Final Exam

Graded

Student

Minnoli Raghavan

Total Points

88 / 100 pts

Question 1

(no title) — **9** / 10 pts

1.1 (no title) — **4** / 5 pts

- + 1 pt Correct Calculation
- ✔ + 4 pts Correct idea
- + 3 pts Almost at the correct idea
- + 2 pts Getting there
- + 1 pt Something related to the class/question
- + 0 pts blank

1.2 (no title) — **5** / 5 pts

- ✔ − 0 pts Correct
- − 1 pt Did not classify all points
- − 1 pt Missed maxes or mins
- − 2 pts Did not apply langrage
- − 1 pt Computational errors
- − 4 pts made very little progress
- − 5 pts didnt write anything

Question 2

(no title)

10 / 10 pts

✔ **+ 3 pts** Correct bounds of integration for spherical. A lot of this was handed to you in the problem, so this amounts to understanding the θ bounds.

+ 1 pt Partial credit for bounds, e.g. θ bounds are correct but some weird discrepancy happened in the other bounds

✔ **+ 3 pts** Correct Jacobian for spherical.

✔ **+ 2 pts** relevant integration work (if the integral computed has no clear relationship to the problem, then rubric element not awarded)

✔ **+ 2 pts** Fully evaluated correct final answer $\frac{\sqrt{3}-1}{3}\pi a^3$.

+ 1 pt Nearly correct final answer, e.g. a minor arithmetic blunder near the end, trigonometric functions were evaluated incorrectly (or not evaluated), or integral itself was slightly wrong. (If the integral was *very* wrong, e.g. did not involve integrating $\sin\phi$, or was wrong and other computational mistakes were made, this rubric element is not awarded.)

+ 4 pts Partial credit for different approach with reasonable progress

+ 2 pts Partial credit for different approach with little progress

+ 0 pts Incorrect

− 1 pt point adjustment

− 2 pts point adjustment

Question 3

(no title) **8 / 10 pts**

- ✔ **+ 2 pts** Show $g = e^{-xy}$

Lagrange multipliers

- ✔ **+ 2 pts** $\nabla g = \lambda \nabla h$
- ✔ **+ 2 pts** Computations
- **+ 1 pt** $g(\pm\sqrt{1/2}, \pm\sqrt{1/8}) = e^{-1/4}$ min
- **+ 1 pt** $g(\pm\sqrt{1/2}, \mp\sqrt{1/8}) = e^{1/4}$ max

- ✔ **+ 2 pts** Critical points on interior, solve $\nabla g = 0$ to get $g(0, 0) = 1$

Point adjustments

- **+ 1 pt** Click here to replace this description.
- **+ 1 pt** Click here to replace this description.
- **− 1 pt** Click here to replace this description.
- **− 1 pt** Click here to replace this description.

- **+ 0 pts** N/A

Question 4

(no title) **8 / 10 pts**

- **+ 10 pts** Perfect
- ✔ **+ 6 pts** Correct Jacobian and Integral
- ✔ **+ 2 pts** Show some Calculation Effort (if your integral is correct)
- **+ 4 pts** Correct Jacobian but Incorrect Integral
- **+ 4 pts** Correct Integral but Incorrect Jacobian
- **+ 1 pt** Show some Calculation Effort (even there is some error in your integral)
- **+ 2 pts** Incorrect but show something Relevant
- **+ 0 pts** Blank

Question 5

(no title) **10 / 10 pts**

- ✔ **− 0 pts** Correct

Question 6

(no title)

10 / 10 pts

✔ **+ 10 pts** Correct (up to minor computational error). Clear indication/justification that the Jacobian is the ratio of the areas.

+ 7 pts Ratio is flipped / got the change of variables backwards, or other significant error with computing Jacobian.

+ 7 pts Significant progress using change of variables. Usually means mistakes in/lacking explanation/justification for why the Jacobian is the answer.

