

1 Kinematics in 1D

1.1 Quantities

1. Position (\vec{x})

- Direction indicates which side of the origin the object is located at
- Subscript specifies condition e.g. \vec{x}_i or \vec{x}_f represents initial or final positions

2. Displacement (\vec{S})

- Represents the change in an object's position, inclusive of direction
- $\vec{S} = \vec{x}_f - \vec{x}_i$
- Distance is the length travelled irrespective of direction
- Distance is a scalar quantity, while displacement is a vector
 $d = |\vec{S}|$

3. Time (Δt)

4. Velocity (\vec{v})

- Rate of change of position
- $\vec{v} = \frac{\Delta \vec{x}}{\Delta t}$
- Usually, \vec{u} represents initial velocity while \vec{v} represents final velocity
- Speed is the scalar quantity associated with velocity

5. Acceleration (\vec{a})

- Rate of change of velocity
- $\vec{a} = \frac{\Delta \vec{v}}{\Delta t}$

1.2 Kinematics Equations

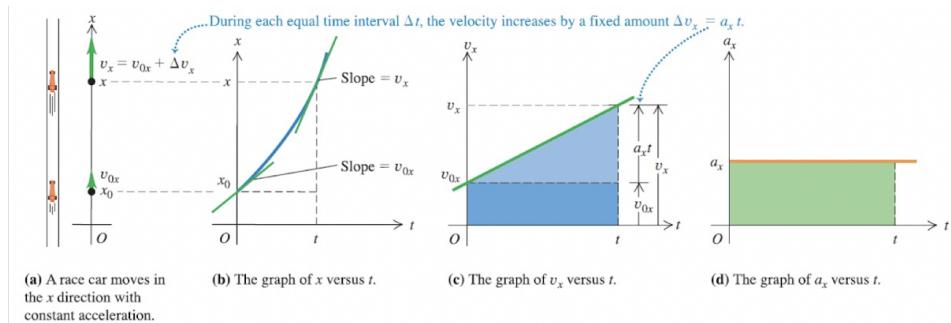
$$1. S = ut + \frac{1}{2}at^2$$

$$2. v = u + at$$

$$3. v^2 = u^2 + 2aS$$

$$4. S = \frac{1}{2}(u + v)t$$

1.3 Motion Diagrams

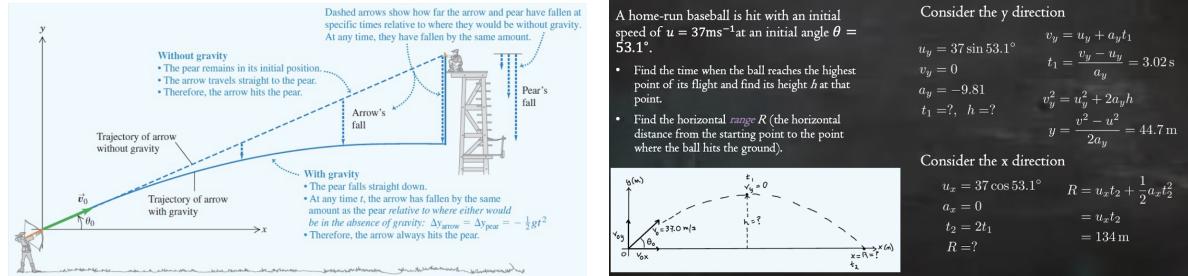
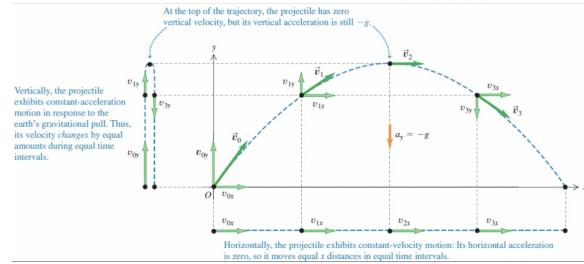


2 Projectile Motion

By resolving vector quantities into their x and y components, 2D motion can be (mostly) treated as two separate 1D motions

- If gravity is the only force acting
 - $a_x = 0$
 - $a_y = -9.81$

- Trajectory of projectile motion is parabolic, and depends on the initial velocity u and the acceleration due to gravity
- Given the initial velocity, we can find the height (H) and range (R) of the trajectory
 - $H = \frac{u^2 \sin^2 \theta}{2g}$
 - $R = \frac{u^2 \sin 2\theta}{2g}$



