Lógica intuicionista

Reglas básicas:

$$\frac{}{\Gamma, \tau \vdash \tau}\, ax$$

$$\frac{\Gamma \vdash \tau \qquad \Gamma \vdash \sigma}{\Gamma \vdash \tau \wedge \sigma}\, \wedge_i$$

$$\frac{\Gamma \vdash \tau \wedge \sigma}{\Gamma \vdash \tau}\, \wedge_{e_1} \qquad \frac{\Gamma \vdash \tau \wedge \sigma}{\Gamma \vdash \sigma}\, \wedge_{e_2}$$

$$\frac{\Gamma \vdash \tau}{\Gamma \vdash \tau \vee \sigma}\, \vee_{i_1} \qquad \frac{\Gamma \vdash \sigma}{\Gamma \vdash \tau \vee \sigma}\, \vee_{i_2}$$

$$\frac{\Gamma \vdash \tau \vee \sigma \qquad \Gamma, \tau \vdash \rho \qquad \Gamma, \sigma \vdash \rho}{\Gamma \vdash \rho}\, \vee_e$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \tau}\, \bot_e$$

$$\frac{\Gamma, \tau \vdash \sigma}{\Gamma \vdash \tau \Rightarrow \sigma}\, \Rightarrow_i$$

$$\frac{\Gamma \vdash \tau \Rightarrow \sigma \qquad \Gamma \vdash \tau}{\Gamma \vdash \sigma}\, \Rightarrow_e$$

$$\frac{\Gamma, \tau \vdash \bot}{\Gamma \vdash \neg\tau}\, \neg_i$$

$$\frac{\Gamma \vdash \tau \qquad \Gamma \vdash \neg\tau}{\Gamma \vdash \bot}\, \neg_e$$

Reglas derivadas:

$$\frac{\Gamma \vdash \tau}{\Gamma \vdash \neg\neg\tau}\, \neg\neg_i \qquad \frac{\Gamma \vdash \tau \Rightarrow \sigma \qquad \Gamma \vdash \neg\sigma}{\Gamma \vdash \neg\tau}\, MT$$

Lógica clásica

Reglas básicas: **Reglas derivadas:**

$$\frac{\Gamma \vdash \neg\neg\tau}{\Gamma \vdash \tau}\, \neg\neg_e \qquad \frac{\Gamma, \neg\tau \vdash \bot}{\Gamma \vdash \tau}\, PBC \qquad \frac{}{\Gamma \vdash \tau \vee \neg\tau}\, LEM$$

Ejercicios:

Con lógica intuicionista

I. *Modus ponens* relativizado:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \rho \vdash \rho \Rightarrow \sigma \Rightarrow \tau}\,\text{ax} \qquad \dfrac{}{\Gamma, \sigma \Rightarrow \tau \vdash \rho}\,\text{ax}}{\Gamma \vdash \sigma \Rightarrow \tau}\Rightarrow_e \qquad \dfrac{\dfrac{}{\Gamma \vdash \rho \Rightarrow \sigma}\,\text{ax} \qquad \dfrac{}{\Gamma, \sigma \vdash \rho}\,\text{ax}}{\Gamma \vdash \sigma}\Rightarrow_e}{\Gamma = \rho \Rightarrow (\sigma \Rightarrow \tau), \rho \Rightarrow \sigma, \rho \vdash \tau}\Rightarrow_e}{(\rho \Rightarrow (\sigma \Rightarrow \tau)), (\rho \Rightarrow \sigma) \vdash (\rho \Rightarrow \tau)}\Rightarrow_i}{(\rho \Rightarrow (\sigma \Rightarrow \tau)) \vdash ((\rho \Rightarrow \sigma) \Rightarrow (\rho \Rightarrow \tau))}\Rightarrow_i}{\vdash (\rho \Rightarrow (\sigma \Rightarrow \tau)) \Rightarrow ((\rho \Rightarrow \sigma) \Rightarrow (\rho \Rightarrow \tau))}\Rightarrow_i
$$

II. Reducción al absurdo

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \rho \Rightarrow \bot}\,\text{ax} \qquad \dfrac{}{\Gamma \vdash \rho}\,\text{ax}}{\Gamma = (\rho \Rightarrow \bot), \rho \vdash \bot}\Rightarrow_e}{(\rho \Rightarrow \bot) \vdash \neg\rho}\neg_i}{\vdash (\rho \Rightarrow \bot) \Rightarrow \neg\rho}\Rightarrow_i
$$

