Exam Applied Stochastic Modeling - Solutions

The solutions are always provisionary

December 19, 2016, 8:45 - 11:30 hours

Exercise 1.

a. Tourists arrive according to an inhomogeneous Poisson process, so the total number of tourists arriving during [0,8] follows a Poisson distribution with mean

$$\int_0^8 \lambda(t)\mathrm{d}t = \int_0^4 12\mathrm{d}t + \int_4^8 8\mathrm{d}t = 4 \times 12 + 4 \times 8 = 80.$$

In the above $\lambda(t)$ denotes the arrival rate at time t.

b. We have an $\mathrm{M}_t/\mathrm{G}/\infty$ queue, for which $m(\tau) = \int_0^\tau \lambda(t)\mathbb{P}(S > \tau - t)\mathrm{d}t$, with S a random variable representing the time of a bike tour. Now, using that S follows an exponential distribution with rate 0.5, we have, for $\tau \in [0, 4]$,

$$m(\tau) \;=\; \int_0^\tau 12e^{-0.5(\tau-t)}\mathrm{d}t \;=\; 12\int_0^\tau e^{-0.5t}\mathrm{d}t \;=\; 24(1-e^{-0.5\tau})$$

For $\tau \in [4, 8]$, we obtain

$$\begin{aligned} m(\tau) &= \int_0^4 12e^{-0.5(\tau-t)}\mathrm{d}t + \int_4^\tau 8e^{-0.5(\tau-t)}\mathrm{d}t \\ &= 24e^{-0.5\tau}(e^2-1) + 16(1-e^{-0.5(\tau-4)}) \;=\; 16 - 24e^{-0.5\tau} + 8e^{-0.5(\tau-4)}. \end{aligned}$$

c. A sketch of $m(\tau)$ can be found in Figure 1. The peak is at time 4, as this is the end of the period with the high arrival rate (and due to the distribution of the bike tour, the load adapts exponentially fast). Furthermore, observe the start-up phase during $[0, 4]$ where the mean number of customers converges exponentially fast to an equilibrium. After time 4, the system adapts to the new (lower) arrival rate.

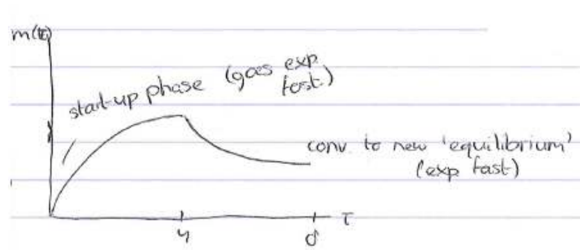

Figure 1: Sketch of $m(\tau)$ for Exercise 1c.

Exercise 2.

a. Define $X(t)$ as the number of customers in the system at time t. Then $\{X(t), t \geq 0\}$ is a continuous time Markov chain on the state space $\{0, 1, \ldots\}$. The transition diagram is given in Figure 2.

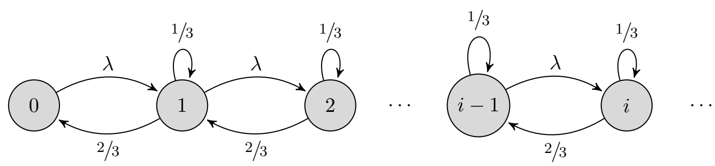

Figure 2: State diagram Exercise 2a.

The balance equations (for sets) are: $\lambda\pi_{i-1} = \frac{2}{3}\pi_i$, for $i = 1, 2, \ldots$, This gives $\pi_i = \frac{3}{2}\lambda\pi_{i-1} = \ldots = \left(\frac{3}{2}\lambda\right)^i \pi_0$. (Note the similarity with the standard M/M/1 queue.) Normalization gives

$$\sum_{i=0}^{\infty} \left(\frac{3}{2}\lambda\right)^i \pi_0 = 1,$$

such that $\pi_0 = 1 - \frac{3}{2}\lambda$ and thus $\pi_i = (1 - \frac{3}{2}\lambda)\left(\frac{3}{2}\lambda\right)^i$. The expected waiting time $\mathbb{E}W_Q$ (for the first service time) is obtained using

$$\mathbb{E}W_Q = \mathbb{E}L_Q \times 1 = \frac{\frac{3}{2}\lambda}{1 - \frac{3}{2}\lambda}.$$

The system is stable for $\lambda < \frac{2}{3}$.

