# CS2040S

# Data Structures and Algorithms

## On the importance of being balanced

Welcome!

# Plan of the Day

## Trees

- Terminology
- Traversals
- Operations

## Balanced Trees

- Height-balanced binary search trees
- AVL trees
- Rotations

# Announcements

Midterm : Thursday March 10, 6:30pm

Location: ~14 different venues (MPSH).

Note: In person, face-to-face

Safe distancing: 48 students / room, spaced

About < 5 people have e-mailed me with conflicts (i.e., other midterms, national service, etc.). So I assume this is a good time for the remaining 645+ of you!

# Part 2

## On the importance of being balanced

# Part 2

**On the importance of being balanced**

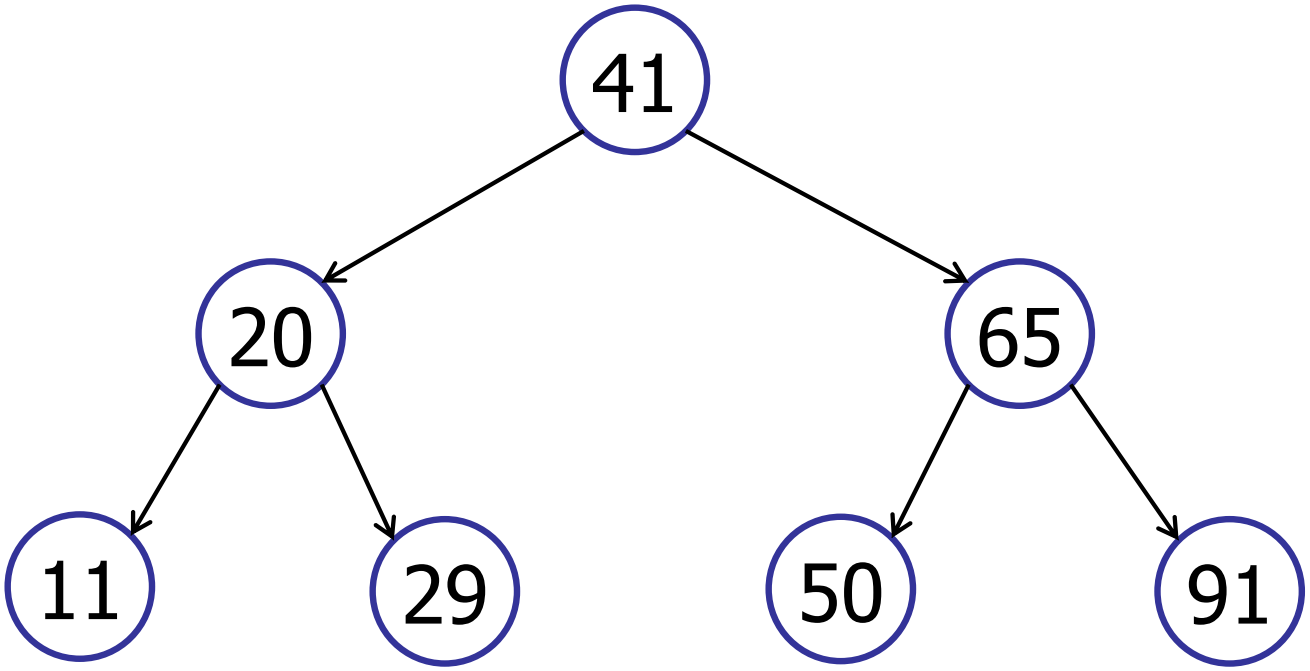
- Height-balanced binary search trees
- AVL trees
- Rotations

# Dictionary Interface

## A collection of (key, value) pairs:

| **interface** | **IDictionary** | |
|---|---|---|
| void | insert(Key k, Value v) | *insert (k,v) into table* |
| Value | search(Key k) | *get value paired with k* |
| Key | successor(Key k) | *find next key > k* |
| Key | predecessor(Key k) | *find next key < k* |
| void | delete(Key k) | *remove key k (and value)* |
| boolean | contains(Key k) | *is there a value for k?* |
| int | size() | *number of (k,v) pairs* |

# Recap: Binary Search Trees




- Two children: v.left, v.right
- Key: v.key
- **BST Property**: all in left sub-tree < key < all in right sub-right

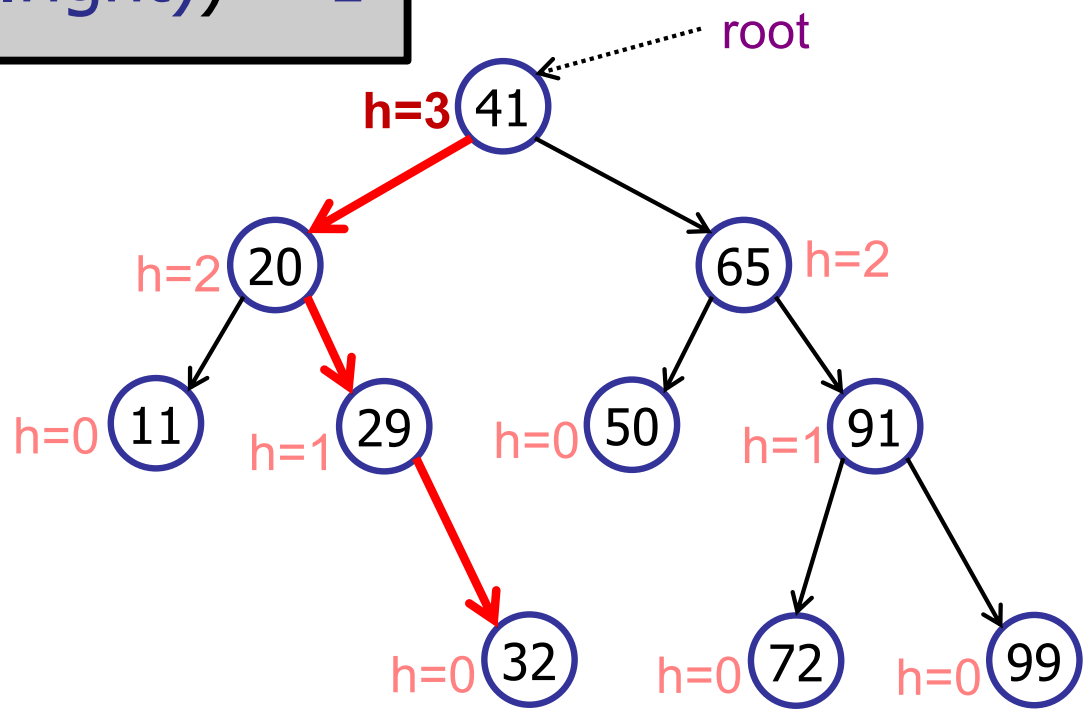
# Binary Search Trees Heights

## Height:

Number of edges on longest path from root to leaf.

$h(v) = 0$ (if v is a leaf)

$h(v) = \max(h(v.left), h(v.right)) + 1$




(For simplicity: h(null) = -1)

# Binary Search Tree

Modifying Operations

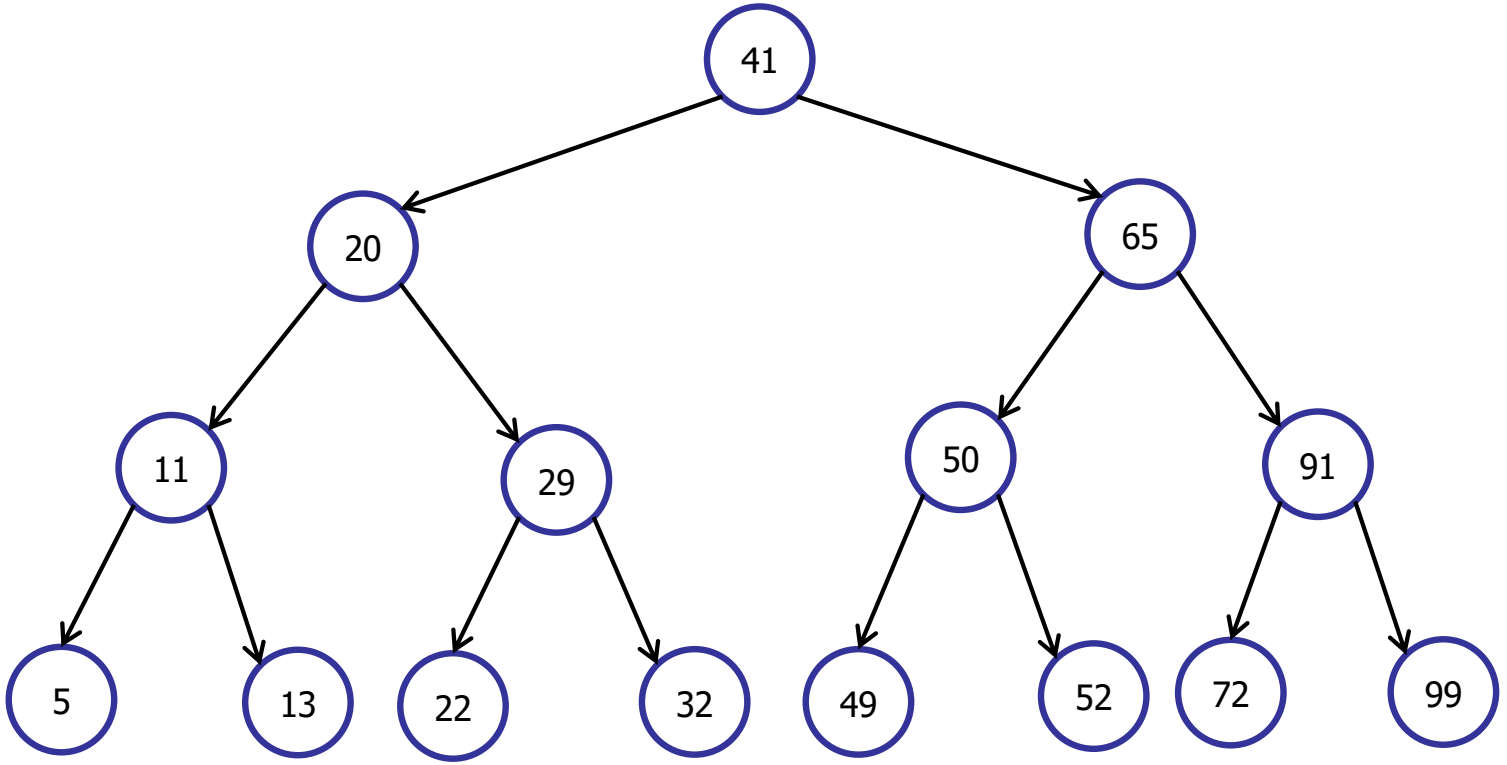
- insert
- delete

Query Operations:

- search
- predecessor, successor
- findMax, findMin
- in-order-traversal

# The Importance of Being Balanced

Operations take O(h) time



What is the largest possible height h?

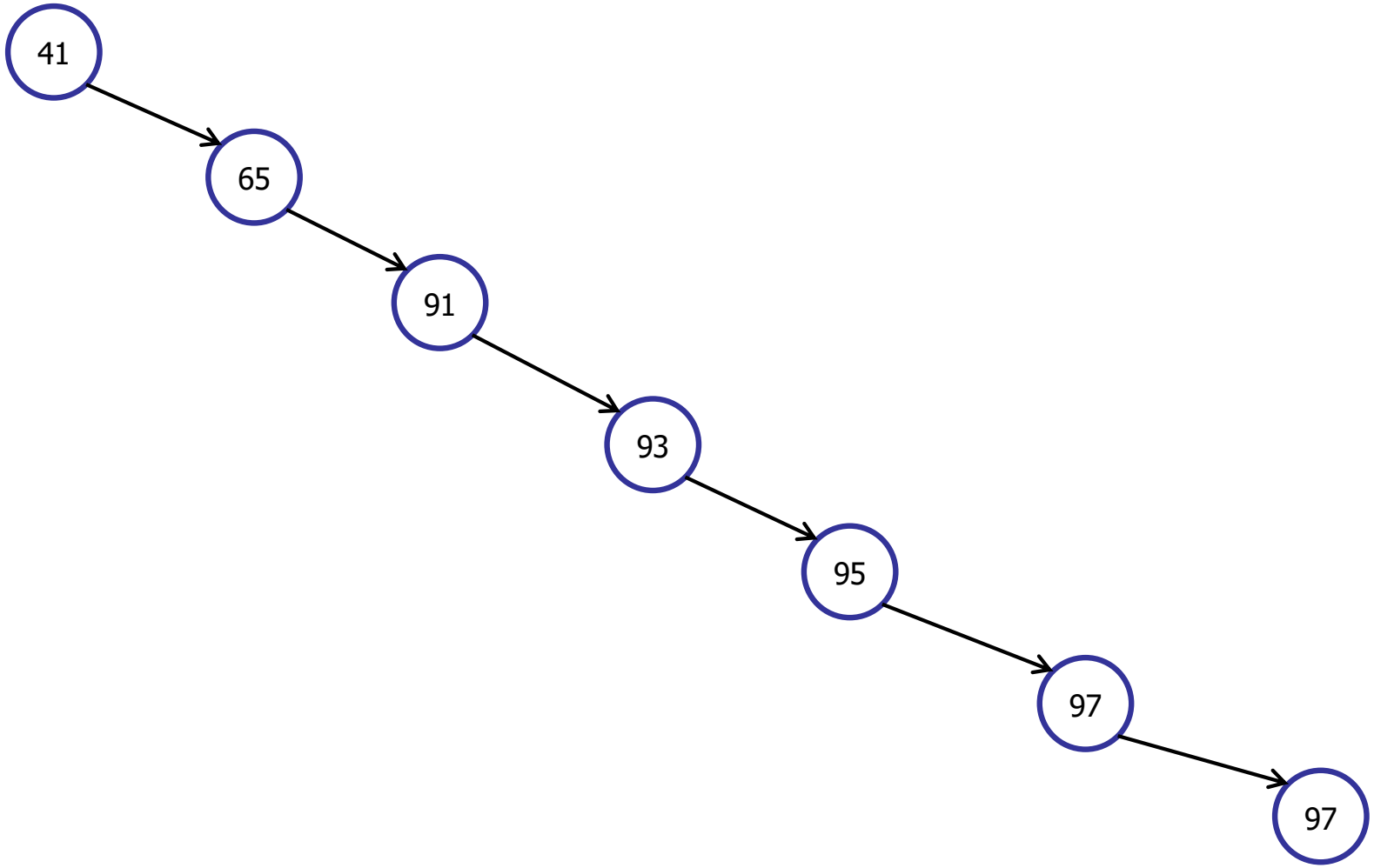
Last time…

1. $\theta(1)$
2. $\theta(\log n)$
3. $\theta(\sqrt{n})$
4. ✓ $\theta(n)$
5. $\theta(n^2)$

# The Importance of Being Balanced

Operations take O(h) time

$h \leq n$



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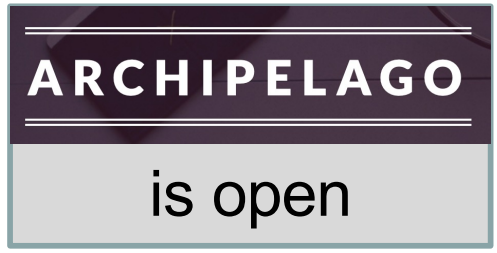

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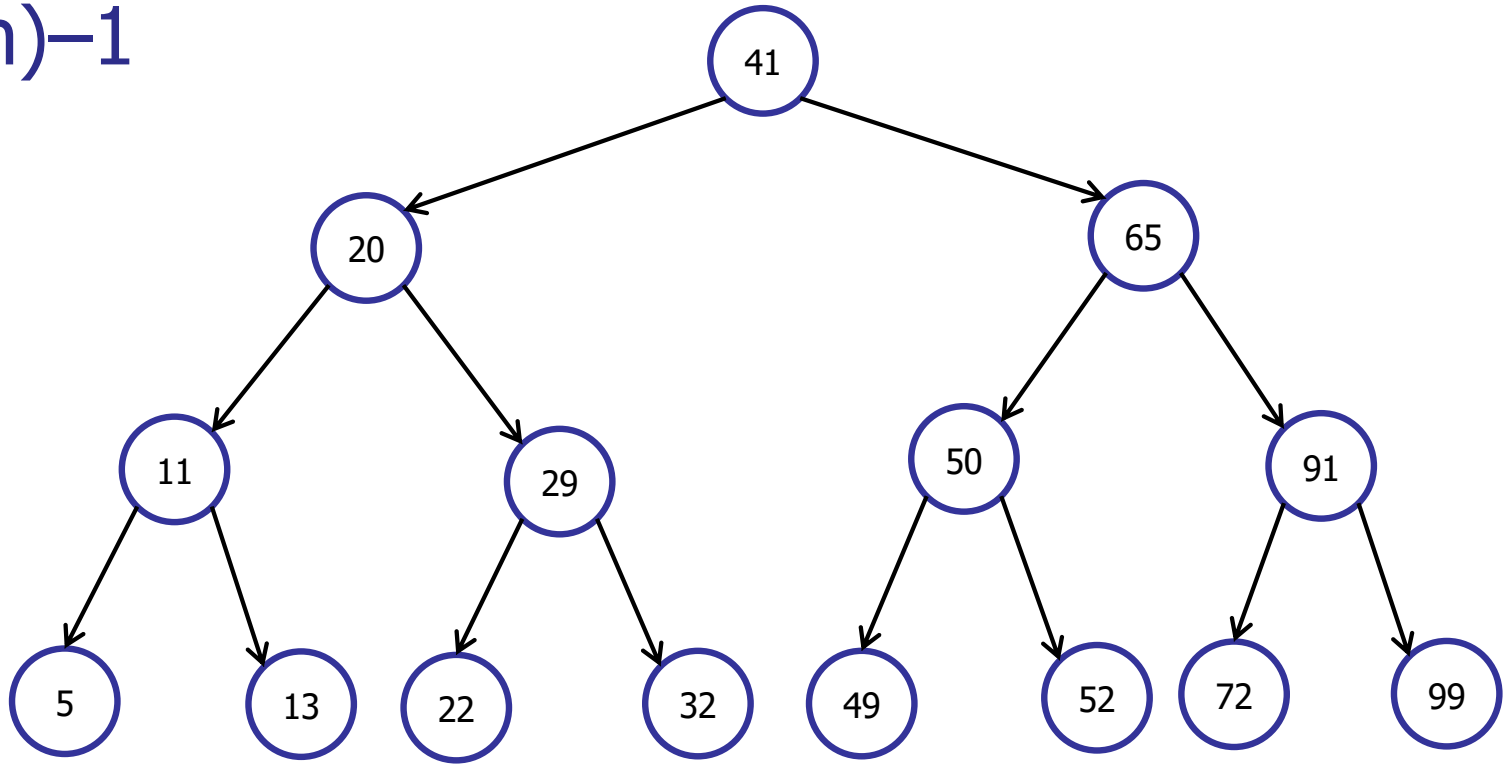
# The Importance of Being Balanced

