Homework 10

1. $x = e^{\alpha t}$ $\quad [\alpha^2 + \gamma\alpha + \omega_0^2]x = 0 \Rightarrow \alpha = \frac{-\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$

$x \neq 0$

a) $\frac{\gamma^2}{4} > \omega_0^2 \Rightarrow$ overdamped

$$x(t) = e^{-\frac{\gamma}{2}t \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\,t} = e^{-2t \pm t\sqrt{3}} = c_1 e^{t(\sqrt{3}-2)} + c_2 e^{t(-\sqrt{3}-2)}$$

$c_1 = 1 - c_2$

$x(0) = 1 = c_1 + c_2$

$(\sqrt{3}+2)c_2 = (\sqrt{3}-2)(1-c_2)$

$v(t) = (\sqrt{3}-2)c_1 e^{t(\sqrt{3}-2)} + (-\sqrt{3}-2)c_2 e^{t(-\sqrt{3}-2)}$

$= (\sqrt{3}-2) - (\sqrt{3}c_2 + \cancel{2c_2})$

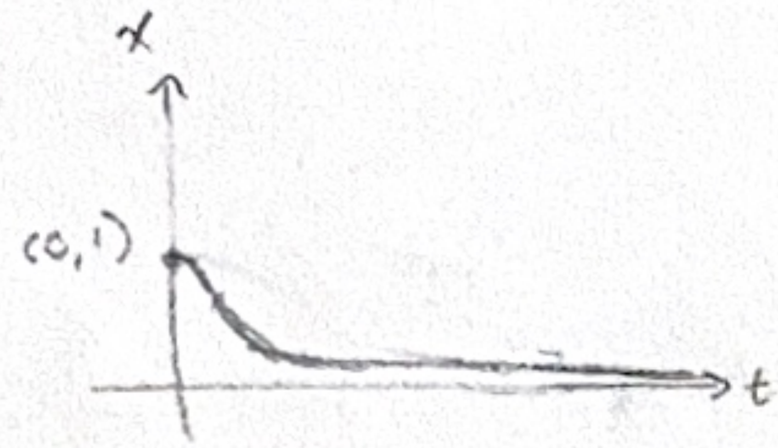
$v(0) = 0 = (\sqrt{3}-2)c_1 + (-\sqrt{3}-2)c_2$

$2\sqrt{3}\,c_2 = \sqrt{3} - 2$

$c_2 = \frac{\sqrt{3}-2}{2\sqrt{3}} = \frac{1}{2} - \frac{1}{\sqrt{3}}$

$$\boxed{x(t) = \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)e^{(\sqrt{3}-2)t} + \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)e^{(-\sqrt{3}-2)t}}$$

$c_1 = \frac{1}{2} + \frac{1}{\sqrt{3}}$



b) $\frac{\gamma^2}{4} = \omega_0^2 \Rightarrow$ critically damped

1 real solution $\alpha = \frac{-\gamma}{2}$

$x = c_1 e^{-\frac{\gamma}{2}t}$ need 2nd constant

$x = c_1 e^{-\frac{\gamma}{2}t} + c_2 t e^{-\frac{\gamma}{2}t}$

$x = c_1 e^{-t} + c_2 t e^{-t}$

check

$a\ddot{x} + b\dot{x} + cx = 0$ for $x = te^{\alpha t}$

$a(2\alpha e^{\alpha t} + \alpha^2 t e^{\alpha t}) + b(e^{\alpha t} + \alpha t e^{\alpha t}) + cte^{\alpha t}$

