

Lógica intuicionista

Reglas básicas:

$$\frac{}{\Gamma, \tau \vdash \tau} ax$$

$$\frac{\Gamma \vdash \tau \quad \Gamma \vdash \sigma}{\Gamma \vdash \tau \wedge \sigma} \wedge_i$$

$$\frac{\Gamma \vdash \tau \wedge \sigma}{\Gamma \vdash \tau} \wedge_{e_1} \quad \frac{\Gamma \vdash \tau \wedge \sigma}{\Gamma \vdash \sigma} \wedge_{e_2}$$

$$\frac{\Gamma \vdash \tau}{\Gamma \vdash \tau \vee \sigma} \vee_{i_1} \quad \frac{\Gamma \vdash \sigma}{\Gamma \vdash \tau \vee \sigma} \vee_{i_2}$$

$$\frac{\Gamma \vdash \tau \vee \sigma \quad \Gamma, \tau \vdash \rho \quad \Gamma, \sigma \vdash \rho}{\Gamma \vdash \rho} \vee_e$$

$$\frac{\Gamma \vdash \perp}{\Gamma \vdash \tau} \perp_e$$

$$\frac{\Gamma, \tau \vdash \sigma}{\Gamma \vdash \tau \Rightarrow \sigma} \Rightarrow_i$$

$$\frac{\Gamma \vdash \tau \Rightarrow \sigma \quad \Gamma \vdash \tau}{\Gamma \vdash \sigma} \Rightarrow_e$$

$$\frac{\Gamma, \tau \vdash \perp}{\Gamma \vdash \neg \tau} \neg_i$$

$$\frac{\Gamma \vdash \tau \quad \Gamma \vdash \neg \tau}{\Gamma \vdash \perp} \neg_e$$

Reglas derivadas:

$$\frac{\Gamma \vdash \tau}{\Gamma \vdash \neg \neg \tau} \neg \neg_i \quad \frac{\Gamma \vdash \tau \Rightarrow \sigma \quad \Gamma \vdash \neg \sigma}{\Gamma \vdash \neg \tau} MT$$

Lógica clásica

Reglas básicas: Reglas derivadas:

$$\frac{\Gamma \vdash \neg \neg \tau}{\Gamma \vdash \tau} \neg \neg_e \quad \frac{\Gamma, \neg \tau \vdash \perp}{\Gamma \vdash \tau} PBC \quad \frac{}{\Gamma \vdash \tau \vee \neg \tau} LEM$$

$$\frac{\frac{\frac{}{\Gamma \vdash \rho \Rightarrow \sigma} \text{ax} \quad \frac{}{\Gamma \vdash \rho} \text{ax}}{\Gamma \vdash \sigma} \Rightarrow_e \quad \frac{}{\Gamma \vdash \neg \sigma} \text{ax}}{\Gamma = (\rho \Rightarrow \sigma), \neg \sigma, \rho \vdash \perp} \neg_e$$

$$\frac{}{(\rho \Rightarrow \sigma), \neg \sigma \vdash \neg \rho} \neg_i$$

$$\frac{}{\rho \Rightarrow \sigma \vdash \neg \sigma \Rightarrow \neg \rho} \Rightarrow_i$$

$$\frac{}{\vdash (\rho \Rightarrow \sigma) \Rightarrow (\neg \sigma \Rightarrow \neg \rho)} \Rightarrow_i$$

VI. Adjunción

Probar $((\rho \wedge \sigma) \Rightarrow \tau) \Leftrightarrow (\rho \Rightarrow \sigma \Rightarrow \tau)$ se reduce a probar

$$((\rho \wedge \sigma) \Rightarrow \tau) \Rightarrow (\rho \Rightarrow \sigma \Rightarrow \tau) \text{ y } (\rho \Rightarrow \sigma \Rightarrow \tau) \Rightarrow ((\rho \wedge \sigma) \Rightarrow \tau)$$

