

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of problems will be selected for grading. Write LEGIBLY on the FRONT side of the page only. Upload your solutions to Canvas. This exam is worth a total of 50 points.

1. Problem 20.1.17.

(a) Prove that if  $X$  is an infinite-dimensional Banach space, then every Hamel basis for  $X$  is uncountable.

Otherwise, if it has a countable (Hamel) basis  $\{e_n\}_{n \in \mathbb{N}}$  that means that

$$X = \bigcup_{N=0}^{\infty} \text{span}(\{x_0, x_1, \dots, x_N\}).$$

Then, by the Baire Category Theorem we know that at least one of the sets

$$E_{N'} := \text{span}(\{x_0, x_1, \dots, x_{N'}\})$$

is not nowhere dense. However, that is a contradiction since each space

$$E_N := \text{span}(\{x_0, x_1, \dots, x_N\})$$

is a finite-dimensional subspace which is thus proper and closed and is thus nowhere dense as noted on page 242 of the textbook.

(b) Let  $X$  be an infinite-dimensional Banach space. Suppose that  $M$  is an infinite-dimensional subspace of  $X$  that has a countable Hamel basis. Prove that  $M$  is a meager subset of  $X$ , and  $M$  is not closed.

Note that by part (a) the fact that  $M$  has a countable Hamel basis implies that  $M$  is not complete. The fact that it is not complete implies that it is not closed since incompleteness means that  $\overline{M} \supsetneq M$  and since  $\overline{M} = \bigcap \{Y \subseteq X : Y \subseteq X \text{ closed and } Y \supseteq M\}$  is the smallest closed set containing  $M$  that also means that  $M$  is not closed.

Furthermore,  $M$  is meager since if  $\{h_n\}_{n \in \mathbb{N}}$  is our Hamel basis for  $M$  then

$$M = \bigcup_{N \in \mathbb{N}} \text{span}(\{h_0, h_1, \dots, h_N\})$$

and since  $\text{span}(\{h_0, h_1, \dots, h_N\})$  is a proper (of course  $\text{span}(\{h_0, h_1, \dots, h_N\}) \subseteq M \subsetneq X$ ) since if  $M = X$  then  $M$  would not have a countable Hamel basis), finite-dimensional subspace for each  $N \in \mathbb{N}$  we have that  $\text{span}(\{h_0, h_1, \dots, h_N\})$  is nowhere dense for all  $N \in \mathbb{N}$ , meaning by definition that  $M$  is meager.

(c) Prove that  $C_c(\mathbb{R})$  is a meager, dense subspace of  $C_0(\mathbb{R})$ .

Indeed we see that  $C_c(\mathbb{R})$  is meager since

$$C_c(\mathbb{R}) = \bigcup_{N \in \mathbb{N}} C_N(\mathbb{R})$$

where  $C_N(\mathbb{R}) := \{f \in C_c(\mathbb{R}) : f(x) = 0 \forall x \in \mathbb{R} \text{ such that } |x| > N\}$ . Note that  $C_N(\mathbb{R})$  is nowhere dense in  $C_0(\mathbb{R})$  for each  $N \in \mathbb{N}$  since  $\overline{C_N(\mathbb{R})}$  has empty interior. Otherwise, there would exist an open ball contained in  $\overline{C_N(\mathbb{R})}$  meaning that for some  $f \in \overline{C_N(\mathbb{R})}$  there exists

$r > 0$  such that  $g \in \overline{C_N(\mathbb{R})}$  for all  $g \in C_0(\mathbb{R})$  with  $\|g - f\| < r$ . However, that fails. Namely, take

$$g(x) = \chi_{\|x\| \leq N} f(x) + \max(0, -r|x - (N+1)| + r) + \max(0, -r|x - (-N-1)| + r) \in C_0(\mathbb{R})$$

and note that

$$(g - f)(x) = \max(0, -r|x - (N+1)| + r) + \max(0, -r|x - (-N-1)| + r)$$

and also that

$$\|g - f\| = \sup_{x \in \mathbb{R}} (\max(0, -r|x - (N+1)| + r) + \max(0, -r|x - (-N-1)| + r)) = r$$

yet clearly  $g \notin \overline{C_N(\mathbb{R})}$  since  $g$  is not a limit points of functions  $f_n \in C_N(\mathbb{R})$  because for any sequence  $\{f_n\}_{n \in \mathbb{N}} \subseteq C_N(\mathbb{R})$  one has that

$$(\lim_{n \rightarrow \infty} f_n)(N+1) = \lim_{n \rightarrow \infty} (f_n(N+1)) = \lim_{n \rightarrow \infty} 0 = 0 < r = g(N+1).$$

Thus,  $C_c(\mathbb{R})$  is a meager subspace.

However, it is a dense subspace since for any  $h \in C_0(\mathbb{R})$  we may find a sequence of functions  $g_n \in C_c(\mathbb{R})$  so that

$$\lim_{n \rightarrow \infty} \|g_n - h\|_u = 0.$$

Namely, for  $n \in \mathbb{N}$  let

$$g_n(x) = \chi_{|x| \leq n} h(x) + \chi_{|x| \geq n+1} 0 + \chi_{-n-1 < x < -n} (h(-n))(x+n+1) + \chi_{n < x < n+1} (-h(n))(x-n-1)$$

