Exam Applied Stochastic Modeling - Solutions

The solutions are always provisionary

December 17, 2018, 8:45 - 11:30 hours

Exercise 1.

a. It follows directly that the mean service time equals $\mathbb{E}S = 1$. The second moment follows from

$$\mathbb{E}S^2 = \int_0^2 \frac{1}{2}u^2 \mathrm{d}u = \frac{1}{2}\frac{1}{3}u^3\Big|_{u=0}^2 = \frac{4}{3}.$$

Consequently, the residual service time upon arrival reads

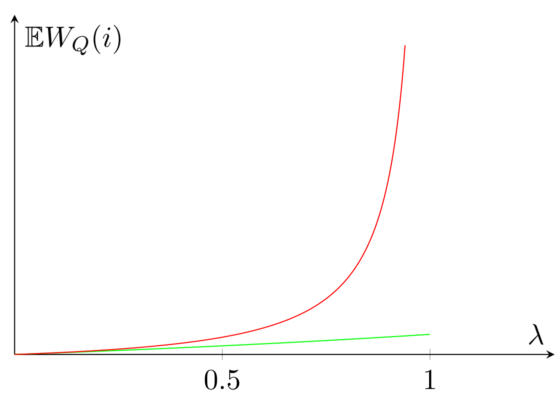
$$\mathbb{E}R = \frac{\lambda \mathbb{E}S^2}{2} = \frac{2}{3}\lambda.$$

Now, using the loads $\rho_1 = {}^1\!/_2\lambda \times {}^1\!/_2 = {}^\lambda\!/_4$, $\rho_2 = {}^1\!/_2\lambda \times {}^3\!/_2 = {}^{3\lambda}\!/_4$, and combining the above, we obtain

$$\begin{aligned} \mathbb{E}W_Q(1) &= \frac{\mathbb{E}R}{1-\rho_1} = \frac{2\lambda/3}{1-\lambda/4} \\ \mathbb{E}W_Q(2) &= \frac{\mathbb{E}R}{(1-\rho_1)(1-\rho_1-\rho_2)} = \frac{2\lambda/3}{(1-\lambda/4)(1-\lambda)} \end{aligned}$$

giving the desired result.

b. See the figure below for a sketch of $\mathbb{E}W_Q(1)$ (green line) and $\mathbb{E}W_Q(2)$ (red line). Both functions are increasing and convex.



For $\lambda \to 1$, it holds that $\mathbb{E}W_Q(1) = {}^8\!/_9 < \infty$, whereas $\mathbb{E}W_Q(2) \to \infty$ for $\lambda \to 1$. Class 1 is only affected by the load of class 1 (ρ_1) and the residual service time; as ρ_1 is strictly smaller than 1 (in fact, at most ${}^1\!/_4$), the waiting time of class 1 remains bounded. Class 2 is affected by the total load $\rho_1 + \rho_2 = \lambda$, which converges to 1 such that the total number of customers tends to grow large.

Exercise 2.

a. The system is stable for $\alpha < \mu$.

b. Let $X(t)$ denote the number of customers at time t. The transition diagram of the birth-and-death process $X(t)_{t\geq 0}$ is given in Figure 1. The distribution of the number of customers in the system follows from the balance equations (for sets): $4\pi(0) = \mu\pi(1)$ and $\alpha\pi(i-1) = \mu\pi(i)$, for $i = 2, 3, \ldots$. The first equation yields $\pi(1) = {}^4\!/\!_\mu\, \pi(0)$, whereas the combination gives

$$\pi(i) = \left(\frac{\alpha}{\mu}\right)^{i-1} \pi(1) = \left(\frac{\alpha}{\mu}\right)^{i-1} \frac{4}{\mu}\pi(0), \qquad i = 1, 2, \ldots.$$

Using normalization, we obtain that

$$\pi(0) = \left[1 + \sum_{i=1}^{\infty} \left(\frac{\alpha}{\mu}\right)^{i-1} \frac{4}{\mu}\right]^{-1} = \left[1 + \frac{4}{\mu - \alpha}\right]^{-1} = \frac{\mu - \alpha}{\mu - \alpha + 4}.$$

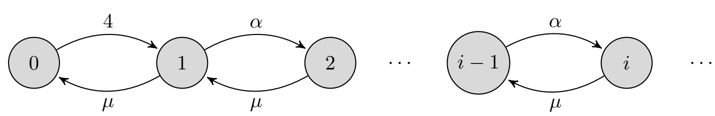


Figure 1: State diagram Exercise 2b.

