Introduction to Social Choice Theory

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What Is Social Choice Theory Trying to Accomplish?

Goal: *Given the individual preferences of agents, how to aggregate these so as to obtain a social preference.*

- Prime example: voting
- Non-strategic setting
- Impossibility results (Arrow, Muller-Satterthwaite)

Voting Mechanisms

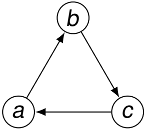
3	5	7	6
a	a	b	c
b	c	d	b
c	b	c	d
d	d	a	a

Definition:

- *Plurality voting*: The candidate which is top ranked by most voters is selected.
- *Majority voting*: The candidate which is top ranked by majority of voters is selected.
- *Cumulative voting*: Each voter has *k votes which can be cast arbitrarily (e.g., all on one candidate). The candidate with most votes is selected.*
- *Approval voting: Each voter can cast one single vote for as many of the candidates. The candidate with most votes is selected (voters cannot rank candidates).*
- *An alternative a is a Condorcet winner if for every other alternative b there is a majority preferring a to b.*

Condorcet Paradox

8	6	7
a	c	b
b	a	c
c	b	a



The Condorcet Paradox: A Condorcet winner does not always exist.

Condorcet Paradox

		number of agents						
		3	5	7	9	11		$\to \infty$
number of alternatives	3	.056	.069	.075	.078	.080	...	.088
	4	.111	.139	.150	.156	.160	...	.176
	5	.160	.200	.215	.230	.251	...	.2.51
	6	.202	.255	.258	.284	.294	...	.315
	7	.239	.299	.305	.342	.243	...	.369
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
	$\to \infty$	1	1	1	1	1	...	1

Probability $p(x, y)$ of no Condorcet winner for x alternatives and y agents.

(from: Moulin (1988), page 230)

The Borda Rule

Definition *(The Borda Rule)*: Given a finite set of alternatives X and strict individual preferences, for each ballot, each alternative is given one point for every other alternative it is ranked above. The alternatives are then ranked proportional to the number of points they aggregate.

1	5	5	3	2
a	a	c	b	b
d	d	b	a	d
b	c	a	d	c
c	b	d	c	a

8	6	7
a	c	b
b	a	c
c	b	a

Remark: In the left case, the Borda rule selects the Condorcet loser!

Social Choice Rules: A Formal Model

Definition: Be $N = \{1, 2, \ldots, n\}$ a set of players, O a set of alternatives, and L_{-} a class of preference relations over O.

- *Social Choice Function (scf)*: $C : L_{-}^{N} \rightarrow O$
- *Social Choice Correspondence (scc)*: $C : L_{-}^{N} \rightarrow 2^{O}$
- *Social Welfare Function (swf)*: $W : L_{-}^{N} \rightarrow L_{-}$
- *Social Welfare Correspondence (swc)*: $W : L_{-}^{N} \rightarrow 2^{L_{-}}$

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Definition *(Condorcet Condition)*: An outcome $o \in O$ is a Condorcet winner if $\forall o' \in O : \#(o \succ o') \geq \#(o' \succ o)$. A social choice function satisfies the Condorcet condition if it always picks a Condorcet winner when one exists.

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Definition *(Smith set)***:** The Smith set is the smallest non-empty set $S \subseteq O$ such that $\forall o \in S, \forall o' \notin S : \#(o \succ o') \geq \#(o' \succ o)$, i.e., each member of S beats every other candidate outside S in a pairwise election. **The Smith set exits always**. When the Condorcet winner exists, then the Smith set is a singleton consisting of the Condorcet winner.

Borda cannot always select one winner

Example:

$\succsim_1$	$\succsim_2$	$\succsim_3$
a	b	c
b	c	a
c	a	b

Question: Who is the Borda winner?

Borda cannot always select one winner

Example:

$\succsim_1$	$\succsim_2$	$\succsim_3$
a	b	c
b	c	a
c	a	b

x	count
a	$2 + 0 + 1 = 3$
b	$1 + 2 + 0 = 3$
c	$0 + 1 + 2 = 3$

Question: Who is the Borda winner?

