Introduction to Game Theory (2)

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Mixed Strategies and Expected Utility

Definition: Let (N, A, u) be a strategic game. Then:

- $\Delta(A_i)$ is the set of *mixed strategies*, *i.e.*, set of all probability distributions over A_i.
- $\Delta(A) = \Delta(A_1) \times \cdots \times \Delta(A_n)$, set of *mixed strategy profiles*.
- Expected utility of mixed strategies $s \in \Delta(A)$ is defined as follows:

$$u_i(s) = \sum_{a \in A} \left(u_i(a) \cdot \prod_{j \in N} s_j(a_j) \right)$$

where a is a pure strategy profile, a_j is the strategy of player j in a, and $s_j(a_j)$ is the probability value assigned to a_j by s_j.

Note: We identify a (pure) strategy 'a' with the mixed strategy 's' for which $s(a) = 1$.

Mixed Strategies and Expected Utility

$$u_i(s) = \sum_{a \in A} \left(u_i(a) \cdot \prod_{j \in N} s_j(a_j) \right)$$

	A_q	B_{1-q}
A_p	1, 1	0, 0
B_{1-p}	0, 0	1, 1

$$s = \langle (A_p, B_{1-p}) \, , \; (A_q, B_{1-q}) \rangle$$

$$\begin{aligned} u_{row}(s) \quad &= \quad \textstyle\sum_{a \in A} \left(u_{row}(a) \cdot \prod_{j \in N} s_j(a_j) \right) \\ & \quad\quad 1 * (p * q) \; + \\ & \quad\quad 0 * (p * (1 - q)) \; + \\ & \quad\quad 0 * ((1 - p) * q) \; + \\ & \quad\quad 1 * ((1 - p) * (1 - q) \\ &= \quad 2pq - p - q + 1 \end{aligned}$$

The Prisoner's Dilemma

> *Two suspects are taken into custody and separated. The district attorney is certain that they are guilty of a specific crime, but he does not have adequate evidence to convict them at a trial. He points out to each prisoner that each has two alternatives: to confess to the crime the police are sure they have done, or not to confess. If they will both do not confess, then the district attorney states he will book them on some very minor trumped up charge such as petty larceny and illegal possession of a weapon, and they will both receive minor punishment; if they both confess they will be prosecuted, but he will recommend less than the most severe sentence; but if one confesses and the other does not, then the confessor will receive lenient treatment for turning state's evidence whereas the latter will get "the book" slapped on him.*
>
> (Luce and Raiffa, 1957, p. 95)

The Prisoner's Dilemma

	NotConfess	*Confess*
NotConfess	2, 2	0, 3
Confess	3, 0	1, 1

Pareto Efficiency

Definition: An outcome $o \in O$ is *(weakly) Pareto efficient* if there is no outcome that is strictly better for all players, *i.e.*, if

there is no $o' \in O$ such that for all $i \in N$: $o' \succ_i o$

Definition: A mixed strategy profile $s \in \Delta(A)$ is *(weakly) Pareto efficient* if there is no mixed strategy profile that is strictly better for all players, *i.e.*, if

there is no $s' \in \Delta(A)$ such that for all $i \in N$: $u_i(s') > u_i(s)$

Pareto Efficiency

	NotConfess	*Confess*
NotConfess	$2, 2$	$0, 3$
Confess	$3, 0$	$1, 1$

Which are the Pareto efficient outcomes?

Pareto Efficiency

	NotConfess	Confess
NotConfess	2, 2	0, 3
Confess	3, 0	1, 1

Which are the Pareto efficient outcomes?

Dominance

Definition: A pure strategy a_i for player i *(strongly) dominates* another strategy a'_i if for any strategies of the opponents, a_i leads to a more preferable outcome than a'_i, *i.e.*, if:

$$\text{for all } b \in A: \quad (b_1, \ldots, a_i, \ldots, b_n) \succ_i (b_1, \ldots, a'_i, \ldots, b_n).$$

Dominance

	NotConfess	*Confess*
NotConfess	2, 2	0, 3
Confess	3, 0	1, 1

Which are the strongly dominant strategy profiles?

Dominance

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Which are the strongly dominant strategy profiles?

Dominance

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Which are the strongly dominant strategy profiles?

Dominance

Definition: A *mixed* strategy s_i for player i *(strongly) dominates* another mixed strategy s'_i if for any mixed strategies of the opponents, s_i has a greater expected utility than s'_i, *i.e.*, if:

$$\text{for all } t_{j \neq i} \in \Delta(A_j): \quad u_i(t_1, \ldots, s_i, \ldots, t_n) > u_i(t_1, \ldots, s'_i, \ldots, t_n).$$

A mixed strategy s_i of player i that (strongly) dominates all other mixed strategies of i is called a *strongly dominant* strategy for player i. A mixed strategy profile $(s_1, \ldots, s_n)$ is called a *(strongly) dominant mixed strategy equilibrium* if s_i is (strongly) dominant strategy for player i for every $i = 1, \ldots, n$.

Definition: A *mixed* strategy s_i for player i *(strongly) dominates* a pure strategy s'_i if for any strategies of the opponents, s_i has a greater expected utility than s'_i, *i.e.*, if:

$$\text{for all } t \in A: \quad u_i(t_1, \ldots, s_i, \ldots, t_n) > u_i(t_1, \ldots, s'_i, \ldots, t_n).$$

Dominance

	left	*right*
top	0, 3	3, 0
middle	3, 0	0, 3
bottom	1, 1	1, 1

Iterated Elimination of Dominated Strategies

Procedure of iterated elimination of dominated strategies:

- Eliminate one after another actions of player that are (weakly or strongly) dominated, until this is no longer possible
- If only one payoff profile remains, we say the game is *dominance solvable*.

