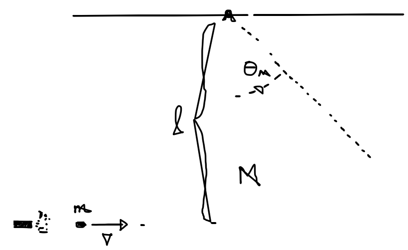
Problem 1. Bang. 25 pts



A rifle fires a point-like bullet of mass m with velocity $\vec{v} = v\hat{i}$ towards a pendulum. The pendulum is a thin rod of length ℓ and mass $M = 2m$, with moment of inertia $I = \frac{1}{12}M\ell^2$ about its center, which hangs from a frictionless pivot. The pendulum is initially at rest, and the bullet collides completely inelastically with the bottom of the pendulum, sticking to it. The resulting bullet-pendulum system swings up under the influence of gravity g.

a) What quantities of the bullet-pendulum system are conserved *during the instant of the collision*?

The angular momentum $\vec{L}$ and the momentum P^y.

b) What quantities of the bullet-pendulum system are conserved during the entire duration of the problem?

There are no conserved quantities. (However, energy is conserved *after* the collision)

c) What is the maxima angle θ_M that the pendulum swings up to? Your answer may depend on v, ℓ, m, g. Hint: you can phrase your answer in the form $1 - \cos(\theta_M) = \cdots$.

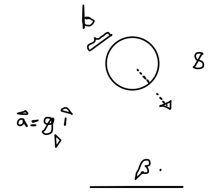
The moment of inertia of the combined systems about the pivot is $I_T = I + M(\ell/2)^2 + m\ell^2 = \frac{5}{3}m\ell^2$. Because angular momentum is conserved during the collision, just after the collision $\ell mv = I_T\dot{\theta}$. The kinetic energy is thus $E = \frac{1}{2}I_T\dot{\theta}^2 = \frac{1}{2I_T}(\ell mv)^2$. If the pendulum swings to θ, the change in gravitational potential energy is $\Delta U = g(1-\cos(\theta))(M\ell/2 + m\ell)$. So at the maximum $(1-\cos(\theta))2m\ell g = \frac{1}{2I_T}(\ell mv)^2$. Simplifying, $1 - \cos(\theta_M) = \frac{3v^2}{20g\ell}$

d) The pendulum will proceed to swing back and forth. If $\theta_M \ll 1$, what is the period T of the oscillation?

$I_T\ddot{\theta} = \tau^z = -g\sin(\theta)(M\ell/2 + m\ell) = -2gm\ell\sin(\theta)$. So for small oscillations, $\ddot{\theta} + \frac{2gm\ell}{\frac{5}{3}m\ell^2}\theta = 0$.

So $\omega_0 = \sqrt{\frac{6g}{5\ell}}$, and $T = 2\pi/\omega_0 = 2\pi\sqrt{\frac{5\ell}{6g}}$.

Problem 2. Rollin'. 25 pts.



A disk of mass M, radius b, and moment of inertia $I_0 = \frac{1}{2}Mb^2$ rolls down an incline plane of angle $\beta > \pi/4$ without slipping. The incline plane is being pushed so that it has a *fixed* acceleration $\vec{a} = a\hat{i}$; in particular suppose $a = g$. The disk is released from rest relative to the plane, and falls under the influence of gravity g.

When the disk has rolled a distance s along the ramp, what is its angular velocity $\dot{\theta}$? Your answer may depend on M, b, g, s, β. You can refer to K.K. Ex. 7.16, pg 275, if you find it helpful.

It is convenient to solve this problem by working in an accelerating frame and introducing a fictitious force so that bodies accelerate as $\vec{g} = -g\hat{i} - g\hat{j}$. Since gravity now points at 45°, with magnitude $g' = \sqrt{2}g$, we can tilt the whole picture by 45°. Gravity now points downwards again, but the effective slope is reduced by $\beta' = \beta - \frac{\pi}{4}$. Furthermore, the body "drops" by a reduced amount $h' = s\sin(\beta - \pi/4)$ in this frame. The problem is now identical to KK 7.16: the velocity is $V = \sqrt{\frac{4g'h'}{3}}$, and $\dot{\theta} = -V/b$. So $\dot{\theta} = -\sqrt{\frac{4\sqrt{2}gs\sin(\beta-\pi/4)}{3}}/b$. The sign reverses for $\beta < \pi/4$.

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Problem 3. Slidin'. 20 pts. A particle of mass m moves in 1D under the influence of a potential $U(x) = U_0\left(\frac{x}{x_0} - \frac{1}{3}\left(\frac{x}{x_0}\right)^3\right)$, $U_0, x_0 > 0$ and friction $F_f = -b\dot{x}$ for $b > 0$.

a) Make a rough sketch of the potential $U(x)$, making sure to capture the existence of any minima and maxima and the behavior as $x \to -\infty$, $x \to \infty$.

b) Which x are stable and unstable equilibrium points? Give expressions for these x in terms of U_0, x_0.