Remark: The Borda rule does not always provide a social choice function, but a social choice correspondence.

Other Voting Methods

Definition *(Plurality with Elimination)***:** Each voter casts a single vote for their most-preferred candidate. The candidate with the fewest votes is eliminated. Each voter who cast a vote for the eliminated candidate cast a new vote for the candidate he most prefers among the remaining candidates. This process is eliminated until only one candidate remains.

Definition *(Pairwise Elimination)***:** Voters are given in advance a schedule for the order in which pairs of candidates will be compared. Given two candidates (and based on each voter's preference ordering) determine the candidate that each voter prefers. The candidate who is preferred by a minority of voters is eliminated, and the next pair of non-eliminated candidates in the schedule is considered. Continue until only one candidate remains.

Sensitivity of Voting Methods

8	6	7
a	c	b
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- Sensitivity to a losing candidate: remove c and check Majority, Borda, and condorcet methods.
- Sensitivity to the agenda setter: compare $a.b.c$ and $a.c.b$ agenda's in pairwise elimination method.

Social Welfare Functions

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- W has the *Pareto property* if

$$o_1 \succ_i o_2 \text{ for all } i \in N \text{ implies } o_1 \succ_{W(\succ_1,\ldots,\succ_n)} o_2$$

Intuition: If alternative o_1 is unanimously preferred to alternative o_2, o_1 should be ranked higher than o_2 in the social ordering.

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- W is *dictatorial* if there is some $i \in N$ such that *for all preference profiles* $\succ$:

  $$o_1 \succ_i o_2 \text{ implies } o_1 \succ_{W(\succ_1,\ldots,\succ_n)} o_2$$

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- A social welfare function has an *unrestricted domain* if it defines a social ordering for *all* preference profiles.

Independence of Irrelevant Alternatives

Definition *(Independence of Irrelevant Alternatives)***:** For preference profiles $\succ, \succ' \in L_{-}^{N}$ and alternatives $o_1, o_2 \in O$:

$$o_1 \succ_{W(\succ)} o_2 \Leftrightarrow o_1 \succ_{W(\succ')} o_2, \quad \text{whenever} \quad o_1 \succ_i o_2 \Leftrightarrow o_1 \succ'_i o_2, \text{ for all } i \in N$$

Intuition: The social preference of two alternatives only depends on the *relative ordering* of these two alternatives in the individual preference relations.

Independence of Irrelevant Alternatives

Ordering according to Borda scores

1	5	5	3	2
a	a	c	b	b
d	d	b	a	d
b	c	a	d	c
c	b	d	c	a

$a \succ_{Borda} b$

1	5	5	3	2
a	a	c	b	b
b	c	b	a	c
c	b	a	c	a
d	d	d	d	d

$b \succ_{Borda} a$

Approval voting satisfies the independence of irrelevant alternatives criterion.

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Remark: IIA captures a consistency property of social choice rules. Lack of such consistency enables strategic manipulation.

Arrow's Impossibility Theorem

Theorem *(Arrow, 1951)*: *For* $|O| \geq 3$*, any social welfare function with unrestricted domain satisfying the Pareto property and Independence of Irrelevant Alternatives is dictatorial.*

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Remark: No hope for general social welfare functions, but what about social choice functions?

Social Choice Function

Definition *(Weak Pareto efficiency)***:** A social choice function C is weakly Pareto efficient if, for any preference profile $\succ \in \underline{L}^n$, if there exists a pair of outcomes o_1 and o_2 such that $\forall i \in N : \; o_1 \succ_i o_2$, then $C(\succ) \neq o_2$.

Intuition: Social choice function does not select any outcome that is dominated by other outcomes for all agents.

Definition *(Monotonicity)***:** A social choice function C is *monotonic* if for all $o \in O$ and for any preference profile $\succ \in \underline{L}^N$ with $C(\succ) = o$, then for any other preference profile $\succ'$ with the property that $\forall i \in N, \; \forall o' \in O : \; o \succ'_i o'$ if $o \succ_i o'$, it must be that $C(\succ') = o$.

