2021 H2 Math PROMO Compilation - APGP (2 Qns with Ans)

April 12, 2022

1 [NJC/PROMO/9758/2021/Q1]

A curve $y=\frac{2}{{{x}^{2}}-3}$ undergoes, in succession, the following transformations, where a and b are positive constants. A: A translation of a units in the negative x-direction. B: A reflection about the y-axis. C: A scaling parallel to the y-axis with scale factor of b. The equation of the resultant curve is $y=\frac{6}{{{x}^{2}}-10x+22}$. Find the values of a and b. [3]

SOLUTION

2 $y=\frac{2}{{{x}^{2}}-3}\xrightarrow{A}y=\frac{2}{{{(x+a)}^{2}}-3}\xrightarrow{B}y=\frac{2}{{{(-x+a)}^{2}}-3}$ $\xrightarrow{C x+a)}^{2}}-3}\Rightarrow y=\frac{2b}{{{x}^{2}}-2ax+{{a}^{2}}-3}$ Comparing with$y=\frac{6}{{{x}^{2}}-10x+22},$ $\begin{align} & -2a=-10\Rightarrow a=5 \\ & b=3 \\ \end{align}$

Alternatively, $y=\frac{6}{{{x}^{2}}-10x+22}=\frac{6}{{{(5-x)}^{2}}-3}\Rightarrow \frac{y}{3}=\frac{2}{{{(5-x)}^{2}}-3}$ $\therefore a=5,\text{ }b=3$

3 [NJC/PROMO/9758/2021/Q10]

The curve ${{C}_{1}}$ has equation

\[{{(x-1)}^{2}}-{{(y-4)}^{2}}=1.\]

(i) Sketch ${{C}_{1}},$ labelling clearly the equations of any asymptotes and the coordinates of any vertices and the points where the curve crosses the x- and y-axes. [3]

The curve ${{C}_{2}}$ with equation

\[y=\frac{{{x}^{2}}+ax}{x-1},\]

where a is a constant, has two turning points.

(ii) Find the range of possible values of a, showing your working clearly. [3]

It is further given that ${{C}_{1}}$ does not intersect ${{C}_{2}}.$

(iii) By finding the equation of the oblique asymptote of ${{C}_{2}}$ in terms of a, find the value of a exactly. [2]
(iv) Assuming now that a is the value you have found in part (iii), sketch ${{C}_{2}}$ on the same diagram in part (i), labelling clearly the equations of any asymptotes and the coordinates of any turning points and the points where the curve crosses the x- and y-axes. [3]

SOLUTION

4 (i), (iv)

(ii) At the stationary points of ${{C}_{1}},$ Thus, for ${{C}_{1}}$ to have 2 turning points, $\begin{align} & {{(-2)}^{2}}-4(1)(-a)>0 \\ & 4+4a>0 \\ & a>-1 \end{align}$ (iii) ${{C}_{1}}$ is a hyperbola with vertices lying on the horizontal line \[y=4.\] ${{C}_{2}}:$$y=\frac{{{x}^{2}}+ax}{x-1}=x+(a+1)+\frac{a+1}{x-1}$ Through observation from the sketch in part (i), we see that the only way for the two curves to have no point of intersection is for both curves to share the same asymptote with positive gradient. Thus the oblique asymptote of ${{C}_{1}}$ with positive gradient is \[y-4=(1)(x-1)\Rightarrow y=x+3.\] Hence, \[a+1=3\Rightarrow a=2.\]

Alternatively, sub \[x=1,\] \[y=4\]into the equation$y=x+(a+1)$ to obtain $4=1+(a+1)\Rightarrow a=2$

5 [NJC/PROMO/9758/2021/Q11]

(a) Find $\int{\frac{x}{{{x}^{4}}+6{{x}^{2}}+9}\text{ d}x}.$ [3]

(b) Find $\int{{{x}^{2}}\ln \left(x+2 \right)}\text{ d}x.$ [3]

(c) Using the substitution $u=\frac{1}{x}$, show that

\[\int_{\sqrt{2}}^{2}{\frac{1}{x\sqrt{{{x}^{2}}-2}}}\text{ d}x=\frac{\sqrt{2}}{k}\text{ }\!\!\pi\!\!\text{ }\],

where k is a constant to be determined. [5]

