Problem Set 6

1.

$E = \frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2$

$E = \frac{1}{2}\frac{M^2}{m}v_2^2 + \frac{1}{2}Mv_2^2$

$E = \frac{1}{2}Mv_2^2\left(\frac{M}{m}+1\right)$

$\frac{E_M}{E} = \frac{1}{\frac{M}{m}+1} = \boxed{\frac{m}{M+m}}$

$mv_1 = Mv_2$

$v_1 = \frac{M}{m}v_2$

$\frac{1}{2}Mv_2^2 = E_M$

$m \gg M$ The additive M becomes negligible, so the limit approaches 1 therefore all the energy goes to the railway car

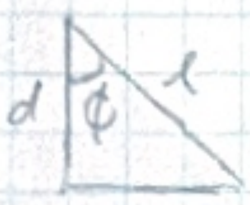
$m \ll M$ The additive m becomes negligible, so the limit approaches $\frac{m}{M}$. Since M is much larger, it approaches 0, implying that all the energy goes to the person

(5.3) 2. a) $mv = (M+m)V \Rightarrow \boxed{V = \frac{m}{M+m}v}$

b) $\frac{1}{2}\cancel{(m+M)}V^2 = \cancel{(m+M)}gh$

$V = \sqrt{2gh}$

$\boxed{v = \frac{M+m}{m}\sqrt{2gl(1-\cos\phi)}}$



$d = l\cos\phi$

$h = l - d = l(1-\cos\phi)$

(5.5) 3. a) $\frac{dL}{dt} = 0$ if constant

$\frac{d}{dt}(mr^2\dot\theta) = 2mr\dot\theta\dot r + mr^2\ddot\theta = mr\underbrace{(2\dot\theta\dot r + r\ddot\theta)}_{\text{tangential acceleration}}$

since no force in tangential direction, $\frac{d}{dt}(mr^2\dot\theta) = 0$

$\Downarrow$

L is constant

b) $a_{tan} = 0 = 2\dot\theta\dot r + r\ddot\theta$

$\frac{\ddot\theta}{\dot\theta} = \frac{\dot\omega}{\omega} = -\frac{2\dot r}{r}$

$\int\frac{\dot\omega}{\omega}dt = -2\int\frac{\dot r}{r}dt = \int\frac{d\omega}{\omega} = -2\int\frac{dr}{r} = \ln\left(\frac{\omega}{\omega_0}\right) = \ln\left(\frac{r_0^2}{r^2}\right) \Rightarrow \omega = \frac{r_0^2\omega_0}{r^2}$

$\Delta K_{rad} = \frac{1}{2}m(v^2 - v_0^2)$

$\Delta K_{tan} = \frac{1}{2}m(r\omega)^2 = \frac{1}{2}m\left(\frac{r_0^4\omega_0^2}{r^2} - \frac{r_0^4\omega_0^2}{r_0^2}\right)$

$W = \int_{r_0}^{r} m(\ddot r - r\dot\theta^2)\,dr$

$W_{rad} = \int_{r_0}^{r} m\frac{d\dot r}{dt}dr = m\int_{t_0}^{t}\frac{d\dot r}{dt}\frac{dr}{dt}dt$

$= \int_{t_0}^{t}\frac{d}{dt}\left(\frac{1}{2}m\dot r^2\right)dt = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$

$W_{tan} = \int_{r_0}^{r} mr\dot\theta^2\,dr = -\int_{r_0}^{r} m\frac{r_0^4\omega_0^2}{r^3}dr = +\frac{1}{2}m\left(\frac{r_0^4\omega_0^2}{r^2} - \frac{r_0^4\omega_0^2}{r_0^2}\right)$

$\boxed{\Delta K = \Delta K_{tan} + \Delta K_{rad} = W_{tan} + W_{rad} = W}$

$W_{rad} = \Delta K_{rad}$

$W_{tan} = \Delta K_{tan}$

(5.8) 4 a) $\Delta E = \frac{1}{2}k(x_0 - \Delta x)^2 - \frac{1}{2}k(x_0)^2 = \frac{1}{2}k(\Delta x^2 + 2\Delta x x_0) \approx -k\Delta x x_0 = -W$