Caso 1

$$\frac{\frac{\frac{\Gamma \vdash (\rho \wedge \sigma) \Rightarrow \tau}{\Gamma \vdash (\rho \wedge \sigma) \Rightarrow \tau} \text{ax} \quad \frac{\frac{\Gamma \vdash \rho}{\Gamma \vdash \rho} \text{ax} \quad \frac{\Gamma \vdash \sigma}{\Gamma \vdash \sigma} \text{ax}}{\Gamma \vdash (\rho \wedge \sigma)} \wedge_i}{\Gamma = (\rho \wedge \sigma) \Rightarrow \tau, \rho, \sigma \vdash \tau} \Rightarrow_e$$

Caso 2

$$\frac{\frac{\frac{\Gamma \vdash \rho \Rightarrow \sigma \Rightarrow \tau}{\Gamma \vdash \rho \Rightarrow \sigma \Rightarrow \tau} \text{ax} \quad \frac{\frac{\Gamma \vdash \rho \wedge \sigma}{\Gamma \vdash \rho} \wedge_{e_1} \quad \frac{\frac{\Gamma \vdash \rho \wedge \sigma}{\Gamma \vdash \rho \wedge \sigma} \text{ax}}{\Gamma \vdash \sigma} \wedge_{e_2}}{\Gamma \vdash \sigma \Rightarrow \tau} \Rightarrow_e \quad \frac{\Gamma = (\rho \Rightarrow \sigma \Rightarrow \tau), (\rho \wedge \sigma) \vdash \tau}{(\rho \Rightarrow \sigma \Rightarrow \tau) \vdash ((\rho \wedge \sigma) \Rightarrow \tau)} \Rightarrow_i}{\vdash (\rho \Rightarrow \sigma \Rightarrow \tau) \Rightarrow ((\rho \wedge \sigma) \Rightarrow \tau)} \Rightarrow_i$$

IX. Conmutatividad (\wedge)

$$\frac{\frac{\frac{\rho \wedge \sigma \vdash \rho \wedge \sigma}{\rho \wedge \sigma \vdash \rho} \text{ax} \quad \frac{\frac{\rho \wedge \sigma \vdash \rho \wedge \sigma}{\rho \wedge \sigma \vdash \sigma} \text{ax}}{\rho \wedge \sigma \vdash \sigma \wedge \rho} \wedge_{e_1} \quad \wedge_{e_2}}{\vdash (\rho \wedge \sigma) \Rightarrow (\sigma \wedge \rho)} \Rightarrow_i$$

X. Asociatividad (\wedge)

$$((\rho \wedge \sigma) \wedge \tau) \Leftrightarrow (\rho \wedge (\sigma \wedge \tau))$$
$$(\Rightarrow)$$

$$\frac{\frac{\frac{\Gamma \vdash (\rho \wedge \sigma) \wedge \tau}{\Gamma \vdash \rho \wedge \sigma} \text{ax}}{\Gamma \vdash \rho} \wedge_{e_1} \quad \frac{\frac{\frac{\Gamma \vdash (\rho \wedge \sigma) \wedge \tau}{\Gamma \vdash \rho \wedge \sigma} \text{ax}}{\Gamma \vdash \sigma} \wedge_{e_2} \quad \frac{\frac{\Gamma \vdash (\rho \wedge \sigma) \wedge \tau}{\Gamma \vdash \tau} \text{ax}}{\Gamma \vdash \sigma \wedge \tau} \wedge_i}{\frac{\Gamma = ((\rho \wedge \sigma) \wedge \tau) \vdash (\rho \wedge (\sigma \wedge \tau))}{\vdash ((\rho \wedge \sigma) \wedge \tau) \Rightarrow (\rho \wedge (\sigma \wedge \tau))} \Rightarrow_i} \wedge_i$$