Intuition on monotonicity: Improving one alternative should not influence the relative ordering of other alternatives.

Social Choice Function

Theorem *(Muller-Satterthwaite, 1977)*: *For $|O| \geq 3$, any social choice function C that is weakly Pareto efficient and monotonic is dictatorial.*

Consider Plurality scores which is not dictatorial and satisfies weak Pareto efficiency.

3	2	2
a	b	c
b	c	b
c	a	a

$$C(\succ_{plurality}) = a$$

3	2	2
a	b	b
b	c	a
c	a	c

$$C(\succ_{plurality}) = b$$

Restrictions

Definition: An alternative x is a *weak Condorcet winner* if there is no other alternative b for which there is some majority preferring b to a,

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Theorems: In case of the following restrictions, the existence of a weak Condorcet winner is guaranteed.

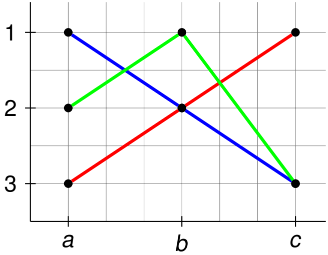
- Single-peaked preferences (Black, 1948)
- Dichotomous preferences (Inada, 1964)

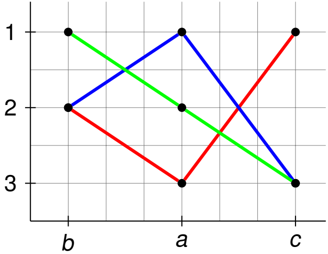
Single-peaked Preferences

Definition: Given a predetermined linear ordering of the alternative set O, a preference relation $\succ_i$ is *single-peaked* if there exists a point $p \in O$ (its peak) such that for all $o_1, o_2 \in O$ such that $p \geq o_1 > o_2$ or $o_2 > o_1 \geq p$ then $o_1 \succ_i o_2$.

Intuition: Each player prefers points closer to their peaks over points that are more distant.

Single-peaked Preferences





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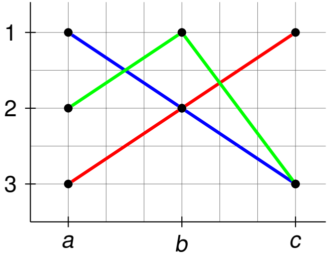

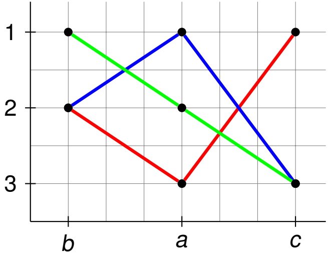

$\succ_1$	$\succ_2$	$\succ_3$
c	a	b
b	b	a
a	c	c

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Intuition: Each player prefers points closer to their peaks over points that are more distant.

Definition: Given a profile of single-peaked preferences $\succ_1, \ldots, \succ_n$ with peaks $p_1, \ldots, p_n$ (with respect to a predetermined linear ordering on alternatives), the *median voter rule* selects the median among some superset of the peaks.

Theorem *(Black, 1948)*: *The median voter rule selects a weak Condorcet winner.*

Dichotomous Preferences and Approval Voting

Preferences $\succ_i$ over O are dichotomous if there is a non-empty set $O' \subseteq O$ such that

- $o' \succ_i o$, for all $o' \in O'$ and $o \in O \setminus O'$
- $o'_1 \sim_i o'_2$, for all $o'_1, o'_2 \in O'$
- $o_1 \sim_i o_2$, for all $o_1, o_2 \in O \setminus O'$

Intuition: Agents with dichotomous preferences only distinguish between "good" and "bad" alternatives

Theorem *(Inada, 1964)***:** *If all agents have dichotomous preferences, a weak Condorcet winner is guaranteed to exist.*

Remark: *Approval voting* is a voting mechanism in which each voter can submit zero or one vote to each candidate. The candidates with most votes win. This voting rule always selects a weak Condorcet winner.