Introduction to Representation Theory: MIT OpenCourseWare 18.712
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[1.20] Let V be a nonzero finite-dimensional representation of an algebra A. Show that it has an irreducible subrepresentation. Then show by example that this does not always hold for infinite-dimensional representations.

Either V is irreducible or it is not. If not, then it has some proper non-trivial subrepresentation V_1 of dimension $dim(V_1) < dim(V)$. We continue in this fashion finding subrepresentations

$$V_i \subsetneq V_{i-1} \subsetneq V_1 \subsetneq V$$

until V_i is irreducible which we know is possible for the following reason. We note that any representation of dimension 1 is irreducible and indeed the dimension decreases by at least one at each step, or written precisely that $dim(V_i) \leq dim(V_{i-1}) - 1$, meaning that $1 \leq dim(V_i) \leq dim(V) - i$. So, indeed provided that $dim(V) > 0$ this process will make sense and terminate since $1 \leq dim(V) - i$ means that $i \leq dim(V) - 1 < \infty$.

[1.21] Problem 1.21. Let A be an algebra over a field k. The center Z(A) of A is the set of all elements $z \in A$ which commute with all elements of A. For example, if A is commutative then $Z(A) = A$.

(a) Show that if V is an irreducible finite-dimensional representation of A then any element $z \in Z(A)$ acts in V by multiplication by some scalar $\chi_V \in k$. Show that $\chi_V : Z(A) \to k$ is a homomorphism. It is called the central character of V.

This makes intuitive sense since the center of the $GL_n(\mathbb{R})$ for instance is the set of scalar multiples of the identity. Now I provide a formal proof.

First, I note that we also need the field k to be algebraically closed otherwise taking $A = \mathbb{C}$ (as an $\mathbb{R}$-algebra) we note that $Z(A) = A$. Then, one notes that $V = A$ is a 2-dimensional representation over $k = \mathbb{R}$ (not algebraically closed). Indeed, taking the regular representation and the element $g = 1 + i = (1, 1) \in Z(A)$, we note that g acts on an element $v = (a, b) \in V$ by

$$(a, b) \mapsto^g (a - b, a + b),$$

and clearly taking $(a, b) = (0, 1)$ we see that

$$(a - b, a + b) = (-1, 1) \neq \lambda(0, 1)$$

for any $\lambda \in k = \mathbb{R}$.

However, assuming that k is algebraically closed, we proceed to prove the statement. Indeed, we cite Corollary 1.17, noting that for any $z \in Z(A)$ we have that $\rho(z)$ is an intertwining operator within $\rho(A)$ since for any $a \in A$ to verify $\rho(z)$ is an intertwining operator we need to show that

$$(\rho(z))(\rho(a)v)) = \rho(a)(\rho(z))(v),$$

for all $v \in V$ and all $a \in A$.
Indeed, we note that

$$(\rho(z))(\rho(a)v)) = \rho(za)v = \rho(az)v = \rho(a)\rho(z)v.$$

Then Corollary 1.17 gives the result.

(b) Show that if V is an indecomposable finite-dimensional representation of A then for any $z \in Z(A)$ the operator $\rho(z)$ by which z acts in V has only one eigenvalue $\chi_V(z)$, equal to the scalar by which z acts on some irreducible subrepresentation of V. Thus $\chi_V : Z(A) \to k$ is a homomorphism, which is again called the central character of V.

First, I show that if ρ has an eigenvalue then it is unique and then I will show at the end that it has an eigenvalue, which is clearly true by virtue of k being algebraically closed.

Suppose there exist ρ_1, ρ_2 such that

$$\rho(z)v_1 = \lambda_1 v_1$$

and

$$\rho(z)v_2 = \lambda_2 v_2$$

for some $\lambda_1, \lambda_2 \in k$ and $v_1, v_2 \in V$ with $v_1, v_2 \neq 0$.
Now, for this fixed $z \in A$ let $W = \{v \in V : \rho(z)v = \lambda_1 v\}$. This is a vector subspace.

I would like to show that W is a subrepresentation, namely that for all $w \in W$ and all $a \in A$ we have that $\rho(a)w \in W$.

Assume not. Assume that there exists $w \in W$ such that $\rho(a)w \notin W$ meaning that

$$\rho(z)(\rho(a)w) \neq \lambda_1(\rho(a)w).$$

Then note that $a = z^{-1}az$ where $z \in Z(A)$ is the same fixed z from above.

So, $\rho(a) = \rho(z^{-1}az)$ meaning that

$$\rho(a)w = \rho(z^{-1}az)w.$$

Now multiplying on both sides by $\rho(z)$ we get

$$\rho(z)\rho(a)w = \rho(z)\rho(z^{-1}az)w$$

which means that

$$\rho(z)\rho(a)w = \rho(z)\rho(z^{-1}a)\rho(z)w = \rho(z)\rho(z^{-1}a)\lambda_1 w = \rho(zz^{-1})\rho(a)\lambda_1 w$$

$$= \lambda_1\rho(e)\rho(a)w = \lambda_1\rho(a)w.$$

We now have a contradiction since the above implies that

$$\rho(z)(\rho(a)w) = \lambda_1(\rho(a)w),$$

which means by definition that $\rho(a)w \in W$. So, indeed W is a subrepresentation.

Now, either W is irreducible or it is not. If not, then as shown in Exercise 1.20, we know that it contains some irreducible representation W'. Now, indeed we have shown that for any $z \in V$ one has that $\rho(z)$ has only one eigenvalue equal to the scalar by which z acts on W'.