$$(\leftarrow)$$

$$\begin{array}{c}
\frac{\frac{\frac{}{\Gamma \vdash \rho \wedge (\sigma \wedge \tau)}{\text{ax}}}{\Gamma \vdash \rho} \wedge_{e_1} \quad \frac{\frac{\frac{}{\Gamma \vdash \rho \wedge (\sigma \wedge \tau)}{\text{ax}}}{\Gamma \vdash \sigma \wedge \tau} \wedge_{e_2} \quad \frac{\frac{}{\Gamma \vdash \rho \wedge (\sigma \wedge \tau)}{\text{ax}}}{\Gamma \vdash \sigma \wedge \tau} \wedge_{e_2}}{\Gamma \vdash \sigma} \wedge_{e_1} \quad \frac{\frac{}{\Gamma \vdash \rho \wedge (\sigma \wedge \tau)}{\text{ax}}}{\Gamma \vdash \tau} \wedge_{e_2}}{\Gamma \vdash \rho \wedge \sigma} \wedge_i \\
\frac{\Gamma = (\rho \wedge (\sigma \wedge \tau)) \vdash ((\rho \wedge \sigma) \wedge \tau)}{\vdash (\rho \wedge (\sigma \wedge \tau)) \Rightarrow ((\rho \wedge \sigma) \wedge \tau)} \Rightarrow_i
\end{array}$$

XI. Conmutatividad (\vee)

$$\begin{array}{c}
\frac{}{(\rho \vee \sigma) \vdash \rho \vee \sigma} \text{ax} \quad \frac{\frac{}{(\rho \vee \sigma), \rho \vdash \rho} \text{ax}}{(\rho \vee \sigma), \rho \vdash \sigma \vee \rho} \vee_{i_2} \quad \frac{\frac{}{(\rho \vee \sigma), \sigma \vdash \sigma} \text{ax}}{(\rho \vee \sigma), \sigma \vdash \sigma \vee \rho} \vee_{i_1}}{(\rho \vee \sigma) \vdash \sigma \vee \rho} \vee_e \\
\frac{}{\vdash (\rho \vee \sigma) \Rightarrow (\sigma \vee \rho)} \Rightarrow_i
\end{array}$$

XII. Asociatividad (\vee)

$$((\rho \vee \sigma) \vee \tau) \Leftrightarrow (\rho \vee (\sigma \vee \tau))$$

$$(\Rightarrow)$$

$$\begin{array}{c}
\frac{\frac{}{\Gamma \vdash (\rho \vee \sigma) \vee \tau} \text{ax} \quad \frac{\frac{}{\Sigma \vdash \rho \vee \sigma} \text{ax} \quad \frac{\frac{}{\Sigma, \rho \vdash \rho} \text{ax}}{\Sigma, \rho \vdash \rho \vee (\sigma \vee \tau)} \vee_{i_1} \quad \frac{\frac{}{\Sigma, \sigma \vdash \sigma} \text{ax}}{\Sigma, \sigma \vdash \sigma \vee \tau} \vee_{i_1} \quad \frac{}{\Sigma, \sigma \vdash \rho \vee (\sigma \vee \tau)} \vee_{i_2}}{\Sigma = \Gamma, (\rho \vee \sigma) \vdash \rho \vee (\sigma \vee \tau)} \vee_e \quad \frac{\frac{}{\Gamma, \tau \vdash \tau} \text{ax}}{\Gamma, \tau \vdash \sigma \vee \tau} \vee_{i_2}}{\Gamma, \tau \vdash \rho \vee (\sigma \vee \tau)} \vee_e \\
\frac{\Gamma = ((\rho \vee \sigma) \vee \tau) \vdash (\rho \vee (\sigma \vee \tau))}{\vdash ((\rho \vee \sigma) \vee \tau) \Rightarrow (\rho \vee (\sigma \vee \tau))} \Rightarrow_i
\end{array}$$

