Linear Representations of Finite Groups by Serre
Exercises completed by Caitlin Beecham

Exercise 2.5
Let ρ be a linear representation with character χ. Show that the number of times that ρ contains the unit representation is equal to $(\chi|1) = \frac{1}{g}\sum_{s\in G}\chi(s)$.

We note by Theorem 2 that every representation is a direct sum of irreducible representations. Thus, we have that

$$V = \oplus_{i=1}^{m} k_i W_i.$$

where $W_i \neq W_j$ for $i \neq j$ and W_i is irreducible and $k_i \in \mathbb{N}_{\geq 1}$ for all $i \in [m]$. Then, if χ is the character of V and χ_i is the character of W_i for each $i \in [m]$ Proposition 2 tells us that $\chi = \sum_{i\in[m]} k_i\chi_i$. Thus,

$$(\chi \mid 1) = \sum_{i\in[m]} k_i(\chi_i \mid 1) = \sum_{i\in[m]} k_i\chi_{W_i=1},$$

which is by definition the number of times V contains the unit representation.

Exercise 2.6
Let X be a finite set on which G acts. Let ρ be the corresponding permutation representation and let χ be its character.

(a) The set Gx of images under G of an element $x \in X$ is called an *orbit*. Let c be the number of distinct orbits. Show that c is equal to the number of times that ρ contains the unit representation 1; deduce from this that $(\chi \mid 1) = c$. In particular, if G is transitive (i.e., if $c = 1$), ρ can be decomposed into $1 \oplus \theta$ and θ does not contain the unit representation. If ψ is the character of θ, we have $\chi = 1+\psi$ and $(\psi \mid 1) = 0$.

We show that c is equal to the number of times that ρ contains the unit representation 1 in a few steps.

- Note that an elementary theorem in algebra tells us that the orbits $\{Gx : x \in X\}$ partition the set X we know that if $\{x_i : i \in [N]\}$ is a set of representatives for the orbits, then

$$X = \cup_{i\in[N]} Gx_i.$$

- Then, by definition of the permutation representation which has basis

$$B = \{e_{x_j} : j \in [|X|]\}$$

I claim that

$$V = \bigoplus_{i\in[N]} \text{span}(\{\rho_g(e_{x_i}) : g \in G\}) =: \bigoplus_{i\in[N]} W_i.$$

(Note that the W_i are not irreducible but are stable under the action of G).

  - Namely, each W_i is stable under the action of G since

$$\rho_g(w_i) = \rho_g(\sum_{h\in G} c_h\rho_h(e_{x_i})) = \sum_{h\in G} c_h\rho_{gh}(e_{x_i}) = \sum_{k\in G} c_{g^{-1}k}\rho_k(e_{x_i}) \in W_i.$$

  - Also, we can now show that the W_i span V or more precisely that

$$V \subseteq \bigoplus_{i\in[N]} W_i$$

by noting that $V = \text{span}(\{e_{x_j} : j \in [|X|]\})$ and for all $j \in [|X|]$ there exists a unique $i(j) \in [N]$ such that $x_j \in Gx_{i(j)}$ which implies that $e_{x_j} = g(j)x_{i(j)} \in W_i$ for some $g(j) \in G$. Then, since any $v \in V$ has the form $\sum_{j\in[|X|]} c_{x_j}e_{x_j}$ we know that

$$v = \sum_{j\in[|X|]} c_{x_j}e_{x_j} = \sum_{j\in[|X|]} c_{x_j}g(j)e_{x_{i(j)}} = \sum_{j\in[N]} e_{x_j} \sum_{\{k\in[|X|]:i(k)=j\}} c_{x_k}g(k)$$

Then the fact that the above is a finite linear combination of elements of W_i for $i \in [N]$ proves the claim.

– Finally we show that

$$W_i \cap W_j = \{0\}$$

for all $i, j \in [N]$ with $i \neq j$.
This follows from the fact that

$$\rho_g(e_{x_i}) = e_{gx_i} \in B$$

for all $g \in G$ and all $i \in [|X|]$. Namely, if we denote

$$Gx_i = \{x_1^i, x_2^i, \dots, x_{n_i}^i\} \text{ such that } x_1^i = x_i$$

then that implies that

$$\{\rho_g(e_{x_i}) : g \in G\} = \{e_{x_1^i}, e_{x_2^i}, \dots, e_{x_{n_i}^i}\} =: S_i,$$