and note that since  $h \in C_0(\mathbb{R})$  meaning that  $\lim_{|x| \rightarrow \infty} |h(x)| = 0$  we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|g_n - h\|_u &= \lim_{n \rightarrow \infty} \|(\chi_{|x| \leq n} h(x) + \chi_{|x| \geq n+1} 0 + \chi_{-n-1 < x < -n} (h(-n))(x+n+1) \\ &\quad + \chi_{n < x < n+1} (-h(n))(x-n-1)) - h(x)\| \\ &= \lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} |\chi_{|x| \leq n} 0 + \chi_{|x| \geq n+1} (-h(x)) + \chi_{-n-1 < x < -n} ((h(-n))(x+n+1) - h(x)) \\ &\quad + \chi_{n < x < n+1} ((-h(n))(x-n-1) - h(x))| \\ &= \lim_{n \rightarrow \infty} (\max(\sup_{x \in \mathbb{R}: \|x\| \geq n+1} |(-h(x))|, \sup_{-n-1 < x < -n} |(h(-n))(x+n+1) - h(x)|, \\ &\quad \sup_{n < x < n+1} |(-h(n))(x-n-1) - h(x)|)) \\ &= \max\left(\lim_{n \rightarrow \infty} \left(\sup_{x \in \mathbb{R}: \|x\| \geq n+1} |(-h(x))|\right), \lim_{n \rightarrow \infty} \left(\sup_{-n-1 < x < -n} |(h(-n))(x+n+1) - h(x)|\right), \right. \\ &\quad \left. \lim_{n \rightarrow \infty} \left(\sup_{n < x < n+1} |(-h(n))(x-n-1) - h(x)|\right)\right) \end{aligned}$$

but since  $\lim_{|x| \rightarrow \infty} |h(x)| = 0$  that means that for all  $\epsilon > 0$  there exists  $M(\epsilon) \in \mathbb{N}$  such that  $|h(x)| < \epsilon$  for all  $x \in \mathbb{R}$  with  $|x| \geq M(\epsilon)$ . Thus, continuing on we see

$$\begin{aligned}
&\leq \max\left(\epsilon, \lim_{n \rightarrow \infty} \left( \sup_{-n-1 < x < -n} |(h(-n))(x + n + 1)| + |h(x)| \right), \right. \\
&\quad \left. \lim_{n \rightarrow \infty} \left( \sup_{n < x < n+1} |(-h(n))(x - n - 1)| + |h(x)| \right) \right) \\
&\leq \max\left(\epsilon, \lim_{n \rightarrow \infty} \left( \sup_{-n-1 < x < -n} |(h(-n))| + |h(x)| \right), \lim_{n \rightarrow \infty} \left( \sup_{n < x < n+1} |(-h(n))| + |h(x)| \right) \right) \\
&\quad (\text{since } 0 \leq |x + n + 1| \leq 1 \text{ for } x \in [-n - 1, -n] \text{ and } 0 \leq |x - n - 1| \leq 1 \text{ for } x \in [n, n + 1]) \\
&\leq \max(\epsilon, 2\epsilon, 2\epsilon) \\
&\leq 2\epsilon,
\end{aligned}$$

but now since  $\epsilon > 0$  was arbitrary we see that

$$\lim_{n \rightarrow \infty} \|g_n - h\|_u = 0,$$

proving that  $C_c(\mathbb{R})$  is dense in  $C_0(\mathbb{R})$ .

2. Let  $X$  and  $Y$  be Banach spaces. Prove that  $A \in \mathcal{B}(X, Y)$  is surjective if and only if  $\text{range}(A)$  is not a meager subset of  $Y$ .

Hint: Lemma to the Open Mapping Theorem.

First, note that if  $A \in \mathcal{B}(X, Y)$  is surjective, then  $\text{range}(A) = Y$  then by the Baire Category Theorem since  $Y$  is a complete metric space we know that  $Y = \text{range}(A)$  is not a meager subset of  $Y$ .

Now, assume that  $\text{range}(A)$  is not a meager subset of  $Y$ . That means in particular that  $\text{range}(A)$  is not nowhere dense meaning that  $Y \setminus \overline{\text{range}(A)}$  is not dense in  $Y$  meaning that there exists a ball  $B_s^Y(y_0) \subseteq Y \setminus (Y \setminus \overline{\text{range}(A)}) = \overline{\text{range}(A)}$ .

Now, one has that

$$\sup_{z \in B_s^Y(y_0)} \|z\| \leq \sup_{z \in B_s^Y(y_0)} (\|y_0\| + \|z - y_0\|) \leq \|y_0\| + s.$$

Furthermore, the inverse mapping tells us that  $\|A^{-1}\| < \infty$  which means that for all  $z \in B_s^Y(y_0) \cap \text{range}(A)$  we have

$$\|A^{-1}(z)\| \leq \|A^{-1}\| \|z\| \leq \|A^{-1}\| (\|y_0\| + s) =: R$$

meaning that

$$B_s^Y(y_0) \cap \text{range}(A) \subseteq A(B_R^X(0))$$

and thus

$$B_s^Y(y_0) \subseteq \overline{B_s^Y(y_0) \cap \text{range}(A)} \subseteq \overline{A(B_R^X(0))}.$$

Now, from  $X$  construct an auxilliary map  $B : X \rightarrow X$  defined by  $B(x) = Rx$ .

Now, let  $A' = A \circ B$  which means that

$$A'x = A(B(x)) = A(Rx)$$

Now, note that

$$A'(B_1^X(0)) = A(B(B_1^X(0))) = A(B_R^X(0)),$$

which implies that  $A'$  satisfies the hypotheses of Lemma 20.3.3 since

$$\overline{A'(B_1^X(0))} = \overline{A(B_R^X(0))} \supseteq B_s^Y(y_0)$$

which then implies that  $A'(B_1^X(0))$  contains an open ball  $B_r^Y(0)$  for some radius  $r > 0$ .

But now since

$$B_r^Y(0) \subseteq A'(B_1^X(0)) = A(B_R^X(0))$$

We simply note that

$$\text{range}(A) = \cup_{n=1}^{\infty} A(B_{nR}^X(0))$$

and likewise

$$Y = \cup_{n=1}^{\infty} B_{nr}^Y(0)$$

Now, I claim that  $Y \subseteq \text{range}(A)$  which can be shown by noting that

Observation A: For any  $y \in Y$  we have that  $y \in B_{n'r}^Y(0)$  for some  $n' \in \mathbb{N}$ .

Now, I claim that

$$B_{n'r}^Y(0) \subseteq A(B_{mR}^X(0)) \text{ for some } m \in \mathbb{N}. \quad (0.1)$$

Indeed the above holds if and only if

$$A^{-1}(B_{n'r}^Y(0)) \subseteq A^{-1}(A(B_{mR}^X(0))) = B_{mR}^X(0), \quad (0.2)$$

so we show that.