$$(\Leftarrow)$$

$$\begin{array}{c}
\frac{\frac{}{\Gamma \vdash \rho \vee (\sigma \vee \tau)} \text{ax} \quad \frac{\frac{}{\Gamma, \rho \vdash \rho} \text{ax}}{\Gamma, \rho \vdash \rho \vee \sigma} \vee_{i_1} \quad \frac{\frac{}{\Gamma, \rho \vdash \rho \vee \sigma} \text{ax}}{\Gamma, \rho \vdash (\rho \vee \sigma) \vee \tau} \vee_{i_1} \quad \frac{\frac{}{\Sigma \vdash \sigma \vee \tau} \text{ax} \quad \frac{\frac{}{\Sigma, \sigma \vdash \sigma} \text{ax}}{\Sigma, \sigma \vdash \rho \vee \sigma} \vee_{i_2}}{\Sigma = \Gamma, (\sigma \vee \tau) \vdash (\rho \vee \sigma) \vee \tau} \vee_e \quad \frac{\frac{}{\Sigma, \tau \vdash \tau} \text{ax}}{\Sigma, \tau \vdash (\rho \vee \sigma) \vee \tau} \vee_{i_2}}{\Gamma = (\rho \vee (\sigma \vee \tau)) \vdash ((\rho \vee \sigma) \vee \tau)} \vee_e \\
\frac{\Gamma = (\rho \vee (\sigma \vee \tau)) \vdash ((\rho \vee \sigma) \vee \tau)}{\vdash (\rho \vee (\sigma \vee \tau)) \Rightarrow ((\rho \vee \sigma) \vee \tau)} \Rightarrow_i
\end{array}$$

Con lógica clásica:

I. Absurdo clásico

$$\begin{array}{c}
\frac{\frac{}{\Gamma \vdash \neg \tau \Rightarrow \perp} \text{ax} \quad \frac{}{\Gamma \vdash \neg \tau} \text{ax}}{\Gamma = \{\neg \tau \Rightarrow \perp, \neg \tau\} \vdash \perp} \Rightarrow_e \\
\frac{}{\neg \tau \Rightarrow \perp \vdash \tau} \text{PBC} \\
\frac{}{\vdash (\neg \tau \Rightarrow \perp) \Rightarrow \tau} \Rightarrow_i
\end{array}$$

II. Ley de Peirce

$$\begin{array}{c}
 \frac{}{\Gamma, \neg\tau, \tau \vdash \tau} \text{ax} \quad \frac{}{\Gamma, \neg\tau, \tau \vdash \neg\tau} \text{ax} \\
 \hline
 \frac{}{\Gamma, \neg\tau, \tau \vdash \perp} \neg_e \\
 \frac{}{\Gamma, \neg\tau, \tau \vdash \rho} \perp_e \\
 \frac{}{\Gamma, \neg\tau \vdash (\tau \Rightarrow \rho) \Rightarrow \tau} \text{ax} \quad \frac{}{\Gamma, \neg\tau \vdash (\tau \Rightarrow \rho)} \Rightarrow_i \\
 \hline
 \frac{}{\Gamma, \neg\tau \vdash \tau} \Rightarrow_e \quad \frac{}{\Gamma, \neg\tau \vdash \neg\tau} \text{ax} \\
 \hline
 \frac{}{\Gamma, \neg\tau \vdash \perp} \neg_e \\
 \frac{}{\Gamma = ((\tau \Rightarrow \rho) \Rightarrow \tau) \vdash \tau} \text{PBC} \\
 \hline
 \frac{}{\vdash ((\tau \Rightarrow \rho) \Rightarrow \tau) \Rightarrow \tau} \Rightarrow_i
 \end{array}$$

III. Tercero excluido

Esto se puede probar con PBC pero ya tenemos dado LEM.

$$\frac{}{\vdash \tau \vee \neg\tau} \text{LEM}$$

IV. Consecuencia milagrosa

$$\begin{array}{c}
 \frac{}{\Gamma \vdash \neg\tau \Rightarrow \tau} \text{ax} \quad \frac{}{\Gamma \vdash \neg\tau} \text{ax} \\
 \hline
 \frac{}{\Gamma \vdash \tau} \Rightarrow_e \quad \frac{}{\Gamma \vdash \neg\tau} \text{ax} \\
 \hline
 \frac{}{\Gamma = \{(\neg\tau \Rightarrow \tau), \neg\tau\} \vdash \perp} \neg_e \\
 \hline
 \frac{}{\Gamma = \{(\neg\tau \Rightarrow \tau), \neg\tau\} \vdash \perp} \text{PBC} \\
 \hline
 \frac{}{(\neg\tau \Rightarrow \tau) \vdash \tau} \Rightarrow_i \\
 \hline
 \frac{}{\vdash (\neg\tau \Rightarrow \tau) \Rightarrow \tau} \Rightarrow_i
 \end{array}$$

