Ejercicio 5

Demostrar en deducción natural que las siguientes fórmulas son teoremas sin usar principios de razonamiento clásicos salvo que se indique lo contrario. Recordemos que una fórmula σ es un teorema si y sólo si vale $\vdash \sigma$:

I. Modus ponens relativizado:

$$\frac{\frac{\Gamma}{\Gamma,\rho\vdash\rho\Rightarrow\sigma\Rightarrow\tau}\underset{}{\text{ax}}\quad\overline{\Gamma,\sigma\Rightarrow\tau\vdash\rho}\underset{}{\text{ax}}\underset{}{\text{ax}}\quad\overline{\Gamma\vdash\rho\Rightarrow\sigma}\underset{}{\text{ax}}\quad\overline{\Gamma,\sigma\vdash\rho}\underset{}{\text{ax}}\underset{}{\text{ax}}\\ \frac{\Gamma\vdash\rho\Rightarrow\sigma}{\Gamma\vdash\sigma\Rightarrow\sigma}\underset{}{\Rightarrow_{e}}$$

$$\frac{\Gamma\vdash\rho\Rightarrow\sigma}{\Gamma\vdash\sigma\Rightarrow}\underset{}{\Rightarrow_{e}}$$

$$\frac{\Gamma\vdash\rho\Rightarrow\sigma}{\Gamma\vdash\sigma\Rightarrow}\underset{}{\Rightarrow_{e}}$$

$$\frac{\Gamma\vdash\rho\Rightarrow\sigma\Rightarrow\sigma}{\Gamma\vdash\sigma\Rightarrow\Rightarrow}\underset{}{\Rightarrow_{e}}$$

$$\frac{\Gamma\vdash\rho\Rightarrow\sigma\Rightarrow\sigma}{\Gamma\vdash\sigma\Rightarrow\Rightarrow}\underset{}{\Rightarrow_{e}}$$

$$\frac{\Gamma\vdash\rho\Rightarrow\sigma\Rightarrow\sigma}{\Gamma\vdash\sigma\Rightarrow\Rightarrow}\underset{}{\Rightarrow_{e}}$$

$$\frac{\Gamma\vdash\rho\Rightarrow\sigma\Rightarrow\sigma\Rightarrow\sigma}{\Gamma\vdash\sigma\Rightarrow\Rightarrow}\underset{}{\Rightarrow_{e}}$$

II. Reducción al absurdo

$$\frac{\frac{\Gamma \vdash \rho \Rightarrow \bot}{\Gamma \vdash \rho} \xrightarrow{\text{ax}} \frac{}{\Gamma \vdash \rho} \xrightarrow{\text{ax}}}{\frac{\Gamma \vdash \rho \Rightarrow \bot}{(\rho \Rightarrow \bot) \vdash \neg \rho} \xrightarrow{\neg_{i}}} \Rightarrow_{e}$$

III. Introducción de la doble negación

IV. Eliminación de la triple negación

$$\frac{\frac{\overline{\Gamma \vdash \rho}}{\Gamma \vdash \neg \neg \rho} \stackrel{\text{ax}}{\neg \neg_{i}} \frac{}{\Gamma \vdash \neg \neg \neg \rho} \underset{\neg}{\text{ax}} \frac{}{\Gamma \vdash \neg \neg \neg \rho} \underset{\neg}{\text{ax}} \frac{}{\Gamma \vdash \neg \neg \neg \rho} \underset{\neg}{\text{ax}} \frac{}{\neg \neg \neg \rho \vdash \neg \rho} \underset{\rightarrow}{\neg \neg}_{i} \frac{}{\neg \neg \neg \rho \vdash \neg \rho} \underset{\rightarrow}{\Rightarrow}_{i}$$

V. Contraposición

$$\begin{array}{c|c} \hline \Gamma \vdash \rho \Rightarrow \sigma & \text{ax} & \hline \Gamma \vdash \rho \\ \hline \hline \hline \Gamma \vdash \sigma & \Rightarrow_e & \hline \Gamma \vdash \neg \sigma \\ \hline \hline \hline \hline \Gamma \vdash \sigma & \Rightarrow_e & \hline \hline \Gamma \vdash \neg \sigma \\ \hline \hline \hline \hline \hline \Gamma \vdash (\rho \Rightarrow \sigma), \neg \sigma, \rho \vdash \bot \\ \hline \hline \hline (\rho \Rightarrow \sigma), \neg \sigma \vdash \neg \rho \\ \hline \hline \rho \Rightarrow \sigma \vdash \neg \sigma \Rightarrow \neg \rho \\ \hline \hline \vdash (\rho \Rightarrow \sigma) \Rightarrow (\neg \sigma \Rightarrow \neg \rho) \\ \hline \hline \end{array} \Rightarrow_i$$

