Ejercicio 5

Demostrar en deducción natural que las siguientes fórmulas son teoremas sin usar principios de razonamiento clásicos salvo que se indique lo contrario. Recordemos que una fórmula σ es un teorema si y sólo si vale $\vdash \sigma$:

I. *Modus ponens* relativizado:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \rho \vdash \rho \Rightarrow \sigma \Rightarrow \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \sigma \Rightarrow \tau \vdash \rho$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma \vdash \sigma \Rightarrow \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \Rightarrow \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \sigma \vdash \rho$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma \vdash \sigma$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma = \rho \Rightarrow (\sigma \Rightarrow \tau), \rho \Rightarrow \sigma, \rho \vdash \tau$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$(\rho \Rightarrow (\sigma \Rightarrow \tau)), (\rho \Rightarrow \sigma) \vdash (\rho \Rightarrow \tau)$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$(\rho \Rightarrow (\sigma \Rightarrow \tau)) \vdash ((\rho \Rightarrow \sigma) \Rightarrow (\rho \Rightarrow \tau))$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\rho \Rightarrow (\sigma \Rightarrow \tau)) \Rightarrow ((\rho \Rightarrow \sigma) \Rightarrow (\rho \Rightarrow \tau))$}
\end{prooftree}
$$

II. Reducción al absurdo

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \Rightarrow \bot$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma = (\rho \Rightarrow \bot), \rho \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$(\rho \Rightarrow \bot) \vdash \neg \rho$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\rho \Rightarrow \bot) \Rightarrow \neg \rho$}
\end{prooftree}
$$

III. Introducción de la doble negación

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\rho, \neg \rho \vdash \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\rho, \neg \rho \vdash \neg \rho$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\rho, \neg \rho \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$\rho \vdash \neg \neg \rho$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash \rho \Rightarrow \neg \neg \rho$}
\end{prooftree}
$$

IV. Eliminación de la triple negación

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho$}
\RightLabel{$\neg\neg_i$}
\UnaryInfC{$\Gamma \vdash \neg \neg \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg \neg \neg \rho$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma = \neg \neg \neg \rho, \rho \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$\neg \neg \neg \rho \vdash \neg \rho$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash \neg \neg \neg \rho \Rightarrow \neg \rho$}
\end{prooftree}
$$

V. Contraposición

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \Rightarrow \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma \vdash \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg \sigma$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma = (\rho \Rightarrow \sigma), \neg \sigma, \rho \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$(\rho \Rightarrow \sigma), \neg \sigma \vdash \neg \rho$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\rho \Rightarrow \sigma \vdash \neg \sigma \Rightarrow \neg \rho$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\rho \Rightarrow \sigma) \Rightarrow (\neg \sigma \Rightarrow \neg \rho)$}
\end{prooftree}
$$

VI. Adjunción

Probar $((\rho \land \sigma) \Rightarrow \tau) \Leftrightarrow (\rho \Rightarrow \sigma \Rightarrow \tau)$ se reduce a probar

$((\rho \land \sigma) \Rightarrow \tau) \Rightarrow (\rho \Rightarrow \sigma \Rightarrow \tau)$ y $(\rho \Rightarrow \sigma \Rightarrow \tau) \Rightarrow ((\rho \land \sigma) \Rightarrow \tau)$

Caso 1

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash (\rho \land \sigma) \Rightarrow \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \sigma$}
\RightLabel{$\land_i$}
\BinaryInfC{$\Gamma \vdash (\rho \land \sigma)$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma = (\rho \land \sigma) \Rightarrow \tau, \rho, \sigma \vdash \tau$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$((\rho \land \sigma) \Rightarrow \tau), \rho \vdash (\sigma \Rightarrow \tau)$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$((\rho \land \sigma) \Rightarrow \tau) \vdash \rho \Rightarrow (\sigma \Rightarrow \tau)$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash ((\rho \land \sigma) \Rightarrow \tau) \Rightarrow (\rho \Rightarrow \sigma \Rightarrow \tau)$}
\end{prooftree}
$$