Operations take O(h) time

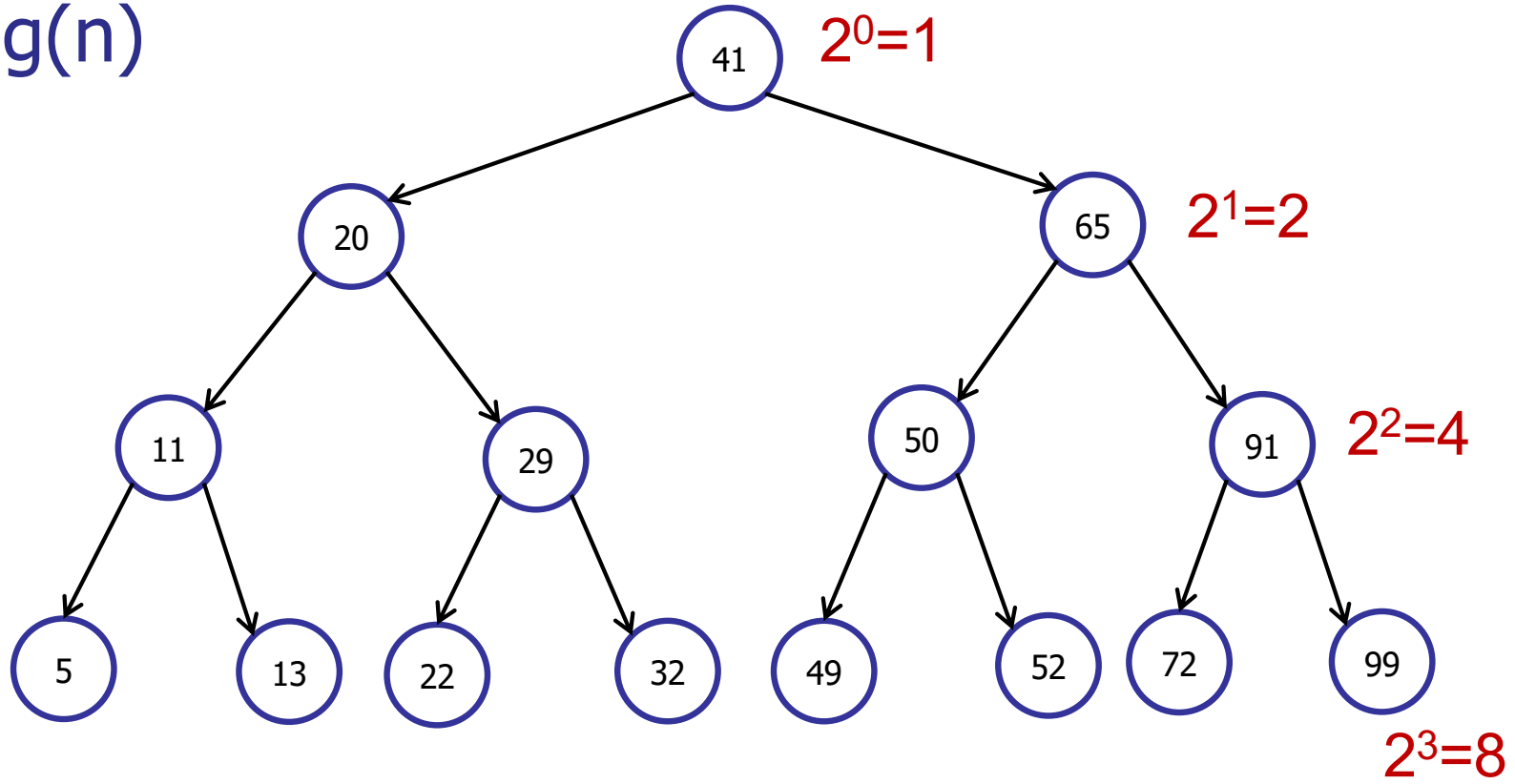
- $h \geq \log(n)-1$



# The Importance of Being Balanced

Operations take O(h) time

$h+1 \geq \log(n)$




$n \leq 1 + 2 + 4 + \ldots + 2^h$

$\leq 2^0 + 2^1 + 2^2 + \ldots + 2^h < 2^{h+1}$

# The Importance of Being Balanced

Operations take O(h) time

$\log(n) - 1 \leq h \leq n$

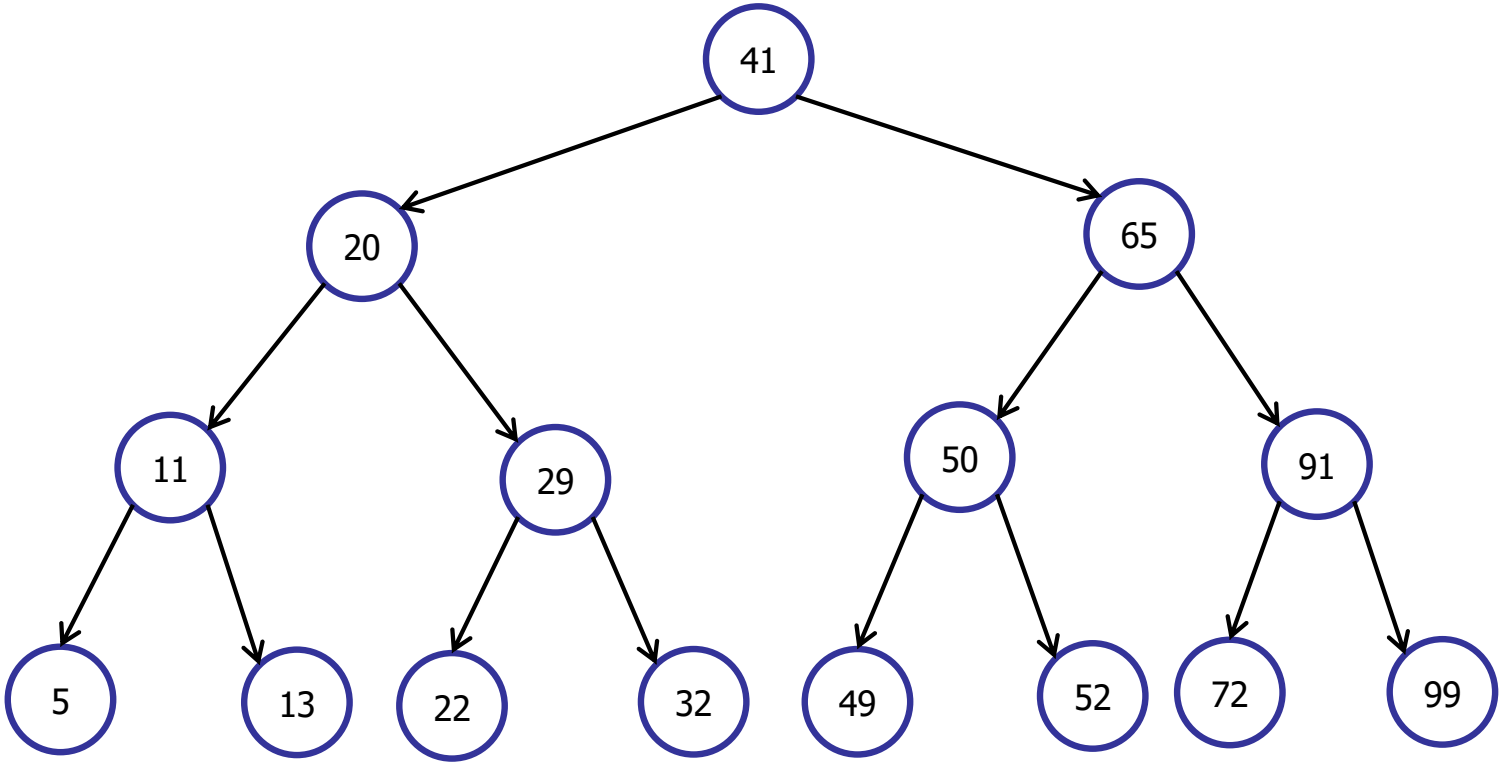
Key definition

A BST is balanced if $h = O(\log n)$

On a balanced BST: all operations run in $O(\log n)$ time.

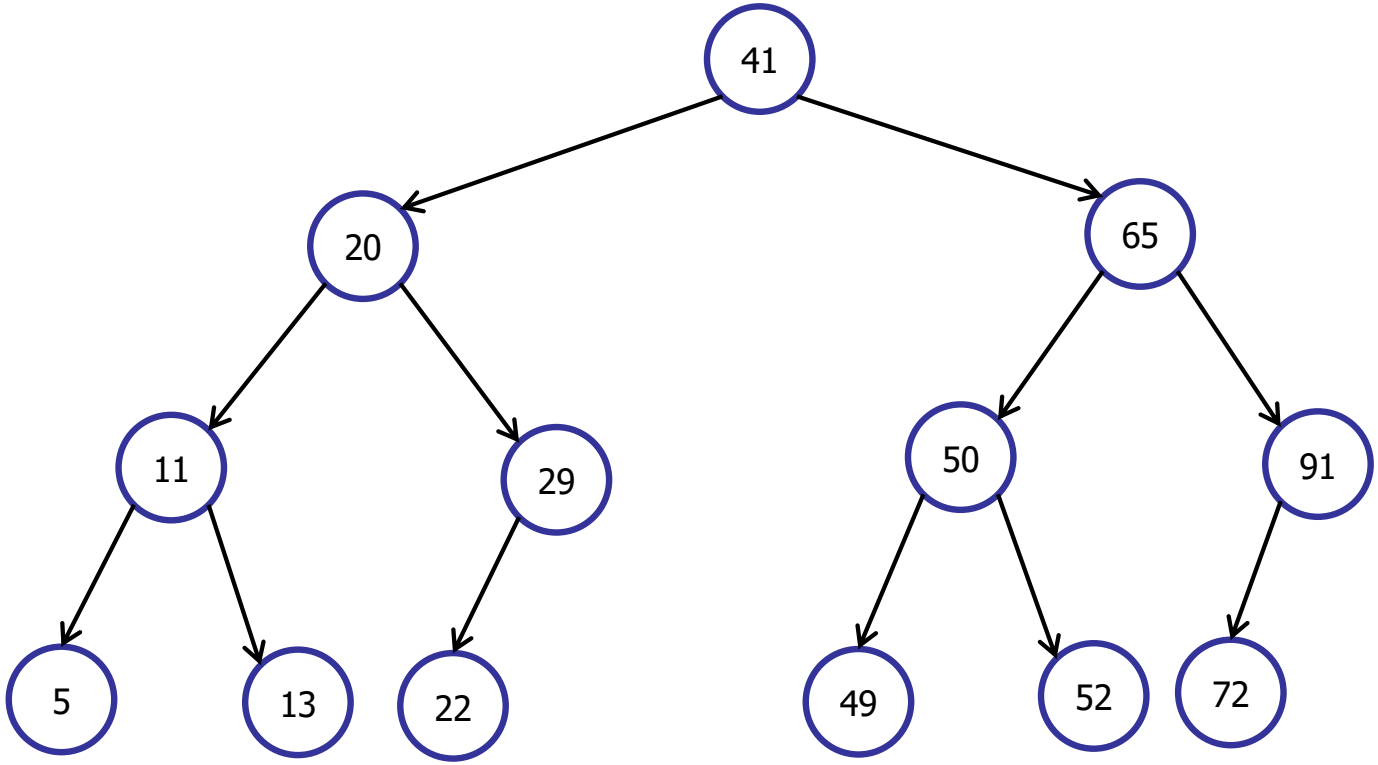
# The Importance of Being Balanced

Perfectly balanced:



# The Importance of Being Balanced

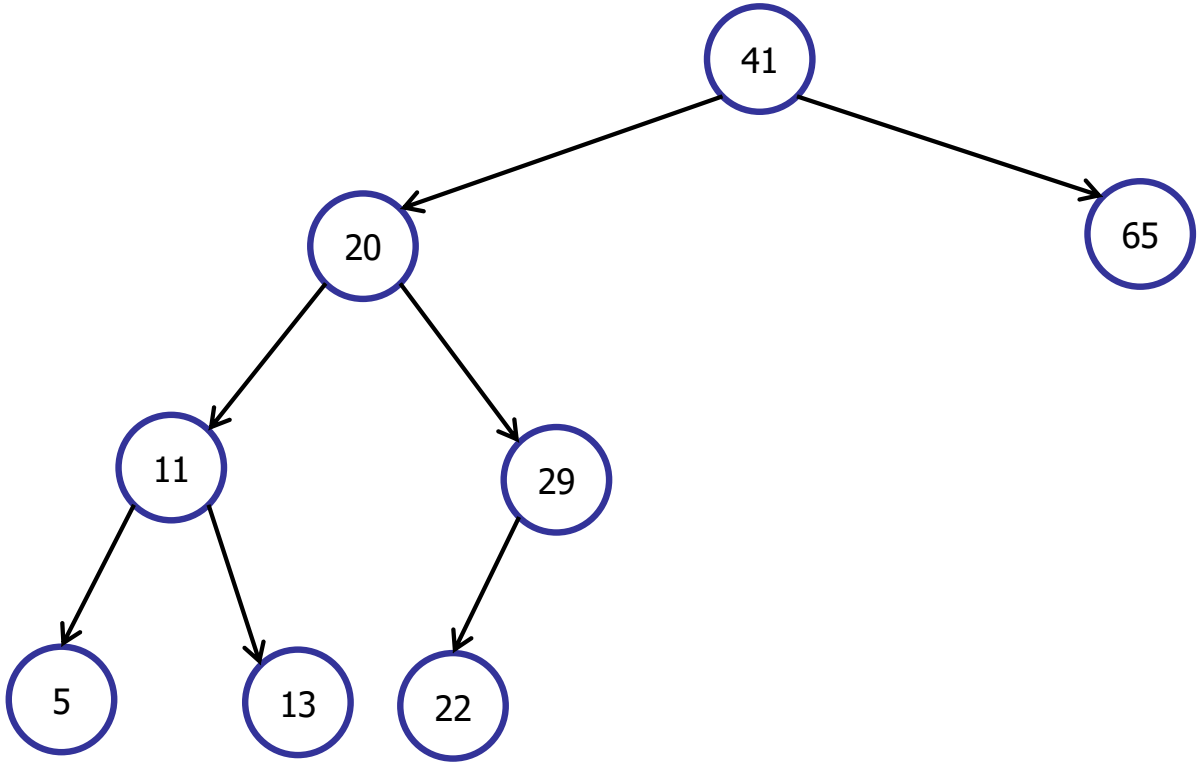
Almost perfectly balanced:




Every subtree has (almost) the same number of nodes.

# The Importance of Being Balanced

Not perfectly balanced:




Left tree has 6, right tree has 1.

# Balanced Search Trees

Many different flavors of balanced search trees

- AVL trees (Adelson-Velsii & Landis, 1962)
- B-trees / 2-3-4 trees (Bayer & McCreight, 1972)
- BB[$\alpha$] trees (Nievergelt & Reingold 1973)
- Red-black trees (see CLRS 13)
- Splay trees (Sleator and Tarjan 1985)
- Treaps (Seidel and Aragon 1996)
- Skip Lists (Pugh 1989)
- Scapegoat Trees (Anderson 1989)

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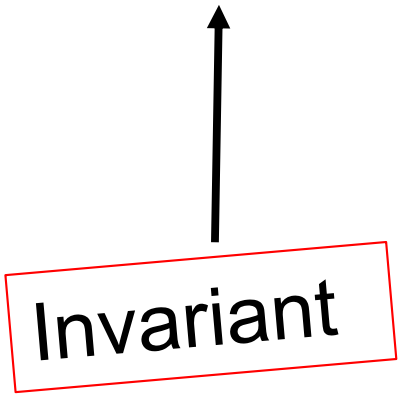
# The Importance of Being Balanced

How to get a balanced tree:

- Define a good property of a tree.
- Show that if the good property holds, then the tree is balanced.
- After every insert/delete, make sure the good property still holds. If not, fix it.



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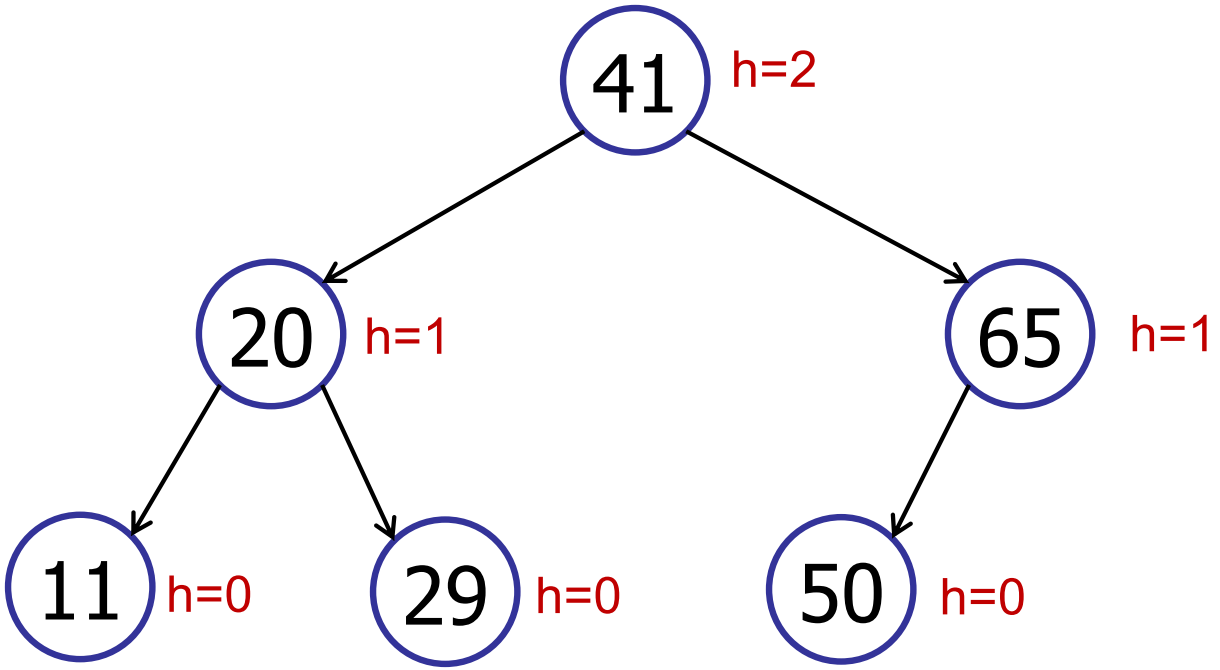
- In every node v, store height:

$$v.height = h(v)$$



# AVL Trees [Adelson-Velskii & Landis 1962]

Step 0: Augment

- In every node v, store height:

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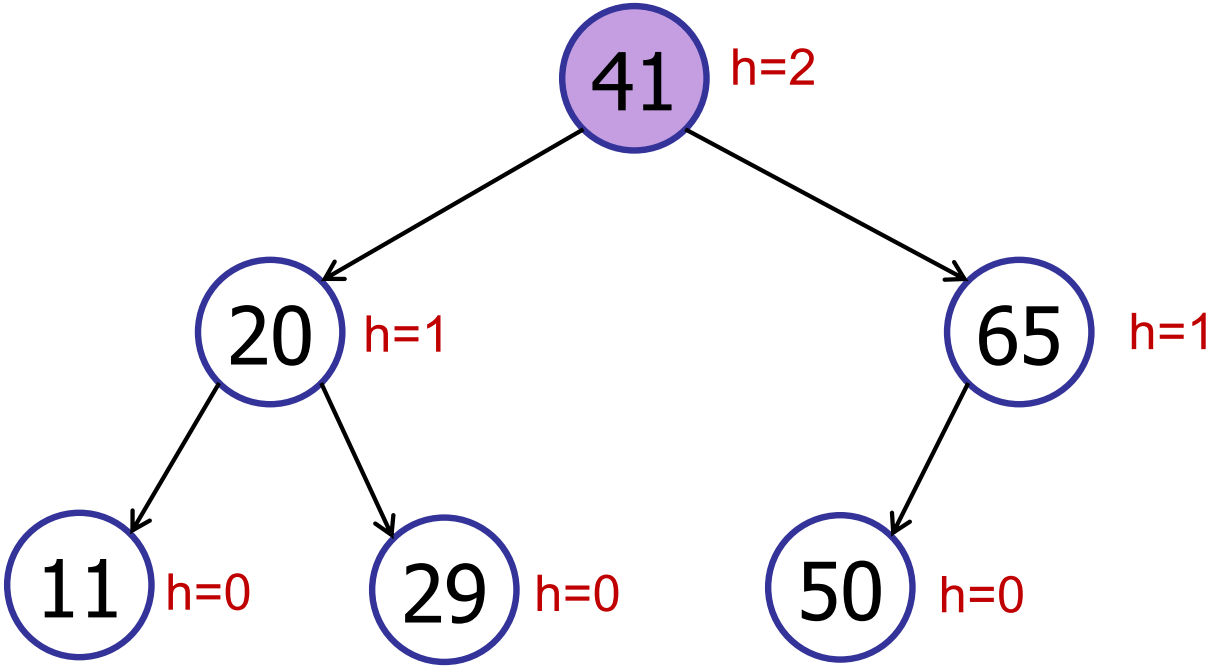
- On insert & delete update height:

```
insert(x)
        if (x < key)
                left.insert(x)
        else right.insert(x)
        height = max(left.height, right.height) + 1
```

# Binary Search Trees

insert(27)