$W = 4fx_0 = +k\Delta x x_0 \Rightarrow \boxed{\Delta x = \frac{4f}{k}}$ constant bc not dependent on x_0

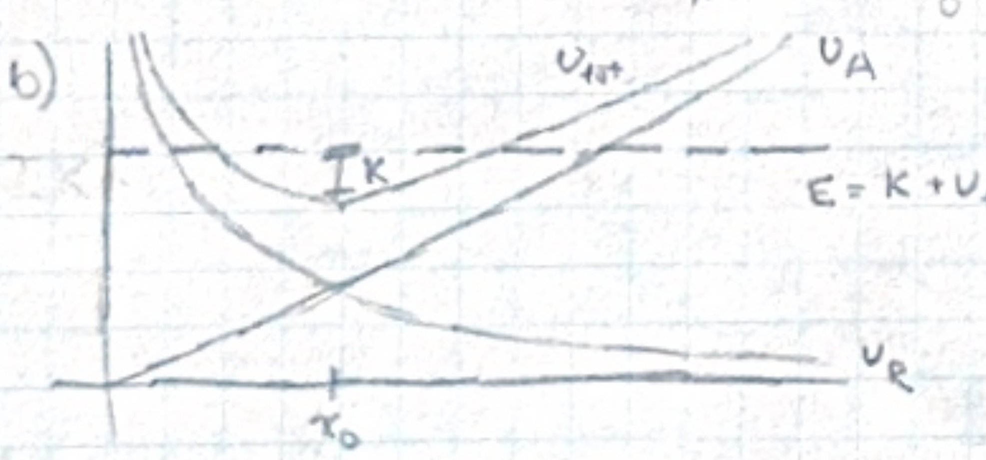
b) $n\Delta x = x_0 \Rightarrow \boxed{n = \frac{x_0 k}{4f}}$

(5.13) 5. a) $U = -G\frac{mM}{r} \Rightarrow \boxed{U_{tot} = -2G\frac{mM}{\sqrt{x^2+a^2}}}$

b) $K_f - K_i = U_i - U_f$ $\quad \frac{1}{2}m(v^2 - v_i^2) = 2GMm\left(\frac{1}{\sqrt{a^2}} - \frac{1}{\sqrt{10a^2}}\right) \Rightarrow v^2 - v_i^2 = 4\frac{GM}{a}\left(1 - \frac{1}{\sqrt{10}}\right) \Rightarrow v = \sqrt{v_i^2 + 4\left(1-\frac{1}{\sqrt{10}}\right)\frac{GM}{a}}$

(5.14) 6. a) $F = \frac{A}{x^2} - B$ $\quad U(x) = \int_x^\infty \frac{A}{x^2}dx + \int_0^x B\,dx = +\frac{A}{x} + Bx = \boxed{Bx + \frac{A}{x}}$

b)



$F = 0 \Rightarrow \frac{A}{x^2} = B \Rightarrow \frac{A}{x} = Bx$

$U_a = U_R$

$K = E - U_{tot} = E - U_a - U_R$

(5.7) 7. $mg\cos\theta - N = m\frac{v^2}{r}$

$T = 2N\cos\theta - Mg$

$mgr = \frac{1}{2}mv^2 + mgr\cos\theta$

$m\frac{v^2}{r} = 2mg(1-\cos\theta) = mg\cos\theta - N$

$\Rightarrow N = mg(3\cos\theta - 2)$

Edge case: $T = 0$

$2N\cos\theta = Mg$

$2m(3\cos\theta - 2)\cos\theta = M$

Edge case $\Rightarrow$ max $\cos\theta$ ...