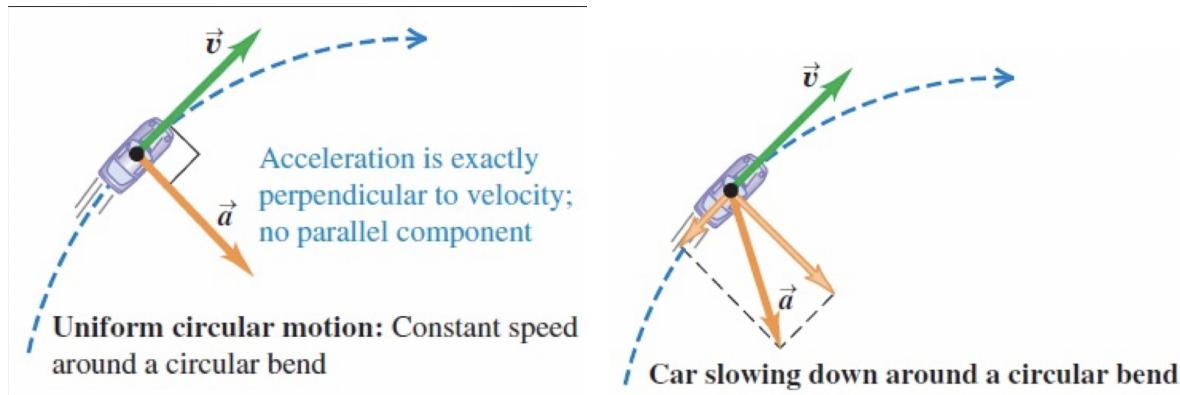
3 Circular Motion

Since velocity is a vector quantity, velocity changes as an object changes direction to travel in a circle, and an acceleration must exist by definition (even if speed is constant)

3.1 Centripetal Acceleration

The acceleration that keeps an object moving in a circle without changing its speed

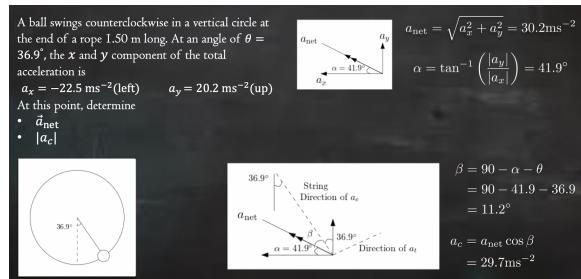
- $a_c = \frac{v^2}{r}$
- Directed towards the center of the circle
- Since \vec{v} is tangent to the circle, and Δv points toward the center of the circle, therefore, a_c is perpendicular to \vec{v}
- Since perpendicular vectors are independent, therefore, centripetal acceleration does not affect the speed of the object in the tangential direction
- However, a_c gives the object a velocity component in the perpendicular direction, thus changing the direction of \vec{v}



3.2 Non-Uniform Circular Motion

When the speed of the object is changing, net acceleration is **not** perpendicular to velocity

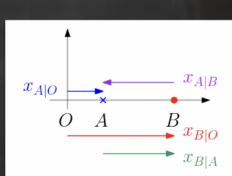
- When tangential acceleration a_t is non-zero, v no longer remains constant
- Therefore, centripetal acceleration a_c is also not constant
- We can split a_{net} into its components - tangential and perpendicular to the circle - to find a_t and a_c respectively



4 Relative Motion

- Velocity and displacement are relative quantities
- When a reference frame is not mentioned, Earth is generally assumed to be the reference frame

- For relative position (and velocity) questions, we use subscripts to identify the reference frame
 $\vec{x}_{\text{object}|\text{ref frame}}$ $\vec{v}_{\text{object}|\text{ref frame}}$
- For instance, in the diagram
 - $\vec{x}_{A|O}$ is the position of A from origin O (\vec{x}_A)
 - $\vec{x}_{B|O}$ is the position of B from origin O (\vec{x}_B)
 - $\vec{x}_{B|A}$ is the relative position of B from A
 - $\vec{x}_{A|B}$ is the relative position of A from B
 Note the different direction between $\vec{x}_{B|A}$ and $\vec{x}_{A|B}$
- Given an origin O, relative position can be found
 $\vec{x}_{A|B} = \vec{x}_{A|O} - \vec{x}_{B|O} = \vec{x}_{A|O} + \vec{x}_{O|B}$