Finally, we show that $\rho(z)$ actually has an eigenvalue $\lambda \in k$. Since k is algebraically closed, indeed it does since all roots of the characteristic polynomial belong to k.

So, indeed we have shown the desired result where λ which exists as argued above is an eigenvalue and we take $W_0 = \{w \in V : \rho(z)v = \lambda z\}$. Either W_0 is irreducible or it is not. If not we follow the same procedure above Equation we find W_i irreducible such that $\rho(z)w = \lambda w$ for all $w \in W_i$.

(c) <u>Does $\rho(z)$ in (b) have to be a scalar operator?</u>

No, it does not. For instance, take $A = \mathbb{Z}/2\mathbb{Z}a + \mathbb{Z}/2\mathbb{Z}b$ as a $\mathbb{Z}/2\mathbb{Z}$ algebra where a, b are indeterminants and we declare $ab = ba$ and that $\overline{0} = 0a + 0b$. Then, define $\rho : A \to GL(\mathbb{R}^2)$ by

$$\rho(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\rho(1a + 0b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\rho(1a + 1b) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since A is commutative one has that $Z(A) = A$. So, let $z = 1a + 1b$. We see that

$$\rho(1a + 1b) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is not a scalar operator since

$$\rho(1a + 1b)(1, 1)^T = (0, 1)^T$$

but

$$(1, 1)^T \neq c(0, 1)^T$$

for any $c \in \mathbb{Z}/2\mathbb{Z}$.

I provide another slightly different example in which A is an algebra over an infinite field. Namely, take $A = \mathbb{Q} + \mathbb{Q}x$ where x is an indeterminant and we stipulate that $x^2 = 0$. (So really this is just the ring $\mathbb{Q}[x]/(x^2)$).

Then, let $\rho : A \to GL(\mathbb{R}^2)$ be defined by

$$\rho(a + bx) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Indeed, we see that

$$\begin{aligned} \rho((a + bx)(c + dx)) &= \rho(ac + (ad + bc)x) \\ &= \begin{pmatrix} ac & ad + bc \\ 0 & ac \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \\ &= \rho(a + bx)\rho(c + dx), \end{aligned}$$

which shows that ρ is a homomorphism of algebras.

However, $\rho(1 + x)$ is not a scalar operator since in fact, similar to before, we obtain

$$\rho(1 + x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and again

$$\rho(1 + x)(0, 1)^T = (1, 1)^T \neq \alpha(0, 1)^T$$

for any $\alpha \in \mathbb{Q}$.

Now, say we require k to be algebraically closed and ask whether one can still find a counterexample. Indeed we can. Take $A = \mathbb{C}[x]/(x^n)$ as a $\mathbb{C}$ algebra and let $\rho : \mathbb{C}[x]/(x^n) \to \mathbb{C}^n$ be defined by

$$\rho(x) = J(2, n).$$

First, I verify that ρ is a homomorhism of algebras. Namely, I note

$$\begin{aligned} \rho((a + bi)(c + di)) &= \rho(ac - bd + (ad + bc)i) \\ &= \begin{pmatrix} (ac - bd) & (ad + bc) \\ -(ad + bc) & (ac - bd) \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \end{aligned}$$

Once again $Z(\mathbb{C}) = \mathbb{C}$. Take $z = i$. Then,

$$\rho(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

and note that

$$\rho(i)(1,0)^T = (0,-1)^T \neq \alpha(1,0)^T$$

for any $\alpha \in \mathbb{C}$.
We must also verify however that $\rho, \mathbb{C}^2$ is an indecomposable representation.

It suffices to show that it is irreducible, which it is. Otherwise, there would exist some non-trivial proper subrepresentation $W \subseteq V$, which must be of dimension $dim(W) = 1$ since $dim(V) = 2$. So, it must be of the form $W = span(w)$ such that

$$\rho(a + bi)w \in W$$

for all $a, b \in \mathbb{R}$, meaning that

$$\rho(a + bi)w = (\alpha + \beta i)w$$

for some $\alpha, \beta \in \mathbb{R}$. Then, writing $w = (w_1, w_2)^T$ for some $w_1, w_2 \in \mathbb{C}$ gives

$$\begin{aligned} \rho(a + bi)(w_1, w_2)^T &= (\alpha + \beta i)(w_1, w_2)^T \\ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} (w_1, w_2)^T &= (\alpha + \beta i)(w_1, w_2)^T \\ \begin{pmatrix} aw_1 + bw_2 \\ -bw_1 + aw_2 \end{pmatrix} &= \begin{pmatrix} (\alpha + \beta i)w_1 \\ (\alpha + \beta i)w_2)^T \end{pmatrix} \end{aligned}$$

which means that

$$aw_1 + bw_2 = \alpha w_1 + \beta i w_2$$

and

$$-bw_1 + aw_2 = \alpha w_2 + \beta i w_2.$$

The above statement must hold for all $a, b \in \mathbb{R}$ so take $(a, b) = (0, 1)$ meaning $a + bi = i$).
If $(a, b) = (a, 0)$ then

$$\rho(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and indeed the eigenvalues of $\rho(i)$ are the roots of $x^2 + 1$ which are $\lambda_1 = i$ and $\lambda_2 = -i$ with corresponding eigenvectors $v_1 = (-i, 1)^T$ and $v_2 = (i, 1)^T$, which we note form a basis for $\mathbb{C}^2$. If ρ is reducible, that would require some non-trivial stable subspace $U \subseteq V$, which would need to be one-dimensional. Then, the requirement that $dim(U) = 1$ implies that $U = Span(u)$ for some $u \in V$.

