Chapter 4

Propositional Calculus

1. Introduction

Propositional calculus, or propositional logic, is concerned with those forms of reasoning for which the validity of the conclusions is independent of the structure of the propositions and follows uniquely from the truth or falsity of these propositions. As with predicate calculus, which we study in Chapter 5, two complementary and linked aspects have to be taken into account:

1. The syntactic aspect: this is simply a matter of specifying a formal system, as defined in Chapter 3, within which deductions can be made, constituting the theorems of propositional calculus.
2. The semantic aspect: this concerns the interpretation of the formulae and consists of the analysis of 'formulae that are always true,' called tautologies.

There are many ways in which a formal system for the syntactic aspect can be defined. The one we have chosen has the advantage of allowing us to arrive reasonably quickly at the completeness theorem (Proposition 9) but the disadvantage of limiting the language to the connectives $\neg$ and $\rightarrow$. For the study of the semantic aspect we extend the language by including the further connectives $\vee$, $\wedge$, $\leftrightarrow$.

The link between these two aspects is provided by the proof that the formulae that are tautologies (that is, that are semantically valid) are simply those that are theorems (that is, that are syntactically valid). This is the completeness theorem, one consequence of which, proved as Corollary 1 to Proposition 9, is that the decision problem for propositional calculus admits of a definite positive answer.

2. Definitions and some theorems

Throughout Chapters 4 and 5 Po is used to denote propositional calculus and Pr is used to denote predicate calculus. We define propositional calculus Po as the formal system $(\Sigma_{Po}, F_{Po}, A_{Po}, R_{Po})$:

- $\Sigma_{Po} = \{p_0, p_1, \ldots, p_n, \ldots\} \cup \{\neg, \rightarrow, (,)\}$
  where the symbols p_i are called the propositional variables, atomic propositions or atoms.

- F_{Po} = the smallest set of formulae such that:

  $\forall$ i: $p_i \in F_{Po}$ and
  $\forall A \in F_{Po}, \forall B \in F_{Po}$: $\neg A \in F_{Po}$, $(A \to B) \in F_{Po}$

  For all $i \geqslant 0$ p_i is in F_{Po} and for all A in F_{Po} and B in F_{Po} $\neg A$ is in F_{Po} and $(A \to B)$ is in F_{Po}.

- A_{Po} = the set of formulae of any of the following three forms:

  SA_1: $(A \to (B \to A))$
  SA_2: $((A \to (B \to C)) \to ((A \to B) \to (A \to C)))$
  SA_3: $(\neg A \to \neg B) \to (B \to A)$

  where A, B, C $\in F_{Po}$.

  The expressions SA_1, SA_2 and SA_3 are called axioms (schemata): to each corresponds an infinite set of axioms, for example corresponding to SA_1 are the axioms:

  $(p_0 \to (p_1 \to p_0))$ where A is p_0 and B is p_1
  $((p_2 \to p_0) \to ((p_1 \to p_0) \to (p_2 \to p_0)))$ where A is $(p_2 \to p_0)$ and B is $(p_1 \to p_0)$, etc.

  We sometimes omit the outer pair of parentheses in these formulae; we have done so in SA_3, but not in SA_1 and SA_2.

- R_{Po} = {m.p.} meaning that there is only one rule (*modus ponens*):

  $$A, (A \to B) \vdash_{\text{m.p.}} B$$

  As we are only concerned with the system Po in this chapter, we omit it from deduction statements and write simply $A_1, A_2, \ldots, A_n \vdash B$.

The following is an example of a deduction in Po.

f_1: $((p_0 \to ((p_1 \to p_0) \to p_0)) \to ((p_0 \to (p_1 \to p_0)) \to (p_0 \to p_0)))$	(SA_2)
f_2: $(p_0 \to ((p_1 \to p_0) \to p_0))$	(SA_1)
f_3: $((p_0 \to (p_1 \to p_0)) \to (p_0 \to p_0))$	(m.p. f_1, f_2)
f_4: $(p_0 \to (p_1 \to p_0))$	(SA_1)
f_5: $(p_0 \to p_0)$	(m.p. f_3, f_4)

Thus $(p_0 \to p_0)$ is a theorem of Po:

$(p_0 \to p_0) \in T_{Po}$

which we can write:

$\vdash (p_0 \to p_0)$

Any well-formed formulae can be substituted for p_0 and p_1 in this deduction; so we have the following propositions.

Proposition 1: for all $A \in F_{Po}$: $(A \to A) \in T_{Po}$ which can be written as $\vdash (A \to A)$

In what follows, A, B, C, A_i, B_i and C_i all represent well-formed formulae in Po.

