Linear Representations of Finite Groups by Serre
Exercises completed by Caitlin Beecham

Exercise 2.1
Let χ and χ' be characters of two representations. Prove the formulas

$$(\chi + \chi')^2_\sigma = \chi^2_\sigma + \chi'^2_\sigma + \chi\chi'$$

and

$$(\chi + \chi')^2_\alpha = \chi^2_\alpha + \chi'^2_\alpha + \chi\chi'.$$

We use proposition 3. Namely, note

$$\begin{aligned}
(\chi + \chi')^2_\sigma(s) &= \frac{1}{2}((\chi + \chi')(s))^2 + (\chi + \chi')(s^2)) \\
&= \frac{1}{2}(\chi^2(s) + \chi'^2(s) + 2\chi(s)\chi'(s) + \chi(s^2) + \chi'(s^2)) \\
&= \frac{1}{2}(\chi^2(s) + \chi(s^2)) + \frac{1}{2}(\chi'^2(s) + \chi'(s^2)) + \chi(s)\chi'(s) \\
&= (\chi^2_\sigma + \chi'^2_\sigma + \chi\chi')(s).
\end{aligned}$$

The proof for the alternating square is analogous.

Exercise 2.2
Let X be a finite set on which G acts. Let ρ be the corresponding permutation representation. Let χ_X be the character of ρ. Let $s \in G$. Show that $\chi_X(s)$ is the number of elements of X fixed by s.

This is almost a tautology. However, I provide a formal proof. The fact that ρ is a permutation representation means that for every $s \in G$ we see that ρ_s is a permutation matrix. In particular, we order the elements of X as $x_1, x_2, \ldots, x_n$. Then, say $sx_i =: x_{s,i}$. We associate e_i with x_i. Now, we define the matrix ρ_s by

$$\rho_s(e_i) = e_{s,i}$$

and we note that indeed since the map $\pi_s : x \mapsto sx$ is a permutation of X, that ρ_s is a permutation matrix. Now, we examine the diagonal of ρ_s.
We see that $(\rho_s)^i_i = (\rho_s(e_i))^i = 1$ if $\rho_s(e_i) = e_i$ and $= 0$ otherwise. So, $(\rho_s)^i_i = 1$ if $sx_i = x_i$ and $= 0$ otherwise. Thus,

$$Tr(\rho_s) = \sum_{i \in [|X|]} (\rho_s)^i_i = \sum_{i \in [|X|] : sx_i = x_i} 1 = \#\{i \in [|X|] : sx_i = x_i\}.$$

Exercise 2.3
Let $\rho : G \to GL(V)$ be a linear representation with character χ and let V' be the dual of V, i.e. the space of linear forms on V. For $x \in V$, $x' \in V'$ let $\langle x, x' \rangle$ denote the value of the linear form x' at x. Show that there exists a unique linear representation $\rho' : G \to GL(V')$ such that

$$\langle \rho_s x, \rho'_s x' \rangle = \langle x, x' \rangle$$

for $s \in G, x \in V, x' \in V'$. This is called the contragradient (or dual) representation of ρ; its character is χ^*.

Let ρ'_s be defined by $\rho'_s(x') = x' \circ \rho_s^{-1}$ and note that

$$\langle \rho_s(x), \rho'_s(x') \rangle = \langle \rho_s(x), x' \circ \rho_s^{-1} \rangle = (x' \circ \rho_s^{-1})(\rho_s(x)) = x'(x) = \langle x, x' \rangle$$

as desired.

Exercise 2.4
Let $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ be two representations with characters χ_1, χ_2. Let $W = Hom(V_1, V_2)$ be the vector space of linear mappings $f : V_1 \to V_2$. For $s \in G$ and $f \in W$, let $\rho_s f = \rho_{2,s} \circ f \circ \rho_{1,s}^{-1}$; so $\rho_s f \in W$. Show that this defines a linear representation $\rho : G \to GL(W)$, and that its character is $\chi_1^* \cdot \chi_2$. This representation is isomorphic to $\rho_1' \otimes \rho_2$, where ρ_1' is the contragradient of ρ_1.

