Chain Rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Implicit

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-F_x}{F_y}$$

for $F(x,y)=0$

$F(x, f(x))$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \qquad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

for $F(x,y,z)=0$

$F(x,y,f(x,y))$

Cartesian

$$L = \int \sqrt{dx^2+dy^2} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Polar

$x = r\cos\theta \qquad r^2 = x^2+y^2$

$y = r\sin\theta \qquad \tan\theta = \frac{x}{y}$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

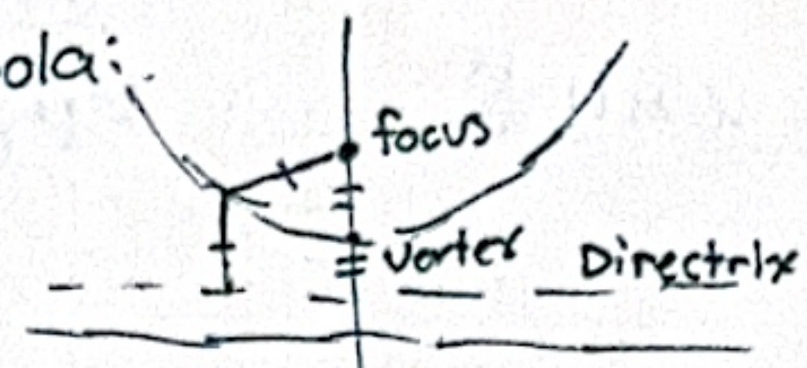
$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Conic Sections

Circle: $(x-h)^2 + (y-k)^2 + (z-\ell)^2 = r^2$

w/ center (h,k,ℓ)

Parabola:



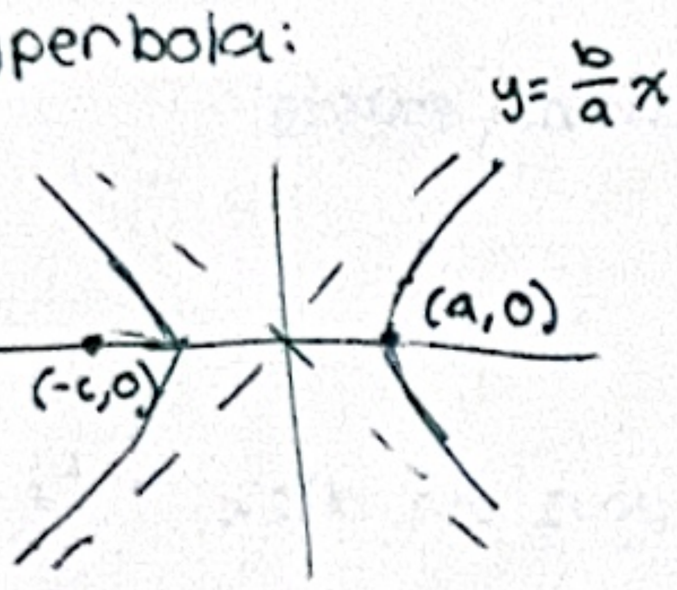
for $P(x,y)$ on parabola

$|PF| = |PD|$

$\sqrt{x^2+(y-p)^2} = |y-p|$

$\Rightarrow x^2 = 4py$

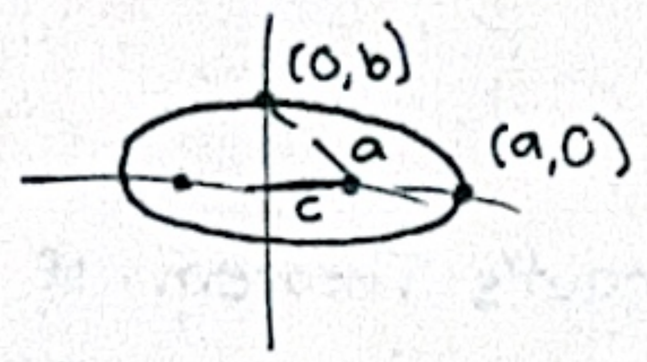
Hyperbola:



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

c = focus

Ellipse:



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a^2 = b^2 + c^2$$

Vector Rules

$\vec{a}+\vec{b} = \vec{b}+\vec{a} \qquad \vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c} \qquad c(\vec{a}+\vec{b}) = c\vec{a}+c\vec{b} \qquad (c\vec{a})(\vec{a}) = (c)(d\vec{a}) \qquad \vec{a}(c+d) = c\vec{a}+d\vec{a}$

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

Dot Product

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 = |\vec{a}||\vec{b}|\cos\theta \qquad \vec{a}\cdot\vec{a} = |\vec{a}|^2 \qquad \vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} \qquad (c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

Projections: $\text{comp}_{\vec{a}}\vec{b} = \vec{b}\cdot\hat{a} \qquad \text{proj}_{\vec{a}}\vec{b} = (\vec{b}\cdot\hat{a})\hat{a}$

Cross Product

$$\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = |\vec{a}||\vec{b}|\sin\theta$$

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$\vec{a}\times\vec{b} = -\vec{b}\times\vec{a} \qquad (c\vec{a})\times\vec{b} = c(\vec{a}\times\vec{b}) = \vec{a}\times(c\vec{b}) \qquad \vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c} \qquad (\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}\ \vec{b}\times\vec{c}$

$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ $\Leftarrow$ Triple product: volume of parallelepiped

Vector Lines/Planes

$\vec{r} = \vec{r}_0 + \vec{v}t = (1-t)\vec{r}_0 + \vec{r}_1 t$ $\quad x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct \quad \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \quad L = \int_\alpha^\beta |\vec{r}'(t)|dt$

Types of vector problems

Line - Plane intersection:
Plug parametric line into cartesian plane

Closest point on plane:
Use line-plane to find point $\Rightarrow$ distance formula
or $d^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ where $z = f(x,y)$ $\Rightarrow$ D test

Vector to Cartesian Plane

$(x,y,z)\cdot\vec{n} = (x_0,y_0,z_0)\cdot\vec{n}$

$\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

$\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$ $\quad \frac{d}{dt}(\vec{u}\times\vec{v}) = \vec{u}'\times\vec{v} + \vec{u}\times\vec{v}'$

$\frac{d}{dt}[\vec{u}\cdot\vec{v}] = \vec{u}'\cdot\vec{v} + \vec{u}\cdot\vec{v}'$ $\quad \frac{d}{dt}(\vec{u}(f(t))) = f'(t)\,\vec{u}'(f(t))$

Vector Calculus

$\lim \vec{r}(t) = (\lim f(t), \lim g(t), \lim h(t))$ $\quad \vec{r}'(t) = (f'(t), g'(t), h'(t))$ $\quad \int r(t)dt = (\int f(t), \int g(t), \int h(t))$

Limits

For lim to exist, must be same from all paths

Check $\overbrace{y=0,\ x=0}^{\text{axes}}$, $y=x$
↳ squeeze theorem if probably exists

Polynomials continuous on $\mathbb{R}^2$

Rational functions continuous on domain

Partials

$f_x = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$

Clairaut's Theorem: If f_{xy} and f_{yx} continuous on disk D containing (a,b), $f_{xy} = f_{yx}$

Tangent Plane

$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$z = f(x_0,y_0) + f_x(x-x_0) + f_y(y-y_0)$

for $V(x,y,z)$

$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$

Directional Derivative: $\hat{u} = (a,b)$

$D_{\hat{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ah, y_0+bh) - f(x_0,y_0)}{h}$

$D_{\hat{u}} = f_x a + f_y b$ $\quad \vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}$

$D_{\hat{u}} = \vec{\nabla}f\cdot\hat{u} = f_x\cos\theta + f_y\sin\theta = |\vec{\nabla}f||\hat{u}|\cos\theta$

↗ max when $\vec{\nabla}f \parallel \hat{u}$

Level sets

$F(x(t), y(t), z(t)) = k$

$\vec{\nabla}F\cdot\vec{r}'(t) = 0$

Tangent Plane: same w/ $\Delta F = 0$

$F_x(x-x_0) + F_y(y-y_0) + F_z(z-z_0) = 0$

D Test

$D = f_{xx}f_{yy} - (f_{xy})^2$

If $f_x = f_y = 0$ or DNE, critical point

$D<0$ saddle

$D>0$ $f_{xx}>0$ min

$D>0$ $f_{xx}<0$ max

$D=0$ who knows?