Problem Set 1

1 a) $\frac{dh}{dt} = \frac{dh}{dx}\frac{dx}{dt} \Rightarrow \frac{d}{dt}\left(h(x(t))\right) = \frac{dh}{dx}\frac{dx}{dt}$

b) $\frac{dh}{dt} = \left(\frac{1}{x\ln(10)}\right)\left(\beta\cos(\beta t)\right) = \frac{\beta\cos(\beta t)}{\ln(10)\sin(\beta t)} = \frac{\beta}{\ln(10)}\cot(\beta t)$

c) $\frac{d^2h}{dt^2} = \frac{d}{dt}\left(h'(x(t))\,x'(t)\right) = h''(x(t))\left(x'(t)\right)^2 + h'(x(t))\,x''(t)$

2. a) $\frac{dz}{dt} = \frac{d}{dt}\left(f(t)g(t)\right) = f(t)g'(t) + f'(t)g(t)$

b) $\frac{dz}{dt} = e^{\gamma t}\left(\delta e^{\delta t}\right) + \gamma e^{\gamma t}e^{\delta t} = e^{\gamma t}e^{\delta t}(\delta+\gamma) = e^{(\delta+\gamma)t}(\delta+\gamma)$

c) $z(t) = e^{\gamma t}e^{\delta t} = e^{(\gamma+\delta)t} \Rightarrow \frac{dz}{dt} = (\gamma+\delta)e^{(\gamma+\delta)t}$

3. a) $\sin(x) \approx x - \frac{x^3}{3!}$ $\quad \sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$

b) $\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ $\quad \cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$

c) $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$ $\quad e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

d) $e^{ix} \approx 1 + ix + \frac{-x^2}{2!} + \frac{-ix^3}{3!} + \frac{x^4}{4!} = 1 + ix + \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!}$

$\cos(x) + i\sin(x) = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} = e^{ix}$ $\quad \therefore e^{ix} = \cos(x) + i\sin(x)$

e) $e^{ix} + e^{-ix}$ odd power $\to 0$ even power $\to$ double $\quad \cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$

4. $\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \Rightarrow \cos^2(x) = \frac{1}{4}\left(e^{2ix} + 2e^{ix-ix} + e^{-2ix}\right) = \frac{1}{4}\left(\cos(2x) + i\sin(2x) + \cos(-2x) + i\sin(-2x) + 2\right)$

$\cos^2(x) = \frac{1}{4}\left(2\cos(2x) + 2\right) = \frac{1+\cos(2x)}{2}$ $\quad \therefore \cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$

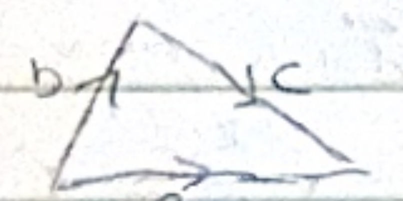
5. $\frac{dy}{dx} - by(x) = a \Rightarrow p = -b \quad P = -bx \quad \int\left(e^{-bx}\frac{dy}{dx} - be^{-bx}y\right)dx = \int e^{-bx}a\,dx$

product rule $\quad e^{-bx}y = e^{-bx}\frac{-a}{b} + C_0 \Rightarrow y = \frac{-a}{b} + C_1$

6. 

$\vec{f} = \vec{a} + \vec{b}$ $\quad \cos(90) = 0 = \frac{\vec{e}\cdot\vec{f}}{|\vec{e}||\vec{f}|} \Rightarrow \vec{e}\cdot\vec{f} = 0 = (\vec{a}_1+\vec{b}_1)(\vec{d}_1-\vec{c}_1) + (\vec{a}_2+\vec{b}_2)(\vec{d}_2-\vec{c}_2)$

$\vec{e} = \vec{d} - \vec{c}$ $\quad = \cancel{(\vec{a}_1\vec{d}_1} - \cancel{\vec{c}_1\vec{b}_1} + \cancel{\vec{b}_1\vec{d}_1} - \cancel{\vec{b}_1\vec{c}_1}) + (-\cancel{\vec{a}_2\vec{c}_2} + \cancel{\vec{b}_2\vec{d}_2} + \vec{a}_2\vec{b}_2 - \cancel{\vec{b}_2\vec{c}_2}) = 0$

7. 

