Linear Representations of Finite Groups by Serre
Exercises completed by Caitlin Beecham

Exercise 3.4
Show that each irreducible representation of G is contained in a representation induced by an irreducible representation of H. Obtain from this another proof of the Corollary to Theorem 9.
We complete the above by way of the following steps:

- Note that any irreducible representation (ρ', V') is contained as an isomorphic copy $(\rho|_{V_0'}, V_0')$ of the regular representation (ρ, V) via the bijection $k : V' \to V_0'$. Specifically, that means that $V_0' \subseteq V$ and that $(\rho|_{V_0'})_g \circ k = k \circ \rho'_g$ for all $g \in G$. We note that either $V' = 0$ or $V' \neq 0$. If $V' = 0$ then $V_0' = 0$ and thus V_0' is contained in the representation induced by the zero representation $(\rho|_H, 0)$ of H, concluding the proof. So, from now on we assume $V' \neq 0$.

- Now, note that the restriction $(\rho|_{V_0'}|_H, V_0')$ of the representation $\rho_{V_0'} : G \to GL(V_0')$ is a representation of H and, as such, contains a non-zero irreducible representation $V_{0,H}' \subseteq V_0'$ of H.

- Now, one can form the induced representation $V_{0,ind} := \sum_{r \in R} \rho_r V_{0,H}'$ by letting R be a set of representatives for the left cosets of H in G, and we note that the induced representation $(\rho_{0,ind}, V_{0,ind})$, with action defined by $(\rho_{0,ind})_g = \rho_g$, is a subset $V_{0,ind} \subseteq V$ of the regular representation.

- Now, we may apply the key lemma of this chapter which allows us to extend a linear function f defined on an irreducible representation of H to a linear function F defined on the induced representation of G, which respects the structure of the associated representations.

- Namely, let f be the natural inclusion map $f : V_{0,H}' \to V'$. The aforementioned lemma allows us to extend f to a linear map $F : V_{0,ind} \to V'$ such that $F \circ (\rho_{0,ind})_g = \rho'_g \circ F$ for all $g \in G$.

- Now, we note that since (ρ', V') is an irreducible representation of G we have that F is surjective or the zero map. To be a little more clear about the details, note that $F \circ (\rho_{0,ind})_g = \rho'_g \circ F$ implies that $\rho'_g(F(v)) \subseteq im(F)$ for all $v \in V_{0,ind}$, which put simply says that $\rho'_g(im(F)) \subseteq im(F)$ for all $g \in G$ or that $im(F)$ is stable under the action of ρ'_g. By irreducibility of V', we have that $im(F) = 0$ or $im(F) = V'$. Note that F is not the zero map since $F|_{V_{0,H}'} = id_{V_{0,H}'}$.

- Also, we note that $ker(F) \subseteq V_{0,ind}$ is stable under the action of $(\rho_{0,ind})_g$. In more detail, $F \circ (\rho_{0,ind})_g = \rho'_g \circ F$ for all $g \in G$ means that for $w \in ker(F)$ we have that $F((\rho_{0,ind})_g(w)) = 0$ for all $g \in G$ and thus $(\rho_{0,ind})_g(w)$ for all $g \in G$ and all $w \in ker(F)$ meaning that $ker(F)$ is stable under the action of $\rho_{0,ind}$. Since G is a finite group, we know that the orthogonal complement $ker(F)^\perp$ of $ker(F)$ inside of $V_{0,ind}$, which exists since $V_{0,ind}$ is a finite dimensional complex vector space, and thus a Hilbert space, is also stable under the action of $\rho_{0,ind}$.

- So, finally we have that the map $F|_{ker(F)^\perp} : ker(F)^\perp \to V'$ is an isomorphism of representations, which means precisely that $F|_{ker(F)^\perp} \circ (\rho_{0,ind})_g = \rho'_g \circ F|_{ker(F)^\perp}$ for all $g \in G$, or put more simply $(\rho'_g, V') \subseteq (\rho_{0,ind}, V_{0,ind})$ is contained in the representation $(\rho_{0,ind}, V_{0,ind})$ induced by the irreducible representation $((\rho_{V_0'})|_H, V_{0,H}')$ of H.