(An alternative is to interpret the system as a special type of queueing network, consisting of one single-server queue.)

b. Observe that the service time S of an arbitrary customer follows a hyperexponential distribution. Let S_1 and S_2 be exponentially distributed with rates 1 and 2, respectively. We then have

$$\begin{aligned} \mathbb{E}S &= \frac{\lambda}{\lambda + \lambda/3}\mathbb{E}S_1 + \frac{\lambda/3}{\lambda + \lambda/3}\mathbb{E}S_2 = \frac{3}{4} \times 1 + \frac{1}{4} \times \frac{1}{2} = \frac{7}{8} \\ \mathbb{E}S^2 &= \frac{3}{4} \times \frac{2}{1^2} + \frac{1}{4} \times \frac{2}{2^2} = \frac{13}{8} \end{aligned}$$

The system under consideration is thus an M/H_2/1 queue with total arrival rate $\lambda + \frac{\lambda}{3} = \frac{4}{3}\lambda$ and load $\rho = \frac{4}{3}\lambda \times \frac{7}{8} = \frac{7}{6}\lambda$. For $\lambda < \frac{6}{7}$,

$$\mathbb{E}W_Q = \frac{\lambda\mathbb{E}S^2}{2(1-\rho)} = \frac{\frac{4}{3}\lambda \times \frac{13}{8}}{2(1 - \frac{7}{6}\lambda)} = \frac{13\lambda}{2(6 - 7\lambda)}.$$

c. We now have an M/G/1 priority queue. Using b, we get $\mathbb{E}R = \frac{\lambda\mathbb{E}S^2}{2} = \frac{4}{3}\lambda \times \frac{13}{8 \cdot 2} = \frac{13}{12}\lambda$. Moreover, for the load of the two classes, $\rho_1 = \frac{\lambda}{3} \times \frac{1}{2} = \frac{\lambda}{6}$ and $\rho_2 = \lambda \times 1$. Substituting these parameters, yields

$$\begin{aligned} \mathbb{E}W_Q(1) &= \frac{\mathbb{E}R}{1 - \rho_1} = \frac{\frac{13}{12}\lambda}{1 - \frac{\lambda}{6}} \\ \mathbb{E}W_Q(2) &= \frac{\mathbb{E}R}{(1-\rho_1)(1-\rho_1-\rho_2)} = \frac{\frac{13}{12}\lambda}{(1 - \frac{\lambda}{6})(1 - \frac{7}{6}\lambda)} \end{aligned}$$

Class 1 (unsatisfied customers) improves in terms of expected waiting time, but class 2 not.

Exercise 3.

a. See Figure 3.1 in the notes of Koole (page 44).

b. Regeneration epochs are the moments when the subway arrives. The function given describes the function sketched in a (and is the time until arrival of the next subway). Let S denote a generic interarrival time between two subways. Now, the mean waiting time can be determined as (also, see p. 44 of the notes of Koole)

$$\frac{\mathbb{E}[\text{cost per cycle}]}{\mathbb{E}[\text{cycle length}]} = \frac{1}{\mathbb{E}S}\mathbb{E}\left[\int_0^S S - t \, \mathrm{d}t\right] = \frac{1}{\mathbb{E}S}\mathbb{E}\left[\int_0^S t \, \mathrm{d}t\right] = \frac{\mathbb{E}S^2}{2\mathbb{E}S}$$

c. Using the renewal reward theorem again, we now obtain the following long-run average cost (with $F_S(\cdot)$ the distribution function of the interarrival times)

$$\frac{1}{\mathbb{E}S}\mathbb{E}\left[\int_0^S (S-t)^2 \, \mathrm{d}t\right] = \frac{1}{\mathbb{E}S}\mathbb{E}\left[\int_0^S t^2 \, \mathrm{d}t\right] = \frac{1}{\mathbb{E}S}\mathbb{E}\left[\int\int_0^s t^2 \mathrm{d}t\mathrm{d}F_S(s)\right] = \frac{\mathbb{E}S^3}{3\mathbb{E}S}.$$

d. Yes, the regeneration epochs are the moments when even numbered subways arrive.

Exercise 4.

a. Let λ denote the arrival rate and μ the service rate, such that $\rho = \lambda/(2\mu)$. The transition diagram for the number of customers in het system (having limiting distribution p_i) can be found in Figure 3.

