1 Vectors

1.1 Dot Product

The dot product is defined as

$$\vec{A} \cdot \vec{B} = AB\cos(\theta) = A_1B_1 + A_2B_2 + A_3B_3 + \cdots \tag{1.1.1}$$

where θ is the angle between $\vec{A}$ and $\vec{B}$ when drawn tail to tail

$$\begin{aligned}\vec{A} \cdot \vec{B} =& A \text{ times the projections of } \vec{B} \text{ onto } \vec{A}\\ =& A \text{ times the projections of } \vec{A} \text{ onto } \vec{B}\end{aligned}$$

1.2 Cross Product (Vector Product)

The cross product is defined as

$$\vec{C} = \vec{A} \times \vec{B} = AB\sin(\theta) \tag{1.2.1}$$

and is non-commutative.

1.2.1 Right Hand Rule

The resultant vector $\vec{C}$ from crossing $\vec{A}$ (thumb) with $\vec{B}$ (index) is the direction of the palm (middle)

2 Kinematics

2.1 Velocity and Acceleration

$$\vec{v} = \frac{dr}{dt} = \frac{d(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{dt} \tag{2.1.1}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{\mathbf{i}} + \frac{dv_y}{dt}\hat{\mathbf{j}} + \frac{dv_z}{dt}\hat{\mathbf{k}} \tag{2.1.2}$$

$$= \frac{d^2\vec{r}}{dt^2} \tag{2.1.3}$$

2.2 Kinematics Equations

$$\vec{r} = \vec{r_0} + \vec{v_0}t + \frac{1}{2}\vec{a}t^2 \tag{2.2.1}$$

$$\vec{r} = \vec{r_0} + \frac{1}{2}(\vec{v_0} + \vec{v})t \tag{2.2.2}$$

$$\vec{v} = \vec{v_0} + \vec{a}t \tag{2.2.3}$$

$$\vec{v}^{\,2} = \vec{v_0}^{\,2} + 2\vec{a} \cdot \Delta\vec{x} \tag{2.2.4}$$

3 Polar Coordinates

$$r = \sqrt{x^2 + y^2} \tag{3.0.1}$$

$$\theta = \arctan(\frac{y}{x}) \tag{3.0.2}$$

$$x = r\cos(\theta) \tag{3.0.3}$$

$$y - r\sin(\theta) \tag{3.0.4}$$

Defining the unit vectors,

$$\hat{\mathbf{r}}(\theta) = \cos(\theta)\hat{\mathbf{i}} + \sin(\theta)\hat{\mathbf{j}} \tag{3.0.5}$$

$$\hat{\boldsymbol{\theta}}(\theta) = -\sin(\theta)\hat{\mathbf{i}} + \cos(\theta)\hat{\mathbf{j}} \tag{3.0.6}$$

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}} \tag{3.0.7}$$

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\,\dot{\theta}\hat{\mathbf{r}} \tag{3.0.8}$$

Giving velocity and acceleration in polar coordinates:

$$\vec{v} = \dot{r}\hat{\mathbf{r}} + r\,\dot{\theta}\hat{\boldsymbol{\theta}} \tag{3.0.9}$$

$$\vec{a} = (\ddot{r} - r(\dot{\theta})^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}} \tag{3.0.10}$$

3.1 Unform Circular Motion

$$\dot{\vec{r}} = 0 \tag{3.1.1}$$

$$\ddot{\vec{\theta}} = 0 \tag{3.1.2}$$

$$\vec{v} = \omega r\,\hat{\boldsymbol{\theta}} \tag{3.1.3}$$

$$\vec{a} = -\omega r^2\hat{\mathbf{r}} = -\frac{v^2}{r}\hat{\mathbf{r}} \tag{3.1.4}$$

4 Taylor Series

General Form:

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots \tag{4.0.1}$$

$$f(a+x) = f(a) + f'(a)x + f'(a)\frac{x^2}{2!} + \cdots \tag{4.0.2}$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots = x \tag{4.0.3}$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots = 1 - \frac{1}{2}x^2 \tag{4.0.4}$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots \tag{4.0.5}$$

$$\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + \cdots \tag{4.0.6}$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots \tag{4.0.7}$$

Differentials

$$\Delta f = f(x + \Delta x) - f(x) \approx f'(x)\Delta x \tag{4.0.8}$$

$$\frac{df}{dx} \approx \frac{\Delta f}{\Delta x} \tag{4.0.9}$$

5 Newton's Laws

1. Inertial systems exist
2. $\vec{F} = m\vec{a}$
3. $\vec{F_{ba}} = -\vec{F_{ab}}$

5.1 Ficticious Forces

When $\vec{R}$ is the vector from the origin of an inertial system to a new system,

$$\vec{r'} = \vec{r} - R \tag{5.1.1}$$

$$\vec{F}_{\text{apparent}} = \vec{F}_{\text{true}} - M\ddot{\vec{R}} \tag{5.1.2}$$

$$\vec{F}_{\text{apparent}} = \vec{F}_{\text{true}} + \vec{F}_{\text{fictitious}} \tag{5.1.3}$$