Proposition 2: if $A_1, A_2, \ldots, A_{n-1} \vdash (A_n \to B)$, then $A_1, A_2, \ldots, A_{n-1}, A_n \vdash B$

Proof

Let $C_1, C_2, \ldots, C_p$ be a deduction for $(A_n \to B)$ from $A_1, A_2, \ldots, A_{n-1}$. Then $C_1, C_2, \ldots, C_p, A_n, B$ is a deduction of B from $A_1, A_2, \ldots, A_n$. For the $(p + 1)$th formula in the deduction is A_n which is an hypothesis, and the $(p + 2)$th follows by applying m.p. to the pth and $(p + 1)$th. The converse is less evident (see Proposition 3).

$\triangle$

Proposition 3: if $A_1, A_2, \ldots, A_{n-1}, A_n \vdash B$, then $A_1, A_2, \ldots, A_{n-1} \vdash (A_n \to B)$ (deduction theorem)

Proof

The statement $A_1, A_2, \ldots, A_{n-1}, A_n \vdash B$ implies that there is a deduction for B from the hypotheses $A_1, A_2, \ldots, A_{n-1}, A_n$. Let k be the length of this chain.

If $k = 1$ there are three possibilities:

1. B is an axiom; then:

   f_1: B (axiom)
   f_2: $(B \to (A_n \to B))$ (SA_1)
   f_3: $(A_n \to B)$ (m.p. f_1, f_2)

   is a deduction of $(A_n \to B)$, and therefore $A_1, A_2, \ldots, A_{n-1} \vdash (A_n \to B)$.
2. B is one of the hypotheses $A_1, A_2, \ldots, A_{n-1}$; then:

   f_1: B (hypothesis)
   f_2: $(B \to (A_n \to B))$ (SA_1)
   f_3: $(A_n \to B)$ (m.p. f_1, f_2)

   is again a deduction of $(A_n \to B)$ from $A_1, A_2, \ldots, A_{n-1}$.
3. B is the hypothesis A_n; then $\vdash (A_n \to B)$ follows from Proposition 1.
   Therefore $A_1, A_2, \ldots, A_{n-1} \vdash (A_n \to B)$ is true for all chains of length $k = 1$.

Now suppose the result holds for all chain lengths $k < k_0$; we show that it then holds for $k = k_0$. There are now four possibilities:

(a) B is an axiom;
(b) B is one of the hypotheses $A_1, A_2, \ldots, A_{n-1}$;
(c) B is the hypothesis A_n;
(d) B follows by applying m.p. to $(C \to B)$ and C.

Cases (a), (b) and (c) can be treated as cases 1, 2 and 3 for $k = 1$; for case (d) if $(C \to B)$ and C are formulae in the deduction of B from $A_1, A_2, \ldots, A_n$, then:

$A_1, A_2, \ldots, A_n \vdash (C \to B)$ with fewer than k_0 steps
$A_1, A_2, \ldots, A_n \vdash C$ also with fewer than k_0 steps

It then follows from the assumption that:

$A_1, A_2, \ldots, A_{n-1} \vdash (A_n \to (C \to B))$ and
$A_1, A_2, \ldots, A_{n-1} \vdash A_n \to C$

and using SA_2 we have:

$$A_1, A_2, \ldots, A_{n-1} \vdash ((A_n \to (C \to B)) \to ((A_n \to C) \to (A_n \to B)))$$

Finally, applying m.p. twice:

$$A_1, A_2, \ldots, A_{n-1} \vdash (A_n \to B)$$ △

Examples

We have now shown that

$A_1, A_2, \ldots, A_{n-1}, A_n \vdash B$ iff
$A_1, A_2, \ldots, A_{n-1} \vdash (A_n \to B)$

To illustrate the use of this proposition, we see that in order to prove:

$$\vdash (A \to (B \to C)) \to (B \to (A \to C))$$

we need only to prove that:

$$(A \to (B \to C)) \vdash (B \to (A \to C))$$

for which it suffices to prove that:

$$(A \to (B \to C)), B \vdash (A \to C)$$

and therefore that:

$$(A \to (B \to C)), B, A \vdash C$$

which is trivial, for:

f_1: $A \to (B \to C)$ (hypothesis)
f_2: B (hypothesis)
f_3: A (hypothesis)

f_4: $B \to C$ (m.p. f_1, f_3)
f_5: C (m.p. f_2, f_4)

Use of the proposition can avoid the need to write out a complete deduction in detail, which can involve many lines.