$= (2a\alpha + b)e^{\alpha t} + (a\alpha^2 + b\alpha + c)te^{\alpha t}$

$= 0$

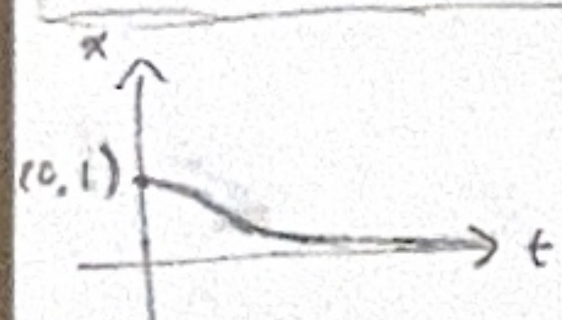
$c_2 = c_1 = 1$

$x(0) = 1 = c_1$

$\boxed{x(t) = e^{-t}(1+t)}$

$v(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}$

$v(0) = 0 = c_2 - c_1$



c) $\frac{\gamma^2}{4} < \omega_0^2 \Rightarrow$ underdamped

$\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = \sqrt{\frac{3}{4}}$

$\alpha = \frac{\gamma^2}{2} \pm i\omega$

$x = X_0 e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t + \phi\right)$

$x(0) = 1 = X_0\cos(\phi)$

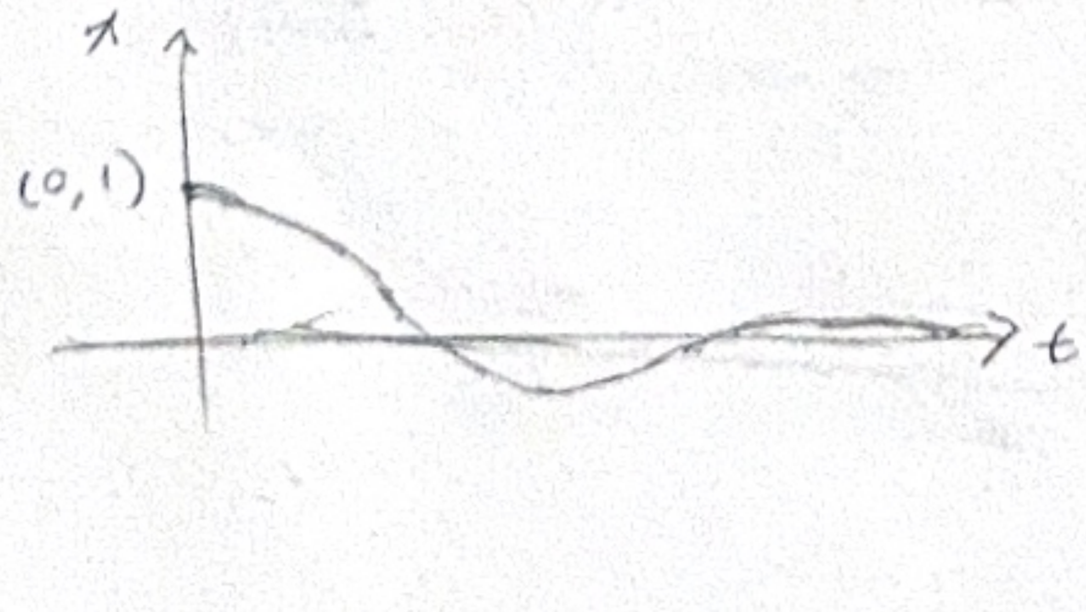
$v(t) = -\frac{1}{2}X_0 e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t + \phi\right) - X_0 e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t + \phi\right)\frac{\sqrt{3}}{2}$

$v(0) = 0 = -\frac{1}{2}X_0(\cos\phi + \sqrt{3}\sin\phi)$

$X_0 = \frac{1}{\cos\phi} \Rightarrow 0 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\tan\phi \Rightarrow \tan\phi = \frac{-1}{\sqrt{3}} \Rightarrow \phi = \frac{5\pi}{6}$

$X_0 = \frac{1}{\cos\left(\frac{5\pi}{6}\right)} = \frac{-2}{\sqrt{3}}$

$$\boxed{x = \frac{-2}{\sqrt{3}} e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t + \frac{5\pi}{6}\right)}$$