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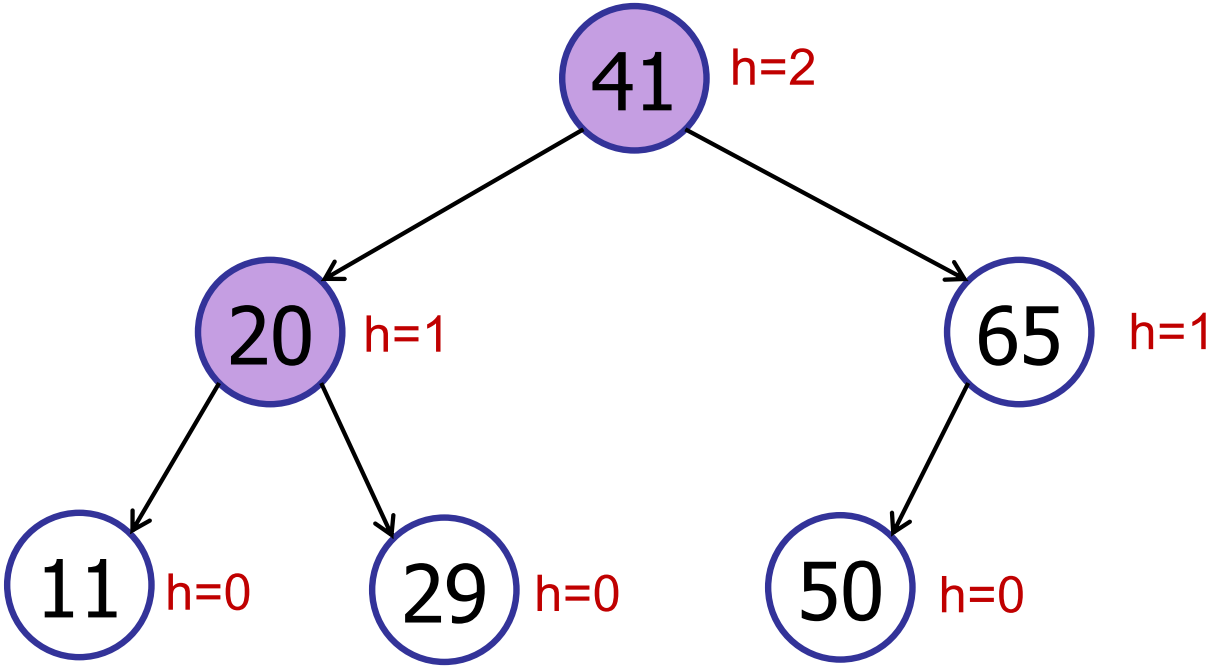

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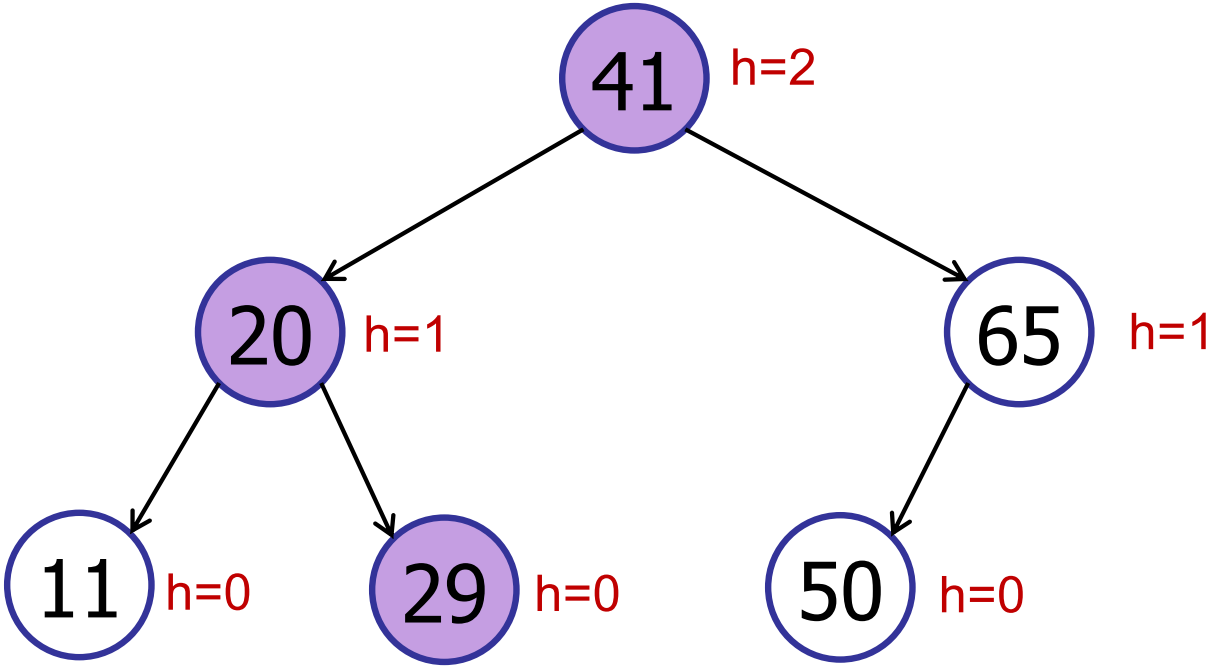

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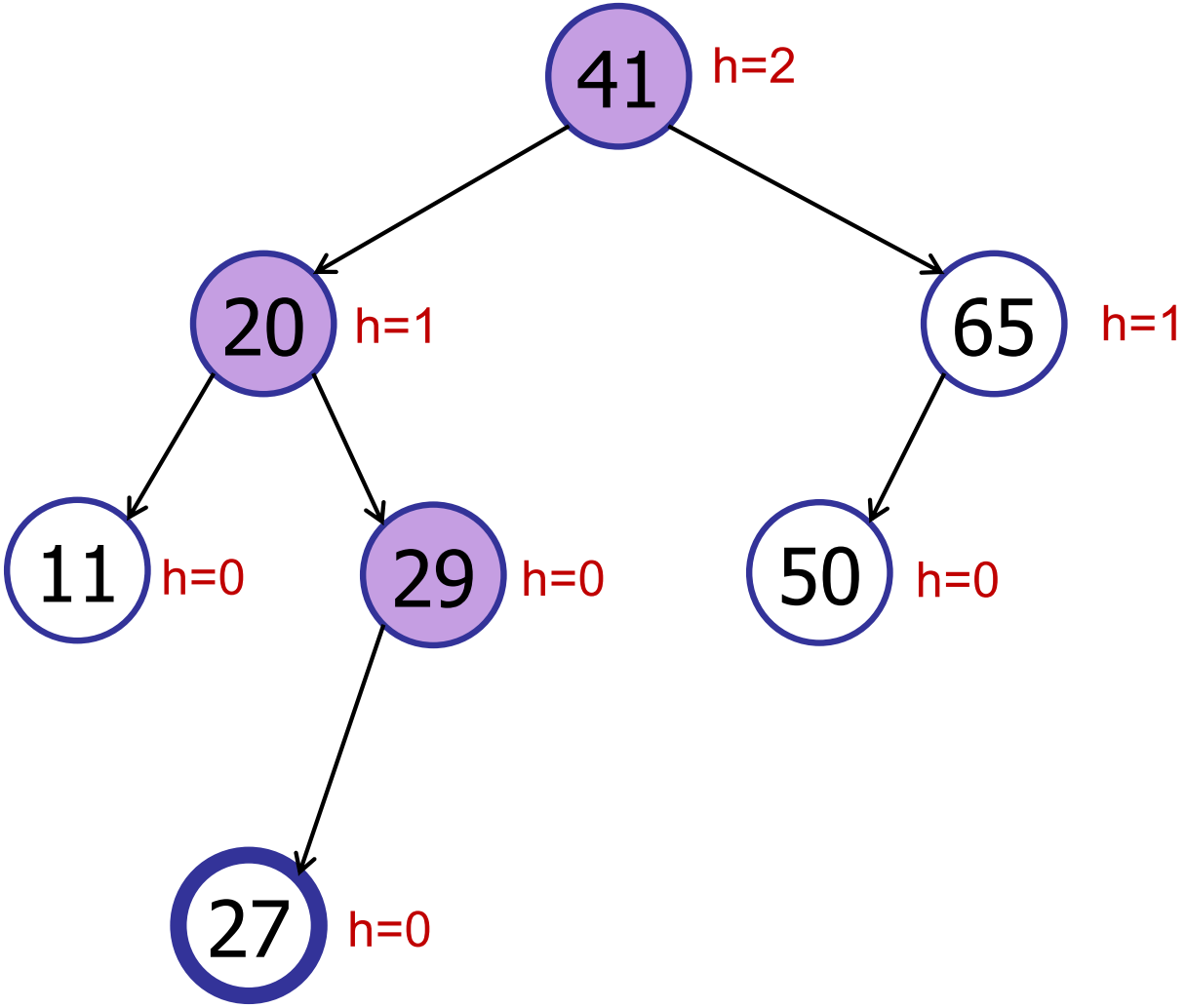

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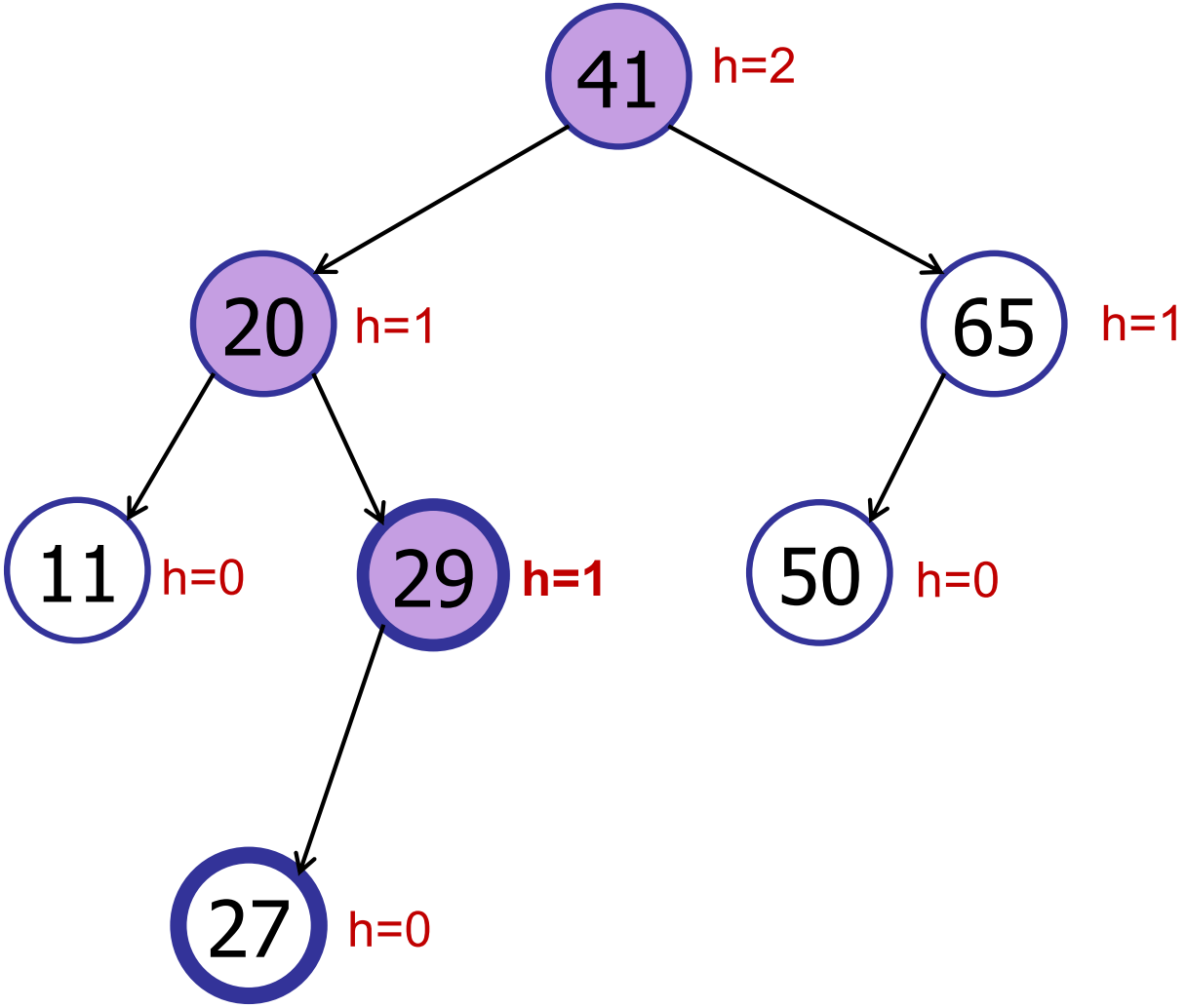

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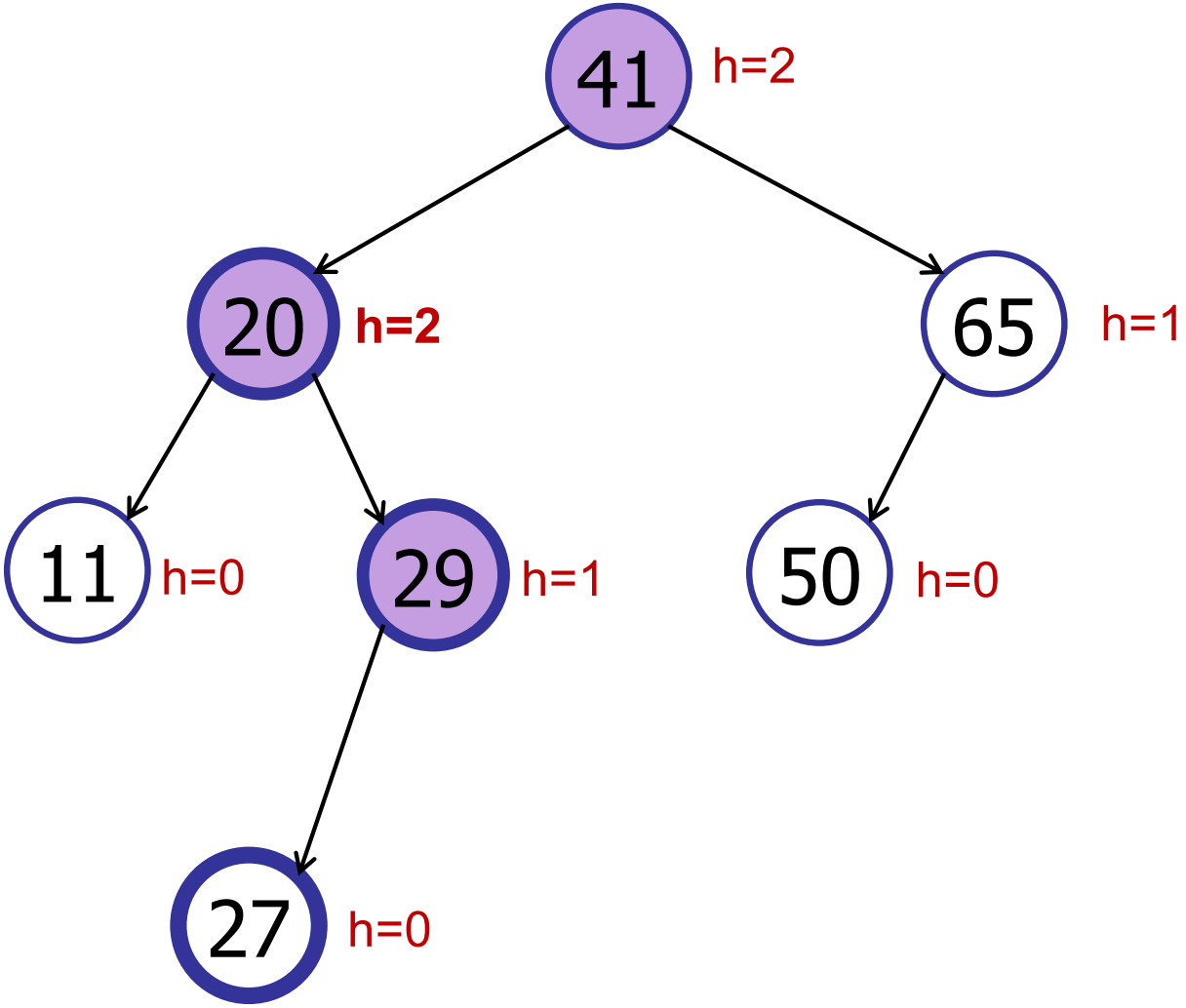

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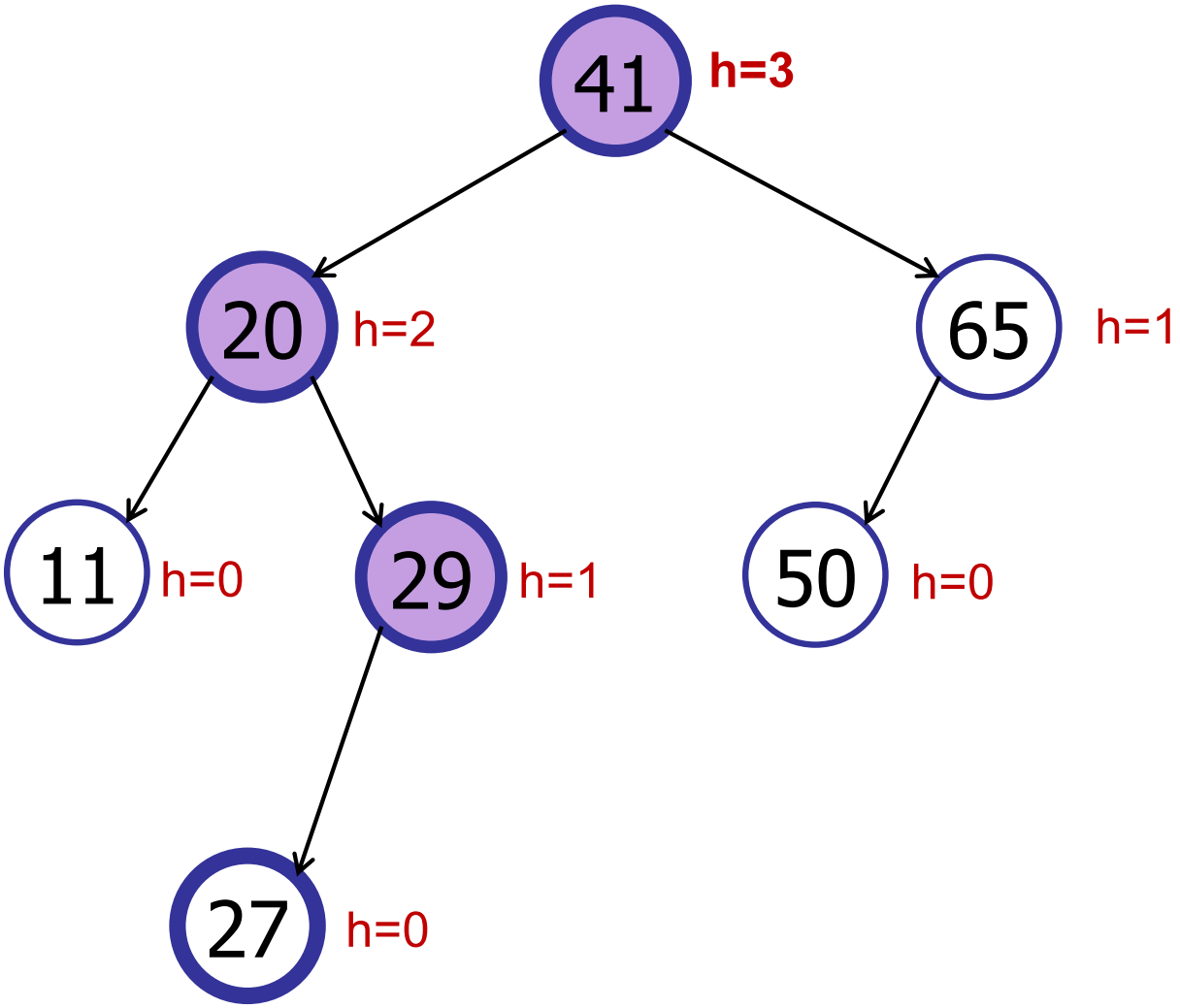