- As velocity and displacement are related as

$$\vec{v} = \frac{\Delta \vec{x}}{t},$$

we expect velocity to obey a similar equation

$$\vec{v}_{A|B} = \vec{v}_{A|O} - \vec{v}_{B|O} = \vec{v}_{A|O} + \vec{v}_{O|B}$$

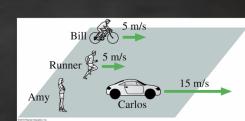
- Therefore for instance,

$$\vec{v}_{R|B} = \vec{v}_{R|O} - \vec{v}_{B|O} = 0$$

$$\vec{v}_{R|C} = \vec{v}_{R|O} - \vec{v}_{C|O} = -10\text{ms}^{-1}$$

and

$$\vec{v}_{A|R} = \vec{v}_{A|O} - \vec{v}_{R|O} = -5\text{ms}^{-1}$$



The engine of a boat drives it across a river that is 1800m wide. The velocity of the boat relative to the water is 4.0ms^{-1} directed perpendicular to the current. The velocity of the water relative to the shore is 2.0ms^{-1} .

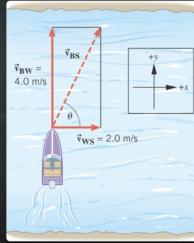
- What is the velocity of the boat relative to the shore?
- How long does it take for the boat to cross the river?

$$v_{W|S} = 2\text{ms}^{-1}$$

$$v_{B|W} = 4\text{ms}^{-1}$$

$$v_{B|S} = ?$$

$$t = ?$$



$$\vec{v}_{B|W} = \vec{v}_{B|S} - \vec{v}_{W|S}$$

$$\vec{v}_{B|S} = \vec{v}_{B|W} + \vec{v}_{W|S}$$

$$|v_{B|S}| = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

$$\theta = \tan^{-1} \left(\frac{v_{B|W}}{v_{W|S}} \right) = \tan^{-1} \left(\frac{4}{2} \right) = 63.4^\circ$$

Note that the displacement we have is along the y axis only. Therefore we need only the y-component of $v_{B|S}$ to calculate the time taken.

$$v_{B|S_y} = v_{B|S} \sin 63.5 = 4\text{ms}^{-1}$$

$$t = \frac{\Delta y}{v_{B|S_y}} = \frac{1800}{4} = 450\text{s}$$

5 Forces

5.1 Weight / Gravitational Force

Force that an object on or near the surface of an astronomical body experiences.

- Non-contact force
- $W = mg$
- Acts towards the center of the astronomical body

5.2 Tension

Force an object experiences due to a string/rope pulling at it.

- Contact force
- No fixed equation
- Acts along the string/rope (in both directions, depending on the object we are analysing)

5.3 Normal Reaction Force (N)

Force that is exerted on an object by a surface.

- Contact force
- No fixed equation
- Acts perpendicular to the surface

5.4 Friction

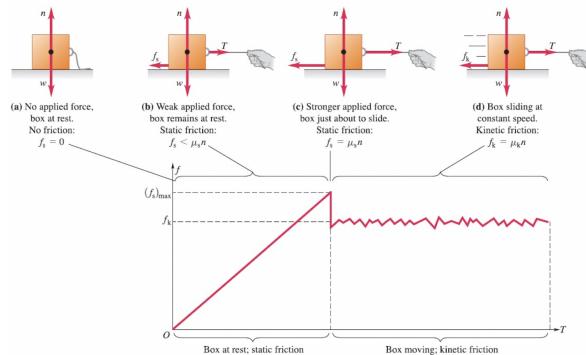
Force that resists an object's motion (kinetic friction, f_k) or tendency to move (static friction, f_s)

- Contact force
- Acts parallel to the surface, opposing the direction of motion
- f_s varies from 0 to a maximum value, $f_{s_{max}}$.

Before $f_{s_{max}}$ is reached, $f_s = -F_{\text{applied}}$.

