Chapter 5

First-order Predicate Calculus

1. Introduction

As soon as we embark on handling general properties that are at all complex, or relations between objects, we find that we need to process statements whose truth depends on the variables: for example, 'X gives Y to Z.' Such statements are called predicates and their theory, which is a generalization of propositional calculus, is called predicate calculus.

Mathematicians developed predicate calculus to meet their own particular needs; its immense power of expression is shown by the fact that it has enabled them to construct representations—for example, by means of a first-order axiomatization of set theory—of all the objects and concepts that they use. It is because of its very general ability to represent and manipulate knowledge that computer scientists have taken up this calculus; it is used, in particular, by many expert systems and in several AI languages, for example PROLOG and SNARK.

In this chapter we develop first-order predicate calculus in a rather general form, with the use of predicate and function symbols of any arity (number of arguments). After a fairly strict fixing of the syntax—which we shall find useful in Chapters 6 and 7—we continue as we did for propositional calculus. A full definition of the formal system Pr (where Pr denotes predicate calculus) is given in Section 3, the semantics is developed in Section 4 and the links between the two aspects are given in Section 5. Sections 6 and 7 show how, starting with Pr, other formal systems can be developed and give some results that are relevant to decision problems for formal systems.

2. Terms, atoms, formulae: free and bound variables

A. Basic Elements

These are standard and all disjoint denumerable sets:

V
C
F_j $(j \in \mathbb{N} - \{0\})$
P_j $(j \in \mathbb{N})$

The elements of V are called *variables*; they are always denoted by the lower case letters x, y, z, u, v and w, which sometimes carry suffixes.

The elements of C are called constants; they are always denoted by the lower case letters a, b, c, d and e, which again sometimes carry suffixes.

The elements of F_j are called *j-ary function symbols*; they are always denoted by the lower case letters f, g and h, and sometimes carry suffixes.

The elements of P_j are called *j-ary relational symbols* or *j-ary predicate symbols*; they are always denoted by the lower case letters p, q, r and s, which sometimes carry suffixes.

[Note that C is regarded as F_0 sometimes.]

B. Terms

The set of *first-order predicate calculus terms*, written *Term*, is the smallest set of formulae defined on the alphabet:

$$V \cup C \cup (\cup F_j) \cup \{(\} \cup \{)\} \cup \{,\}$$

such that:

1. $V \subset$ Term
2. $C \subset$ Term
3. for $t_1, t_2, \ldots, t_j \in$ Term, if $f \in F_j$, then $f(t_1, t_2, \ldots, t_j) \in$ Term

Examples of terms are:

x; a; $h(a)$; $g(a, x, f(x, a))$; $f(f(f(x_1, x_2), x_3), x_4)$

To put this more explicitly, suppose the symbol a, a constant, represents a person called Alan, and the function symbol f is to be interpreted as 'father of,' then:

f(a) represents the person who is the father of Alan

and:

f(f(a)) represents Alan's paternal grandfather

Note that the lower case letter t, sometimes with suffixes, is reserved for terms.

C. Atoms

The set of *atoms* or *atomic formulae*, of first-order calculus, written *Atom*, is the smallest set of formulae defined on the alphabet:

$$V \cup C \cup (\cup F_j) \cup (\cup P_j) \cup \{(\} \cup \{)\} \cup \{,\}$$

such that:

for $t_1, t_2, \ldots, t_j \in$ Term, if $p \in P_j$, then $p(t_1, t_2, \ldots, t_j) \in$ Atom

Examples of atoms

p; q(g(x, y)); r(x, a, s(x, f(c)))

Note that if p is a o-ary predicate, we write p rather than p( ) for the atom it forms. Continuing with the example just given, suppose q stands for the unary predicate 'is a father,' then:

q(a) represents the statement 'Alan is a father'

If p′ is the binary predicate 'is the father of,' then p′(a, b) represents the statement 'Alan is the father of the person denoted by b' (Bernard, say).

A point to be noted here is the difference between the meanings of the word 'father' according to whether it is used as a function, a unary predicate or a binary predicate.

D. Formulae

The set of formulae of first-order predicate calculus, written F_{Pr}, is the smallest set of formulae defined on the alphabet:

$$\Sigma_{Pr} = V \cup C \cup (\cup F_j) \cup (\cup P_j) \cup \{(\} \cup \{)\} \cup \{,\} \cup \{\neg, \rightarrow, \forall\}$$

such that:

1. Atom $\subset F_{Pr}$
2. if A is in F_{Pr} and B is in F_{Pr} and x is in V, then:
   $\neg$ A, (A $\rightarrow$ B) and $\forall$ x A are in F_{Pr}

Examples of formulae are:

p
$(p_1 \rightarrow p_2)$
$(\neg\, q(x, f(x)) \rightarrow r(x))$
$\forall x\ \neg\, r(x)$
$\forall x\, \forall y\, (p(x) \rightarrow (r(y) \rightarrow \forall z\, s(x, y, z)))$

Continuing with the same example, we can have the following formulae:

$\forall x\, p'(f(x), x)$

meaning 'for all x, the father of x is the father of x' and:

$(p'(a, b) \rightarrow p'(a, c))$

meaning 'if Alan is the father of Bernard, then Alan is the father of (say) Charles,' a statement that follows, for example, from the fact that Bernard and Charles are brothers.

The upper case letters A, B and C, sometimes with suffixes, are used to denote formulae.

E. Enumeration of Σ_{Pr}

This alphabet, which by hypothesis is denumerable, has a standard enumeration, fixed once and for all. We can assume that this is enumeration such that the set of numbers that enumerates each of the following sets:

V
C
F_j, $j \in \mathbb{N} - \{0\}$
P_j, $j \in \mathbb{N}$

is recursive.

F. Extension of the Vocabulary

In writing out the formulae we shall sometimes use the further symbols $\exists$, $\leftrightarrow$, $\vee$, $\wedge$. These can be regarded simply as shorthand ways of writing certain relations:

$\exists\, x\, A$	represents	$\neg \forall\, x \neg A$
$(A \vee B)$	represents	$(\neg A \rightarrow B)$
$(A \wedge B)$	represents	$\neg (A \rightarrow \neg B)$
$(A \leftrightarrow B)$	represents	$\neg ((A \rightarrow B) \rightarrow \neg (B \rightarrow A))$

A short study of these equivalences will show that the extensions to the vocabulary resulting from the use of these extra symbols have just the meaning that one would normally attach to the statements. For example, to say that there is an x having the property A is exactly the same as saying that it is not true that every x has the property not-A.

