Personal Notes Giving a Big Picture Overview of Schur and Diamond's "A First Course in Modular Forms"

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- Modular Curves and Theory of Riemann Surfaces:
  - Modular forms give rise to fundamental domains, by which we can quotient the complex plane and compactify to get a Riemann surface. At the end of the day, all of these Riemann surfaces are all complex tori of some genus. The fact that these are complex tori is important since the fact that there are intuitively several bands along which to integrate gives an intuitive, visual understanding of the Jacobian. I obscured some details there, but the idea is that the Jacobian is formed as a quotient involving the dual space of a space of differential forms and a homology group. The dual space to a space of differential forms can crudely be understood as a space of integration operators, since one can integrate differential forms to get a constant, roughly speaking. Also, any algebraic topology student will recognize the idea of traversing (or integrating) an integral number of times around bands of an n-holed torus as looking at the homology of such a surface. Here the denominator in our useful isomorphic idea of the Jacobian is exactly this idea of integrating an integral number of times around each band and all that is left in the numerator, so to speak, is integration around each band a fractional amount. So one sees a clear bijection between the Riemann surface and its Jacobian since specifying a fractional traversal along each band specifies a location on the surface. I realize that all this is very, very vague but I think it should hopefully answer the faculty's question of which material in the book I have covered.
- Modular Curves and Differential Geometry:
  - Depending on the specific matrix group one considers (e.g. $\Gamma_0(N), \Gamma_1(N)$ for various N) the quotienting and compactification of the fundamental domain to "get a torus" is at-best complicated and does not preserve meaningful algebraic structure in terms of the bands and at-worst is downright inaccurate. In the more complicated cases for which our choice of matrix group results in many elliptic points, the careful construction of various atlases with charts specifying local coordinates at these elliptic points of order 2 and 3 becomes necessary. This is one place where differential geometry makes its appearance and cannot be done away with easily.
  - It, of course, appears as I said before through brief mention of differential forms, during the discussion of equivalences between the Jacobian, the Picard group, and the surface itself, during the discussion of Abel's theorem.
- Operator Theoretic material related to Hecke Operators:
  - The space of modular forms decomposes into the space of cusp forms and the quotient space of the space of modular forms by cusp forms, known as the Eisenstein space.
  - The space of cusp forms is broken into the mutually orthogonal (with respect to the Petersson inner product) spaces of oldforms and newforms, which are each stable under the action of the various Hecke operators. Newforms are of the most interest since the requisite modular forms f, found to satisfy the representation theoretic version of the Modularity Conjecture, end up being newforms that are able to be systematically constructed due to the fact that the space of newforms has a meaningful "canonical basis" of simultaneous eigenfunctions.
- Algebraic Number Theoretic material:
  - The fields whose inherent machinery I enjoy the most are representation theory and algebraic number theory. Thus, I was quite happy to find that considering solutions over the algebraic closure of the rationals brings into serious play the theory of rings of integers and prime factorization of ideals in Dedekind domains, and in the section on Galois representations draws more serious connections by introducing the notions of inertia groups, decomposition groups, and Frobenius elements which seem

to extend the notion of working modulo a prime to modulo a prime ideal. Of course, it is then due to the fact that such Frobenius elements are dense in relevant Galois groups, that one obtains the Modularity Theorem in the way that Weil actually proved it.