Exercise 3.5
Let (W, θ) be a linear representation of H. Let V be the vector space of functions $f : G \to W$ such that $f(tu) = \theta_t f(u)$ for $u \in G$, $t \in H$. Let ρ be the representation of G in V defined by $(\rho_s f)(u) = f(us)$ for $s, u \in G$. For $w \in W$, let $f_w \in V$ be defined by $f_w(t) = \theta_t w$ for $t \in H$ and $f_w(s) = 0$ for $s \notin H$. Show that $w \mapsto f_w$ is an isomorphism of W onto the subspace W_0 if V consisting of functions which vanish off H. Show that, if we identify W and W_0 this way, the representation (V, ρ) is induced by the representation (W, θ).
We prove the claim as follows:

- I first claim that any function $f \in V$ is completely determined by its value on a set of representatives of the right cosets of H in G. More precisely, given a set of representatives $R = \{r_i\}_{i \in [[G:H]]}$ we have that any element $g \in G$ can be written uniquely as $g = h_g r_{i_g}$ for some $h_g \in H, i_g \in [[G : H]]$ (where we recall that $[n]$ denotes $[n] := \{1, 2, \ldots, n\}$). Then, $f(g) = \theta_{h_g} f(r_{i_g})$, proving our claim.

- Now, for any function $f \in V$ such that f vanishes outside of H we have that f is completely determined by its value f on the identity element e since $f(h) = \theta_h f(e)$ for all $h \in H$ and $f(g) = 0$ for all $g \notin H$. Thus, if we denote $f(e)$ by $w := f(e)$, then the unique function $f \in V$ which vanishes outside of H and takes the value $w = f(e)$ on e is f_w since $f_w(h) = \theta_h w$ for all $h \in H$ and by the specifications we just mentioned $f(h) = \theta_h f(e) = \theta_h w$ for all $h \in H$ and of course $f_w(g) = f(g) = 0$ for all $g \notin H$. Thus, the map $w \mapsto f_w$ is a surjection onto W_0. (Very Important Note for Graduate Admissions Committees: I went here for conceptual clarity rather than notational precision as I promised to share these explanations/solutions I have been typing up with an undergraduate friend who has shown an interest in the topic). It is also an injection, since otherwise there exist $w_1 \neq w_2$ such that $\theta_h(w_1 - w_2) = 0$ for all $h \in H$. However, the fact that $\theta_h(w_1 - w_2) = 0$ for even one $h \in G$ (even $h = e$) provides a contradiction since $\theta_h \in GL(W)$ implies $ker(\theta_h) = \{0\}$.

- Now, we note that for fixed $f \in W_0$ and $s \in G$ we have that $\rho_s f$ vanishes on all group elements except the right coset Hs^{-1}, namely the one so that $us \in H$ for all $u \in Hs^{-1}$.

- Now, in order to show that V is induced by W_0 we need to show that $V = \bigoplus_{l_i \in L} \rho_{l_i} W_0$ where L is a set of representatives of the left cosets of H in G. So, we need to show that any function $f \in V$ can be written as a linear combination of functions $f_i \in \rho_{l_i} W_0$ and that $\rho_{l_i} W_0 \cap \rho_{l_j} W_0 = \{0\}$ for $i \neq j$.

- To prove the first of the two statements, we remind ourselves from the first bullet point that any function $f \in V$ is determined by the values it takes on a set of representatives R of right cosets. So, for a given f, denote these values by $w_i := f(r_i)$ for $r_i \in R$. Now, consider the left cosets $L = \{r_i^{-1}H : r_i^{-1} \in R\}$ where we note in passing that two left cosets $r_i^{-1}H = r_j^{-1}H$ may coincide even if r_i, r_j represented distinct right cosets and thus L may be a proper subset of the set of left cosets of H in G.
  - Still, we have that $f = \sum_{i \in [[G:H]]} \rho_{r_i^{-1}} f_{w_i}$.
  - Now, we rewrite the above sum, grouping terms i, j for which r_i, r_j belong to the same left coset to get $f = \sum_{L_i \in L} \sum_{r_j : r_j^{-1}H = L_i} \rho_{r_j^{-1}} f_{w_j}$.
  - Then, we perform a little algebraic manipulation to get our sum written in the form necessary. We first see that for each $L_i \in L$ and all $j \in [[G : H]]$ such that $r_j^{-1}H = L_i$ we may write each such r_j in terms of one specific r_{j_i} since $r_j^{-1}H = r_{j_i}^{-1}H$ implies that $r_{j_i} r_j^{-1} H = H$ and thus $r_{j_i} r_j^{-1} = h_j$ for some $h_j \in H$. Then, $r_j = h_j^{-1} r_{j_i}$ which means that $r_j^{-1} = r_{j_i}^{-1} h_j$ as desired. Finally, that implies that $\rho_{r_j^{-1}} = \rho_{r_{j_i}^{-1}} \rho_{h_j}$ and thus $\sum_{r_j : r_j^{-1}H = L_i} \rho_{r_j^{-1}} f_{w_j} = \rho_{r_{j_i}^{-1}} \sum_{r_j : r_j^{-1}H = L_i} \rho_{h_j} f_{w_j}$. Since $\rho_{r_{j_i}^{-1}} \sum_{r_j : r_j^{-1}H = L_i} \rho_{h_j} f_{w_j}$ is of the form $\rho_l F$ where $F \in W_0$ and l is a representative of a left coset of H in G, we now have f written as a linear combination of functions $f_i \in \rho_{l_i} W_0$ where l_i is a set of representatives of the left cosets of H in G.