+ 2 pts Something like "change of variables does not change the area of something enclosed."

+ 4 pts Some progress using change of variables. For example, computed the Jacobian.

+ 4 pts Some progress computing integrals directly. At minimum set up one area integral correctly.

+ 0 pts No significant progress / No valid justification.

Common mistakes (I didn't indicate whether you made these, but they are for your reference)

+ 0 pts Computing arc length integrals. Or computing $\int_0^{2\pi} R(\theta)\mathrm{d}\theta$. Or other integral that computes something that is not area.

+ 0 pts The idea that the transformation doesn't change the area.

+ 0 pts If you had two different answers, it may have lowered your score, unless it was clear that you scratched one out.

Question 7

(no title) **10** / 10 pts

Divergence

✔ **+ 10 pts** Correct (applies divergence theorem, uses div(curl)=0).

+ 7 pts Applies the divergence theorem and made progress computing the resulting integral, or insufficient justification.

+ 4 pts Some progress (e.g. applies divergence theorem or finds that $\mathrm{div} G = 0$)

Stokes

+ 10 pts Correct (applies Stokes theorem by breaking the surface up into two pieces, showing cancellation of boundary integrals).

+ 8 pts Uses Stokes theorem by saying the boundary is empty.

+ 3 pts Uses Stokes to set up a valid line integral for some part of the surface.

Direct computation

+ 10 pts Correct complete computation.

+ 7 pts Incorrect complete computation.

+ 5 pts Good progress. (e.g. sets up a valid integral to compute $\iint_S \vec{G} \cdot \vec{n}\, \mathrm{d}S$ by finding $r_\theta \times r_z$)

+ 2 pts Some progress (e.g. computes curl F)

+ 0 pts No significant progress / no justification.

Comments (some might apply to you even if I forgot to select them)

+ 0 pts Applies divergence theorem to F

+ 0 pts It's not actually a sphere

+ 0 pts The fact that the normals point in opposite directions doesn't tell you that the integral cancels out over the two hemispheres. For example, find the flux of the the vector field <0,0,z> across the sphere

Question 8

(no title) 7 / 10 pts

✔ **+ 5 pts** Stokes' Thereom

✔ **+ 5 pts** Correctly evaluates line integral to 6π

+ 5 pts Uses Stokes' backwards or divergence theorem to change to surface integral over disc of radius $\sqrt{3}$ and correctly evaluates surface integral to 6π

+ 3 pts Uses divergence theorem on non-closed surface.

+ 1 pt Something somewhat relevant

− 1 pt Minor error

✔ **− 3 pts** Major error

+ 2 pts Largely incorrect or irrelevant integral calculations

+ 1 pt Correct curl

+ 0 pts Nothing of relevance

Question 9

(no title) **10** / 10 pts

✔ **+ 10 pts** Correct

Gradient of $g(x, y, z) = xf(y/x) - z$ or equivalent

+ 2 pts (Product rule, chain rule) $g_x = f(y/x) - (y/x)f'(y/x)$

+ 2 pts (Chain rule) $g_y = f'(y/x)$

+ 1 pt $g_z = -1$

Tangent plane

+ 1 pt An arbitrary point $(x_0, y_0, x_0 f(y_0/x_0))$ **on the surface**

+ 2 pts Use the gradient **at the arbitrary point above** as a normal vector to derive a correct plane equation

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+ 2 pts Conclude that $(0, 0, 0)$ is always on the plane given previous steps are correct/almost correct.

+ 0 pts No relevant progress

− 1 pt Point adjustment

− 2 pts Point adjustment

Question 10

(no title) 6 / 10 pts

+ 10 pts Correct

✔ **+ 2 pts** Correct ∇f

✔ **+ 2 pts** Correct $\nabla \cdot \nabla f$

✔ **+ 3 pts** Divergence Theorem

+ 3 pts Correct surface with explanation

✔ **− 1 pt** Extremely minor error, insufficient explanation

− 2 pts Minor error

− 3 pts Major error

+ 1 pt Something relevant

+ 0 pts Click here to replace this description.