Note that $u \in \{v_1, v_2\}$. Otherwise, if $u = c_1u_1 + c_2u_2 = ((-c_1 + c_2)i, c_1 + c_2)^T$ for some $c_1, c_2 \in \mathbb{C}$ with $c_1, c_2 \neq 0$, then

$$\begin{aligned} \rho(i)(u) &= \rho(i)(((-c_1 + c_2)i, c_1 + c_2)^T) \\ &= c_1(\rho(i)(u_1)) + c_2(\rho(i)(u_2)) \\ &= c_1\lambda_1 u_1 + c_2\lambda_2 v_2 \\ &= c_1(i(-i, 1)^T) + c_2(-i(i, 1)^T) \\ &= c_1(1, i)^T + c_2(1, -i)^T \\ &= (c_1 + c_2, (c_1 - c_2)i)^T. \end{aligned}$$

If $span(U)$ is stable under the action of A, then that would imply that

$$\begin{pmatrix} c_1 + c_2 \\ (c_1 - c_2)i \end{pmatrix} \in Span\left(\begin{pmatrix} (-c_1 + c_2)i \\ c_1 + c_2 \end{pmatrix} \right)$$

meaning that

$$\begin{pmatrix} c_1 + c_2 \\ (c_1 - c_2)i \end{pmatrix} = \begin{pmatrix} d(-c_1 + c_2)i \\ d(c_1 + c_2) \end{pmatrix}$$

for some $d \in \mathbb{C}$.
Then, that implies that

$$c_1 + c_2 = di(-c_1 + c_2) \tag{1}$$
$$c_1 - c_2 = -di(c_1 + c_2). \tag{2}$$

Then, adding the above equations gives

$$2c_1 = -2c_1 di$$

implying that $c_1 = 0$, which cannot happen by assumption, or that $1 = -di$ meaning that $d = i$.
Now, substituting back into our original pair of equations 1 and 2 gives

$$c_1 + c_2 = c_1 - c_2$$
$$c_1 - c_2 = c_1 + c_2$$

which then implies that $c_2 = -c_2$ or that $c_2 = 0$, which contradicts our assumption that $u = c_1v_1 + c_2v_2$ with $c_1, c_2 \neq 0$.

Thus, $U = Span(v_1)$ or $U = Span(v_2)$. However, neither subspace is stable under the action of A. Namely, take $a + bi = 2 + i$. Then,

$$\rho(2 + i) = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

so that

$$\begin{aligned} \rho(2 + i)v_1 &= \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} v_1 \\ &= \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} (-i, 1)^T \\ &= \begin{pmatrix} 1 - 2i \\ 2 - i \end{pmatrix} \end{aligned}$$

Now, if we were to have $\rho(2 + i)w_1 \in W_1$ we would need $r + si \in \mathbb{C}$ such that

$$1 - 2i = -i(r + si)$$

and

$$2 - i = r + si$$

which means we would have $r = 2$ and $s = -1$ implying that

$$1 - 2i = -i(2 - i) = -2i + i^2 = -2i - 1,$$

which is a contradiction. So, indeed we see that this representation is in fact irreducible and consequently also indecomposable, which means this is a counterexample. So, even if we require the field k to be algebraically closed, $\rho(z)$ is not necessarily a scalar operator.

[1.22]
Let A be an associative algebra, and V a representation of A. By $End_A(V)$ one denotes the algebra of all homomorphisms of representations $V \to V$. Show that $End_A(A) = A^{op}$, the algebra A with opposite multiplication. I assume that $\rho(a)(b) = ab$ for all $a, b \in A$ is the regular representation.

We want to construct a bijection between $\{\psi \in GL(A) : \psi(\rho(a)b)) = \rho(a)\psi(b)\forall a, b \in A\}$ (Condition (*)) and A. Let's try

$$\tau : \langle A, \cdot_{op}\rangle \to GL(A)$$

$$\tau : a \mapsto \rho(a)$$

for each $a \in A$. Now, is this a homomorphism of algebras? Yes, namely we have that $\tau(c_1a + c_1b) = \rho(c_1a + c_2b) = c_1\rho(a) + c_2\rho(b)$. Also, note that $\rho(a)$ satisfies Condition $(*)$ since $(\rho(\rho(a)b))(c) = \rho(ab)(c) = (ab) * c = a * (bc) = a * \rho(b)(c) = \rho(a)(\rho(b)(c))$. Also, we have that for all $c \in A$ that $\tau(ab)(c) = \rho(ab)(c) = (ab) * (c) = \rho(a)(\rho(b)(c)) = \tau(a)(\tau(b)c)$. Now since multiplication in the endomorphism ring of two endomorphisms f, g is defined by $(f * g)(c) = g(f(c))$ we have that $\tau(ab) = \tau(b) * \tau(a)$ proving the claim.

[1.23]
Prove the following "Infinite-Dimensional Version of Schur's Lemma": Let A be an algebra over $\mathbb{C}$ and V be an irreducible representation of A with at most countable basis. Then any homomorphism of representations $\phi : V \to V$ is a scalar operator.