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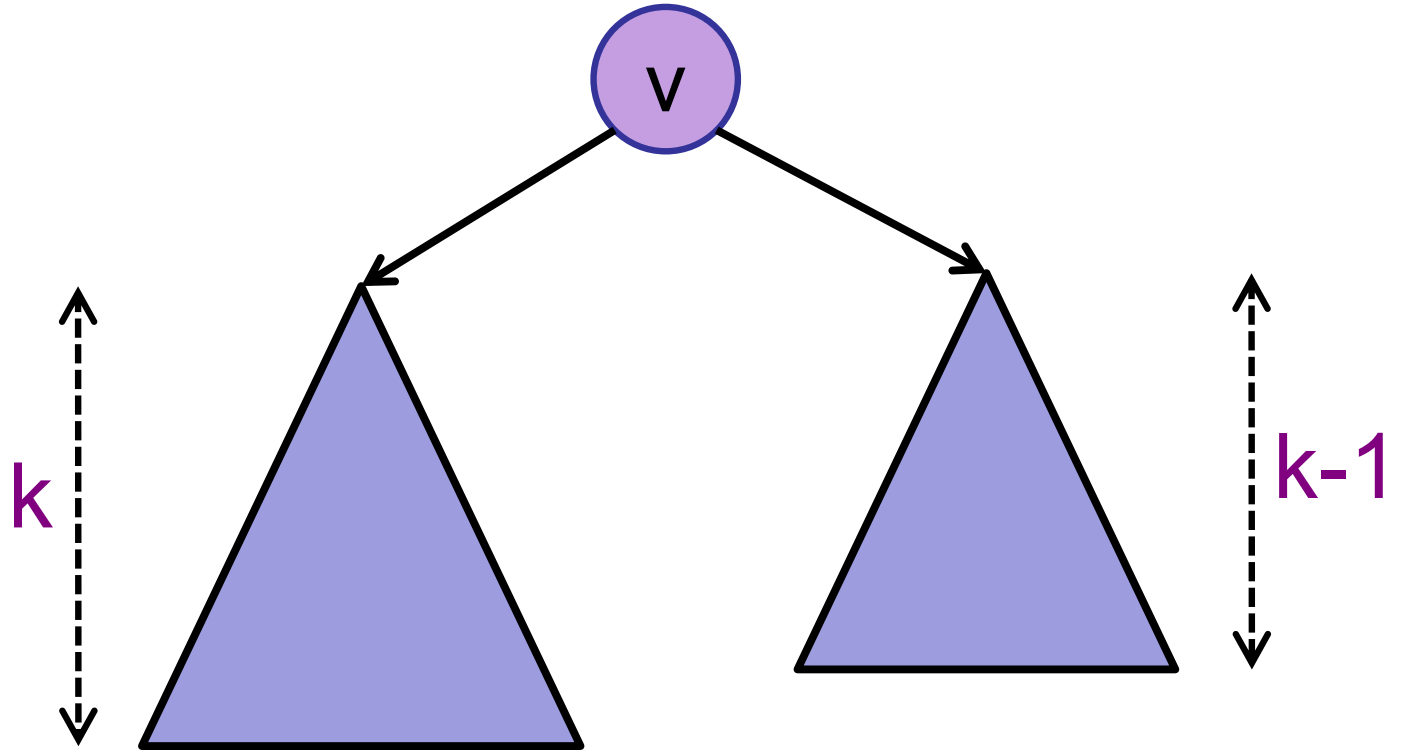
# AVL Trees [Adelson-Velskii & Landis 1962]

Key definition

## Step 1: Define Invariant

- A node v is **height-balanced** if:

$$|v.left.height - v.right.height| \leq 1$$



# AVL Trees [Adelson-Velskii & Landis 1962]

Step 1: Define Invariant

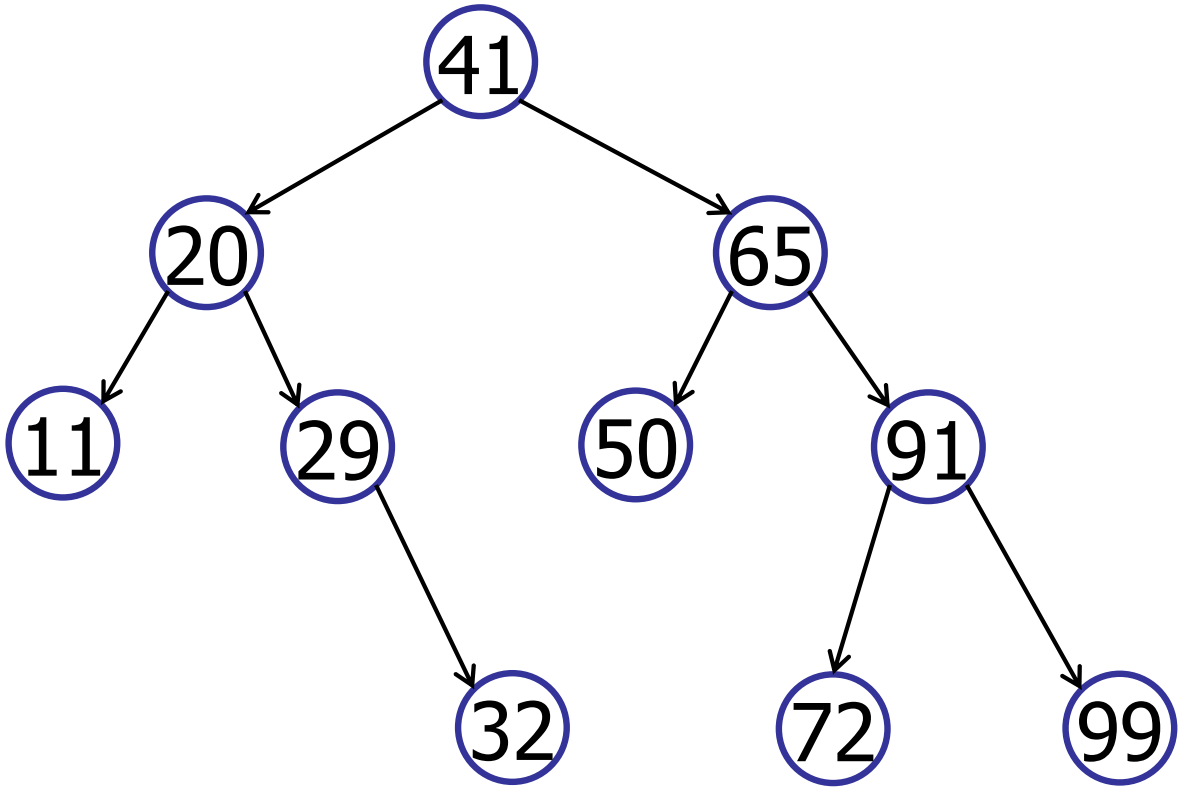
- A node v is height-balanced if:

$$|v.left.height - v.right.height| \leq 1$$

- A binary search tree is height balanced if **every** node in the tree is height-balanced.

# Is this tree height-balanced?

1. Yes
2. No
3. I'm confused.

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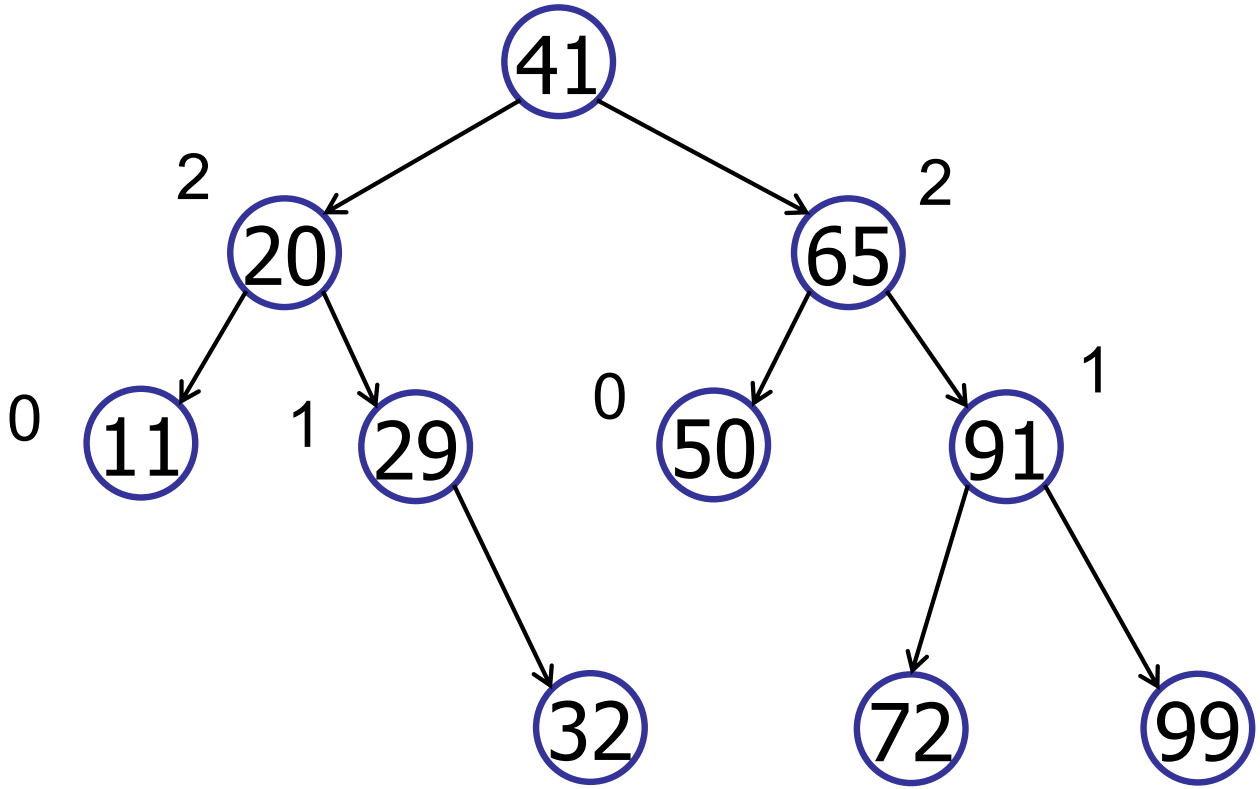

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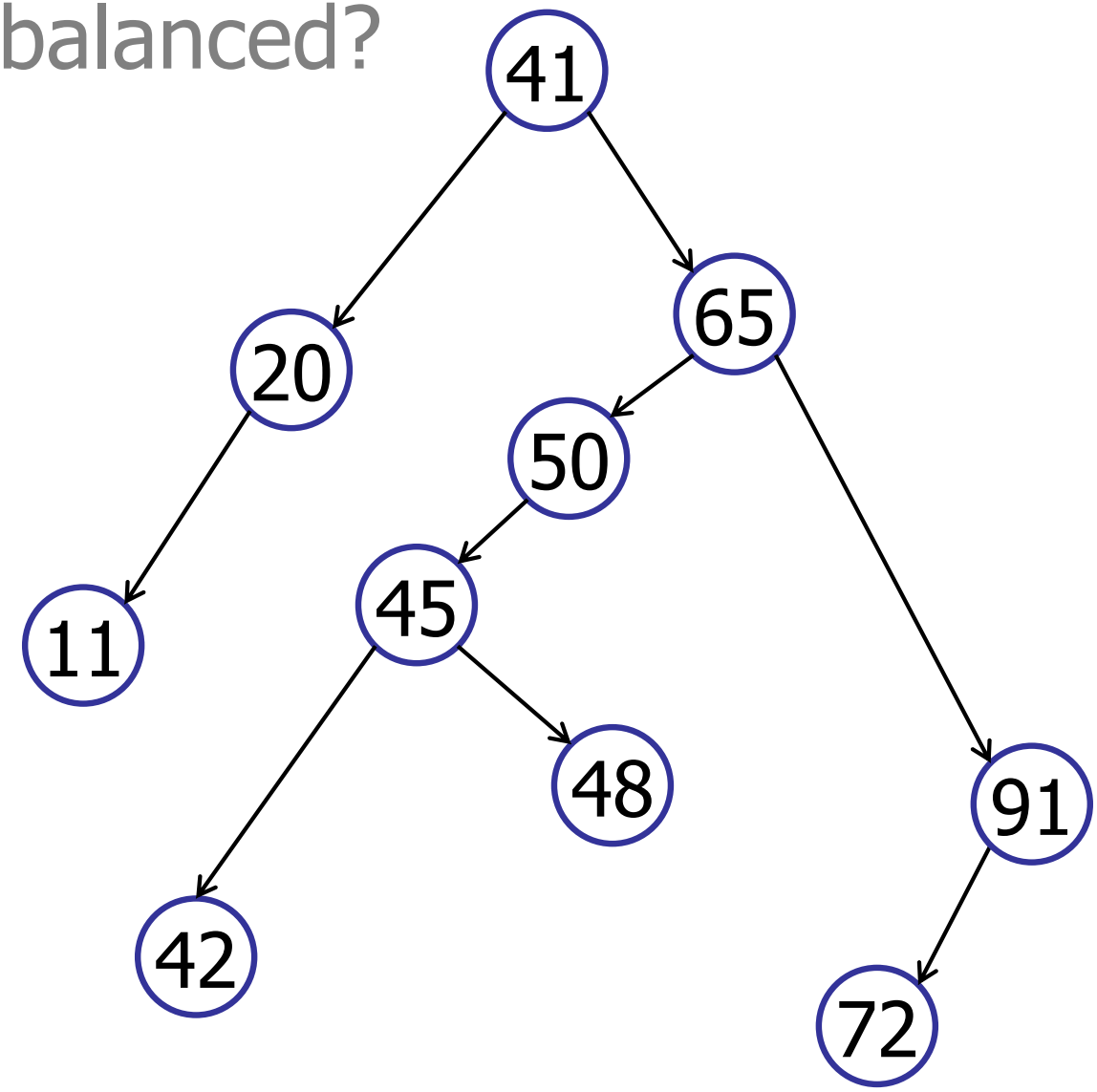


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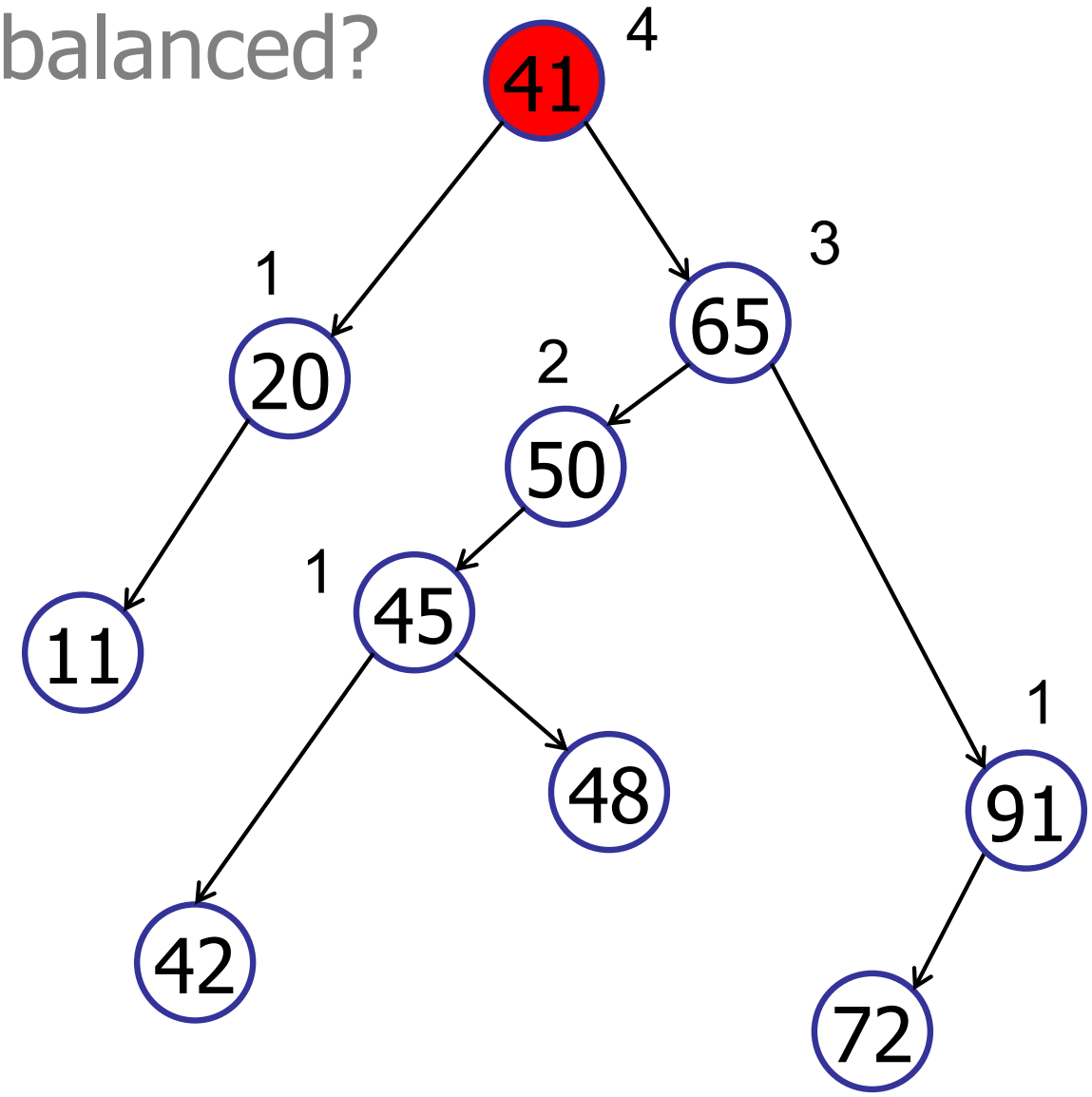

# Height-Balanced Trees

Claim:

A height-balanced tree with n nodes has **at most** height $h < 2\log(n)$.

# Height-Balanced Trees

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A height-balanced tree with $n$ nodes has **at most** height $h < 2\log(n)$.

$\Leftrightarrow h/2 < \log(n)$

$\Leftrightarrow 2^{h/2} < 2^{\log(n)}$

$\Leftrightarrow 2^{h/2} < n$

A height-balanced tree with height $h$ has **at least** $n > 2^{h/2}$ nodes

# Height-Balanced Trees

Strategy:

Assume tree is height-balanced with height h.

Show that it has **at least** $n > 2^{h/2}$ nodes.

➔ Since every height-balanced tree with height h has **at least** $n > 2^{h/2}$ nodes, we know that every height-balanced tree with n nodes has height $h < 2\log(n)$.

If height were bigger, it would need to have more nodes!

# Height-Balanced Trees

Proof:

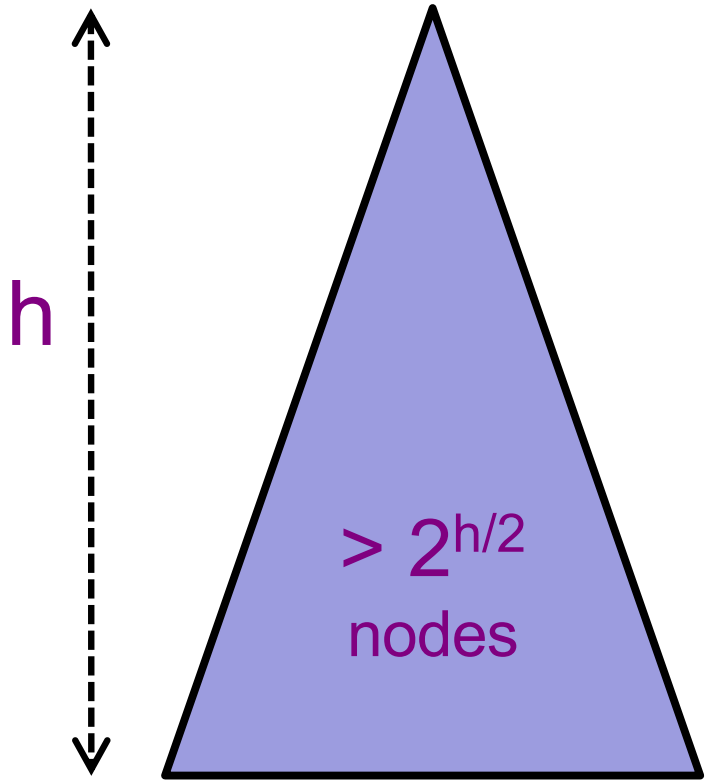
Let $n_h$ be the minimum number of nodes in a height-balanced tree of height h.