SOLUTION

6 (a) $\int{\frac{x}{{{x}^{4}}+6{{x}^{2}}+9}\text{ d}x}$ $\begin{align} & =\frac{1}{2}\int{\frac{2x}{{{\left({{x}^{2}}+3 \right)}^{2}}}\te d}x} \\ & =\frac{1}{2}\times \frac{{{\left({{x}^{2}}+3 \right)}^{-1}}}{-1}+c \\ & =-\frac{1}{2\left({{x}^{2}}+3 \right)}+c \end{align}$ (b) \[\begin{align} & \int{{{x}^{2}}\ln \left(x+2 \right)}\text{ d}x \\ & =\frac{{{x}^{3}}}{3}\ln \left(x+2 \right)-\int{\frac{{{x}^{3}}}{3}\left(\frac{1}{x+2} \right)}\text{ d}x \\ & =\frac{{{x}^{3}}}{3}\ln \left(x+2 \right)-\frac{1}{3}\int{\frac{{{ d}x \\ & =\frac{{{x}^{3}}}{3}\ln \left(x+2 \right)-\frac{1}{3}\int{{{x}^{2}}-2x+4-\frac{8}{x+2}}\text{ d}x \\ & =\frac{{{x}^{3}}}{3}\ln \left(x+2 \right)-\frac{1}{3}\left[\frac{{{x}^{3}}}{3}-\frac{2{{x}^{2}}}{2}+4x-8\ln \left(x+2 \right) \right]+c \\ & =\frac{{{x}^{3}}}{3}\ \left(x+2 \right)-\frac{1}{3}\left[\frac{{{x}^{3}}}{3}-{{x}^{2}}+4x-8\ln \left(x+2 \right) \right]+c\text{ or} \\ & \frac{{{x}^{3}}}{3}\ln \left(x+2 \right)-\frac{{{x}^{3}}}{9}+\frac{{{x}^{2}}}{3}-\frac{4}{3}x+\frac{8}{3}\ln \left(x+2 \right)+c\text{ or} \\ & \left(\frac{{{x}^{3}}+8}{3} \right)\ln \left(x+2 \right)-\frac{{{x}^{3}}}{9}+\frac{{{x}^{2}}}{3}-\frac{4}{3}x+c\ \text{or} \\ & \frac{1}{9}\left[3\left({{x}^{3}}+8 \right)\ln \left(x+2 \right)-{{x}^{3}}+3{{x}^{2}}-12x \right]+c \end{align}\]

(c) $\text{Let }u=\frac{1}{x}\Rightarrow x=\frac{1}{u}\Rightarrow \frac{\text{d}x}{\text{d}u}=-\frac{1}{{{u}^{2}}}$ $\text{when }x=\sqrt{2},\text{ }u=\frac{\sqrt{2}}{2};\text{ when }x=2,\text{ }u=\frac{1}{2}$ \[\begin{align} & \int_{\sqrt{2}}^{2}{\f 2}}}\text{ d}x=\int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}}{\frac{u}{\sqrt{\frac{1}{{{u}^{2}}}-2}}}\left(-\frac{1}{{{u}^{2}}} \right)\text{ d}u \\ & =\int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}}{\frac{u}{\sqrt{\frac{1-2{{u}^{2}}}{{{u}^{2}}}}}}\left(-\frac{1}{{{u}^{2}}} \right)\text{ d}u \\ & =-\int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}}{\frac{1}{\sqrt{1-2{{u}^{2}}}}}\text{ d}u \\ & =-\frac{1}{\sqrt{2}}\int_{\fr {{\left(\sqrt{2}u \right)}^{2}}}}}\text{ d}u \\ & =-\frac{1}{\sqrt{2}}\left[{{\sin }^{-1}}\left(\sqrt{2}u \right) \right]_{\frac{\sqrt{2}}{2}} \\ & =-\frac{1}{\sqrt{2}}\left[\frac{\text{ }\!\!\pi\!\!\text{ }}{4}-\frac{\text{ }\!\!\pi\!\!\text{ }}{2} \right] \\ & =\frac{1}{\sqrt{2}}\left(\frac{\text{ }\!\!\pi\!\!\text{ }}{4} \right) \\ & =\frac{\sqrt{2}}{8}\text{ }\!\!\pi\!\!\text{ } \end{align}\]

7 [NJC/PROMO/9758/2021/Q12]

In the diagram below, a light source is placed at the point P with coordinates $(-2,\ -4,\ -2).$ A rectangular glass prism is placed such that the top of the prism is closer to point P than the bottom of the prism.

It is given that the top of the prism is a part of the plane with equation

$\mathbf{r}=\left(\begin{matrix} -2 \\ 3 \\ -2 \\ \end{matrix} \right)+\lambda \left(\begin{matrix} 1 \\ 2 \\ -3 \\ \end{matrix} \right)+\mu \left(\begin{matrix} 3 \\ -4 \\ 1 \\ \end{matrix} \right),$

where $\lambda \text{ and }\mu $ are parameters.

(i) Show that this plane has a Cartesian equation of the form $x+y+z=d$ for some constant d to be determined. [3]

A ray of light is sent in direction $3\mathbf{i}+6\mathbf{j}+2\mathbf{k}$ from the light source at P. The light ray enters the prism at point Q which lies on the top of the prism, as shown in the diagram below.

(ii) Find the exact coordinates of Q. [3]

The light ray emerges from the prism at point R with coordinates \[\left(c,\ \frac{53}{11},\,\frac{91}{11} \right),\] as shown in the diagram below.

It is known that the plane PQR is perpendicular to the top of the prism.

(iii) Show that $c=\frac{87}{11}.$ [3]

Snell's Law states that $\sin \theta =k\sin \phi ,$ where k is the refractive index of the prism, is the acute angle between the normal to the top of the prism and PQ, and is the acute angle between the normal to the top of the prism and QR.