Proposition 4: the following are theorems of Po

1. $((A \to B) \to ((B \to C) \to (A \to C)))$,
2. $(B \to ((B \to C) \to C))$,
3. $(\neg B \to (B \to C))$,
4. $(\neg \neg B \to B)$,
5. $(B \to \neg \neg B)$,
6. $((A \to B) \to (\neg B \to \neg A))$,
7. $(B \to (\neg C \to \neg (B \to C)))$,
8. $((B \to A) \to ((\neg B \to A) \to A))$.

Proof

The proofs are left as an exercise for the reader.

△

3. Interpretation of the formulae of Po′

We now define a new set of formulae, whose alphabet is that of Po extended by adding the further connectives v, ∧, ↔ ; these are read 'or', 'and,' 'is equivalent to', respectively:

$\Sigma_{Po'} = \{p_0, p_1, \ldots, p_n, \ldots\} \cup \{\neg, \to, \vee, \wedge, \leftrightarrow, (,)\}$
$F_{Po'}$ = the smallest set of formulae such that:

$\forall$ i: $p_i \in F_{Po'}$ and
$\forall A \in F_{Po'}, \forall B \in F_{Po'}$: $\neg A$, $(A \to B)$, $(A \vee B)$, $(A \wedge B)$, $(A \to B)$ all $\in F_{Po'}$

Clearly,

$F_{Po} \subset F_{Po'}$

We do not consider extra rules and axioms for Po′ but see Exercises.

In the following A, B, C, A_i, B_i and C_i all stand for well-formed formulae of $F_{Po'}$.

We call an *interpretation* (evaluation, realization or assignment) of $F_{Po'}$ any mapping:

i: $\{p_0, p_1, \ldots, p_n, \ldots\} \to \{\mathbf{T}, \mathbf{F}\}$

where **T** denotes true and **F** denotes false.

The mapping is extended to cover the whole of $F_{Po'}$ by means of the following:

$$\begin{aligned}
i(\neg A) &= \neg\,[i(A)]\\
i((A \rightarrow B)) &= \rightarrow [i(A), i(B)]\\
i((A \vee B)) &= \vee\,[i(A), i(B)]\\
i((A \wedge B)) &= \wedge\,[i(A), i(B)]\\
i((A \leftrightarrow B)) &= \leftrightarrow [i(A), i(B)]
\end{aligned}$$

where the expressions on the right are interpreted as follows:

X Y	$\rightarrow$ [X, Y]	v [X, Y]	$\wedge$ [X, Y]	$\leftrightarrow$ [X, Y]
T T	**T**	**T**	**T**	**T**
T F	**F**	**T**	**F**	**F**
F T	**T**	**T**	**F**	**F**
F F	**T**	**F**	**F**	**T**

also:

$\neg\,[\mathbf{T}] = \mathbf{F},\ \neg\,[\mathbf{F}] = \mathbf{T}$

For example, if i is an interpretation of $F_{Po'}$ such that:

$i[p_0] = \mathbf{T},\ i[p_1] = \mathbf{F},\ i[p_2] = \mathbf{T}$

then

$$\begin{aligned}
i[(((p_0 \rightarrow p_1) \vee \neg p_2))] &= \vee\,[i(p_0 \rightarrow p_1), i(\neg p_2)]\\
&= \vee\,[\rightarrow [i(p_0), i(p_1)],\ \neg\,[i(p_2)]]\\
&= \vee\,[\rightarrow [\mathbf{T}, \mathbf{F}],\ \neg\,[\mathbf{T}]]\\
&= \vee\,[\mathbf{F}, \mathbf{F}] = \mathbf{F}
\end{aligned}$$

The following definitions involve the concept of interpretation; all the formulae named belong to the set $F_{Po'}$.

1. Formula A is a *tautology* if $i[A] = \mathbf{T}$ for any interpretation i. This is written $\models A$.
2. Formula B is a *consequence* of A if $i[B] = \mathbf{T}$ whenever $i[A] = \mathbf{T}$. This is written $A \models B$.
3. Formula B is a consequence of a set of formulae $\mathscr{A}$ if $i[B] = \mathbf{T}$ whenever $i[A] = \mathbf{T}$ for all $A \in \mathscr{A}$. This is written $\mathscr{A} \models B$.
4. Two formulae A, B are *equivalent* if $A \models B$ and $B \models A$. This is written $A \equiv B$.
5. A formula A is *satisfiable* or *consistent* if there is an interpretation i such that $i[A] = \mathbf{T}$.
6. A set of formulae $\mathscr{A}$ is satisfiable or consistent if there is an interpretation i such that $i[A] = \mathbf{T}$ for all $A \in \mathscr{A}$. Such an interpretation is then called a *model* of $\mathscr{A}$.

