Linear Representations of Finite Groups by Serre
Exercises completed by Caitlin Beecham

Exercise 3.1
Show directly, using Schur's Lemma, that each irreducible representation of an abelian group, finite or not, has degree 1.
Consider an irreducible representation (ρ, V) of G. Now, G abelian implies that (for fixed $g \in G$) $\rho_g \rho_h = \rho_h \rho_g$ for all $h \in G$. Thus, we may apply Schur's lemma to conclude that $\rho_g = \lambda_g I$ is a scalar multiple of the identity. This applies for all $g \in G$ meaning that $\rho_g = \lambda_g I$ for all $g \in G$. However, this provides a contradiction unless $dim(V) = 1$ since otherwise $span(\{e_1\}) \subsetneq V$ is a proper non-trivial subrepresentation.

Exercise 3.2
Let ρ be an irreducible representation of G of degree n and character χ; let C be the center of G (i.e. the set of $s \in G$ such that $st = ts$ for all $t \in G$), and let c be its order.

(a) Show that ρ_s is a homothety for each $s \in C$. [Use Schur's Lemma]. Deduce from this that $|\chi(s)| = n$ for all $s \in C$.

Well, for $s \in C$ we have that $\rho_s \rho_h = \rho_h \rho_s$ for all $h \in G$ which implies by Schur's lemma that $\rho_s = \lambda_s I$ is a scalar multiple of the identity for all $s \in C$. The proof of Proposition 1 part (ii) on page 11 shows that $|\lambda_s| = 1$ (here $||$ denotes complex modulus). Thus, $|\chi(s)| = |\lambda(s)|n = n$.

(b) Prove the inequality $n^2 \leq g/c$.
Note that $\sum_{s \in G} |\chi(s)|^2 = g$ and $\sum_{s \in C} |\chi(s)|^2 + \sum_{s \in G \setminus C} |\chi(s)|^2 = cn^2 + \sum_{s \in G \setminus C} |\chi(s)|^2$. So, $cn^2 \leq g$ which proves the claim.

(c) Show that, if ρ is faithful (i.e. $\rho_s \neq 1$ for $s \neq 1$), the group C is cyclic.
It suffices to show that $\{\lambda_s : s \in C\}$ is cyclic. Note that since C is a group we have that $\{\rho_s : s \in C\}$ is a group and thus $\{\lambda_s : s \in C\} \subseteq \{z \in \mathbb{C} : |z| = 1\}$ is also a group. I claim that any finite subgroup of $\{z \in \mathbb{C} : |z| = 1\}$ is cyclic. Of course, any finite group is finitely generated, so assume for contradiction that $\{\lambda_s : s \in C\}$ has a generating set of minimum size $\langle \lambda_{s_1}, \ldots, \lambda_{s_j} \rangle$ of at least two elements ($j \geq 2$). Then, note that $\lambda_{s_1} = \zeta_{ord(s_1)}^{r_1}$ for some r_1 coprime to $ord(s_1)$. Likewise $\lambda_{s_2} = \zeta_{ord(s_2)}^{r_2}$ for some r_2 coprime to s_2. Then, note that we have that $\lambda_{s_1} = \zeta_{ord(s_1)ord(s_2)}^{r_1 ord(s_2)}$ and $\lambda_{s_2} = \zeta_{ord(s_1)ord(s_2)}^{r_2 ord(s_1)}$. Now, by Bezout's identity there exist $a, b \in \mathbb{Z}$ such that $ar_1 ord(s_2) + br_2 ord(s_1) = gcd(r_1 ord(s_2), r_2 ord(s_1))$. Furthermore, for all $m \in \mathbb{Z}$ we have that $mgcd(r_1 ord(s_2), r_2 ord(s_1)) = mar_1 ord(s_2) + mbr_2 ord(s_1)$ meaning that $\zeta_{ord(s_1)ord(s_2)}^{mgcd(r_1 ord(s_2), r_2 ord(s_1))} = \zeta_{ord(s_1)ord(s_2)}^{mar_1 ord(s_2) + mbr_2 ord(s_1)} = (\zeta_{ord(s_1)ord(s_2)}^{r_1 ord(s_2)})^{ma} (\zeta_{ord(s_1)ord(s_2)}^{r_2 ord(s_1)})^{mb}$ or in other words that

$$\langle \zeta_{ord(s_1)ord(s_2)}^{gcd(r_1 ord(s_2), r_2 ord(s_1))} \rangle \subseteq \langle \zeta_{ord(s_1)ord(s_2)}^{r_1 ord(s_2)}, \zeta_{ord(s_1)ord(s_2)}^{r_2 ord(s_1)} \rangle = \langle \lambda_{s_1}, \lambda_{s_2} \rangle.$$