c. If $\alpha = 0$, then the process is similar to that of an M/M/1/1 queue. Regeneration epochs can be the moments at which the queue becomes empty (just after the service completion)[1]. We have that the expected cycle length is $\mathbb{E}T = \frac{1}{4} + \frac{1}{\mu}$. The costs per cycle is the idle time per cycle, hence $\mathbb{E}[\text{costs per cycle}] = \frac{1}{4}$. Using the renewal reward theorem, we find

$$\mathbb{P}(\text{idle}) = \frac{\mathbb{E}[\text{costs per cycle}]}{\mathbb{E}T} = \frac{1/4}{1/4 + 1/\mu} = \frac{\mu}{4 + \mu}.$$

d. First note that we can use the same regeneration epochs as in part c. Also, we still have that $\mathbb{E}[\text{costs per cycle}] = \frac{1}{4}$. It remains to determine the expected cycle length[2] , which can be found by conditioning on the idle time:

$$\begin{aligned}
\mathbb{E}T &= \int_{u=0}^{t} \left(u + \frac{1}{\mu}\right) \times 4e^{-4u}\mathrm{d}u + \int_{u=t}^{\infty} \left(u + \frac{1}{2\mu}\right) \times 4e^{-4u}\mathrm{d}u \\
&= \int_{u=0}^{\infty} u \times 4e^{-4u}\mathrm{d}u + \frac{1}{\mu}\left(1 - e^{-4t}\right) + \frac{1}{2\mu}e^{-4t} \;=\; \frac{1}{4} + \frac{1}{2\mu}\left(2 - e^{-4t}\right),
\end{aligned}$$

since the integral on the second line equals $1/4$. Hence, by the renewal reward theorem again, we have

$$\mathbb{P}(\text{idle}) = \frac{\mathbb{E}[\text{costs per cycle}]}{\mathbb{E}T} = \frac{1/4}{1/4 + \frac{1}{2\mu}\left(2 - e^{-4t}\right)}.$$

Exercise 3.

a. Let $N(t)_{t\geq 0}$ denote the arrival process of customers (i.e. a Poisson process with rate 10

[1] There are many other regeneration epochs possible in this case.

[2] An alternatigve is to write $\mathbb{E}T = \mathbb{E}A + \mathbb{P}(A \leq t)\frac{1}{\mu} + \mathbb{P}(A > t)\frac{1}{2\mu}$, with A denoting the interarrival time.

starting at time 0). We appy thinning of Poisson processes as follows: for an arrival at time $t \in [0, \tau]$, it is considered to be of type 1 if the service time is larger then $\tau - t$ (denoted as process $N_1(t)_{t\in[0,\tau]}$) and of type 2 otherwise. Then, for $t \in [0, \tau]$, $N_1(t)_{t\in[0,\tau]}$ is a Poisson proces with rate $10e^{-(\tau-t)}$.

Now, the number of customers present at time τ equals $N_1(0, \tau)$, with $N_1(s, t)$ the number of type 1 arrivals during $[s, t]$. Note that $N_1(0, \tau)$ is a Poisson random variable with rate

$$m(\tau) = \int_0^\tau 10e^{-(\tau-t)}\mathrm{d}t = 10\int_0^\tau e^{-t}\mathrm{d}t = 10\left(1 - e^{-\tau}\right),$$

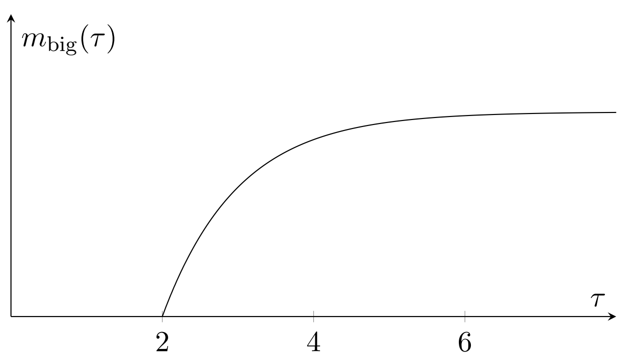
giving the desired result.

b. Observe that a customer arriving at time $t \in [0, \tau]$ is still present at time τ if the service time exceeds $\tau - t$. Hence, we may use the same splitting as in part a. If a customer at time τ is big, it must have arrived before time $\tau - 2$. Thus, the number of big customers at time τ equals $N_1(0, \tau - 2)$, for $\tau \geq 2$. Note that $N_1(0, \tau - 2)$ is a Poisson random variable with rate

$$m_{\text{big}}(\tau) = \int_0^{\tau-2} 10e^{-(\tau-t)}\mathrm{d}t = 10\; e^{-\tau}e^{+t}\big|_{t=0}^{\tau-2} = 10\left(e^{-2} - e^{-\tau}\right),$$

giving the desired result.

c. Please see the figure below for a sketch of $m_{\text{big}}(\tau)$. The figure displays that the queue gradually increases until it reaches its equilibirium, i.e., reflecting the startup of the system.



Exercise 4.

a. The routing equations are $\gamma_1 = \lambda + p_1\gamma_1$ and $\gamma_2 = p_2\gamma_1$. This gives $\gamma_1 = \lambda/(1-p_1)$ and $\gamma_2 = p_2\lambda/(1-p_1)$. The system is stable if $\gamma_1/3 < 1$ and $\gamma_2/3 < 1$. As $\gamma_2 < \gamma_1$, we need that $\lambda/(1-p_1) < 3$ for the system to be stable.

