Modal Logics for Agents

Basic Concepts

John-Jules Meyer
Universiteit Utrecht
Intelligent Systems Group

Overview

- Basic Modal Logic
- Relevant Forms of Modal Logic
  - Dynamic logic
  - Temporal logic
  - Epistemic logic
  - Deontic logic
  - Context logic
  - …

Overview (ctd)

- Applications
  - Agent logics such as BDI-CTL
  - Common Knowledge & Joint/Collective Intentions
  - …

Basic Modal Logic

- Expresses 'intensional' (context/situation-sensitive) notions such as
  - Knowledge
  - Belief
  - Obligation
  - Action
  - Time

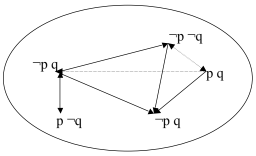
Modal language

- Propositional logic
- Extended with modal operators
  - $\Box\phi$: it is *necessary* that ϕ
  - $\Diamond\phi$: it is *possible* that ϕ

Semantics: Kripke models

- Kripke model $M=\langle S,\pi,R\rangle$
  - S is a set of *worlds* (*states*)
  - π is a truth assignment function
    - $\pi : S \times AT \rightarrow \{tt, ff\}$
  - $R \subseteq S \times S$ is an *accessibility relation*

Kripke models



Interpretation

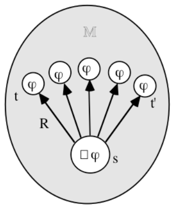
- Like classical propositional logic
- But now relative to a model and a world (state):

  M,s ⊨ ϕ
- E.g.
  - M,s ⊨ p ⇔ π(s, p) = tt
  - M,s ⊨ $\phi \wedge \psi$ ⇔ M,s ⊨ ϕ and M,s ⊨ ψ

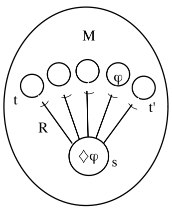
Interpretation of □ and ◇

- M,s ⊨ □ϕ ⇔ M,t ⊨ ϕ for *every* t such that R(s,t)
- M,s ⊨ ◇ϕ ⇔ M,t ⊨ ϕ for *some* t such that R(s,t)

Interpretation of □



Interpretation of ◇



Validity in modal logic

- ϕ is *valid* in a model M=⟨S,π,R⟩, (denoted M ⊨ ϕ) ⇔

  M,s ⊨ ϕ for all s ∈ S.
- ϕ is *valid* (denoted ⊨ ϕ) ⇔

  M ⊨ ϕ for all Kripke models M.
- Sometimes we need validity wrt subclasses of models:
- ϕ is *valid* wrt class $\mathscr{C}$ (denoted $\mathscr{C}$ ⊨ ϕ) ⇔

  M ⊨ ϕ for all Kripke models M ∈ $\mathscr{C}$.

Basic Modal Logic: system $K_{(m)}$

- As usual in logic we can try to axiomatize validities.

- *Axioms:*
  - All (or enough) propositional tautologies
  - $(\Box\varphi \wedge \Box(\varphi \to \psi)) \to \Box\psi$ (K axiom)

- *Rules:*
  - $\varphi, \varphi \to \psi / \psi$ (modus ponens)
  - $\varphi / \Box\varphi$ (necessitation rule)

Caution!!

- NB. Distinguish the *necessitation rule*

$$\varphi / \Box\varphi$$

from the *invalid* assertion

$$\varphi \to \Box\varphi$$

Derivability in K

- A *derivation* of a formula φ is a finite sequence of formulas $\varphi_1, \varphi_2, \ldots, \varphi_n = \varphi$, where each φ_i, for $1 \leq i \leq n$, is either an instance of the axioms (or rather axiom schemes), or the conclusion of one of the rules of which the premises have been derived already, i.e. appear as φ_j in the sequence with $j < i$.
- When we can derive an epistemic formula φ by using the axioms and rules of $\mathbf{K}_{(m)}$, we write $\mathbf{K}_{(m)} \vdash \varphi$

System $K_{(m)}$

- System $K_{(m)}$ is *sound and complete*, i.e.

$$\models \varphi \;\Leftrightarrow\; \mathbf{K}_{(m)} \vdash \varphi$$

- This means that exactly all valid modal assertions can be obtained by derivations in system $K_{(m)}$