with

$$B = \bigsqcup_{i \in [N]} S_i.$$

– Then, one has that $W_i = \text{span}(S_i)$, and since $\{S_i : i \in [N]\}$ are pairwise disjoint subsets of basis vectors that means that $W_i \cap W_j = \{0\}$ for all $i \neq j$ since otherwise if there exists non-zero $w \in W_i \cap W_j$ for some $i \neq j$ then that means that $w = \sum_{k \in [n_i]} c_k e_{x_k^i} = \sum_{l \in [n_j]} b_l e_{x_l^j}$ where $\{c_k : k \in [n_i], c_k \neq 0\} \neq \emptyset$ and $\{b_l : l \in [n_j], b_l \neq 0\} \neq \emptyset$. Thus, one has that

$$0 = \sum_{k \in [n_i]} c_k e_{x_k^i} - \sum_{l \in [n_j]} b_l e_{x_l^j}$$

a contradiction since $\{e_{x_k^i} : k \in [n_i]\} \sqcup \{e_{x_l^j} : l \in [n_j]\}$ is a linearly independent set.

- Finally, I claim that each W_i contains the unit representation exactly once. Namely, note that for each $i \in [N]$ one has that the subspace $U_i = \text{span}(u_i)$ spanned by the vector

$$u_i := \sum_{g \in G} \rho_g(e_{x_i})$$

is invariant under G since

$$\rho_h(u_i) := \rho_h(\sum_{g \in G} \rho_g(e_{x_i})) = \sum_{g \in G} \rho_{hg}(e_{x_i}) = \sum_{k \in G} \rho_k(e_{x_i}) = u_i$$

for all $h \in G$ and furthermore for $u = cu_i \in \text{span}(u_i)$ for some $c \in F$ one has by linearity that $\rho_h(u) = c\rho_h(u_i) = cu_i \in \text{span}(u_i)$, meaning the subspace is stable and thus each W_i such that $i \in [N]$ contains the unit representation at least once.

- Also, each orbit contains the unit representation at most once. Otherwise, suppose that for some orbit, $Gx_i = \{x_1^i, x_2^i, \dots, x_{n_i}^i\}$, there exists a vector $u \in \text{span}(\{e_{x_k^i} : k \in [n_i]\})$ such that $\text{span}(u) \neq \text{span}(u_i)$ and

$$\rho_g(u) = u$$

for all $g \in G$. Then, $u \in S_i$ implies that

$$u = \sum_{j \in [n_i]} \alpha_j e_{x_j^i}$$

for some $\alpha_j \in F$ and since x_1^i, x_j^i belong to the same orbit for all $j \in [n_i]$ that implies that for all $j \in [2 : n_i]$ there exists $g_j \in G$ such that $g_j x_1^i = x_j^i$ which then implies that $\rho_{g_j} e_{x_1^i} = e_{x_j^i}$, and thus

$$u = (\sum_{j \in [n_i]} \alpha_j \rho_{g_j}) e_{x_i},$$

where we of course are using the fact that $x_i = x_1^i$ by definition. Now, we will group the terms of u that appear by the image $e_x = \rho_{g_j}(e_{x_i})$. So, we have

$$u = \sum_{x_k^i \in Gx_i} \sum_{j \in [n_i] : g_j x_i = x_k^i} \alpha_j \rho_{g_j}(e_{x_i}) = \sum_{x_k^i \in Gx_i} (\sum_{j \in [n_i] : g_j x_i = x_k^i} \alpha_j) e_{x_k^i}.$$

Now, for all $g \in G$ we have that

$$\rho_g(u) = \sum_{x_k^i \in Gx_i} (\sum_{j \in [n_i] : g_j x_i = x_k^i} \alpha_j) \rho_g(e_{x_k^i}) = \sum_{x_k^i \in Gx_i} (\sum_{j \in [n_i] : g_j x_i = x_k^i} \alpha_j) e_{g x_k^i}$$

and since $\rho_g(u) = u$ for all $g \in G$ we have that

$$\sum_{j \in [n_i] : g_j x_i = x_k^i} \alpha_j = \sum_{j \in [n_i] : g_j x_i = g x_k^i} \alpha_j$$

for all $k \in [|X|]$ and all $g \in G$, and since for all $k' \in [n_i]$ there exists $g \in G$ such that $gx_k = x_{k'}$ that means that

$$\sum_{j \in [n_i] : g_j x_i = x_k} \alpha_j = \sum_{j \in [n_i] : g_j x_i = x_{k'}} \alpha_j$$

for all $k, k' \in [|X|]$ and thus, if we denote

$$A_k = \sum_{j \in [n_i] : g_j x_i = x_k^i} \alpha_j$$

then that means

$$A_k = A_k'$$

for all $k, k' \in [|X|]$ and since

$$u = \sum_{x_k^i \in Gx_i} A_k e_{x_k^i}$$

that implies that

$$u = A_1 \sum_{x_k^i \in Gx_i} e_{x_k^i}$$

Also, note that by similarly grouping the terms of u_i we have

$$u_i = \sum_{x_k^i \in Gx_i} \sum_{j \in [n_i] : g_j x_i = x_k^i} \rho_{g_j}(e_{x_i}) = \sum_{x_k^i \in Gx_i} |\mathrm{Stab}_G(x_k^i)| e_{x_k^i},$$