Namely, for any  $y \in B_{n'r}^Y(0)$  we see that  $\|A^{-1}y\| \leq \|A^{-1}\| \|y\| \leq n'r \|A^{-1}\|$ , meaning that

$$A^{-1}(B_{n'r}^Y(0)) \subseteq B_{nr'\|A^{-1}\|}^X(0)$$

So, if one takes  $m \in \mathbb{N}$  such that

$$mR \geq n'r\|A^{-1}\|$$

so say

$$m = \lceil \frac{n'r\|A^{-1}\|}{R} \rceil$$

then we have that

$$B_{nr'\|A^{-1}\|}^X(0) \subseteq B_{mR}^X(0)$$

which shows Equation 0.2 and thus the claim Equation 0.1 which along with Observation A shows that  $Y \subseteq \text{range}(A)$ .

3. Problem 20.2.10, parts (a)–(d).

Let  $\{x_n\}_{n \in \mathbb{N}}$  be a sequence in a Banach space  $X$ . Fix  $1 \leq p \leq \infty$ , and suppose that for every  $\mu \in X^*$  we have that

$$T(\mu) = (\mu(x_n))_{n \in \mathbb{N}} \in \ell^{p'}.$$

We call  $T$  the *analysis operator* associated with the sequence  $\{x_n\}_{n \in \mathbb{N}}$ . Prove the following statements.

(a)  $T: X^* \rightarrow \ell^{p'}$  is bounded and linear.

To show linearity we simply note that

$$\begin{aligned} T(a\mu + b\nu) &= ((a\mu + b\nu)(x_n))_{n \in \mathbb{N}} = (a(\mu(x_n)) + b(\nu(x_n)))_{n \in \mathbb{N}} = a(\mu(x_n))_{n \in \mathbb{N}} + b(\nu(x_n))_{n \in \mathbb{N}} \\ &= aT(\mu) + bT(\nu). \end{aligned}$$

To show boundedness we will apply the Uniform Boundedness Principle.

Note that as stated on page 106 of the textbook, since  $X$  is a Banach space we know that  $X^*$  is as well and of course  $\ell^{p'}$  is a normed space. Now, note that we have a sequence of operators  $A_i: X^* \rightarrow \ell^{p'}$  for  $i \in \mathbb{N}$  defined by

$$A_i(\mu) = (\chi_{k \leq i} \mu(x_k))_{k \in \mathbb{N}}.$$

Now, note that since  $\|(\chi_{k \leq i} \mu(x_k))_{k \in \mathbb{N}}\|_{\ell^{p'}} \nearrow \|(\mu(x_k))_{k \in \mathbb{N}}\|_{\ell^{p'}}$  as  $i \rightarrow \infty$ , we have that

$$\sup_{i \in \mathbb{N}} \|A_i \mu\|_{\ell^{p'}} = \lim_{i \rightarrow \infty} \|A_i \mu\|_{\ell^{p'}} = \|\lim_{i \rightarrow \infty} A_i \mu\|_{\ell^{p'}} = \|T\mu\|_{\ell^{p'}} < \infty$$

So, the Uniform Boundedness Principle implies that

$$\begin{aligned} \sup_{i \in \mathbb{N}} \|A_i\| &= \sup_{\|\mu\|=1} \sup_{i \in \mathbb{N}} \|A_i \mu\|_{\ell^{p'}} = \sup_{\|\mu\|=1} \lim_{i \rightarrow \infty} \|A_i \mu\|_{\ell^{p'}} = \sup_{\|\mu\|=1} \|\lim_{i \rightarrow \infty} A_i \mu\|_{\ell^{p'}} \\ &= \sup_{\|\mu\|=1} \|T\mu\|_{\ell^{p'}} = \|T\| < \infty. \end{aligned}$$

(b) If  $1 \leq p < \infty$  then the series  $\sum_{n=1}^{\infty} c_n x_n$  converges for each sequence  $(c_n)_{n \in \mathbb{N}} \in \ell^p$ , and the *synthesis operator*  $Uc = \sum_{n=1}^{\infty} c_n x_n$  is a bounded map of  $\ell^p$  into  $X$ .

Hint: Hahn–Banach (Corollary 19.1.4).

Denote

$$\sum_{n=1}^N c_n x_n =: S_N.$$

We show convergence of  $\sum_{n \in \mathbb{N}} c_n x_n$  through the following steps:

- We note that for each  $N \in \mathbb{N}$  we may obtain a sequence of operators

$$(\pi(S_N))_{N \in \mathbb{N}} \subseteq B(X^*, F)$$

via the map  $\pi : X \rightarrow X^{**}$  given in the book.

- By definition of  $\pi$  we note that

$$\pi(S_N)(\mu) = \sum_{n=1}^N c_n \mu(x_n).$$

- Then, we know by Holder's Inequality that

$$\sum_{n=1}^{\infty} |c_n \mu(x_n)| = \|(c_n \mu(x_n))_{n \in \mathbb{N}}\|_{\ell^1} \leq \|c_n\|_{\ell^p} \|\mu(x_n)_{n \in \mathbb{N}}\|_{\ell^{p'}} < \infty,$$

Now, since  $F$  is complete we also know by Theorem 1.2.8 that

$$\sum_{n=1}^{\infty} c_n \mu(x_n)$$

converges.

- Thus,

$$\lim_{N \rightarrow \infty} \pi(S_N)(\mu) = \sum_{n=1}^{\infty} c_n \mu(x_n) \neq \infty$$

exists for all  $\mu \in X^*$  which means we can define a function  $S : X^* \rightarrow F$  by

$$S\mu = \sum_{n=1}^{\infty} c_n \mu(x_n).$$

- Then, the Banach-Steinhaus Theorem implies that

$$S \in B(X^*, F).$$

- Now, we further show that

$$\pi(S_N) \rightarrow S$$

in operator norm.