VI. Adjunción

Probar $((\rho \land \sigma) \Rightarrow \tau) \Leftrightarrow (\rho \Rightarrow \sigma \Rightarrow \tau)$ se reduce a probar

$$((\rho \land \sigma) \Rightarrow \tau) \Rightarrow (\rho \Rightarrow \sigma \Rightarrow \tau) \lor (\rho \Rightarrow \sigma \Rightarrow \tau) \Rightarrow ((\rho \land \sigma) \Rightarrow \tau)$$

Caso 1

$$\frac{\Gamma \vdash (\rho \land \sigma) \Rightarrow \tau}{\Gamma \vdash (\rho \land \sigma) \Rightarrow \tau} \text{ax} \quad \frac{\overline{\Gamma \vdash \rho} \text{ax}}{\Gamma \vdash (\rho \land \sigma)} \xrightarrow{\bigwedge_{i}} \xrightarrow{\bigcap_{i} \vdash (\rho \land \sigma)} \xrightarrow{\bigwedge_{i}} \frac{\Gamma = (\rho \land \sigma) \Rightarrow \tau, \rho, \sigma \vdash \tau}{((\rho \land \sigma) \Rightarrow \tau), \rho \vdash (\sigma \Rightarrow \tau)} \Rightarrow_{i} \frac{((\rho \land \sigma) \Rightarrow \tau) \vdash \rho \Rightarrow (\sigma \Rightarrow \tau)}{\vdash ((\rho \land \sigma) \Rightarrow \tau) \Rightarrow (\rho \Rightarrow \sigma \Rightarrow \tau)} \Rightarrow_{i}$$

Caso 2

$$\frac{\frac{\Gamma \vdash \rho \Rightarrow \sigma \Rightarrow \tau}{\Gamma \vdash \rho \land \sigma} \overset{\text{ax}}{\wedge_{e_1}}}{\frac{\Gamma \vdash \rho \land \sigma}{\Gamma \vdash \rho} \Rightarrow_e} \xrightarrow{\frac{\Gamma \vdash \rho \land \sigma}{\Gamma \vdash \sigma} \land_{e_2}} \frac{\frac{}{\Gamma \vdash \rho \land \sigma} \overset{\text{ax}}{\wedge_{e_2}}}{\frac{\Gamma \vdash \sigma \Rightarrow \tau}{\Gamma \vdash \sigma} \Rightarrow_e} \Rightarrow_e} \frac{\frac{\Gamma \vdash \rho \land \sigma}{\Gamma \vdash \sigma} \Rightarrow_e}{\frac{(\rho \Rightarrow \sigma \Rightarrow \tau) \vdash ((\rho \land \sigma) \Rightarrow \tau)}{\vdash (\rho \Rightarrow \sigma \Rightarrow \tau) \Rightarrow ((\rho \land \sigma) \Rightarrow \tau)} \Rightarrow_i}$$

VII. de Morgan (I)

Preguntar

Queremos probar $\neg(\rho \lor \sigma) \Leftrightarrow (\neg \rho \land \neg \sigma)$:

 (\Longrightarrow)