7. Two sets of formulae are *equivalent* if they have identical models.
8. A formula A is *unsatisfiable* or *inconsistent* if $i[A] = \mathbf{F}$ for every interpretation i. It is easily shown that A is inconsistent iff $\neg$ A is a tautology.
9. A set of formulae $\mathscr{A}$ is unsatisfiable or inconsistent if for every interpretation i there is an $A \in \mathscr{A}$ such that $i[A] = \mathbf{F}$. Otherwise expressed, there is no model for $\mathscr{A}$.

These concepts of consistency and inconsistency are related to those defined in Chapter 3 (Section 1) by the following result:

Let $\mathscr{A}$ be a set of formula in Po and $S_{\mathscr{A}}$ the formal system obtained by adding $\mathscr{A}$ to the set of axioms of Po. Then $\mathscr{A}$ is consistent in the above sense if and only if $S_{\mathscr{A}}$ is consistent in the sense of Chapter 3.

The proof follows easily from those given in Section 4.

Proposition 5: if A and B are formulae of F_{Po},
(a) $\models (A \rightarrow B)$ iff $A \models B$
(b) $\models (A \leftrightarrow B)$ iff $A \equiv B$
(c) If $\models A$ and $\models (A \rightarrow B)$, then $\models B$
(d) $\models (A \wedge B)$ iff $\models A$ and $\models B$
(e) If $\models A$ or $\models B$, then $\models A \vee B$

Proof

The proofs are left as an exercise for the reader.

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Proposition 6: for any formula $A \in F_{Po}$, if $\vdash A$, then $\models A$, i.e. those formulae of F_{Po} that are theorems (true syntactically) are also tautologies (true semantically)

Proof

The proof is by induction on the length of the deduction that gives A.

Suppose $\vdash A$. If the length n of the deduction is 1, A is an axiom; and since all axioms are tautologies this means that $\models A$.

Suppose now that the result is true for all theorems for which the deduction is of length $< n$. The last link in any chain of length n that gives A is either:

an axiom; when $\models A$ follows from the result for $n = 1$

or:

an application of *modus ponens* to a pair of formulae $(B \rightarrow A)$ and B that occur before A in the deduction.

In the second case the two formulae are theorems that can be established by deductions of length less than n, and are therefore tautologies. It follows from Proposition 5(c) that A is a tautology.

△

Proposition 7: let A be a tautology of $F_{Po'}$ that involves the propositional variables $p_0, p_1, \ldots, p_n$, and let $A_0, A_1, \ldots, A_n$ be any formulae of $F_{Po'}$. Then the formula A′ obtained by replacing each p_i by A_i is a tautology

Proof

Let i be any interpretation of $F_{Po'}$. In order to calculate i[A] we first calculate $i[A_0], \ldots, i[A_n]$ and then use the values found in the expression for A. But as A is a tautology the value of i[A] is **T**, whatever the interpretation of its components; so the value i[A′] calculated in this way is also **T**, and A′ is therefore a tautology also.

△

Proposition 8: for any $A \in F_{Po'}$ there are formulae B_1, B_2, B_3, B_4 and B_5 equivalent to A such that

(a) B_1 uses only the connectives $\vee$, $\neg$
(b) B_2 uses only $\wedge$, $\neg$
(c) B_3 uses only $\rightarrow$, $\neg$
(d) B_4 uses only $\vee$, $\wedge$, $\neg$ and is of the form $D_1 \wedge D_2 \wedge \ldots \wedge D_m$ where each D_i is of the form $p_1 \vee p_2 \vee \ldots \vee p_m \vee \neg q_1 \vee \neg q_2 \vee \ldots \vee \neg q_n$
(e) B_5 uses only $\vee$, $\wedge$, $\neg$ and is of the form $D_1 \vee D_2 \vee \ldots \vee D_1$ where each D is of the form $p_1 \wedge p_2 \wedge \ldots \wedge p_m \wedge \neg q_1 \wedge \neg q_2 \wedge \ldots \wedge \neg q_n$

B_4 and B_5 are called conjunctive normal and disjunctive normal forms, respectively.

