CS 173
a.k.a. Discrete Structures

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1 Logic

Converse: $p \to q$ becomes $q \to p$. It is not logically equivalent.
Contrapositive: $p \to q$ becomes $\neg q \to \neg p$. Logically equivalent.
Negation implies: $\neg(p \to q) \equiv \neg p \vee q$
De Morgan's Law $\neg(p \wedge q) \equiv \neg p \vee \neg q$.
$\neg(p \vee q) \equiv \neg p \wedge \neg q$.

2 Sets

Definition: A set is an unordered collection of objects.
Cardinality: Is how many objects there are in a set: $A = \{1, 3, 4\}$ then $|A| = 3$.
Cartesian Product: A x B= $\{(x, y)|x \in A$ and $y \in B\}$.
Size of set union: $|A \cup B| = |A| + |B|$.
Product rule: if $|A| = n, |B| = q$ then $|A$x $B| = nq$.

3 Number theory

Divisibility a divides b if $b = an$ for some integer n. The short hand for a divedes b is $a|b$.
Prime numbers an integer $q \geq 2$ is prime if the only positive factors of q are q and 1. An integer $q \geq 2$ is composite if it is not prime.

Fundamental Theorem of Arithmetics: Every integer ≥ 2 can be written as the product of one or more prime factors. Except for the order in which you write the factors, this prime factorization is unique.
GCD: Greatest Common Divisor. The shorthand is gcd(a,b). It is done my taking the prime factors and xracting hte shared factors.
LCM: Least Common Multiple. $lcm(a, b) = \frac{ab}{gcd(a,b)}$. If two integers share no factors, $gcd(a, b) = 1$ then they are called **relatively prime**.

Congruence mod k: if k is any positive integer, two integers a and b are congruent mod k (written $a \equiv b(modk)$) if $k|(a - b)$, thus $a - b = kn$.

4 Relations

A relation R on a set A is a subset of A^2, thus R is a set of ordered pairs of elements from A.
Reflexive: Every element is related to itself. $\forall x \in A, xRx$.
Irreflexive: No element is related to itself. $\forall x \in A, x \not R x$.
Symmetric: $\forall x, y \in A, xRy \to yRx$.
Antisymmetric: $\forall x, y \in A$ with $x \neq y, ifxRy$, then $y \not R x$.
$\forall x, y \in A, ifxRy \wedge yRx$, then $x = y$.
Transitive: $\forall a, b, c \in A, aRb \wedge bRc \to aRc$.
Partial order: A relation that is transitive, reflexive and antisymmetric.
Linear (total) order: is a partial order in which every pair of elements are

comparable. That is, $\forall x, y, x \geq y \vee y \geq x$.
Strict partial order: is a relation that is irreflexive, antisymmetric and transitive.
Equivalence relation: is a relation that is reflexive, symmetric and transitive.
Equivalence class: Is the set of all elements related to x. $[x]_R = \{y \in A | xRy\}$.

5 Functions

Onto

Definition: A function $f : A \to B$ is *onto* or *surjective* if its image is its whole co-domain. If every element in the co-domain if mapped in the domain. Or, equivalently,

$$\forall y \in B, \exists x \in A, f(x) = y$$

Say you want to compose the function $f : A \to B$ and $g : B \to C$. Then $gof : A \to C$.

One-To-One

A function is $one-to-one$ or $injective$ if it never assigns two input values to the same output value. Its formal definition is:

$$\forall x, y \in A, x \neq y \to f(x) \neq f(y)$$

or, by taking the contrapositive,

$$forallx, y \in A, f(x) = f(y) \to x = y$$

6 Graphs

Graphs are a very general class of objects. A graph consists of a set of nodes V and a set of edges E. Two nodes connected by an edge are called **neighbors** or **adjacent**.
Undirected graph: when edges can be crossed in both directions.
Simple graph: If it has neither multiple edges nor loop edges.
Degrees: the degree of a vertex v, written $deg(v)$ is the number of edges which have v as an endpoint.
Handshaking Theorem: $\sum_{v \in V} deg(v) = 2|E|$.
Complete graph:K_n is a graph in which every vertex is connected to another vertex.
Cycle: C_n. It is a graph that is only a cycle.
Wheels: W_n is just like C_n except that it has an additional central "hub" node which is connected to all the others.
Isomorphism: An isomorphism from G_1 to G_2 is a bijection $f : V_1 \to V_2$ such that vertices a and b are joined by an edge if and only if $f(a)$ and $f(b)$ are joined by an edge.
Walk: a walk of legth k from bertex a to vertex b is a sequence of vertices and a sequence of edge that connects them.
Closed walk: if the starting and ending verticies are the same. Otherwise is **open**.