This is a Jackson network and the stationary distribution is thus of product form:

$$\begin{aligned}
\pi(n_1, n_2) &= \left(1 - \frac{\gamma_1}{3}\right)\left(\frac{\gamma_1}{3}\right)^{n_1}\left(1 - \frac{\gamma_2}{3}\right)\left(\frac{\gamma_2}{3}\right)^{n_2} \\
&= \left(1 - \frac{\lambda}{3(1-p_1)}\right)\left(\frac{\lambda}{3(1-p_1)}\right)^{n_1}\left(1 - \frac{p_2\lambda}{3(1-p_1)}\right)\left(\frac{p_2\lambda}{3(1-p_1)}\right)^{n_2}.
\end{aligned}$$

b. Let $X_i(t)$ denote the number of customers at station i at time t. The transition diagram of the Markov process $(X_1(t), X_2(t))_{t\geq 0}$ is depicted in Figure 2. The balance equations are, for $n_2 \geq 1$,

$$\begin{aligned}
(\lambda + 6)\pi(n_1, n_2) &= \lambda\pi(n_1 - 1, n_2) + 3\pi(n_1 + 1, n_2 - 1) + 3\pi(n_1, n_2 + 1), \quad n_1 \geq 1 \\
(\lambda + 6)\pi(0, n_2) &= 6\pi(0, n_2 + 1) + 3\pi(1, n_2 - 1).
\end{aligned}$$

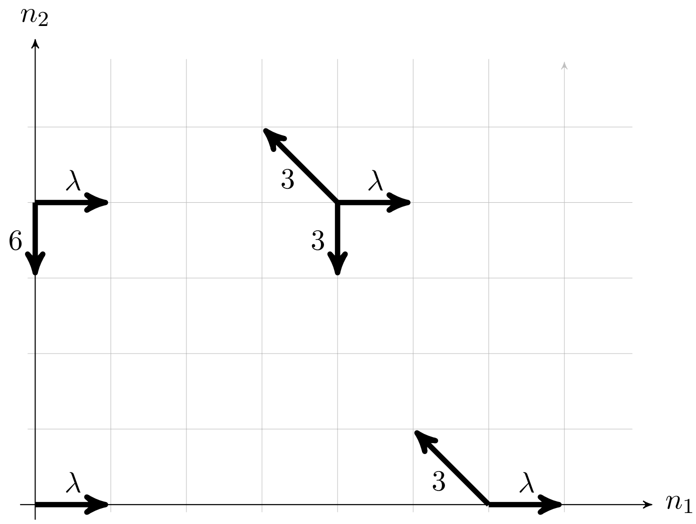


Figure 2: State diagram for Exercise 4b. Only outgoing transitions are shown.

Exercise 5.
a. The three terms of $P(S)$ correspond to the three different sales options: (i) $K - S$ is the deterministic number of items sold to the trader, (ii) $\mathbb{E}\min(D, S)$ is the expected sales that they try to sell themselves, and (iii) $\mathbb{E}(S - D)^+$ are the expected number of unsold items that go to the auction.
b. Assume for now that the demand is continuous and let $F_D(\cdot)$ denote the distribution function. Note that $\mathbb{E}\min(D, S) = S - \mathbb{E}(S - D)^+$. Hence, the expected income can also be written as

$$P(S) = p_2 K - (p_2 - p_1)S + (v - p_1)\mathbb{E}(S - D)^+.$$

Taking derivatives with respect to S yields $P'(S) = p_1 - p_2 + (v - p_1)F_D(S)$, since ${}^{\mathrm{d}}/_{\mathrm{d}S}\,\mathbb{E}(S - D)^+ = F_D(S)$. Setting $P'(S) = 0$ (and noting this provides the unique maximum) gives $F_D(S) = {}^{(p_1 - p_2)}/_{(p_1 - v)}$, or

$$S^* = F_D^{-1}\left(\frac{p_1 - p_2}{p_1 - v}\right).$$

If the demand is discrete, we may use marginal arguments. If the organization tries to sell the Sth item themselves, the expected income is $p_1(1 - F_D(S)) + vF_D(S) = p_1 - (p_1 - v)F_D(S)$. If this items is sold to the trader, the income is p_2. Hence, the organization should try to sell themselves as long as $p_1 - (p_1 - v)F_D(S) \geq p_2$, i.e. S^* is the largest integer that satisfies this equation.
c. If $v > p_2$ it is always more profitable to try to sell themselves, as the value at the auction is also larger than the value obtained from the trader. Hence, $S^* = K$ in this case. Observe that the ${}^{(p_1 - p_2)}/_{(p_1 - v)} > 1$ (if $v \leq p_1$), so that the inverse in that point is not well defined.