# Exam Applied Stochastic Modeling
## 18 December 2017, 8:45-11:30 hours

This exam consists of **5** problems, each consisting of several questions. All answers should be motivated, including calculations, formulas used, etc. It is allowed to use 1 sheet of paper (or 2 sheets written on one side) with **hand-written** notes. The use of a calculator is allowed.

For each question it is indicated in brackets how many points can be obtained. The grade for the exam is then given by: # points/4 + 1.

**Exercise 1.** Consider the M/M/$\infty$ queue with an infinite number of servers and a Poisson arrival process with rate $\lambda$ and exponential service times with rate $\mu$.
a. [3 pt.] Draw the state diagram with the transition rates and use this to derive the equilibrium number of customers in the system.

Consider the same queueing model, but now suppose that the arrival process starts in the distinct past and ends at time 0; that is, the arrival rate $\lambda(t)$ equals $\lambda$ for $t \in (-\infty, 0]$ and $\lambda(t) = 0$ for $t > 0$.
b. [4 pt.] Let $m(\tau)$ be the mean number of customers in the system at time $\tau$. Verify that $m(0) = \frac{\lambda}{\mu}$ and that $m(\tau) = \frac{\lambda}{\mu}e^{-\mu\tau}$, for $\tau > 0$. Make a sketch of $m(\tau)$ as a function of $\tau > 0$ and explain its behavior.

**Exercise 2.** Consider an M/G/1 priority queue with three types of customers. The arrival rates are $1/6$, $1/6$, and $1/18$ for types 1, 2, and 3, respectively. The service times are *exactly* 1, 2, and 3 for types 1, 2, and 3, respectively.
a. [1 pt.] Argue, without calculations, which class should have highest and second highest priority to minimize the aggregate waiting time.
b. [4 pt.] Verify that the mean service time of an arbitrary customer is $\mathbb{E}S = 12/7$. For the priorities chosen in a, determine the expected waiting time for each type.
c. [3 pt.] Suppose that the service times of type 2 turn out to follow an exponential distribution with a mean of 2 (instead of exactly 2). Does this affect the priorities chosen in part a? Determine the expected waiting time of type 1.

**Exercise 3.** An open queueing network consists of two queues, both having a single server. New customers arrive to queue 1 according to a Poisson process with rate $\lambda$. After being served at queue 1, customers go to queue 2 with probability $p$ and leave the system with probability $1-p$. The service times at queue 1 are exponential with rate 2 and at queue 2 exponential with rate 1.
a. [2 pt.] Formulate the routing equations for this system. For which value of $\lambda$ is the system stable (in terms of $p$)?
b. [3 pt.] Assume for now that $\lambda = 1$. Give the stationary distribution of the joint number of customers and the expected waiting time for queue 2. Make a sketch of the expected waiting time at queue 2 as a function of $p \in (0,1)$ and explain what happens when $p$ goes to 1.

When there are at least 2 customers at queue 1, the service rate at queue 1 is increased to 3. As soon as there is 1 customer again at queue 1, the service rate is changed back to 2.
c. [4 pt.] Model the number of customers at queue 1 as a birth-and-death process and derive its marginal stationary distribution. What is the impact on queue 2?

**Exercise 4.** Teams of professionals continuously alternate between working on projects and having breaks. More precisely, the work schedule is as follows. The team starts to work on a project. If the first project is finished within 2 hours, they start working on a new project after which they have a break. If the first project takes more than 2 hours, they have a break directly after the first project. A break takes 30 minutes, after which the schedule repeats. The time for a project is assumed to be exponentially distributed with rate $\mu$ (per hour).
a. [3 pt.] Give regeneration epochs and use renewal theory to show that the fraction of time the team is working on projects equals

$$\frac{2 - e^{-2\mu}}{2 - e^{-2\mu} + \mu/2}.$$

b. [2 pt.] Determine the fraction of time that the team is working on projects if $\mu$ goes to 0 and explain that result. Do the same for $\mu \to \infty$.

**Exercise 5.** Consider a continuous-review continuous-product deterministic inventory model with holding costs, order costs, and *lead time* $L = 1$ (EOQ model). Demand arrives continuously at rate 2 per time unit. The order costs are 18 per order and holdings costs are 2 per product per time unit.
a. [3 pt.] Compute the optimal reorder level, reorder size and the corresponding average costs per time unit.

The manager uses a different order strategy. A reorder is placed when the inventory becomes zero, whereas the reorder size is $Q$. Orders that are placed when inventory is zero are lost (lost sales). The manager uses a cost of $q$ per lost sale.
b. [4 pt.] Make a sketch of the development of the inventory level over time and define regeneration epochs. Argue that the cycle length is $1 + Q/2$ and determine the average costs per time unit.