Equilibria occur for $\frac{dU}{dx} = 0 = \frac{U_0}{x_0}\left(1 - \left(\frac{x}{x_0}\right)^2\right)$. So $x = \pm x_0$ are equilibrium positions. For stability, we check $\frac{d^2U}{dx^2} = -2\frac{U_0}{x_0^2}\frac{x}{x_0}$. For $x = -x_0$, $U'' > 0$ (potential minima), so it is stable. For $x = x_0$, $U'' < 0$ (minima) so it is unstable.

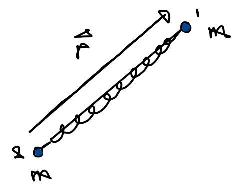
c) Suppose the particle is undergoing *small* oscillations about the stable equilibrium point. For what range of $0 \le b < b_*$ will the small oscillations be underdamped? b_* may depend on U_0, x_0, m.

The effective spring constant for stable minima is the curvature of the potential, $k = 2\frac{U_0}{x_0^2}$. So $\omega_0 = \sqrt{k/m} = \sqrt{2\frac{U_0}{mx_0^2}}$. With $\gamma = b/m$, the motion is underdamped for $\gamma < 2\omega_0$, giving $b < 2m\sqrt{2\frac{U_0}{mx_0^2}}$.

d) The particle is released from rest at $x(t = 0) = -2x_0$. Will the particle escape to $x \to \infty$? Explain.

The particle will not escape. Note that the potential energy of the starting point is equal to the potential energy at the $x = x_0$ maxima: $U(-2x_0) = U(x_0)$. So in the *absence* of friction, the particle would have just enough energy to reach this hump. But with friction, it will lose energy, so by the time $x \to x_0$ it will run out of energy and roll back down.

Problem 4. Do-si-do . 30 pts.



Two point particles at $\vec{r}_1(t), \vec{r}_2(t)$ each have mass m and are attached by a massless spring with spring constant k and equilibrium length $\ell = 0$. There are no other forces in the problem, and the initial conditions are such that all motion is in the plane of the page. As usual, define the relative displacement $\vec{r} = \vec{r}_1 - \vec{r}_2$, $r = |\vec{r}|$, $\vec{r} = r\,\hat{r}$.

a) Suppose that at time $t = 0$s the system has initial conditions $\vec{r}_1(0) = -3\hat{\imath}$m, $\vec{v}_1(0) = 2\hat{\jmath}$ m/s, $\vec{r}_2(0) = 3\hat{\imath}$ m, $\vec{v}_2(0) = -4\hat{\jmath}$ m/s. What is the location of the center of mass $\vec{R}(t)$ for subsequent times? Your answer may depend on m, k, t and numerical constants.

The CM motion proceeds in a line because there are no external forces: $\vec{R}(t) = -\hat{\jmath}$m/s.

b) Find an equation of motion for the relative distance r of the form

$$\mu\ddot{r} = f_{\text{eff}}(r) \tag{1}$$

where μ and f_{eff} should be given in terms of m, k, r and the initial relative angular momentum L. You do not need to derive everything from scratch.

Define $\mu = \frac{m^2}{m+m} = m/2$. Then $\mu\ddot{r} = -kr + \frac{L^2}{\mu r^3}$.

c) What is the radius $r_* = |\vec{r}|$ and period T of a *circular* orbit of angular momentum L? Your answers may depend on m, k and L.

A circular orbit occurs for $f_{\text{eff}} = 0$, giving $r_* = [\frac{2L^2}{mk}]^{1/4}$. To find the period, note that $L = \mu r^2\dot{\theta}$ so $\dot{\theta} = \frac{L}{\mu r_*^2} = \sqrt{2\frac{k}{m}}$. $T = 2\pi/\dot{\theta} = 2\pi\sqrt{\frac{m}{2k}}$

d) Does the problem have any unbound orbits? (Unbound meaning orbits in which $r \to \infty$ at long times.)

No, because the potential energy grows like r^2, it would require infinite energy for the two masses to unbind.

e) Suppose the initial conditions are such that $\vec{R}_{CM}(t) = 0$, but with an initial separation $r \neq r_*$. Does the resulting orbit in the 2D plane exactly repeat itself, as for the gravitational Kepler problem, or is it aperiodic? Give a calculation or argument to support your claim.

The motion will be exactly periodic - this isn't true for central force motion in general, but is a special property of $f = -kr$. To see this, suppose we forget about using polar coordinate and use Cartesian instead (though it is possible to derive this in polar language too). Letting the relative displacement be $\vec{r} = x\hat{\imath} + y\hat{\jmath}$, the equations of motion are

$$\mu\ddot{x} = -kx \tag{2}$$

$$\mu\ddot{y} = -ky \tag{3}$$

This is just the equation for two decoupled harmonic oscillators! Hence the general solution is $x = A_x\cos(\omega t + \phi_x)$, $y = A_y\cos(\omega t + \phi_y)$ for the *same* frequency $\omega = \sqrt{k/\mu}$. Because the x, y both oscillate at the same ω, the motion is perfectly periodic.