(iv) Find the value of k. [3]

(v) Find the exact thickness of the prism measured in the direction of the normal at Q. [2]

SOLUTION

8 (i) Let the plane that contains the top of the prism be ${{p}_{1}}.$ $\begin{align} & \left(\begin{matrix} 1 \\ 2 \\ -3 \\ \end{matrix} \right)\times \left(\begin{matrix} 3 \\ -4 \\ 1 \\ \end{matrix} \right)=\left(\begin{matrix} (2)(1)-(-4)(-3) \\ (-3)(3)-(1)(1) \\ (1)(-4)-(2)(3) \\ \end{matrix} \right) \\ & =\left(\begin{matrix} -10 \\ -10 \\ -10 \\ \end{matrix} \right) \end{align}$ Thus, a vector normal to ${{p}_{1}}$ is $-\frac{1}{10}\left(\begin{matrix} -10 \\ -10 \\ -10 \\ \end{matrix} \right)=\left(\begin{matrix} 1 \\ 1 \\ 1 \\ \end{matrix} \right).$ Hence a Cartesian equation of ${{p}_{1}}$ is $\begin{align} & \mathbf{r}\cdot \left(\begin{matrix} 1 \\ 1 \\ 1 \\ \end{matrix} \right)=\left(

\begin{matrix} -2 \\ 3 \\ -2 \\ \end{matrix} \right)\cdot \left(\begin{matrix} 1 \\ 1 \\ 1 \\ \end{matrix} \right) \\ & =-2+3-2=-1 \\ & x+y+z=-1 \end{align}$ (ii) Since Q lies on l,

\[\overset{\xrightarrow{{}}}{\mathop{OQ}}\,=\left(\begin{matrix} -2+3q \\ -4+6q \\ -2+2q \\ \end{matrix} \right)\] for some \[q\in \mathbb{R}.\]

Since Q lies on ${{p}_{1}},$ \[\begin{align} & (-2+3q)+(-4+6q)+(-2+2q)=-1 \\ & -8+11q=-1 \\ & q=\frac{7}{11} \end{align}\]

Therefore, \[\overset{\xrightarrow{{}}}{\mathop{OQ}}\,=\left(\begin{matrix} -2+3\left(\tfrac{7}{11} \right) \\ -4+6\left(\tfrac{7}{11} \right) \\ -2+2\left(\tfrac{7}{11} \right) \\ \end{matrix} \right)=\left(\begin{matrix} -{1}/{11}\; \\ -{2}/{11}\; \\ -{8}/{11}\; \\ \end{matrix} \right)\]

Therefore, the coordinates of Q are \[\left(-\frac{1}{11},\ -\frac{2}{11},\,-\frac{8}{11} \right).\]

(iii) $\left(\begin{matrix} 3 \\ 6 \\ 2 \\ \end{matrix} \right)\times \left(\begin{matrix} 1 \\ 1 \\ 1 \\ \end{matrix} \right)=\left(\begin{matrix} (6)(1)-(2)(1) \\ (2)(1)-(3)(1) \\ (3)(1)-(6)(1) \\ \end{matrix} \right)=\left(\begin{matrix} 4 \\ -1 \\ -3 \\ \end{matrix} \right)$ Thus $\left(\begin{matrix} 4 \\ -1 \\ -3 \\ \end{matrix} \right)$ is a normal vector to the plane PQR. Therefore, $\overset{\xrightarrow{{}}}{\mathop{QR}}\,\ \cdot \left(\begin{matrix} 4 \\ -1 \\ -3 \\ \end{matrix} \right)=0$ $\begin{align} & \left(\begin{matrix} c-\left(-\tfrac{1}{11} \right) \\ \tfrac{53}{11}-\left(-\tfrac{2}{11} \right) \\ \tfrac{91}{11}-\left(-\tfrac{8}{11} \right) \\ \end{matrix} \right)\cdot \left(\begin{matrix} 4 \\ -1 \\ -3 \\ \end{matrix} \right)=0 \\ & \left(\begin{matrix} c+\tfrac{1}{11} \\ 5 \\ 9 \\ \end{matrix} \right)\cdot \left(\begin{matrix} 4 \\ -1 \\ -3 \\ \end{matrix} \right)=0 \\ & 4c+\tfrac{4}{11}-5-27=0 \\ & 4c=\frac{348}{11} \\ & c=\frac{87}{11} \end{align}$ (iv) \[\begin{align} & \cos \theta =\left| \frac{\left(\begin{matrix} 3 \\ 6 \\ 2 \\ \end{matrix} \right)\cdot \left(\begin{matrix} 1 \\ 1 \\ 1 \\ \end{matrix} \right)}{\sqrt{{{3}^{2}}+{{6}^{2}}+{{2}^{2}}}\sqrt{{{1}^{2}}+{{1}^{2}}+{{1 \right| \\ & =\left| \frac{3+6+2}{7\sqrt{3}} \right| \\ & =\frac{11}{7\sqrt{3}}\Rightarrow \theta ={{\cos }^{-1}}\left(\frac{11}{7\sqrt{3}} \right)=24.870{}^\circ \end{align}\]

\[\begin{align} & \cos \phi =\left| \frac{\left(\begin{matrix} 8 \\ 5 \\ 9 \\ \end{matrix} \right)\cdot \left(\begin{matrix} 1 \\ 1 \\ 1 \\ \end{matrix} \right)}{\sqrt{{{8}^{2}}+{{5}^{2}}+{{9}^{2}}}\sqrt{{{1}^{2}}+{{1}^{2}}+{{1}^{2}}}} \right| \\ & =\left| \frac{8+5+9}{\sqrt{510}} \right|=\frac{22}{\sqrt{510}} \\ & \theta ={{\cos }^{-1}}\left(\frac{22}{\sqrt{510}} \right)=13.049{ \end{align}\]