$f_{s_{max}} = \mu_s N$, μ_s is the coefficient of static friction

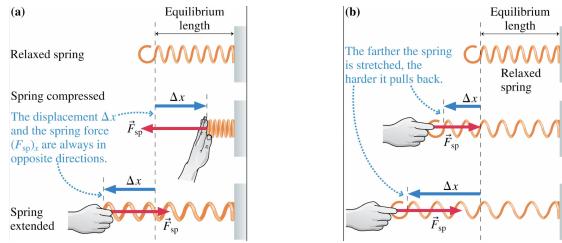
$f_k = \mu_k N$, μ_k is the coefficient of kinetic friction



5.5 Spring / Elastic Force

Force that attempts to restore an object back to its equilibrium length

- Contact force
- $\vec{F} = -k\Delta\vec{x}$
 - k is the spring / proportionality constant
 - $\Delta\vec{x}$ is a vector representing the object's stretch/compression
- Acts in the opposite direction of $\Delta\vec{x}$



6 Newton's Laws of Motion

6.1 Inertia

Every object continues its state of motion in a straight line unless acted on by an external net force

- Theoretically speaking, applying an initial force on an object will allow the object to move forever
- The quantity influencing inertia is the object's mass
- The greater its mass, the more difficult it is to change its state of motion

6.2 $\sum \vec{F}_{net} = ma$

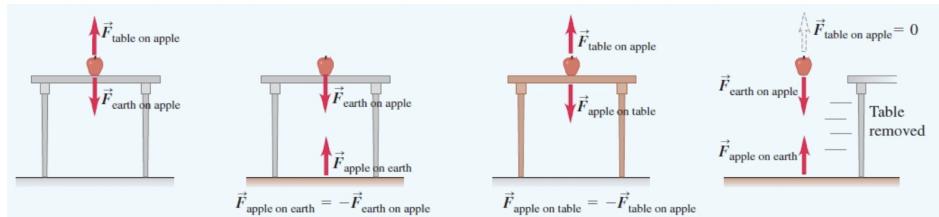
The rate of change of velocity is directly proportional to net force, and is in the direction of the net force.

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \dots = \sum \vec{F}_{net}$$

6.3 Action-Reaction Pairs

Every action has an equal an opposite reaction. The action and reaction must

- Act on different objects
- Have equal magnitude
- Be in opposite directions



7 Equilibrium Dynamics

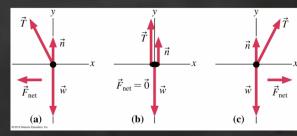
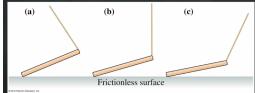
1. Sketch the system
2. Choose a body

- Draw its free body diagram (FBD)
 - Include only forces acting on the body
 - Exclude forces exerted by the body
- Determine if the object is accelerating (and in what direction)
- Choose an appropriate axis (align the positive direction with the direction of acceleration)
- Resolve the forces into components
- Apply Newton's second law for each component

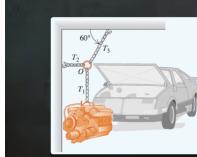
$$\sum \vec{F}_x = ma_x \quad \sum \vec{F}_y = ma_y$$

3. Repeat for next body

A rod is free to slide on a frictionless sheet of ice. One end of the rod is lifted by a string. If the rod is at rest, which diagram shows the correct angle of the string?



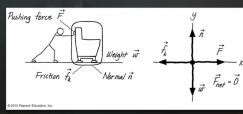
A car engine with weight w hangs from a chain that is linked at point O to two other chains, one fastened to the ceiling and the other to the wall. Find the tension in each of the three chains, assuming that w is given and the weights of the chains themselves are negligible.



- Note that there's two bodies that one could think of here:
- Engine
 - O-ring

$$\begin{aligned} \sum F_{y\text{-ring}} &= 0 \\ T_3 \sin 60^\circ - T_1 &= 0 \\ T_3 &= \frac{T_1}{\sin 60^\circ} \\ &= \frac{T_1}{\sin 60^\circ} \\ &= 1.155w \\ \sum F_x &= 0 \\ T_3 \cos 60^\circ - T_2 &= 0 \\ T_2 &= T_3 \cos 60^\circ \\ &= 0.577w \end{aligned}$$

Carol wants to move her 32 kg sofa to a different room in the house. She places "sofa sliders," slippery disks with $\mu_k = 0.080$, on the carpet, under the feet of the sofa. She then pushes the sofa at a steady 0.40 m s^{-1} across the floor. How much force does she apply to the sofa?