G. Reduction of Σ_{Pr} to a Finite Alphabet

This can be done as follows. Instead of $V \cup C \cup (\cup F_j) \cup (\cup P_j)$ we take the finite set of symbols $\{v, c, f, p, I, +\}$ and write:

1. for the variables:
   v, vI, vII, vIII, etc.
2. for the constants:
   c, cI, cII, cIII, etc.
3. for the j-ary functional symbols:
   $\text{f}\underbrace{+\ +\ \ldots\ +}_{\text{(j times)}}$, $\text{f}\underbrace{+\ +\ \ldots\ +}_{\text{(j times)}}\text{I}$, $\text{f}\underbrace{+\ +\ \ldots\ +}_{\text{(j times)}}\text{II}$, etc.
4. for the j-ary predicate symbols:
   $\text{p}\underbrace{+\ +\ \ldots\ +}_{\text{(j times)}}$, $\text{p}\underbrace{+\ +\ \ldots\ +}_{\text{(j times)}}\text{I}$, $\text{p}\underbrace{+\ +\ \ldots\ +}_{\text{(j times)}}\text{II}$, etc.

using superscripts for example q(g(x, y)) becomes
p+ I(f++I(v, vI))
—a string constructed from the finite set {v, c, f, p, I, + }

By this means, or by using any equivalent method, it becomes possible to apply computer processing techniques to predicate calculus; and in particular, to implement the algorithms proposed here not only for restricted parts of this but for the full generality of predicate calculus.

If we proceed in this way the required recursivity of the enumeration of Σ_{Pr} can be ensured by adopting a principle of 'increasing length, alphabetical order if the lengths are equal.'

H. Higher Order Systems

In first-order predicate calculus there is only one type of object and the quantifiers bear only on these. Predicate calculi with objects of higher type, that is, of order greater than 1, can be envisaged, in which quantifiers can bear on predicates as well as on objects, as for example in the formula:

$$\forall\, p\, \exists\, q\, \forall\, x : p(x,x) \leftrightarrow q(x)$$

This, however, is outside the scope of this study.

I. Bound and Free Variables

The variables of a formula are the set of elements of V that appear in the formula, written var(A).

The *bound variables* of a formula A, written vb(A), are a subset of var(A) and defined recursively as follows:

1. If $A \in$ Atom, then $vb(A) = \emptyset$
2. If A is of the form $(B \rightarrow C)$, then $vb(A) = vb(B) \cup vb(C)$
3. If A is of the form $\neg\, B$, then $vb(A) = vb(B)$
4. If A is of the form $\forall\, x\, B$, then $vb(A) = vb(B) \cup \{x\}$

The *free variables*, of a formula A written vf(a), are also a subset of var(A) and defined recursively as follows:

1. If $A \in$ Atom, then $vf(A) = var(A)$
2. If A is of the form $(B \rightarrow C)$, then $vf(A) = vf(B) \cup vf(C)$
3. If A is of the form $\neg\, B$, then $vf(A) = vf(B)$
4. If A is of the form $\forall\, x\, B$, then $vf(A) = vf(B) - \{x\}$

A formula that has no free variables is called a *closed* formula.

Clearly, $vb(A) v\, vf(A) = var(A)$ for any formula A.

Examples

$A = (p(f(x, y)\ v\ \forall\, z\ r(a, z)$

$var(A) = \{x, y, z\}$

$$\begin{aligned} vb(A) &= \{z\} \\ vf(A) &= \{x, y\} \end{aligned}$$

$$\begin{aligned} B &= (\forall\, x\; p(x, y, z) \vee \forall z\, (p(z) \rightarrow r(z))) \\ var(B) &= \{x, y, z\} \\ vb(B) &= \{x, z\} \\ vf(B) &= \{y, z\} \end{aligned}$$

Note that z is both free and bound in B.

$C = \forall\, x\, \exists\, y\, (p(x, y) \rightarrow \forall\, z\; r(x, y, z))$
is a closed formula.

J. Renaming and Substitution

Let A(x) be a formula containing x as a free variable, and let t be a term. We denote by A(t) the formula derived from A(x) by substituting t for x wherever the latter occurs in A(x), after having first changed the names of the bound variables in A so that x, if it appeared also as a bound variable, is no longer bound, and that any bound variables in t are no longer bound in A.

When $x \notin vf(A)$ we can write:

$A = A(x) = A(t)$

Example

$$\begin{aligned} A(x) &= (p(x) \vee \forall\, y\, \exists\, x\; r(x, y)) \\ t &= f(y, u) \end{aligned}$$

To derive A(t) we first change the names of the bound variables y and x, x because it is the variable to be substituted for and y because it occurs in t getting:

$(p(x) \vee \forall\, z_1\, \exists\, z_2\; r(z_2, z_1))$

The substitution then gives:

$A(t) = (p(f(y, u)) \vee \forall\, z_1\, \exists\, z_2\; r(z_2, z_1))$

3. The formal system Pr of first-order predicate calculus

This is defined as follows:

1. Σ_{Pr}: the alphabet defined in Section 2
2. F_{Pr}: the set of formulae defined in Section 2
3. A_{Pr}: the subset of formulae from F_{Pr} having one of the following forms:

SA_1: $(A \rightarrow (B \rightarrow A))$
SA_2: $((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$

SA_3: $((\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A))$
SA_4: $(\forall x\, A(x) \rightarrow A(t))$
SA_5: $((D \rightarrow B) \rightarrow (D \rightarrow \forall x\, B))$

where A, B, C and D are any formulae of F_{Pr} and D does not have x as a free variable, x is a variable and t is a term

4. R_{Pr} is the generalization rule and modus ponens rule {m.p., g}

$$\text{m.p.: } A, A \rightarrow B \vdash B$$
$$\text{g: } A \vdash \forall x\, A$$

for all A, B $\in F_{Pr}$ and all variables x.

The possibility of deducing B from the hypotheses $A_1, A_2, \ldots, A_n$ is written:

$$A_1, A_2, \ldots, A_n \vdash_{Pr} B$$

or simply:

$$A_1, A_2, \ldots, A_n \vdash B$$

As an example of a deduction suppose we wish to prove that:

$$\forall x \forall y\, p(x, y) \vdash \forall z\, p(z, z)$$

The sequence is as follows:

$f_1 : \forall x \forall y\, p(x,y)$	(hypothesis)
$f_2 : (\forall x \forall y\, p(x,y) \rightarrow \forall y\, p(z,y))$	(SA_4)
$f_3 : \forall y\, p(z,y)$	(m.p. with f_1, f_2)
$f_4 : (\forall y\, p(z,y) \rightarrow p(z,z))$	(SA_4)
$f_5 : p(z,z)$	(m.p. with f_3, f_4)
$f_6 : \forall z\, p(z,z)$	(g)

The set of theorems of Pr—that is, of formulae that can be deduced from the axioms alone, without any additional hypotheses—is denoted by T_{Pr}. We have the following propositions:

Proposition 1: for all $A \in F_{Pr}$, $(A \rightarrow A) \in T_{Pr}$

Proposition 2: if $A_1, A_2, \ldots, A_{n-1} \vdash (A_n \rightarrow B)$, then $A_1, A_2, \ldots, A_{n-1}, A_n \vdash B$

Proposition 3: if $A_1, A_2, \ldots, A_n$ are closed formulae, then if $A_1, A_2, \ldots, A_{n-1}, A_n \vdash B$; $A_1, A_2, \ldots, A_{n-1} \vdash (A_n \rightarrow B)$ (deduction theorem)

The proofs of Propositions 1 and 2 are exactly the same as those for the Propositions 1 and 2 in Chapter 4; also the proof of Proposition 3 is the same as those for Proposition 3 in Chapter 4 but there is now a fifth case (e) in addition to (a), (b), (c) and (d). This is dealt with as follows.