Proof

(a) Repeated use of the equivalences:

$$(A \rightarrow B) \equiv (\neg A \vee B)$$
$$(A \wedge B) \equiv \neg(\neg A \vee \neg B)$$
$$(A \leftrightarrow B) \equiv ((A \rightarrow B) \wedge (B \rightarrow A)) \equiv ((A \wedge B) \vee (\neg A \wedge \neg B))$$

enables any formula to be transformed into an equivalent formula in which the only connectives are $\neg$ and $\vee$.

(b) and (c) are proved similarly.
(d) The required transformation can be made by using the above three equivalences and in addition:

$$\neg (A \vee B) \equiv (\neg A \wedge \neg B)$$
$$\neg (A \wedge B) \equiv (\neg A \vee \neg B)$$
$$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$$
$$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$$
$$\neg \neg A \equiv A$$

(e) is proved similarly.

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Any formula of the form $p_1 \vee p_2 \vee \ldots \vee p_m \vee \neg q_1 \vee \neg q_2 \vee \ldots \vee \neg q_n$ is called a *clause*; (set of clauses $\simeq$ conjunction) so Proposition 8(d) can be stated

'any set of formulae is equivalent to a set of clauses'.

(1) Replace each formula A of the set by its normal conjunctive form: $D_1 \wedge D_2 \wedge \ldots \wedge D_m$.
(2) Remove all '$\wedge$'.

For example, take the formula $((p_0 \vee p_1) \rightarrow p_2) \wedge (p_0 \leftrightarrow p_3)$. This gives rise to the following equivalent formulae:

$$(\neg (p_0 \vee p_1) \vee p_2) \wedge ((p_0 \rightarrow p_3) \wedge (p_3 \rightarrow p_0))$$
$$((\neg p_0 \wedge \neg p_1) \vee p_2) \wedge ((\neg p_0 \vee p_3) \wedge (\neg p_3 \vee p_0))$$
$$(\neg p_0 \vee p_2) \wedge (\neg p_1 \vee p_2) \wedge (\neg p_0 \vee p_3) \wedge (\neg p_3 \vee p_0)$$

Thus the original formula is equivalent to the set of clauses:

$$\neg p_0 \vee p_2$$
$$\neg p_1 \vee p_2$$
$$\neg p_0 \vee p_3$$
$$\neg p_3 \vee p_0$$

This transformation into clause form is very important; it is used in Chapter 6.

4. Some consequences

Proposition 9: the completeness theorem for propositional calculus: for all $A \in F_{Po}$, if $\models A$, then $\vdash A$ i.e. every tautology of F_{Po} is a theorem of Po

This and Proposition 6 together state that $\models A$ iff $\vdash A$, which means that the formal system defined at the beginning of this chapter provides a correct model for propositional calculus. Other models can of course be constructed, having different axioms and different inference rules but in every case the correctness has to be proved.

It is important to understand that the truth of this theorem (i.e. the appropriateness of the model) is not obvious *a priori*; in fact, if some of the axioms are removed from Po the resulting formal system no longer has the property that every tautology is a theorem, although the converse remains true, the proof of Proposition 6 remaining valid.

Lemma

Let $A \in F_{Po}$ and let $p_1, p_2, \ldots, p_k$ be the propositional variables in A. Let i be an interpretation of Po

Define:

$$B_m = p_m \text{ if } i[p_m] = \mathbf{T}$$
$$= \neg\, p_m \text{ if } i[p_m] = \mathbf{F}$$

and let A′ be the formula defined by:

$$A' = A \text{ if } i[A] = \mathbf{T}$$
$$= \neg\, A \text{ if } i[A] = \mathbf{F}$$

Then:

$$B_1, B_2, \ldots, B_k \vdash A'$$

For example if: $A = (p_1 \rightarrow p_2) \rightarrow p_3$; $i[p_1] = \mathbf{T}$, $i[p_2] = \mathbf{F}$, $i[p_3] = \mathbf{F}$

then:

$$B_1 = p_1, B_2 = \neg\, p_2, B_3 = \neg\, p_3, i[(p_1 \rightarrow p_2) \rightarrow p_3] = \mathbf{T}$$

and:

$$A' = (p_1 \rightarrow p_2) \rightarrow p_3$$

The lemma then gives:

$$p_1, \neg\, p_2, \neg\, p_3 \vdash ((p_1 \rightarrow p_2) \rightarrow p_3)$$

Proof

We prove the lemma by induction on the number of connectives in A; this number is called the *complexity* of A.

