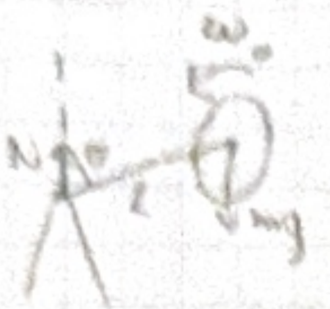


$\vec{\tau} = mgl\sin\theta\,\hat{\phi}$ $\qquad \vec{L} = I_0\vec{\omega}_0$ $\qquad$ If a vector $\vec{u}$ is rotating at angular vel $\vec{\Omega}$, $\frac{d\vec{u}}{dt} = \vec{\Omega}\times\vec{u}$

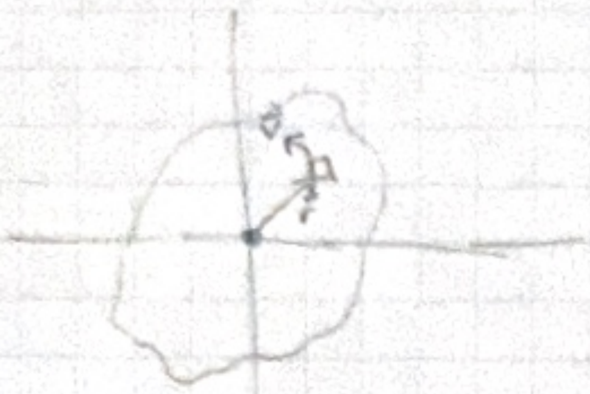
$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{\Omega}\times\vec{L}$ $\qquad \nearrow \vec{L}$

$\Omega L_\perp = |\tau|$

$\perp$ to axis of rotation

$\Omega L\sin\theta = mgl\sin\theta$

$\Omega = \frac{mgl}{L} = \frac{mgl}{I_0\omega_0}$ $\qquad$ Good approx when $\omega_0 \gg \Omega$



$\vec{L} = \int dm\, \vec{r}\times\vec{v}$ $\qquad \vec{v} = \vec{\omega}\times\vec{r}$

$= \int dm\, \vec{r}\times(\vec{\omega}\times\vec{r})$

$= \int dm\left[r^2\vec{\omega} - (\vec{r}\cdot\vec{\omega})\vec{r}\right]$ ← Angular momentum no longer parallel to angular velocity

$L_x = \int dm\left[r^2\omega_x - (x\omega_x + y\omega_y + z\omega_z)x\right]$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \underbrace{\int dm \begin{pmatrix} r^2-x^2 & -xy & -xz \\ -xy & r^2-y^2 & -yz \\ -xz & -yz & r^2-z^2 \end{pmatrix}}_{I \text{ Inertia Tensor}} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Spectral Theorem

There always exists a choice of axes $\hat{1}\ \hat{2}\ \hat{3}$

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \qquad I_1 = \int r_\perp^2\, dm$$

distance to $\perp$ axis

$\rho(x,y,z) = \rho(r,z)$

$I_{xy} = \int dm\,(-xy)$

$= \int dx\,dy\,dz\,\rho(x,y,z)(-xy)$

$= \int dz\, r\,dr\,d\theta\,\rho(r,z)(-r^2\sin\theta\cos\theta)$

$= \int \rho(r,z) r^3\, dr\, dz\, \frac{1}{2}\underbrace{\int \sin(2\theta)\,d\theta}_{0} = 0$

$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt}\left(I_1\omega_1\hat{1} + I_2\omega_2\hat{2} + I_3\omega_3\hat{3}\right)$

$= I_1\dot{\omega}_1\hat{1} + I_2\dot{\omega}_2\hat{2} + I_3\dot{\omega}_3\hat{3}$
$+ I_1\omega_1(\vec{\omega}\times\hat{1}) + I_2\omega_2(\vec{\omega}\times\hat{2}) + \ldots$ $\qquad \frac{d\hat{1}}{dt} = \vec{\omega}\times\hat{1}$

$= \left(\frac{d\vec{L}}{dt}\right)_{body} + (\vec{\omega}\times\vec{L})$

$$\vec{L} = I\vec{\omega} = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$K = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$

$L_1 = I_1\omega_1$

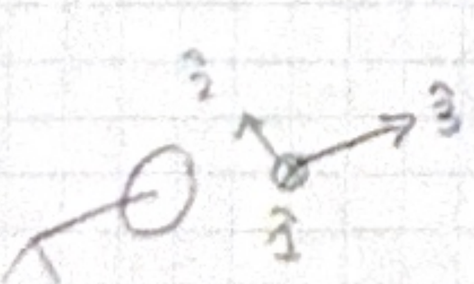
$L_2 = I_2\omega_2$

$L_3 = I_3\omega_3$

Setting Axes

More general $\left\{ \frac{d\hat{1}}{dt} \neq \vec{\omega}\times\hat{1} \right.$ $\qquad$ Allows torque to always be along 1 axis

$= (\vec{\omega} - \omega_0\hat{3})\times\hat{1}$



$\omega = \omega_0\hat{3} + \Omega\hat{z}$ $\qquad \hat{z} = \cos\theta\,\hat{3} + \sin\theta\,\hat{2}$

$\omega = \underbrace{(\omega_0 + \cos\theta)}_{\omega_3}\hat{3} + \underbrace{(\Omega\sin\theta)}_{\omega_2}\hat{2}$

$L_2 = I_2\omega_2$

$L_3 = I_3\omega_3$

$I_1 = I_2 = I$

$\tau_1 = I_1\dot{\omega}_1 + (\Omega\hat{z}\times\vec{L})_1$

$0 = I_2\dot{\omega}_2 + (\Omega\hat{z}\times\vec{L})_2$

$0 = \underbrace{I_3\dot{\omega}_3}_{\omega_3 \text{ constant}} + \underbrace{(I_2 - I_1)}_{0}\omega_1\omega_2$

$\omega_3 = \omega_0 + \Omega\cos\theta$

ω_0 constant if θ constant

$L_z = \cos\theta\, L_3 + \sin\theta\, L_2 = I_3\omega_3\cos\theta + I_2\omega_2\sin\theta$

ω_2 constant $\omega_2 = \Omega\sin\theta \Rightarrow \Omega$ constant

$Mgl\sin\theta = I_1\overset{0}{\dot{\omega}_1} + (\Omega\hat{z}\times\vec{L})_1$

$Mgl\sin\theta = \Omega(\cos\theta\,\hat{3} + \sin\theta\,\hat{2})\times(I_2\omega_2\hat{2} + I_3\omega_3\hat{3})$

$= \Omega\,(I_3\omega_3\sin\theta - I_2\underbrace{\omega_2}_{\Omega\sin\theta}\cos\theta)\hat{1}$

$mgl = \Omega\,(I_3\omega_3 + I_2\omega_2\,\Omega\cos\theta)$