Some (non)theorems in K

- $\models \Box(\phi \wedge \psi) \leftrightarrow (\Box\phi \wedge \Box\psi)$
- $\models \Box(\phi \vee \psi) \leftarrow (\Box\phi \vee \Box\psi)$
- $\not\models \Box(\phi \vee \psi) \to (\Box\phi \vee \Box\psi)$
- $\models \Diamond(\phi \vee \psi) \leftrightarrow (\Diamond\phi \vee \Diamond\psi)$
- $\models \Diamond(\phi \wedge \psi) \to (\Diamond\phi \wedge \Diamond\psi)$
- $\not\models \Diamond(\phi \wedge \psi) \leftarrow (\Diamond\phi \wedge \Diamond\psi)$
- $\not\models \Box\neg\phi \to \neg\Box\phi$
- $\not\models \Box\neg\phi \leftarrow \neg\Box\phi$

Application: Dynamic Logic

- An example of an (indexed) version of system K is dynamic logic, where the $\Box$ modality is associated with the execution results of a program / action
- $\Box_\alpha$, normally written $[\alpha]$

Dynamic Logic

- Syntax
  - Operator $[\alpha]$ with reading:
  - $[\alpha]\varphi$: after execution of α it holds (nec.) that φ
  - $<\alpha>\varphi = \neg[\alpha]\neg\varphi$
- Semantics
  - Accessibility relation R_α for every action α
    - $R_{\alpha;\beta} = R_\alpha \circ R_\beta$
    - $R_{\alpha+\beta} = R_\alpha \cup R_\beta$
    - $R_{\alpha*} = R_\alpha*$

Dynamic Logic

- Interpretation formulas
  - $M,s \models [\alpha]\varphi \Leftrightarrow$ for all s' with $R_\alpha(s,s')$: $M,s' \models \varphi$
  - $M,s \models <\alpha>\varphi \Leftrightarrow$ for some s' with $R_\alpha(s,s')$: $M,s' \models \varphi$

Dynamic Logic

- Basic property (K)
  - $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$

  Structure of actions
  - $[\alpha_1 ; \alpha_2]\varphi \leftrightarrow [\alpha_1]([\alpha_2]\varphi)$
  - $[\alpha_1 + \alpha_2]\varphi \leftrightarrow [\alpha_1]\varphi \wedge [\alpha_2]\varphi$
  - $[\alpha^*]\varphi \to \varphi$
  - $[\alpha^*]\varphi \to [\alpha][\alpha^*]\varphi$
  - $[\alpha^*](\varphi \to [\alpha]\varphi) \to (\varphi \to [\alpha^*]\varphi)$

Special properties of accessibility relations

- R is *reflexive* if $\forall s \in S\ (s, s) \in R$.
- R is *transitive* if $\forall s, t, u \in S$: $(s, t) \in R$ & $(t, u) \in R \Rightarrow (s, u) \in R$.
- R is *symmetrical* if $\forall s, t \in S$: $(s, t) \in R \Rightarrow (t, s) \in R$.
- R is *euclidean* if $\forall s, t, u \in S$: $(s, t) \in R$ & $(s, u) \in R \Rightarrow (t, u) \in R$.
- R is *serial* if $\forall s \in S\ \exists t \in S\ (s, t) \in R$.
- R is an *equivalence relation* if R is reflexive, transitive and symmetrical

Special classes of models

- $\mathcal{T}_{(m)}$ is the class of all *reflexive* Kripke models with m agents.
- $\mathcal{S4}_{(m)}$ is the class of all *reflexive-transitive* Kripke models with m agents.
- $\mathcal{S5}_{(m)}$ is the class of all Kripke models with m agents with accessibility relations that are *equivalence relations*.
- $\mathcal{KD}_{(m)}$ is the class of all Kripke models with m agents with *serial* accessibility relations
- $\mathcal{KD45}_{(m)}$ is the class of all Kripke models with m agents with *serial, transitive and euclidean* accessibility relations

Systems T, S4, S5, KD, KD45

- $T_{(m)} = K_{(m)}$ + axiom $\{\Box_i\varphi \to \varphi\}$
- $S4_{(m)} = T_{(m)}$ + axiom $\{\Box_i\varphi \to \Box_i\Box_i\varphi\}$
- $S5_{(m)} = S4_{(m)}$ + axiom $\{\neg\Box_i\varphi \to \Box_i\neg\Box_i\varphi\}$
- $KD_{(m)} = K_{(m)}$ + axiom $\{\neg\Box_i\bot\}$
- $KD45_{(m)} = K_{(m)}$ + axioms $\{\neg\Box_i\bot$, $\Box_i\varphi \to \Box_i\Box_i\varphi$, $\neg\Box_i\varphi \to \Box_i\neg\Box_i\varphi\}$