First, I show that D is at most countable dimensional. To do so, I exhibit a countable spanning set of an even larger space namely $End(V)$.
Namely, if $\{w_n\}_{n\in\mathbb{N}}$ is a countable basis for V (we may assume the basis is countably infinite since if it is finite there is nothing to show). Then consider the set $S = \cup_{i\in\mathbb{N}} \cup_{j\in\mathbb{N}} f^i_j$ where $f^i_j \in End(V)$ is the endomorphism defined by $f^i_j : v_i \mapsto v_j$ and $f^i_j : v_k \mapsto 0$ if $k \neq i$. Then, note that any endomorphism $g : V \to V$ is defined by $g(v_i)$ for each $i \in \mathbb{N}$ and since v_i spans V one has that $g(v_i) = \sum_{j\in\mathbb{N}} a^i_j v_j$ and thus $g = \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} a^i_j f^i_j$. Thus since $|S| = | \cup_{n\in\mathbb{N}} \mathbb{N}| = |\mathbb{N}|$ one has that $dim(End(V)) \leq |\mathbb{N}|$ and since $End_A(V) \subseteq End(V)$ the claim follows.
Then, assume that ϕ is not a scalar and as the hint suggests I show that $\mathbb{C}(\phi)$ is a transcendental extension of $\mathbb{C}id \subseteq D$. Otherwise it is algebraic meaning that there exists $p(x) = \sum_{i=0}^{n}(a_i id)x^i$ such that $p(\phi) = 0$. However, since $\mathbb{C}$ is alebgraically closed we know that $p(x)$ splits into linear factors $p(x) = \prod_{i=0}^{n}(x - \lambda_i id)$ meaning that $\phi = \lambda_i id$ for some $i \in [0 : n]$, a contradiction. Thus, $\mathbb{C}(\phi)$ is a transcendental extension and is thus uncountably infinite-dimensional as vector space over $\mathbb{C}$. However, that provides a contradiction as $\mathbb{C}(\phi) \subseteq D$ which is at most countable dimensional. Thus, ϕ is a scalar operator.

[1.24]
Let $A = k[x_1, \ldots, x_n]$ and $I \neq A$ be any ideal in A containing all homogenous polynomials of degree $\geq N$. Show that A/I is an indecomposable representation of A.

Note that $1 + I \in A/I$ is cyclic and in particular $\rho(A)(1 + I) = A/I$ meaning that A/I is not decomposable since if it were of the form $A/I = V \oplus W$ with V, W non-empty subrepresentations meaning that $\rho(a)V \subseteq V$ and $\rho(a)W \subseteq W$. Now, without loss of generality one has that $1 + I \in V$ and then however since $W \neq \emptyset$ there exists $f + I \in W$ and then $\rho(f)(1 + I) = f + I \in W$, a contradiction since $1 + I \in V$.

[1.25]
Let $V \neq 0$ be a representation of A. We say that a vector $v \in V$ is cyclic if it generates V, i.e., $Av = V$. A representation admitting a cyclic vector is said to be cyclic. Show that

(a) V is irreducible if and only if all nonzero vectors of V are cyclic.

If all nonzero $v \in V$ are cyclic, then V is irreducible. otherwise, if there existed a subrepresentation W with $0 \subsetneq W \subsetneq V$, then taking any $w \in W$ we have that $\{\rho(a)w : a \in A\} = V \supsetneq W$ a contradiction to W a subrepresentation. Now, to show the converse we show that if some non-zero vector $v \in V$ is not cyclic then, V is reducible. In particular, if one has such v let $W = Av$ and note that W is closed under the action of A and is thus a subrepresentation. Since $Av \neq V$ and $v \neq 0$ implying $Av \neq 0$ we have that $0 \neq W \subsetneq V$ is a subrepresentation.

(b) V is cyclic if and only if it is isomorphic to A/I, where I is a left ideal in A.

Define the map $\psi_v : A \to V$ by $\psi_v(a) = \rho(a)(v)$. Note that by the Ring Isomorphsm Theorem we have that $Im(\psi_v) \cong A/ker(\psi_v)$. Since $Im(\psi_v) = V$ and since $ker(\psi_v)$ is an ideal by definition of a ring isomorphism we have that $V \cong A/I$ where $I = ker(\psi_v)$ if V is cyclic. Now, if $V \cong A/I$, then

(c) <u>Give an example of an indecomposable representation which is not cyclic.</u>

Note that one has an obvious isomorphism $\phi : A^* \to \mathbb{R}^3$ given by $\phi(f) = (f(1), f(x), f(y))^T$. Then, by definition of ρ one can write

$$\rho(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho(x) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\rho(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that A^* is not cyclic since if one did have a generator f with $\phi(f) = (f(1), f(x), f(y))^T$ then of course one would need $(f(1), f(x), f(y))^T \neq (0,0,0)^T$, and then one has that $\rho(1)(\phi(f)) = (f(1), f(x), f(y))^T$, $\rho(x)(\phi(f)) = (f(x), 0, 0)^T$, and $\rho(y)(\phi(f)) = (f(y), 0, 0)^T$.
Also, note that $A = \{k_1 + k_2 x + k_3 y : k_i \in k \forall i \in [3]\}$. Thus,

$$\{\rho(a) : a \in A\} = \{ \begin{pmatrix} k_1 & 0 & 0 \\ k_2 & k_1 & 0 \\ k_3 & 0 & k_1 \end{pmatrix} : k_i \in k \forall i \in [3]\}.$$

