

Demostrar en deducción natural que las siguientes fórmulas son teoremas sin usar principios de razonamiento clásicos salvo que se indique lo contrario. Recordemos que una fórmula σ es un teorema si y sólo si vale $\vdash \sigma$:

$$\frac{\frac{\frac{\frac{\Gamma, \rho \vdash \rho \Rightarrow \sigma \Rightarrow \tau}{\Gamma \vdash \sigma \Rightarrow \tau} \text{ax}}{\Gamma \vdash \sigma \Rightarrow \tau} \Rightarrow_e \quad \frac{\frac{\frac{\Gamma, \sigma \Rightarrow \tau \vdash \rho}{\Gamma \vdash \sigma} \text{ax}}{\Gamma \vdash \sigma} \Rightarrow_e}{\Gamma \vdash \sigma} \Rightarrow_e}{\Gamma = \rho \Rightarrow (\sigma \Rightarrow \tau), \rho \Rightarrow \sigma, \rho \vdash \tau} \Rightarrow_i}{\frac{(\rho \Rightarrow (\sigma \Rightarrow \tau)), (\rho \Rightarrow \sigma) \vdash (\rho \Rightarrow \tau)}{(\rho \Rightarrow (\sigma \Rightarrow \tau)) \vdash ((\rho \Rightarrow \sigma) \Rightarrow (\rho \Rightarrow \tau))} \Rightarrow_i}{\vdash (\rho \Rightarrow (\sigma \Rightarrow \tau)) \Rightarrow ((\rho \Rightarrow \sigma) \Rightarrow (\rho \Rightarrow \tau))} \Rightarrow_i$$
$$\frac{\frac{\frac{\Gamma \vdash \rho \Rightarrow \perp}{\Gamma = (\rho \Rightarrow \perp), \rho \vdash \perp} \neg_i}{(\rho \Rightarrow \perp) \vdash \neg \rho} \Rightarrow_i}{\vdash (\rho \Rightarrow \perp) \Rightarrow \neg \rho} \Rightarrow_e$$
$$\frac{\frac{\frac{}{\rho, \neg\rho \vdash \rho} \text{ax} \quad \frac{}{\rho, \neg\rho \vdash \neg\rho} \text{ax}}{\frac{}{\rho, \neg\rho \vdash \perp} \neg_e} \neg_i}{\frac{}{\rho \vdash \neg\neg\rho} \Rightarrow_i} \vdash \rho \Rightarrow \neg\neg\rho$$
$$\frac{\frac{\frac{\text{ax}}{\Gamma \vdash \rho}}{\Gamma \vdash \neg\neg\rho} \neg\neg_i \quad \frac{\text{ax}}{\Gamma \vdash \neg\neg\rho}}{\Gamma = \neg\neg\rho, \rho \vdash \perp} \neg_e \quad \frac{\neg\neg\rho \vdash \neg\rho}{\vdash \neg\neg\rho \Rightarrow \neg\rho} \Rightarrow_i$$
$$\frac{\frac{\frac{}{\Gamma \vdash \rho \Rightarrow \sigma} \text{ax} \quad \frac{}{\Gamma \vdash \rho} \text{ax}}{\Gamma \vdash \sigma} \Rightarrow_e \quad \frac{}{\Gamma \vdash \neg \sigma} \text{ax}}{\Gamma = (\rho \Rightarrow \sigma), \neg \sigma, \rho \vdash \perp} \neg_e$$

$$\frac{}{(\rho \Rightarrow \sigma), \neg \sigma \vdash \neg \rho} \neg_i$$

$$\frac{}{\rho \Rightarrow \sigma \vdash \neg \sigma \Rightarrow \neg \rho} \Rightarrow_i$$

$$\frac{}{\vdash (\rho \Rightarrow \sigma) \Rightarrow (\neg \sigma \Rightarrow \neg \rho)} \Rightarrow_i$$

Probar $((\rho \wedge \sigma) \Rightarrow \tau) \Leftrightarrow (\rho \Rightarrow \sigma \Rightarrow \tau)$ se reduce a probar