We need to show that this map respects the group operation, namely that

$$\rho_{st}(f) = \rho_s(\rho_t(f)).$$

Well,

$$\begin{aligned}
\rho_{st}(f) &= \rho_{2,st} \circ f \circ \rho_{1,st}^{-1} \\
&= \rho_{2,st} \circ f \circ \rho_{1,(st)^{-1}} \\
&= \rho_{2,s}\rho_{2,t} \circ f \circ \rho_{1,t^{-1}s^{-1}} \\
&= \rho_{2,s}\rho_{2,t} \circ f \circ \rho_{1,t^{-1}}\rho_{1,s^{-1}} \\
&= \rho_{2,s} \circ (\rho_{2,t} \circ f \circ \rho_{1,t^{-1}}) \circ \rho_{1,s^{-1}} \\
&= \rho_s(\rho_{2,t} \circ f \circ \rho_{1,t^{-1}}) \\
&= \rho_s(\rho_t(f)).
\end{aligned}$$

We first show that $\rho \cong \rho_1' \otimes \rho_2$ by explicitly contructing an isomorphism.
Let $\tau : V_1^* \otimes V_2 \to Hom(V_1, V_2)$ be defined by $\tau(v_1', v_2) = (v_1 \mapsto v_1'(v_1)v_2)$.
Then, we note

$$\begin{aligned}
\tau((\rho_1' \otimes \rho_2)(v_1', v_2)) &= \tau(\rho_1'(v_1') \otimes \rho_2(v_2)) \\
&= (v_1 \mapsto \rho_1'(v_1')\rho_2(v_2)) \\
&= (v_1 \mapsto v_1'(\rho_1^{-1}(v_1))\rho_2(v_2))
\end{aligned}$$

and also note

$$\begin{aligned}
\rho_s(\tau(v_1', v_2)) &= \rho_{2,s} \circ (\tau(v_1', v_2)) \circ \rho_{1,s}^{-1} \\
&= \rho_{2,s} \circ (w \mapsto v_1'(w)v_2)) \circ \rho_{1,s}^{-1} \\
&= \rho_{2,s} \circ (v_1 \mapsto \rho_{1,s}^{-1}v_1 \mapsto v_1'(\rho_{1,s}^{-1}(v_1))v_2) \\
&= \rho_{2,s} \circ (v_1 \mapsto v_1'(\rho_{1,s}^{-1}(v_1))v_2) \\
&= (v_1 \mapsto v_1'(\rho_{1,s}^{-1}(v_1))\rho_{2,s}(v_2))
\end{aligned}$$

So, we have constructed an isomorphism as desired.

To show that its character is $\chi_1^* \cdot \chi_2$, we simply cite the fact that $\chi(\rho_1' \otimes \rho_2) = \chi(\rho_1') \cdot \chi(\rho_2)$. However, we also need to show that $\chi(\rho_1') = \chi(\rho_1)^*$.
To do so, we note that

$$\chi(\rho_1') = Tr(\rho_1') = \sum_i \lambda_i(\rho_1').$$

We now examine the eigenvalues of ρ_1'. Indeed, the eigenvalue, eigenvector pairs are (k, v_1') where $k \in K$ and $v_1' \in V_1'$ such that $\rho_1'(v_1') = kv_1'$, and since $v_1' : V_1 \to K$ is a map that implies that these maps differ by a constant on all inputs $v_1 \in V_1$, or more precisely that $\rho_1'(v_1')(v_1) = kv_1'(v_1)$ for all $v_1 \in V_1$.
Recall

$$\rho_1'(v_1')(v_1) = v_1'(\rho_1^{-1}(v_1))$$

meaning that

$$v_1'(\rho_1^{-1}(v_1)) = kv_1'(v_1) = kv_1'(v_1)$$

for all $v_1 \in V_1$. So certainly if $(\hat{k}, v_1)$ is an eigenpair of ρ_1, then it suffices to take $v_1' : V_1 \to K$ such that $v_1'(cv_1) = cv_1$ for all $c \in K$ and $v_1'(u_1) = 0$ for all $u_1 \perp v_1$.