Case (e)

B is obtained by applying Rule g to a formula $B = \forall x\, C$.
By hypothesis:

$$A_1, A_2, \ldots, A_{n-1} \vdash (A_n \to C)$$

By SA_5:

$$A_1, A_2, \ldots, A_{n-1} \vdash ((A_n \to C) \to (A_n \to \forall x\, C))$$

It follows by applying *modus ponens* that:

$$A_1, A_2, \ldots, A_{n-1} \vdash (A_n \to \forall x\, C)$$

which covers this case, and the inductive proof is completed as before.

△

If we have: $C' \subset C;\ F'_j \subset F_j;\ P'_j \subset P_j$

and define the alphabet $\Sigma_{Pr'}$ by:

$$\Sigma_{Pr'} = V \cup C' \cup (\cup F'_j) \cup (\cup P'_j) \cup \{(\} \cup \{)\} \cup \{,\} \cup \{\neg, \to, \forall\}$$

The sets Term′, Atom′, and $F_{Pr'}$ can then be defined with respect to $\Sigma_{Pr'}$ in exactly the same way as Term, Atom and F_{Pr} are defined with respect to Σ_{Pr}; and similarly the formal system Pr′.

Example

If:

$$C' = C,\ F'_j = \emptyset,\ P'_j = P_j$$

we have what is called first-order predicate calculus without function symbols.

4. Interpretations

A. Definitions

Let C', F'_j, P'_j and Σ_{Pr} be defined as above; we say that an interpretation of $\Sigma_{Pr'}$ consists of:

1. a non-empty set S, called the domain for the interpretation
2. an element $\bar{c} \in S$ for each $c \in C'$
3. a mapping $\bar{f}: S^j \to S$ for each $f \in F'_j$
4. a mapping $\bar{p}: S^j \to \{\mathbf{T}, \mathbf{F}\}$ for each $p \in P'_j$

Given these, we associate:

1. with every term t having j variables, a mapping t of $S^j \to S$, as follows:
   (a) if t is a variable, $\bar{t}$ is the identity mapping $S \to S$
   (b) if t is a constant c, $\bar{t} = \bar{c}$
   (c) if t is of the form $f(t_1, t_2, \ldots, t_j)$, $\bar{t} = \bar{f}(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_j)$ (composition of mappings)
2. with every formula having j free variables, a mapping $\bar{A}$ of $S^j \to \{\mathbf{T}, \mathbf{F}\}$ as follows:
   (a) if $A = p(t_1, t_2, \ldots, t_j)$ is an atom:
   $\bar{A} = \bar{p}(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_j)$
   (b) if A is of the form $\neg B$, then for all $(s_1, s_2, \ldots, s_j) \in S^j$:
   $$\bar{A}(s_1, s_2, \ldots, s_j) = \mathbf{T} \text{ if } \bar{B}(s_1, s_2, \ldots, s_j) = \mathbf{F}$$
   $$= \mathbf{F} \qquad = \mathbf{T}$$
   (c) if A is of the form $(B \to C)$, then for all $(s_1, s_2, \ldots, s_j) \in S^j$:
   $$\bar{A}(s_1, s_2, \ldots, s_j) = \mathbf{F} \text{ if } \bar{B}(s_1, s_2, \ldots, s_j) = \mathbf{T} \text{ and } \bar{c}(s_1, s_2, \ldots, s_j) = \mathbf{F}$$
   $$= \mathbf{T} \text{ otherwise}$$
   (d) if A is of the form $\forall x\, B(x, y_1, y_2, \ldots, y_j)$, then for all $(s_1, s_2, \ldots, s_j) \in S^j$:
   $$\bar{A}(s_1, s_2, \ldots, s_j) = \mathbf{T} \text{ if for all } s \in S,\ \bar{B}(s, s_1, s_2, \ldots, s_j) = \mathbf{T}$$
   $$= \mathbf{F} \text{ otherwise}$$

Given any interpretation, use of these rules enables any closed formula to be associated with one or other of the symbols **T**, **F**. We sometimes write i[t], i[A] instead of $\bar{t}$, $\bar{A}$ for these interpretations of terms t or formulae A. If A contains free variables and i[A] is the constant function whose value is always **T**, we write i[A] = **T**.

B. Example 1

Let:

$C' = \emptyset$ (no constant symbols)
$F'_1 = \{f\}$
$F'_j = \emptyset$ for $j \geqslant 2$ (only one function symbol)
$P'_2 = \{p\}$
$P'_j = \emptyset$ for $j \neq 2$ (only one relation symbol)

One of the formulae of Pr′ is:

$$\forall x\, (p(x, f(x)) \to p(f(x), x))$$

Consider this with the interpretation $i = (S, \bar{f}, \bar{p})$ where:

$S = \{s_1, s_2, s_3\}$
$\bar{f}$: $s_1 \to s_2$; $s_2 \to s_3$; $s_3 \to s_1$
$\bar{p}$: $(s_1, s_2) \to \mathbf{T}$; $(s_2, s_1) \to \mathbf{T}$
$(x, y) \to \mathbf{F}$ for all other couples $(x, y) \in S^2$

This associates with A = p(x, f(x)) a function for S to {**T**, **F**} defined by:

$\bar{A}(s_1) = \bar{p}(s_1, \bar{f}(s_1)) = \bar{p}(s_1, s_2) = \mathbf{T}$
$\bar{A}(s_2) = \bar{p}(s_2, \bar{f}(s_2)) = \bar{p}(s_2, s_3) = \mathbf{F}$
$\bar{A}(s_3) = \bar{p}(s_3, \bar{f}(s_3)) = \bar{p}(s_3, s_1) = \mathbf{F}$

Similarly, with B = p(f(x), x) is associated a function defined by:
$\bar{B}(s_1) = \mathbf{T}$, $\bar{B}(s_2) = \mathbf{F}$, $\bar{B}(s_3) = \mathbf{F}$
Then with the formula:
$C = (p(x, f(x)) \rightarrow p(f(x), x))$
is associated the function:
$\bar{C}(s_1) = \bar{C}(s_2) = \bar{C}(s_3) = \mathbf{T}$
Therefore:
$i[\forall\ x\ (p(x, f(x)) \rightarrow p(f(x), x))] = \mathbf{T}$
It is easily seen that this formula is not true for all interpretations.