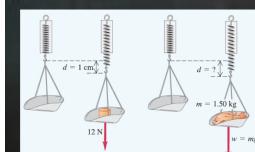
$$m = 32 \text{ kg}$$

$$\mu_k = 0.08$$

$$F = ?$$

$$\begin{aligned} \sum F_y &= 0 & \sum F_x &= 0 \\ F - f_k &= 0 & F - f_k &= 0 \\ n = w &= mg & F = f_k &= \mu_k n \\ &= 313.91 \text{ N} & &= \mu_k n \\ & & &= 25 \text{ N} \end{aligned}$$

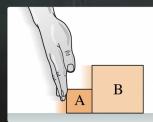
A spring balance used to weigh fish is built with a spring that stretches 1.00 cm when a 12.0 N weight is placed in the pan. When the 12.0 N weight is replaced with a 1.50 kg fish, what distance does the spring stretch?



$$\begin{aligned} \sum F &= 0 & \sum F &= 0 \\ 12 - kx &= 0 & mg - kx &= 0 \\ k &= \frac{12}{0.01} & x &= \frac{mg}{k} \\ &= 1200 \text{ N m}^{-1} & &= 0.0123 \text{ m} \end{aligned}$$

You exert a force F on body A which has a mass of 2.00kg. Body B is in contact with body A, which has a mass of 3.00kg. The coefficient of kinetic friction for bodies A and B respectively are 0.20 and 0.30. Determine

- The F needed to keep the body moving at constant velocity
- The force that B exerts on A
- The force that A exerts on B



$$\begin{aligned} m_A &= 2.00 \text{ kg} \\ m_B &= 3.00 \text{ kg} \\ \mu_A &= 0.2 \\ \mu_B &= 0.3 \end{aligned}$$

Treating both bodies as one

$$\begin{aligned} \sum F_x &= 0 & f_{k_B} &\leftarrow \text{A, B} & \rightarrow F \\ F - f_{k_A} - f_{k_B} &= 0 & f_{k_A} & \\ F &= f_{k_A} + f_{k_B} & & \\ &= \mu_{k_A} m_A g + \mu_{k_B} m_B g & & \\ &= 12.8 \text{ N} & & \end{aligned}$$

Just body A

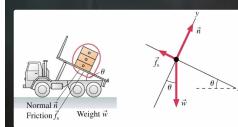
$$\begin{aligned} n_A &\leftarrow A & \sum F_x &= 0 \\ f_{k_A} &\leftarrow B & F - n_A - f_{k_A} &= 0 \\ & & n_A &= F - f_{k_A} \\ & & &= 8.83 \text{ N} \end{aligned}$$

where n_A is the normal force on A due to B

Note that n_B and n_A are action-reaction pairs, and therefore will have the same magnitude

A 50.0 kg steel file cabinet is in the back of a clump truck. The truck's bed, also made of steel, is slowly tilted.

- What is the magnitude of the friction force on the cabinet when the bed is tilted 20° ?
- At what angle will the file cabinet begin to slide? (given $\mu_s = 0.8$)



$$\begin{aligned} m &= 50 \text{ kg} & \sum F_x &= 0 \\ \theta &= 20^\circ & mg \sin \theta - f_s &= 0 \\ f_s &=? & f_s &= mg \sin \theta \\ & & &= 168 \text{ N} \end{aligned}$$

$$\begin{aligned} m &= 50 \text{ kg} & \sum F_x &= 0 \\ \theta &=? & mg \sin \theta - f_{s\text{max}} &= 0 \\ f_{s\text{max}} &= mg \sin \theta & f_{s\text{max}} &= mg \sin \theta \\ \mu_s m g \cos \theta &= 0 & \mu_s m g \cos \theta &= mg \sin \theta \\ n - mg \cos \theta &= 0 & \mu_s &= \tan \theta \\ n &= mg \cos \theta & \mu_s &= \tan^{-1} \mu_s \\ & & &= 39^\circ \end{aligned}$$

8 Non-Equilibrium Dynamics