Suppose $n = 0$, so that $A' = B_1$. There are two possibilities:

1. $i[p_1] = \mathbf{T}$, so $B_1 = p_1$ and $A' = A$
   therefore:
   $B_1 \vdash A'$
2. $i[p_1] = \mathbf{F}$, so $B_1 = \neg\, p_1$ and $A' = \neg\, p_1$
   therefore again:
   $B_1 \vdash A'$

Suppose now the lemma to be true for all formulae of complexity $n < n_0$. There are two possibilities:

1. A is of the form $\neg$ C; again there are two possibilities:
   (a) $i[A] = \mathbf{T}$, so $A' = A$, $i[C] = \mathbf{F}$, $C' = \neg C = A = A'$
   But C' is of complexity $n_0 - 1$, so by hypothesis:
   $B_1, B_2, \ldots, B_k \vdash C'$
   and since in this case $A' = C'$ we have:
   $B_1, B_2, \ldots B_k \vdash A'$
   (b) $i[A] = \mathbf{F}$, so $A' = \neg A$, $i[C] = \mathbf{T}$, $C' = C$
   and therefore:
   $A' = \neg \neg C$
   By hypothesis:
   $B_1, B_2, \ldots, B_k \vdash C$
   so by Proposition 4:
   $B_1, B_2, \ldots, B_k \vdash (C \rightarrow \neg \neg C)$
   and by *modus ponens*:
   $B_1, B_2, \ldots, B_k \vdash \neg \neg C$
   which is:
   $B_1, B_2, \ldots, B_k \vdash A'$
2. A is of the form $(B \rightarrow C)$, and again there are two possibilities:
   (a) $i[A] = \mathbf{F}$, so $i[B] = \mathbf{T}$, $i[C] = \mathbf{F}$, $C' = \neg C$, $B' = B$,
   $A' = \neg A = \neg (B \rightarrow C)$
   By hypothesis:
   $B_1, B_2, \ldots, B_k \vdash \neg C$, $B_1, B_2, \ldots, B_k \vdash B$
   From these and Proposition 4(7) we have:
   $B_1, B_2, \ldots, B_k \vdash (B \rightarrow (\neg C \rightarrow \neg (B \rightarrow C)))$
   whence by two applications of *modus ponens* we get:
   $B_1, B_2, \ldots, B_k \vdash A'$
   (b) $i[A] = \mathbf{T}$
   The proof is left as an exercise for the reader. $\triangle$

We have now proved that if the lemma holds for all formulae of complexity $n < n_0$ it holds for complexity n_0; and as we have proved it for $n = 0$ (i.e. $n < 1$) it is true for all $n \geqslant 0$.

We can now prove Proposition 9.

Proof of Proposition 9

Let A be a tautology with propositional variables $p_1, p_2, \ldots, p_k$. There are 2^k different interpretations of this set of k variables, for each of which the application of the lemma gives a deduction (or 'result') of the type:

$$B_1, B_2, \ldots, B_k \vdash A$$

In particular, corresponding to the pair of interpretations:

$$i[p_1] = i[p_2] = \ldots = i[p_{k-1}] = i[p_k] = \mathbf{T}$$
$$i[p_1] = i[p_2] = \ldots = i[p_{k-1}] = \mathbf{T},\ i[p_k] = \mathbf{F}$$

we have the respective deductions:

$p_1, p_2, \ldots, p_{k-1}, p_k \vdash A$
$p_1, p_2, \ldots, p_{k-1}, \neg p_k \vdash A$

Thus we have the following from the deduction theorem:

$p_1, p_2, \ldots, p_{k-1} \vdash (p_k \to A)$
$p_1, p_2, \ldots, p_{k-1} \vdash (\neg p_k \to A)$

whence from Proposition 4(8) we have:

$p_1, p_2, \ldots, p_{k-1} \vdash ((p_k \to A) \to ((\neg p_k \to A) \to A))$

and by *modus ponens*:

$p_1, p_2, \ldots, p_{k-1} \vdash A$

Proceeding in the same way we can show that

$p_1, p_2, \ldots, p_{k-2} \vdash A$

and so on, until finally:

$p_1 \vdash A$
$\vdash A$

which is the result to be proved. △

We now prove two corollaries to Proposition 9.

