \infty$. We know that since $S_N y_n \to 0$ as $N \to \infty$ we have that there exists $M \in \mathbb{N}$ such that for all $N \geq M$ one has that $||S_N y_n|| < \epsilon/4$. Also, since $||e - y_n|| \to 0$ as $n \to \infty$ we have that there exists $P \in \mathbb{N}$ such that for all $n \geq P$ we have that $||e - y_n|| < \frac{\epsilon}{4}$. Then, that implies that for $n \geq P$ and $N \geq M$ we have that $||S_N e|| = ||S_N y_n + S_N(e - y_n)|| < \frac{\epsilon}{2}$. Finally, since $S_N Px \to PPx = Px$ as $N \to \infty$ we know there exists $Q \in \mathbb{N}$ such that for all $N \geq Q$ we have $||S_N Px - Px|| < \epsilon/2$. Putting that all together we get that for all $N \geq max(P,Q)$ (and $n \geq P$) that $||S_N x - Px|| = ||S_N Px - Px + S_N e|| \leq ||S_N Px - Px|| + ||S_N e|| < \epsilon$ meaning that $S_N x \to Px$ as $N \to \infty$.

4. Problem 16.4.19. Let H and K be Hilbert spaces, and suppose that $A \in \mathcal{B}(H,K)$. Prove the following statements.

(a) $A^*A \in \mathcal{B}(H)$ and $AA^* \in \mathcal{B}(K)$ are self-adjoint and positive.

First, note that since $||A|| = ||A^*||$ one has that $A^* \in \mathcal{B}(K,H)$. Then, note that since $||A^*A|| \leq ||A^*||||A||$ and also $||AA^*|| \leq ||A||||A^*||$ one has that $A^*A \in \mathcal{B}(H)$ and $AA^* \in \mathcal{B}(K)$. They are self adjoint since for all $h_1, h_2 \in H$one has that $\langle A^*(Ah_1), h_2\rangle = \langle Ah_1, Ah_2\rangle = \langle h_1, A^*Ah_2\rangle$. Likewise for all $k_1, k_2 \in K$ one has that $\langle A(A^*k_1), k_2\rangle = \langle A^*k_1, A^*k_2\rangle = \langle k_1, AA^*k_2\rangle$ proving the claim.
Finally, we note that by definition A^*A is positive if and only if $\langle A^*Ah, h\rangle \geq 0$ for all $h \in H$. Now, note that by self adjointness and conjugate symmetry of the dot product one has that $\langle A^*Ah, h\rangle = \langle h, A^*Ah\rangle = \overline{\langle A^*Ah, h\rangle}$ meaning that $\langle A^*Ah, h\rangle \in \mathbb{R}$. Then, note that $\langle A^*Ah, h\rangle = \langle Ah, Ah\rangle = ||Ah||^2 \geq 0$. Likewise one has that $\langle AA^*h, h\rangle = \langle A^*h, A^*h\rangle = ||A^*h||^2 \geq 0$.

(b) A^*A is positive definite $\iff$ A is injective $\iff$ A^* has dense range.

We first show that A^*A positive definite implies that A is injective. In particular since A is linear we need only show that $ker(A) = 0$. So, consider arbitrary $h \in H$ with $h \neq 0$ and note that $\langle A^*Ah, h\rangle > 0$ since A^*A is positive definite and $\langle A^*Ah, h\rangle = \langle Ah, Ah\rangle = ||Ah||^2 > 0$, meaning that $Ah \neq 0$ proving injectivity.
Now, we suppose A is injective and show that A^* has dense range. Suppose for contradiction that the range of A^* was not dense in H. Then, that would imply there exists $h \in H$ with $h \neq 0$ such that $\langle A^*k, h\rangle = 0$ for all $k \in K$. However, then note that $0 = \langle k, Ah\rangle$ for all $k \in K$. However, that then implies that $Ah = 0$, a contradiction to injectivity.
Now, we assume A^* has dense range and aim to show that A is injective. Otherwise, if A were not injective, one would have $h \neq 0$ such that $Ah = 0$. Then, that would imply that $0 = \langle Ah, k\rangle$ for all $k \in K$ which then by definition of the adjoint implies that $0 = \langle h, A^*k\rangle$ for all $k \in K$. However, that then provides a contradiction to $\{A^*k : k \in K\}$ being dense in H.
Finally, we assume that A is injective and show that A^*A is positive definite. For arbitrary $h \in H$ with $h \neq 0$ we consider $\langle A^*Ah, h\rangle = \langle Ah, Ah\rangle = ||Ah||^2$ and now since A is injective that implies that $Ah \neq 0$ which implies $||Ah||^2 > 0$, concluding the proof.