Caso 2

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \Rightarrow \sigma \Rightarrow \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \land \sigma$}
\RightLabel{$\land_{e_1}$}
\UnaryInfC{$\Gamma \vdash \rho$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma \vdash \sigma \Rightarrow \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \land \sigma$}
\RightLabel{$\land_{e_2}$}
\UnaryInfC{$\Gamma \vdash \sigma$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma = (\rho \Rightarrow \sigma \Rightarrow \tau), (\rho \land \sigma) \vdash \tau$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$(\rho \Rightarrow \sigma \Rightarrow \tau) \vdash ((\rho \land \sigma) \Rightarrow \tau)$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\rho \Rightarrow \sigma \Rightarrow \tau) \Rightarrow ((\rho \land \sigma) \Rightarrow \tau)$}
\end{prooftree}
$$

VII. de Morgan (I)

Preguntar

Queremos probar $\neg(\rho \lor \sigma) \Leftrightarrow (\neg\rho \land \neg\sigma)$:

$(\Longrightarrow)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \rho \vdash \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \rho \vdash \rho$}
\RightLabel{$\lor_{i_1}$}
\UnaryInfC{$\Gamma, \rho \vdash (\rho \lor \sigma)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \rho \vdash \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \rho, \sigma \vdash \rho$}
\RightLabel{$\lor_e$}
\TrinaryInfC{$\Gamma, \rho \vdash \neg\rho$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, \rho \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$\Gamma \vdash \neg\rho$}
\AxiomC{}
\UnaryInfC{$\Gamma, \sigma \vdash \tau$}
\AxiomC{}
\UnaryInfC{$\Gamma, \sigma \vdash \neg\tau$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, \sigma \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$\Gamma \vdash \neg\sigma$}
\RightLabel{$\land_i$}
\BinaryInfC{$\Gamma = \neg(\rho \lor \sigma) \vdash (\neg\rho \land \neg\sigma)$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash \neg(\rho \lor \sigma) \Rightarrow (\neg\rho \land \neg\sigma)$}
\end{prooftree}
$$

$(\Longleftarrow)$

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\vdash (\neg\rho \land \neg\sigma) \Rightarrow \neg(\rho \lor \sigma)$}
\end{prooftree}
$$

VIII. de Morgan (II)

IX. Conmutatividad ($\land$)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\rho \land \sigma \vdash \rho \land \sigma$}
\RightLabel{$\land_{e_1}$}
\UnaryInfC{$\rho \land \sigma \vdash \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\rho \land \sigma \vdash \rho \land \sigma$}
\RightLabel{$\land_{e_2}$}
\UnaryInfC{$\rho \land \sigma \vdash \sigma$}
\RightLabel{$\land_i$}
\BinaryInfC{$\rho \land \sigma \vdash \sigma \land \rho$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\rho \land \sigma) \Rightarrow (\sigma \land \rho)$}
\end{prooftree}
$$

X. Asociatividad ($\land$)

$((\rho \land \sigma) \land \tau) \Leftrightarrow (\rho \land (\sigma \land \tau))$

$(\Longrightarrow)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash (\rho \land \sigma) \land \tau$}
\RightLabel{$\land_{e_1}$}
\UnaryInfC{$\Gamma \vdash \rho \land \sigma$}
\RightLabel{$\land_{e_1}$}
\UnaryInfC{$\Gamma \vdash \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash (\rho \land \sigma) \land \tau$}
\RightLabel{$\land_{e_1}$}
\UnaryInfC{$\Gamma \vdash \rho \land \sigma$}
\RightLabel{$\land_{e_2}$}
\UnaryInfC{$\Gamma \vdash \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash (\rho \land \sigma) \land \tau$}
\RightLabel{$\land_{e_2}$}
\UnaryInfC{$\Gamma \vdash \tau$}
\RightLabel{$\land_i$}
\BinaryInfC{$\Gamma \vdash \sigma \land \tau$}
\RightLabel{$\land_i$}
\BinaryInfC{$\Gamma = ((\rho \land \sigma) \land \tau) \vdash (\rho \land (\sigma \land \tau))$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash ((\rho \land \sigma) \land \tau) \Rightarrow (\rho \land (\sigma \land \tau))$}
\end{prooftree}
$$

