Proposition 1. *As $N \to \infty$ and $t \to \infty$, the ratio of $\mathbf{X}_b$ to $\mathbf{X}_i$, $i \neq b$ in solutions to the system with positive coefficients, initial conditions $\mathbf{X} = \mathbf{0}$, and $B_i \neq B_j$ for all $i \neq j$*

$$\frac{d\mathbf{X}_i}{dt} = (A_i + B_i\mathbf{X}_i)\left(N - \sum_{j=1}^{J}\mathbf{X}_j\right) - \frac{sd_i\mathbf{X}_i}{K + L_i\mathbf{X}_i}, i \in 1, 2, \ldots, J \tag{1}$$

approaches ∞ where $\mathbf{X}_b$ is the component of $\mathbf{X}$ with the largest value of B_i in (1).

Proof. We begin by noting that for solutions to (1) starting at $\mathbf{X} = \mathbf{0}$, $\mathbf{X}_i \geq 0$ for all i for all t. This is because when $\mathbf{X}_i = 0$, (1) becomes $\frac{d\mathbf{X}_i}{dt} = A_i$, which is positive. Additionally, we note that for solutions to (1) starting at $\mathbf{X} = \mathbf{0}$, $\sum_{j=1}^{J}\mathbf{X}_j < N$ for all t. This is because, when $\sum_{j=1}^{J}\mathbf{X}_j = N$, (1) becomes

$$\frac{d\mathbf{X}_i}{dt} = -\frac{sd_i\mathbf{X}_i}{K + L_i\mathbf{X}_i} \tag{2}$$

which is non-positive for all values of i because all coeffcients are positive and $\mathbf{X}_i \geq 0$ by the earlier note. (2) is negative for at least one component of $\mathbf{X}$ because the only way for it to be zero for all values of i is if $\mathbf{X} = \mathbf{0}$, which is impossible because we set $\sum_{j=1}^{J}\mathbf{X}_j = N$ and $N > 0$. Thus $\sum_{j=1}^{J}\mathbf{X}_j < N$ for all t.

Next, we see that (1) is asymptotically equivalent to

$$\frac{d\mathbf{X}_i}{dt} = (A_i + B_i\mathbf{X}_i)\left(N - \sum_{j=1}^{J}\mathbf{X}_j\right) \tag{3}$$

as $N \to \infty$ with initial condition $\mathbf{X} = \mathbf{0}$ because

$$\lim_{N\to\infty} \frac{(A_i + B_i\mathbf{X}_i)\left(N - \sum_{j=1}^{J}\mathbf{X}_j\right) - \frac{sd_i\mathbf{X}_i}{K+L_i\mathbf{X}_i}}{(A_i + B_i\mathbf{X}_i)\left(N - \sum_{j=1}^{J}\mathbf{X}_j\right)} = \tag{4}$$

$$\lim_{N\to\infty} 1 - \frac{sd_i\mathbf{X}_i}{(K + L_i\mathbf{X}_i)(A_i + B_i\mathbf{X}_i)(N - \sum_{j=1}^{J}\mathbf{X}_j)} = 1 \tag{5}$$

and $\sum_{j=1}^{J} < N$ by the earlier note which implies that $N - \sum_{j=1}^{J}\mathbf{X}_j \neq 0$. Thus as $N \to \infty$, the solutions to (1) approach those of (3). Looking at (3), we see that the derivative of each component has the common factor $N - \sum_{j=1}^{J}\mathbf{X}_j$. Therefore, the solution curve to (3) starting at $\mathbf{X} = \mathbf{0}$ has the same shape as that of

$$\frac{d\mathbf{X}_i}{dt} = (A_i + B_i\mathbf{X}_i). \tag{6}$$

for $\sum_{j=1}^{J}\mathbf{X}_j < N$. Using separation by parts, the solution to (6) with initial condition $\mathbf{X} = \mathbf{0}$ is

$$\mathbf{X}_i(t) = \frac{A_i}{B_i}\left(e^{B_i t} - 1\right) \tag{7}$$

Denote the component of $\mathbf{X}_i(t)$ with the largest value of B_i as component b. We next note that if $B_i > B_j$, the limit of the ratio between any two components $\mathbf{X}_i(t)$ and $\mathbf{X}_j(t)$ as $t \to \infty$ is

$$\lim_{t\to\infty} \frac{\frac{A_i}{B_i}\left(e^{B_i t} - 1\right)}{\frac{A_j}{B_j}\left(e^{B_j t} - 1\right)} = \infty \tag{8}$$

Therefore, the ratio between $\mathbf{X}_b(t)$ and any other component of $\mathbf{X}(t)$ must approach ∞ as $t \to \infty$. As $N \to \infty$, the time for which $\sum_{j=1}^{J} \mathbf{X}_j < N$ also approaches ∞. As (3) and (6) have solutions which follow the same curve while $\sum_{j=1}^{J} \mathbf{X}_j < N$, this means that the same conclusion about ratios of components holds for solutions to (3) as $N \to \infty$ and $t \to \infty$.

Thus, as solutions with the initial condition $\mathbf{X} = \mathbf{0}$ to (1) approach those of (3) which in turn approach those of (6) as $N \to \infty$, the ratio of $\mathbf{X}_b$ in solutions to (1) to any other component must approach ∞ as $N \to \infty$ and $t \to \infty$.

□

Corollary 1. *There exist values of N for which the component $\mathbf{X}_b$, which is defined as in the above proof, of the solution to (1) with initial condition $\mathbf{X} = \mathbf{0}$ is larger than any other component as $t \to \infty$.*

Proof. Assume for contradiction that there were no values of N for which the corollary held true. Then, there would not exist a value of N for which $\mathbf{X}_b > \mathbf{X}_i$ for $i \neq b$ as $t \to \infty$. Therefore, the ratio between $\mathbf{X}_b$ and any other component could not approach ∞ as $N \to \infty$ and $t \to \infty$ because there would be no values of N for which $\mathbf{X}_b$ were greater than any other component. This contradicts the above proof. Thus, the corollary holds true. □