$$\frac{\frac{\overline{\Gamma,\rho\vdash\rho}}{\Gamma,\rho\vdash\rho}\overset{\mathrm{ax}}{}}{\frac{\overline{\Gamma,\rho\vdash\rho}}\overset{\mathrm{ax}}{}} \frac{\overline{\Gamma,\rho\vdash\rho}\overset{\mathrm{ax}}{}}{\Gamma,\rho\vdash\rho}\overset{\mathrm{ax}}{}}{\frac{\Gamma,\rho\vdash\rho}}\overset{\mathrm{ax}}{} \frac{\overline{\Gamma,\rho\vdash\rho}}{}^{-1}\overset{\mathrm{ax}}{}} \frac{\overline{\Gamma,\rho\vdash\rho}}{\frac{\Gamma,\rho\vdash\rho}}\overset{\mathrm{ax}}{}} \frac{\overline{\Gamma,\rho\vdash\rho}}{\frac{\Gamma,\sigma\vdash\rho}}\overset{\mathrm{ax}}{}}{\frac{\Gamma,\sigma\vdash\rho}}\overset{\mathrm{ax}}{}}{\frac{\Gamma,\sigma\vdash\rho}}\overset{\mathrm{ax}}{}}{\frac{\Gamma,\sigma\vdash\rho}}\overset{\mathrm{ax}}{}}{\frac{\Gamma,\sigma\vdash\rho}}\overset{\mathrm{ax}}{}}_{-e}$$

$$\frac{\Gamma,\rho\vdash\rho}{\Gamma,\sigma\vdash\rho}\overset{\mathrm{ax}}{}}{\frac{\Gamma,\sigma\vdash\rho}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}{\frac{\Gamma,\sigma\vdash\rho}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax}}{}}\overset{\mathrm{ax$$

VIII. de Morgan (II)

IX. Conmutatividad (\land)

$$\frac{\frac{\rho \wedge \sigma \vdash \rho \wedge \sigma}{\rho \wedge \sigma \vdash \rho} \underset{\wedge}{\text{ax}}}{\frac{\rho \wedge \sigma \vdash \rho \wedge \sigma}{\rho \wedge \sigma \vdash \rho} \underset{\wedge}{\wedge}_{e_{2}}} \frac{\text{ax}}{\rho \wedge \sigma \vdash \rho \wedge \sigma} \underset{\wedge}{\wedge}_{e_{2}}}{\frac{\rho \wedge \sigma \vdash \sigma \wedge \rho}{\vdash (\rho \wedge \sigma) \Rightarrow (\sigma \wedge \rho)}} \underset{i}{\Rightarrow_{i}}$$

X. Asociatividad (\wedge)

$$((\rho \wedge \sigma) \wedge \tau) \Leftrightarrow (\rho \wedge (\sigma \wedge \tau))$$

 (\Longrightarrow)

$$\frac{\frac{\Gamma \vdash (\rho \land \sigma) \land \tau}{\Gamma \vdash (\rho \land \sigma) \land \tau} \land_{e_1}}{\frac{\Gamma \vdash (\rho \land \sigma) \land \tau}{\Gamma \vdash \rho} \land_{e_1}} \xrightarrow{\frac{\Gamma \vdash (\rho \land \sigma) \land \tau}{\Gamma \vdash \sigma} \land_{e_2}} \frac{\exists x}{\frac{\Gamma \vdash (\rho \land \sigma) \land \tau}{\Gamma \vdash \tau} \land_{i}} \land_{e_2}}{\frac{\Gamma \vdash \sigma \land \tau}{\Gamma \vdash \sigma \land \tau} \land_{i}}$$

$$\frac{\Gamma = ((\rho \land \sigma) \land \tau) \vdash (\rho \land (\sigma \land \tau))}{\vdash ((\rho \land \sigma) \land \tau) \Rightarrow_{i}} \land_{i}$$

 (\Longleftrightarrow)

$$\frac{\frac{\Gamma \vdash \rho \land (\sigma \land \tau)}{\Gamma \vdash \rho \land (\sigma \land \tau)} \land_{e_{1}}}{\frac{\Gamma \vdash \rho \land \sigma \land \tau}{\Gamma \vdash \sigma} \land_{e_{1}}} \land_{e_{2}} \frac{\frac{\Gamma \vdash \rho \land (\sigma \land \tau)}{\Gamma \vdash \sigma \land \tau} \land_{e_{2}}}{\frac{\Gamma \vdash \rho \land \sigma}{\Gamma \vdash \tau} \land_{e_{1}}} \frac{}{\frac{\Gamma \vdash \sigma \land \tau}{\Gamma \vdash \tau} \land_{e_{2}}} \land_{e_{2}}}{\frac{\Gamma \vdash \rho \land (\sigma \land \tau)) \vdash ((\rho \land \sigma) \land \tau)}{\vdash (\rho \land (\sigma \land \tau)) \Rightarrow ((\rho \land \sigma) \land \tau)}} \Rightarrow_{i}$$