III. Introducción de la doble negación

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\rho, \neg\rho \vdash \rho}\,\text{ax} \qquad \dfrac{}{\rho, \neg\rho \vdash \neg\rho}\,\text{ax}}{\rho, \neg\rho \vdash \bot}\neg_e}{\rho \vdash \neg\neg\rho}\neg_i}{\vdash \rho \Rightarrow \neg\neg\rho}\Rightarrow_i
$$

IV. Eliminación de la triple negación

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \rho}\,\text{ax}}{\Gamma \vdash \neg\neg\rho}\neg\neg_i \qquad \dfrac{}{\Gamma \vdash \neg\neg\neg\rho}\,\text{ax}}{\Gamma = \neg\neg\neg\rho, \rho \vdash \bot}\neg_e}{\neg\neg\neg\rho \vdash \neg\rho}\neg_i}{\vdash \neg\neg\neg\rho \Rightarrow \neg\rho}\Rightarrow_i
$$

V. Contraposición

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \rho \Rightarrow \sigma}\,\text{ax} \qquad \dfrac{}{\Gamma \vdash \rho}\,\text{ax}}{\Gamma \vdash \sigma}\Rightarrow_e \qquad \dfrac{}{\Gamma \vdash \neg\sigma}\,\text{ax}}{\Gamma = (\rho \Rightarrow \sigma), \neg\sigma, \rho \vdash \bot}\neg_e}{(\rho \Rightarrow \sigma), \neg\sigma \vdash \neg\rho}\neg_i}{\rho \Rightarrow \sigma \vdash \neg\sigma \Rightarrow \neg\rho}\Rightarrow_i}{\vdash (\rho \Rightarrow \sigma) \Rightarrow (\neg\sigma \Rightarrow \neg\rho)}\Rightarrow_i
$$

VI. Adjunción

Probar $((\rho \wedge \sigma) \Rightarrow \tau) \Leftrightarrow (\rho \Rightarrow \sigma \Rightarrow \tau)$ se reduce a probar

$((\rho \wedge \sigma) \Rightarrow \tau) \Rightarrow (\rho \Rightarrow \sigma \Rightarrow \tau)$ y $(\rho \Rightarrow \sigma \Rightarrow \tau) \Rightarrow ((\rho \wedge \sigma) \Rightarrow \tau)$

Caso 1

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash (\rho \wedge \sigma) \Rightarrow \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \sigma$}
\RightLabel{$\wedge_i$}
\BinaryInfC{$\Gamma \vdash (\rho \wedge \sigma)$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma = (\rho \wedge \sigma) \Rightarrow \tau, \rho, \sigma \vdash \tau$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$((\rho \wedge \sigma) \Rightarrow \tau), \rho \vdash (\sigma \Rightarrow \tau)$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$((\rho \wedge \sigma) \Rightarrow \tau) \vdash \rho \Rightarrow (\sigma \Rightarrow \tau)$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash ((\rho \wedge \sigma) \Rightarrow \tau) \Rightarrow (\rho \Rightarrow \sigma \Rightarrow \tau)$}
\end{prooftree}
$$

Caso 2

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \Rightarrow \sigma \Rightarrow \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \wedge \sigma$}
\RightLabel{$\wedge_{e_1}$}
\UnaryInfC{$\Gamma \vdash \rho$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma \vdash \sigma \Rightarrow \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \wedge \sigma$}
\RightLabel{$\wedge_{e_2}$}
\UnaryInfC{$\Gamma \vdash \sigma$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma = (\rho \Rightarrow \sigma \Rightarrow \tau), (\rho \wedge \sigma) \vdash \tau$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$(\rho \Rightarrow \sigma \Rightarrow \tau) \vdash ((\rho \wedge \sigma) \Rightarrow \tau)$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\rho \Rightarrow \sigma \Rightarrow \tau) \Rightarrow ((\rho \wedge \sigma) \Rightarrow \tau)$}
\end{prooftree}
$$