In order for A^* to be cyclic we would need $f \in A^*$ such that $\rho(A)f = A^*$. Now, if one had such f then certainly $f \not\equiv 0$. If $f(1) = 0$ then note that $(1,0,0)^T \notin \rho(A)f$ and thus f is not a cyclic vector. Now, if another $v = (f(1), f(x), f(y))^T$ with $f(0) \neq 0$ is a generator then note

$$\{\rho(a)f : a \in A\} = \{ \begin{pmatrix} k_1 & 0 & 0 \\ k_2 & k_1 & 0 \\ k_3 & 0 & k_1 \end{pmatrix} \begin{pmatrix} f(1) \\ f(x) \\ f(y) \end{pmatrix} : k_i \in k \forall i \in [3]\} = \{ \begin{pmatrix} k_1 f(1) \\ k_2 f(1) + k_1 f(x) \\ k_3 f(1) + k_1 f(y) \end{pmatrix} : k_i \in k \forall i \in [3]\}.$$

If $g \in \rho(A)f$ for arbitrary $g \in A^*$ then note that $k_1 = g(1)/f(1)$ meaning that $k_2 = (g(x) - g(1)/f(1)f(x))/f(1)$ and $k_3 = (g(y) - g(1)/f(1)f(y))/f(1)$ which I think would show that A^* is cyclic simply taking $f \in A^*$ to be $f : 1 \mapsto 1$, $f : x \mapsto 0$, $f : y \mapsto 0$ then for any $g \in A^*$ we simply take $k_1 = g(1), k_2 = g(x), k_3 = g(y)$. I may need to come back and review this. However, for now I'll think of another example.
Let

$$M = \{M_{a,b,c} := \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{R}\}$$

Note that M is an algebra since M is closed under matrix multiplication. Also, note that if one takes the representation $V = \mathbb{R}^3$ with $\rho(M)v = Mv$ then one notes that V is indecomposable since otherwise it has the form $V = U \oplus W$ with U irreducible. Now, without loss of generality $dim(U) = 1$. One wishes to find all 1-dimensional subrepresentations of V which would be of the form $U = \{k(x,y,z)^T : k \in \mathbb{R}\}$ for some fixed $(x,y,z)^T \in \mathbb{R}^3$ which would imply that

$$\{\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \{\begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}$$

for all $a, b, c \in \mathbb{R}$ meaning

$$\{\begin{pmatrix} ay + bz \\ cz \\ 0 \end{pmatrix} = \{\begin{pmatrix} \lambda_{a,b,c} x \\ \lambda_{a,b,c} y \\ \lambda_{a,b,c} z \end{pmatrix}$$

meaning $\lambda_{a,b,c} z = 0$ which implies that $\lambda_{a,b,c} = 0$ or $z = 0$. First I handle the case in which $z \neq 0$ which implies that $\lambda_{a,b,c} = 0$ which implies that $cz = 0$ meaning $c = 0$, a contradiction since the above equation in fixed (x,y,z) but varied $\lambda_{a,b,c}$ must hold for all $a, b, c \in \mathbb{R}$.
If $z = 0$ we have that $cz = \lambda_{a,b,c} y = 0$ and also as long as $x \neq 0$ we have $\lambda_{a,b,c} = \frac{ay}{x}$ and thus $M(x,y,0)^T = (\frac{ay}{x} x, \frac{ay}{x} y, 0)^T = (ay, cz, 0)^T = (ay, 0, 0)^T$ which implies that $\frac{ay}{x} y = 0$ meaning $y^2 = 0$ meaning $y = 0$ (since $\mathbb{R}$ has no zero divisors) since this holds for all $a \in \mathbb{R}$ (provided $z = 0$) which then means that $(x,y,z)^T = (x,0,0)^T$ with $x \neq 0$. Indeed this is an irreducible subrepresentation. However, V does not decompose as $U \oplus W$ since $M_{1,1,1}(0,1,0)^T = (1,0,0)^T \in U$ but $(0,1,0)^T \notin U$.
Finally, note that $V = \mathbb{R}^3$ is not cyclic since $(0,0,1)^T \in V$ but $(0,0,1)^T \neq M_{a,b,c}(x,y,z)^T$ for all $a, b, c, x, y, z \in \mathbb{R}$.

[1.26]

Let A be the Weyl algebra, generated by two elements x, y with the relation

$$yx - xy - 1 = 0.$$

(a) If char$k = 0$, what are the finite-dimensional representations of A? What are the two-sided ideals in A?

If $char(k) = 0$ then there are no finite-dimensional representations of A since $yx - xy - 1 = 0$ implies that $\chi(\rho(yx) - \rho(xy) - I) = \chi(\rho(yx)) - \chi(\rho(xy)) - \chi(I) = -\chi(I) = 0$ implies that $\chi(I) = dim(V) = 0$ meaning $dim(V) = 0$ meaning that $V = 0$.
For the second part, consider a non-zero ideal $I \subseteq A$, meaning there exists $p(x,y) \in I$.