\[k=\frac{\sin \theta }{\sin \phi }=\frac{\sin 24.870{}^\circ }{\sin 13.049{}^\circ }=1.86\text{ (to 3 s}\text{.f}\text{.)}\]
(v) The thickness of prism is the length of projection of QR onto the normal. Therefore, \[\begin{align} & \text{Thickness of prism}=\left| \frac{\overset{\xrightarrow{{}}}{\mathop{QR}}\,\ \cdot \left(\begin{matrix} 1 \\ 1 \\ 1 \\ \end{matrix} \right)}{\sqrt{{{1}^{2}}+{{1}^{2}}+{{1}^{2}}}} \right| \\ & =\left| \frac{\left(\begin{matrix} 8 \\ 5 \\

9 \\ \end{matrix} \right)\ \cdot \left(\begin{matrix} 1 \\ 1 \\ 1 \\ \end{matrix} \right)}{\sqrt{3}} \right| \\ & =\left| \frac{8+5+9}{\sqrt{3}} \right| \\ & =\frac{22}{\sqrt{3}} \\ & =\frac{22\sqrt{3}}{3}\text{ units} \end{align}\] Alternatively, \[\begin{align} & \text{Thickness of prism}=QR\cos \phi \\ & =\sqrt{{{8}^{2}}+{{5}^{2}}+{{9}^{2}}}\times \frac{22}{\sqrt{510}} \\ & =\sqrt{170}\times \frac{22}{\sqrt{510}} \\ & =\frac{22}{\sqrt{3}} \\ & =\frac{22\sqrt{3}}{3}\text{ units} \end{align}\]

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9 [NJC/PROMO/9758/2021/Q2]

A candy shop is having a Halloween sale. The items that are on promotion are chocolate bars, gummy bears and lollipops. There is a 20% discount for every chocolate bar purchased, a $2 discount for every 3 bags of gummy bears purchased and every 6 lollipops can be purchased at the price of 5 lollipops.

Hannah, Jo and Pete are preparing for a Halloween party. The table below shows the total bill and the number of chocolate bars, the number of bags of gummy bears and the number of lollipops bought from the candy shop. Chocolate Bar Gummy bears Lollipops Total Bill ($) Jo 5 3 36 65.20 Hannah 4 14 24 119.84 Pete 17 5 20 89.12

Calculate the original selling price for each of a chocolate bar, a bag of gummy bears and a lollipop. [4]

SOLUTION

10 Let $x, $y and $z be the cost of a chocolate bar, a bag of gummy bears and a lollipop respectively.

$\begin{align} & 5(0.8)x+(3y-2)+30z=65.20 \\ & 4x+3y+30z=67.20\text{ ----- (1)} \\ & 4(0.8)x+(14y-8)+20z=119.84 \\ & 3.2x+14y+20z=127.84\text{ ----- (2) } \\ & \text{17(0}\text{.8)}x+(5y-2)+17z=89.12\text{ } \\ & \text{ 13}\text{.6}x+5y+17z\t }=91.12\text{ ----- (3) } \end{align}$

From GC, $x=2.70,\text{ }y=6.80\text{ and }z=1.20$ The cost of a chocolate bar, a bag of gummy bears and a lollipop is $2.70, $6.80 and $1.20 respectively.

11 [NJC/PROMO/9758/2021/Q3]

The diagram shows a sketch of the curve $y=\text{f}(x).$ The curve cuts the x-axis at \[\left(0,\ \text{0} \right)\], \[\left(2,\ \text{0} \right)\] and \[\left(4,\ \text{0} \right).\] It has a stationary point \[\left(3,\ -\frac{5}{2} \right)\] and asymptotes with equations \[x=1\] and \[y=2.\]

Sketch, on separate diagrams, the curves with the following equations, stating the equations of any asymptotes and the coordinates of any turning points and any points where the curves cross the x- and y-axes.

(a) $y=\left| \text{f}(x) \right|,$ and [2]

(b) $y=\frac{1}{\text{f}(x)}.$ [3]

SOLUTION

12 (a)

(b)

13 [NJC/PROMO/9758/2021/Q4]

(a) 4 (i) Solve the inequality $20x\ge \frac{17{{x}^{2}}+81x-6}{5-3x},$ leaving your answer in exact form. [4]

(b) (ii) Hence solve the inequality $\frac{20}{x}\ge \frac{17+81x-6{{x}^{2}}}{5{{x}^{2}}-3x}$ exactly. [3]

SOLUTION

14 (i) \[20x\ge \frac{17{{x}^{2}}+81x-6}{5-3x}\text{ and }x\ne \frac{5}{3}\] \[\begin{align} & \frac{17{{x}^{2}}+81x-6}{5-3x}-20x\le 0 \\ & \frac{17{{x}^{2}}+81x-6-20x\left(5-3x \right)}{5-3x}\le 0 \\ & \frac{17{{x}^{2}}+81x-6-100x+60{{x}^{2}}}{5-3x}\le 0 \\ & \frac{77{{x}^{2}}-19x-6}{5-3x}\le 0 \\ & \frac{\left(7x-3 \right)\left(11x+2 \right)}{5-3x}\le 0 \end{align}\]