Show:

$n_h > 2^{h/2}$

Note:

$n_h > n_{h-1}$

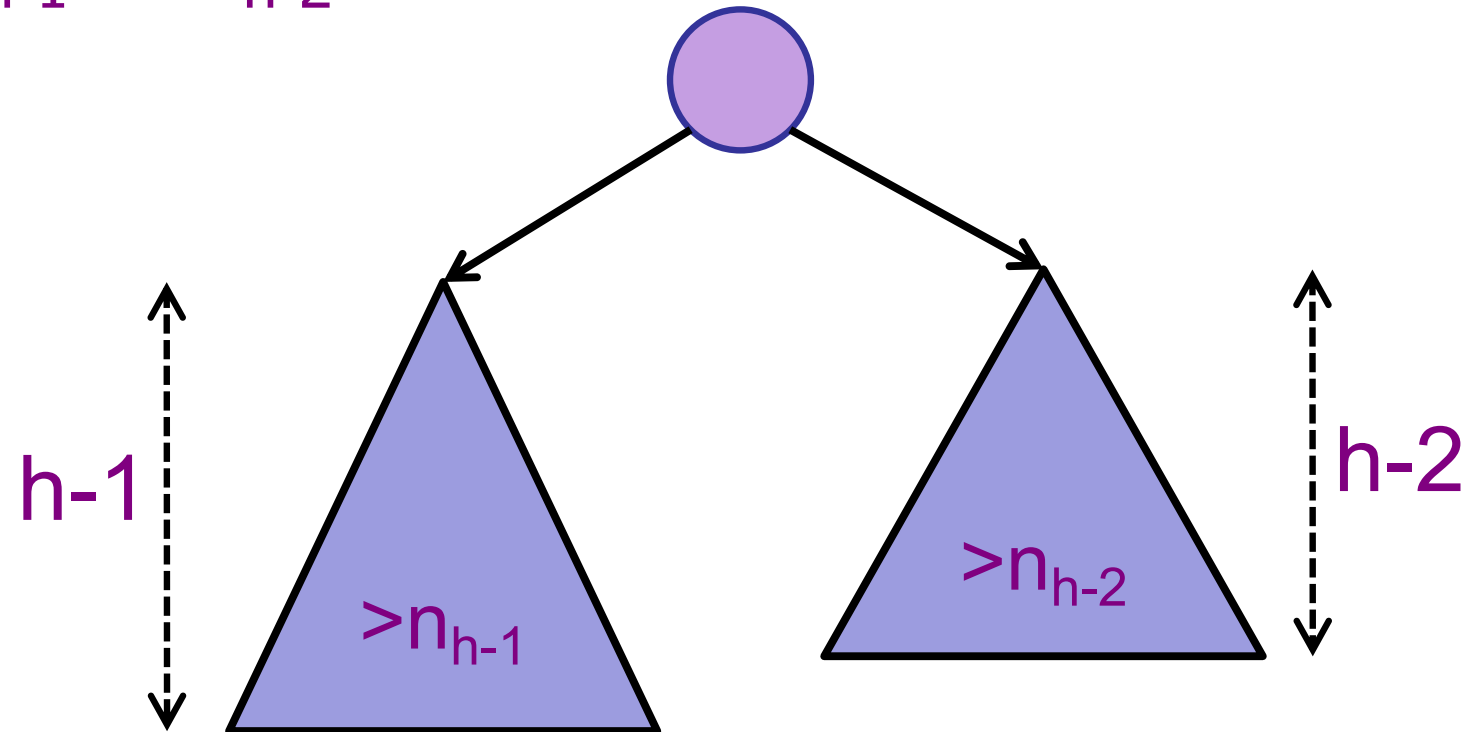


# Height-Balanced Trees

Proof:

Let $n_h$ be the minimum number of nodes in a height-balanced tree of height h.

$n_h \geq 1 + n_{h-1} + n_{h-2}$



# Height-Balanced Trees

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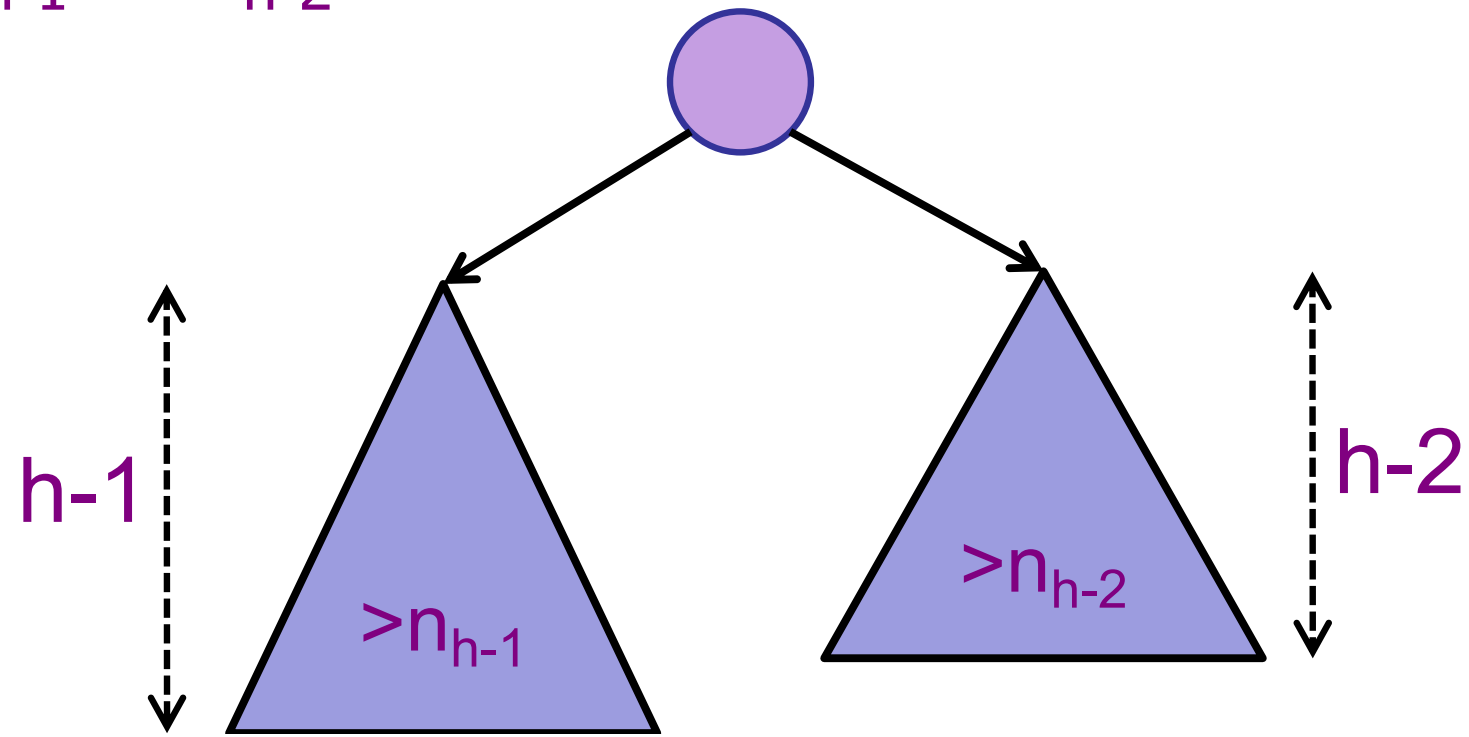
$\geq 2n_{h-2}$



# Height-Balanced Trees

Proof:

Let $n_h$ be the minimum number of nodes in a height-balanced tree of height $h$.

$$n_h \geq 1 + n_{h-1} + n_{h-2}$$

$$\geq 2n_{h-2}$$

$$\geq 4n_{h-4}$$

$$\geq 8n_{h-6}$$

$$\geq \ldots$$

How many times?

Base case:
$n_0 = 1$

# Height-Balanced Trees

Proof:

Let $n_h$ be the minimum number of nodes in a height-balanced tree of height $h$.

$$n_h \geq 1 + n_{h-1} + n_{h-2}$$

$$\geq 2^1 n_{h-2}$$

$$\geq 2^2 n_{h-4}$$

$$\geq 2^3 n_{h-6}$$

$$\geq \ldots \geq 2^k n_0$$

What is k?

Base case:

$n_0 = 1$

# Height-Balanced Trees

Proof:

Let $n_h$ be the minimum number of nodes in a height-balanced tree of height $h$.

$$n_h \geq 1 + n_{h-1} + n_{h-2}$$

$$\geq 2n_{h-2}$$

$$\geq 2^{h/2} n_0$$

$$\geq 2^{h/2}$$

Base case:
$n_0 = 1$

# Height-Balanced Trees

Claim:

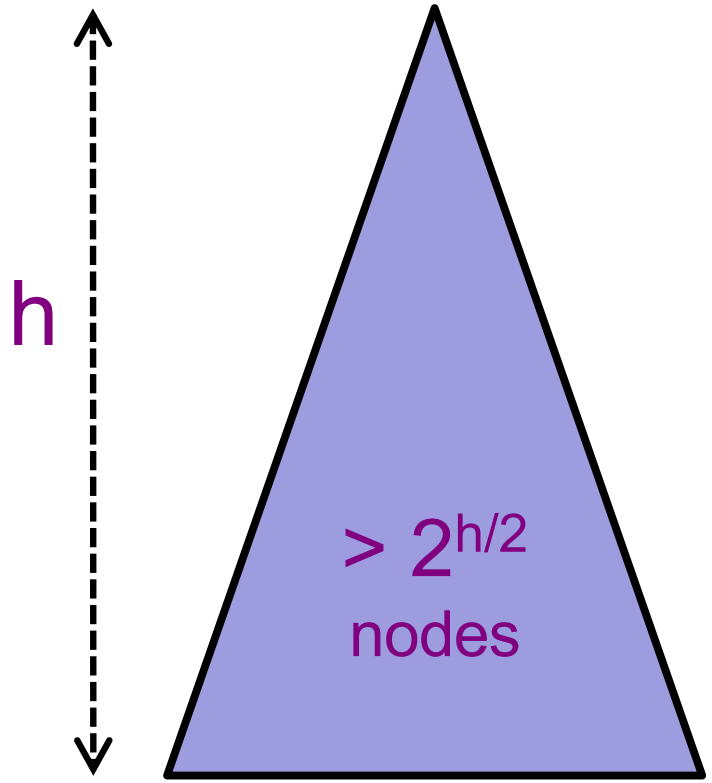
A height-balanced tree with n nodes has height $h < 2\log(n)$.

Show:

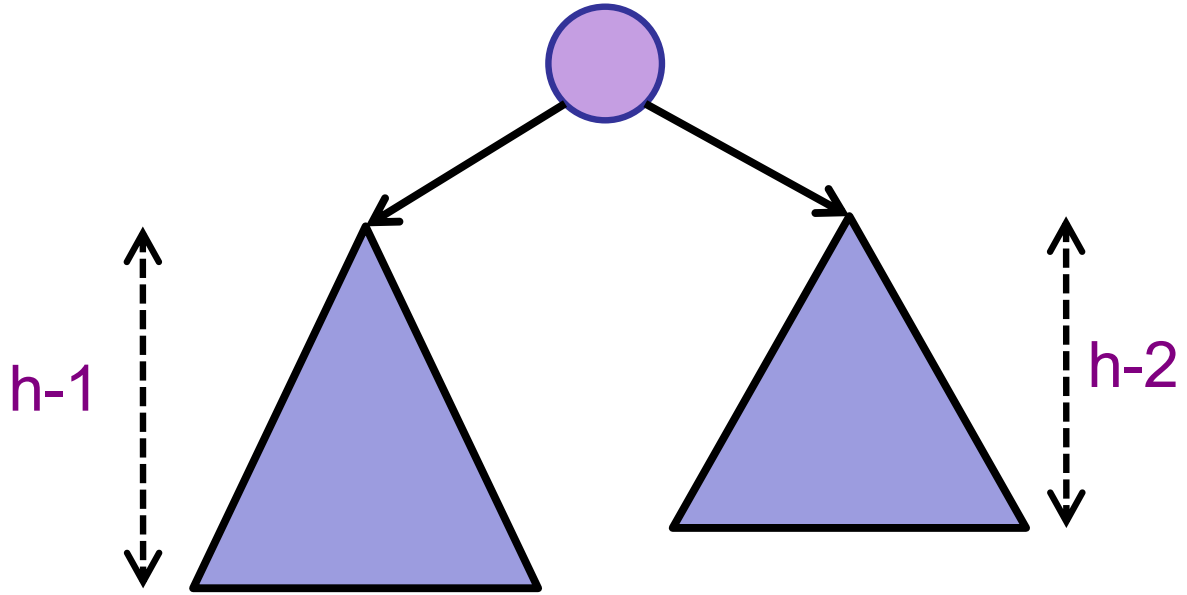
$n > 2^{h/2}$

$\Leftrightarrow$

$h < 2\log(n)$



# Height-Balanced Trees




Show (induction):

$F_n$ = $n^{th}$ Fibonacci number

$n_h = F_{h+2} - 1 \cong \phi^{h+1}/\sqrt{5} - 1$ (rounded to nearest int)

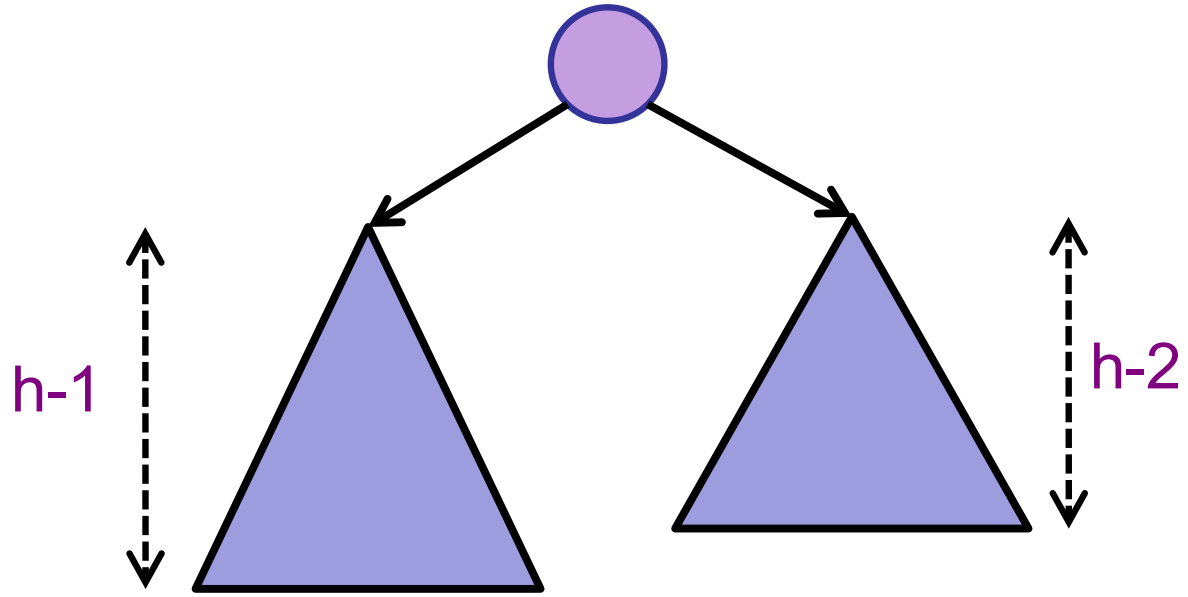
$h \cong \log(n) / \log(\phi)$ $\quad\quad$ $\phi \cong 1.618$

$h \cong 1.44 \log(n)$

# Height-Balanced Trees

Claim:

A height-balanced tree is balanced, i.e., has height $h = O(\log n)$.



# AVL Trees [Adelson-Velskii & Landis 1962]

Step 0: Augment

Step 1: Define Balance Condition ← invariant

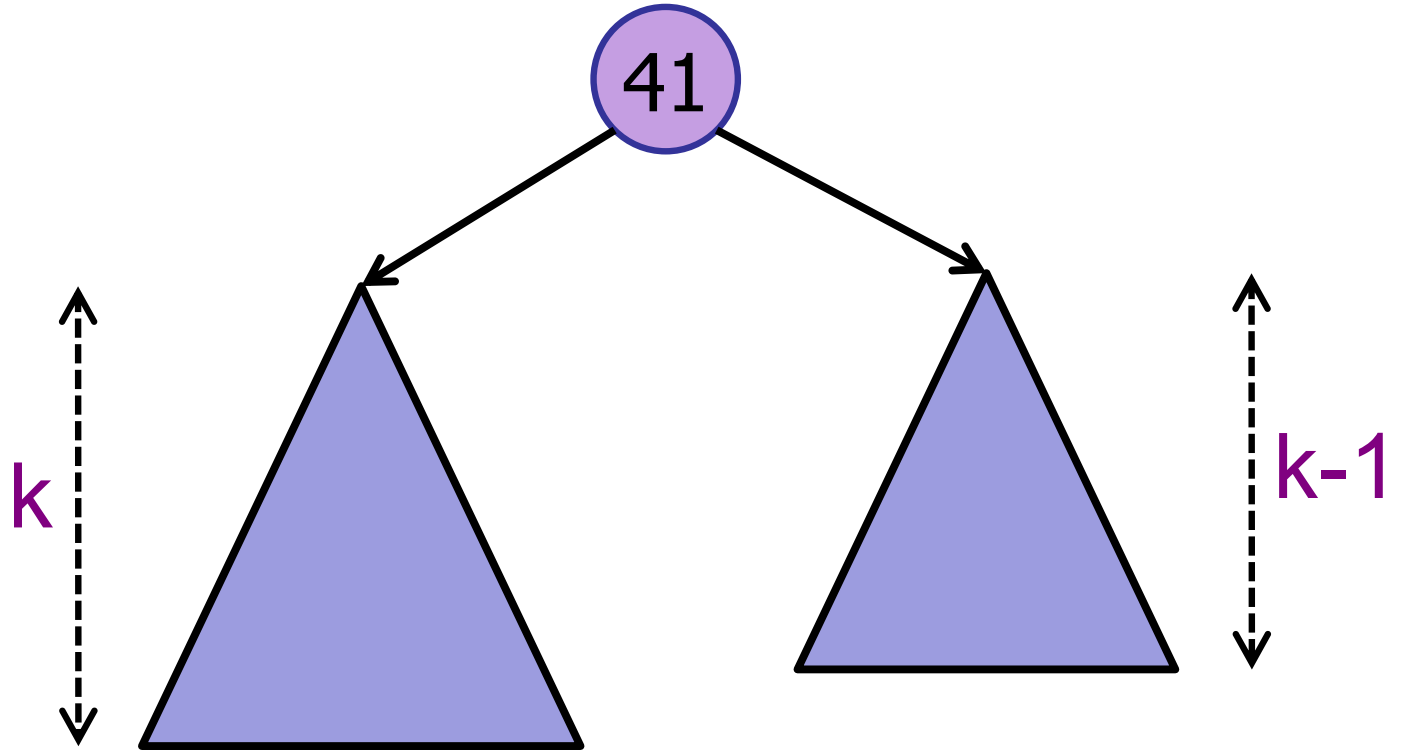
Step 2: Maintain Balance

# It’s good that we don’t have to

Balance perfectly

# AVL Trees [Adelson-Velskii & Landis 1962]

Step 2: Show how to maintain height-balance



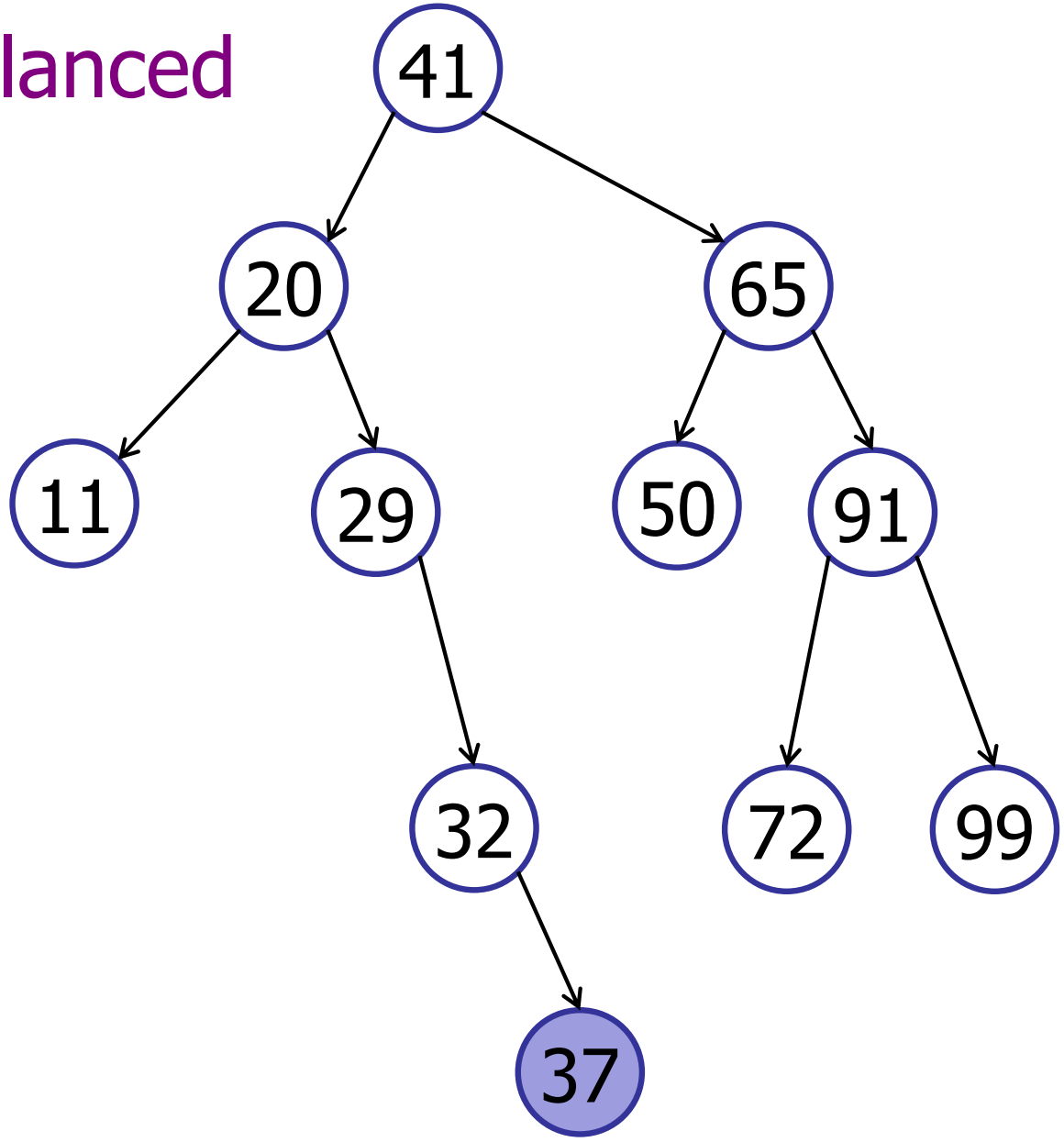
# Inserting in an AVL Tree

Before insertion, balanced

insert(37)