Final - Math 53, Fall 2023
J.A. Sethian
Wed., Dec. 13, 2023

YOUR NAME IS Minnoli Raghavan

YOUR TA'S NAME IS Xianglong Ni

YOUR SECTION NUMBER IS 101

- There are ten problems. Each is worth ten points, for a total of 100 points.
- Hunt and choose. Look through the exam, find the easy ones, and do them!
- Do all work on the Exam.
- If you need more space, continue on to the back (and say there's more there).
- An answer with no explanation (except for the true/false questions) will receive no credit.
- Write legibly.
- Stay Calm: Show us what you know.
- **PUT A BOX AROUND YOUR ANSWERS!!!!!!**

A formula: If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ then

$$\vec{u} \times \vec{v} = (u_2 * v_3 - u_3 * v_2, -(u_1 * v_3 - u_3 * v_1), u_1 * v_2 - u_2 * v_1)$$

Another formula:

$$\text{determinant} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (ad - bc)$$

and another formula:

$$\text{determinant} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - ge)$$

Possibly useful hint: If $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$, then

$$\nabla \times \vec{F} = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})$$

Question 1a

Find the points on the surface $xyz = 1$ such that the tangent plane is parallel to surface $x + 2y + 3z = 0$.

$\vec{n} = \langle 1, 2, 3 \rangle$

normal to $f(x,y,z) = xyz$ level set is $\nabla f = \langle yz, xz, xy \rangle$

when the planes are parallel, the normals are parallel, so $\nabla f = \lambda \vec{n}$ and $xyz = 1$

$x \neq 0$
$y \neq 0$
$z \neq 0$

$$\begin{cases} yz = \lambda \\ xz = 2\lambda \\ xy = 3\lambda \\ xyz = 1 \end{cases}$$

$yz = \frac{x}{2} z \rightarrow 2y = x$

$yz = y\frac{x}{3} \rightarrow 3z = x$

$x\left(\frac{x}{2}\right)\left(\frac{x}{3}\right) = 1 \rightarrow x = \sqrt[3]{6} \quad y = \frac{\sqrt[3]{6}}{2} \quad z = \frac{\sqrt[3]{6}}{3}$

$\boxed{\left\langle \sqrt[3]{6}, \frac{\sqrt[3]{6}}{2}, \frac{\sqrt[3]{6}}{3} \right\rangle}$

Question 1b:

Find the extreme (maximal and minimal) values of the function $x^2 + y^2 + z^2$ such that $x^4 + y^4 + z^4 = 1$. Be sure to give the input points for these extreme values.

Lagrange multipliers

let $f(x,y,z) = x^2 + y^2 + z^2$
and $g(x,y,z) = x^4 + y^4 + z^4$

$\nabla f = \langle 2x, 2y, 2z \rangle = \lambda \langle 4x^3, 4y^3, 4z^3 \rangle = \lambda \nabla g$

$$\begin{cases} x = \lambda 2x^3 \\ y = \lambda 2y^3 \\ z = \lambda 2z^3 \\ x^4 + y^4 + z^4 = 1 \end{cases}$$

		$f(x,y,z)$
if $x = y = 0$ then $z = 1$	$(0,0,1)$	1
if $x = z = 0$ then $y = 1$	$(0,1,0)$	1
if $y = z = 0$ then $x = 1$	$(1,0,0)$	1
if $x = 0$, $y \& z \neq 0$ then $y^2 = z^2 = \frac{1}{\sqrt{2}}$	$\left(0, \pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right)$	$\sqrt{2}$
if $x \& y \& z \neq 0$ then $x^2 = y^2 = z^2 = \frac{1}{\sqrt{3}}$	$\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\frac{1}{\sqrt[4]{3}}, \pm\frac{1}{\sqrt[4]{3}}\right)$	$\sqrt{3}$

same if any one of x y or z is 0

max

min

Question 2:

Find the volume of the ball $\rho \leq a$ (where a is a number) that lies between the cones $\phi = \pi/6$ and $\phi = \pi/3$. Here, the cones are described in spherical coordinates.

$$\int_0^{2\pi}\int_{\pi/6}^{\pi/3}\int_0^a \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/6}^{\pi/3} \frac{a^3}{3}\sin\phi \, d\phi\, d\theta$$

$$= \int_0^{2\pi} \frac{a^3}{3}\Big[-\cos\phi\Big]_{\pi/6}^{\pi/3} d\theta \qquad -\frac{1}{2}-\left(-\frac{\sqrt{3}}{2}\right)$$

$$= \frac{a^3}{3}\left(\frac{\sqrt{3}-1}{2}\right)\int_0^{2\pi} d\theta$$

$$= \boxed{\frac{a^3\pi}{3}(\sqrt{3}-1)}$$

Question 3:

$f(t)$ $\quad$ $t = x^3 - y^3$

Let $z = f(x^3 - y^3)$, where f is a twice differentiable function.

Let $g(x, y) = e^{-xy} + y^2 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y}$. $= e^{-xy} + y^2 \frac{dz}{dt}\frac{\partial t}{\partial x} + x^2 \frac{dz}{dt}\frac{\partial t}{\partial y} = e^{-xy} + y^2(3x^2)\frac{dz}{dt} + x^2(-3y^2)\frac{dz}{dt}$

Find the extreme (maximal and minimal) values of g on the region $x^2 + 4y^2 \le 1$. $= e^{-xy}$

Be sure to give the input values for these extreme values.

$g(x,y) = e^{-xy}$

$g_x = -ye^{-xy}$ $\quad$ critical pt @ $(0,0)$

$(g_x = g_y = 0$ @ $(0,0))$

$g_y = -xe^{-xy}$

$g(0,0) = 1$

on $x^2 + 4y^2 = 1$ $\quad$ let $h(x) = x^2 + 4y^2$

$\nabla g = \lambda \nabla h$

$x \,\&\, y \neq 0$ $\quad \begin{cases} -ye^{-xy} = \lambda 2x \\ -xe^{-xy} = \lambda 8y \\ x^2 + 4y^2 = 1 \end{cases}$ $\quad -4y^2 e^{-xy} = -xye^{-xy}$

$4y = x$

$16y^2 + 4y^2 = 1 \quad y = \pm\sqrt{\frac{1}{20}}$

$\downarrow$

candidates: $\left(4\sqrt{\frac{1}{20}}, \sqrt{\frac{1}{20}}\right)$

$\left(-4\sqrt{\frac{1}{20}}, -\sqrt{\frac{1}{20}}\right)$

$g = e^{-\frac{4}{20}}$

Max: @ $(0,0)$ $g(0,0) = 1$

Min: @ $\pm\left(4\sqrt{\frac{1}{20}}, \sqrt{\frac{1}{20}}\right)$ $g = e^{-\frac{1}{5}}$

Question 4:

Use the transformation $x = u^2$, $y = v^2$, $z = w^2$ to find the volume of the region bounded by $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ and the coordinate planes.

$u + v + w = 1$

$$dV = \left| \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} \right| du\,dv\,dw = \left| \det \begin{bmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{bmatrix} \right| du\,dv\,dw$$

$$= 8uvw$$

$$\int_0^1 \int_0^{1-w} \int_0^{1-v-w} 8uvw \, du\,dv\,dw$$

$$= 4 \int_0^1 \int_0^{1-w} (1-v-w)^2 vw \, dv\,dw$$

$$= 4 \int_0^1 \int_0^{1-w} (1 - v - w - v + v^2 + wv - w + vw + w^2) vw \, dv\,dw$$

$$= 4 \int_0^1 \int_0^{1-w} (vw - 2v^2w - 2vw^2 + v^3w + 2v^2w^2 + vw^3) \, dv\,dw$$

$$= 4 \int_0^1 \frac{w}{2}(1-w)^2 - \frac{2}{3}w(1-w)^3 - w^2(1-w)^2 + \frac{w}{4}(1-w)^4 + \frac{2}{3}w^2(1-w)^2 + \frac{w^3}{2}(1-w) \, dw$$