– In particular, note that

$$\|S - \pi(S_N)\|_{op} = \sup_{\|\mu\|=1} |(S - \pi(S_N))(\mu)|.$$

– So, for arbitrary  $\mu \in X^*$  with  $\|\mu\| = 1$  we compute

$$\begin{aligned} |(S - \pi(S_N))(\mu)| &= \left| \sum_{n=1}^{\infty} c_n \mu(x_n) - \sum_{n=1}^N c_n \mu(x_n) \right| \\ &= \left| \sum_{n=N+1}^{\infty} c_n \mu(x_n) \right|. \end{aligned}$$

– Then, we use Holder's Inequality which says that for  $1 \leq p \leq \infty$  whenever  $(a_n)_{n \in \mathbb{N}} \in \ell^p$  and  $(b_n)_{n \in \mathbb{N}} \in \ell^{p'}$  then one has that  $\|(a_n b_n)_{n \in \mathbb{N}}\|_1 \leq \|(a_n)_{n \in \mathbb{N}}\|_{\ell^p} \|(b_n)_{n \in \mathbb{N}}\|_{\ell^{p'}}$ .

In our particular case, we have the sequences

$$(a_n^N)_{n \in \mathbb{N}} = (\chi_{n \geq N} c_n)_{n \in \mathbb{N}} \in \ell^p$$

and

$$(b_n^N)_{n \in \mathbb{N}} = (\chi_{n \geq N} |\mu(x_n)|)_{n \in \mathbb{N}} \in \ell^{p'}$$

where the above are in  $\ell^p$  and  $\ell^{p'}$  respectively since

$$\begin{aligned} \|(\chi_{n \geq N} c_n)_{n \in \mathbb{N}}\|_{\ell^p} &= \left( \sum_{n=1}^{\infty} |\chi_{n \geq N} c_n|^p \right)^{1/p} \\ &\leq \left( \sum_{n=1}^{\infty} |c_n|^p \right)^{1/p} \\ &= \|(c_n)_{n \in \mathbb{N}}\|_{\ell^p} < \infty, \end{aligned}$$

and

$$\begin{aligned} \|(\chi_{n \geq N} |\mu(x_n)|)_{n \in \mathbb{N}}\|_{\ell^{p'}} &= \left( \sum_{n=1}^{\infty} (\chi_{n \geq N} |\mu(x_n)|)^{p'} \right)^{1/p'} \\ &\leq \left( \sum_{n=1}^{\infty} |\mu(x_n)|^{p'} \right)^{1/p'} \\ &= \|(\mu(x_n))_{n \in \mathbb{N}}\|_{\ell^{p'}} \\ &= \|T(\mu)\|_{\ell^{p'}} \text{ (by definition of } T) \\ &\leq \|T\| \|\mu\| \text{ (by definition of operator norm)} \\ &= \|T\| \text{ (since } \|\mu\| = 1) \\ &< \infty \text{ (by part A).} \end{aligned}$$

due to the main hypothesis of the problem statement.

So, we have that

$$\begin{aligned}
\|a_n^N b_n^N\|_1 &= \sum_{n=N}^{\infty} |c_n| |\mu(x_n)| \leq \| (a_n^N)_{n \in \mathbb{N}} \|_{\ell^p} \| (b_n^N)_{n \in \mathbb{N}} \|_{\ell^{p'}} \\
&= \| (\chi_{n \geq N} c_n)_{n \in \mathbb{N}} \|_{\ell^p} \| (\chi_{n \geq N} |\mu(x_n)|)_{n \in \mathbb{N}} \|_{\ell^{p'}} \\
&\leq \| (\chi_{n \geq N} c_n)_{n \in \mathbb{N}} \|_{\ell^p} \|T\| \\
&= \left( \sum_{n=N}^{\infty} |c_n|^p \right)^{1/p} \|T\|.
\end{aligned}$$

All in all, we have concluded that

$$\left| \sum_{n=N}^{\infty} c_n \mu(x_n) \right| \leq \left( \sum_{n=N}^{\infty} |c_n|^p \right)^{1/p} \|T\|$$

for all  $\mu \in X^*$  with  $\|\mu\| = 1$  which means that

$$\sup_{\|\mu\|=1} \left| \sum_{n=N}^{\infty} c_n \mu(x_n) \right| \leq \left( \sum_{n=N}^{\infty} |c_n|^p \right)^{1/p} \|T\|.$$

and since  $\|(c_n)_{n \in \mathbb{N}}\|_{\ell^p} = \left( \sum_{n=0}^{\infty} |c_n|^p \right)^{1/p} < \infty$  we have that

$$\|(c_n)_{n \in \mathbb{N}}\|_{\ell^p}^p = \sum_{n=0}^{\infty} |c_n|^p < \infty$$

meaning that

$$\sum_{n=N}^{\infty} |c_n|^p \rightarrow 0$$

as  $N \rightarrow \infty$  or more precisely, that for all  $\epsilon > 0$  there exists  $N(\epsilon) \in \mathbb{N}$  such that

$$\sum_{n=M}^{\infty} |c_n|^p < \epsilon$$

for all  $M \geq N(\epsilon)$ .

– Finally, to show that

$$\sup_{\|\mu\|=1} \left| \sum_{n=N}^{\infty} c_n \mu(x_n) \right| \rightarrow 0$$

as  $N \rightarrow \infty$ , fix  $\epsilon > 0$ . Now, let  $\epsilon' := (\frac{\epsilon}{\|T\|})^p$ . Now, for  $M \geq N(\epsilon')$  we have that

$$\sum_{n=M}^{\infty} |c_n|^p < \epsilon' = \left( \frac{\epsilon}{\|T\|} \right)^p$$

meaning that

$$\left( \sum_{n=M}^{\infty} |c_n|^p \right)^{1/p} \|T\| < \left( \left( \frac{\epsilon}{\|T\|} \right)^p \right)^{1/p} \|T\| = \epsilon$$



for all  $M \geq N(\epsilon')$  thus proving the statement that  $\sup_{\|\mu\|=1} |\sum_{n=N}^{\infty} c_n \mu(x_n)| \rightarrow 0 \rightarrow 0$  as  $N \rightarrow \infty$ .