IX. Conmutatividad ($\wedge$)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\rho \wedge \sigma \vdash \rho \wedge \sigma$}
\RightLabel{$\wedge_{e_1}$}
\UnaryInfC{$\rho \wedge \sigma \vdash \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\rho \wedge \sigma \vdash \rho \wedge \sigma$}
\RightLabel{$\wedge_{e_2}$}
\UnaryInfC{$\rho \wedge \sigma \vdash \sigma$}
\RightLabel{$\wedge_i$}
\BinaryInfC{$\rho \wedge \sigma \vdash \sigma \wedge \rho$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\rho \wedge \sigma) \Rightarrow (\sigma \wedge \rho)$}
\end{prooftree}
$$

X. Asociatividad ($\wedge$)

$((\rho \wedge \sigma) \wedge \tau) \Leftrightarrow (\rho \wedge (\sigma \wedge \tau))$

$(\Longrightarrow)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash (\rho \wedge \sigma) \wedge \tau$}
\RightLabel{$\wedge_{e_1}$}
\UnaryInfC{$\Gamma \vdash \rho \wedge \sigma$}
\RightLabel{$\wedge_{e_1}$}
\UnaryInfC{$\Gamma \vdash \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash (\rho \wedge \sigma) \wedge \tau$}
\RightLabel{$\wedge_{e_1}$}
\UnaryInfC{$\Gamma \vdash \rho \wedge \sigma$}
\RightLabel{$\wedge_{e_2}$}
\UnaryInfC{$\Gamma \vdash \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash (\rho \wedge \sigma) \wedge \tau$}
\RightLabel{$\wedge_{e_2}$}
\UnaryInfC{$\Gamma \vdash \tau$}
\RightLabel{$\wedge_i$}
\BinaryInfC{$\Gamma \vdash \sigma \wedge \tau$}
\RightLabel{$\wedge_i$}
\BinaryInfC{$\Gamma = ((\rho \wedge \sigma) \wedge \tau) \vdash (\rho \wedge (\sigma \wedge \tau))$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash ((\rho \wedge \sigma) \wedge \tau) \Rightarrow (\rho \wedge (\sigma \wedge \tau))$}
\end{prooftree}
$$

$(\Longleftarrow)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \rho \land (\sigma \land \tau)}\text{ax}}{\Gamma \vdash \rho}\land_{e_1} \quad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash \rho \land (\sigma \land \tau)}\text{ax}}{\Gamma \vdash \sigma \land \tau}\land_{e_2}}{\Gamma \vdash \sigma}\land_{e_1}}{\Gamma \vdash \rho \land \sigma}\land_i \quad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash \rho \land (\sigma \land \tau)}\text{ax}}{\Gamma \vdash \sigma \land \tau}\land_{e_2}}{\Gamma \vdash \tau}\land_{e_2}}{\dfrac{\Gamma = (\rho \land (\sigma \land \tau)) \vdash ((\rho \land \sigma) \land \tau)}{\vdash (\rho \land (\sigma \land \tau)) \Rightarrow ((\rho \land \sigma) \land \tau)}\Rightarrow_i}\land_i$$

XI. Conmutatividad ($\lor$)

$$\dfrac{\dfrac{\dfrac{}{(\rho \lor \sigma) \vdash \rho \lor \sigma}\text{ax} \quad \dfrac{\dfrac{}{(\rho \lor \sigma), \rho \vdash \rho}\text{ax}}{(\rho \lor \sigma), \rho \vdash \sigma \lor \rho}\lor_{i_2} \quad \dfrac{\dfrac{}{(\rho \lor \sigma), \sigma \vdash \sigma}\text{ax}}{(\rho \lor \sigma), \sigma \vdash \sigma \lor \rho}\lor_{i_1}}{(\rho \lor \sigma) \vdash \sigma \lor \rho}\lor_e}{\vdash (\rho \lor \sigma) \Rightarrow (\sigma \lor \rho)}\Rightarrow_i$$