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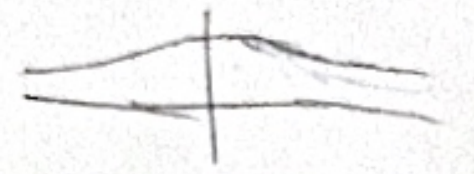
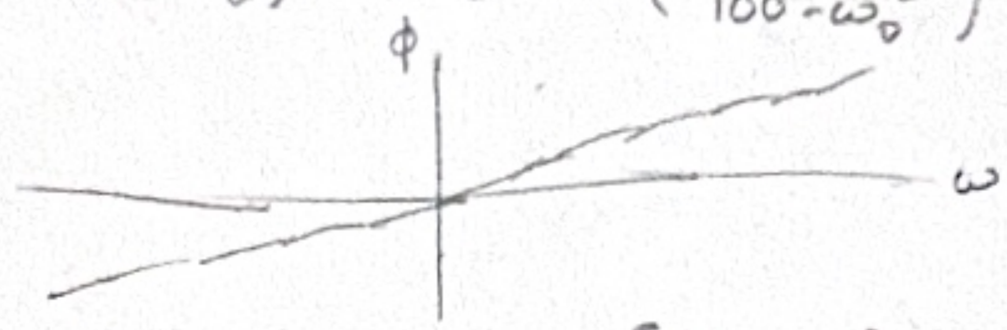
2. $X_o = \frac{F_o}{m}\frac{1}{\sqrt{(\omega_o^2-\omega^2)^2+(\omega\gamma)^2}}$ $\quad \phi = \arctan\left(\frac{\omega\gamma}{\omega_o^2-\omega^2}\right)$ $\quad Q = \frac{\omega_o}{\gamma}$

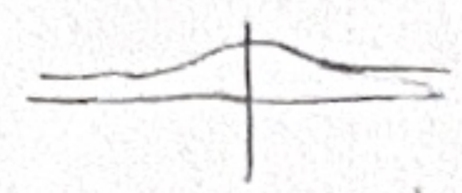
$A = \frac{1\{f_o \cos(\omega 0)\}}{\sqrt{(100+\omega^2)^2+(\omega\gamma)^2}}$ $\quad \phi = \arctan\left(\frac{\omega\gamma}{100-\omega^2}\right)$ $\quad Q = \frac{10}{\gamma} \Rightarrow \gamma = \frac{10}{Q}$

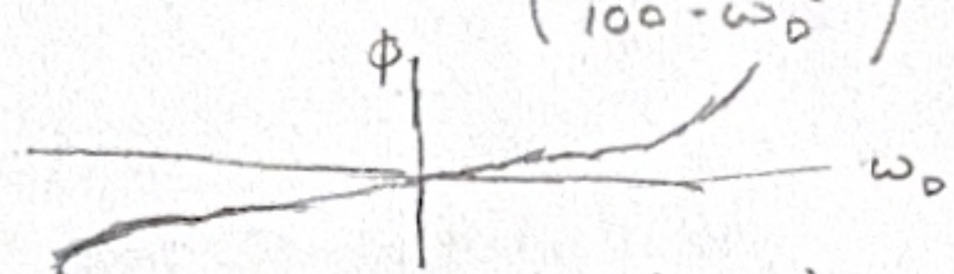
a) $\gamma = 10$ $\quad A(\omega_o) = \frac{1}{\sqrt{(100+\omega_o^2)^2+(10\omega_o)^2}}$ $\quad \phi(\omega_o) = \arctan\left(\frac{10\omega_o}{100-\omega_o^2}\right)$



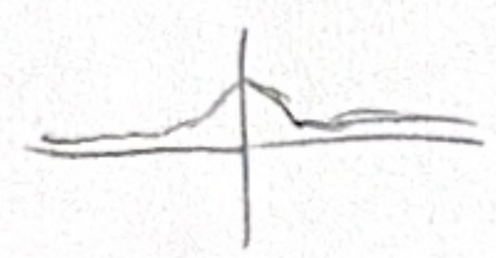


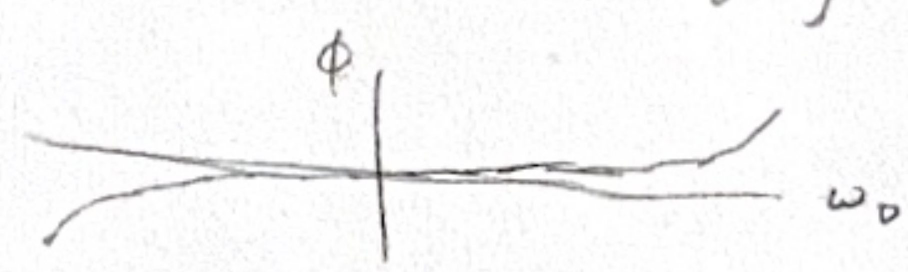
b) $\gamma = 5$ $\quad A(\omega_o) = \frac{1}{\sqrt{(100+\omega_o^2)^2+(5\omega_o)^2}}$ $\quad \phi(\omega_o) = \arctan\left(\frac{5\omega_o}{100-\omega_o^2}\right)$





c) $\gamma = 1$ $\quad A(\omega_o) = \frac{1}{\sqrt{(100+\omega_o^2)^2+\omega_o^2}}$ $\quad \phi(\omega_o) = \arctan\left(\frac{\omega_o}{100-\omega_o^2}\right)$