C. Example 2

Let:

$C' = \{a, b, c\}$	(three constants to represent three people, Alan, Bernard, Charles)
$F'_1 = \{f\}$	(one function symbol to represent the function 'father of')
$F'_j = \emptyset$ for $j \geqslant 2$	
$P'_i = \{p\}$	(one unary predicate to represent 'is a father' and one binary 'is the father of')
$P'_2 = \{p'\}$	
$P'_j = \emptyset$ for $j \geqslant 3$	

A possible interpretation of $\Sigma_{Pr'}$ is a real-life situation in which each constant represents a different person and the predicates state what is true and what is false concerning each person. We could take, for example:

$\underset{\bar{}}{i} = (S, \bar{a}, \bar{b}, \bar{c}, \bar{f}, \bar{p}, \bar{p}')$

where:

S = {Alan, Bernard, Charles, ancestor}
$\bar{a}$ = Alan, $\bar{b}$ = Bernard, $\bar{c}$ = Charles
$\bar{f}$: $\bar{b} \rightarrow \bar{a}$: $\bar{c} \rightarrow \bar{a}$; $\bar{a} \rightarrow$ ancestor; ancestor $\rightarrow$ ancestor
$\bar{p}(\bar{a}) = \mathbf{T}$; $\bar{p}(\bar{b}) = \mathbf{F}$; $\bar{p}(\bar{c}) = \mathbf{F}$; $\bar{p}$(ancestor) = **T**
$\bar{p}'(x, y) = \mathbf{T}$ iff ($x = \bar{a}$ and $y = \bar{b}$) or ($x = \bar{a}$ and $y = \bar{c}$) or (x = ancestor and $y = \bar{a}$) or (x = ancestor and y = ancestor)

In this interpretation the formulae

$\forall x\ \exists y\ p(y, x)$
$\forall x\ p\ (f(x), x)$
$\forall x\ (\exists y p\ (x, y) \rightarrow p'\ (x))$ are true.

D. Some Comments

Once fixed, the interpretation determines which of the values **T** or **F** should be assigned to any closed formula, however complicated it may be; and this is achieved by performing computations with its *constituent functions*. This method, by which the finding of the truth value of a formula is reduced to a simple calculation, enables us to treat the concept of 'truth' in a rigorous manner and in particular to ask the questions:

1. Are those formulae that can be proved to be theorems of the formal system Pr always true? (i.e. associated with **T** by every interpretation)
2. Are the theorems of Pr exactly the formulae that are always true?

However, it must not be thought that the definition of 'truth' given here for the formulae of Pr is arbitrary: a moment's reflexion will show that it is the only one possible and that any formula that can be shown to be 'true' in this sense for any interpretation whatever corresponds to a statement that any reasonable person, using the symbols $\forall$, $\rightarrow$ and $\neg$ in their usual sense, would consider true in the ordinary sense.

E. Tautologies, Consequences, etc.

A *tautology* is any formula $A \in F_{Pr}$ such that $i[A] = \mathbf{T}$ for any interpretation i. This is written $\models A$. Examples are:

1. $\models (p(x) \vee \neg\, p(x))$
2. $\models (\forall\, x\, (p(x) \wedge q(x)) \leftrightarrow (\forall\, y\, p(y) \wedge \forall\, z\, q(z)))$
3. $\models (\exists\, y\, \forall\, x\, r(z, x, y) \rightarrow \forall\, x\, \exists\, y\, r(z, x, y))$
4. $\models (\forall\, x\, (p(x) \rightarrow p(f(x))) \rightarrow (\forall\, y(p(y) \rightarrow p(f(fy))))$

It follows from the definition that any formula A containing free variables $x_1, x_2, \ldots, x_n$ is a tautology if and only if the closed formula $\forall\, x_1 \forall\, x_2, \ldots, \forall\, x_n\, A$ is a tautology.

In contrast to the case for propositional calculus, the number of different interpretations for a Pr formula is not finite, so in predicate calculus there is no equivalent to the truth table. Therefore the question of whether or not a given formula is a tautology cannot be settled by enumerating the possible interpretations but must be tackled by a reasoning process.

The following definitions are predicate calculus analogues of those for propositional calculus given in Chapter 4 (Section 3). All the formulae belong to the set F_{Pr}.

1. Tautology, written $\models A$, has already been defined.
2. B is a *consequence* of A if $i[B] = \mathbf{T}$ for every interpretation i for which $i[A] = \mathbf{T}$. This is written $A \models B$.
3. B is a consequence of a set of formulae $\mathscr{A}$ if $i[B] = \mathbf{T}$ for every interpretation i for which $i[A] = \mathbf{T}$ for every $A \in \mathscr{A}$. This is written $\mathscr{A} \models B$.

4. A is *satisfiable* or *consistent* if there is an interpretation i for which i[A] = **T**.
5. A set of formulae $\mathscr{A}$ is satisfiable or consistent if there is an interpretation i such that i[A] = **T** for all A $\in \mathscr{A}$. This interpretation is called a *model* of $\mathscr{A}$.
6. A is *unsatisfiable* or *inconsistent* if there is no interpretation for which i[A] = **T**. It is easily seen that A is unsatisfiable iff $\neg$ A is a tautology when A is a closed formula.
7. A set of formulae $\mathscr{A}$ is unsatisfiable or inconsistent if there is no interpretation such that i[A] = **T** for all A $\in \mathscr{A}$.

Examples are:

1. the formula $\forall$ x p(x, x) is a consequence of the pair of formulae:

   $\forall$ x $\forall$ y (q(x, y) $\rightarrow$ p(x, y))
   $\forall$ z q(z, z)

2. the following set is satisfiable:

   $\forall$ x $\forall$ y (p(x, x) $\rightarrow$ (p(y, y) $\rightarrow$ p(x, y)))
   $\forall$ z p(z, z)

   It can be shown that if S is any given non-empty set there is only one model for this set of formulae having S as base, and that this is the model (s, $\bar{p}$) defined by $\bar{p}(s_1, s_2)$ = **T** for all $s_1, s_2 \in$ S.

3. The following set of formulae is unsatisfiable:

   $\exists$ a p(a)
   $\forall$ x (p(x) $\rightarrow$ $\neg$ r(x))
   $\forall$ y ($\neg$ r(y) $\rightarrow$ q(y))
   $\forall$ z (q(z) $\rightarrow$ $\neg$ p(z))

   but every subset of three formulae from the four of the set is satisfiable.