Otherwise, $p(x,y)$ is of course a sum of terms $p(x,y) = \sum_{i=0}^{N} a_i \prod_{j=0}^{n_i} x^{r_j^i} y^{s_j^i} =: \sum_{i=0}^{N} t_i(x,y)$ where $r_j^i, s_j^i \in \{0,1\}$ and $r_j^i \neq s_j^i$. I claim that we may write $p(x,y)$ in the form $p(x,y) = \sum_{i=0}^{M} b_i y^{s_i} x^{r_i}$. I prove so by induction on the quantity $M := \max_{i \in [0:N]}(w(t_i(x,y))$ where
$w(t_i(x,y)) := \sum_{j \in [1:n_i] : s_j^i = 1 \text{ and } r_l^i = 1 \text{ for some } l < j} |\{l < j : r_l^i = 1\}|$ i.e. the sum over all $y's$ that appear not grouped to the left in the ith term or the number of $x's$ that appear before them. Note for clarity that M is a function of our expression of $p(x,y)$. Indeed $p(x,y)$ is not changing throughout the proof below. So, in our base case where $M = 0$ there is nothing to prove. So, we may assume that $M \geq 1$. Then, we note that $p(x,y) = \sum_{i \in [0:N] : w(t_i(x,y)) = 0} t_i(x,y) + \sum_{i \in [0:N] : w(t_i(x,y)) \neq 0} t_i(x,y)$. Now, note that for each $i \in [0:N]$ such that $w_(t_i(x,y)) \neq 0$ we know that $t_i(x,y) = a_i y^{d_i} x^{e_i} xy s_i(x,y)$ where $s_i(x,y) = \prod_{k \in [0:K_i]} x^{\alpha_k} y^{\beta_k}$ and $d_i, e_i \in \mathbb{N}_0$. Then, indeed $p(x,y) = \sum_{i \in [0:N] : w(t_i(x,y)) = 0} t_i(x,y) + \sum_{i \in [0:N] : w(t_i(x,y)) \neq 0} a_i y^{d_i} x^{e_i} xy s_i(x,y)$, but now note that $a_i y^{d_i} x^{e_i} xy s_i(x,y) = a_i y^{d_i} x^{e_i} yx s_i(x,y) - a_i y^{d_i} x^{e_i} s_i(x,y)$ and that
$w(a_i y^{d_i} x^{e_i} yx s_i(x,y)) = w(a_i y^{d_i} x^{e_i} xy s_i(x,y)) - 1$ and $w(a_i y^{d_i} x^{e_i} s_i(x,y)) \leq w(a_i y^{d_i} x^{e_i} xy s_i(x,y)) - 1$. Thus, for our new expression we have

$$\begin{aligned} M\Big(\sum_{i \in [0:N] : w(t_i(x,y)) = 0} t_i(x,y) + \sum_{i \in [0:N] : w(t_i(x,y)) \neq 0} (a_i y^{d_i} x^{e_i} yx s_i(x,y) - a_i y^{d_i} x^{e_i} s_i(x,y))\Big) \\ = M\Big(\sum_{i \in [0:N] : w(t_i(x,y)) = 0} t_i(x,y) + \sum_{i \in [0:N] : w(t_i(x,y)) \neq 0} t_i(x,y)\Big) - 1. \end{aligned}$$

So, indeed we have shown by induction that we may write $p(x, y)$ in the form $p(x, y) = \sum_{i=0}^{M} b_i y^{s_i} x^{r_i}$. Now, letting $p_0(x, y) = \sum_{i=0}^{M} b_i y^{s_i} x^{r_i}$ (Note here when we write $p(x, y)$ we mean using the exact formal expression specified) iterate the following process.

Then, note that $xp(x, y) \in I$ and $-p(x, y)x \in I$. Let $P(x, y) = xp(x, y) - p(x, y)x$. I claim firstly that we can write $P(x, y) = \sum_{i=0}^{M'} B_i y^{s'_i} x^{r'_i} = \sum_{i=0}^{M'} S_i(x, y)$ and that $max\{s'_i : i \in [0 : M']]\} < max\{s_i : i \in [0 : M]\}$ (i.e. the max power of y appearing in any term goes down by at least 1) meaning that after finitely iterations of the process one obtains a polynomial solely in x.
I show this by looking term by term. Consider the term $S_i(x, y) = b_i y^{s_i} x^{r_i}$. Either $s_i = 0$ in which case $xS_i(x, y) - S_i(x, y)x = 0$. Otherwise if $s_i \neq 0$ then note that $xb_i y^{s_i} x^{r_i} - b_i y^{s_i} x^{r_i} x = b_i xy^{s_i} x^{r_i} - b_i y^{s_i - 1} yxx^{r_i} = b_i xy^{s_i} x^{r_i} - b_i y^{s_i-1} xyx^{r_i} - b_i y^{s_i-1} x^{r_i} = b_i xy^{s_i} x^{r_i} - b_i y^{s_i-2} xy^2 x^{r_i} - b_i y^{s_i-1} x^{r_i} - b_i y^{s_i-1} x^{r_i} = b_i xy^{s_i} x^{r_i} - b_i y^{s_i-3} xy^3 x^{r_i} - b_i y^{s_i-1} x^{r_i} - b_i y^{s_i-1} x^{r_i} - b_i y^{s_i-1} x^{r_i} = b_i xy^{s_i} x^{r_i} - b_i yxy^{s_i-1} x^{r_i} - \sum_{l=1}^{s_i} b_i y^{s_i-1} x^{r_i} = b_i(xy - yx)y^{s_i-1} x^{r_i} - s_i b_i y^{s_i-1} x^{r_i} = -b_i y^{s_i-1} x^{r_i} - s_i b_i y^{s_i-1} x^{r_i}$. So, indeed we see that

$$xp(x, y) - p(x, y)x = 0 + \sum_{i \in [0:M']: s_i \neq 0} (-b_i y^{s_i-1} x^{r_i} - s_i b_i y^{s_i-1} x^{r_i})$$

concluding the proof that $max\{s'_i : i \in [0 : M']]\} < max\{s_i : i \in [0 : M]\}$ and also, since we assumed that $p(x, y) \notin k[x]$ to start with that means that $xp(x, y) - p(x, y)x \neq 0$, concluding the proof.