$-\frac{2}{11}\le x\le \frac{3}{7}\text{ or}~~x>\frac{5}{3}$ (ii) Replacing x with \[\frac{1}{x}\] in the original inequality, \[\begin{align} & 20\left(\frac{1}{x} \right)\ge \frac{17{{\left(\frac{1}{x} \right)}^{2}}+81\left(\frac{1}{x} \right)-6}{5-3\left(\frac{1}{x} \right)}\times \frac{{{x}^{2}}}{{{x}^{2}}} \\ & \frac{20}{x}\ge \frac{17+81x-6{{x}^{2}}}{5{{x}^{2}}-3x} \end{align}\]

Thus, from the first part,

$\begin{align} & -\frac{2}{11}\le \frac{1}{x}\le \frac{3}{7}\text{ or}~\ \frac{1}{x}>\frac{5}{3} \\ & x\le -\frac{11}{2}\text{ or }x\ge \frac{7}{3}\text{ or}~\ 0<x<\frac{3}{5} \\ \end{align}$

15 [NJC/PROMO/9758/2021/Q5]

\[\text{f}(x)=\left\{ \begin{matrix} {{\left(x-2 \right)}^{2}}, & 0\le x<2 \\ 2x-4, & 2\le x\le 4 \\ \end{matrix} \right.\]

and that $\text{f}(x)=\text{f}(x+4)$ for all real values of x. (i) State the value of $\text{f}(21).$ [1]

(ii) Sketch the graph of $y=\text{f}(x)$ for $-6\le x\le 6.$ [2]

The function g is defined by

$\text{g}(x)=\sqrt{x-4},\quad 4\le x\le 20.$

(iii) Find fg in a similar form as f. [4]

SOLUTION

16 (i) $\text{f}(21)=\text{f}(5\times 4+1)=\text{f}(1)={{(1-2)}^{2}}=1.$

(ii)

(iii) When $0\le \text{g}\left(x \right)<2$, $4\le x<8$. This corresponds to $\text{fg}(x)=\text{f}\left(\sqrt{x-4} \right)={{\left(\sqrt{x-4}-2 \right)}^{2}}$. When $2\le \text{g}\left(x \right)\le 4$, $8\le x\le 20$. This corresponds to $\text{fg}(x)=\text{f}\left(\sqrt{x-4} \right)=2\sqrt{x-4}-4$.

Therefore, $\text{fg}(x)=\left\{ \begin{matrix} {{\left(\sqrt{x-4}-2 \right)}^{2}}\text{, } & 4\le x<8 \\ 2\sqrt{x-4}-4,\text{ } & 8\le x\le 20 \\ \end{matrix} \right.$

17 [NJC/PROMO/9758/2021/Q6]

A curve ${{C}_{1}}$ has equation

\[{{x}^{2}}+2{{y}^{2}}=100\]

and a curve ${{C}_{2}}$ has parametric equations

\[x=2{{\text{e}}^{-t}}-4{{\text{e}}^{2t}},\text{ }y=3{{\text{e}}^{-t}}+{{\text{e}}^{2t}}.\] (i) On the same diagram, sketch ${{C}_{1}}$ and ${{C}_{2}},$ labelling the coordinates of the points where both curves cross the x- and y-axes. [5]

(ii) Show that ${{C}_{2}}$ has a Cartesian equation of the form ${{\left(ax+by \right)}^{2}}\left(cx+dy \right)=k$

for some integer constants a, b, c, d and k to be determined. [3]

SOLUTION

18 (i) For ${{C}_{2}},$ when \[x=0,\]

\[\begin{align} & 2{{\text{e}}^{-t}}-4{{\text{e}}^{2t}}=0 \\ & 4{{\text{e}}^{2t}}=2{{\text{e}}^{-t}} \\ & {{\text{e}}^{3t}}=\frac{1}{2} \\ & t=\frac{1}{3}\ln \frac{1}{2}=-\frac{1}{3}\ln 2 \end{align}\]

Therefore,

\[y=3{{\text{e}}^{\frac{1}{3}\ln 2}}+{{\text{e}}^{-\frac{2}{3}\ln 2}}=4.41\text{ (to 3 s}\text{.f}\text{.)}\]

(ii) \[x=2{{\text{e}}^{-t}}-4{{\text{e}}^{2t}}(1)\] \[y=3{{\text{e}}^{-t}}+{{\text{e}}^{2t}}(2)\] $2\times (2)-3\times (1):2y-3x=14{{\text{e}}^{2t}}(3)$ $\begin{align} & 4\times (2)+1\times (1):4y+x=14{{\text{e}}^{-t}} \\ & {{\text{e}}^{t}}=\frac{14}{x+4y} \\ \end{align}$ Substituting (4) into (3), $2y-3x=14{{\left(\frac{14}{x+4y} \right)}^{2}}$ ${{\left(x+4y \right)}^{2}}\left(2y-3x \right)=2744$

19 [NJC/PROMO/9758/2021/Q7]

Due to intense rainfall, the Bukit Teemah canal is often filled to the brim, which causes the surrounding areas to be prone to flooding. The Ministry of Environment is looking into redesigning the canal to improve the flow of water by maximising the cross-sectional area, $A\text{ }{{\text{m}}^{2}},$of the canal.