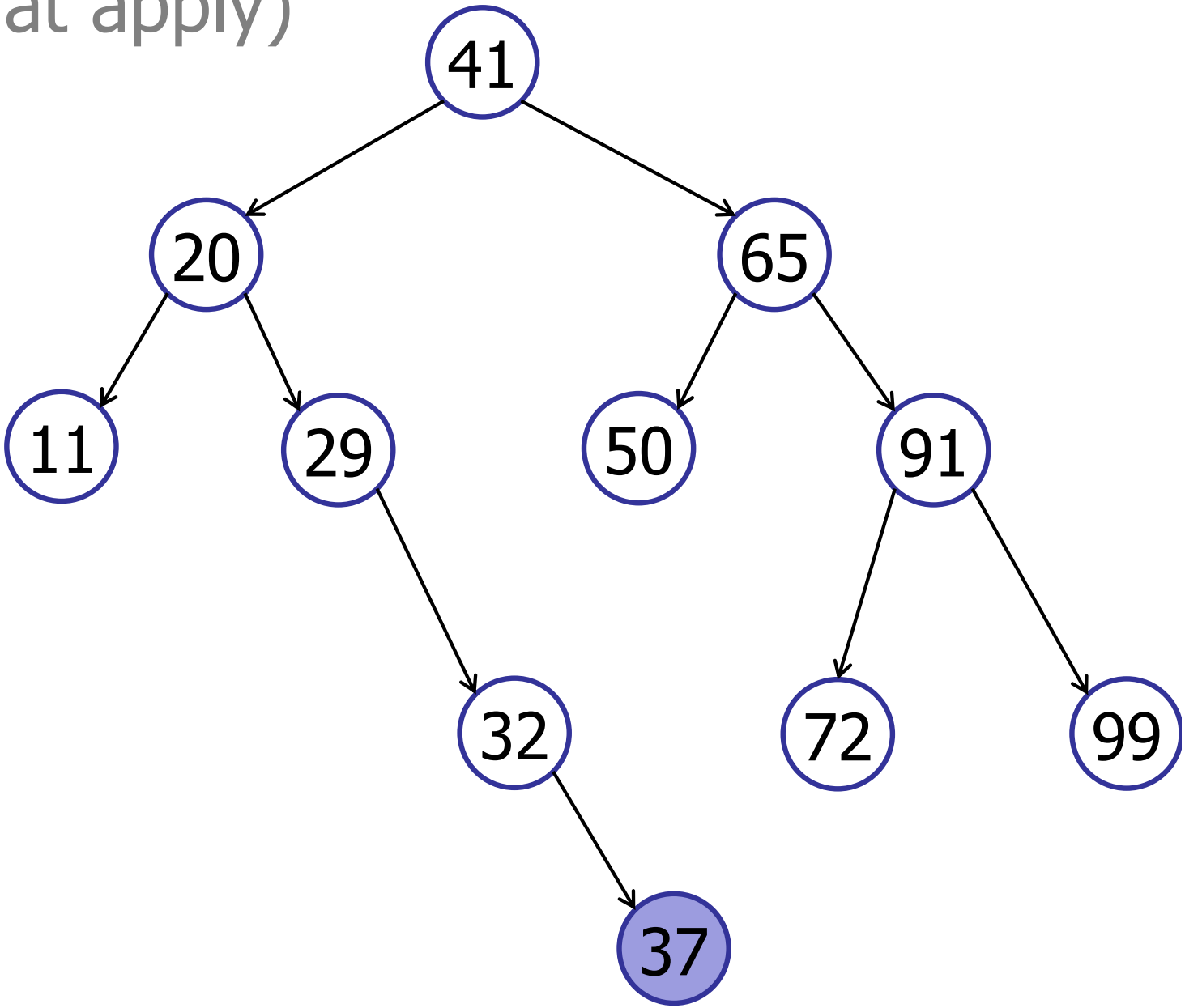
No longer balanced after insertion!

Need to rebalance!



# Which nodes need rebalancing? (click all that apply)

1. 41
2. 20
3. 11
4. 29
5. 32
6. 37
7. 65



# Which nodes need rebalancing? (click all that apply)

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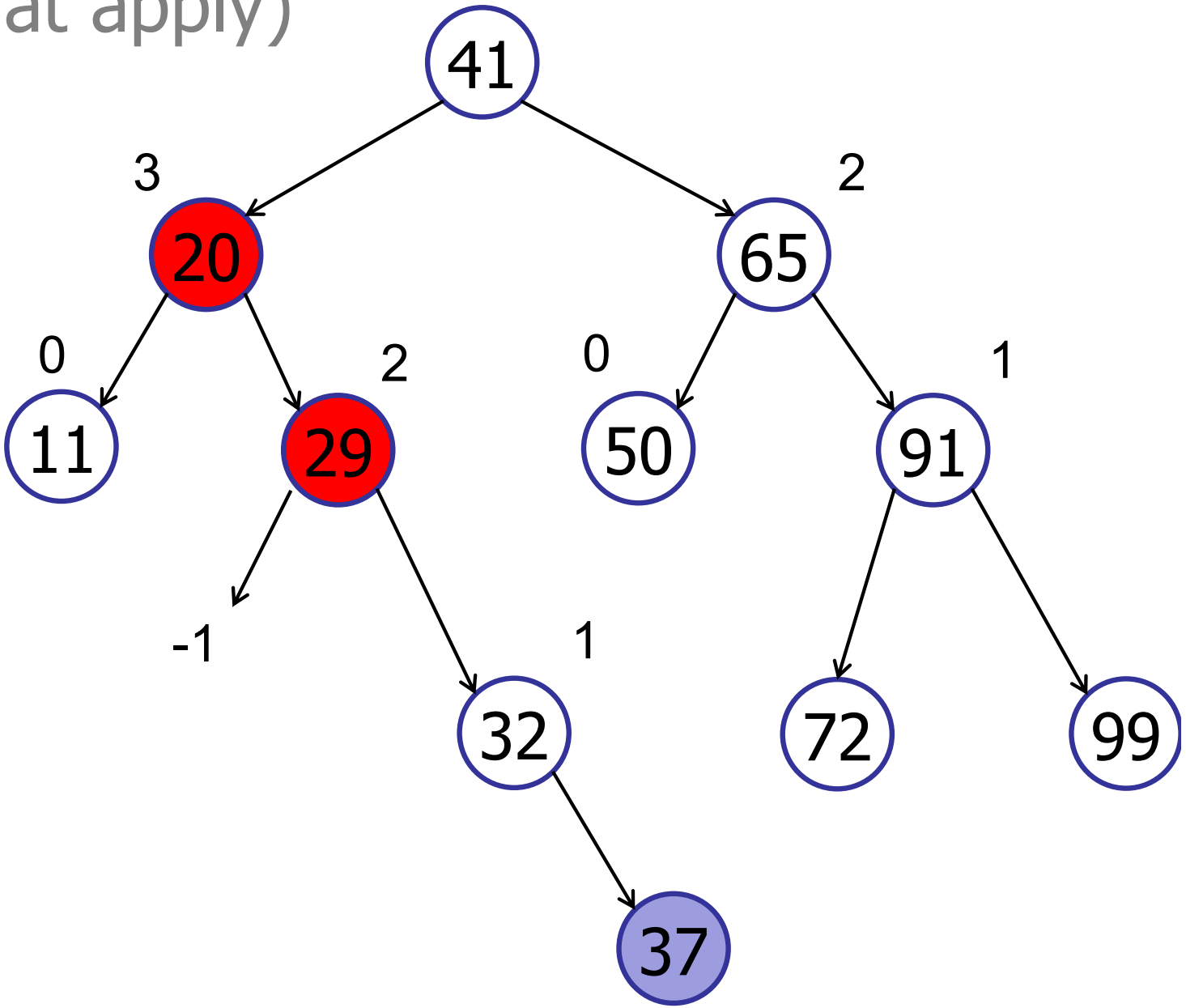
6. 37
7. 65

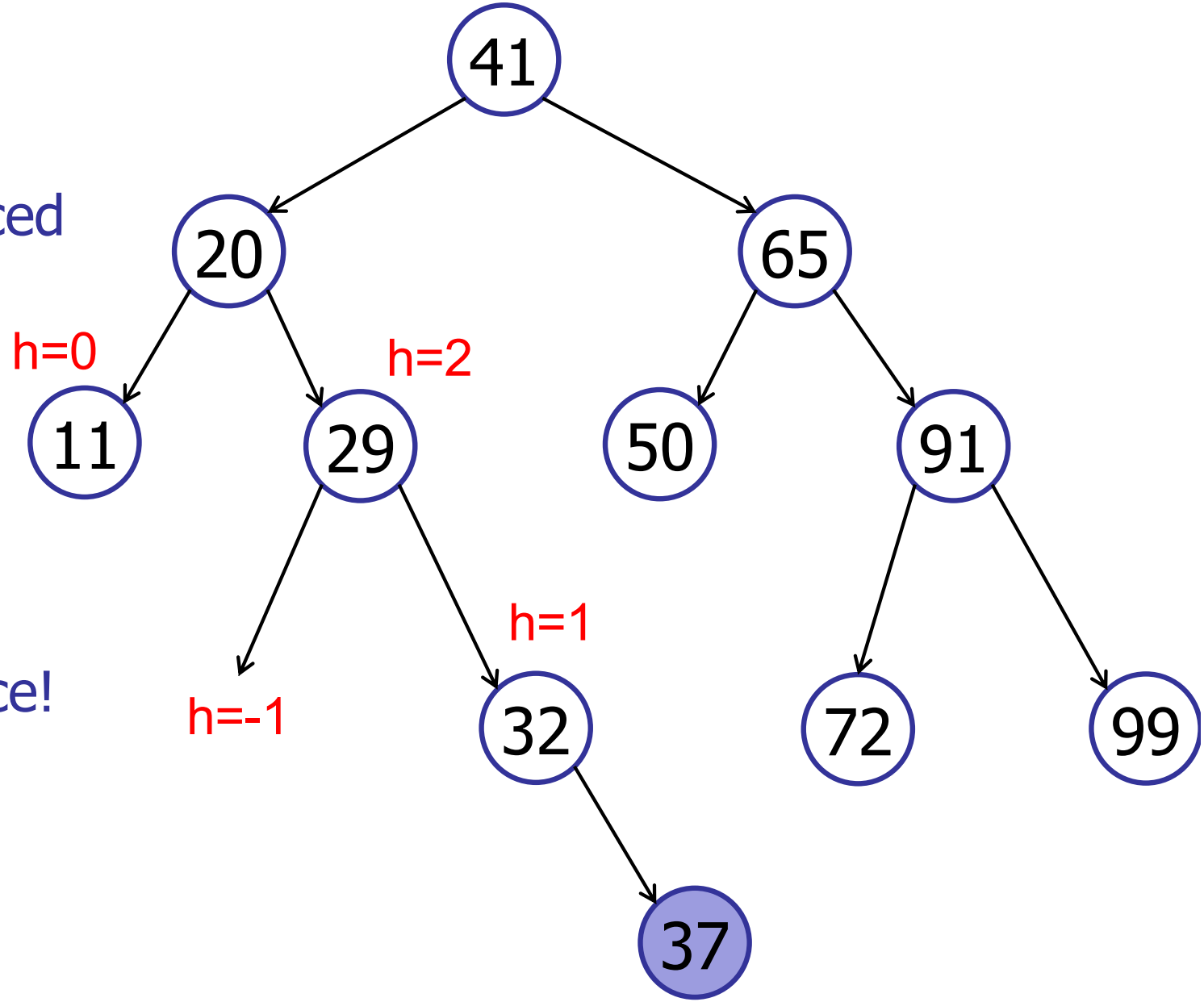


# Inserting in an AVL Tree

insert(37)

No longer balanced after insertion!

Need to rebalance!



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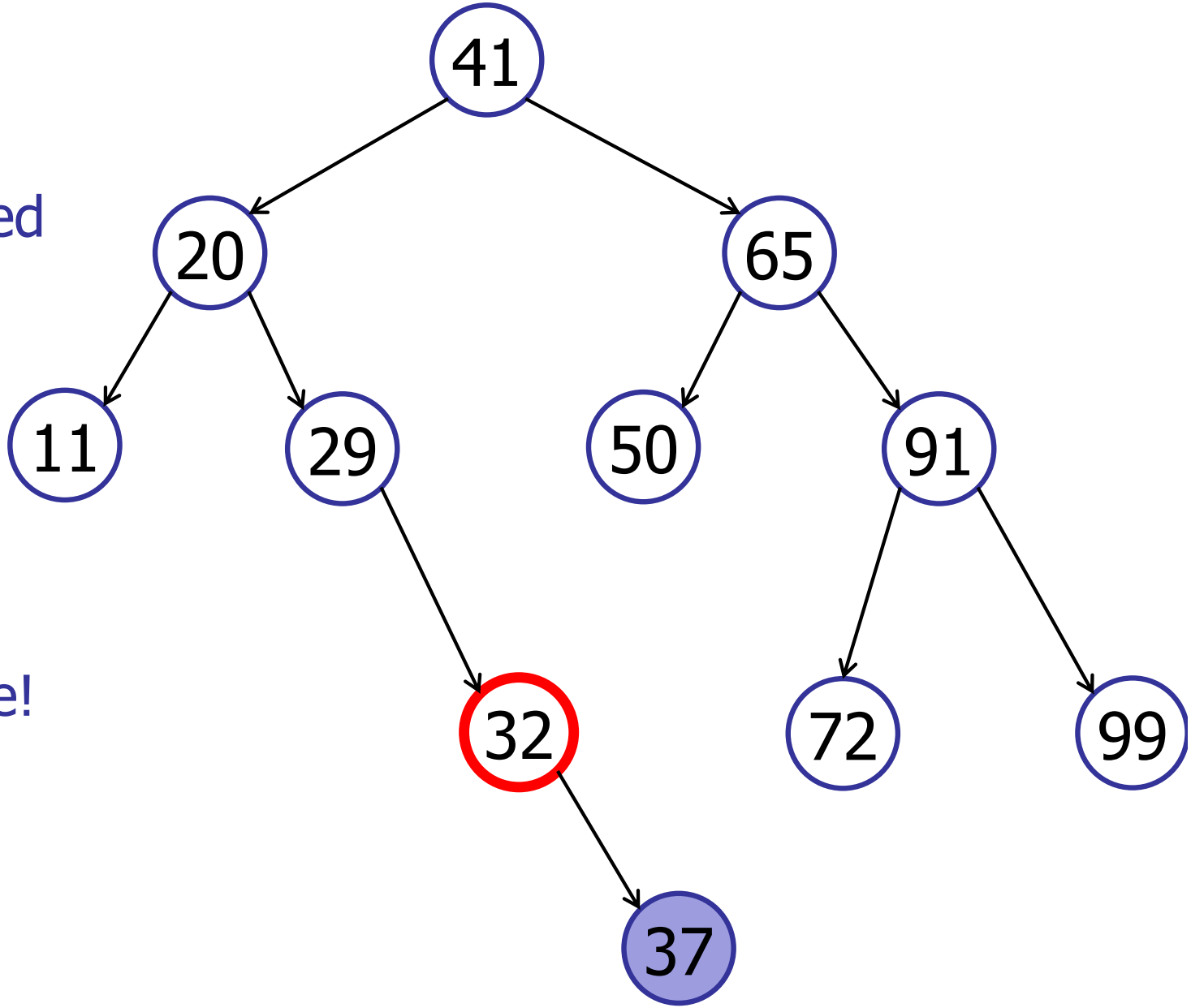

# Inserting in an AVL Tree

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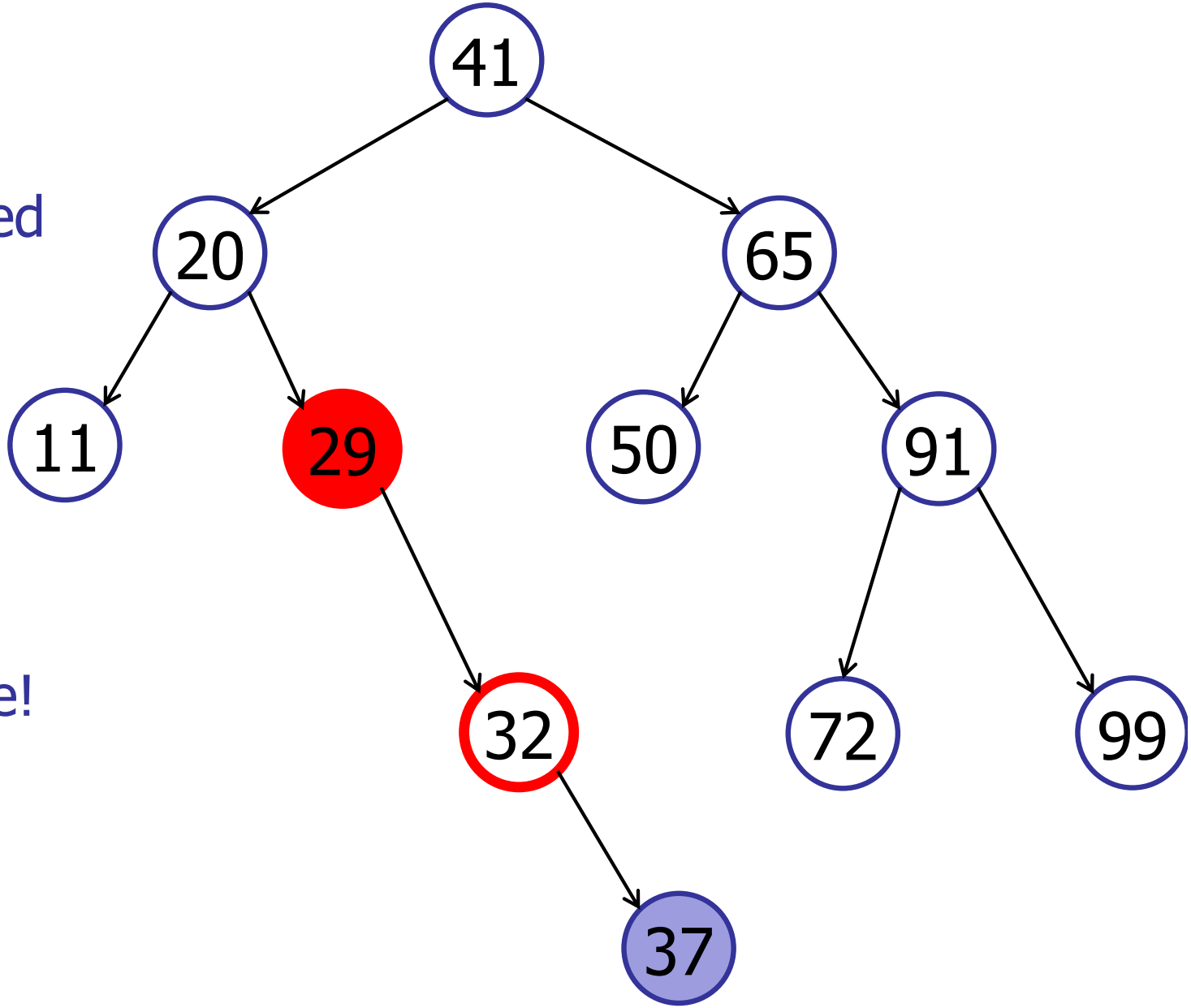