- Representation Theoretic material:
  - Of course, the actual proof Weil gave for the Modularity Theorem was done via representation theory. Namely, he proved that $a_p(E) = a_p(f)$, for all but finitely many primes, by showing that the formulas for trace and determinant of two operators, one relating to the elliptic curve E and the other to a newform f, agree on a dense subset of the relevant Galois group, and thus are equal as functions since they are continuous. So, the importance is clear, but I will add that the reason the topic stands out to me is my strong interest in and decent knack for the field.
  - It is very interesting here to explore the representation theoretic ideas brought into play here. Namely, a representation theorist would take great care to consider the inherent structure of a group and why it might lend itself to certain almost canonical representations. In such a vein, there might be canonical properties associated with representations of a given group. Namely, when considering representations of the group of symmetries of the square, it seems that to an intelligent human capable of nuanced understanding (not a machine for detecting whether representations are reducible and putting them into blocks) that a canonical choice of dimension for such a representation would be two, and the associated matrices would simply be the collection of the appropriate rotation and reflection matrices. The canonical choice of field in this case also seems clear: namely the smallest field containing all the entries in the intuitively obvious rotation and reflection matrices. So, in this case, I suppose it would be the rationals. However, there are some choices made in the development of representation theoretic parts of this field that perplex me. Namely, I can accept that the group of torsion points of an elliptic curve in $(\mathbb{Z}/N\mathbb{Z})^2$ and I certainly understand why there is a natural p-adic structure in the set of p^k torsion points for k in the integers, which is what the Tate Module captures. What I don't understand is why the relevant two-dimensional Galois representations lie over the contrived field noted in the literature, and why they are a-priori two dimensional. (I will add that it seems to me that use of such a contrived field is what guarantees an irreducible two-dimensional representation, but what I think I am missing is what such a choice captures about the structure of the curve and its additive group of torsion points and whether the structure is translated in meaningful ways by these Galois representations).
  - It is also my tentative understanding that these Galois groups act on the torsion points, but if so, I am unclear as to how. I think it may be that torsion points can be embedded as roots of unity via the Weil pairing and thus a member of the Galois group which permutes elements of C, including roots of unity, might permute these torsion points (which are members of $(\mathbb{Z}/N\mathbb{Z})^2$). However, then, it is unclear to me, what, structurally, such a member of the Galois group permuting the roots of unity, might translate to visually and algebraically in terms of permuting torsion points on the relevant elliptic curve. Again, let me emphasize that my strategy so far has been to take a bird's eye view of the subject, so I am allowing myself to develop tentative ideas that may be incorrect with the understanding that they will later be corrected should I choose definitively to specialize in this area.
- Algebraic Geometry material:
  - There are other statements in terms of algebraic curves and varieties, whose difference is exactly that curves have dimension one over the appropriate field while varieties may have dimension two or more. However, Schur and Diamond restrict attention to algebraic curves in the interest of appealing to students with little background in algebraic geometry. If I were admitted to a program with available courses in algebraic geometry, I would hope to deepen my understanding of varieties and also schemes, which unlike varieties account for the fact that some roots of polynomials are multiple roots, a fact which is unfortunately obscured in the theory of varieties since the associated vanishing ideals are always radical ideals.
  - I am also learning briefly about the relation between varieties and function fields. Of course, the fact that Riemann surfaces are isomorphic if and only if they have the same function field was instrumental in at least one proof given in the textbook. I believe that fact was used in the handling of the finitely many primes at which an elliptic curve may have bad reduction.
  - I was also happy to see concrete motivation in the field of modular forms for abstract practices in algebraic geometry. In particular, the practices of homogenizing and dehomogenizing ideals came up, due to the importance of working with non-singular curves, namely those for which the associated gradients are nowhere zero. I noticed that adding a homogenizing variable could in theory turn a singular curve in two variables into a non-singular variety in three variables.

- Arithmetic Geometry Material:
  - The Eichler-Shimura relation could be viewed as the culmination of Schur and Diamond. However, a deeper understanding of pullbacks and pushforwards of Jacobians (whose composition may or may not multiply a stated quantity by a given integer of important consequence) seems needed to make sense of such a result.
  - I will also mention that I have a decently deep understanding about some topics, like Jacobians and the Picard group, but that there are still other holes in my knowledge that need to be filled. Namely, it would be nice to study results such as the Riemann-Roch and Mordell-Weil Theorems in depth in a formal class setting, complete with assignments and tests. I have found some loosely relevant arithmetic geometry courses online (through MIT OCW for instance), but the focus is too computational.