$6\cos\theta - 2 = 0$

$\boxed{\cos\theta = \frac{1}{3}}$

$\boxed{\theta = 70^\circ}$

Quadratic formula

$3\cos^2\theta - 2\cos\theta + \frac{M}{2m} > 0$

$\cos\theta_{max} > \frac{1}{3} + \sqrt{\frac{1}{9} - \frac{M}{6m}}$

$\frac{1}{3} > \frac{1}{3} + \sqrt{\frac{1}{9} - \frac{M}{6m}}$

$\frac{1}{9} > \frac{M}{6m}$

$\boxed{m > \frac{3}{2}M}$

(6.1) 8. $U = 2G\frac{Mm}{\sqrt{x^2+a^2}}$ $\frac{dU}{dx} = +G\frac{Mmx}{(x^2+a^2)^{3/2}}$ $\frac{d^2U}{dx^2} = \frac{2GMm}{(x^2+a^2)^{3/2}} - \frac{6GMmx^2}{(x^2+a^2)^{5/2}}$

$U(x) = U(0) + U'(0)x + \frac{1}{2}U''(0)x^2 = -2G\frac{Mm}{a} + \frac{1}{2}\left(2\frac{GMm}{a^3}\right)x^2$

$U(x) = \frac{1}{2}kx^2$ $k = 2\frac{GMm}{a^3}$ $\omega = \sqrt{\frac{k}{m}} = \boxed{\sqrt{\frac{2GM}{a^3}}}$

) 9 $V_0 = \sqrt{2gh}$ $V_{cm} = \frac{M\sqrt{2gh} - m\sqrt{2gh}}{M+m} \approx \frac{M\sqrt{2gh}}{M} = \sqrt{2gh}$

$M(0) + m(-2v_0) = M(0) + m(2v_0)$ CM reference frame

$v_f = 3v_0$

$\frac{9v_0^2}{2g} = \boxed{h' = 9h}$

6.6) 10. $Mv_0 = 2Mv_1 = 3Mv_2$

$P_0 = 2P_1 = 3P_2$

$E_0 = CE_1$ $\frac{4P_1^2}{2M} = C\frac{P_1^2}{M}$ $E_0 = 2E_1$

$E_1 = CE_2$ $\frac{9P_2^2}{3M} = C\frac{P_2^2}{M}$ $E_1 = 3E_2$

$E_0 = 6E_2$

$\boxed{E_2 = E_0/6}$

6.8) 11. $mv_0 - MV = \frac{MV_1}{\sqrt{2}}$

$\frac{MV_1}{\sqrt{2}} = \frac{mv_0}{2} = mv_0 - MV$

① $V = \frac{1}{2}\frac{m}{M}v_0$ ② $V_1 = \frac{1}{\sqrt{2}}\frac{m}{M}v_0$

$\frac{1}{2}mv_0^2 + \frac{1}{2}MV^2 = \frac{1}{2}m\left(\frac{v_0}{2}\right)^2 + \frac{1}{2}MV_1^2$

$\frac{3}{4}m\cancel{v_0^2} + \frac{1}{4}\frac{m^2}{M}\cancel{v_0^2} = \frac{1}{2}\frac{m^{\cancel{2}}}{M}\cancel{v_0^2}$

$\boxed{\frac{m}{M} = 3}$

6.14) 12. a) $F = \frac{\Delta P}{\Delta T} = \frac{2mv}{2l/v} = \boxed{\frac{mv^2}{l}}$

b) $v_0' = v + V = v_1'$

$v_1 = v + 2V$ $\Delta v = 2V$

$\frac{\Delta v}{\Delta T} = \boxed{\frac{dv}{dt} = \frac{2V}{2x/v} = \frac{vV}{x}}$

$\frac{dv}{dx} = \frac{\frac{dv}{dt}}{\frac{dx}{dt}} = -\frac{1}{v}\frac{dv}{dt} = \boxed{\frac{-V}{x}}$

$\int_{v_0}^{v}\frac{dv}{v} = -\int_{x_0}^{x}\frac{dx}{x} = \ln\left(\frac{v}{v_0}\right) = \ln\left(\frac{x_0}{x}\right)$ $\boxed{v(x) = \frac{v_0 l}{x}}$

c) $F = \frac{dp}{dt} = m\frac{dv}{dt} = \boxed{\frac{mvV}{x}}$

or $F_{av} = \frac{1}{\Delta T}\int_0^{\Delta T} m\frac{dv}{dt}dt = \frac{mv}{2x}\int_0^{2V} dv = \boxed{\frac{mvV}{x}}$