$\ddot{x} + \omega^2 x = 0$

$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$

Dampened:

$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0$ linear equation

x_1, x_2 solutions

$F = -\frac{dV_1(x)}{dx}$ $\Rightarrow$ $c_1 x_1 + c_2 x_2$ also a solution

$$\underbrace{\left[\frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2\right]}_{L} x = 0$$

$L(c_1 x_1 + c_2 x_2) = c_1 L x_1 + c_2 L x_2 = 0$

$x = e^{\alpha t}$

$Lx = [\alpha^2 + \gamma\alpha + \omega_0^2] x = 0$

$\Rightarrow \cancel{x = 0}, \ \alpha^2 + \gamma\alpha + \omega_0^2 = 0$

$\alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$

1. $\frac{\gamma^2}{4} > \omega_0^2 \longrightarrow$ overdamped

$\alpha = -\frac{\gamma}{2} \pm \beta$ $\left(\beta = \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right)$

general solution:

$x(t) = e^{-\frac{\gamma}{2}t}\left[c_1 e^{\beta t} + c_2 e^{-\beta t}\right]$

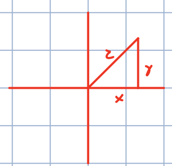
2. $\omega_0^2 > \frac{\gamma^2}{4}$ underdamped

$\alpha = \frac{\gamma}{2} \pm i\omega$ $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

$x(t) = e^{-\frac{\gamma}{2}t}\left[c_1 e^{i\omega t} + c_2 e^{-i\omega t}\right]$

$z = x + iy$ $x = \mathrm{Re}(z) = \frac{z + z^*}{2}$

$z^* = x - iy$ $y = \mathrm{Im}(z) = \frac{z - z^*}{2i}$



$z = re^{i\varphi}$

$|z| = r = \sqrt{x^2 + y^2}$

$|z|^2 = (x + iy)(x - iy) = zz^*$

$z^* = re^{-i\varphi}$ $zz^* = r^2 e^{i(\phi - \varphi)}$

$= r^2$

3. $\frac{\gamma^2}{4} = \omega_0^2$ critically damped

$c_1 e^{-\frac{\gamma}{2}t} + c_2 t e^{-\frac{\gamma}{2}t}$

$x = e^{-\frac{\gamma}{2}t} u(t)$

$\ddot{u} + \omega^2 u = 0 \Rightarrow \ddot{u} = 0$

$= c_1 + c_2 t$

$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}$

$\ddot{x} = \gamma\dot{x} = \omega_0 x = f(t)$

$Lx = f$

x_1, x_2 are solutions $Lx_1 = f$, $Lx_2 = f$

$L(x_1 - x_2) = Lx_1 - Lx_2 = 0$

$x_1 - x_2$ is a solution to $\underbrace{Lx = 0}_{\text{homogenous}}$

A general solution of $Lx = f$:

$x(t) = x_p(t) + x_h(t)$

$Lx_h = 0$

$x_1 = e^{-\frac{\gamma}{2}t}\left[c_1 e^{i\omega t} + c_2 e^{-i\omega t}\right]$

$Lx = f_1 + f_2$

$Lx = f\cos(\Omega t)$

$Lx = fe^{i\Omega t}$

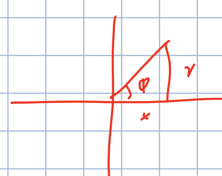
$\left[\frac{d^2}{dt^2} + \gamma\frac{d}{dt} + \omega_0^2\right]x = fe^{i\Omega t}$

$x = Ae^{i\Omega t}$

$\left[-\Omega^2 + i\gamma\Omega + A_1^2\right] = A\cancel{e^{i\Omega t}} = f\cancel{e^{i\Omega t}}$

$|A|e^{i\phi} = A = \frac{f}{\omega_0^2 + \Omega^2 + i\gamma\Omega} = \frac{\omega_0^2 - \Omega^2 - i\gamma\Omega}{(\omega_0^2 - \Omega^2)^2 + \gamma^2\Omega^2}$

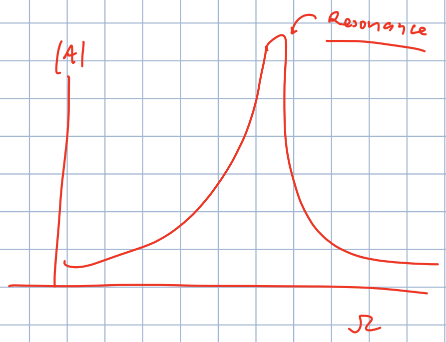
$\phi = \tan^{-1}\left(\frac{y}{x}\right) \longrightarrow |A| = \frac{f}{\sqrt{(\omega_0^2 - \Omega^2)^2 + \gamma^2\Omega^2}}$



$x_p(t) = |A| e^{i(\Omega t + \phi)}$

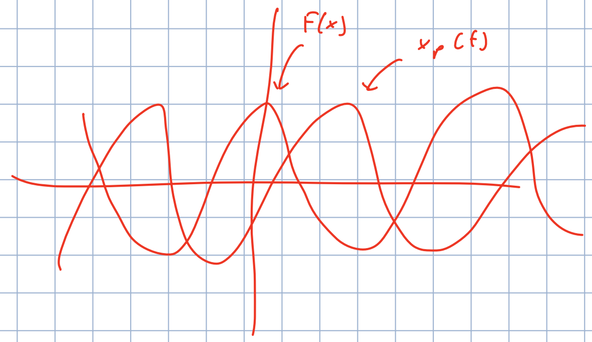
$\phi = -\tan^{-1}\left(\frac{\gamma \Omega}{\omega_0^2 - \Omega^2}\right)$

where $|A| = \frac{f}{\sqrt{(\omega_0^2 - \Omega^2) - \gamma^2 \Omega^2}}$



$\frac{\gamma^2}{4} << \omega_0^2$

$Q >> 1$



Real Solution

$x_p(t) = |A| = \cos(\Omega t + \phi)$