The cross-section of the canal has sides of fixed lengths CD FG IJ 4 m, DE HI 3 m and EF GH 0.5 m. Also, the vertical depth DL IK s m and \[\angle CDL=\angle JIK=\theta \] radians.

(i) Show that \[A=40\cos \theta +8\sin 2\theta +2\]. [2]

(ii) Use differentiation to find the value of which gives a maximum value of A. [4]

The National Water Agency conducts regular inspections on the water quality in the canal. During one such inspection, an officer transfers water from the canal into a plastic container (as shown in the diagram below) at a constant rate of 162$\text{c}{{\text{m}}^{3}}$ per second.

The plastic container is in the shape of a hollow circular cone with fixed radius 15 cm and fixed height 35 cm. After t seconds, the depth of water in the container is h cm and the top surface of the water has a radius of r cm.

(iii) Find the rate at which h is increasing at the instant when h 21. [4]

SOLUTION

20 (i) \[A=\left(3+4+3 \right)s+4\left(0.5 \right)+2\left(\frac{1}{2} \right)\left(4 \right)\left(s \right)\sin \theta \] \[=10s+2+4s\sin \theta \] $\text{Since }\cos \theta =\frac{s}{4}\Rightarrow s=4\cos \theta ,$ $\begin{align} & A=10\left(4\cos \theta \right)+2+4\sin \theta \left(4\cos \theta \right) \\ & =40\cos \theta +8\left(2\sin \theta \cos \theta \right)+2 \\ & =40\cos \theta +8\sin 2\theta +2 \end{align}$ (ii) $\frac{\text{d}A}{\text{d}\theta }=40\left(-\sin \theta \right)+8\left(\cos 2\theta \right)\left(2 \right)$ $=16\cos 2\theta -40\sin \theta $

$\text{For }A\text{ to be maximum, }\frac{\text{d}A}{\text{d}\theta }=0,$ $\begin{align} & 16\cos 2\theta -40\sin \theta =0 \\ & 16\left(1-2{{\sin }^{2}}\theta \right)-40\sin \theta =0 \\ & 32{{\sin }^{2}}\theta +40\sin \theta -16=0 \\ & 4{{\sin }^{2}}\theta +5\sin \theta -2=0 \\ \end{align}$ $\begin{align} & \sin \theta =\frac{-5\pm \sqrt{25-4\left(4 \right)\left(-2 \right)}}{8} \\ & =\frac{-5\pm \sqrt{57}}{8} \end{align}$ $\begin{align} & \text{Since }0<\theta <\frac{\text{ }\!\!\pi\!\!\text{ }}{2},\text{ }\sin \theta >0, \\ & \therefore \sin \theta =\frac{-5+\sqrt{57}}{8}=0.31873 \\ \end{align}$ $\theta =0.32439$

$\begin{align} & \frac{{{\text{d}}^{2}}A}{\text{d}{{\theta }^{2}}}=16\left(-\sin 2\theta \right)\left(2 \right)-40\cos \theta \\ & =-32\sin 2\theta -40\cos \theta \end{align}$

$\begin{align} & \text{When }\theta =0.32439, \\ & \frac{{{\text{d}}^{2}}A}{\text{d}{{\theta }^{2}}}=-32\sin \left(0.64878 \right)-40\cos \left(0.32439 \right) \\ & =-57.249<0 \end{align}$

Thus, A is a maximum when $\theta =0.324\ \text{(to 3 s}\text{.f}\text{.)}\text{.}$

(iii) Let the volume of water in the container be V$\text{c}{{\text{m}}^{3}}.$ $\begin{align} & V=\frac{1}{3}\text{ }\!\!\pi\!\!\text{ }{{r}^{2}}h \\ & \text{Using similar triangles, }\frac{r}{h}=\frac{15}{35},\text{ } \\ & \therefore r=\frac{3}{7}h \\ \end{align}$ $\begin{align} & V=\frac{1}{3}\text{ }\!\!\pi\!\!\text{ }{{\left(\frac{3}{7}h \right)}^{2}}h \\ & =\frac{3\text{ }\!\!\pi\!\!\text{ }}{49}{{h}^{3}} \\ & \frac{\text{d}V}{\text{d}h}=\frac{3\text{ }\!\!\pi\!\!\text{ }}{49}\left(3 \right){{h}^{2}} \\ & =\frac{9\text{ }\!\!\pi\!\!\text{ }}{49}{{h}^{2}} \end{align}$ \[\begin{align} & \frac{\text{d}h}{\text{d}t}=\frac{\text{d}h}{\text{d}V} \frac{\text{d}V}{\text{d}t} \\ & =\frac{49}{9\text{ }\!\!\pi\!\!\text{ }{{h}^{2}}}\left(162 \right) \\ & =\frac{882}{\text{ }\!\!\pi\!\!\text{ }{{h}^{2}}} \end{align}\] \[\begin{align} & \text{When }h=21, \\ & \frac{\text{d}h}{\text{d}t}=\frac{882}{\text{ }\!\!\pi\!\!\text{ }{{\left(21 \right)}^{2}}}=0.637\text{ or }\frac{2}{\text{ }\!\!\pi\!\!\text{ }} \\ \end{align}\] h is increasing at a rate of 0.637 or $\frac{2}{\text{ }\!\!\pi\!\!\text{ }}\text{ cm per second}\text{.}$