but by the orbit stabilizer theorem

$$|Gx_k^i| = |G| / |\mathrm{Stab}_G(x_k^i)|$$

and since $Gx_k^i = Gx_i$ for all $k \in [n_i]$ that implies that $|\mathrm{Stab}_G(x_k^i)| = |\mathrm{Stab}_G(x_i)|$ for all $k \in [n_i]$ and thus

$$u_i = |\mathrm{Stab}_G(x_i)| \sum_{x_k^i \in Gx_i} e_{x_k^i},$$

but then that implies that

$$\mathrm{span}(\sum_{x_k^i \in Gx_i} e_{x_k^i}) = \mathrm{span}(u) = \mathrm{span}(u_i),$$

a contradiction.

(b) Let G act on the product $X \times X$ by means of the formula

$$s(x, y) = (sx, sy).$$

Show that the character of the corresponding permutation representation is equal to χ^2.

Let $\hat{\rho}$ denote the representation on the product $X \times X$ and $\hat{\chi}$ denote the corresponding character and recall from Exercise 2.2 that for any given permutation representation $\chi(s)$ is the number of elements $x \in X$ that are fixed by $s \in G$. Now, consider $X^s = \{x \in X : sx = x\}$. We then have that $X^s \times X^s = \{(x, y) \in X \times X : sx = x, sy = y\} = (X \times X)^s$. Of course then $\hat{\chi}(s) = |X^s \times X^s| = |X^s|^2 = (\chi(s))^2$, which concludes the proof.

(c) Suppose the G is transitive on X and that X has at least two elements. We say that G is doubly transitive if, for all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$, there exists $s \in G$ such that $x' = sx$ and $y' = sy$. Prove the equivalence of the following properties:

(1) G is doubly transitive.

(2) The action of G on $X \times X$ has two orbits, the diagonal and its complement.

(3) $(\chi^2|1) = 2$.

(4) The representation defined in (a) is irreducible.

Clearly (1) implies (2), namely since considering the action of s on (x, y) we have exactly two cases, namely either $x \neq y$ or $x = y$.

If $x = y$ then $s(x, x) = (sx, sx)$ and in fact doubly transitive implies transitive since for any $u \in X$ with $u \neq x$ we may find s such that $sx = u$. Namely, we may set a dummy variable $w \neq x$ and then require s such that $s(x, w) = (u, sw)$ which exists since $sw \neq u = sx$ by basic facts of a group action (namely cancellation). So, indeed for any $x \in X$ and any $u \in X$ with $u \neq x$ we have $s \in G$ such that $s(x, x) = (u, u)$. Of course if $u = x$ we take $s = e$.

If $x \neq y$, then for any $x' \neq y'$ we can find $s \in G$ such that $s(x, y) = (x', y')$. Thus, the complement of the diagonal is indeed an orbit.

Now, clearly (2) implies (1) since the complement of the diagonal being an orbit is equivalent to (1). Thus (1) and (2) are equivalent.

Now, (3) says $(\chi^2|1) = 2$. By part (a) this means that the number of orbits of the action of G on $X \times X$ is 2. Of course, we know that the diagonal is one orbit since as argued above doubly transitive implies transitive. Then, the remains of $X \times X$ must be exactly the other orbit. Thus, (3) implies (2) and of course (2) implies (3), meaning they are equivalent.

Finally, (4) is equivalent to (3) since as noted in part (a) we have that $\chi = 1 + \psi$ where ψ is the character of θ. Now, $\chi^2 = (1+\psi)^2 = 1 + 2\psi + \psi^2$ and since $2 = (\chi^2|1) = (1|1) + 2(\psi|1) + (\psi^2|1) = 1 + 2(0) + (\psi^2|1)$ (where we are using $(\psi|1) = 0$ from part (a)) that gives $(\psi^2|1) = 1$.
Now, by definition $(\psi^2|1) = \frac{1}{g}\sum_{s\in G} \psi^2(s) = \frac{1}{g}\sum_{s\in G} \psi(s)\psi(s)^* = (\psi|\psi)$. Then, Theorem 5 says that $(\psi|\psi) = 1$ if and only if the representation (θ, V) is irreducible. So, indeed (3) implies (4) and also (4) implies (3) since Theorem 5 was an equivalence. Thus, (3) is equivalent to (4) and finally we have that (1), (2), (3), and (4) are all equivalent.