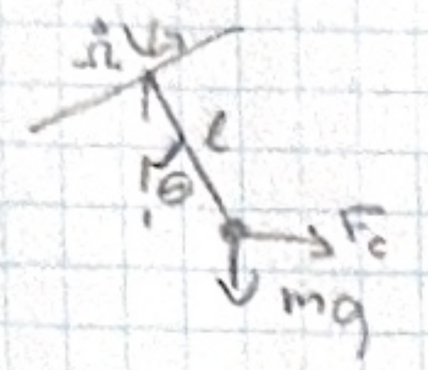
Problem Set 09

(9.12) 1.




$$\boxed{\omega = \sqrt{\frac{g}{l} - \Omega^2}}$$

$$F_c = Mr\Omega^2 = Ml\Omega^2\theta$$
$$\tau = Mgl\sin\theta - F_c l\cos\theta$$
$$\tau = Mgl\theta - Ml^2\Omega^2\theta$$
$$-Ml\ddot{\theta} = Mgl\theta - Ml^2\Omega^2\theta$$
$$-\ddot{\theta} = \left(\frac{g}{l} - \Omega^2\right)\theta$$

(10.3) 2.

$$F = \frac{-k}{r^3} \qquad u(r) = \int_\infty^r \frac{-k}{r^3}\,dr = \frac{-c}{r^2}$$

Radial motion uniform

$\Downarrow$

Radial Force 0

$\Downarrow$

$u_{eff} = 0$

$$u_{eff} = \frac{L^2}{2mr^2} - \frac{c}{r^2} = \frac{1}{r^2}\left(\frac{L^2}{2m} - c\right)$$
$$u_{eff} = 0 \rightarrow c = \frac{L^2}{2m}$$
$$L^2 = (mr^2\dot{\theta})^2 = 2cm$$
$$\dot{\theta} = \sqrt{\frac{2c}{m}}\frac{1}{r^2}$$

uniform radial velocity

$$\int d\theta = \sqrt{\frac{2c}{m}}\int\frac{dt}{r^2} = \sqrt{\frac{2c}{m}}\int\frac{dt}{dr}\frac{dr}{r^2}$$
$$\boxed{\theta(r) = \sqrt{\frac{2c}{m}}\frac{1}{v}\int_{r_0}^{r}\frac{dr}{r^2} = \sqrt{\frac{2c}{m}}\frac{1}{v}\left(\frac{1}{r_0} - \frac{1}{r}\right)}$$

(10.4) 3. $u_{eff} = \frac{-A}{r^n} + \frac{L^2}{2mr^2}$

$u_{eff}$ has minimum at $r = r_0$ (circular orbit)

$$\left.\frac{du}{dr}\right|_{r=r_0} = \frac{nA}{r_0^{n+1}} - \frac{L^2}{mr_0^3} = 0 \qquad \frac{nA}{r_0^{n+1}} = \frac{L^2}{mr_0^3}$$
$$\frac{d^2u}{dr^2} = \frac{-n(n+1)A}{r_0^{n+2}} + \frac{3L^2}{mr_0^4} > 0$$
$$\frac{-(n+1)L^2}{mr_0^4} + \frac{3L^2}{mr_0^4} > 0$$
$$3 \geq n+1 \rightarrow \boxed{n \leq 2 \quad n \neq 0}$$

(10.6) 4. $u(r) = \int_0^r Kr^4\,dr = \frac{1}{5}Kr^5$

$$u_{eff} = \frac{1}{5}Kr^5 + \frac{L^2}{2mr^2}$$

circular motion

$$0 = \left.\frac{du}{dr}\right|_{r=r_0} = Kr_0^4 - \frac{L^2}{mr_0^3}$$
$$\boxed{r_0 = \left(\frac{L^2}{Km}\right)^{1/7}}$$
$$\boxed{E = \frac{1}{5}Kr_0^5 + \frac{L^2}{2mr_0^2}}$$

Taylor Expansion

$$u_{eff} = u_{eff}(r_0) + 0 + \frac{1}{2}\left.\frac{d^2u}{dr^2}\right|_{r=r_0}(r^2) + \ldots \approx u_{eff}(r_0) + \frac{1}{2}\left(4Kr_0^3 + \frac{3L^2}{mr_0^4}\right)r^2 = u_{eff}(r_0) + \frac{1}{2}\left(4K\left(\frac{L^2}{Km}\right)^{3/7} + \frac{3L^2}{m}\left(\frac{Km}{L^2}\right)^{4/7}\right)r^2$$
$$u_{eff} = u_{eff}(r_0) + \frac{1}{2}\underbrace{\left(7K\left(\frac{L^2}{Km}\right)^{3/7}\right)}_{k = 7K\left(\frac{L^2}{Km}\right)^{3/7}}r^2$$

$$\boxed{\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{7K(L^2/Km)^{3/7}}{m}}}$$