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$$\frac{\frac{\frac{\Gamma \vdash (\rho \wedge \sigma) \Rightarrow \tau}{\Gamma \vdash (\rho \wedge \sigma) \Rightarrow \tau} \text{ax} \quad \frac{\frac{\Gamma \vdash \rho}{\Gamma \vdash \rho} \text{ax} \quad \frac{\Gamma \vdash \sigma}{\Gamma \vdash \sigma} \text{ax}}{\Gamma \vdash (\rho \wedge \sigma)} \wedge_i}{\Gamma \vdash (\rho \wedge \sigma) \Rightarrow \tau} \Rightarrow_e$$

$$\frac{\frac{\frac{\text{ax}}{\Gamma \vdash \rho \Rightarrow \sigma \Rightarrow \tau}}{\Gamma \vdash \sigma \Rightarrow \tau} \quad \frac{\frac{\frac{\text{ax}}{\Gamma \vdash \rho \wedge \sigma}}{\Gamma \vdash \rho} \wedge_{e_1} \quad \frac{\frac{\text{ax}}{\Gamma \vdash \rho \wedge \sigma}}{\Gamma \vdash \sigma} \wedge_{e_2}}{\Gamma \vdash \sigma} \Rightarrow_e \Rightarrow_e \quad \frac{\Gamma = (\rho \Rightarrow \sigma \Rightarrow \tau), (\rho \wedge \sigma) \vdash \tau}{(\rho \Rightarrow \sigma \Rightarrow \tau) \vdash ((\rho \wedge \sigma) \Rightarrow \tau)} \Rightarrow_i}{\vdash (\rho \Rightarrow \sigma \Rightarrow \tau) \Rightarrow ((\rho \wedge \sigma) \Rightarrow \tau)} \Rightarrow_i$$

Preguntar

 (\Rightarrow)

$$\frac{\frac{\frac{\frac{\frac{\frac{\Gamma, \rho \vdash \rho}{\text{ax}}}{\Gamma, \rho \vdash (\rho \vee \sigma)}{\vee_{i_1}}}{\Gamma, \rho \vdash \neg \rho}{\text{ax}}}{\Gamma, \rho \vdash \neg \rho}{\neg_e} \quad \frac{\frac{\frac{\frac{\Gamma, \sigma \vdash \tau}{\text{ax}}}{\Gamma, \sigma \vdash \neg \tau}{\neg_e}}{\Gamma, \sigma \vdash \perp}{\neg_i} \quad \frac{\frac{\frac{\Gamma, \sigma \vdash \neg \tau}{\text{ax}}}{\Gamma, \sigma \vdash \neg \sigma}{\neg_e}}{\Gamma \vdash \neg \sigma}{\neg_i} \quad \frac{\frac{\Gamma, \rho \vdash \perp}{\Gamma \vdash \neg \rho}{\neg_i}}{\Gamma \vdash \neg \sigma}{\neg_i} \quad \frac{\Gamma = \neg(\rho \vee \sigma) \vdash (\neg \rho \wedge \neg \sigma)}{\vdash \neg(\rho \vee \sigma) \Rightarrow (\neg \rho \wedge \neg \sigma)}{\Rightarrow_i}$$

$$\vdash (\neg \rho \wedge \neg \sigma) \Rightarrow \neg(\rho \vee \sigma)$$

IX. Conmutatividad (\wedge)

$$\frac{\frac{\frac{}{\text{ax}}{\rho \wedge \sigma \vdash \rho \wedge \sigma}}{\rho \wedge \sigma \vdash \rho} \wedge_{e_1} \quad \frac{\frac{\frac{}{\text{ax}}{\rho \wedge \sigma \vdash \rho \wedge \sigma}}{\rho \wedge \sigma \vdash \sigma} \wedge_{e_2}}{\rho \wedge \sigma \vdash \sigma \wedge \rho} \wedge_i}{\vdash (\rho \wedge \sigma) \Rightarrow (\sigma \wedge \rho)} \Rightarrow_i$$

X. Asociatividad (\wedge)

$$((\rho \wedge \sigma) \wedge \tau) \Leftrightarrow (\rho \wedge (\sigma \wedge \tau))$$

(\Rightarrow)

$$\frac{\frac{\frac{\frac{}{\text{ax}}{\Gamma \vdash (\rho \wedge \sigma) \wedge \tau}}{\Gamma \vdash \rho \wedge \sigma} \wedge_{e_1}}{\Gamma \vdash \rho} \wedge_{e_1} \quad \frac{\frac{\frac{\frac{}{\text{ax}}{\Gamma \vdash (\rho \wedge \sigma) \wedge \tau}}{\Gamma \vdash \rho \wedge \sigma} \wedge_{e_1} \quad \frac{\frac{}{\text{ax}}{\Gamma \vdash \sigma}}{\Gamma \vdash \sigma} \wedge_{e_2}}{\Gamma \vdash \sigma \wedge \tau} \wedge_i}{\Gamma \vdash (\rho \wedge \sigma) \wedge (\sigma \wedge \tau)} \wedge_i}{\Gamma = ((\rho \wedge \sigma) \wedge \tau) \vdash (\rho \wedge (\sigma \wedge \tau))} \Rightarrow_i$$

