Right now, these notes connect with prior knowledge described in [Recap] in order to answer [Objective] and are written in Obsidian md. They will later grow independent and become html. Before that, please see `.pdf` of obsidian looks.

differential-forms-in-chalk

Recap

So we know how to apply wedge and differentiation operators to differential forms.
Then, we get a number from a differentiation form by integrating it.

Objective

Both 1-forms and 2-forms correspond to basis vectors in $\mathbb{R}^3$. 1-forms are componenty; 2-forms are surfacey.
The question: *How?*

To answer this question, we have to know differential forms more than as "mathematical objects."

A differential forms is a *function*.

- takes in vectors
- outputs a function (which we treat as a number)

An n-form takes in n vectors.

Definition: 1-form as a function that takes in 1 vector

There are 3 options + 1 example to explain what this function does.

Option 1. Formula.

Formula left hand side:

$$\overbrace{dx_{\underbrace{k}}}^{\omega}\;\overbrace{\underbrace{(a_1\hat{e}_1 + a_2\hat{e}_2 \dots + a_n\hat{e}_n)}_{\text{general form of a vector in } \mathbb{R}^n}}^{\text{its input}}$$

$k \in (1, 2, \dots, n)$.

There are n 1-forms in $\mathbb{R}^n$
and what they do are pretty symmetric.

Formula right hand side:

$$a_k.$$

The dx_k function picks out the kth component in its input.

Option 2. I tell you this function is linear. I give you the base cases.

Every dx_k is a linear function.

What happens when each dx_k takes in a unit vector:

$$\overbrace{dx_k}^{\omega}\underbrace{\overbrace{(\hat{e}_i)}^{\text{input}}}_{\text{Only a subset of vectors in } \mathbb{R}^n} = \underbrace{\begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}}_{\text{Looks like orthogonality!}}$$

Or δ_k^i

Option 3. This function is a projection.

dx_k means projection onto e_k.

$dx_k(\mathbf{v})$ is the kth component of $\mathbf{v}$.

Note following options 1, 2, 3.

If we want to find $\omega(\mathbf{v})$ where ω is not a basis 1-form ($\omega = dx_k$) but a linear combination of basis 1-forms,

$$\omega = \sum_i c_i \, dx_i,$$

then

$$\omega(\mathbf{v}) = \sum_i c_i \, dx_i(\mathbf{v}).$$

Option 4 (Example). Find $dx_1(< 8, 15, 17 >)$. Ans: 8.

Find $dx_2(< 8, 15, 17 >)$.

> Ans: 15.

Find $dx_3(< 8, 15, 17, 1 >)$.

> Ans: 17.

Find $dx_4(< 8, 15, 17, 1 >)$.

> Ans: 1.

Find $\omega(< 8, 15, 17 >)$ where $\omega = dx_1 + dx_2$.

> Ans: $8 + 15 = 23$

Find $\omega(< 8, 15, 17 >)$ where $\omega = 3\ dx_1 + 7\ dx_2$.

> Ans: $3 \cdot 8 + 7 \cdot 15 = 129$

Answer: 1-forms are component-y basis elements.

Dictionary

Vector in $\mathbb{R}^3$: $\vec{x} = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3$

number

basis vector

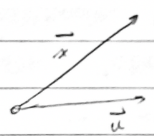
basis 1-form

1-form : $\omega = a_1\, dx_1 + a_2\, dx_2 + a_3\, dx_3$

The associated 1-form of a vector is called its "work form."

Suppose $\vec{x}$ represents a force.

pushing on this point



while an objects gets displaced by $\vec{u}$.

Work done on the object :

$$\text{Work} = \vec{x} \cdot \vec{u}$$
$$= \omega(\vec{u})$$

Reiterate:
If ω is the work form (1-form counterpart) of $\mathbf{x}$, then

$$\mathbf{x} \cdot \mathbf{u} = \omega(\mathbf{u}).$$

Definition: 2-form as a function that takes in 2 vectors.

3 options + 3 examples to explain what this function does.

Option 1. Formula.

Formula left hand side:

$$\overbrace{dx_i \wedge dx_j}^{\omega} \; \overbrace{(\vec{u}, \vec{v})}^{\text{input : 2 vectors in } \mathbb{R}^n}$$

$$i, j \in (1, 2, \dots n)$$

(i, j) order matters !

Formula right hand side:

$$\begin{vmatrix} dx_i(\vec{u}) & dx_i(\vec{v}) \\ dx_j(\vec{u}) & \underbrace{dx_j(\vec{v})}_{\substack{\text{1-form acting on} \\ \text{1 vector} \\ \longrightarrow \text{ number}}} \end{vmatrix}$$

This function is a determinant.
We rely on a matrix of 1-forms acting on 1 vector.

Mnemonics

- For a matrix to be determinant-able, it has to be square.

$$\left.\underbrace{\begin{vmatrix} dx_i(\vec{u}) & dx_i(\vec{v}) \\ dx_j(\vec{u}) & dx_j(\vec{v}) \end{vmatrix}}_{\substack{\text{2 columns} \\ \because \\ \text{2 vectors}}}\right\} \substack{\text{2 rows} \\ \because \\ \text{2-form} \\ \text{has 2} \\ \text{1-forms}}$$

Thus 2-forms require 2 vectors for input.

- If there are identical rows, the determinant is 0.
  For $\omega = dx_k \wedge dx_k$,

$$\omega(\vec{u}, \vec{v}) = \begin{pmatrix} u_k & v_k \\ u_k & v_k \end{pmatrix},$$

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so it has identical rows.
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Option 2. Generalize a linear function to take more than 1 input.

If we add an input, we can go from *linear* to *multilinear*.
[image relating with determinant properties]

- vector addition:
- scalar multiplications:

When there are multiple inputs, should the order of inputs matter?
Yes:

$$\omega(\mathbf{u}, \mathbf{v}) = -\omega(\mathbf{u}, \mathbf{v}).$$

The sign is alternated when the order of two inputs is switched. This flavour of "order matters" is called *alternating*.

In summary, the 2-form is an *multilinear alternating* function. (Make your own base cases from the formula given option 1.)

Option 3. This function is a projection.

$dx_i \wedge dx_j$ means projection onto the plane spanned by $e_i,\ e_j$.

$dx_i \wedge dx_j(\mathbf{u}, \mathbf{v})$ is the area of the parallelogram spanned by the projections of $\mathbf{u}, \mathbf{v}$ onto $e_i,\ e_j$.

Option 4 (Example). Find $dx_1 \wedge dx_2(< 8, 15, 17 >, < 3, 4, 5 >)$. Ans: -13.

Answer: 2-forms are component-y basis elements.

Dictionary

[image]

The associated 2-form of a vector is called its "flux form."

[image]