Furthermore

$$\lambda_{s_1} = (\zeta_{ord(s_1)ord(s_2)}^{gcd(r_1 ord(s_2), r_2 ord(s_1))})^{\frac{r_1 ord(s_2)}{gcd(r_1 ord(s_2), r_2 ord(s_1))}}$$

and

$$\lambda_{s_2} = (\zeta_{ord(s_1)ord(s_2)}^{gcd(r_1 ord(s_2), r_2 ord(s_1))})^{\frac{r_2 ord(s_1)}{gcd(r_1 ord(s_2), r_2 ord(s_1))}},$$

implying that

$$\lambda_1, \lambda_2 \in \zeta_{ord(s_1)ord(s_2)}^{gcd(r_1 ord(s_2), r_2 ord(s_1))}$$

which means

$$\langle \zeta_{ord(s_1)ord(s_2)}^{gcd(r_1 ord(s_2), r_2 ord(s_1))} \rangle = \langle \lambda_{s_1}, \lambda_{s_2} \rangle.$$

So, we have obtained a contradiction, namely that

$$\langle \lambda_{s_1}, \lambda_{s_2}, \lambda_{s_3}, \ldots, \lambda_{s_j} \rangle = \langle \zeta_{ord(s_1)ord(s_2)}^{gcd(r_1 ord(s_2), r_2 ord(s_1))}, \lambda_{s_3}, \ldots, \lambda_{s_j} \rangle,$$

which is impossible since we said that $\lambda_{s_1}, \lambda_{s_2}, \lambda_{s_3}, \ldots, \lambda_{s_j}$ was a generating set for $\{\lambda_s : s \in C\}$ of minimum size. Thus, C is cyclic.

Exercise 3.3
Let G be an abelian group of order g and let $\hat{G}$ be the set of irreducible characters of G. If χ^1, χ^2 belong to $\hat{G}$, the same is true of their product $\chi^1\chi^2$. Show that this makes $\hat{G}$ an abelian group of order g; the group $\hat{G}$ is called the dual of the group G. For $x \in G$ the mapping $\psi_x : \chi \mapsto \chi(x)$ is an irreducible character of $\hat{G}$ and so an element of the dual $\hat{\hat{G}}$ of $\hat{G}$. Show that the map of G into $\hat{\hat{G}}$ thus obtained is an injective homomorphism; conclude (by comparing the order of the two groups) that it is an isomorphism.

Note that since the number of non-isomorphic irreducible representations of g is the number of conjugacy classes we have that that number is g. Since two representations are isomorphic if and only if they have the same character that means that we have g distinct character functions χ^i for $i \in [g]$. Also $\hat{G}$ is abelian since $(\chi^i\chi^j)(g) := \chi^i(g)\chi^j(g) = \chi^j(g)\chi^i(g) =: (\chi^j\chi^i)(g)$. Note that we have a map $\psi : G \to \hat{\hat{G}}$ defined by $\psi : x \mapsto \psi_x$. To show that it is a homomorphism we must show that $\psi(xy) = \psi_x\psi_y$. Well clearly $\psi(xy) = \psi_{xy}$ and by definition ψ_{xy} is the map defined by $\psi_{xy} : \chi \mapsto \chi(xy)$ but $\chi(xy) = trace(\rho_{xy}) = trace(\rho_x\rho_y)$. Now note that $\psi_x\psi_y$ is the map defined by $\psi_x\psi_y : \chi \mapsto \chi(x)\chi(y) = trace(\rho_x)trace(\rho_y)$. However, G abelian implies and χ^i irreducible for all $i \in [g]$ implies by 3.2 (a) that $\rho_x = \lambda_x I$ and $\rho_y = \lambda_y I$ where I is the 1 by 1 identity matrix and of course $\rho_x\rho_y = \lambda_x\lambda_y I$ meaning that $trace(\rho_x) = \lambda_x$, $trace(\rho_y) = \lambda_y$ and $trace(\rho_x\rho_y) = \lambda_x\lambda_y = trace(\rho_x)trace(\rho_y)$ proving that ψ is a homomorphism. Now to prove that ψ is injective we must show that $\psi(x) = \psi(y)$ implies that $x = y$. So assume that $\psi(x) = \psi(y)$. Then by definition we have $\psi_x = \psi_y$ which means that $\psi_x(\chi) = \psi_y(\chi)$ for all $\chi \in \hat{G}$. In other words we have that $\chi(x) = \chi(y)$ for all $\chi \in \hat{G}$. However, that implies that x and y belong to the same conjugacy class or represent the same column in the character table since otherwise their columns would be orthogonal meaning that $\sum_{\chi^i : i \in [g]} \chi^i(x)\overline{\chi^i(y)} = 0$ but $\sum_{\chi^i : i \in [g]} \chi^i(x)\overline{\chi^i(y)} = \sum_{\chi^i : i \in [g]} |\chi^i(x)|^2 = |C_G(x)| = g$ since G is abelian and $g \neq 0$. Then x and y in the same conjugacy class implies that $x = y$ since G is abelian. Finally, we note that $|\hat{\hat{G}}| = g$ meaning ψ is an isomorphism. TODO: explain why $|\hat{\hat{G}}| = g$!!!