$\vec{b} + \vec{c} = \vec{a} \Rightarrow \vec{b}\times(\vec{b}+\vec{c}) = \vec{b}\times\vec{a} = \vec{b}\times\vec{b} + \vec{b}\times\vec{c} = \vec{b}\times\vec{a} \Rightarrow \vec{b}\times\vec{c} = \vec{b}\times\vec{a}$

$\Rightarrow |\vec{b}||\vec{c}|\sin(\pi - a) = |\vec{b}||\vec{a}|\sin(c) \Rightarrow |\vec{c}|\sin(a) = |\vec{a}|\sin(c) \Rightarrow \frac{\sin(a)}{|\vec{a}|} = \frac{\sin(c)}{|\vec{c}|}$

8. a) $\vec{A}\cdot\vec{B} = 3x + 4y - 4z = |\vec{A}||\vec{B}|\cos\left(\frac{\pi}{2}\right) = 0$ $\quad z = 0 \Rightarrow 3x = -4y$ $\quad \vec{B} = 4\hat{i} - 3\hat{j} + 0\hat{k}$ $\quad \hat{B} = \frac{\vec{B}}{|\vec{B}|} = \frac{4}{5}\hat{i} - \frac{3}{5}\hat{j}$

b) $\vec{A}\times\vec{B} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 4 & -4\\ 4 & -3 & 0\end{vmatrix} = (0-12)\hat{i} - (0+16)\hat{j} + (-9-16)\hat{k} = -12\hat{i} - 16\hat{j} - 25\hat{k} = \vec{C}$ $\quad |\vec{C}| = \sqrt{144+256+625} = \sqrt{1025}$

$\hat{C} = \frac{\vec{C}}{|\vec{C}|} = \frac{-12}{\sqrt{1025}}\hat{i} - \frac{16}{\sqrt{1025}}\hat{j} - \frac{25}{\sqrt{1025}}\hat{k}$

c) $\vec{B}\times\vec{C} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -3 & 0\\ 12 & 16 & 25\end{vmatrix} = (-75-0)\hat{i} - (100-0)\hat{j} + (64+36)\hat{k} = -75\hat{i} - 100\hat{j} + 100\hat{k} = -25(3\hat{i} + 4\hat{j} - 4\hat{k}) = -25\vec{A}$

9. $\vec{A}'(t) = \vec{C}_0 \Rightarrow \vec{A}(t) = \vec{C}_0 t + \vec{C}_1$ $\quad \vec{A}(t_1) = \vec{C}_0 t_1 + \vec{C}_1 = \vec{r}_1$ $\quad \vec{A}(t_1+T) = \vec{C}_0(t_1+T) + \vec{C}_1 = \vec{r}_2$ $\quad \vec{A}(t) = \frac{\vec{r}_2 - \vec{r}_1}{T}t + \frac{\vec{r}_1(T+t_1) - \vec{r}_2 t_1}{T}$

$\vec{C}_1 = \vec{r}_1 - \vec{C}_0 t_1$ $\quad \vec{C}_1 = \vec{r}_1 - \vec{C}_0 t_1 \Rightarrow \vec{C}_0(t_1+T) + \vec{r}_1 - \vec{C}_0(t_1) = \vec{r}_2$

$= \vec{r}_1 - \frac{\vec{r}_2 t_1}{T} + \frac{\vec{r}_1 t_1}{T} = \frac{\vec{r}_1(T+t_1) - \vec{r}_2 t_1}{T}$ $\quad \boxed{\vec{A}(t) = \frac{\vec{r}_1(T+t_1-t)}{T} + \frac{\vec{r}_2(t-t_1)}{T}}$

$\vec{C}_0(t_1+T-t_1) = \vec{r}_2 - \vec{r}_1$

$\vec{C}_0 = \frac{\vec{r}_2 - \vec{r}_1}{T}$

10. $\left[\frac{vh}{g}\right] = \frac{m^2}{s}\frac{s^2}{m} = \cancel{\frac{m}{s}}$ $\quad \left[\frac{gh^2}{v^2}\right] = \frac{m^3}{s^2}\frac{s^2}{m^2} = m\checkmark$ $\quad \left[\frac{v^2}{g}\right] = \frac{m^2}{s^2}\frac{s^2}{m} = m\checkmark$ $\quad \left[\sqrt{\frac{v^2h}{g}}\right] = \sqrt{\frac{m^3}{s^2}\frac{s^2}{m}} = m\checkmark$

$\left[\frac{v^2}{g}\sqrt{1+\frac{2gh}{v^2}}\right] = \frac{m^2}{s^2}\frac{s^2}{m} = m\checkmark$ $\quad \left[\cancel{\frac{v^2/g}{1+\frac{2gh}{v^2}}}\right] = \frac{m^2}{s^2}\frac{s^2}{m} = m\checkmark$

$v\to 0,\ L\to 0$ $\quad \Rightarrow \sqrt{\frac{v^2h}{g}}$ or $\frac{v^2}{g}\sqrt{1+\frac{2gh}{v^2}}$ $\quad L \propto t$ bc no a_x $\quad x = v_0 t$

$h\to\infty,\ L\to\infty$ $\quad v\to\infty$ $\quad \cancel{L\propto v}$ $\quad L\propto v^2\checkmark$ $\quad t^2 \propto y$ bc constant a_y $\quad y = at^2 \Rightarrow L\propto v^2$

$v^2 \propto t^2$ bc constant a_y $\quad v = at$