Proposition 4: if $\mathscr{A} \subset F_{Pr}$ is a set of closed formulae of Pr and B a closed formula of Pr, then $\mathscr{A} \models$ B iff $\mathscr{A} \cup \{\neg$ B$\}$ is unsatisfiable

Proof

1. To prove the condition necessary suppose $\mathscr{A} \models$ B and let i be an interpretation. Then if:

   i[A] = **T** for all A $\in \mathscr{A}$
   i[B] = **T** by definition

Therefore:

$i[\neg B] = \mathbf{F}$
and i is not a model of $\mathscr{A} \cup \{\neg B\}$.
If $i[A] = \mathbf{F}$ for at least one formula of $\mathscr{A}$
i is not a model of $\mathscr{A} \cup \{\neg B\}$.
In either case i is not a model of $\mathscr{A} \cup \{\neg B\}$ and $\mathscr{A} \cup \{\neg B\}$ is therefore unsatisfiable.

2. To prove the condition sufficient suppose $\mathscr{A} \cup \{\neg B\}$ is unsatisfiable. Let i be such that:

   $i[A] = \mathbf{T}$ for all $A \in \mathscr{A}$

   Then since $\mathscr{A} \cup \{\neg B\}$ has no model it is impossible that:

   $i[\neg B] = \mathbf{T}$

   Therefore:

   $i[B] = \mathbf{T}$

   Hence:

   $A \models B$

△

This proposition is only a simple consequence of the definitions. It is nevertheless important, because it establishes the validity of a proof by contradiction (*reductio ad absurdum*): to prove $\mathscr{A} \models B$ it is necessary and sufficient to prove that $\mathscr{A} \cup \{\neg B\}$ has no model.

Proposition 5: For any formula $A \in F_{Pr}$, if $\vdash A$, then $\models A$, i.e. formulae of F_{Pr} that are theorems are also tautologies: syntactic validity implying semantic validity

Proof

The proof follows the same lines as that given for Proposition 6 for propositional calculus in Chapter 4, taking into account the generalization rule of inference and the new system of axioms.

△

5. Some consequences

Propositions 9, 10, 11 and 12 in Chapter 4 can be adapted to hold for predicate calculus but the proofs are more difficult and are given here; the reader may consult Boolos and Jeffrey (1974), Huet (1975), Kleene (1967), Largeault (1972), and Yasuhara (1971). They are stated as follows.

Proposition 6: for all formulae $A \in F_{Pr}$, if $\models A$, then $\vdash A$ (completeness theorem)

Proposition 7: let $\mathscr{A}$ be a set of formulae of F_{Pr}, then if every finite subset $\mathscr{A}' \subset \mathscr{A}$ is satisfiable, $\mathscr{A}$ is satisfiable (compactness theorem)

Proposition 8: let $\mathscr{A}$ be a set of formulae of F_{Pr} and $B \in F_{Pr}$, then if $\mathscr{A} \models B$ there is a finite subset $\mathscr{A}'$ of $\mathscr{A}$ such that $\mathscr{A}' \models B$ (finiteness theorem)

Proposition 9: if $\mathscr{A} \subset F_{Pr}$ and $\mathscr{A} \in F_{Pr}$, then $\mathscr{A} \models \mathscr{A}$ iff $\mathscr{A} \vdash A$ (generalized completeness theorem)

Proposition 10: let $\mathscr{A} \subset F_{Pr}$ and B be a tautology, then $\mathscr{A} \vdash \neg B$ iff there is no model for $\mathscr{A}$.

Proof

Proposition 10 can be deduced from Proposition 9 as follows. First suppose $\mathscr{A} \vdash \neg B$, then $\mathscr{A} \models \neg B$ and it follows that $i[\neg B] = T$ for every interpretation such that $i[A] = T$ for every $A \in \mathscr{A}$. But since B is a tautology $i[B] = T$ for all interpretations and therefore there cannot be an interpretation such than $i[A] = T$ for all $A \in \mathscr{A}$, that is, there is no model for $\mathscr{A}$. The condition is therefore sufficient.

If $\mathscr{A}$ has no model, then $\mathscr{A} \models \neg B$ and therefore $\mathscr{A} \vdash \neg B$, and the condition is necessary. △

As stated in Chapter 4 for propositional calculus, it is easily shown that the formal system obtained by adding $\mathscr{A}$ to the axioms of Pr is consistent if and only if $\mathscr{A}$ is consistent, these terms being defined as in Chapter 3. Furthermore it is coherent if and only if there is no tautology B such that $F \vdash \neg B$. Thus Proposition 10 shows that the idea of consistency given in this chapter agrees well with that of Chapter 3.

Proposition 11: if $\mathscr{A} \subset F_{Pr}$ is such that there are models for $\mathscr{A}$, then there is at least one denumerable model (that is, a model having a denumerable domain) (Lowenheim–Skolem Theorem)

This theorem shows in particular that the property of denumerability cannot be expressed in terms of first-order predicate calculus formulae.

6. Axiomatic theory

A. Definitions

An *axiomatic theory* TA is any formal system such that:

1. its alphabet is of the form $\Sigma_{Pr'}$ (see Section 3);
2. its set of well-formed formulae is F_{Pr}, defined on $\Sigma_{Pr'}$;
3. its set of axioms contains all the axioms of Pr, called the logical axioms of the theory, any others being called the non-logical axioms;
4. its rules of inference are m.p. and g.

Any interpretation i such that i[A] = **T** for every axiom A is called a model of TA.

B. Application of the results of Section 5

The set of theorems of an axiomatic theory TA is, by definition, the set of all formulae that can be deduced from the axioms (logical or non-logical); it follows from Proposition 9 that it can also be defined as the set of all formulae that are true for every interpretation that satisfies the axioms, that is, for any model. This equivalence means that everything that is true is deducible and everything that is deducible is true; while it is not self-evident—recall that we have not given a proof for Proposition 9—it is what justifies the use of the idea of an axiomatic theory in first-order predicate calculus. Not only are there several ways of formalizing a predicate calculus of order > 1, the equivalence of syntactic and semantic validity does not in general hold for higher orders.

If $\mathscr{A}$ denotes the set of non-logical axioms of an axiomatic theory TA, the various definitions and theorems concerning the consistency of the theory can be summarized by saying that all the following properties are equivalent:

1. There is no formula A of TA such that $\vdash_{\overline{TA}}$ A and $\vdash_{\overline{TA}} \neg$ A.
2. TA is consistent.
3. TA is non-contradictory.
4. TA is coherent.
5. There are formulae of TA that are not theorems.
6. $\mathscr{A}$ is consistent.
7. $\mathscr{A}$ is satisfiable.
8. $\mathscr{A}$ has models.
9. $\mathscr{A}$ has denumerable models.