21 [NJC/PROMO/9758/2021/Q8]

(a) A curve C has parametric equations

$x=2\theta -\sin 2\theta ,\text{ }y=1-\cos 2\theta ,$ for $0\le \theta \le \frac{\text{ }\!\!\pi\!\!\text{ }}{2}$.

Find the exact area of the region bounded by C, the line $x=\text{ }\!\!\pi\!\!\text{ }$ and the x-axis. [5]

(b) The region bounded by the curve $y=\tan x$, the line $y=1$ and the y-axis is rotated about the x-axis through $2\text{ }\!\!\pi\!\!\text{ }$ radians. Find the exact volume of the solid formed. [5]

SOLUTION

22 (a) Area of the bounded region \[\begin{align} & =\int_{0}^{\text{ }\!\!\pi\!\!\text{ }}{y}\text{ d}x \\ & =\int_{0}^{\frac{\text{ }\!\!\pi\!\!\text{ }}{2}}{\left(1-\cos 2\theta \right)\left(\frac{\text{d}x}{\text{d}\theta } \right)}\text{ d}\theta \\ & =\int_{0}^{\frac{\text{ }\!\!\pi\!\!\text{ }}{2}}{\left(1-\cos 2\theta \right)\left(2-2\cos 2\theta \right)}\text{ d}\theta \\ & =2\int_{0}^{\frac{\text{ }\!\!\pi\!\!\text{ }}{2}}{{{\left(1-\cos 2\theta \right)}^{2}}}\text{ d}\theta \\ & =2\int_{0}^{\frac{\text{ }\!\!\pi\!\!\text{ }}{2}}{1-2\cos 2\theta +{{\cos }^{2}}2\theta }\text{ d}\theta \\ & =\int_{0}^{\frac{\text{ }\!\!\pi\!\!\text{ }}{2}}{2-4\cos 2\theta +\left(\cos 4\theta +1 \right)}\text{ d}\theta \\ & =\int_{0}^{\frac{\text{ }\!\!\pi\!\!\text{ }}{2}}{3-4\cos 2\theta +\cos 4\theta }\text{ d}\theta \\ & \text{=}\left[3\theta -2\sin 2\theta +\frac{1}{4}\sin 4\theta \right]_{0}^{\frac{\text{ }\!\!\pi\!\!\text{ }}{2}} \\ & =\left[\frac{\text{3 }\!\!\pi\!\!\text{ }}{2}-\sin \text{ }\!\!\pi\!\!\text{ }+\frac{1}{4}\sin 2\text{ }\!\!\pi\!\!\text{ } \right]=\frac{3\text{ }\!\!\pi\!\!\text{ }}{2}\text{ unit}{{\text{s}}^{2}} \end{align}\]

(b) When $\tan x=1,$ $x=\frac{\text{ }\!\!\pi\!\!\text{ }}{4}$ Volume of solid formed \[\begin{align} & =\text{ }\!\!\pi\!\!\text{ }\left({{1}^{2}} \right)\left(\frac{\text{ }\!\!\pi\!\!\text{ }}{4} \right)-\text{ }\!\!\pi\!\!\text{ }\int_{0}^{\frac{\text{ }\!\!\pi\!\!\text{ }}{4}}{{{y}^{2}}}\text{ d}x \\ & =\frac{{{\text{ }\!\!\pi\!\!\text{ }}^{2}}}{4}-\text{ }\!\!\pi\!\!\text{ }\int_{0}^{\frac{\text{ }\!\!\pi\!\!\text{ }}{4}}{{{\tan }^{2}}x}\text{ d}x \\ & =\frac{{{\text{ }\!\!\pi\!\!\text{ }}^{2}}}{4}-\text{ }\!\!\pi\!\!\text{ }\int_{0}^{\frac{\text{ }\!\!\pi\!\!\text{ }}{4}}{{{\sec }^{2}}x-1}\text{ d}x \\ & =\frac{{{\text{ }\!\!\pi\!\!\text{ }}^{2}}}{4}-\text{ }\!\!\pi\!\!\text{ }\left[\tan x-x \right]_{0}^{\frac{\text{ }\!\!\pi\!\!\text{ }}{4}} \\ & =\frac{{{\text{ }\!\!\pi\!\!\text{ }}^{2}}}{4}-\text{ }\!\!\pi\!\!\text{ }\left[\tan \frac{\text{ }\!\!\pi\!\!\text{ }}{4}-\frac{\text{ }\!\!\pi\!\!\text{ }}{4} \right] \\ & =\frac{{{\text{ }\!\!\pi\!\!\text{ }}^{2}}}{2}-\text{ }\!\!\pi\!\!\text{ unit}{{\text{s}}^{3}} \\ \end{align}\]

23 [NJC/PROMO/9758/2021/Q9]

Relative to the origin O, the points A, B and C have position vectors \[2\mathbf{u}-3\mathbf{v},\]\[\mathbf{u}+2\mathbf{v}\]ar \[3\mathbf{u}-2\mathbf{v}\]respectively, where u and v are two non-zero and non-parallel vectors. (i) Show that the area of triangle ABC is given by $3\left| \mathbf{u}\times \mathbf{v} \right|.$ [3]

It is given that $\mathbf{u}=2\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $\mathbf{v}=3\mathbf{i}-4\mathbf{j}-p\mathbf{k},$ where p is a constant.