XI. Conmutatividad (∨)

$$\frac{\frac{}{(\rho \vee \sigma) \vdash \rho \vee \sigma} \operatorname{ax} \quad \frac{\overline{(\rho \vee \sigma), \rho \vdash \rho} \operatorname{ax}}{(\rho \vee \sigma), \rho \vdash \sigma \vee \rho} \vee_{i_{2}} \quad \frac{\overline{(\rho \vee \sigma), \sigma \vdash \sigma} \operatorname{ax}}{(\rho \vee \sigma), \sigma \vdash \sigma \vee \rho} \vee_{i_{1}}}{\frac{(\rho \vee \sigma) \vdash \sigma \vee \rho}{\vdash (\rho \vee \sigma) \Rightarrow (\sigma \vee \rho)}} \Rightarrow_{i}$$

XII. Asociatividad (V)

$$((\rho \vee \sigma) \vee \tau) \Leftrightarrow (\rho \vee (\sigma \vee \tau))$$

 (\Longrightarrow)

$$\frac{\frac{\overline{\Sigma,\rho\vdash\rho}\text{ ax}}{\Sigma,\rho\vdash\rho}\text{ ax}}{\frac{\overline{\Sigma,\rho\vdash\rho}\text{ ax}}{\Sigma,\rho\vdash\rho\vee(\sigma\vee\tau)}\vee_{i_{1}}}\frac{\frac{\overline{\Sigma,\sigma\vdash\sigma}\text{ ax}}{\Sigma,\sigma\vdash\sigma\vee\tau}\vee_{i_{1}}}{\frac{\overline{\Sigma,\sigma\vdash\sigma}\text{ bold}}{\Sigma,\sigma\vdash\rho\vee(\sigma\vee\tau)}\vee_{i_{2}}}\frac{\frac{\overline{\Gamma,\tau\vdash\tau}\text{ ax}}{\Gamma,\tau\vdash\tau}\vee_{i_{2}}}{\frac{\Gamma,\tau\vdash\sigma\vee\tau}{\Gamma,\tau\vdash\sigma\vee\tau}\vee_{i_{2}}}\frac{\frac{\overline{\Gamma,\tau\vdash\tau}\text{ ax}}{\Sigma,\sigma\vdash\rho\vee(\sigma\vee\tau)}\vee_{i_{2}}}{\frac{\Gamma=((\rho\vee\sigma)\vee\tau)\vdash(\rho\vee(\sigma\vee\tau))}{\vdash((\rho\vee\sigma)\vee\tau)\Rightarrow(\rho\vee(\sigma\vee\tau))}}\Rightarrow_{i}$$

 (\longleftarrow)

$$\frac{\frac{\frac{1}{\Gamma,\rho\vdash\rho}\operatorname{ax}}{\frac{\Gamma,\rho\vdash\rho\vee\sigma}{\vee_{i_{1}}}}}{\frac{\Gamma,\rho\vdash\rho\vee\sigma}{\Gamma,\rho\vdash(\rho\vee\sigma)\vee\tau}\vee_{i_{1}}} \xrightarrow{\frac{\frac{\Sigma,\sigma\vdash\sigma}{\Sigma,\sigma\vdash\rho\vee\sigma}\vee_{i_{2}}}{\frac{\Sigma,\sigma\vdash\rho\vee\sigma}{\vee_{i_{2}}}\vee_{i_{1}}}} \xrightarrow{\frac{\Sigma,\tau\vdash\tau}{\Sigma,\tau\vdash(\rho\vee\sigma)\vee\tau}\vee_{i_{2}}} \frac{\operatorname{ax}}{\frac{\Sigma,\sigma\vdash\rho\vee\sigma}{\Sigma,\sigma\vdash(\rho\vee\sigma)\vee\tau}\vee_{i_{1}}} \xrightarrow{\frac{\Sigma,\tau\vdash\tau}{\Sigma,\tau\vdash(\rho\vee\sigma)\vee\tau}\vee_{i_{2}}} \times \frac{\operatorname{ax}}{\frac{\Sigma,\tau\vdash\tau}{\Sigma,\tau\vdash(\rho\vee\sigma)\vee\tau}\vee_{i_{2}}} \times \operatorname{ax}} \times \operatorname{ax} \times \operatorname{ax}$$