# Inserting in an AVL Tree

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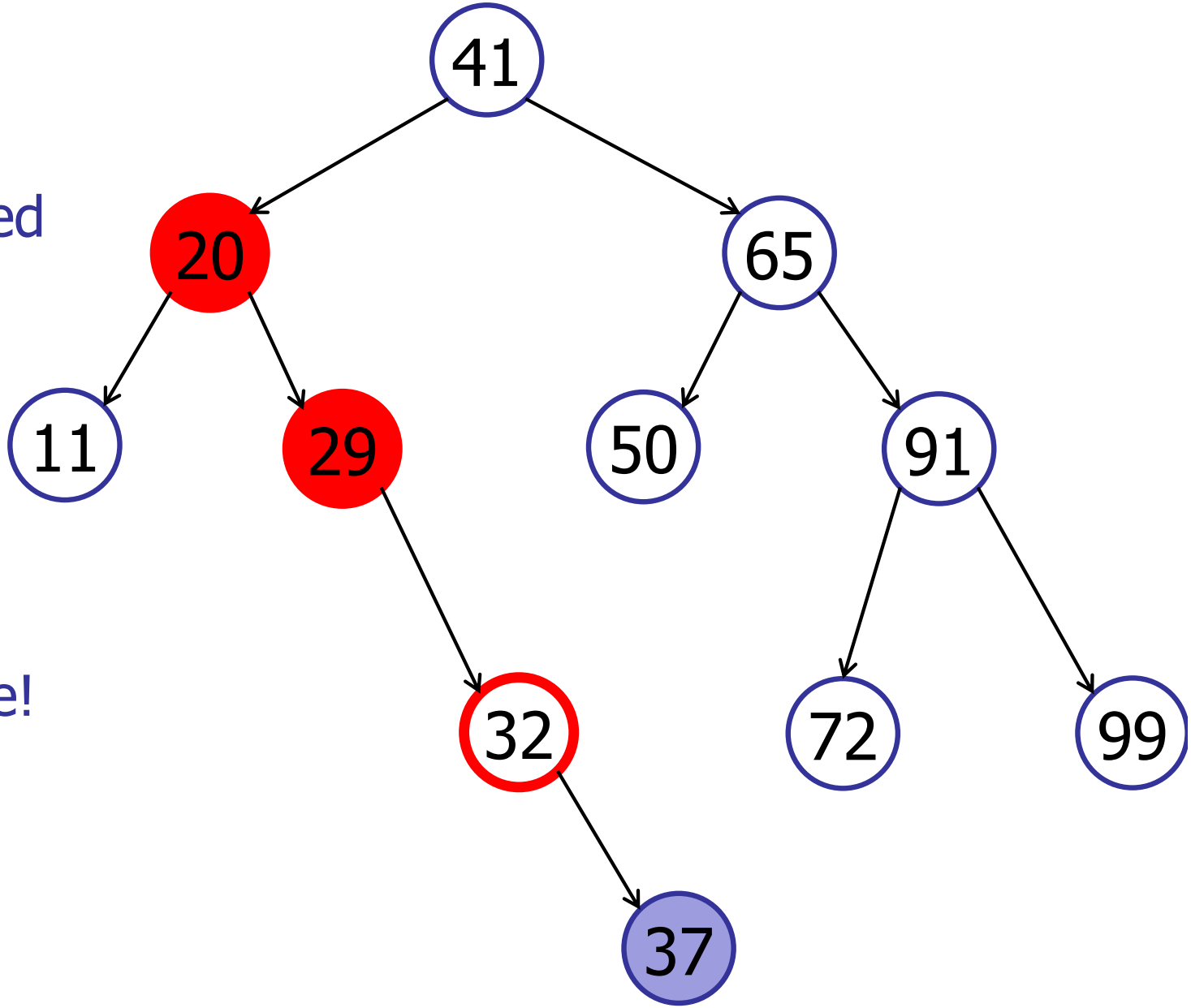
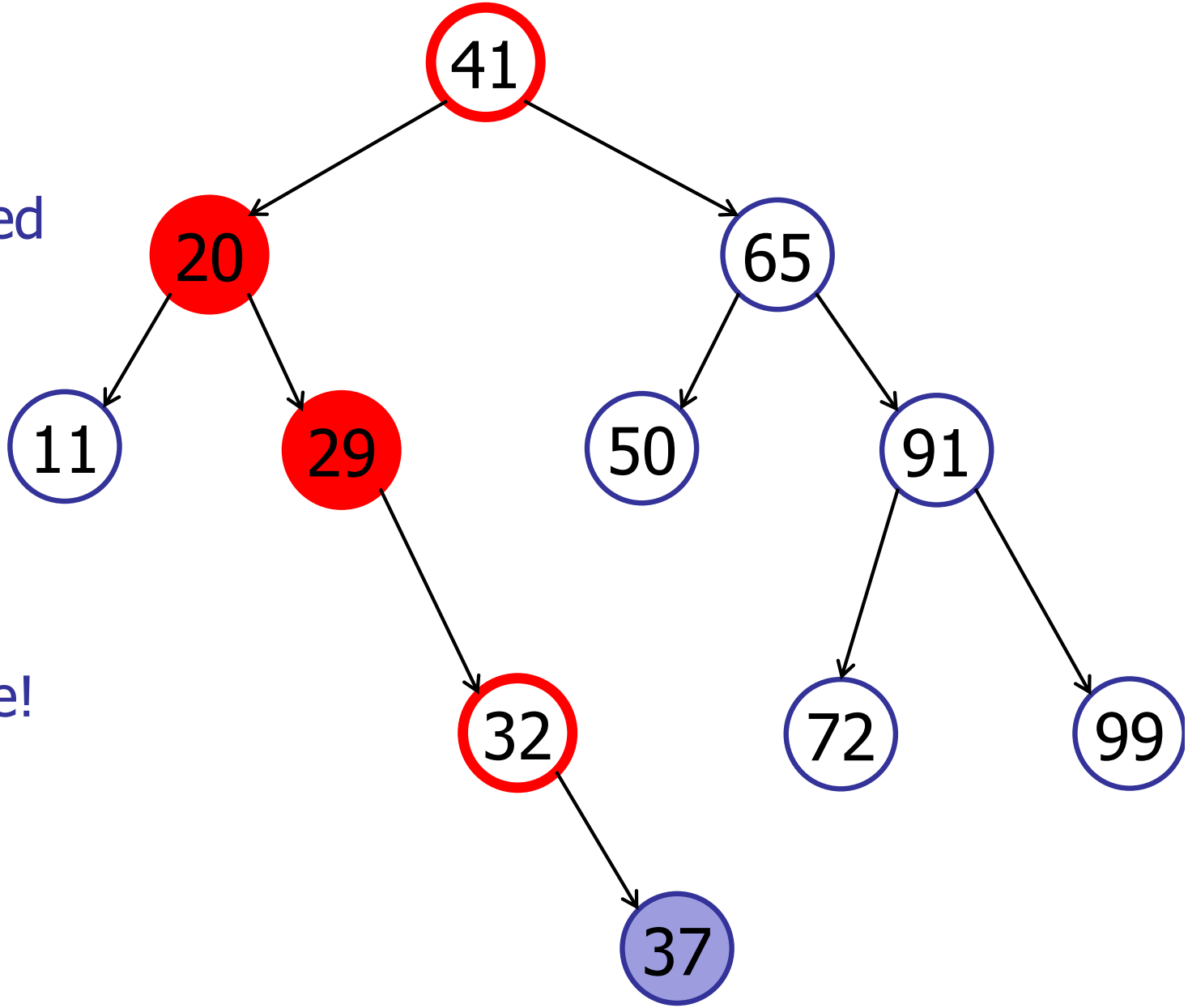


# Inserting in an AVL Tree

insert(37)

No longer balanced after insertion!

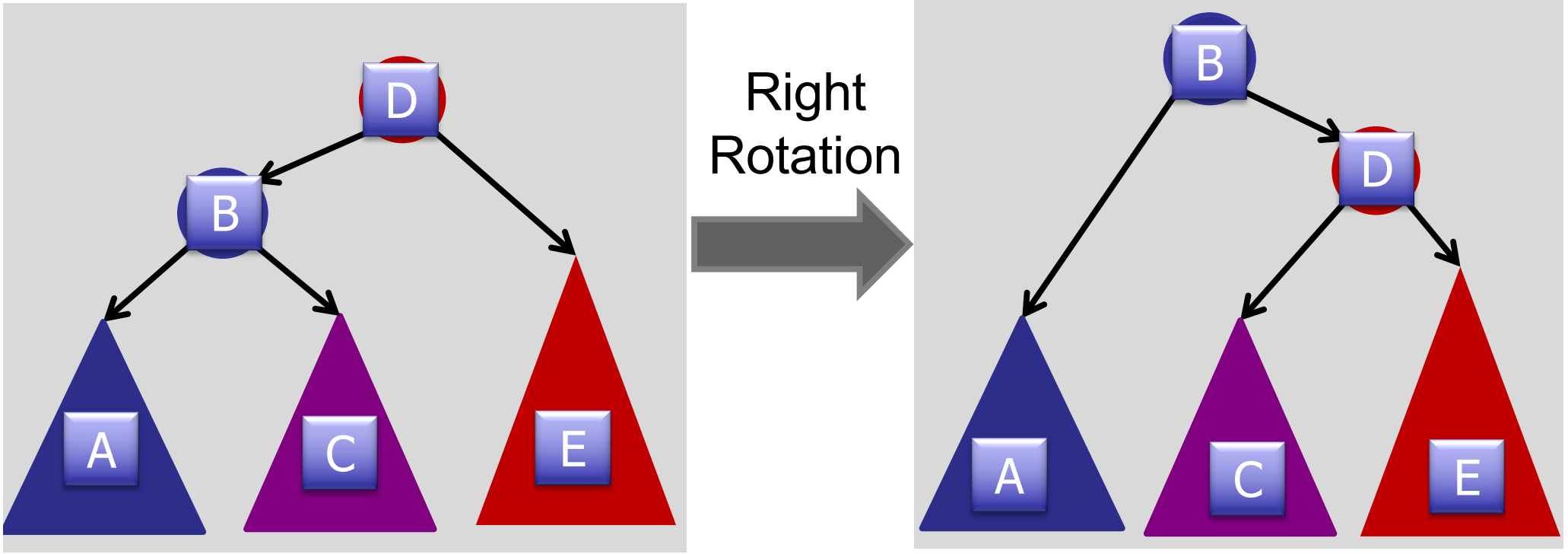
Need to rebalance!



# Trick to rebalance the tree

Tree rotation!

# Tree Rotations

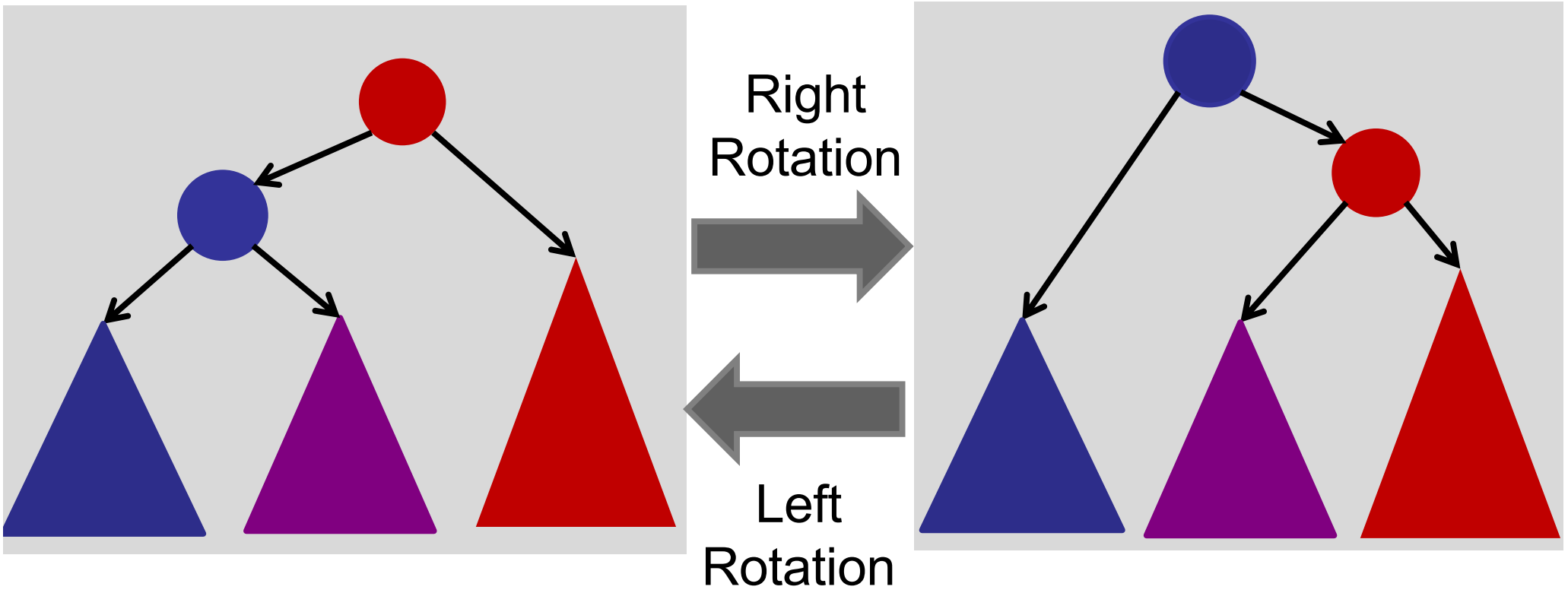



A < B < C < D < E

Rotations maintain ordering of keys.

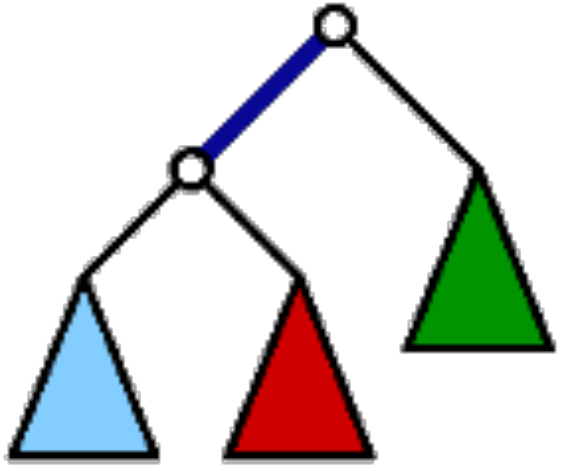
⇒ Maintains BST property.

# Tree Rotations

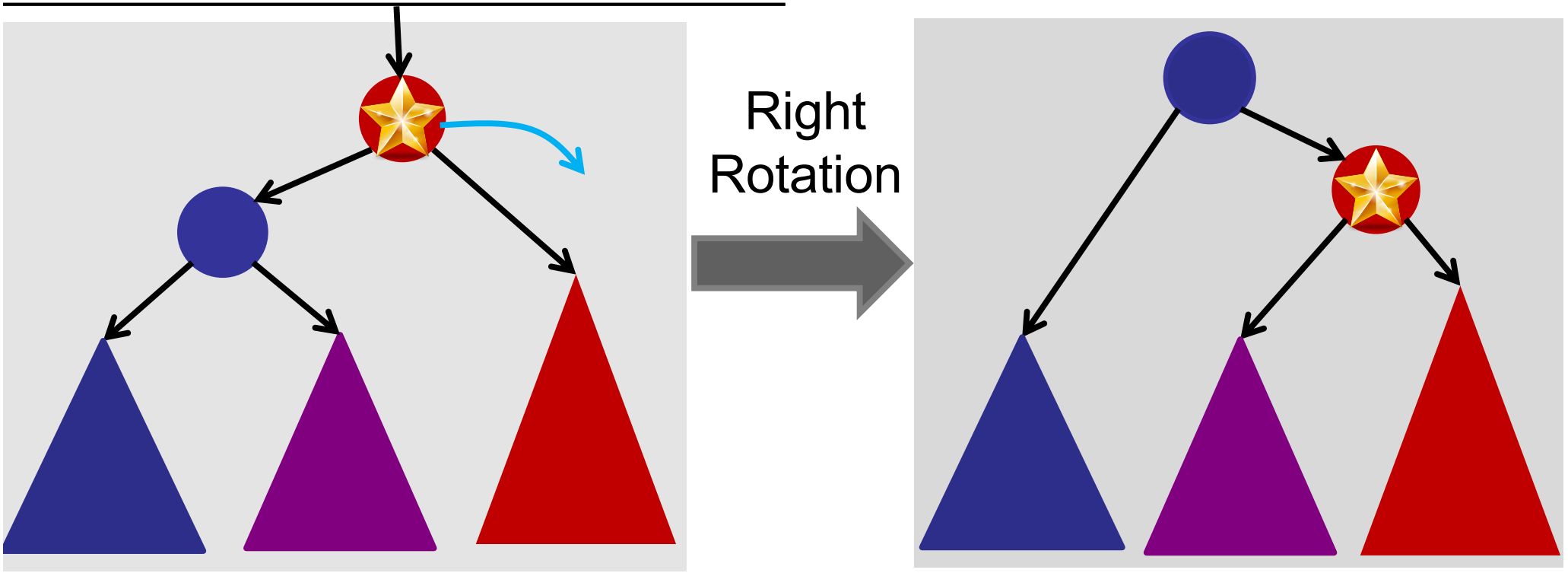


# Wait….

What is a left rotation and what is a right rotation!?