Path: A path is an open walk in which no vertex is used more than one.
Cycle: Is a closed walk with at least three vertices in which no vertex is used more than once except that the starting and ending vertices are the same. **Connected Graph:** A graph is connected if there is a walk between every pair of nodes.
Cut edge: An edge that if its cut will increase the number of connected componets.
Diameter: Is the maximum distance between any pair of nodes in hte graph.
Euler circuits: When there is a closed walk that uses each edge of the graph exactly once. It is possible whenever the degree of every node in the graph is ever.
Bipartite graph: Is a graph that can be colored with two colors.

7 Induction

Hypercube: A n-cube or a hypercube Q_n os the graph of the of the corners and edges of an n-dimentional cube.

8 Trees

Definition: Finite, non-empty set of nodes plus a set of edges. each edge connects each node to its **parent** node.
Root: A node with no parent.
Child: if x is the parent of y, then y is called a child of x. It z is also a child of x, then y and x are **siblings**.
Leaf node is a node that has no children.
Internal node: a node that does have a children.
Levels: The nodes of a tree can be organized into levels, based on how many edges away from the root htey are.
Height: maximum level of any of its node, or maximum length of a path from the root to a leaf.
m-ary trees: A m-ary tree is the restriction on how many child a node can have.
Heap property: if, for every node X in the tree, the value of X is at least as big as the value in each of X's children.

9 Big-O

Definition: f(x) is O(g(x)) if and only if there are positive real numbers c, k such that $0 \leq f(x) \leq cg(x), \forall x \geq k$.
Omega f(x): $g(x)$ is $\Omega(f(x))$ if and only if $f(x)$ is $O(g(x))$. Which means that if $g(x) \succeq f(x)$.
Theta g(x): f(x) is $\Theta(g(x))$ if and only if $g(x)$ is $O(f(x))$ and $f(x)$ is $O(g(x))$. Therefore is the equivalence relation.

10 Algorithms

Linear search

Go through the array, find the minimum value.
Average Case: $O(n)$.
Best Case: $O(n)$.

Binary search

Given a sorted array, returns the index of the given value.
Running time: $O(\log n)$
.

Insertion sort

Devide the input array into two pieces and orders it.
Running time: $O(n^2)$.

Mergesort

Mergesort devides the big input list into smaller pieces, and hten it merges them back together. It is a very stable and very predictable algorithm.
Running time: $O(n \log_n)$.

Katatsuba's algorithm

First fast algorithm for multiplying big numbers together. It works by breaking up numbers and them multiplieng them together, using the recurrence $T(n) = 3T(\frac{n}{2}) + O(n)$, which gives you $O(n^{\log_2 3})$.

11 Sets of Sets

Powerset: Is th set containing all subsets of A.
Partition: when we divide a base set A into non-overlapping subsets, the result is called a *partition*.
Binomial Theorem: Let x and y be varibales and let n be any natural number. Then

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

12 Countability

Cardinality: Two sets A and B have the same cartinality ($|A| = |B|$) if an only if there is a bijection from A to B.
Countable: An infinite set A is *countable* or $countablyinfinite$ if there is a bijection from $\mathbb{N}$ onto A.
Cantor Schroeder Bernstein Theorem: $|A| \leq |B|$ if and only if there is a one-to-one function from A to B. Then if $|A| \leq |B|$ and $|B| \leq |A|$, $|A| = |B|$.

13 Planar Graphs

Definition: A planar graph is a graph which can be drawn in hte plane without any edges crossing.

Degree of a face is the length of its boundary, how many edges it has.
Euler's formula: $v - e + f = 2$.
Handshaking theorem for faces: sum of the faces degrees is also $2e$.
Free tree: Any connected graph with no cycles.
Corollay of Euler's formula: In a graph where $v \geq 3, e \leq 3v - 6$.
Subdivision: In a graph G a *subdivision* of another graph F is G just like F except that you've divided up some of F's edges by adding vertces in the middle of them.
Homeomorphic: Two graphs are *homeomorphic* if one is a subdivision of another, or they are both subdivision of some third graph.
Kuratowski's theorem: A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.