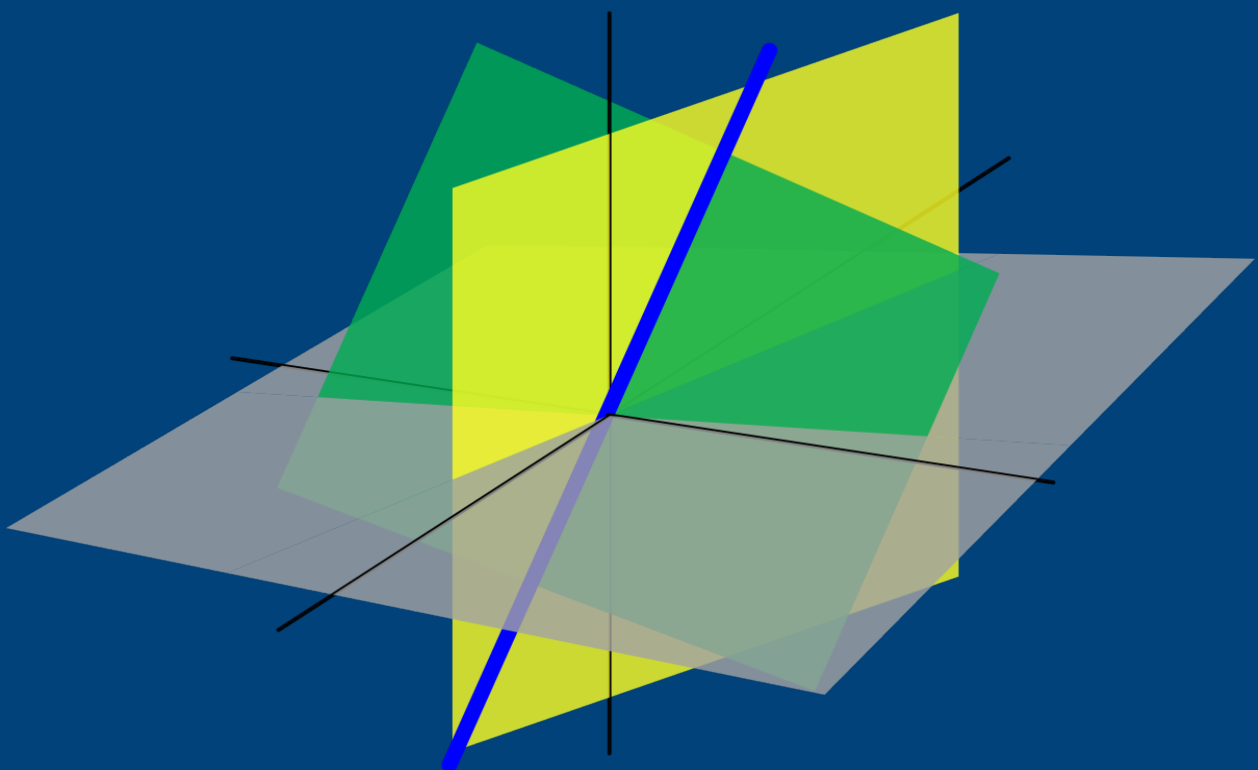
Linear Algebra Done Right

Lecture Notes

"It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment."

CARL FRIEDRICH GAUSS



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1 Vector Spaces

1.1 $\mathbb{R}^n$ and $\mathbb{C}^n$

Definition 1.1 $\mathbb{R}$ denotes the field of real numbers.

Some nonconstant polynomials with real coefficients have no real zeroes. Example: the equation:

$$x^2 + 1 = 0$$

has no real solutions. Thus, we invent a solution called i, such that $i^2 = -1$.

Definition 1.2 Complex Numbers

- A complex number is an ordered pair (a, b), where $a, b \in \mathbb{R}$, but we will write this as $a + bi$.
- The set of all complex numbers is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

- Addition and multiplication on $\mathbb{C}$ are defined as follows

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Note 1.1 If $a \in \mathbb{R}$, we identify $a + 0i$ with the real number a. Thus we think of $\mathbb{R}$ as a subset of $\mathbb{C}$. We also usually write $0 + bi$ as just bi, and we usually write $0 + 1i$ as just i. From the definition of multiplication above, we have that $i^2 = -1$.

Note 1.2 Properties of Complex Arithmetic: $\forall \alpha, \beta \in \mathbb{C}$

- Commutativity

$$\alpha + \beta = \beta + \alpha \text{ and } \alpha\beta = \beta\alpha$$

- Associativity

$$(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda) \text{ and } (\alpha\beta)\lambda = \alpha(\beta\lambda)$$

- Identities

$$\lambda + 0 = \lambda \text{ and } \lambda 1 = \lambda$$

- Additive Inverse

$$\text{For every } \alpha \in \mathbb{C} \text{ there exists a unique } \beta \in \mathbb{C} \text{ such that } \alpha + \beta = 0$$

- Multiplicative Inverse

$$\text{For every } \alpha \in \mathbb{C} \setminus \{0\}, \text{ there exists a unique } \beta \in \mathbb{C} \text{ such that } \alpha\beta = 1$$

- Distributivity

$$\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$$

Definition 1.3 $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$

Elements of $\mathbb{F}$ are sometimes called scalars. We call it $\mathbb{F}$ because those are both fields.