$\frac{w^3}{2} - w^2 + \frac{w}{2} -$

$\frac{1}{4} - \frac{1}{3} + \frac{1}{4} = \frac{1}{2} - \frac{1}{3}$

$x = u^2 \quad y = v^2 \quad z = (1-u-v)^2$

$\vec{r}_u = \langle 2u, 0, -2(1-u-v) \rangle \qquad \vec{r}_v = \langle 0, 2v, -2(1-u-v) \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 4v(1-u-v), 4u(1-u-v), 0 \rangle$

Question 6:

Let C_1 be a simple closed curve in the $s-t$ plane given by

$$s = R(\theta)\cos\theta \quad t = R(\theta)\sin\theta \quad R(\theta) = 1 + \frac{1}{10}\sin(16\pi\theta) \quad 0 \le \theta \le 2\pi$$

Let α be a fixed number, and consider the transformation

$$x(s,t) = 8(\cos\alpha)s - (\sin\alpha)t + 6$$

$$y(s,t) = 8(\sin\alpha)s + (\cos\alpha)t + 7$$

Use this change of variables to transform C_1 into a new closed curve C_2.
Let A_1 be the area of C_1, and A_2 be the area of C_2.
What is $\frac{A_2}{A_1}$? Show the details of your calculation to explain your answer.

$$dA_2 = \left|\det\begin{bmatrix}\frac{\partial x}{\partial s} & \frac{\partial x}{\partial t}\\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t}\end{bmatrix}\right| dA_1$$

$$\frac{A_2}{A_1} = \left|\det\begin{bmatrix}\frac{\partial x}{\partial s} & \frac{\partial x}{\partial t}\\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t}\end{bmatrix}\right| = \left|\det\begin{bmatrix}8(\cos\alpha) & -\sin\alpha\\ 8\sin\alpha & \cos\alpha\end{bmatrix}\right|$$

$$= 8\cos^2\alpha + 8\sin^2\alpha$$

$$= \boxed{8}$$

Question 7:

Let S be the closed surface given by

$$x = R(z)\cos\theta \quad y = R(z)\sin\theta \quad z = z \quad R(z) = (1-z^2) \quad 0 \le \theta \le 2\pi, \quad -1 \le z \le 1$$

Let $\vec{F} = (2xy^3, xy^4z, z^3 + 8x)$

Let $\vec{G} = \nabla \times \vec{F}$

Find the flux of $\vec{G}$ through the surface S, that is, find

$$\iint_S \vec{G} \cdot \vec{n}\, dS$$

Since S is a closed surface and $\vec{G}$ comes from a curl, the flux will be 0. This can be shown with the Divergence Theorem:

$$\underbrace{\iint_S \vec{G}\cdot\vec{n}\,dS = \iiint_E (\nabla\cdot\vec{G})\,dV}_{S=\partial E} = \iiint_E \nabla\cdot(\nabla\times\vec{F})\,dV$$

$$= \iiint_E 0\,dV$$

Since $\nabla\cdot(\nabla\times\vec{F})$ is always 0.

Question 8:

outward

Let S be the part of the sphere $x^2 + y^2 + (z-1)^2 = 4$ above the xy plane.

Let $\vec{F}$ be the vector field $(-y, x, 0)$.

Let $\vec{G} = \nabla \times \vec{F}$.