XII. Asociatividad ($\lor$)

$((\rho \lor \sigma) \lor \tau) \Leftrightarrow (\rho \lor (\sigma \lor \tau))$

$(\Longrightarrow)$

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (\rho \lor \sigma) \lor \tau}\text{ax} \quad \dfrac{\dfrac{}{\Sigma \vdash \rho \lor \sigma}\text{ax} \quad \dfrac{\dfrac{}{\Sigma, \rho \vdash \rho}\text{ax}}{\Sigma, \rho \vdash \rho \lor (\sigma \lor \tau)}\lor_{i_1} \quad \dfrac{\dfrac{\dfrac{}{\Sigma, \sigma \vdash \sigma}\text{ax}}{\Sigma, \sigma \vdash \sigma \lor \tau}\lor_{i_1}}{\Sigma, \sigma \vdash \rho \lor (\sigma \lor \tau)}\lor_{i_2}}{\Sigma = \Gamma, (\rho \lor \sigma) \vdash \rho \lor (\sigma \lor \tau)}\lor_e \quad \dfrac{\dfrac{\dfrac{}{\Gamma, \tau \vdash \tau}\text{ax}}{\Gamma, \tau \vdash \sigma \lor \tau}\lor_{i_2}}{\Gamma, \tau \vdash \rho \lor (\sigma \lor \tau)}\lor_{i_2}}{\dfrac{\Gamma = ((\rho \lor \sigma) \lor \tau) \vdash (\rho \lor (\sigma \lor \tau))}{\vdash ((\rho \lor \sigma) \lor \tau) \Rightarrow (\rho \lor (\sigma \lor \tau))}\Rightarrow_i}\lor_e$$

$(\Longleftarrow)$

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \rho \lor (\sigma \lor \tau)}\text{ax} \quad \dfrac{\dfrac{\dfrac{}{\Gamma, \rho \vdash \rho}\text{ax}}{\Gamma, \rho \vdash \rho \lor \sigma}\lor_{i_1}}{\Gamma, \rho \vdash (\rho \lor \sigma) \lor \tau}\lor_{i_1} \quad \dfrac{\dfrac{}{\Sigma \vdash \sigma \lor \tau}\text{ax} \quad \dfrac{\dfrac{\dfrac{}{\Sigma, \sigma \vdash \sigma}\text{ax}}{\Sigma, \sigma \vdash \rho \lor \sigma}\lor_{i_2}}{\Sigma, \sigma \vdash (\rho \lor \sigma) \lor \tau}\lor_{i_1} \quad \dfrac{\dfrac{}{\Sigma, \tau \vdash \tau}\text{ax}}{\Sigma, \tau \vdash (\rho \lor \sigma) \lor \tau}\lor_{i_2}}{\Sigma = \Gamma, (\sigma \lor \tau) \vdash (\rho \lor \sigma) \lor \tau}\lor_e}{\dfrac{\Gamma = (\rho \lor (\sigma \lor \tau)) \vdash ((\rho \lor \sigma) \lor \tau)}{\vdash (\rho \lor (\sigma \lor \tau)) \Rightarrow ((\rho \lor \sigma) \lor \tau)}\Rightarrow_i}\lor_e$$

Con lógica clásica:

I. Absurdo clásico

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg\tau \Rightarrow \bot}\text{ax} \quad \dfrac{}{\Gamma \vdash \neg\tau}\text{ax}}{\Gamma = \{\neg\tau \Rightarrow \bot, \neg\tau\} \vdash \bot}\Rightarrow_e}{\neg\tau \Rightarrow \bot \vdash \tau}\text{PBC}}{\vdash (\neg\tau \Rightarrow \bot) \Rightarrow \tau}\Rightarrow_i$$

II. Ley de Peirce

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg\tau \vdash (\tau \Rightarrow \rho) \Rightarrow \tau}\,\text{ax} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg\tau, \tau \vdash \tau}\,\text{ax} \quad \dfrac{}{\Gamma, \neg\tau, \tau \vdash \neg\tau}\,\text{ax}}{\Gamma, \neg\tau, \tau \vdash \bot}\,\neg_e}{\Gamma, \neg\tau, \tau \vdash \rho}\,\bot_e}{\Gamma, \neg\tau \vdash (\tau \Rightarrow \rho)}\,\Rightarrow_i}{\Gamma, \neg\tau \vdash \tau}\,\Rightarrow_e \quad \dfrac{}{\Gamma, \neg\tau \vdash \neg\tau}\,\text{ax}}{\dfrac{\Gamma, \neg\tau \vdash \bot}{\Gamma = ((\tau \Rightarrow \rho) \Rightarrow \tau) \vdash \tau}\,\text{PBC}}\,\neg_e$$

$$\dfrac{\Gamma = ((\tau \Rightarrow \rho) \Rightarrow \tau) \vdash \tau}{\vdash ((\tau \Rightarrow \rho) \Rightarrow \tau) \Rightarrow \tau}\,\Rightarrow_i$$