Now we discuss the idea of a "list." To understand the idea, here some examples of simple sets we have already seen in other mathematics:

- The set $\mathbb{R}^2$, which you can think of as a plane, is the set of all ordered pairs of real numbers:

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

- The set $\mathbb{R}^3$, which you can think of as ordinary space, is the set of all ordered triples of real numbers:

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

Definition 1.4 A *list* of *length n* is an ordered collection of n numbers separated by commas and surrounded by parenthesis.

$$\text{i.e. } (x_1, \ldots, x_n)$$

Two lists are equal if an only if they have the same length and the same elements.

Here are some examples of lists from sets we are familiar with:

1. $(7, 3)$ is a list of length 2. Thus $(7, 3) \in \mathbb{R}^2$.
2. $(5, 9, -2)$ is a list of length 3. Thus $(5, 9, -2) \in \mathbb{R}^3$

Definition 1.5 $\mathbb{F}^n$ is the set of all lists of length n of elements of $\mathbb{F}$:

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \ldots, n\}$$

Elements of $\mathbb{F}^n$ are often called *points* or *vectors*.

It does not matter if these sets have geometric sense. We can manipulate them algebraically. This is where the name linear <u>algebra</u> comes from.

Definition 1.6 Addition in $\mathbb{F}^n$ is defined by adding the corresponding coordinates:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

Definition 1.7 The product of a number $\lambda \in \mathbb{F}$ and a vector $\mathbb{F}^n$ is defined by multiplying each coordinate of the vector by λ:

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

Single letters can denote elements of $\mathbb{F}^n$ efficiently. You can say $x + y = z$ instead of saying e.g.

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (z_1, \ldots, z_n)$$

Definition 1.8 Let 0 denote the list of length n whose coordinates are all 0:

$$0 = (0, \ldots, 0)$$

It should always be clear from context which 0 you're talking about. For example: we have the following:

Theorem If $x \in \mathbb{F}^n$, then $0x = 0$.

The 0 on the LHS is a scalar in $\mathbb{F}$. The 0 on the RHS is a vector in $\mathbb{F}^n$.

1.2 Definition of a Vector Space

The motivation for the definition of a vectors space comes from the properties of addition and scalar multiplication in $\mathbb{F}^n$:

- Addition is commutative, associative, and has an identity.
- Every element has an additive inverse.
- Scalar multiplication is associative.
- Scalar multiplication by 1 acts as expected.
- Addition and scalar multiplication are connected by distributive properties.

First, let us define what addition/scalar multiplication is.

Definition 1.9
- An *addition* on a set V is a function that assigns an element $u + w \in V$ to each pair of elements $u, w \in V$
- A *scalar multiplication* on a set V is a function that assigns an element $\lambda u \in V$ to each $\lambda \in \mathbb{F}$ and each $u \in V$

Example 1.1 Suppose V is the set of real valued functions on the interval $[0, 1]$. For $f, g \in V$ and $\lambda \in \mathbb{R}$, define $f + g$ and λf by:

$$(f + g)(x) = f(x) + g(x)$$

and

$$(\lambda f)(x) = \lambda f(x)$$

Thus $f + g \in V$ and $\lambda f \in V$.

Now, we can define a vector space V. These are based off the properties of $\mathbb{F}^n$:

Definition 1.10 A *vector space* is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

- $u + w = w + u$ for all $u, w \in V$
- $(u + v) + w = u + (v + w)$ and $(ab)u = a(bu)$ for all $u, v, w \in V$ and all $a, b \in \mathbb{F}$
- There exists $0 \in V$ such that $u + 0 = u$ for all $u \in V$
- For every $u \in V$, there exists $w \in V$ such that $u + w = 0$
- $1u = u$ for all $u \in V$
- $a(u + w) = au + aw$ and $(a + b)u = au + bu$ for all $a, b \in \mathbb{F}$ and all $u, w \in V$

Example 1.2 Vector Spaces:

- $\mathbb{F}^n$ with the usual operations of addition and scalar multiplications is a vector space.
- $\mathbb{F}^\infty$ is defined to be the set of all sequences of elements of $\mathbb{F}$:

  $$\{(x_1, x_2, \ldots) : x_j \in \mathbb{F} \text{ for } j = 1, 2, \ldots\}$$

  Addition and scalar multiplication are also defined coordinate-wise. This is also a vector space.
- More generally, if S is a set, let F^S denote the set of functions from S to $\mathbb{F}$. For $f, g \in \mathbb{F}^S$, the sum $f + g \in \mathbb{F}^S$ is the function defined by:

  $$(f + g)(x) = f(x) + g(x)$$

  for all $x \in S$. For $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^S$, the product $\lambda f \in \mathbb{F}^S$ is the function defined by

  $$(\lambda f)(x) = \lambda f(x)$$

  for all $x \in S$. With these definitions, $\mathbb{F}^S$ becomes a vector space.

Our first theorem then follows:

Theorem 1.1 If V is a vector space, $\forall u \in V$, $0u = 0$.

Proof For arbitrary $u \in V$, we have:

$$\begin{aligned} 0u &= (0+0)u \\ &= 0u + 0u \end{aligned}$$

Adding the additive inverse of $0u$, denoted $-0u$, to both sides of the equation above gives:

$$\begin{aligned} 0u + (-0u) &= 0u + 0u + (-0u) \\ 0 &= 0u \end{aligned}$$

as desired. ■

Advantages of the abstract approach to vector spaces:

- Can apply what was done in multiple new situations.
- Stripping away inessential properties leads to greater understanding.

If V is a vector space, it would be incorrect to prove that $0u = 0$ for $u \in V$ by writing: Let $u = (x_1, \ldots, x_n)$, thus...

Note 1.3 An element of V is not necessarily of the form $(x_1, \ldots, x_n)$.