C. Examples

Example 1

Theory of equivalence relations TEQ, here:

1. there is only one predicate symbol $p \in P_2$, no constants or function symbols
2. the non-logical axioms are:

$$\forall x\ p(x, x)$$
$$\forall x\ \forall y\ (p(x, y) \rightarrow p(y, x))$$
$$\forall x\ \forall y\ \forall z\ ((p(x, y) \wedge p(y, z)) \rightarrow p(x, z))$$

An example of a theorem is:

$$\forall x\ \forall y\ \forall z\ \forall u\ (((p(x, y) \wedge p(y, z)) \wedge \neg\ p(x, u)) \rightarrow \neg\ p(z, u))$$

A model for TEQ is:

1. domain S
2. a mapping p: $S^2 \to \{\mathbf{T}, \mathbf{F}\}$

   such that:

   $\bar{p}(x, y) = \mathbf{T}$

   is an equivalence relation, in the usual sense, on S

Example 2

A real-life situation where in the interests of easy reading we abandon the conventions given at the beginning of the chapter; in addition, this will enable us to use a notation closer to that of PROLOG.

We want to construct a 'theory' for a family group of 11 people for whom the parent-child relations are as shown in Fig. 5.

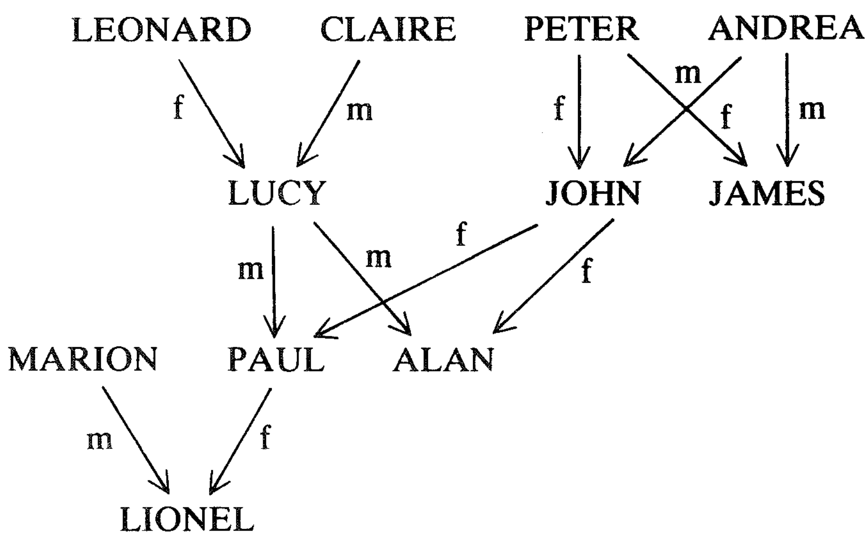

Fig. 5. Parent–child relationships for a family group of 11 people (m = mother, f = father).

One way of constructing a relevant theory is outlined.

As constants we take a set of symbols that represent each of the people, in this case their first names:

C′ = {Leonard, Claire, Peter, Andrea, Lucy, John, James, Marion, Paul, Alan, Lionel}

There are no function symbols, but several predicates:

unary: masculine
feminine

binary: is-father-of
is-mother-of
is-parent-of
is-grandfather-of

is-grandmother-of
is-grandparent-of

Axioms follow from a description of the group in which the predicates are given their normal meanings; we may take the following:

a1 masculine (Leonard)
a2 masculine (Peter)
a3 masculine (John)
a4 masculine (James)
a5 masculine (Paul)
a6 masculine (Alan)
a7 masculine (Lionel)
a8 feminine (Claire)
a9 feminine (Andrea)
a10 feminine (Lucy)
a11 feminine (Marion)
a12 is-father-of (Leonard, Lucy)
a13 is-mother-of (Claire, Lucy)
a14 is-father-of (Peter, John)
a15 is-father-of (Peter, James)
a16 is-mother-of (Andrea, John)
a17 is-mother-of (Andrea, James)
a18 is-mother-of (Lucy, Paul)
a19 is-mother-of (Lucy, Alan)
a20 is-father-of (John, Paul)
a21 is-father-of (John, Alan)
a22 is-mother-of (Marion, Lionel)
a23 is-father-of (Paul, Lionel)
a24 $\forall$ x $\forall$ y ((is-mother-of(x, y) v is-father-of(x, y)) $\leftrightarrow$ is-parent-of (x, y))
a25 $\forall$ x $\forall$ y (is-grandfather-of(x, y) $\leftrightarrow \exists$ z (is-father-of(x, z) $\wedge$ is-parent-of(z, y)))
a26 $\forall$ x $\forall$ y (is-grandmother-of(x, y) $\leftrightarrow \exists$ z (is-mother-of(x, z) $\wedge$ is-parent-of(z, y)))
a27 $\forall$ x $\forall$ y ((is-grandmother-of(x, y) v (is-grandfather-of(x, y) $\leftrightarrow$ (is-grandparent-of(x, y))))

Among the many theorems that can be proved are:

F_1 is-grandparent-of(Leonard, Paul)
F_2 $\exists$ x (is-parent-of(John, x))
F_3 $\forall$ x $\forall$ y(is-grandparent-of(x, y) $\leftrightarrow$
($\exists$ z(is-father-of(x, z) $\wedge$ is-father-of(z, y))
v $\exists$ z(is-father-of(x, z) $\wedge$ is-mother-of(z, y))
v $\exists$ z(is-mother-of(x, z) $\wedge$ is-father-of(z, y))
v $\exists$ z(is-mother-of(x, z) $\wedge$ is-mother-of(z, y))))

This shows two things:

1. for each of these formulae F_i there is a deduction using only the axioms of Pr together with the axioms a1 to a27, and the final formula is F_i;
2. whatever the interpretation, if:

   $i[A] = \mathbf{T}$ for all $A \in \{a1–a27\}$

   then:

   $i[F_1] = i[F_2] = i[F_3] = \mathbf{T}$

As an interpretation of this theory we might have the following:

1. A domain S, which we could think of as a set of objects among which are representatives of each constant but which could contain other objects as well.
2. An interpretation of the constants, that is, a correspondence between each constant and an object of S: for example, an object Alan $\in$ E corresponding to the constant Alan $\in$ C′.
3. An interpretation of the predicates, that is, a choice of **T** or **F** for each predicate applied to each object of S: for example, for the predicate masculine and the object $s \in S$ a value, **T** or **F**, for masculine(s).

By definition, an interpretation will be a model of the theory if all the axioms are true, that is, if the calculation of i[A] gives the value **T** for every axiom A.

D. Axiomatic Theories with Equality (ATE)

This name is given to any axiomatic theory for which:

1. the alphabet includes the binary predicate symbol (conventionally, $x = y$ is written instead of $= (x, y)$);
2. the axioms include:

   $\forall x \; x = x$

   and:

   $(x = y) \rightarrow (A(x) \leftrightarrow A(y))$

   for all formulae A (having x as free variables).

It can be shown that an ATE has a model if and only if it has a model in which the symbol = is interpreted as equality; such a model is called a standard equality model.

The following are examples of ATEs.