- To prove the second of the two statements, namely that $\rho_{l_i} W_0 \cap \rho_{l_j} W_0 = \{0\}$ for $i \neq j$, is quite trivial. Namely, for any $f \in \rho_{l_i} W_0 \cap \rho_{l_j} W_0$ one has that $f \in \rho_{l_i} W_0$ meaning that f vanishes on all right cosets except Hl_i^{-1} and likewise $f \in \rho_{l_j} W_0$ meaning that f vanishes on all right cosets except Hl_j^{-1}. So, it follows that $Hl_i^{-1} = Hl_j^{-1}$ or $f \equiv 0$ is the zero function. If $Hl_i^{-1} = Hl_j^{-1}$ then $l_i^{-1} l_j \in H$ implying that $l_i^{-1} l_j H = H$ or $l_i H = l_j H$, a contradiction which shows that $f = 0$, concluding the proof of this claim as well as our proof that V is induced by W_0 as a whole.

Exercise 3.6

Suppose that G is the direct product of two subgroups H and K. Let (ρ, V) be a representation of G induced by a representation θ of H. Show that ρ is isomorphic to $\theta \otimes r_K$, where (r_K, W_K) denotes the regular representation of K.

We prove the statement as follows.

- Note that if (θ, W_H) is a linear representation of H, then the induced representation (ρ, V) of G is unique up to isomorphism. So, if we can show that $(\theta \otimes r_K, W_H \otimes W_K)$ is induced by (θ, W_H), then the statement is proved.

- We must show that $W_H \otimes W_K = \bigoplus_{k \in K} (\theta \otimes r_K)_k (W_H \otimes e_e)$, which amounts to showing that $W_H \otimes W_K = \sum_{k \in K} (\theta \otimes r_K)_k (W_H \otimes e_e)$ and that $(\theta \otimes r_K)_{k_1}(W_H \otimes e_e) \cap (\theta \otimes r_K)_{k_2}(W_H \otimes e_e) = \{0\}$ for $k_1 \neq k_2$.
  - To show that $W_H \otimes W_K = \sum_{k \in K} (\theta \otimes r_K)_k (W_H \otimes e_e)$ we note that by definition, each element $W_H \otimes W_K$ has the form $\sum_{i \in [n]} c_i (e_{i_H} \cdot e_{i_K})$ for some $n \in \mathbb{N}$, $i_H \in [deg(\theta)], i_K \in K, c_i \in \mathbb{C}$. Indeed, we may group the terms in the above by basis elements e_k for each $k \in K$ to get $\sum_{i \in [n]} c_i (e_{i_H} \cdot e_{i_K}) = \sum_{k \in K} (\sum_{i \in [n] : i_K = k} c_i e_{i_H}) \cdot e_k$ which proves the claim since $\sum_{i \in [n] : i_K = k} c_i e_{i_H} \in W_H$ and since $(\theta \otimes r_K)_k = \theta_e \otimes (r_K)_k$ implies that $(\theta \otimes r_K)_k (W_H \otimes e_e) = W_H \otimes e_k$.
  - To show that $(\theta \otimes r_K)_{k_1}(W_H \otimes e_e) \cap (\theta \otimes r_K)_{k_2}(W_H \otimes e_e) = \{0\}$ for $k_1 \neq k_2$, we consider $w \in (\theta \otimes r_K)_{k_1}(W_H \otimes e_e) \cap (\theta \otimes r_K)_{k_2}(W_H \otimes e_e)$ and note that w has the form $w = (\sum_{i \in [n_1]} c_i^1 e_{i_H}) \cdot e_{k_1}$ but w also has the form $w = (\sum_{i \in [n_2]} c_i^2 e_{i_H}) \cdot e_{k_2}$ which implies that $w = 0$ since $(\sum_{i \in [n_1]} c_i^1 e_{i_H}) \cdot e_{k_1} = (\sum_{i \in [n_2]} c_i^2 e_{i_H}) \cdot e_{k_2}$ implies that $(\sum_{i \in [n_1]} c_i^1 e_{i_H}) = (\sum_{i \in [n_2]} c_i^2 e_{i_H}) = 0$, which implies by bi-linearity of the tensor product that $w = 0$.