(c) AA^* is positive definite $\iff$ A^* is injective $\iff$ A has dense range.

We simply apply the exact argument above expect switching the roles of A and A^* and also H and K.

5. Problem 16.5.5. We say that a sequence of vectors $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space H is a *tight frame* for H if there exists a number $A > 0$, called the *frame bound*, such that

$$\sum_{n=1}^{\infty}\langle x, x_n\rangle^2 = A\,||x||^2, \qquad \text{for all } x \in H.$$

Every tight frame is a Bessel sequence, and therefore the series

$$Sx = \sum_{n=1}^{\infty}\langle x, x_n\rangle\, x_n$$

converges for every $x \in H$.

Prove that S and $S - AI$ are self-adjoint, where I is the identity operator on H, and use this to show that $S = AI$ (Hint: Corollary 16.4.9). Then prove that the following three statements are equivalent.
First, we show that S is self adjoint by direct computation.

$$\begin{aligned}
\langle Sx, y\rangle &= \langle\sum_{n=1}^{\infty}\langle x, x_n\rangle x_n, y\rangle \\
&= \sum_{n=1}^{\infty}\langle\langle x, x_n\rangle x_n, y\rangle \\
&= \sum_{n=1}^{\infty}\langle x, x_n\rangle\langle x_n, y\rangle \\
&= \sum_{n=1}^{\infty}\langle x, \overline{\langle x_n, y\rangle} x_n\rangle \\
&= \sum_{n=1}^{\infty}\langle x, \langle y, x_n\rangle x_n\rangle \\
&= \langle\sum_{n=1}^{\infty} x, \langle y, x_n\rangle x_n\rangle = \langle x, Sy\rangle,
\end{aligned}$$

for all $x, y \in H$ meaning that $S^* = S$ by uniqueness of the adjoint.
Now, in the case that $\mathbb{F} = \mathbb{C}$ we once again show that $S - AI$ is self-adjoint by direct computation.

$$\begin{aligned}
\langle (S - AI)x, y\rangle &= \langle Sx, y\rangle - \langle Ax, y\rangle \\
&= \langle x, Sy\rangle - \langle x, Ay\rangle \text{ (since } A \in \mathbb{R} \text{ and S self-adjoint)} \\
&= \langle x, (S - AI)y\rangle
\end{aligned}$$

thus proving self-adjointness. Now, note that

$$\begin{aligned}\langle Sx - AIx, x\rangle &= \langle Sx, x\rangle - A\langle x, x\rangle \\ &= \sum_{n=1}^{\infty}\langle\langle x, x_n\rangle x_n, x\rangle - \sum_{n=1}^{\infty}|\langle x, x_n\rangle|^2 \\ &= \sum_{n=1}^{\infty}\langle x, x_n\rangle\langle x_n, x\rangle - \sum_{n=1}^{\infty}|\langle x, x_n\rangle|^2 \\ &= \sum_{n=1}^{\infty}|\langle x, x_n\rangle|^2 - \sum_{n=1}^{\infty}|\langle x, x_n\rangle|^2 = 0.\end{aligned}$$

Now, indeed note that by Theorem 6.4.9 we have that

(a) $||x_n||^2 = A$ for every $n \in \mathbb{N}$.