Corollary 1: the decision problem for propositional calculus Po is solvable: this can be stated T_{Po} is recursive or a program can be written which, for any input formula $A \in F_{Po}$, will state within a finite time whether or not A is a theorem

Proof

To prove this, we first note that since $\vdash A$ (A is a theorem) is equivalent to $\models A$ (A is a tautology), it is sufficient to show that a program can be written to establish whether or not a given $A \in F_{Po}$ is a tautology. Such a program can be written as follows:

1. Find the propositional variables $p_1, p_2, \ldots, p_k$ that occur in the expression for A.
2. For each of the 2^k possible interpretations of these variables calculate i[A].
3. If i[A] = **T** in every case, A is a tautology; otherwise not. △

Corollary 2: For all formulae $B_1, B_2, \ldots, B_n, A \in F_{Po}$
$B_1, B_2, \ldots, B_n \models A$ iff $B_1, B_2, \ldots B_n \vdash A$

Proof

This follows from the fact that all the following relations are equivalent:

$B_1, B_2, \ldots, B_{n-1}, B_n \models A$
$B_1, B_2, \ldots, B_{n-1} \models (B_n \to A)$ (from the definitions)

....

$\models (B_1 \to (B_2 \to \ldots \to (B_n \to A) \ldots))$ (from the definitions)
$\vdash (B_1 \to (B_2 \to \ldots \to (B_n \to A) \ldots))$ (Proposition 9)
$B_1 \vdash (B_2 \to (B_3 \to \ldots \to (B_n \to A) \ldots))$ (Proposition 2)

....

$B_1, B_2, \ldots, B_n \vdash A$ (Proposition 2) Δ

Proposition 10: the compactness theorem

Let $\mathscr{A}$ be a set of formulae of F_{Po}, $\mathscr{A}'$ a finite subset of $\mathscr{A}$ and A a formula of $\mathscr{A}'$. Then if for every finite subset $\mathscr{A}' \subset \mathscr{A}$ there is an interpretation of $\mathscr{A}'$ such that $\forall$ A $\in \mathscr{A}'$, i[A] = **T**, there is also an interpretation of $\mathscr{A}$ such that $\forall$ A $\in \mathscr{A}$, i[A] = **T**.

Proof

For each formula A $\in \mathscr{A}'$ let I[A] be the set of interpretations of $F_{po'}$ for which i[A] = **T**. I[A] is an open subset of $\{\mathbf{T}, \mathbf{F}\}^{\{p_0, p_1, \ldots, p_n \ldots\}}$ (having the topological product), since A involves only a finite number of the p_i. It is also a closed subset because the interpretations that do not satisfy A are those that satisfy $\neg$ A. The hypothesis of the theorem means that every finite intersection of I[A] for A $\in$, $\mathscr{A}$ is non-empty; then since $\{\mathbf{T}, \mathbf{F}\}^{\{p_0, p_1, \ldots, p_n \ldots\}}$ is compact it follows that the intersection of all the I[A] is non-empty, that is, that there is an interpretation i such that:

$\forall$ A $\in \mathscr{A}$: i[A] = **T**

Note that the theorem can be proved without calling on the topological concepts used here. Δ

Proposition 11: the finiteness theorem

Let $\mathscr{A}$ be a set of formulae of $F_{Po'}$ and B $\in F_{Po'}$; then if $\mathscr{A} \models$ B, there is a finite subset $\mathscr{A}'$ of $\mathscr{A}$ such that $\mathscr{A}' \models$ B.

Proof

The statement $\mathscr{A} \models$ B means that there is no interpretation i (of $F_{po'}$) such that:

$\forall$ A $\in \mathscr{A}$: i[A] = **T**

and:

i[$\neg$ B] = **T**

Applying Proposition 10 to $\mathscr{A} \cup \{\neg B\}$ we see that there is a finite set $\mathscr{B}$ such that:

$\mathscr{B} \subset \mathscr{A} \cup \{\neg B\}$

and there is no interpretation i for which:

$\forall A \in \mathscr{B}$: $i[A] = \mathbf{T}$

Now the set $\mathscr{B} \cup \{\neg B\}$ can be written:

$\mathscr{A}' \cup \{\neg B\}$

with $\mathscr{A}' \subset \mathscr{A}$, $\mathscr{A}'$ finite; and it follows that there is no interpretation i such that:

$\forall A \in \mathscr{A}'$: $i[A] = \mathbf{T}$

and:

$i[\neg B] = \mathbf{T}$

Therefore:

$\mathscr{A}' \models B$

△

Proposition 12: let $\mathscr{A}$ be a set of formulae of F_{P_0} and $A \in F_{P_0}$, then $\mathscr{A} \models A$ iff $\mathscr{A} \vdash A$

Here, $\mathscr{A} \vdash A$ means that A can be deduced from the hypotheses involved in the set $\mathscr{A}$ (possibly infinite).