Now, I claim that for arbitrary $P(x) \in k[x]$ note that for sufficiently large $K \in \mathbb{N}$ one has that $yP(x) - P(x)y \in K$. We consider $yx^n - x^n y$ for $n \in \mathbb{N}_0$. If $n = 0$ then $yx^n - x^n y = 0$. If $n = 1$ then $yx^n - x^n y = 1$. Now, note that for any $n \in \mathbb{N}_{\geq 2}$ one has $yx^n - x^n y = yx^n - x^{n-1} xy = yx^n - x^{n-1} yx + x^{n-1} = yx^n - x^{n-2} yxx + 2x^{n-1} = yx^n - xyx^{n-1} + \sum_{i=0}^{n} x^{n-1} = yx^n - xyx^{n-1} + \sum_{i=0}^{n} x^{n-1} = (yx - xy)x^{n-1} + \sum_{i=2}^{n} x^{n-1} = nx^{n-1}$. Thus, one has that $yP(x) - P(x)y = \frac{d}{dx} P(x)$. Now, clearly if $P(x) = A_N x^N + Q(x)$ where $deg(Q) \leq N - 1$ then $\frac{d^N}{dx^N} P(x) = N!A_N$. Thus, $\frac{1}{N!A_N} \frac{d^N}{dx^N} P(x) = 1$, meaning that if one denote $Q(P(X)) = yP(x) - P(x)y =: Q_1(P(x))$ and $Q_n(P(x)) = Q(Q_{n-1}(P(x)))$ then since $\frac{d^N}{dx^N} P(x) = Q_N(P(X)) \in I$ one has that $1 = \frac{1}{N!A_N} \frac{d^N}{dx^N} P(x) \in I$ meaning that $I = A$. Thus the only non-zero two-sided ideal of A is $I = A$.

(b) Suppose for the rest of the problem that char$k = p$. What is the center of A?

Note that as above $yx^p - x^p y = px^{p-1} = 0$ which implies $yx^p = x^p y$. Also, clearly $xx^p = x^p x$ which implies that $x^p \in Z(A)$. Now, note that there is something close to symmetry between x, y in the given relation. Namely, one has that $(x, y) = (a, b)$ satisfy the relation $ba - ab - 1 = 0$ and so does $(y, -x) = (a, b)$. So, as shown throughout part (a) one has for a, b satisfying the given relation that $ba^p - a^p b = pa^{p-1} = 0$ meaning for $(a, b) = (y, -x)$ one has that $(-x)y^p - y^p(-x) = py^{p-1} = 0$ meaning that $xy^p = y^p x$ and of course $yy^p = y^p y$ implying that $y^p \in Z(A)$.

(c) Find all irreducible finite-dimensional representations of A.

Note that since $\rho(x^p)$ (which I simply denote by x^p when the use is clear from context) is an intertwining operator we know that it is a scalar operator and likewise for $\rho(y^p)$. Thus one has that $\rho(A)v = span\{x^i y^j v : i, j \in [0 : p-1]\}$ and since $y^j v \in span(v)$ one has that $\rho(A) = span\{x^i v : i \in [0 : p-1]\}$. Then, y part (a) of Question 1.25 we know that every non-zero $v \in V$ is cyclic meaning that $Av = V$ for the v meaning that $V = span\{x^i v : i \in [0 : p-1]\}$. Finally note that $\{x^i v : i \in [0 : p-1]\}$ is a linear independent set since otherwise $dim(V) = dim(span\{x^i v : i \in [0 : p-1]\}) < p$ which provides a contradiction since then $\chi(I) < p$ meaning $\chi(I) \notin \mathbb{Z}p$ unless $\chi(I) = 0$ contradicting $0 = \chi(xy) - \chi(yx) = \chi(I)$ since $V \neq 0$.

[1.27]
Let q be a nonzero complex number, and A be the $q-$Weyl algebra over $\mathbb{C}$ generated by $x^{\pm 1}$ and $y^{\pm 1}$ with defining relations $xx^{-1} = x^{-1}x = 1$, $yy^{-1} = y^{-1}y = 1$, and $xy = qyx$.

(a) What is the center of A for different q? If q is not a root of unity, what are the two-sided ideals in A?

Note that $c \in Z(A)$ if and only if $xc = cx$ and $yc = cy$. Clearly if $q = 1$ then A is abelian meaning that $Z(A) = A$. Now say that $q^n = 1$ and $q^s \neq 1$ for $0 \leq s < n$ for some $n \in \mathbb{N}$. We see that if $c = \prod_{i=0}^{r} x^{s_i} y^{t_i}$ is a monomial in x, y with $s_i, t_i \in \mathbb{N}_{\geq 0}$ for each $i \in [0:r]$ then note that $c \in Z(A)$ implies that $xc = cx$ implying that $n \mid \sum_{i \in [0:r]} t_i$ and then $yc = cy$ implies that $n \mid \sum_{i \in [0:r]} t_i$. Thus, $Z(A) = \langle \prod_{i=0}^{r} x^{s_i} y^{t_i} : r \in \mathbb{N}, s_i, t_i \in \mathbb{N}_{\geq 0}, n \mid \sum_{i=0}^{r} t_i, n \mid \sum_{i=0}^{r} s_i \rangle$. (Here $\langle\rangle$ means algebra generation meaning finite linear combinations of these terms).