$(\Longleftarrow)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \land (\sigma \land \tau)$}
\RightLabel{$\land_{e_1}$}
\UnaryInfC{$\Gamma \vdash \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \land (\sigma \land \tau)$}
\RightLabel{$\land_{e_2}$}
\UnaryInfC{$\Gamma \vdash \sigma \land \tau$}
\RightLabel{$\land_{e_1}$}
\UnaryInfC{$\Gamma \vdash \sigma$}
\RightLabel{$\land_i$}
\BinaryInfC{$\Gamma \vdash \rho \land \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \land (\sigma \land \tau)$}
\RightLabel{$\land_{e_2}$}
\UnaryInfC{$\Gamma \vdash \sigma \land \tau$}
\RightLabel{$\land_{e_2}$}
\UnaryInfC{$\Gamma \vdash \tau$}
\RightLabel{$\land_i$}
\BinaryInfC{$\Gamma = (\rho \land (\sigma \land \tau)) \vdash ((\rho \land \sigma) \land \tau)$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\rho \land (\sigma \land \tau)) \Rightarrow ((\rho \land \sigma) \land \tau)$}
\end{prooftree}
$$

XI. Conmutatividad ($\lor$)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(\rho \lor \sigma) \vdash \rho \lor \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(\rho \lor \sigma), \rho \vdash \rho$}
\RightLabel{$\lor_{i_2}$}
\UnaryInfC{$(\rho \lor \sigma), \rho \vdash \sigma \lor \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(\rho \lor \sigma), \sigma \vdash \sigma$}
\RightLabel{$\lor_{i_1}$}
\UnaryInfC{$(\rho \lor \sigma), \sigma \vdash \sigma \lor \rho$}
\RightLabel{$\lor_e$}
\TrinaryInfC{$(\rho \lor \sigma) \vdash \sigma \lor \rho$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\rho \lor \sigma) \Rightarrow (\sigma \lor \rho)$}
\end{prooftree}
$$

XII. Asociatividad ($\lor$)

$((\rho \lor \sigma) \lor \tau) \Leftrightarrow (\rho \lor (\sigma \lor \tau))$

$(\Longrightarrow)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash (\rho \lor \sigma) \lor \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Sigma \vdash \rho \lor \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Sigma, \rho \vdash \rho$}
\RightLabel{$\lor_{i_1}$}
\UnaryInfC{$\Sigma, \rho \vdash \rho \lor (\sigma \lor \tau)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Sigma, \sigma \vdash \sigma$}
\RightLabel{$\lor_{i_1}$}
\UnaryInfC{$\Sigma, \sigma \vdash \sigma \lor \tau$}
\RightLabel{$\lor_{i_2}$}
\UnaryInfC{$\Sigma, \sigma \vdash \rho \lor (\sigma \lor \tau)$}
\RightLabel{$\lor_e$}
\TrinaryInfC{$\Sigma = \Gamma, (\rho \lor \sigma) \vdash \rho \lor (\sigma \lor \tau)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \tau \vdash \tau$}
\RightLabel{$\lor_{i_2}$}
\UnaryInfC{$\Gamma, \tau \vdash \sigma \lor \tau$}
\RightLabel{$\lor_{i_2}$}
\UnaryInfC{$\Gamma, \tau \vdash \rho \lor (\sigma \lor \tau)$}
\RightLabel{$\lor_e$}
\TrinaryInfC{$\Gamma = ((\rho \lor \sigma) \lor \tau) \vdash (\rho \lor (\sigma \lor \tau))$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash ((\rho \lor \sigma) \lor \tau) \Rightarrow (\rho \lor (\sigma \lor \tau))$}
\end{prooftree}
$$