3. (11.7)

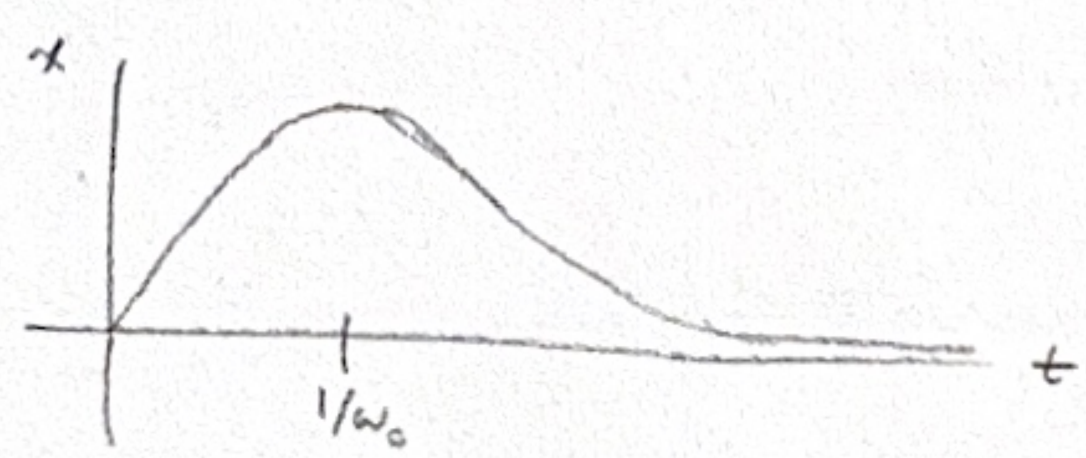
a) $x = (A+Bt)e^{-\omega_o t}$ $\quad \dot{x} = (B-\omega_o(A+Bt))e^{-\omega_o t}$ $\quad \ddot{x} = (-2B\omega_o+\omega_o^2(A+Bt))e^{-\omega_o t}$

$$\ddot{x}+2\omega_o\dot{x}+\omega_o^2x = e^{-\omega_o t}\left(\cancel{-2B\omega_o}+\cancel{\omega_o^2(A+Bt)}+\cancel{2\omega_o}(B-\omega_o\cancel{(A+Bt)})+\omega_o^2\cancel{(A+Bt)}\right)$$

$$= e^{-\omega_o t}(0) = 0 \checkmark$$

b) $x(0) = 0 = A$ $\quad \dot{x}(0) = \frac{I}{m} = B$

$x = \frac{I}{m}te^{-\omega_o t}$



$\left.\frac{dx}{dt}\right|_{t_{max}} = \frac{I}{m}e^{-\omega_o t_{max}}(1-\omega_o t_{max}) = 0 \quad \Rightarrow t_{max} = \frac{1}{\omega_o}$

4. (11.9)

$x = X_o\cos(\omega t+\phi)$ $\quad F = f_o\cos(\omega t)$

$v = -\omega X_o\sin(\omega t+\phi)$

In phase $\rightarrow$ $-\sin(\omega t+\phi) = \cos(\omega t)$ $\quad \phi = -\frac{\pi}{2} \pm 2\pi k \quad k \in \mathbb{Z}$