– So, putting all that together we get that

$$\|S - \pi(S_N)\|_{op} = \sup_{\|\mu\|=1} |(S - \pi(S_N))(\mu)| = \sup_{\|\mu\|=1} \left| \sum_{n=N+1}^{\infty} c_n \mu(x_n) \right| \rightarrow 0$$

as  $N \rightarrow \infty$ .

- So, the above fact that  $(\pi(S_N))_{N \in \mathbb{N}} \subseteq B(X^*, F)$  is convergent implies that it is Cauchy.
- Then, since  $\pi : X \rightarrow B(X^*, F)$  is a linear isometry that implies that  $(S_N)_{N \in \mathbb{N}} \subseteq X$  is Cauchy.
  - Proof of the claim that  $(\pi(S_N))_{N \in \mathbb{N}}$  Cauchy implies that  $(S_N)_{N \in \mathbb{N}}$  is also Cauchy:
  - Note that for all  $n, m \in \mathbb{N}$  one has that

$$\|\pi(S_n) - \pi(S_m)\|_{op} = \|\pi(S_n - S_m)\|_{op} = \|S_n - S_m\|_X.$$

- Now,  $(\pi(S_N))_{N \in \mathbb{N}}$  Cauchy means that for all  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $n, m \geq N$  one has that

$$\|\pi(S_n) - \pi(S_m)\|_{op} < \epsilon.$$

However, that also implies that for all  $n, m \geq N$  one has that

$$\|S_n - S_m\|_X < \epsilon,$$

proving that  $(S_N)_{N \in \mathbb{N}}$  is also Cauchy.

- Finally, note that since  $X$  is complete the fact that  $(S_N)_{N \in \mathbb{N}}$  is Cauchy means that  $(S_N)_{N \in \mathbb{N}}$  is convergent, meaning that

$$\lim_{N \rightarrow \infty} S_N = \sum_{n=1}^{\infty} c_n x_n \in X$$

exists.

Now, to show boundedness note that

$$\begin{aligned}
\|Uc\| &= \left\| \sum_{n=1}^{\infty} c_n x_n \right\| \\
&= \sup_{\|\mu\|=1} \left| \mu \left( \sum_{n=1}^{\infty} c_n x_n \right) \right| \\
&= \left| \mu' \left( \sum_{n=1}^{\infty} c_n x_n \right) \right| \text{ (for some } \mu' \in X^* \text{ with } \|\mu'\| = 1 \text{ by 19.1.4)} \\
&= \left| \sum_{n=1}^{\infty} c_n \mu'(x_n) \right| \text{ (by linearity and continuity of } \mu') \\
&\leq \sum_{n=1}^{\infty} |c_n| |\mu'(x_n)| \text{ (by the triangle inequality)} \\
&= \|(c_n \mu'(x_n))_{n \in \mathbb{N}}\|_1 \\
&\leq \|(c_n)_{n \in \mathbb{N}}\|_{\ell^p} \|(\mu'(x_n))_{n \in \mathbb{N}}\|_{\ell^{p'}} \text{ (by Holder's Inequality)} \\
&= \|(c_n)_{n \in \mathbb{N}}\|_{\ell^p} \|T(\mu')\|_{\ell^{p'}} \\
&\leq \|(c_n)_{n \in \mathbb{N}}\|_{\ell^p} \|T\| \|\mu'\| \\
&= \|(c_n)_{n \in \mathbb{N}}\|_{\ell^p} \|T\| \text{ since } \|\mu'\| = 1.
\end{aligned}$$

So, indeed we see that  $U$  is a bounded linear operator with  $\|U\| \leq \|T\|$ .

(c) If  $p = \infty$  and  $(c_n)_{n \in \mathbb{N}} \in \ell^\infty$ , then the series  $\sum_{n=1}^{\infty} c_n x_n$  converges *weakly*. That is,  $\sum_{n=1}^{\infty} c_n \mu(x_n)$  converges for each  $\mu \in X^*$ . However, show by example that  $\sum c_n x_n$  need not converge in the norm of  $X$ .

In particular, note that the problem hypothesis states that  $T(\mu) = (\mu(x_n))_{n \in \mathbb{N}} \in \ell^1$ . Then once again Holder's inequality implies that

$$\left| \sum_{n=1}^{\infty} c_n \mu(x_n) \right| \leq \sum_{n=1}^{\infty} |c_n \mu(x_n)| = \|(c_n \mu(x_n))_{n \in \mathbb{N}}\|_1 \leq \|(c_n)_{n \in \mathbb{N}}\|_{\infty} \|\mu(x_n)_{n \in \mathbb{N}}\|_1 < \infty,$$

which means by Theorem 1.2.8 that

$$\sum_{n=1}^{\infty} c_n \mu(x_n)$$

converges since  $F \in \{\mathbb{R}, \mathbb{C}\}$  is complete.

Likewise for the tails

$$\|(\chi_{n \geq N} c_n \mu(x_n))_{n \in \mathbb{N}}\|_1 \leq \|(\chi_{n \geq N} c_n)_{n \in \mathbb{N}}\|_{\infty} \|\chi_{n \geq N} \mu(x_n)_{n \in \mathbb{N}}\|_1$$

$$\leq \| (c_n)_{n \in \mathbb{N}} \|_{\infty} \| \chi_{n \geq N} \mu(x_n)_{n \in \mathbb{N}} \|_1 = \| (c_n)_{n \in \mathbb{N}} \|_{\infty} \left( \sum_{n=N}^{\infty} |\mu(x_n)| \right).$$

Once again we show that the tails go to zero by noting that

$$\left| \sum_{n=N}^{\infty} c_n \mu(x_n) \right| \leq \sum_{n=N}^{\infty} |c_n| |\mu(x_n)| = \| (\chi_{n \geq N} c_n \mu(x_n))_{n \in \mathbb{N}} \|_1 \leq \| (c_n)_{n \in \mathbb{N}} \|_{\infty} \left( \sum_{n=N}^{\infty} |\mu(x_n)| \right).$$

Now, since  $\sum_{n=1}^{\infty} |\mu(x_n)| < \infty$  that implies that for all  $\epsilon > 0$  there exists  $M(\epsilon) \in \mathbb{N}$  such that  $\sum_{n=K}^{\infty} |\mu(x_n)| < \epsilon$  for all  $K \geq M(\epsilon)$ .

