# Ejercicio 6

Demostrar en deducción natural que vale $\vdash \sigma$ para cada una de las siguientes fórmulas. Para estas fórmulas es imprescindible usar lógica clásica:

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## I. Absurdo clásico

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg\tau \Rightarrow \bot$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg\tau$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma = \{\neg\tau \Rightarrow \bot, \neg\tau\} \vdash \bot$}
\RightLabel{PBC}
\UnaryInfC{$\neg\tau \Rightarrow \bot \vdash \tau$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\neg\tau \Rightarrow \bot) \Rightarrow \tau$}
\end{prooftree}
$$

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## II. Ley de Peirce

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg\tau \vdash (\tau \Rightarrow \rho) \Rightarrow \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg\tau, \tau \vdash \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg\tau, \tau \vdash \neg\tau$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, \neg\tau, \tau \vdash \bot$}
\RightLabel{$\bot_e$}
\UnaryInfC{$\Gamma, \neg\tau, \tau \vdash \rho$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\Gamma, \neg\tau \vdash (\tau \Rightarrow \rho)$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma, \neg\tau \vdash \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg\tau \vdash \neg\tau$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, \neg\tau \vdash \bot$}
\RightLabel{PBC}
\UnaryInfC{$\Gamma = ((\tau \Rightarrow \rho) \Rightarrow \tau) \vdash \tau$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash ((\tau \Rightarrow \rho) \Rightarrow \tau) \Rightarrow \tau$}
\end{prooftree}
$$

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## III. Tercero excluido

Esto se puede probar con PBC pero ya tenemos dado LEM.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{LEM}
\UnaryInfC{$\vdash \tau \vee \neg\tau$}
\end{prooftree}
$$

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## IV. Consecuencia milagrosa

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg\tau \Rightarrow \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg\tau$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma \vdash \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg\tau$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma = \{(\neg\tau \Rightarrow \tau), \neg\tau\} \vdash \bot$}
\RightLabel{PBC}
\UnaryInfC{$(\neg\tau \Rightarrow \tau) \vdash \tau$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\neg\tau \Rightarrow \tau) \Rightarrow \tau$}
\end{prooftree}
$$

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## V. Contraposición clásica

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg\rho \Rightarrow \neg\tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg\rho$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma \vdash \neg\tau$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma = \{(\neg\rho \Rightarrow \neg\tau), \tau, \neg\rho\} \vdash \bot$}
\RightLabel{PBC}
\UnaryInfC{$(\neg\rho \Rightarrow \neg\tau), \tau \vdash \rho$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$(\neg\rho \Rightarrow \neg\tau) \vdash (\tau \Rightarrow \rho)$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\neg\rho \Rightarrow \neg\tau) \Rightarrow (\tau \Rightarrow \rho)$}
\end{prooftree}
$$

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## VI. Análisis de casos

No compila completo, habría que cambiar **esta línea**

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg\rho \vdash \tau \Rightarrow \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg\rho, \neg\tau \vdash \neg\tau \Rightarrow \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg\rho, \neg\tau \vdash \neg\tau$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma, \neg\rho, \neg\tau \vdash \rho$}
\RightLabel{$\neg_e$}
\UnaryInfC{$\Gamma, \neg\rho, \neg\tau \vdash \bot$}
\RightLabel{PBC}
\UnaryInfC{$\Gamma, \neg\rho \vdash \tau$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma, \neg\rho \vdash \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg\rho \vdash \neg\rho$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, \neg\rho \vdash \bot$}
\RightLabel{PBC}
\UnaryInfC{$\Gamma = \{(\tau \Rightarrow \rho), (\neg\tau \Rightarrow \rho)\} \vdash \rho$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$(\tau \Rightarrow \rho) \vdash (\neg\tau \Rightarrow \rho) \Rightarrow \rho$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\tau \Rightarrow \rho) \Rightarrow ((\neg\tau \Rightarrow \rho) \Rightarrow \rho)$}
\end{prooftree}
$$

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## VII. Implicación vs disyunción

$(\tau \Rightarrow \rho) \Leftrightarrow (\neg\tau \vee \rho)$

$(\Longrightarrow)$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$(\tau \Rightarrow \rho), \neg(\neg\tau \vee \rho) \vdash$}
\AxiomC{}
\UnaryInfC{$(\tau \Rightarrow \rho), \neg(\neg\tau \vee \rho) \vdash$}
\RightLabel{$\neg_e$}
\BinaryInfC{$(\tau \Rightarrow \rho), \neg(\neg\tau \vee \rho) \vdash \bot$}
\RightLabel{PBC}
\UnaryInfC{$(\tau \Rightarrow \rho) \vdash (\neg\tau \vee \rho)$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\tau \Rightarrow \rho) \Rightarrow (\neg\tau \vee \rho)$}
\end{prooftree}
$$

$(\Longleftarrow)$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$(\neg\tau \vee \rho) \Rightarrow (\tau \Rightarrow \rho)$}
\end{prooftree}
$$