$-\sin(\omega t)\cos(\phi) - \cos(\omega t)\sin(\phi) = \cos(\omega t)$

$-\sin\omega t\cos\phi = \cos\omega t(1+\sin\phi)$

$\omega t \to 0,\pi \Rightarrow \sin\phi = -1$

$\omega t \to \pm\frac{\pi}{2} \Rightarrow \cos\phi = 0$

5. (11.11) $x = X_0 \cos(\omega t + \phi)$ $\quad v = -\omega X_0 \sin(\omega t + \phi)$ $\quad$ Time average of $\sin^2 = \cos^2 = \frac{1}{2}$ by integration or $\sin^2 + \cos^2 = 1$

$$E_{av} = \frac{1}{2}k x_{av}^2 + \frac{1}{2} m v_{av}^2 = \frac{1}{2}k\frac{1}{2}X_0^2 + \frac{1}{2}m\frac{1}{2}\omega^2 X^2 \approx \frac{1}{4}kX_0^2 + \frac{1}{4}m\omega_0^2 X_0^2 = \frac{1}{2}kX_0^2$$

$$\omega_0^2 = \frac{k}{m}$$

$$E_{diss} = \frac{P_{av}}{t_{ram}} = \frac{vF}{\frac{1}{\omega}} = \frac{bX_0^2\omega^2 \sin^2(\omega t + \phi)}{1/\omega} = \frac{1}{2}bX_0^2\omega$$

$$\gamma = \frac{b}{m} \qquad \frac{k}{\omega_0} = m\omega_0$$

$$\frac{E_{av}}{E_{diss}} = \frac{\frac{1}{2}kX_0^2}{\frac{1}{2}bX_0^2\omega} = \frac{k}{b\omega} \approx \frac{k}{b\omega_0} = \frac{\omega_0}{\gamma} = Q$$

6. (11.14)

a) $x(t) = x_a(t) + x_b(t)$ $\qquad \ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$ for free damped oscillator

$= x_a(t) + \ddot{x}_b + \gamma\dot{x} + \omega_0^2 x$

$x(t) = x_a(t) + 0$ $\quad$ so if $x_a(t)$ is a solution, $x(t)$ is a solution

b) $x_a = X_0\cos(\omega t + \phi)$

$x_b = Ae^{-\gamma/2 t}\cos(\omega_1 t + \theta)$

$\omega_1 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$

$x = X_0\cos(\omega t + \phi) + Ae^{-\gamma/2 t}\cos(\omega_1 t + \theta)$

$x(0) = 0 = X_0\cos(\phi) + A\cos(\theta)$

$v(t) = -\omega X_0 \sin(\omega t + \phi) - Ae^{-\gamma/2 t}\left(\gamma/2 \cos(\omega_1 t + \theta) + \sin(\omega_1 t + \theta)\,\omega_1\right)$

$v(0) = 0 = -\omega X_0 \sin\phi - A\left(\frac{\gamma}{2}\cos(\theta) + \omega_1 \sin(\theta)\right)$

$A\cos\theta = -X_0\cos\phi$

$A\sin\theta = \left(-\omega X_0 \sin\phi - A\cos\theta\,\frac{\gamma}{2}\right)\frac{1}{\omega_1}$

$A\sin\theta = \frac{X_0}{\omega_1}\left(\frac{\gamma}{2}\cos\phi - \omega\sin\phi\right)$

$\cos(a+b) = \cos a\cos b - \sin a \sin b$

$x = X_0\cos(\omega t + \phi) + e^{-\gamma/2 t}\left(A\cos\theta\cos(\omega_1 t) - A\sin\theta\sin(\omega_1 t)\right)$

$$\boxed{x(t) = X_0\left(\cos(\omega t + \phi) - e^{-\gamma/2 t}\left(\cos\phi\cos(\omega_1 t) + \frac{\sin(\omega_1 t)}{\omega_1}\left(\frac{\gamma}{2}\cos\phi - \omega\sin\phi\right)\right)\right)}$$

c) At resonance

$\phi = \frac{\pi}{2}$; $\omega = \omega_0$; $X_0 = \frac{F_0}{m\omega_0\gamma}$

$\cos\phi = 0$

$\sin\phi = 1$

$x(t) = \frac{-F_0}{m\omega_0\gamma}\sin(\omega t) + e^{-\gamma/2 t}\frac{\omega_0}{\omega_1}X_0 \sin(\omega_1 t)$

$x(t) = -\frac{F_0}{m\omega_0\gamma}\sin(\omega t) + \frac{F_0}{m\omega_1\gamma}e^{-\gamma/2 t}\sin(\omega_1 t)$