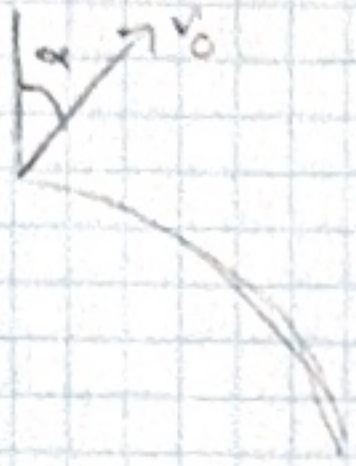
(10.7) 5. $\Delta K = K_f - K_i = \frac{1}{2}m\underbrace{(\vec{v} + \Delta\vec{v})^2}_{expand} - \frac{1}{2}mv^2$

$$\boxed{\Delta K = m(\vec{v}\cdot\Delta\vec{v}) + \frac{1}{2}m\Delta v^2}$$

Same $\Delta v$ causes more $\Delta K$ if $\Delta\vec{v} \parallel \vec{v}$

$\Delta K$ larger if $\vec{v}$ larger (at the perihelion)

(10.8) 6.




$$E = \frac{-GMm}{r} + \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2}$$

$\dot{r} = 0$ at max

Total energy constant

$$= -\frac{GMm}{R} + \frac{1}{2}mv_0^2$$

$\ell = mv_0 R\sin\alpha$

$$-\frac{GMm}{R} + \frac{1}{2}mv_0^2 = -\frac{GMm}{r} + \frac{1}{2}mv_0^2\sin^2\alpha\,\frac{R^2}{r^2}$$

$$-\frac{1}{2}G\frac{Mm}{R} = \frac{-GMm}{r} + \frac{1}{2}G\frac{Mm}{R}\sin^2\alpha\,\frac{R^2}{r^2}$$

$$0 = 1 - 2\frac{R}{r} + \sin^2\alpha\left(\frac{R}{r}\right)^2$$

$$0 = \left(\frac{r}{R}\right)^2 - 2\frac{r}{R} + \sin^2\alpha$$

$$\frac{r}{R} = \frac{2 \pm \sqrt{4 - 4\sin^2\alpha}}{2} \rightarrow r = R(1 \pm \cos\alpha)$$

$r > R$

$$\boxed{r = R(1+\cos\alpha)}$$

$R\cos\alpha$ above surface

(10.12) 7 a) $E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}mv^2 - mg\left(\frac{R^2}{r}\right)$

$$\Delta E = E(4R) - E(2R) = \left(\frac{1}{4} - \frac{1}{2}\right)(mgR) = \frac{1}{8}(3000)(9.8)(R) = \boxed{2.35 \cdot 10^{10}\ \text{J}}$$

b) $A = r_a + r_b = 6R$

(ellipse)

$$E = \frac{-1}{6R}mgR^2 = -\frac{1}{6}mgR$$

$$\underbrace{\frac{1}{2}mv_e^2}_{K_a}\ \underbrace{-\frac{1}{2}mgR}_{U_a} = \underbrace{\frac{-1}{6}mgR}_{E}$$

Expected $\begin{cases} v_0 = \sqrt{g\frac{R}{r}} \\ v_a = \sqrt{\frac{gR}{2}} \\ v_b = \sqrt{\frac{gR}{4}} \end{cases}$

$$v_e = \sqrt{\frac{2gR}{3}}$$

$$\Delta v_a = v_e - v_a = \sqrt{\frac{2gR}{3}} - \sqrt{\frac{gR}{2}} = \boxed{864\ \frac{\text{m}}{\text{s}}}$$

$$\underbrace{\frac{1}{2}mv_e^2}_{K_e} = \underbrace{-\frac{1}{6}mgR}_{E} + \underbrace{\frac{1}{4}mgR}_{U_b}$$

$$v_e = \sqrt{\frac{gR}{6}}$$

$$\frac{1}{2}mv_b^2 = \frac{1}{2}mg\frac{R^2}{4R} = \frac{1}{8}mgR$$

$$v_b = \frac{1}{2}\sqrt{gR}$$

$$\Delta v_b = v_b - v_e = \sqrt{\frac{gR}{4}} - \sqrt{\frac{gR}{6}} = \boxed{727\ \frac{\text{m}}{\text{s}}}$$