Proof

The proof follows from the equivalence of the following relations:

1. $\mathscr{A} \models A$
2. By Proposition 11, there exist formulae $B_1, B_2, \ldots, B_n$ in $\mathscr{A}$ such that: $B_1, B_2, \ldots, B_n \models A$
3. By Corollary 2 to Proposition 9, there are formulae $B_1, B_2, \ldots, B_n$ in $\mathscr{A}$ such that:

   $B_1, B_2, \ldots, B_n \vdash A$
4. $\mathscr{A} \vdash A$ (by definition of the notation)

△

Exercises

General Properties of the Formal System Po

State, with reasons, whether or not the system Po has the following properties:

1. coherent;
2. consistent;
3. categorical;
4. saturated;
5. independent (i.e. are the axioms independent?);
6. finitely axiomatizable.

Truth tables

Suppose the formula A involves the propositional variables $p_0, p_1, \ldots, p_n$. The *truth table* for A is a table giving the truth value (**T** or **F**) of A for each of the 2^n possible combinations of the truth values of the p_i, $i = 1, 2, \ldots, n$. The connectives $\rightarrow$, $\neg$, etc. are interpreted according to the rules given in Chapter 3 (Section 3).

For example, for $A = (p_0 \rightarrow p_1) \rightarrow \neg p_1$ the table for A and also for the subformulae $(p_0 \rightarrow p_1)$ and $\neg p_1$ is as follows:

p_0	p_1	$(p_0 \rightarrow p_1)$	$\neg p_1$	A
T	**T**	**T**	**F**	**F**
T	**F**	**F**	**T**	**T**
F	**T**	**T**	**F**	**F**
F	**F**	**T**	**T**	**T**

1. Interpret the concepts 'tautology,' 'consequence,' 'equivalent formulae,' 'satisfiable formula,' 'unsatisfiable formula' by means of truth tables.
2. Construct truth tables for the sets of axioms which correspond to the schemata SA_1, SA_2, and SA_3.
3. Prove Proposition 5.

Consistency

1. Let A be any formula of F_{Po}. For the formal system obtained by adding A and $\neg A$ as axioms to Po show, by using Formula 3 in Proposition 4, that the set T of theorems is F_{Po}.
2. Show that it follows from 1 that any inconsistent system obtained by adding axioms to Po is no longer coherent (see Chapter 3).

Examples on Proposition 8

1. Find formulae B_1, B_2, B_3, B_4 and B_5 equivalent to:

   $A = ((p_0 \to p_1) \to (p_0 \to p_2))$

   having the forms specified in Proposition 8.
2. Similarly for:

   $A = ((\neg (p_0 \leftrightarrow p_1) \vee (p_1 \wedge p_2)) \to p_0)$

A Propositional Calculus with a Single Connective

The connective $*$ is defined as follows:

X	Y	X * Y
T	T	F
T	F	T
F	T	T
F	F	T

Show that every formula of F_{P_0} is equivalent to a formula involving only $*$.

Substitution

1. Give an example of a formula $F(p_0, p_1, \ldots, p_n)$ which is not a tautology but for which there are formulae $A_0, A_1, \ldots, A_n$ such that $F(A_0, A_1, \ldots, A_n)$ is a tautology.
2. Give the necessary and sufficient conditions for 1 to be possible.

Negation in the Propositional Calculus

Let A be a formula of F_{P_0}, using only the connectives $\neg$, $\vee$ and $\wedge$, and A′ the formula obtained from A by:

1. replacing each p_i by $\neg p_i$;
2. replacing each $\vee$ by $\wedge$ and each $\wedge$ by $\vee$;
3. deleting all $\neg\neg$.

For example:

$A = ((\neg p_0 \vee p_1) \wedge \neg p_2)$
$A' = ((p_0 \wedge \neg p_1) \vee p_2)$

Show that A′ is equivalent to $\neg$ A.

Completeness: A System with too Many Theorems

For the system obtained by adding to Po a fourth axiom schema:

SA_4: $((B \to (A \to B)) \to A)$

1. Are the axioms given by this new set tautologies?
2. Characterize the theorems of this new system.

Completeness: A System with too Few Theorems

Let S be the system obtained from Po by replacing the three axiom schemata by the single one:

SA′: $((A \to B) \to (A \to A))$

1. Is every theorem a tautology?
2. Show that $T_S = \{((A \to B) \to (A \to A)) \mid A, B \in F_{Po}\} \cup \{((A \to B) \to (A \to B)) \mid A, B \in F_{Po}\}$
3. Is every tautology of F_{Po} included in T_S?
4. Is the deduction theorem true for this system?

Axioms and rules for Po′

Using the equivalences suggest some axioms/rules to make Po′ a formal system which is sound and complete with respect to the tautologies.