Example 1

Ordered sets

Apart from = there is only one predicate symbol $p \in P_2$; no constants and no functions. The non-logical axioms are:

$\forall x\ p(x, x)$

$\forall x \forall y\ ((p(x, y) \wedge p(y, x)) \rightarrow x = y)$

$\forall x \forall y \forall z((p(x, y) \wedge p(y, z)) \rightarrow p(x, z)))$

Example 2

First-order arithmetic

The only predicate symbol is = ; there is one constant symbol 0 and three functions:

$s \in F_1$, + and . $\in F_2$

Instead of $+(x, y) \cdot (x, y)$ we write $x + y \quad x \cdot y$.

The non-logical axioms are:

a1: $x = y \rightarrow s(x) = s(y)$

a2: $s(x) = s(y) \rightarrow x = y$

a3: $\neg (s(x) = 0)$

a4: $x + 0 = x$

a5: $x + s(y) = s(x + y)$

a6: $x . 0 = 0$

a7: $x . s(y) = (x . y) + x$

a8: $A(0) \wedge (\forall x\ (A(x \rightarrow A(s(x)))) \rightarrow (\forall z\ A(z))$

for all formulae A having x as a free variable

Whilst a1 to a7 are axioms in the strict sense, a8 is an axiom schema; it is a statement of the principle of induction.

It is clear that a natural model of this theory is provided by the interpretation ($\mathbb{N}$, 0, x + 1, + , .), where x + 1 denotes the successor function and + , . denotes addition and multiplication of integers in the normal sense, respectively; this satisfies all the axioms. What is surprising is that there can be models having a non-denumerable domain, and the reason for this is that axiom a8 is not strong enough: it states that the induction principle must hold, not for all properties of subsets of the domain, but for all properties of subsets that can be expressed as a formula, and while the infinity of formulae is denumerable that of the properties of subsets is not.

This shows that the theory of first-order arithmetic does not provide an adequate means for modelling arithmetic in its normal sense, meaning the set of positive integers together with the standard operations. The partial failure is not the consequence of any bad choice of axioms but lies much deeper and is related to the results of Gödel's work mentioned in Chapter 3.

7. Some results concerning decidability and undecidability

We now give, without proof, some results concerning the decision problem for theories based on predicate calculus.

1. The first-order predicate calculus Pr is undecidable. This means that there is no program that, given any formula A ϵ Pr, will determine within a finite time whether or not A is a theorem of Pr; or, what is the same thing, whether or not A is a tautology of Pr.

   The following results give a reasonably precise delineation of the boundary between decidability and undecidability.
2. Any axiomatic theory with equality that has a single binary predicate symbol in addition to $=$ is undecidable.
3. Any axiomatic theory with equality that has a single binary function symbol is undecidable.
4. Any axiomatic theory with equality that has two unary functional symbols is undecidable.
5. Any axiomatic theory with equality that has:

   a finite number of constant symbols
   a single unary function symbol
   a finite number of unary predicates
   no non-logical axioms

   is decidable.
6. The theory of first-order arithmetic is undecidable.
7. The theory of first-order arithmetic without the symbol '.' is decidable (Pressburger's theorem).

Exercises

Substitution with Renaming

First identify the free and bound variables respectively in the following formulae, then make the substitutions required, holding to the principles of renaming already explained.

1. Substitute f(x, y) for z in:

   $\forall x \forall y\ (p(x, a, z) \rightarrow \exists z\ p(x, y, z))$

2. Substitute f(g(z, v, u)) for x in:

   $\forall x \exists v((p(x, v) \rightarrow r(a)) \rightarrow \forall u \forall v\ (p(u,f(x)) \wedge p(v,f(x)))$

Deductions in Pr

Give detailed proofs of the following:

1. $\vdash$A iff $\vdash \forall x$ A
2. $\vdash(\forall x\ A(x) \rightarrow \forall y\ A(y))$
3. $\vdash(\forall x\ A(x) \rightarrow \exists y\ A(y))$

Models

In the following, the set $\mathscr{A}$ consists of three formulae:

A1: $\forall x \forall y \forall z ((p(x, y) \wedge p(y, z)) \rightarrow p(x, z))$
A2: $\forall x (p(a, x) \wedge p(x, b))$
A3: $\forall x\, p(x, f(x))$

1. Suggest a model i of $\mathscr{A}$, that is:

   a domain S
   elements $\bar{a}$, $\bar{b}$ of S
   a mapping $\bar{p}$ of $S \times S$ to $\{\mathbf{T}, \mathbf{F}\}$
   a mapping $\bar{f}$ of S to S

   such that:

   $i[A1] = i[A2] = i[A3] = \mathbf{T}$

   Give in detail the computation of the mapping $S \rightarrow \{\mathbf{T}, \mathbf{F}\}$ corresponding to the formula:

   $\forall y (p(x, y) \rightarrow p(x, f(y)))$

2. Show that the statement:

   A1, A2, A3 $\models \exists x\, p(x, a)$

   is not true.
   Suggest a model i of $\mathscr{A}$ such that:

   $i[\exists x\, p(x, a)] = \mathbf{F}$

Note that such a model is called a counter example for the formula concerned (here $\exists x\, p(x, a)$) and the method is often used to show that a particular formula is not a consequence of some given set of formulae. It can be used to establish the independence of a set of axioms, and thus, for example, to prove that the parallel axiom of plane geometry cannot be deduced from the other axioms, or that in set theory the axiom of choice cannot be deduced from the others.

Tautologies

State for each of the following formulae whether or not it is a tautology:

1. $(\forall x \exists y\, p(x, y) \rightarrow \exists x\, p(x, x))$
2. $(\exists x \forall y\, p(x, y) \rightarrow \exists x\, p(x, x))$
3. $(\exists y \forall x (p(x, y) \rightarrow r(x)) \rightarrow \exists z\, r(z))$
4. $(\forall x \forall y (p(x, y) \vee p(y, x)) \rightarrow \forall x\, p(x, x))$
5. $(\forall x \exists y \forall z\, p(x, y, z) \rightarrow \exists y \forall z\, p(z, y, z))$
6. $((\forall x (p(x) \rightarrow r(x)) \wedge \exists y \neg r(y)) \rightarrow \exists z \neg p(z))$

More Tautologies

Prove the following:

1. $\models (\forall x\, A(x) \rightarrow \exists y\, A(y))$
2. Show that:
   $\models (\forall x\, \forall y\, A(x, y) \leftrightarrow \forall y\, \forall x\, A(x, y))$
   $\models (\exists x\, \exists y\, A(x, y) \leftrightarrow \exists y\, \exists x\, A(x, y))$
   $\models (\exists x\, \forall y\, A(x, y) \leftrightarrow \forall y\, \exists x\, A(x, y))$
3. If $x \notin vf(A)$ show that:
   $\models (\forall x\, (A \vee B(x)) \leftrightarrow (A \vee \forall x\, B(x)))$
   $\models (\forall x\, (A \wedge B(x)) \leftrightarrow (A \wedge \forall x\, B(x)))$
   $\models (\exists x\, (A \vee B(x)) \leftrightarrow (A \vee \exists x\, B(x)))$
   $\models (\exists x\, (A \wedge B(x)) \leftrightarrow (A \wedge \exists x\, B(x)))$
4. Show that:
   $\models ((\forall x\, A(x) \vee \forall x\, B(x)) \rightarrow \forall x\, (A(x) \vee B(x)))$
   $\models ((\forall x\, A(x) \wedge \forall x\, B(x)) \leftrightarrow \forall x\, (A(x) \wedge B(x)))$
   $\models ((\exists x\, A(x) \vee \exists x\, B(x)) \leftrightarrow \exists x\, (A(x) \vee B(x)))$
   $\models (\exists x\, (A(x) \wedge B(x)) \rightarrow (\exists x\, A(x) \wedge \exists x\, B(x)))$
5. If $x \notin vf(B)$ show that:
   $\models ((\forall x\, A(x) \rightarrow B) \leftrightarrow \exists x\, (A(x) \rightarrow B))$
   $\models ((\exists x\, A(x) \rightarrow B) \leftrightarrow \forall x\, (A(x) \rightarrow B))$
   $\models ((B \rightarrow \forall x\, A(x)) \leftrightarrow \forall x\, (B \rightarrow A(x)))$
   $\models ((B \rightarrow \exists x\, A(x)) \leftrightarrow \exists x\, (B \rightarrow A(x)))$

Equivalent Formulae

In the following $A(x_1, x_2, \ldots, x_n)$ and $B(x_1, x_2, \ldots, x_n)$ are formulae whose free variables are included in the set $\{x_1, x_2, \ldots, x_n\}$. A and B are said to be equivalent if for any interpretation i the mappings i[A], i[B] to {T, F} are identical; this is written $A \equiv B$.

1. Show that:

   $A(x_1, x_2, \ldots, x_n) \equiv B(x_1, x_2, \ldots, x_n)$

   iff:

   $\models A(x_1, x_2, \ldots, x_n) \leftrightarrow B(x_1, x_2, \ldots, x_n)$

2. Show that if:

   $A(x_1, x_2, \ldots, x_n) \equiv B(x_1, x_2, \ldots, x_n)$

   and:

   $A'(x_1, x_2, \ldots, x_n) \equiv B'(x_1, x_2, \ldots, x_n)$

   then:

   $\forall x_i\, A(x_1, x_2, \ldots, x_n) \equiv \forall x_i\, B(x_1, x_2, \ldots, x_n)$

and:

$$\neg A(x_1, x_2, \ldots, x_n) \equiv \neg B(x_1, x_2, \ldots, x_n)$$

and:

$$(A(x_1, x_2, \ldots, x_n) \rightarrow A'(x_1, x_2, \ldots, x_n))$$
$$\equiv (B(x_1, x_2, \ldots, x_n) \rightarrow B'(x_1, x_2, \ldots, x_n))$$

3. Show that if a formula B contains a sub-formula $A(x_1, x_2, \ldots, x_n)$, then if A is replaced by an equivalent formula A′ the resulting formula B′ is equivalent to B.

Negation

Show that the following hold for all formulae A and B:

1. Show that:

$$\neg (A \rightarrow B) \equiv (A \wedge \neg B)$$
$$\neg (A \vee B) \equiv (\neg A \wedge \neg B)$$
$$\neg (A \wedge B) \equiv (\neg A \vee \neg B)$$
$$\neg (A \leftrightarrow B) \equiv ((A \wedge \neg B) \vee (\neg A \wedge B))$$
$$\neg \forall x\, A \equiv \exists x\, \neg A$$
$$\neg \exists x\, A \equiv \forall x\, \neg A$$

2. Using the above equivalences and the results from the preceding section find for each of the following an equivalent formula in which all negation symbols appear only immediately before predicate symbols:

$$\neg (\forall x\, p(x) \rightarrow (\exists y\, p(y) \rightarrow r(y) \vee \exists z\, r(z)))$$
$$\neg (\forall x\, (p(x) \leftrightarrow r(x)) \rightarrow \exists y(p(y) \wedge r(y)))$$
$$\neg (\forall x\, \exists y\, \forall z\, r(x, y, z) \rightarrow \exists t\, (p(t) \rightarrow q(t)))$$

Models Having an Infinite Domain

1. Give a set of axioms for which any model has an infinite domain.
2. For a given non-zero integer n give a set of axioms for which the domain of any model has at least n elements.

Extension of a Model

Let $(S, (\bar{f}_0, \bar{f}_1, \ldots, \bar{f}_n, \ldots), (\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_m, \ldots))$ be a model for a set of formulae $\mathscr{A}$; and T a set such that $T \cap S = \emptyset$.

1. Show that there is a model for $\mathscr{A}$ with domain $S \cup T$ of the form: $(S \cup T, (\bar{\bar{f}}_0, \bar{\bar{f}}_1, \bar{\bar{f}}_2, \ldots, \bar{\bar{f}}_n), (\bar{\bar{p}}_0, \bar{\bar{p}}_1, \ldots, \bar{\bar{p}}_m, \ldots))$

such that the $\bar{\bar{f}}_i$ and $\bar{\bar{p}}_i$ extend the $\bar{f}_i$ and $\bar{p}_i$, respectively, that is:

$$\forall\ (s_1, s_2, \ldots, s_j) \in S^j\colon \bar{\bar{f}}_i(s_1, s_2, \ldots, s_j) = \bar{f}(s_1, s_2, \ldots, s_j)$$

and:

$$\bar{\bar{p}}_i(s_1, s_2, \ldots, s_j) = \bar{p}_i(s_1, s_2, \ldots, s_j)$$

2. Deduce that there is no system of axioms $\mathscr{A}$ such that the domain for any model is either finite or denumerable. Define a non-denumerable model for the system proposed in the answer to 1 in the preceding section and one of $(n + 1)$ elements for the answer to 2 in the preceding section.

Models for Theories with Equality

1. For a given non-zero integer n, give a system of axioms having no symbols for either constants or functions and only $=$ as predicate symbol, such that every standard equality model has a domain of precisely n elements.
2. With the same conditions, give a system for which:
   (a) any set having at least two and at most five elements is a domain of standard equality model
   (b) the domain of every standard equality model has at least two and at most five elements
3. State for each of the systems proposed in (a) and (b) whether or not the set of formulae of Pr that are consequences of the system is recursive.

Recursiveness of Sets of Formulae

For each of the following state whether or not the set is recursive:

A = $\{F \in F_{Pr}$ | F a formula of first-order predicate calculus that is not a theorem$\}$
B = $\{F \in F_{Pr}$ | F as before, either F or $\neg$ F is a theorem$\}$
C = $\{F \in F_{Pr}$ | F as before, for which there is a deduction having at least 10^5 formulae, each of length $< 10^5\}$