Alternative formulation of the 5-axiom

- $\neg\Box_i\varphi \rightarrow \Box_i\neg\Box_i\varphi$

Rewrite:

- $\Diamond_i\neg\varphi \rightarrow \Box_i\Diamond_i\neg\varphi$

Substitute ψ for $\neg\varphi$:

- $\Diamond_i\psi \rightarrow \Box_i\Diamond_i\psi$

Substitute φ for ψ:

- $\Diamond_i\varphi \rightarrow \Box_i\Diamond_i\varphi$

Soundness & completeness of T, S4, S5, KD, KD45

- $\mathbf{T}_{(m)} \vdash \varphi \Leftrightarrow \mathcal{T}_{(m)} \vDash \varphi$
- $\mathbf{S4}_{(m)} \vdash \varphi \Leftrightarrow \mathcal{S4}_{(m)} \vDash \varphi$
- $\mathbf{S5}_{(m)} \vdash \varphi \Leftrightarrow \mathcal{S5}_{(m)} \vDash \varphi$

- $\mathbf{KD}_{(m)} \vdash \varphi \Leftrightarrow \mathcal{KD}_{(m)} \vDash \varphi$
- $\mathbf{KD45}_{(m)} \vdash \varphi \Leftrightarrow \mathcal{KD45}_{(m)} \vDash \varphi$

Deontic logic

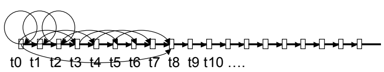
- The system KD is also known as SDL (standard deontic logic)
- Deontic logic is the *logic of obligation, prohibition and permission*, or rather: *the logic of ideal vs actual situations*

Deontic logic

- Prop. Calculus (including MP)
- $(O_i\varphi \wedge O_i(\varphi \rightarrow \psi)) \rightarrow O_i\psi$ (K axiom)
- φ / $O_i\varphi$ (necessitation rule)
- $\neg O_i\bot$ (D-axiom: obligation is consistent)
- $F\varphi \leftrightarrow O\neg\varphi$ (forbidden is obliged to not)
- $P\varphi \leftrightarrow \neg F\varphi$ (permitted is not forbidden)

Temporal Logic

- Basic linear-time logic (LTL)
- Time as accessibility relation
  - Reflexive
  - Transitive



Basic LTL

- Viewed in this way:
  - LTL = $S4_{(1)}$
  - $\Box$ stands for ‘always in the future’
  - By convention the present included in the future

Basic LTL

- Prop. Calculus (including MP)
- $(\Box\varphi \wedge \Box(\varphi \rightarrow \psi)) \rightarrow \Box\psi$ (K axiom)
- φ / $\Box\varphi$ (necessitation rule)
- $\Box\varphi \rightarrow \varphi$ (always implies now)
- $\Box\varphi \rightarrow \Box\Box\varphi$ (always implies always always)

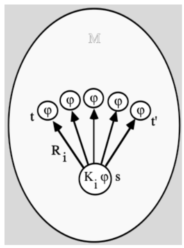
Epistemic & Doxastic Logic

- For knowledge we take the relation R_i :
  - reflexive, (transitive) and euclidean
  - (i.e. an *equivalence* relation)
- For belief we take the relation R_i :
  - serial, transitive and euclidean

Basic Epistemic Logic: $S5_{(m)}$

- Prop. Calculus (including MP)
- $(K_i\varphi \wedge K_i(\varphi \rightarrow \psi)) \rightarrow K_i\psi$ (K axiom)
- φ / $K_i\varphi$ (necessitation rule)
- $K_i\varphi \rightarrow \varphi$ (knowledge is true)
- $K_i\varphi \rightarrow K_iK_i\varphi$ (positive introspection)
- $\neg K_i\varphi \rightarrow K_i\neg K_i\varphi$ (negative introspection)

Interpretation of K_i



Basic Doxastic Logic: $KD45_{(m)}$

- Prop. Calculus (including MP)
- $(B_i\varphi \wedge B_i(\varphi \rightarrow \psi)) \rightarrow B_i\psi$ (K axiom)
- φ / $B_i\varphi$ (necessitation rule)
- $\neg B_i\bot$ (belief is consistent)
- $B_i\varphi \rightarrow B_iB_i\varphi$ (positive introspection)
- $\neg B_i\varphi \rightarrow B_i\neg B_i\varphi$ (negative introspection)

More advanced applications

- Context logic
- Common knowledge & belief
- Collective intentions
- (Extended) LTL
- CTL (tree logic, branching-time)
- BDI-CTL