V. Contraposición clásica

$$\begin{array}{c}
 \frac{}{\Gamma \vdash \tau} \text{ax} \quad \frac{}{\Gamma \vdash \neg\rho \Rightarrow \neg\tau} \text{ax} \quad \frac{}{\Gamma \vdash \neg\rho} \text{ax} \\
 \hline
 \frac{}{\Gamma \vdash \neg\tau} \Rightarrow_e \\
 \hline
 \frac{}{\Gamma = \{(\neg\rho \Rightarrow \neg\tau), \tau, \neg\rho\} \vdash \perp} \neg_e \\
 \hline
 \frac{}{\Gamma = \{(\neg\rho \Rightarrow \neg\tau), \tau, \neg\rho\} \vdash \perp} \text{PBC} \\
 \hline
 \frac{}{(\neg\rho \Rightarrow \neg\tau), \tau \vdash \rho} \Rightarrow_i \\
 \hline
 \frac{}{(\neg\rho \Rightarrow \neg\tau) \vdash (\tau \Rightarrow \rho)} \Rightarrow_i \\
 \hline
 \frac{}{\vdash (\neg\rho \Rightarrow \neg\tau) \Rightarrow (\tau \Rightarrow \rho)} \Rightarrow_i
 \end{array}$$

VI. Análisis de casos

$$\begin{array}{c}
 \frac{}{\Gamma, \neg\rho, \neg\tau \vdash \neg\tau \Rightarrow \rho} \text{ax} \quad \frac{}{\Gamma, \neg\rho, \neg\tau \vdash \neg\tau} \text{ax} \\
 \hline
 \frac{}{\Gamma, \neg\rho, \neg\tau \vdash \rho} \Rightarrow_e \quad \frac{}{\Gamma, \neg\rho, \neg\tau \vdash \neg\rho} \text{ax} \\
 \hline
 \frac{}{\Gamma, \neg\rho, \neg\tau \vdash \perp} \neg_e \\
 \hline
 \frac{}{\Gamma, \neg\rho \vdash \tau \Rightarrow \rho} \text{ax} \quad \frac{}{\Gamma, \neg\rho, \neg\tau \vdash \perp} \text{PBC} \\
 \hline
 \frac{}{\Gamma, \neg\rho \vdash \tau} \Rightarrow_e \\
 \hline
 \frac{}{\Gamma, \neg\rho \vdash \rho}
 \end{array}$$

(*)

$$\begin{array}{c}
\frac{\uparrow}{\Gamma, \neg\rho \vdash \rho} \Rightarrow_e \quad \frac{}{\Gamma, \neg\rho \vdash \neg\rho} \text{ax} \\
\frac{}{\Gamma, \neg\rho \vdash \perp} \neg_e \\
\frac{}{\Gamma = \{(\tau \Rightarrow \rho), (\neg\tau \Rightarrow \rho)\} \vdash \rho} \text{PBC} \\
\frac{(\tau \Rightarrow \rho) \vdash (\neg\tau \Rightarrow \rho) \Rightarrow \rho}{\vdash (\tau \Rightarrow \rho) \Rightarrow ((\neg\tau \Rightarrow \rho) \Rightarrow \rho)} \Rightarrow_i
\end{array}$$

VII. Implicación vs disyunción

$$(\tau \Rightarrow \rho) \Leftrightarrow (\neg\tau \vee \rho)$$

$$(\Rightarrow)$$

$$\begin{array}{c}
\frac{}{(\tau \Rightarrow \rho), \neg(\neg\tau \vee \rho) \vdash} \quad \frac{}{(\tau \Rightarrow \rho), \neg(\neg\tau \vee \rho) \vdash} \neg_e \\
\frac{}{(\tau \Rightarrow \rho), \neg(\neg\tau \vee \rho) \vdash \perp} \text{PBC} \\
\frac{(\tau \Rightarrow \rho) \vdash (\neg\tau \vee \rho)}{\vdash (\tau \Rightarrow \rho) \Rightarrow (\neg\tau \vee \rho)} \Rightarrow_i
\end{array}$$