Exam Applied Stochastic Modeling
19 December 2016, 8:45-11:30 hours

This exam consists of **5** problems, each consisting of several questions. All answers should be motivated, including calculations, formulas used, etc. It is allowed to use 1 sheet of paper (or 2 sheets written on one side) with **hand-written** notes. The use of a calculator is allowed.

For each question it is indicated in brackets how many points can be obtained. The grade for the exam is then given by: # points/4 + 1.

Exercise 1. Tourists for a bike tour through Amsterdam arrive according to an inhomogeneous Poisson process during the time interval $[0, 8]$, corresponding to an 8-hour day. The arrival rate during $[0, 4]$ is estimated to be 12 per hour, whereas the arrival rate during $[4, 8]$ is estimated to be 8 per hour. For simplicity, we assume that tourists arrive one at a time. We also assume that the bike tour takes an exponential time with a mean of 2 hours.
a. [2 pt.] What is the average number of tourists that arrive for a bike tour on an 8-hour day? And what is its distribution?
b. [3 pt.] Determine $m(\tau)$, i.e., the mean number of tourists on a bike tour at time τ, for $\tau \in [0, 8]$.
c. [2 pt.] Make a sketch of $m(\tau)$ and explain its behavior. When is the peak in $m(\tau)$? Can this be explained?

Exercise 2. Consider the M/M/1 system with 1 server, arrival rate λ and exponential service times with rate 1. After a service, a customer is not always completely satisfied. Specifically, with probability $1/3$ the customer that just completed service immediately rejoins the queue (at the back of the line, if there is any). We assume for now that a rejoining customer is indistinguishable from a newly arriving customer (i.e. has the same service time and probability to rejoin after service).
a. [4 pt.] Use a one-dimensional continuous-time Markov chain to determine the distribution of the queue length. Also derive the expected waiting time. When is the system stable?
Suppose that unsatisfied customers do not rejoin (immediately). Instead, there is an additional Poisson arrival process with rate $\lambda/3$ of unsatisfied customers, next to the newly arriving customers. The service times of the unsatisfied customers follow an exponential distribution with rate 2 (new customers still have a an exponential service time with rate 1).
b. [3 pt.] Calculate the first and second moment of the service time of an arbitrary customer. Also calculate the expected waiting time for this situation.
c. [2 pt.] Management decides that unsatisfied customers receive priority. Calculate the expected waiting time for both groups of customers. Is this good for the expected waiting times?

Exercise 3. Passengers for the subway arrive at arbitrary instants at the subway station. Subways arrive according to a renewal process; we denote U_n as the interarrival time between subway n and subway $n+1$ and assume that the U_n are independent and identically distributed. We are interested in the waiting time X_t if we would arrive at time t.
a. [1 pt.] Draw a sample path (i.e. a realization) of the waiting time process X_t over time.
b. [2 pt.] Define regeneration epochs and verify that $f(X_t) = X_t = T_{n+1} - t$, for $t \in (T_n, T_{n+1}]$ and T_n the moment the nth subway arrives, is an appropriate reward structure to determine the mean waiting time. Also derive the mean waiting time using the renewal reward theorem.
c. [2 pt.] To avoid excessive waiting, the manager of the subway system uses the squared waiting time as a cost function for internal cost calculations, i.e. $f(X_t) = (X_t)^2$. Determine the long-run average costs.
d. [1 pt.] Suppose that U_{2n+1} may depend on U_{2n}, $n = 0, 1, \ldots$, but both random variables are independent of U_m for $m = 0, 1, \ldots, 2n-1$ and $m = 2n+2, 2n+3, \ldots$. Is it possible to define a regenerative process? If so, give the regeneration epochs.

Exercise 4. Consider an M/M/2 queue with ρ the load per server.
a. [3 pt.] Draw the state diagram with the transition rates, and show that $p_i = 2\rho^i p_0$, for $i = 1, 2, \ldots$, and $p_0 = \frac{1-\rho}{1+\rho}$.
Consider now an open queueing network of queues with 2 servers per queue. New customers arrive to queue 1 with rate λ. After being served, they go to queue 2 with probability $4/5$; with probability $1/5$ they go back to queue 1. After being served at queue 2, they go back to queue 1 with probability $1/2$ or leave the system with probability $1/2$. The expected service times are 1 and 2 at queues 1 and 2, respectively.
b. [2 pt.] Formulate the routing equations for this system. For which value of λ is the system stable?
c. [2 pt.] Let $\pi(n_1, n_2)$ denote the stationary distribution of the joint number of customers. Give $\pi(0, 0)$ and $\pi(n_1, n_2)$ for $n_1, n_2 \geq 1$ in terms of λ.

Exercise 5. A company sells christmas trees. Chrismas trees have to be purchased well in advance, where the total order costs $h(S)$ depend on the total order quantity S. The selling price is set at 20 euro. In addition, there is a new tax regulation stating that an additional t euro is charged for every tree that is left after the selling season (and there is no income from leftovers). The demand for trees is represented by the random variable D.
a. [2 pt.] Argue that the expected revenue as a function of the order quantity S is

$$P(S) = 20\mathbb{E}D - h(S) - 20\mathbb{E}(D-S)^+ - t\mathbb{E}(S-D)^+$$

b. [2 pt.] Assume that $h(\cdot)$ is an increasing and convex function. Derive an equation with which the optimal purchase quantity S^* can be determined and show that the corresponding solution represents an optimum.
c. [3 pt.] Assume that the order costs are linear and are given by $h(S) = 8S$. Find the optimal order quantity. For which value of t is the probability of leftovers at most 0.1? Is this a realistic target?