(b) $\{x_n\}_{n\in\mathbb{N}}$ is an orthogonal (but not necessarily *orthonormal*) sequence with no zero elements.

(c) $\{x_n\}_{n\in\mathbb{N}}$ is a Schauder basis for H, i.e., for each $x \in H$ there exist unique scalars $c_n(x)$ such that $x = \sum_{n=1}^{\infty} c_n(x)x_n$, where the series converges in the norm of H.

To show the above are equivalent, I first show that (b) implies (a).

In particular, for fixed $n \in \mathbb{N}$ take $y_n = \frac{x_n}{||x_n||}$. Then, $A = A||y_n||^2 = \sum_{m=1}^{\infty}|\langle y_n, x_m\rangle|^2 = |\langle y_n, x_n\rangle|^2 = |\langle \frac{x_n}{||x_n||}, x_n\rangle|^2 = \frac{1}{||x_n||^2}||x_n||^4 = ||x_n||^2$. Similarly, to show the reverse implication that (a) implies (b) note that if one once again considers $y_n = \frac{x_n}{||x_n||}$ and then $A = A||y_n||^2 = \sum_{m=1}^{\infty}|\langle y_n, x_m\rangle|^2 = ||x_n||^2 + \sum_{m\in\mathbb{N}, m\neq n}|\langle y_n, x_m\rangle|^2$ but then by hypothesis one has that $||x_n||^2 = A$ which implies $\sum_{m\in\mathbb{N}, m\neq n}|\langle y_n, x_m\rangle|^2 = 0$ which then implies that $\langle y_n, x_m\rangle = 0$ for all $m \in \mathbb{N}$ with $m \neq n$ thus proving that x_n is orthogonal to x_m for all $m \neq n$ and thus $\{x_n\}_{n\in\mathbb{N}}$ is an orthogonal sequence.

We now show that (a) implies (c). First, scale $\{x_n\}_{n\in\mathbb{N}}$ to get the orthonormal sequence $\{y_n\}_{n\in\mathbb{N}} = \{\frac{x_n}{||x_n||}\}_{n\in\mathbb{N}} = \{\frac{x_n}{\sqrt{A}}\}_{n\in\mathbb{N}}$. Now, note that by Theorem 8.3.7 since A is a frame bound we have for all $y \in H$ that $A||y||^2 = \sum_{n=1}^{\infty}|\langle y, x_n\rangle|^2 = \sum_{n=1}^{\infty}|\langle y, \sqrt{A}y_n\rangle|^2 = A\sum_{n=1}^{\infty}|\langle y, y_n\rangle|^2$ meaning that $||y||^2 = \sum_{n=1}^{\infty}|\langle y, y_n\rangle|^2$ so then by Theorem 8.3.7 that means that for each $x \in H$ there exists a unique sequence of scalars $\{d_n\}_{n\in N}$ such that $x = \sum_{n=1}^{\infty} d_n y_n = \sum_{n=1}^{\infty} d_n \frac{x_n}{\sqrt{A}}$ meaning that there is a unique sequence of scalars which are $\{c_n\}_{n\in\mathbb{N}} = \{\frac{d_n}{\sqrt{A}}\}$ such that $x = \sum_{n=1}^{\infty} c_n x_n$.

Lastly I show that (c) implies (b). Namely, fix $m \in \mathbb{N}$ and now since $Sx_m = Ax_m$ we have that $Ax_m = \sum_{n=1}^{\infty}\langle x_m, x_n\rangle x_m$. Now, (c) implies that $\langle x_m, x_n\rangle = 0$ for all $n \neq m$ and that $\langle x_m, x_m \rangle = A \neq 0$ meaning $||x_m||^2 = A \neq 0$ and thus $x_m \neq 0$ for all $m \in \mathbb{N}$. Thus (c) implies (b).