5.2 Problem Solving Steps

1. Draw force diagrams for each mass
2. Set up coordinates
3. Write equations of motion ($\sum F = Ma$)
4. Write down constraints

5.2.1 Constraints

1. Rope length does not change
2. Mass of rope is 0
3. Normal force forms third-law pair
4. Direction of motion

6 Forces

6.1 Gravitational Force

$$\vec{F_{ba}} = -\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ba} = +\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ab} = -\vec{F_{ab}} \tag{6.1.1}$$

6.1.1 Shell Theorem

Gravitational force of a uniform think spherical shell of mass M and radius R experiences

1. A force equivalent to that if all mass were concentrated in the center, if $r > R$
2. No force if $r < R$

6.1.2 Acceleration

$$\vec{g} = -\frac{GM_e}{R_e^2}\hat{\mathbf{r}} \approx 9.8\,\mathrm{m/s^2} \tag{6.1.2}$$

6.1.3 Weight

$$\vec{W} = -G\frac{M_e m}{R_e^2}\hat{\mathbf{r}} \tag{6.1.3}$$

$$\vec{W} = m\vec{g} \tag{6.1.4}$$

6.2 Electrostatic Force

$$\vec{F}_{ba} = k\frac{q_aq_b}{r^2}\hat{\mathbf{r}}_{ba} \tag{6.2.1}$$

6.3 Frictional Force

For bodies not in relative motion (static):

$$0 \leq f \leq \mu N$$

For bodies in relative motion (kinetic):

$$f = \mu N$$

6.4 Viscosity

$$\vec{F}_v = -C\vec{v} \tag{6.4.1}$$

$$m\frac{dv}{dt} = -Cv \tag{6.4.2}$$

$$\frac{dv}{dt} = -\frac{1}{\tau}v \tag{6.4.3}$$

$$v = v_0 e^{-\frac{t}{\tau}} \tag{6.4.4}$$

$\tau = \frac{m}{C}$ is a characteristic time of the system, such that after a time τ, the velocity will drop by a factor of $\frac{1}{e} \approx 0.37$

The body only travels a distance $v_0\tau$

7 Equilibrium

$$F_n = 0 \tag{7.0.1}$$

$$\tau = 0 \tag{7.0.2}$$

8 Simple Harmonic Motion

Equation of motion is the following 2nd order differential equation

$$M\frac{d^2x}{dt^2} = -kx \tag{8.0.1}$$

$$\frac{d^2x}{dt^2} + \frac{k}{M}x = 0 \tag{8.0.2}$$

$$\omega = \sqrt{\frac{k}{m}} \tag{8.0.3}$$

8.1 Hooke's Law

$$F_s = -k(x - x_0)$$

9 Momentum

Newton's 2nd law using momentum

$$\vec{F} = M\vec{a} \tag{9.0.1}$$

$$\vec{F} = \frac{d}{dt}(M\vec{v}) \tag{9.0.2}$$

$$\vec{F} = \frac{d\vec{P}}{dt} \tag{9.0.3}$$

Dynamics of a system of particles

$$\vec{F}_j = \frac{d\vec{p}_j}{dt} \tag{9.0.4}$$

$$\vec{F}_j^{int} + \vec{F}_j^{ext} = \frac{d\vec{p}_j}{dt} \tag{9.0.5}$$

$$\sum_{j=1}^{N} \vec{F}_j^{int} + \sum_{j=1}^{N} \vec{F}_j^{ext} = \sum_{j=1}^{N} \frac{d\vec{p}_j}{dt} \tag{9.0.6}$$

$$\vec{F}_{ext} = \sum_{j=1}^{N} \frac{d\vec{p}_j}{dt} = \frac{d\vec{P}}{dt} \tag{9.0.7}$$