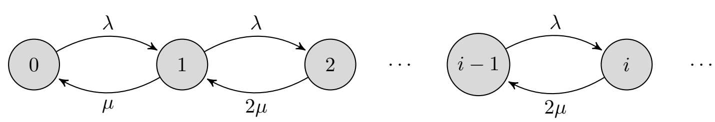

Figure 3: State diagram Exercise 4a.

The balance equations are then as follows: $\lambda p_0 = \mu p_1$ and $\lambda p_{i-1} = 2\mu p_i$, for $i = 2, 3, \ldots$. Expressing in terms of p_0 yields $p_1 = 2\rho p_0$ and, for $i = 2, 3, \ldots$,

$$p_i = \frac{\lambda}{2\mu} p_{i-1} = \rho^{i-1} p_1 = 2\rho^i p_0.$$

Normalization provides

$$p_0 + p_0 \sum_{i=1}^{\infty} 2\rho^i = 1.$$

Working out the summation yields $p_0 \, \frac{1+\rho}{1-\rho} = 1$, and the result follows.

b. Let γ_i represent the effective arrival rate to queue i, $i = 1, 2$. The routing equations are

$$\begin{aligned} \gamma_1 &= \lambda + \frac{1}{5}\gamma_1 + \frac{1}{2}\gamma_2 \\ \gamma_2 &= \frac{4}{5}\gamma_1. \end{aligned}$$

Solving these equations yields $\gamma_1 = \frac{5}{2}\lambda$ and $\gamma_2 = 2\lambda$. The loads of both queues are $\rho_1 = \frac{\gamma_1}{2} = \frac{5}{4}\lambda$ and $\rho_2 = \frac{\gamma_2}{2\cdot 0.5} = 2\lambda$. The network is thus stable for $\lambda < \frac{1}{2}$.

c. For a generalized Jackson network we have $\pi(n_1, n_2) = \pi_1(n_1)\pi_2(n_2)$, with $\pi_i(n_i)$ the marginal distribution of queue i, $i = 1, 2$. Using a, we obtain, for $n_1, n_2 \geq 1$,

$$\begin{aligned}\pi(0,0) &= \frac{1-\rho_1}{1+\rho_1} \times \frac{1-\rho_2}{1+\rho_2} \\ \pi(n_1, n_2) &= \frac{1-\rho_1}{1+\rho_1} 2\rho_1^{n_1} \times \frac{1-\rho_2}{1+\rho_2} 2\rho_2^{n_2}\end{aligned}$$

with ρ_i defined in b.

Exercise 5.

a. The terms of $P(S)$ can be explained as follows. The expected number of sales is $\mathbb{E}[\min(D, S)] = \mathbb{E}D - \mathbb{E}(D - S)^+$; multiplied with the price of 20 gives the expected income. Alternatively, $20\mathbb{E}D$ can be interpreted as the expected income in case of perfect information (no randomness in demand) and $20\mathbb{E}(D - S)^+$ is missed income due to lack of stock. The term $h(S)$ represents the total order cost, and $\mathbb{E}(S - D)^+$ are the expected number of leftovers, for which t is charged per leftover.

b. We should maximize $P(S)$ or minimize the cost $C(S) = h(S) + 20\mathbb{E}(D-S)^+ + t\mathbb{E}(S-D)^+$. Let $F_D(\cdot)$ and $f_D(\cdot)$ be the distribution function and density of S, respectively. Taking the derivative with respect to S yields

$$\frac{\mathrm{d}}{\mathrm{d}S} C(S) = h'(S) - 20(1 - F_D(S)) + tF_D(S) = h'(S) - 20 + (t + 20)F_D(S).$$

Setting the derivative to 0 provides the equation

$$h'(S) + (t + 20)F_D(S) = 20. \tag{1}$$

This is an optimum, as the cost function is convex: $\frac{\mathrm{d}^2}{\mathrm{d}S^2} C(S) = h''(S) + (t + 20)f_D(S) > 0$.

c. Suppose that $h(S) = 8S$, then $h'(S) = 8$ and Equation 1 becomes $8 + (t+20)F_D(S) = 20$. Solving for S provides

$$S^* = F_D^{-1}\left(\frac{12}{t + 20}\right).$$

For given t, the probability of leftovers when ordering the optimal order quantity S^* is

$$\mathbb{P}(D < S^*) = F_D(S^*) = F_D\left(F_D^{-1}\left(\frac{12}{t+20}\right)\right) = \frac{12}{t+20}.$$

Now, solving $\frac{12}{t+20} \leq 0.1$ yields $t \geq 100$.

This is not realistic as the tax would be at least 5 times the selling price.