Now, to show that  $\left| \sum_{n=N}^{\infty} c_n \mu(x_n) \right| \rightarrow 0$  as  $N \rightarrow \infty$ , we fix  $\epsilon > 0$  and now let  $\epsilon_0 = \frac{\epsilon}{\| (c_n)_{n \in \mathbb{N}} \|_{\infty}}$  and now for all  $K \geq M(\epsilon_0)$  we see that

$$\left| \sum_{n=K}^{\infty} c_n \mu(x_n) \right| \leq \| (c_n)_{n \in \mathbb{N}} \|_{\infty} \left( \sum_{n=K}^{\infty} |\mu(x_n)| \right) < \| (c_n)_{n \in \mathbb{N}} \|_{\infty} \epsilon_0 = \| (c_n)_{n \in \mathbb{N}} \|_{\infty} \frac{\epsilon}{\| (c_n)_{n \in \mathbb{N}} \|_{\infty}} = \epsilon,$$

proving that  $\left| \sum_{n=N}^{\infty} c_n \mu(x_n) \right| \rightarrow 0$  as  $N \rightarrow \infty$ .

Now, the above implies that  $\sum_{n=1}^N c_n \mu(x_n)$  is Cauchy since if one fixes  $\epsilon > 0$  and lets  $\epsilon_1 = \frac{\epsilon}{2\| (c_n)_{n \in \mathbb{N}} \|_{\infty}}$ , then for all  $N, P \geq M(\epsilon_1)$  one has that

$$\begin{aligned} \left\| \left( \sum_{n=1}^N c_n \mu(x_n) \right) - \left( \sum_{n=1}^P c_n \mu(x_n) \right) \right\| &= \left\| \left( \sum_{n=1}^{\infty} c_n \mu(x_n) \right) - \left( \sum_{n=1}^{\infty} c_n \mu(x_n) \right) + \left( \sum_{n=1}^N c_n \mu(x_n) \right) - \left( \sum_{n=1}^P c_n \mu(x_n) \right) \right\| \\ &= \left\| \left( \sum_{n=1}^{\infty} c_n \mu(x_n) \right) - \left( \sum_{n=1}^P c_n \mu(x_n) \right) \right\| + \left\| - \left( \sum_{n=1}^{\infty} c_n \mu(x_n) \right) + \left( \sum_{n=1}^N c_n \mu(x_n) \right) \right\| \\ &\leq \left\| \left( \sum_{n=1}^{\infty} c_n \mu(x_n) \right) - \left( \sum_{n=1}^P c_n \mu(x_n) \right) \right\| + \left\| - \left( \sum_{n=1}^{\infty} c_n \mu(x_n) \right) + \left( \sum_{n=1}^N c_n \mu(x_n) \right) \right\| \\ &= \left\| \left( \sum_{n=1}^{\infty} c_n \mu(x_n) \right) - \left( \sum_{n=1}^P c_n \mu(x_n) \right) \right\| + \left\| \left( \sum_{n=1}^{\infty} c_n \mu(x_n) \right) - \left( \sum_{n=1}^N c_n \mu(x_n) \right) \right\| \\ &= \left\| \sum_{n=P+1}^{\infty} c_n \mu(x_n) \right\| + \left\| \sum_{n=N+1}^{\infty} c_n \mu(x_n) \right\| \\ &< \| (c_n)_{n \in \mathbb{N}} \|_{\infty} \left( \sum_{n=P+1}^{\infty} |\mu(x_n)| \right) + \| (c_n)_{n \in \mathbb{N}} \|_{\infty} \left( \sum_{n=N+1}^{\infty} |\mu(x_n)| \right) \\ &< \| (c_n)_{n \in \mathbb{N}} \|_{\infty} \epsilon_1 + \| (c_n)_{n \in \mathbb{N}} \|_{\infty} \epsilon_1 \\ &= \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

thus proving the sequence  $\sum_{n=1}^N c_n \mu(x_n)$  is Cauchy and thus convergent since  $\mathbb{R}, \mathbb{C}$  are both complete.

Now, we give a counterexample to show that  $\sum c_n x_n$  need not converge in the norm of  $X$ . In particular, take  $(c_n)_{n \in \mathbb{N}} = (1)_{n \in \mathbb{N}}$  the sequence of all ones. Now, for  $n = 1$  let  $x_n = e_n \in \ell^2$  and for  $n \geq 2$  let  $x_n = e_n - e_{n-1}$ .

Note that all  $\mu \in (\ell^2)^*$  have the form  $\mu_y \in (\ell^2)^*$  for some  $y \in \ell^2$  with  $\mu_y(x) = \langle x, y \rangle$  for all  $x \in \ell^2$ . (Of course, as mentioned before  $y = (y(n)) := (\mu(e_n))_{n \in \mathbb{N}}$ ).

So, to show that  $\sum_{n=1}^{\infty} \mu(x_n)$  converges for all  $\mu \in (\ell^2)^*$  note that  $\mu = \mu_y$  with  $y \in \ell^2$  defined as above and now we show that the norms of the tails  $|\sum_{n=N}^{\infty} \mu(x_n)| \rightarrow 0$  as  $N \rightarrow \infty$ . In particular note that

$$\begin{aligned} \left| \sum_{n=N}^{\infty} \mu(x_n) \right| &= \left| \sum_{n=N}^{\infty} \langle y, e_n - e_{n-1} \rangle \right| = \left| \sum_{n=N}^{\infty} \langle y, e_n \rangle - \langle y, e_{n-1} \rangle \right| \\ &= \left| \sum_{n=N}^{\infty} \langle y, e_n \rangle - \langle y, e_{n-1} \rangle \right| = |\langle y, e_{N-1} \rangle| = |y(N-1)|. \end{aligned}$$

Now, I claim that since  $y \in \ell^2$  we know that

$$\sum_{n=M}^{\infty} |y(n)|^2 \rightarrow 0$$

as  $M \rightarrow \infty$ , which I also claim means that  $|y(M)| \rightarrow 0$  as  $M \rightarrow \infty$ .

