Proposition 1. *As $N \to \infty$ and $t \to \infty$, the solution $\mathbf{X}(t)$ to the system with positive coefficients, and $B_i \neq B_j$ for all $i \neq j$*

$$\frac{d\mathbf{X}_i}{dt} = (A_i + B_i\mathbf{X}_i)\left(N - \sum_{j=1}^{J}\mathbf{X}_j\right) - \frac{sd_i\mathbf{X}_i}{K + L_i\mathbf{X}_i}, i \in 1, 2, \ldots, J \tag{1}$$

and initial condition $\mathbf{X} = \mathbf{0}$ approaches $\mathbf{X}_b = N$ where b is the component of $\mathbf{X}$ with the largest value of B_i in (1) and all other components of $\mathbf{X}$ approach 0.

Proof. We begin by making the substitution $\mathbf{X} = N\mathbf{P}$. This yields

$$\frac{dN\mathbf{P}_i}{dt} = (A_i + B_iN\mathbf{P}_i)\left(N - \sum_{j=1}^{J}N\mathbf{P}_j\right) - \frac{sd_iN\mathbf{P}_i}{K + L_iN\mathbf{P}_i}. \tag{2}$$

Dividing both sides by N yields

$$\frac{d\mathbf{P}_i}{dt} = (A_i + B_iN\mathbf{P}_i)\left(1 - \sum_{j=1}^{J}\mathbf{P}_j\right) - \frac{sd_i\mathbf{P}_i}{K + L_iN\mathbf{P}_i}. \tag{3}$$

We next note that (3) is asymptotically equivalent to

$$\frac{d\mathbf{P}_i}{dt} = (A_i + B_iN\mathbf{P}_i)\left(1 - \sum_{j=1}^{J}\mathbf{P}_j\right) \tag{4}$$

as $N \to \infty$ because

$$\lim_{N\to\infty}\frac{(A_i + B_iN\mathbf{P}_i)\left(1 - \sum_{j=1}^{J}\mathbf{P}_j\right) - \frac{sd_i\mathbf{P}_i}{K+L_iN\mathbf{P}_i}.}{(A_i + B_iN\mathbf{P}_i)\left(1 - \sum_{j=1}^{J}\mathbf{P}_j\right)} = 1 \tag{5}$$

Thus as $N \to \infty$, the solutions to (3) approach those of (4). Looking at (4), we see that each component has the common factor $1 - \sum_{j=1}^{J}\mathbf{P}_j$. Therefore, for $1 - \sum_{j=1}^{J}\mathbf{P}_j > 0$, the solution curve to (4) is the same as that of

$$\frac{d\mathbf{P}_i}{dt} = (A_i + B_iN\mathbf{P}_i). \tag{6}$$

When $1 - \sum_{j=1}^{J}\mathbf{P}_j = 0$, (4) becomes $\frac{d\mathbf{P}_i}{dt} = 0$. Thus solutions to (4) do not cross the hyperplane described by $1 - \sum_{j=1}^{J}\mathbf{P}_j = 0$. Using separation by parts, the solution to (6) with initial condition $\mathbf{P} = \mathbf{0}$ is

$$\mathbf{P}_i(t) = \frac{A_i}{NB_i}\left(e^{NB_it} - 1\right) \tag{7}$$

Denote the component of $\mathbf{P}_i(t)$ with the largest value of B_i as component b. We next note that if $B_i > B_j$, the limit of the ratio between any two components $\mathbf{P}_i(t)$ and $\mathbf{P}_j(t)$ *for any time* t as $N \to \infty$ is

$$\lim_{N\to\infty} \frac{\frac{A_i}{NB_i}\left(e^{NB_i t} - 1\right)}{\frac{A_j}{NB_j}\left(e^{NB_j t} - 1\right)} = \infty \tag{8}$$

Therefore, the ratio between $\mathbf{P}_b(t)$ and any other component of $\mathbf{P}(t)$ must approach ∞ as $N \to \infty$ at any time t, regardless of the value of $1 - \sum_{j=1}^{J} \mathbf{P}_j$.

As (7) is always increasing in every component with t, it must eventually cross the hyperplane described by $1 - \sum_{j=1}^{J} \mathbf{P}_j = 0$ at some point. This is the same point that solutions to (4) with intial condition $\mathbf{P} = \mathbf{0}$ asymptotically approach. That is, solutions to (4) must approach a constant value and the sum of the components to that solution will approach 1 as $t \to \infty$.

As the sum of the components asymptotically approaches 1 as $t \to \infty$, all components besides b must approach 0 and component b must approach 1 as $N \to \infty$ and $t \to \infty$. If component b did not approach 1, one more other components would approach a non-zero value and the ratio between component b and any of those components would approach a constant value. From (8), we know that this is impossible. Thus, component b must approach 1 which implies that all other components much approach 0 because the sum of the components approaches 1.

As (3) is asymptotically equivalent to (4) as $N \to \infty$, the same holds true for solutions to (3) as $N \to \infty$. Using the relationship $\mathbf{X} = N\mathbf{P}$, we conclude that solutions to (1) with initial condition $\mathbf{X} = \mathbf{0}$ must then approach $\mathbf{X}_b = N$ and all other components equal to 0 as $N \to \infty$ and $t \to \infty$.

□

Corollary 1. *As $N \to \infty$ and $t \to \infty$, the solution to (1) with initial condition $\mathbf{X} = \mathbf{0}$ where there exist two or more values of B_i such that $B_i \geq B_j$ for all $i \neq j$ approaches $\mathbf{X}_i = \frac{A_i}{\sum_{j=1}^{J} A_j} N$*