OPERATOR THEORY HOMEWORK 6

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of problems will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Suppose that X is a normed space and X^* is separable.

(a) Prove that there is a countable set $\{\lambda_n\}_{n\in\mathbb{N}}$ in X^* that is dense in the closed unit sphere

$$D^* = \{\mu \in X^* : \|\mu\| = 1\}.$$

Since X^* is separable, it has a countable dense subset. Call it $\{\alpha_n\}_{n\in\mathbb{N}}$. Now, let $\{\lambda_n\}_{n\in\mathbb{N}} = \{\frac{\alpha_n}{||\alpha_n||}\}_{n\in\mathbb{N}}$. I claim that $\{\lambda_n\}_{n\in\mathbb{N}}$ is dense in D^* and show so using the triangle inequality and continuity of the norm. Namely, for each $\mu \in X^*$ and all $\epsilon > 0$ I would like to find λ_{n_ϵ} such that $||\mu - \lambda_{n_\epsilon}|| < \epsilon$. Well, consider the original set $\{\alpha_n\}_{n\in\mathbb{N}}$ and note that there existed a sequence $\{\alpha_{n_i} : i \in \mathbb{N}\}$ with $\alpha_{n_i} \to \mu$ in norm, or specifically that $||\alpha_{n_i} - \mu|| \to 0$ as $i \to \infty$. So, there exists $m(\epsilon/2) \in \mathbb{N}$ such that $||\alpha_m - \mu|| < \frac{\epsilon}{2}$ for all $m \geq m(\epsilon/2)$. Also, by continuity of the norm, we know that for this sequence, since $||\mu|| = 1$ we have that $||\alpha_{n_i}|| \to 1$ as $i \to \infty$. Then, since $||\alpha_{n_i}|| \to 1$ one has that $||\alpha_{n_i} - \frac{\alpha_{n_i}}{||\alpha_{n_i}||}|| = (1 - \frac{1}{||\alpha_{n_i}||})||\alpha_{n_i}|| \to 0$ as $i \to \infty$. So, there exists $k(\epsilon/2) \in \mathbb{N}$ such that $||\alpha_k - \frac{\alpha_k}{||\alpha_k||}|| < \epsilon/2$ for all $k \geq k(\epsilon/2)$. Now, let $N(\epsilon) = max(m(\epsilon/2), k(\epsilon/2))$. Now, for all $n \geq N(\epsilon)$ we have that $||\frac{\alpha_n}{||\alpha_n||} - \mu|| \leq ||\frac{\alpha_n}{||\alpha_n||} - \alpha_n|| + ||\alpha_n - \mu|| < \epsilon$. So, since by definition $\lambda_n = \frac{\alpha_n}{||\alpha_n||}$ the set is dense in D^*.

(b) For each $n \in \mathbb{N}$, find a unit vector $x_n \in X$ such that $|\lambda_n(x_n)| \geq 1/2$.

Well, since $||\lambda_n|| = 1$ we know that $1 = \sup_{||x||=1} |\lambda_n(x)|$ which means that for all $\epsilon > 0$ there exists x_ϵ^n with $||x_\epsilon^n|| = 1$ such that $|\lambda_n(x_\epsilon^n)| > 1 - \epsilon$. Thus, set $\epsilon = 1/2$ and the result follows by choosing $x_{1/2}^n$. Let $M = \overline{span}\{x_n\}_{n\in\mathbb{N}}$. This is a closed subspace of X, and by Problem 7.4.6 we know that M is separable.

(c) Suppose that $M \neq X$, and use the Hahn–Banach Theorem to derive a contradiction. Conclude that $X = M$, and therefore X is separable.

Note that then by Corollary 19.1.5 to the Hahn-Banach Theorem we know that since there exists $x_0 \in X \setminus M$ that then $x_0' = x_0/||x_0||$ is a unit vector in X and thus there exists a functional $\mu \in X^*$ such that $|\mu(x_0')| = 1$, $\mu|_M = 0$, and $||\mu|| = 1/d$. Now, let $\alpha = d\mu$ which means that $\alpha(x_0') = d$ $\alpha|_M = 0$ and $||\alpha|| = 1$. Now, note that for $\epsilon = 1/4$ there exists $\lambda_k \in \{\lambda_n\}_{n\in\mathbb{N}}$ such that $||\lambda_k - \alpha|| < 1/4$. However, note that

$$||(\lambda_k - \alpha)x_k = ||\lambda_k(x_k) - 0|| \geq 1/2$$

which implies that $||\lambda_k - \alpha|| \geq 1/2$, a contradiction.

2. Exercise 19.2.5. Let X and Y be Banach spaces, and fix a bounded linear operator $A \in \mathcal{B}(X, Y)$.

(a) Choose $\mu \in Y^*$, and define a functional $A^*\mu \colon X \to \mathbf{F}$ by

$$(A^*\mu)(x) \;=\; \mu(Ax), \qquad \text{for } x \in X.$$

Show that $A^*\mu$ is linear and bounded, and therefore $A^*\mu \in X^*$.