(ii) If the area of triangle ABC is $15\sqrt{53}$ square units, find the possible values of p exactly. [4]

(iii) It is given that the angle between vectors u and v is acute. Find this angle, giving your answer in degrees. Explain why there is only one answer. [3]

SOLUTION

24 (i) $\overset{\xrightarrow{{}}}{\mathop{AB}}\,=\left(\mathbf{u}+2\mathbf{v} \right)-\left(2\mathbf{u}-3\mathbf{v} \right)=-\mathbf{u}+5\mathbf{v}$ $\overset{\xrightarrow{{}}}{\mathop{AC}}\,=\left(3\mathbf{u}-2\mathbf{v} \right)-\left(2\mathbf{u}-3\mathbf{v} \right)=\mathbf{u}+\mathbf{v}$ Therefore, area of triangle ABC $\begin{align} & =\frac{1}{2}\left| \overset{\xrightarrow{{}}}{\mathop{AB}}\,\ \times \overset{\xrightarrow{{}}}{\mathop{AC}}\, \right| \\ & =\frac{1}{2}\left| \left(-\mathbf{u}+5\mathbf{v} \right)\ \times \left(\mathbf{u}+\mathbf{v} \right)\ \right| \\ & =\frac{1}{2}\left| -\mathbf{u}\times \mathbf{u}+5\mathbf{v}\times \mathbf{u}-\mathbf{u}\times \mathbf{v}+5\mathbf{v}\times \mathbf{v} \right| \\ & =\frac{1}{2}\left| \mathbf{0}-5\mathbf{u}\times \mathbf{v}-\mathbf{u}\times \mathbf{v}+\mathbf{0} \right| \\ & =\frac{1}{2}\left| -6\mathbf{u}\times \mathbf{v} \right| \\ & =\frac{6}{2}\left| \mathbf{u}\times \mathbf{v} \right| \\ & =3\left| \mathbf{u}\times \mathbf{v} \right|\text{ (shown)} \end{align}$ (ii) $\mathbf{u}\times \mathbf{v}$ $\begin{align} & =\left(\begin{matrix} 2 \\ -1 \\ 2 \\ \end{matrix} \right)\times \left(\begin{matrix} 3 \\ -4 \\ -p \\ \end{matrix} \right) \\ & =\left(\begin{matrix} \left(-1 \right)\left(-p \right)-\left(2 \right)\left(-4 \right) \\ \left(2 \right)\left(3 \right)-\left(2 \right)\left(-p \right) \\ \left(2 \right)\left(-4 \right)-\left(-1 \right)\left(3 \right) \\ \end{matrix} \right) \\ & =\left(\begin{matrix} p+8 \\ 2p+6 \\ -5 \\ \end{matrix} \right) \end{align}$

Since the area of triangle ABC is $15\sqrt{53}$ square units, $\begin{align} & 3\sqrt{{{(p+8)}^{2}}+{{(2p+6)}^{2}}+{{5}^{2}}}= \\ & {{(p+8)}^{2}}+{{(2p+6)}^{2}}+{{5}^{2}}=25\times 53 \\ & 5{{p}^{2}}+40p-1200=0 \\ & {{p}^{2}}+8p-240=0 \\ & (p+20)(p-12)=0 \end{align}$ Therefore, $p=-20$ or $p=12.$ (iii) $\mathbf{u}\cdot \mathbf{v}=\left(\begin{matrix} 2 \\ -1 \\ 2 \\ \end{matrix} \right)\cdot \left(\begin{matrix} 3 \\ -4 \\ -p \\ \end{matrix} \right)=10-2p$ For $p=-20,$ $\mathbf{u}\cdot \mathbf{v}=\underline{10-2(-20)}=50\ \underline{>0}$ For $p=12,$ $\mathbf{u}\cdot \mathbf{v}=\underline{10-2(12)}=-14\ \underline{<0}$

Therefore, $p=-20$ is the only possible case for the angle between u and v to be acute. Let this angle be .

\[\begin{align} & \cos \theta =\frac{\left(\begin{matrix} 2 \\ -1 \\ 2 \\ \end{matrix} \right)\cdot \left(\begin{matrix} 3 \\
-4 \\ 20 \\ \end{matrix} \right)}{\sqrt{{{2}^{2}}+{{1}^{2}}+{{2}^{2}}}\sqrt{{{3}^{2}}+{{4}^{2}}+{{20}^{2}}}} \\ & =\frac{50}{3\sqrt{425}}=
\\ & \theta =36.1{}^\circ \text{ (to 1 d}\text{.p}\text{.)} \end{align}\]