(\Leftarrow)

$$\frac{\frac{\frac{\frac{}{\text{ax}}{\Gamma \vdash \rho \wedge (\sigma \wedge \tau)}}{\Gamma \vdash \rho} \wedge_{e_1} \quad \frac{\frac{\frac{\frac{}{\text{ax}}{\Gamma \vdash \rho \wedge (\sigma \wedge \tau)}}{\Gamma \vdash \sigma \wedge \tau} \wedge_{e_2}}{\Gamma \vdash \sigma} \wedge_{e_1}}{\Gamma \vdash \rho \wedge \sigma} \wedge_i \quad \frac{\frac{\frac{}{\text{ax}}{\Gamma \vdash \rho \wedge (\sigma \wedge \tau)}}{\Gamma \vdash \sigma \wedge \tau} \wedge_{e_2}}{\Gamma \vdash \tau} \wedge_{e_2}}{\Gamma = (\rho \wedge (\sigma \wedge \tau)) \vdash ((\rho \wedge \sigma) \wedge \tau)} \Rightarrow_i$$

XI. Conmutatividad (\vee)

$$\frac{\frac{}{\text{ax}}{(\rho \vee \sigma) \vdash \rho \vee \sigma} \quad \frac{\frac{\frac{}{\text{ax}}{(\rho \vee \sigma), \rho \vdash \rho}}{(\rho \vee \sigma), \rho \vdash \sigma \vee \rho} \vee_{i_2} \quad \frac{\frac{\frac{}{\text{ax}}{(\rho \vee \sigma), \sigma \vdash \sigma}}{(\rho \vee \sigma), \sigma \vdash \sigma \vee \rho} \vee_{i_1}}{(\rho \vee \sigma) \vdash \sigma \vee \rho} \vee_e}{\vdash (\rho \vee \sigma) \Rightarrow (\sigma \vee \rho)} \Rightarrow_i$$

XII. Asociatividad (\vee)

$$((\rho \vee \sigma) \vee \tau) \Leftrightarrow (\rho \vee (\sigma \vee \tau))$$

(\Rightarrow)

$$\frac{\frac{}{\text{ax}}{\Gamma \vdash (\rho \vee \sigma) \vee \tau} \quad \frac{\frac{\frac{}{\text{ax}}{\Sigma \vdash \rho \vee \sigma}}{\Sigma \vdash \rho \vee (\sigma \vee \tau)} \vee_{i_1} \quad \frac{\frac{\frac{\frac{}{\text{ax}}{\Sigma, \sigma \vdash \sigma}}{\Sigma, \sigma \vdash \sigma \vee \tau} \vee_{i_1}}{\Sigma, \sigma \vdash \rho \vee (\sigma \vee \tau)} \vee_{i_2}}{\Sigma = \Gamma, (\rho \vee \sigma) \vdash \rho \vee (\sigma \vee \tau)} \vee_e \quad \frac{\frac{\frac{}{\text{ax}}{\Gamma, \tau \vdash \tau}}{\Gamma, \tau \vdash \sigma \vee \tau} \vee_{i_2}}{\Gamma, \tau \vdash \rho \vee (\sigma \vee \tau)} \vee_e}{\Gamma = ((\rho \vee \sigma) \vee \tau) \vdash (\rho \vee (\sigma \vee \tau))} \Rightarrow_i$$

$$(\leftarrow)$$

$$\frac{\begin{array}{c} \text{ax} \\ \overline{\Gamma, \rho \vdash \rho} \\ \hline \Gamma \vdash \rho \vee (\sigma \vee \tau) \end{array}}{\quad} \text{ax} \quad \frac{\begin{array}{c} \text{ax} \\ \overline{\Gamma, \rho \vdash \rho} \\ \hline \Gamma, \rho \vdash \rho \vee \sigma \end{array} \vee_{i_1} \quad \frac{\text{ax}}{\Sigma \vdash \sigma \vee \tau} \quad \frac{\begin{array}{c} \text{ax} \\ \overline{\Sigma, \sigma \vdash \sigma} \\ \hline \Sigma, \sigma \vdash \rho \vee \sigma \end{array} \vee_{i_2} \quad \frac{\text{ax}}{\Sigma, \tau \vdash \tau} \quad \frac{\begin{array}{c} \text{ax} \\ \overline{\Sigma, \sigma \vdash (\rho \vee \sigma)} \\ \hline \Sigma, \sigma \vdash (\rho \vee \sigma) \vee \tau \end{array} \vee_{i_1} \quad \frac{\text{ax}}{\Sigma, \tau \vdash (\rho \vee \sigma) \vee \tau} \vee_e}{\Sigma = \Gamma, (\sigma \vee \tau) \vdash (\rho \vee \sigma) \vee \tau} \vee_e$$