Note that $A^*\mu(x+y) = \mu(A(x+y)) = \mu(Ax+Ay) = \mu(Ax) + \mu(Ax) = A^*\mu(x) + A^*\mu(y)$ and also $A^*\mu(kx) = \mu(kAx) = k\mu(Ax) = kA^*\mu(x)$ for all $k \in F$ thus proving linearity. Furthermore note that

$$||A^*\mu|| = \sup_{||x||=1} \mu(Ax) \leq \sup_{||x||=1} ||\mu||||Ax|| \leq \sup_{||x||=1} ||\mu||||A|| = ||\mu||||A||$$

thus proving boundedness.

(b) Show that the mapping $A^* \colon \mu \mapsto A^*\mu$ is a bounded linear mapping of Y^* into X^*, and the operator norm of this mapping is $\|A^*\| = \|A\|$. I show that it is bounded by showing that the operator norm satisfies the above equality. Namely, we calculate

$$||A^*|| = \sup_{\mu \in Y^*:||\mu||=1} ||A^*\mu|| = \sup_{\mu \in Y^*:||\mu||=1} \Big(\sup_{x \in X:||x||=1} |A^*\mu(x)| \Big)$$

$$= \sup_{\mu \in Y^*:||\mu||=1} \Big(\sup_{x \in X:||x||=1} |\mu(Ax)| \Big) = \sup_{x \in X:||x||=1} \Big(\sup_{\mu \in Y^*:||\mu||=1} |\mu(Ax)| \Big) = \sup_{x \in X:||x||=1} ||Ax|| = ||A||.$$

where the second to last equality holds by Corollary 19.1.4 to the Hahn-Banach Theorem.

(c) Prove that A^* is the unique operator mapping Y^* into X^* that satisfies

$$\mu(Ax) \;=\; (A^*\mu)(x), \qquad \text{for all } x \in X \text{ and } \mu \in Y^*. \qquad \diamondsuit$$

This is true almost entirely by definition. Assume that there are two operators A^*, B^* which satisfy

$$\mu(Ax) = (A^*(\mu))(x) = (B(\mu))(x)$$

Then, note that $B(\mu) = A^*(\mu) = \mu \circ A$ for all $\mu \in Y^*$ and thus $B = A^*$.

3. Exercise 19.2.12. Let X and Y be Banach spaces, and let $T \colon X \to Y$ be a topological isomorphism. Prove that the adjoint map $T^* \colon Y^* \to X^*$ is a topological isomorphism, and

$$(T^{-1})^* \;=\; (T^*)^{-1}.$$

I first prove that $(T^{-1})^* = (T^*)^{-1}$ and use that to prove that T^* is a topological isomorphism. In particular I show that $((T^{-1})^* \circ T^*)(\mu) = \mu$ and $(T^* \circ (T^{-1})^*)(\alpha) = \alpha$ for all $\mu \in Y^*$ and all $\alpha \in X^*$. Indeed note that $((T^{-1})^*(T^*(\mu))(y) = (T^*(\mu))(T^{-1}(y)) = \mu(T(T^{-1}(y))) = \mu(y)$.

Also, we have that $(T^*((T^{-1})^*(\mu))(y) = ((T^{-1})^*(\mu))(T(y)) = \mu(T^{-1}T(y)) = \mu(y)$.

Now, we show that if $T : X \to Y$ is a topological isomorphism for space some Banach spaces X, Y, then one has that $T^* : Y^* \to X^*$ is a topological isomorphism as well thus proving that T^* is continuous and that $(T^*)^{-1} = (T^{-1})^*$ is continuous by applying the claim to the topological isomorphism $T^{-1} : Y \to X$.

Proof: We must show that for all $\epsilon > 0$ there exists $\delta > 0$ such that $||\alpha - \beta||_{Y^*} < \delta$ implies that $||T^*\alpha - T^*\beta||_{X^*} < \epsilon$. Indeed, note that

$$||T^*\alpha - T^*\beta||_{X^*} = \sup_{||x||=1} |(T^*\alpha - T^*\beta)(x)| = \sup_{||x||=1} |(T^*(\alpha - \beta))(x)|$$

$$= \sup_{||x||=1} |((\alpha - \beta)(T(x)))| \leq \sup_{||x||=1} ||\alpha - \beta||_{Y^*}|T(x)| \leq ||\alpha - \beta||_{Y^*}||T||.$$

So, let $\delta = \epsilon/||T||$ and then indeed the claim follows.

4. Problem 19.2.14. Let M be a closed subspace of a Banach space X. Let $\epsilon\colon M \to X$ be the embedding map defined by $\epsilon(x) = x$ for $x \in M$. Prove that $\epsilon^*\colon X^* \to M^*$ is the restriction map defined by $\epsilon^*\mu = \mu|_M$ for $\mu \in X^*$.

Well, note that $\mu|M(m) = \mu(\epsilon(m))$ for all $m \in M$ and all $\mu \in X^*$ and thus by problem 2(c) we know that $\epsilon^*(\mu) = \mu|M$.