III. Tercero excluido

Esto se puede probar con PBC pero ya tenemos dado LEM.

$$\dfrac{}{\vdash \tau \vee \neg\tau}\,\text{LEM}$$

IV. Consecuencia milagrosa

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg\tau \Rightarrow \tau}\,\text{ax} \quad \dfrac{}{\Gamma \vdash \neg\tau}\,\text{ax}}{\Gamma \vdash \tau}\,\Rightarrow_e \quad \dfrac{}{\Gamma \vdash \neg\tau}\,\text{ax}}{\Gamma = \{(\neg\tau \Rightarrow \tau), \neg\tau\} \vdash \bot}\,\neg_e}{(\neg\tau \Rightarrow \tau) \vdash \tau}\,\text{PBC}}{\vdash (\neg\tau \Rightarrow \tau) \Rightarrow \tau}\,\Rightarrow_i$$

V. Contraposición clásica

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \tau}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma \vdash \neg\rho \Rightarrow \neg\tau}\,\text{ax} \quad \dfrac{}{\Gamma \vdash \neg\rho}\,\text{ax}}{\Gamma \vdash \neg\tau}\,\Rightarrow_e}{\Gamma = \{(\neg\rho \Rightarrow \neg\tau), \tau, \neg\rho\} \vdash \bot}\,\neg_e}{(\neg\rho \Rightarrow \neg\tau), \tau \vdash \rho}\,\text{PBC}}{(\neg\rho \Rightarrow \neg\tau) \vdash (\tau \Rightarrow \rho)}\,\Rightarrow_i}{\vdash (\neg\rho \Rightarrow \neg\tau) \Rightarrow (\tau \Rightarrow \rho)}\,\Rightarrow_i$$

VI. Análisis de casos

$$\dfrac{\dfrac{}{\Gamma, \neg\rho \vdash \tau \Rightarrow \rho}\,\text{ax} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg\rho, \neg\tau \vdash \neg\tau \Rightarrow \rho}\,\text{ax} \quad \dfrac{}{\Gamma, \neg\rho, \neg\tau \vdash \neg\tau}\,\text{ax}}{\Gamma, \neg\rho, \neg\tau \vdash \rho}\,\Rightarrow_e \quad \dfrac{}{\Gamma, \neg\rho, \neg\tau \vdash \neg\rho}\,\text{ax}}{\Gamma, \neg\rho, \neg\tau \vdash \bot}\,\neg_e}{\Gamma, \neg\rho \vdash \tau}\,\text{PBC}}{\Gamma, \neg\rho \vdash \rho}\,\Rightarrow_e$$

$(*)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\uparrow}{\Gamma, \neg\rho \vdash \rho}\Rightarrow_e \quad \dfrac{}{\Gamma, \neg\rho \vdash \neg\rho}\text{ax}}{\Gamma, \neg\rho \vdash \bot}\neg_e}{\Gamma = \{(\tau \Rightarrow \rho), (\neg\tau \Rightarrow \rho)\} \vdash \rho}\text{PBC}}{(\tau \Rightarrow \rho) \vdash (\neg\tau \Rightarrow \rho) \Rightarrow \rho}\Rightarrow_i}{\vdash (\tau \Rightarrow \rho) \Rightarrow ((\neg\tau \Rightarrow \rho) \Rightarrow \rho)}\Rightarrow_i$$

VII. Implicación vs disyunción

$(\tau \Rightarrow \rho) \Leftrightarrow (\neg\tau \vee \rho)$

$(\Longrightarrow)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(\tau \Rightarrow \rho), \neg(\neg\tau \vee \rho) \vdash} \quad \dfrac{}{(\tau \Rightarrow \rho), \neg(\neg\tau \vee \rho) \vdash}}{(\tau \Rightarrow \rho), \neg(\neg\tau \vee \rho) \vdash \bot}\neg_e}{(\tau \Rightarrow \rho) \vdash (\neg\tau \vee \rho)}\text{PBC}}{\vdash (\tau \Rightarrow \rho) \Rightarrow (\neg\tau \vee \rho)}\Rightarrow_i$$