Fact: The strategy profiles that survive iterated elimination of weakly dominated actions may depend on the order of elimination. This is not the case for iterated elimination of strongly dominated actions

Exercise

3, 1	0, 0	0, 0
1, 1	1, 2	5, 0
0, 1	4, 0	0, 0

1, 1	1, 1	0, 0
0, 0	1, 2	1, 2
0, 2	0, 0	0, 3

Best Responses

Definition: A pure strategy a_i is *pure best response* of a player i to a pure strategy profile $(a_1, \ldots, a_n)$ if for all $b_i \in A_i$:

$$(a_1, \ldots, a_i, \ldots, a_n) \succsim_i (a_1, \ldots, b_i, \ldots, a_n)$$

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Definition: A mixed strategy s_i is *(mixed) best response* of a player i to a mixed strategy profile $(s_1, \ldots, s_n)$ if for all $t_i \in \Delta(A_i)$:

$$u_i(s_1, \ldots, s_i, \ldots, s_n) \geq u_i(s_1, \ldots, t_i, \ldots, s_n)$$

Nash Equilibrium

Definition: A pure strategy profile a is a *pure Nash equilibrium* if no player has an incentive to *unilaterally* deviate from a, *i.e.*, if for all players i:

$$\text{for all } b_i \in A_i : \quad a \succsim_i (a_1, \ldots, b_i, \ldots, a_n)$$

2, 2	0, 3
3, 0	1, 1

1, 0	0, 1
0, 1	1, 0

2, 1	0, 0
0, 0	1, 2

Nash Equilibrium

Definition: A mixed strategy profile s is a *Nash equilibrium* if no player has an incentive to *unilaterally* deviate from s, *i.e.*, if for all players i:

$$\text{for all } t_i \in \Delta(A_i): \quad u_i(s) \geq u_i(s_1, \ldots, t_i, \ldots, s_n)$$

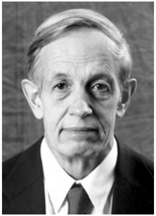
2, 2	0, 3
3, 0	1, 1

1, 0	0, 1
0, 1	1, 0

2, 1	0, 0
0, 0	1, 2

Nash's Theorem

Theorem (*Nash 1950*): Every strategic game with a finite number of pure strategies has a Nash equilibrium in mixed strategies.



Remark: *The proofs are non-constructive and use Brouwer's or Kakutani's fixed point theorems.*

Exercise

$2, 1$	$0, 0$
$0, 0$	$1, 2$

Compute the Nash equilibria of the Battle of the Sexes using both pure and mixed strategies.

Properties of Nash Equilibrium

- Nash equilibrium is perhaps the most important solution concept for non-cooperative games, for which numerous refinements have been proposed.
- Any combination of dominant strategies is a Nash equilibrium.
- Nash equilibria are not generally Pareto efficient.
- Existence in (pure) strategies is not in general guaranteed.
- Nash equilibria are not in general unique (equilibria selection, focal points).
- Nash equilibria are not generally interchangeable.
- Payoffs in different Nash equilibria may vary.

Alternative Characterization of Nash Equilibria

Definition: The *support* of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i \mid s_i(a_i) > 0\}$.

Lemma: A mixed strategy profile s is a Nash equilibrium iff for all players i

- Given $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$, all actions in the support of s_i yield the same expected utility.
- Given $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ no action not in the support of s_i yields a higher expected utility than any action in the support of s_i.

Alternative Characterization of Nash Equilibria

Definition: The *support* of a mixed strategy s_i for a player i is the set of pure strategies $\{a_i \mid s_i(a_i) > 0\}$.

Lemma: A mixed strategy profile s is a Nash equilibrium iff for all players i

- $u_i(s_1, \dots, a_i, \dots, s_n) = u_i(s_1, \dots, b_i, \dots, s_n)$, for all actions $a_i, b_i \in A_i$ in the support of s_i.
- $u_i(s_1, \dots, a_i, \dots, s_n) \geq u_i(s_1, \dots, b_i, \dots, s_n)$, for all actions $a_i, b_i \in A_i$ with a_i in but b_i not in the support of s_i.

The Prisoner's Dilemma

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Iterated Prisoner's Dilemma

- In Prisoner's dilemma is *defect* the dominant strategy.
- Can self-interested agents cooperate? Why?
- Examples from real world: nuclear arm race, public transport
- Shadow of future: cooperation is possible because the game will be played in future again.
- Iterated Prisoner's dilemma is such a scenario.

Axelrod's Tournament (1980)

Robert Axelrod (a political scientist) held a computer tournament designed to investigate how cooperation emerge among self interested agents.

- Computer programs play iterated prisoner's dilemma games against each other.
- Which strategy results in maximum overall payoff?
- Possible strategies followed by the submitted programs:
  - ALLD: always defect
  - ALLC: always cooperate
  - RANDOM: sometime cooperate sometimes defect
  - TIT-FOR-TAT: 1st round Cooperate. Other rounds do what the opponent did at previous round.
  - MAJORITY: 1st round cooperates. Other rounds examines the history of the opponent's actions, counting its total number of defect and cooperates. If opponent defect more often dan cooperate, then defect; otherwise cooperate.
  - JOSS: As TIT-FOR-TAT, except periodically defect.