(⟸)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \vee (\sigma \vee \tau)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \rho \vdash \rho$}
\RightLabel{$\vee_{i_1}$}
\UnaryInfC{$\Gamma, \rho \vdash \rho \vee \sigma$}
\RightLabel{$\vee_{i_1}$}
\UnaryInfC{$\Gamma, \rho \vdash (\rho \vee \sigma) \vee \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Sigma \vdash \sigma \vee \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Sigma, \sigma \vdash \sigma$}
\RightLabel{$\vee_{i_2}$}
\UnaryInfC{$\Sigma, \sigma \vdash \rho \vee \sigma$}
\RightLabel{$\vee_{i_1}$}
\UnaryInfC{$\Sigma, \sigma \vdash (\rho \vee \sigma) \vee \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Sigma, \tau \vdash \tau$}
\RightLabel{$\vee_{i_2}$}
\UnaryInfC{$\Sigma, \tau \vdash (\rho \vee \sigma) \vee \tau$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$\Sigma = \Gamma, (\sigma \vee \tau) \vdash (\rho \vee \sigma) \vee \tau$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$\Gamma = (\rho \vee (\sigma \vee \tau)) \vdash ((\rho \vee \sigma) \vee \tau)$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\rho \vee (\sigma \vee \tau)) \Rightarrow ((\rho \vee \sigma) \vee \tau)$}
\end{prooftree}
$$

Ejercicio 6

Demostrar en deducción natural que vale $\vdash \sigma$ para cada una de las siguientes fórmulas. Para estas fórmulas es imprescindible usar lógica clásica:

I. Absurdo clásico

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg\tau \Rightarrow \bot$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg\tau$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma = \{\neg\tau \Rightarrow \bot, \neg\tau\} \vdash \bot$}
\RightLabel{PBC}
\UnaryInfC{$\neg\tau \Rightarrow \bot \vdash \tau$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\neg\tau \Rightarrow \bot) \Rightarrow \tau$}
\end{prooftree}
$$

II. Ley de Peirce

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg\tau \vdash (\tau \Rightarrow \rho) \Rightarrow \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg\tau, \tau \vdash \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg\tau, \tau \vdash \neg\tau$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, \neg\tau, \tau \vdash \bot$}
\RightLabel{$\bot_e$}
\UnaryInfC{$\Gamma, \neg\tau, \tau \vdash \rho$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\Gamma, \neg\tau \vdash (\tau \Rightarrow \rho)$}
\RightLabel{$\Rightarrow_e$}
\BinaryInfC{$\Gamma, \neg\tau \vdash \tau$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg\tau \vdash \neg\tau$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, \neg\tau \vdash \bot$}
\RightLabel{PBC}
\UnaryInfC{$\Gamma = ((\tau \Rightarrow \rho) \Rightarrow \tau) \vdash \tau$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash ((\tau \Rightarrow \rho) \Rightarrow \tau) \Rightarrow \tau$}
\end{prooftree}
$$

III. Tercero excluido

$\tau \vee \neg\tau$

IV. Consecuencia milagrosa

$(\neg\tau \Rightarrow \tau) \Rightarrow \tau$

V. Contraposición clásica

$(\neg\rho \Rightarrow \neg\tau) \Rightarrow (\tau \Rightarrow \rho)$

VI. Análisis de casos

$(\tau \Rightarrow \rho) \Rightarrow (\neg\tau \Rightarrow \rho) \Rightarrow \rho$

VII. Implicación vs disyunción

$(\tau \Rightarrow \rho) \Leftrightarrow (\neg\tau \vee \rho)$