Find the flux of $\vec{G}$ through the surface S, that is, find

$$\iint_S \vec{G} \cdot \vec{n}\, dS$$

Stokes' thm

$$= \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

$C = \partial S$

$\partial S: \quad x^2 + y^2 = 3 \qquad z = 0$

$$\vec{r}(t) = \langle 3\cos t, 3\sin t, 0 \rangle \quad 0 \le t \le 2\pi$$

$$\int_0^{2\pi} \vec{F} \cdot r'(t)\, dt = \int_0^{2\pi} \langle -3\sin t, 3\cos t, 0\rangle \cdot \langle -3\sin t, 3\cos t, 0 \rangle dt$$

$$= \int_0^{2\pi} \underbrace{(9\sin^2 t + 9\cos^2 t)}_{9}\, dt$$

$$= \boxed{18\pi}$$

Question 9:

Suppose f is a differentiable function of one variable. Show that the tangent planes to the surface $z = xf(\frac{y}{x})$ all pass through the same point.

$$\text{tan. plane @ } (a,b): \; af(\tfrac{b}{a}) + \overbrace{\left(f(\tfrac{b}{a}) + af_x(\tfrac{b}{a})\right)}^{\frac{\partial z}{\partial x}\big|_{(a,b)}}(x-a) + \underbrace{af_y(\tfrac{b}{a})}_{\frac{\partial z}{\partial y}\big|_{(a,b)}}(y-b) = z$$

if $t = \frac{y}{x}$

$$f_x = \frac{df}{dt}\frac{\partial t}{\partial x} = f'(\tfrac{y}{x})\frac{-y}{x^2}$$

$$f_y = \frac{df}{dt}\frac{\partial t}{\partial y} = f'(\tfrac{y}{x})\frac{1}{x}$$

tan. plane: $$z = af(\tfrac{b}{a}) + \left(f(\tfrac{b}{a}) + a(f')(\tfrac{-b}{a^{\cancel{2}}})\right)(x-a) + \cancel{a}(f')(\tfrac{1}{\cancel{a}})(y-b)$$

when $x = y = 0$ it becomes

$$z = af(\tfrac{b}{a}) - af(\tfrac{b}{a}) + bf' - bf' = 0$$

all go through $(0,0,0)$

Question 10:

Let $f(x, y, z) = (\frac{x^2}{2} - \frac{2x^6}{30} - \frac{4y^6}{30} - \frac{6z^6}{30})$

Let S be a closed, non-intersecting surface in R^3 with a clearly defined inside and outside.

Let $A(S)$ be the outward normal flux of ∇f through all of S, that is,

$$A(S) = \iint_S \nabla f \cdot \vec{n} \, dS$$

Among all such surfaces, find the one (that is, write an equation for S) that makes $A(S)$ as big as possible.

$$\nabla f = \left\langle x - \frac{2}{3}x^5, -\frac{4}{5}y^5, -\frac{6}{5}z^5 \right\rangle$$

$$\underset{S=\partial E}{A(S)} = \iiint_E (\nabla \cdot \nabla f) \, dV = \iiint_E (1 - \frac{10}{3}x^4 - 4y^4 - 6z^4) \, dV$$

$$\int_a^b \int_c^d \int_e^f (1 - \frac{10}{3}x^4 - 4y^4 - 6z^4) \, dx \, dy \, dz$$

$$= \int_a^b \int_c^d \left[(1 - 4y^4 - 6z^4)x - \frac{2}{3}x^5\right]_e^f dy \, dz$$

Flux is greatest when the normal to S is in the direction of ∇f

Normal to the level sets of f point in the direction of ∇f

S can be a level set of f

$$\frac{x^2}{2} - \frac{2x^6}{30} - \frac{4x^6}{30} - \frac{6z^6}{30} = C$$

$$\nabla f \cdot \hat{n} = |\nabla f| \overset{1}{|\hat{n}|} \overset{1}{\cos\theta} = |\nabla f|$$