9.1 Center of Mass

$$\vec{F} = \frac{d\vec{P}}{dt} \tag{9.1.1}$$

$$\vec{F} = M\ddot{\vec{R}} = \sum_{j=1}^{N} m_j \ddot{\vec{r}}_j \tag{9.1.2}$$

$$\vec{R} = \frac{1}{M} \sum_{j=1}^{N} m_j \vec{r}_j \tag{9.1.3}$$

Center of mass of an extended body

$$\vec{R} = \frac{1}{M} \int_V \vec{r}\, dm \tag{9.1.4}$$

$$\vec{R} = \frac{1}{M} \int_V \vec{r} \rho \, dV \tag{9.1.5}$$

9.2 Conservation of Momentum

For an isolated system,

$$\vec{F} = \frac{d\vec{P}}{dt} = 0$$

9.3 Impulse

$$\int_0^t \vec{F}\,dt = \vec{P}(t) - \vec{P}(0) \tag{9.3.1}$$

$$\vec{I} = \Delta\vec{P} \tag{9.3.2}$$

9.4 Rockets

For a rocket of mass M moving at velocity $\vec{v}$ expelling a mass of Δm at a relative velocity $\vec{u}$

$$\vec{P}(t) = (M + \Delta m)\vec{v} \tag{9.4.1}$$

$$\vec{P}(t + \Delta t) = M(\vec{v} + \Delta\vec{v}) + \Delta m(\vec{v} + \Delta\vec{v} + \vec{u}) \tag{9.4.2}$$

$$\Delta\vec{P} = M\Delta\vec{v} + \Delta m(\Delta\vec{v} + \vec{u}) \tag{9.4.3}$$

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}}{dt} + \vec{u}\frac{dm}{dt} \tag{9.4.4}$$

$$0 = M\frac{d\vec{v}}{dt} - \vec{u}\frac{dM}{dt} \tag{9.4.5}$$

$$\Delta\vec{v} = -\vec{u}\ln(\frac{M_0}{M_f}) \tag{9.4.6}$$

9.5 Momentum Flow

For one droplet moving with velocity v and stopping at your hand,

$$I = \int F\,dt \tag{9.5.1}$$

$$I = \Delta p \tag{9.5.2}$$

$$I = m(v_f - v) \tag{9.5.3}$$

$$I = -mv \tag{9.5.4}$$

Therefore

$$I_{hand} = mv$$

If the droplets are separated by distance l and T is the time between collisions,

$$F_{av}T = I = mv \tag{9.5.5}$$

$$F_{av} = \frac{mv}{T} = \frac{mv^2}{l} \tag{9.5.6}$$

Extending to a constant flow rate where $[\rho] = \text{kg/m}^3$

$$\dot{\vec{P}} = \rho v^2 A\hat{\mathbf{v}} \tag{9.5.7}$$

The flux density $\vec{J}$ is defined as

$$\vec{J} = \rho v^2 \hat{\mathbf{v}}$$

Therefore,

$$\dot{\vec{P}} = (\vec{J} \cdot \vec{A})\hat{\mathbf{v}}$$

10 Energy

Deriving the Work-Energy Theorem

$$F(x) = m\frac{dv}{dt} = \tag{10.0.1}$$

$$\int_{x_a}^{x_b} F(x)\,dx = m\int_{x_a}^{x_b} \frac{dv}{dt}dx \tag{10.0.2}$$

$$= m\int_{t_a}^{t_b} \frac{dv}{dt}v\,dt \tag{10.0.3}$$

$$= m\int_{t_a}^{t_b} \frac{d}{dt}(\frac{1}{2}v^2)dt \tag{10.0.4}$$

$$= \frac{1}{2}mv^2\Big|_{t_a}^{t_b} \tag{10.0.5}$$

$$W_{ba} = K_b - K_a = \int_{r_a}^{r_b} \vec{F}\cdot dr = \int_{t_a}^{t_b} \vec{F}\cdot\vec{v}\,dt \tag{10.0.6}$$

10.1 Power

$$\frac{\Delta W}{\Delta t} = \vec{F} \cdot \frac{\Delta \vec{r}}{\Delta t} \tag{10.1.1}$$

$$\frac{dW}{dt} = \vec{F} \cdot \vec{v} \tag{10.1.2}$$

10.2 Conservative Forces

Total mechanical energy does not change, given that only conservative forces act on the system

$$\int_{r_a}^{r_b} \vec{F} \cdot d\vec{r} = -U(\vec{r}_b) + U(\vec{r}_a) = K_b - K_a \tag{10.2.1}$$

$$K_a + U_a = K_b + U_b \tag{10.2.2}$$

Additionally,

$$\oint \vec{F} \cdot d\vec{r} = 0$$

10.3 Potential Energy and Force

$$F(x) = -\frac{dU}{dx} \tag{10.3.1}$$

10.4 Non-Conservative Forces

$$\vec{F} = \vec{F}^c + \vec{F}^{nc} \tag{10.4.1}$$

$$W_{ba}^{tot} = -U_b + U_a + W_{ba}^{nc} \tag{10.4.2}$$

$$-U_b + U_a + W_{ba}^{nc} = K_b - K_a \tag{10.4.3}$$

$$(K_b + U_b) - (K_a + U_a) = W_{ba}^{nc} \tag{10.4.4}$$

$$\Delta E = W_{ba}^{nc} \tag{10.4.5}$$