In particular, for fixed  $\epsilon > 0$  we want to find  $M \in \mathbb{N}$  such that  $|y(m)| < \epsilon$  for all  $m \geq M$ . Well, note that for all  $M \in \mathbb{N}$  we have that

$$|y(M)|^2 < \sum_{n=M}^{\infty} |y(n)|^2.$$

So, if we set  $\epsilon' = \epsilon^2$  then since  $\sum_{n=M}^{\infty} |y(n)|^2 \rightarrow 0$  we know there exists  $N(\epsilon')$  such that

$$\left| \sum_{n=P}^{\infty} |y(n)|^2 \right| < \epsilon'$$

for all  $P \geq N(\epsilon')$  and thus that gives that our desired  $M$  is  $M = N(\epsilon')$  and we have that

$$|y(P)|^2 < \sum_{n=P}^{\infty} |y(n)|^2 < \epsilon' = \epsilon^2$$

for all  $P \geq M$  and thus

$$|y(P)| < \epsilon$$

for all  $P \geq M$  which means  $|Y(N-1)| \rightarrow 0$  and thus  $|\sum_{n=N}^{\infty} \mu(x_n)| \rightarrow 0$  as  $N \rightarrow \infty$  meaning that  $\sum_{n=1}^{\infty} \mu(x_n)$  converges (since that means that  $\sum_{n=1}^M \mu(x_n)$  is Cauchy and thus convergent since  $F \in \{\mathbb{R}, \mathbb{C}\}$  is complete).

Now, however I claim that  $\sum_{n=1}^{\infty} x_n$  does not converge in the norm of  $\ell^2$ . Namely, to have convergence one would need that (Condition A) for all  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that for all  $M \geq N$  one has that

$$|\sum_{n=M}^{\infty} x_n| < \epsilon.$$

However, we note that for all  $P \in \mathbb{N}$

$$\|\sum_{n=P}^{\infty} x_n\|_{\ell^2} = \|\sum_{n=P}^{\infty} e_n - e_{n-1}\|_{\ell^2} = \|e_{P-1}\|_{\ell^2} = 1.$$

Thus, for  $\epsilon < 1$  there does not exist the desired  $N \in \mathbb{N}$  such that Condition A holds.

(d) If  $1 \leq p < \infty$  then  $U^*: X^* \rightarrow (\ell^p)^*$  and  $T = U^*$  in the sense of identifying  $\ell^{p'}$  with  $(\ell^p)^*$ . Specifically, for each  $\mu \in X^*$  the linear functional on  $\ell^p$  that is determined by the sequence  $T\mu \in \ell^{p'}$  equals the linear functional  $U^*\mu \in (\ell^p)^*$ .

Note that  $U^*: X^* \rightarrow (\ell^p)^*$  is the unique operator satisfying

$$(U^*\mu)(\{y_n\}_{n \in \mathbb{N}}) = \mu(U(\{y_n\}_{n \in \mathbb{N}}))$$

for all  $\mu \in X^*$  and all  $\{y_n\}_{n \in \mathbb{N}} \in \ell^p$ .

Now, note that if one regards  $T\mu = (\mu(x_n))_{n \in \mathbb{N}} \in \ell^{p'}$  as a functional on  $\ell^p$  via the action

$$(T\mu)(\{y_n\}_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} y_n \mu(x_n)$$

we notice also by linearity and continuity of  $\mu$  that

$$\mu(U(\{y_n\}_{n \in \mathbb{N}})) = \mu(\sum_{n=1}^{\infty} y_n x_n) = \sum_{n=1}^{\infty} y_n \mu(x_n).$$

Thus, we see that

$$(U^*\mu)(\{y_n\}_{n \in \mathbb{N}}) = (T\mu)(\{y_n\}_{n \in \mathbb{N}})$$

for all  $\mu \in X^*$  and all  $(\{y_n\}_{n \in \mathbb{N}}) \in \ell^p$  proving that  $T = U^*$ .

4. Exercise 20.5.7. Let  $\{x_n\}_{n \in \mathbb{N}}$  be a minimal sequence in a Banach space  $X$ , and let  $\{\alpha_n\}_{n \in \mathbb{N}}$  be its biorthogonal sequence in  $X^*$ .

Prove that the following four statements are equivalent.

Remark: You can assume without proof that a biorthogonal sequence  $\{\alpha_n\}_{n \in \mathbb{N}}$  exists in

$X^*$ . This sequence satisfies  $\alpha_m(x_n) = \delta_{mn}$  for  $m, n \in \mathbb{N}$ .

- (a)  $\{x_n\}_{n \in \mathbb{N}}$  is a Schauder basis for  $X$ .
- (b)  $S_N x \rightarrow x$  for each  $x \in X$ .
- (c)  $\{x_n\}_{n \in \mathbb{N}}$  is complete, and  $\sup \|S_N x\| < \infty$  for each  $x \in X$ .
- (d)  $\{x_n\}_{n \in \mathbb{N}}$  is complete, and  $\sup \|S_N\| < \infty$ .