(b) For which q does this algebra have finite-dimensional representations?

Note that if there exists a finite-dimensional representation then since $xy = qyx$ one has that $det(xy) = q^{dim(V)}(det(yx)) = q^{dim(V)} det(xy)$ meaning that $q^{dim(V)} = 1$ or that q is a root of unity of order $ord(q)$ such that $ord(q) \mid dim(V)$, meaning it is necessary that q be a root of unity. Indeed we show in part (c) we show that the condition that q be a root of unity is also sufficient.

(c) Find all finite-dimensional irreducible representations of A for such q.

Say that q is an nth root of unity. Now, I claim that of $v \in V$ is an eigenvector of x then $\{v, yv, y^2v, y^3v, \ldots, y^{n-1}v\}$ is a basis for V. Note that by Problem 1.25 (a) one has that v is cyclic meaning that $\{x^i y^j v : i, j \in \mathbb{Z}\}$ is a spanning set, but note that $x^i y^j v = q^{ij} y^j x^i v = q^{ij} \lambda^i y^j v$ (where $xv = \lambda v$) meaning since $\lambda \neq 0$ (by the fact that $\rho(x)$ is invertible) that $\{y^i v : i \in \mathbb{Z}\}$ is a spanning set but since any $a \in Z(A)$ acts as a scalar we have that $\{y^i v : i \in [0 : n-1]\}$ is a spanning set since if $i \notin [0 : n-1]$ one has that if $i = ni' + i'''$ (where $i''' \in [0 : n-1]$ and $i' \in \mathbb{Z}$) one has that $y^i v = y^{ni'+i'''} v = y^{ni'} y^{i'''} v = \alpha y^{i'''} v$ since $y^{ni'} \in Z(A)$ implies $y^{ni'}$ acts as a scalar. Finally, I claim that $\{v, yv, y^2v, y^3v, \ldots, y^{n-1}v\}$ is a linearly independent set. Otherwise $dim(V) = dim(span\{v, yv, y^2v, y^3v, \ldots, y^{n-1}v\}) < ord(q)$ a contradiction to the observation in part (b) that $ord(q) \mid dim(V)$.

Definition [1.37] Let (V_i, x_h) and (W_i, y_h) be representations of the quiver Q. A homomorphism $\phi : (V_i) \to (W_i)$ of quiver representations is a collection of maps $\phi_i : V_i \to W_i$ such that $y_h \circ \phi_{h'} = \phi_{h''} \circ x_h$ for all $h \in E$.

[1.38] Let A be a $\mathbb{Z}_+$-graded algebra, i.e., $A = \oplus_{n \geq 0} A[n]$, and $A[n] \cdot A[m] \subseteq A[n+m]$. If $A[n]$ is finite-dimensional, it is useful to consider the Hilbert series $h_A(t) = \sum dimA[n]t^n$) the generating function of dimensions of $A[n]$). Often, this series converges to a rational function, and the answer is written in the form of such function. For example, if $A = k[x]$ and $deg(x^n) = n$ then

$$h_A(t) = 1 + t + t^2 + \cdots + t^n + \cdots = \frac{1}{1-t}.$$

Find the Hilbert series of:

(a) $A = k[x_1, \ldots, x_m]$ (where the grading is by degree of polynomials);

Note that $dim(A[n]) = \frac{m^n}{n!}$ since $m^n = |\{(j_1, j_2, \ldots, j_m) : j_i \in \mathbb{N}_{\geq 0}$ and $\sum_{i=1}^{m} j_i = n\}|$ and thus $h_A(t) = \sum_{n=0}^{\infty} \frac{m^n}{n!} t^n$.

(b) $A = k\langle x_1, \ldots, x_m \rangle$ (the grading is by length of words);

Note that $dim(A[n]) = m^n$ since $m^n = |\{x_{i_1} x_{i_2} \ldots x_{i_n}$ such that $i_n \in [1:m]\}|$ meaning that $h_A(t) = \sum_{n=0}^{\infty} m^n t^n$.

(c) A is the exterior (=Grassmann) algebra $\wedge_k[x_1, \ldots, x_m]$, generated over some field k by $x_1, \ldots, x_m$ with the defining relations $x_i x_j + x_j x_i = 0$ and $x_i^2 = 0$ for all i, j (the grading is by degree).

Note that $dim(A[n]) = \binom{m}{n} = \frac{m!}{(m-n)!n!}$ meaning that $h_A(t) = \sum_{n=0}^{\infty} \frac{m!}{(m-n)!n!} t^n = \sum_{n=0}^{m} \frac{m!}{(m-n)!n!} t^n$.

(d) A is the path algebra P_Q of a quiver Q (the grading is defined by $deg(p_i) = 0, deg(a_h) = 1$).

Note that if M_Q is the adjacency matrix of Q (note it may not by symmetric) then $dim(A[n]) = f(M_Q^n)$ where $f(M_Q^n) = \sum_{i=0}^{|V|} \sum_{j=0}^{|V|} (M_Q^n)_j^i$.