Note that (a) implies (b). Since  $\{x_n\}_{n \in \mathbb{N}}$  a Schauder basis means that for any  $x \in X$  there exists a unique sequence of scalars  $c_n$  such that

$$x = \sum_{n=1}^{\infty} c_n x_n$$

and the fact that  $\{\alpha_n\}_{n \in \mathbb{N}}$  is biorthogonal with each  $\alpha_n$  continuous we have that

$$\alpha_m(x) = \alpha_m\left(\sum_{n=1}^{\infty} c_n x_n\right) = \sum_{n=1}^{\infty} c_n \alpha_m(x_n) = c_m \alpha_m(x_m) = c_m \quad (0.3)$$

meaning that

$$x = \sum_{n=1}^{\infty} \alpha_n(x) x_n = \lim_{N \rightarrow \infty} S_N x$$

thus showing (a)  $\implies$  (b).

We also have that (b)  $\implies$  (a). In particular, (b) states that

$$x = \sum_{n=1}^{\infty} \alpha_n(x) x_n$$

for all  $x \in X$ . Furthermore, we note that if one has another expression

$$x = \sum_{n=1}^{\infty} c_n x_n$$

for some  $c_n \in \mathbb{F}$  then as shown in Equation 0.3 we obtain  $c_n = \alpha_n(x)$  which shows uniqueness and thus proves that (b) implies (a).

Now, we show that (b)  $\implies$  (d). Fix  $x \in X$ .

We know that

$$x = \sum_{n=1}^{\infty} \alpha_n(x) x_n = \lim_{N \rightarrow \infty} S_N x,$$

which shows firstly that  $\{x_n\}_{n \in \mathbb{N}}$  is complete. Furthermore, it so happens that  $\|S_N\| < \infty$  for each  $N \in \mathbb{N}$  since

$$\begin{aligned} \|S_N(x)\| &= \left\| \sum_{n=1}^N \alpha_n(x) x_n \right\| \leq \sum_{n=1}^N |\alpha_n(x)| \|x_n\| \leq \sum_{n=1}^N \|\alpha_n\| \|x\| \|x_n\| \\ &\leq (N+1) \|x\| \max_{n \in \{0,1,\dots,N\}} \|x_n\| \max_{n \in \{0,1,\dots,N\}} \|\alpha_n\| \end{aligned}$$

meaning that

$$\|S_N\| \leq (N+1) \|x\| \max_{n \in \{0,1,\dots,N\}} \|x_n\| \max_{n \in \{0,1,\dots,N\}} \|\alpha_n\| < \infty.$$

Now, the Banach-Steinhaus Theorem tells us that

$$\|id\| \leq \sup_{N \in \mathbb{N}} \|S_N\| < \infty$$

which concludes our proof that  $(b) \implies (d)$ .

Now, we show that  $(d) \implies (c)$  by noting that

$$\sup_{N \in \mathbb{N}} \|S_N x\| \leq \sup_{N \in \mathbb{N}} \|S_N\| \|x\| = \|x\| \sup_{N \in \mathbb{N}} \|S_N\|.$$

Now, (d) states that  $M := \sup_{N \in \mathbb{N}} \|S_N\| < \infty$  which then gives that

$$\sup_{N \in \mathbb{N}} \|S_N x\| \leq M \|x\| < \infty$$

for all  $x \in X$  which proves (c).

Likewise I show that  $(c) \implies (d)$ . Note that (c) says  $\sup_{N \in \mathbb{N}} \|S_N x\| < \infty$ . Then, we may apply the Uniform Boundedness Principle to obtain

$$\sup_{N \in \mathbb{N}} \|S_N\| < \infty.$$

Finally we show that  $(d) \implies (b)$ . Namely, for fixed  $\epsilon > 0$  we would like to find  $P(\epsilon) \in \mathbb{N}$  such that  $\|S_N x - x\| < \epsilon$  for all  $N \geq P(\epsilon)$ . So, denote  $M := \sup_{N \in \mathbb{N}} \|S_N\|$ . Now, note that since  $\{x_n\}_{n \in \mathbb{N}}$  is complete, for fixed  $\epsilon' = \frac{\epsilon}{2 \max(M,1)}$  there exist constants  $\{d_n(\epsilon')\}_{n \in \{0,1,\dots,m(\epsilon')\}}$  such that the approximation  $\hat{x} = \sum_{n=1}^{m(\epsilon')} d_n(\epsilon') x_n$  satisfies

$$\|\hat{x} - x\| < \epsilon'.$$

Furthermore, note that

$$S_N \hat{x} = \hat{x}$$

since for  $N \geq m(\epsilon')$  one has that

$$\begin{aligned}
S_N \hat{x} &= \sum_{n=1}^N \alpha_n(\hat{x}) x_n \\
&= \sum_{n=1}^N \alpha_n \left( \sum_{k=0}^{m(\epsilon')} d_k(\epsilon') x_k \right) x_n \\
&= \sum_{n=1}^N \sum_{k=0}^{m(\epsilon')} d_k(\epsilon') \alpha_n(x_k) x_n \\
&= \sum_{k=0}^{m(\epsilon')} d_k(\epsilon') \sum_{n=1}^N \alpha_n(x_k) x_n \\
&= \sum_{k=0}^{m(\epsilon')} d_k(\epsilon') \alpha_k(x_k) x_k \\
&= \sum_{k=0}^{m(\epsilon')} d_k(\epsilon') x_k = \hat{x}.
\end{aligned}$$

Then, by the triangle inequality we have that

$$\begin{aligned}
\|S_N x - x\| &= \|S_N x - S_N \hat{x} + S_N \hat{x} - \hat{x} + \hat{x} - x\| \\
&\leq \|S_N x - S_N \hat{x}\| + \|S_N \hat{x} - \hat{x}\| + \|\hat{x} - x\| \\
&= \|S_N(x - \hat{x})\| + \|\hat{x} - x\| \\
&\leq M\|x - \hat{x}\| + \|\hat{x} - x\| \\
&< M\epsilon' + \epsilon' = M \frac{\epsilon}{2\max(M, 1)} + \frac{\epsilon}{2\max(M, 1)} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

thus proving that  $S_N x \rightarrow x$  as  $N \rightarrow \infty$ .

Of course, the above means that I have shown TFAE since there is a path of implication (using the implications I proved) from  $i$  to  $j$  for all  $i, j \in \{a, b, c, d, e\}$ .