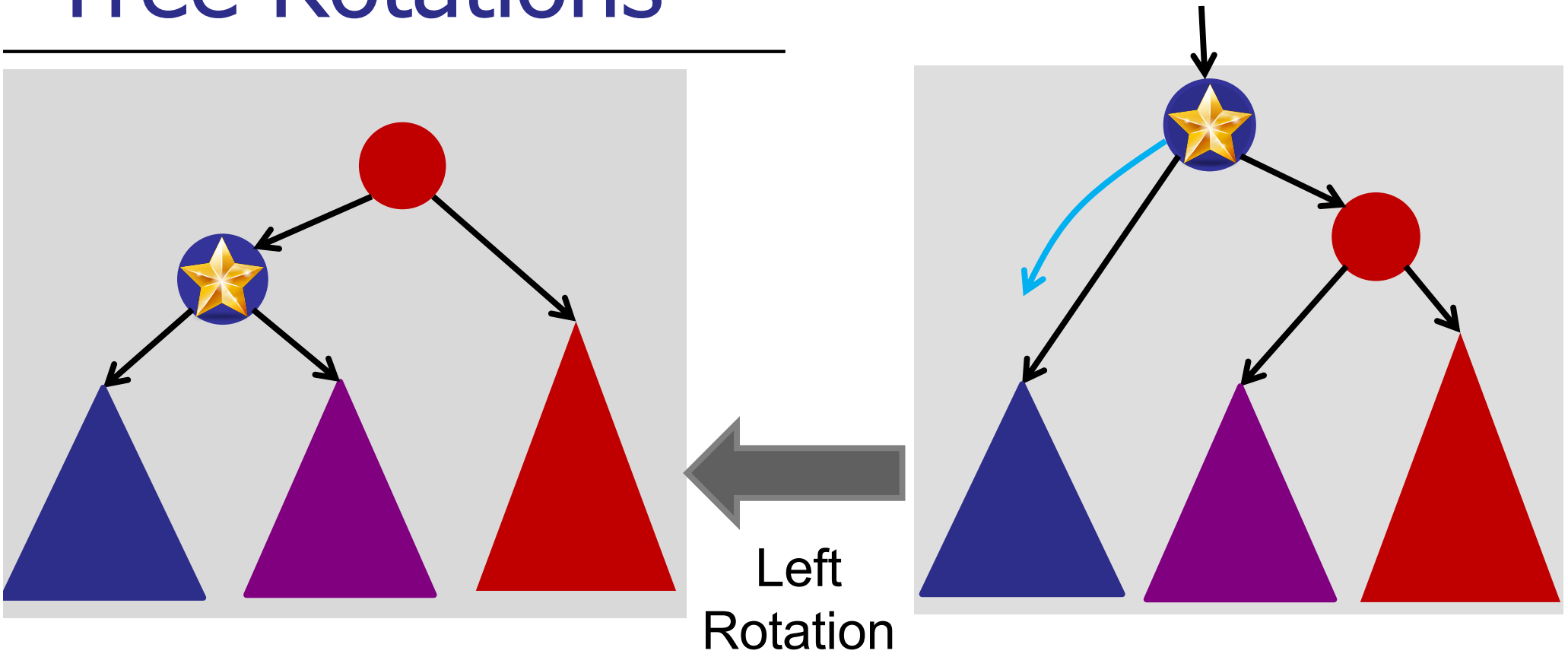


# The way to remember it




The root of the subtree moves right

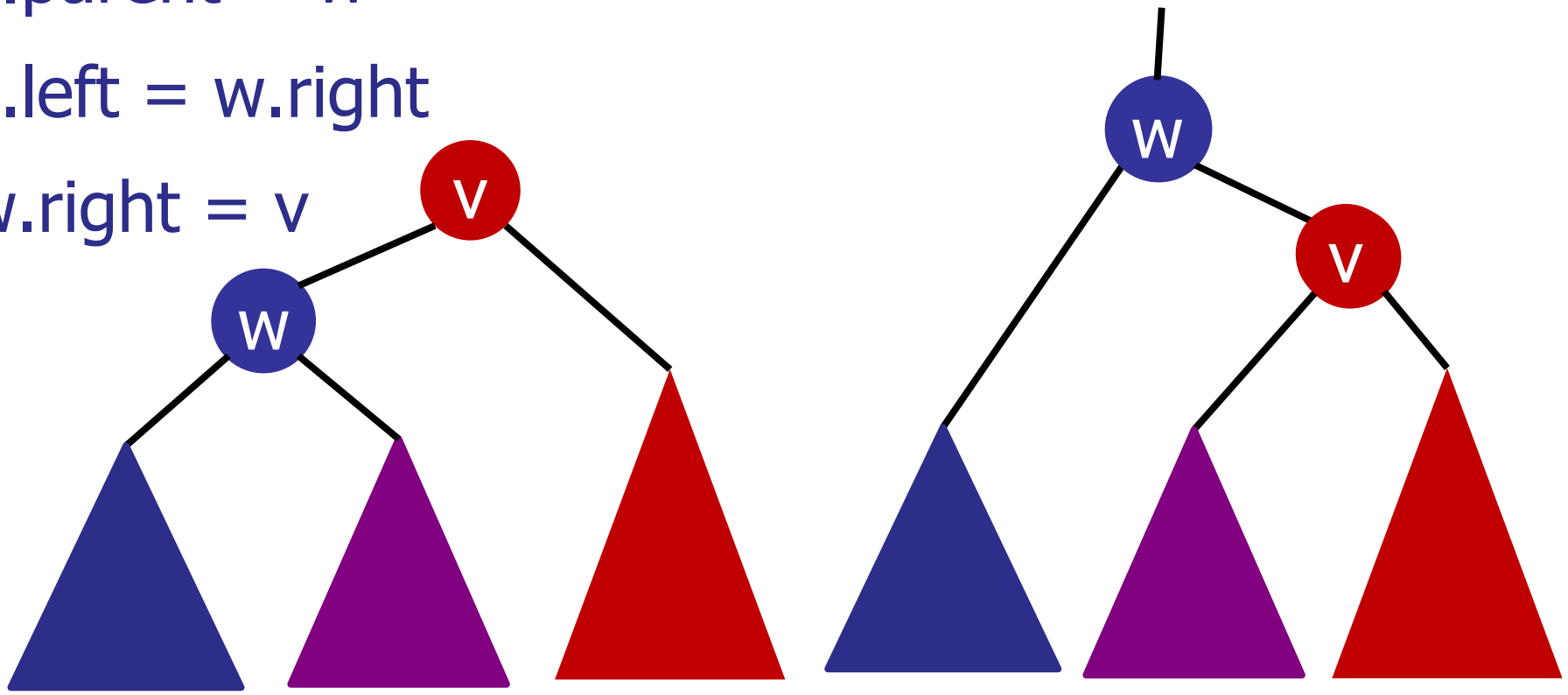
# Tree Rotations



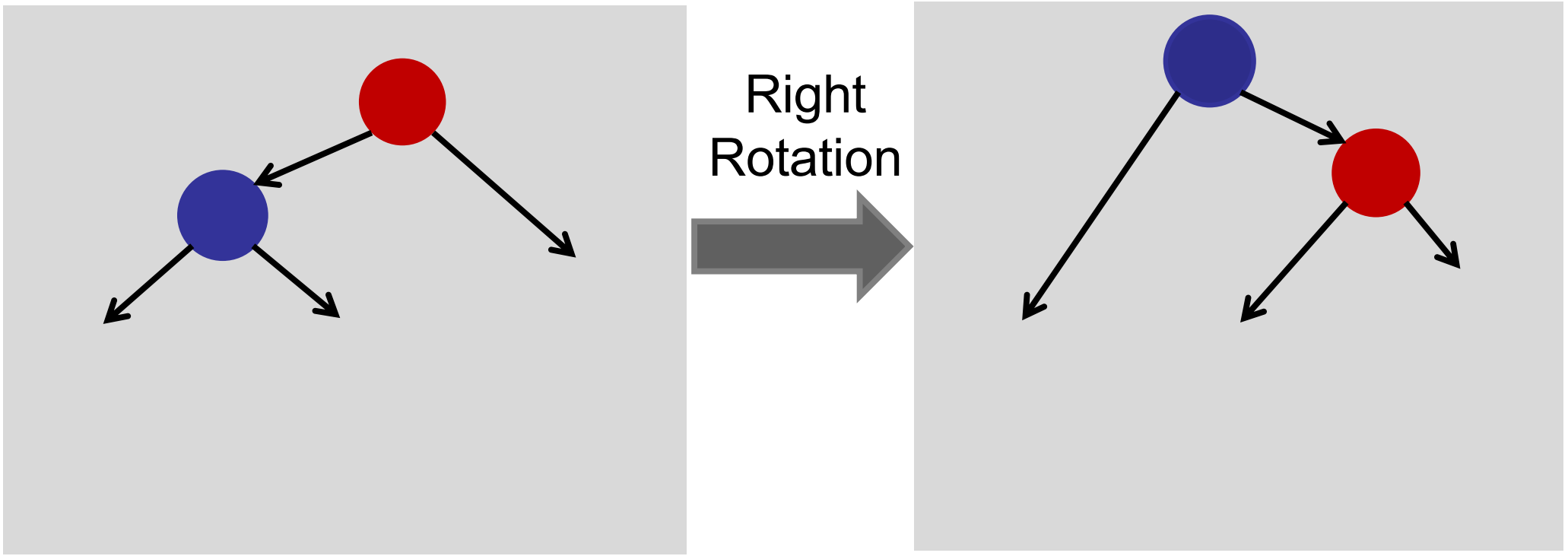

The root of the subtree moves left

# Rotations

```
right-rotate(v)          // assume v has left != null
    w = v.left
    w.parent = v.parent
    v.parent = w
    v.left = w.right
    w.right = v
```



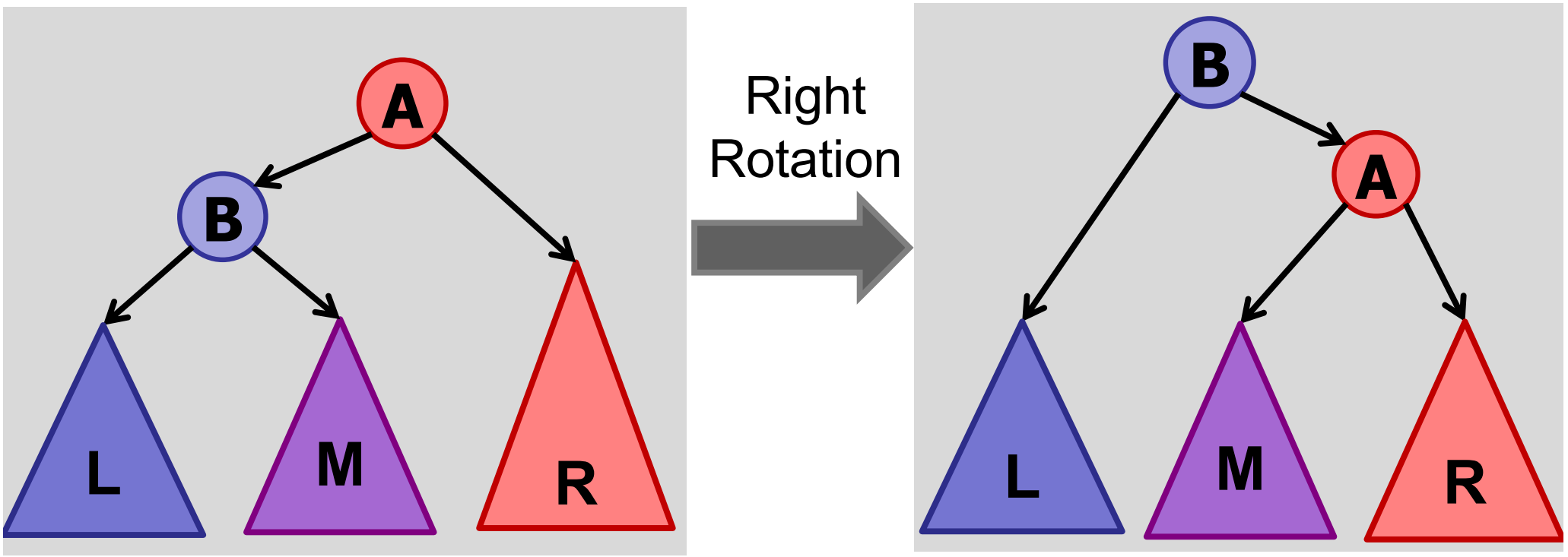
# Tree Rotations




rotate-right requires a left child

rotate-left requires a right child

# Tree Rotations




After insert:

Use tree rotations to restore balance.

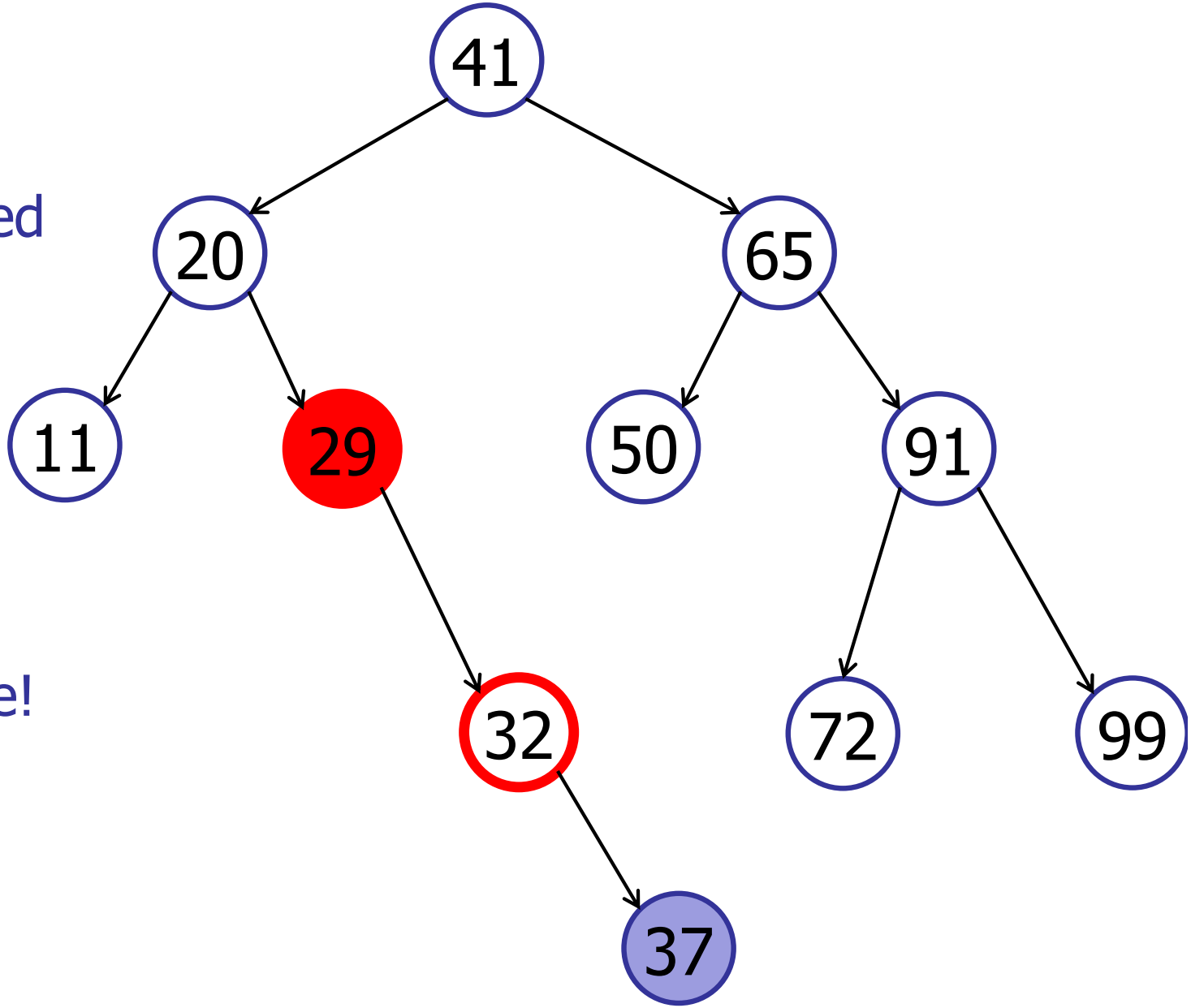
Height is out-of-balance by 1

# Inserting in an AVL Tree

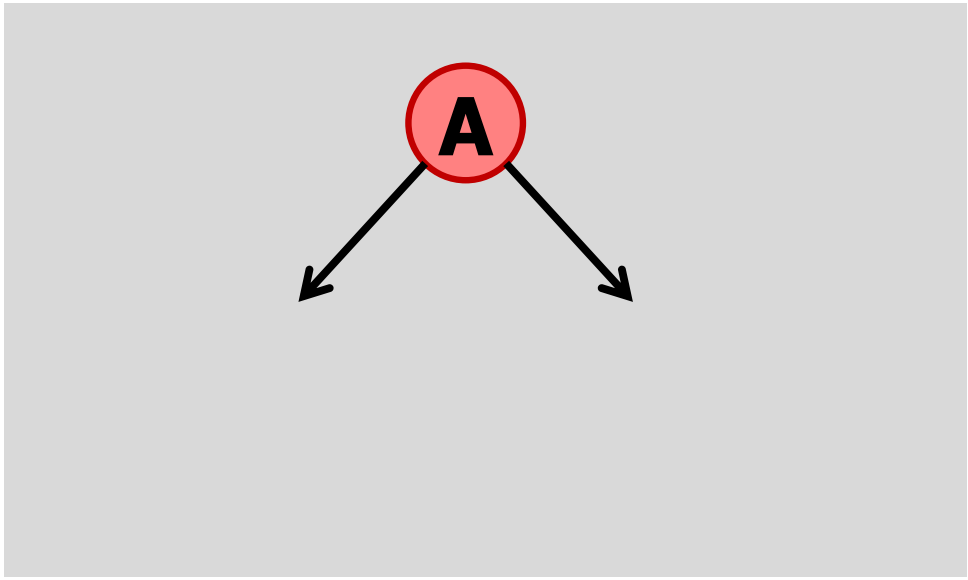
insert(37)

No longer balanced after insertion!

Need to rebalance!



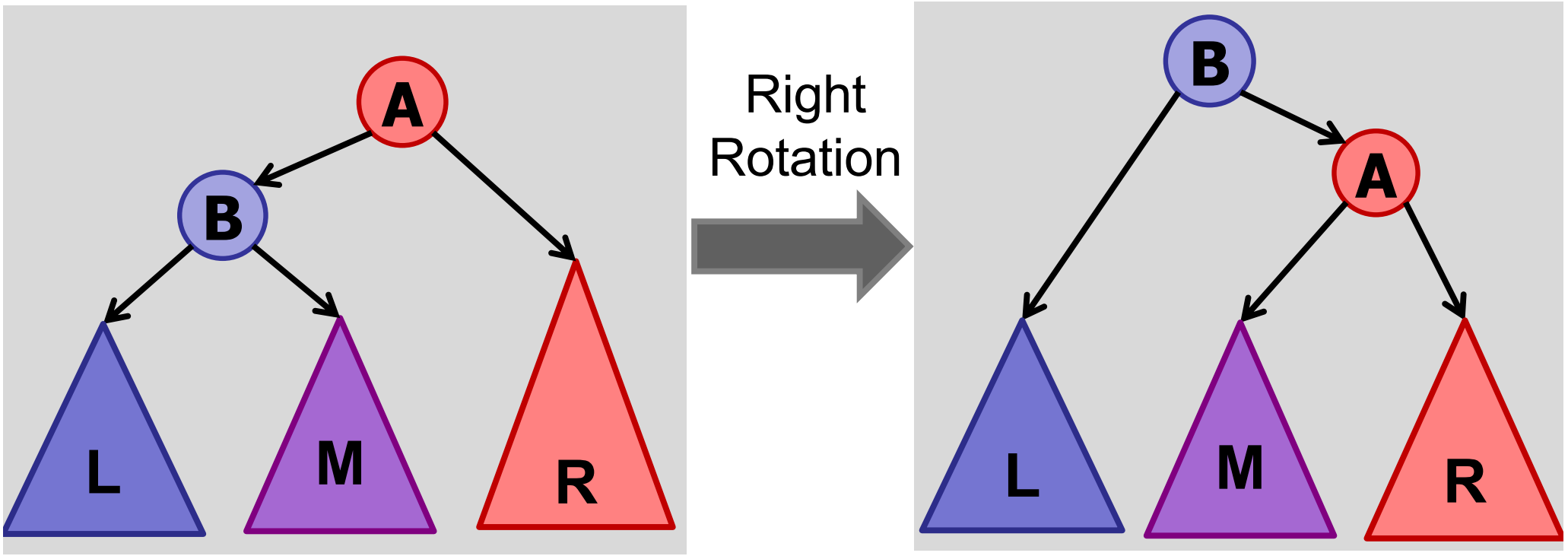
# Tree Rotations




A is **LEFT-heavy** if left sub-tree has larger height than right sub-tree.

A is **RIGHT-heavy** if right sub-tree has larger height than left sub-tree.

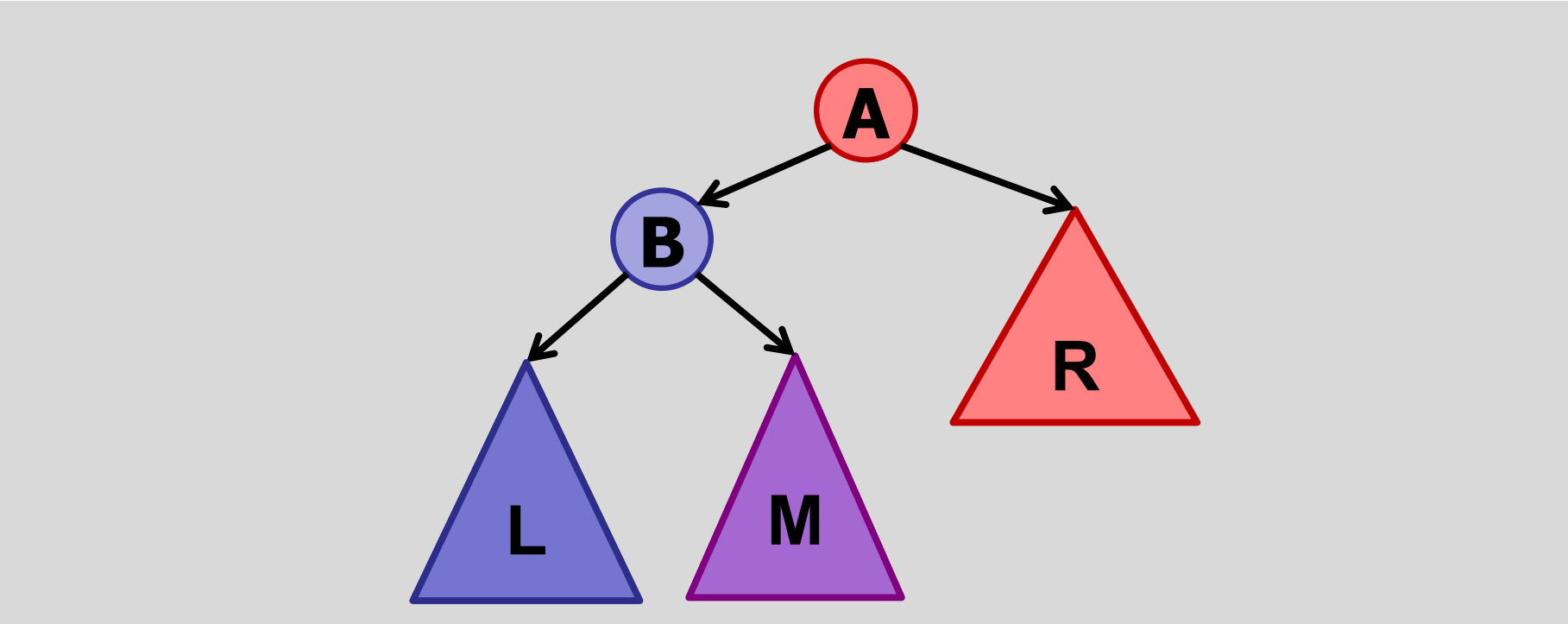
# Tree Rotations




Use tree rotations to restore balance.

After insert, start at bottom, work your way up.

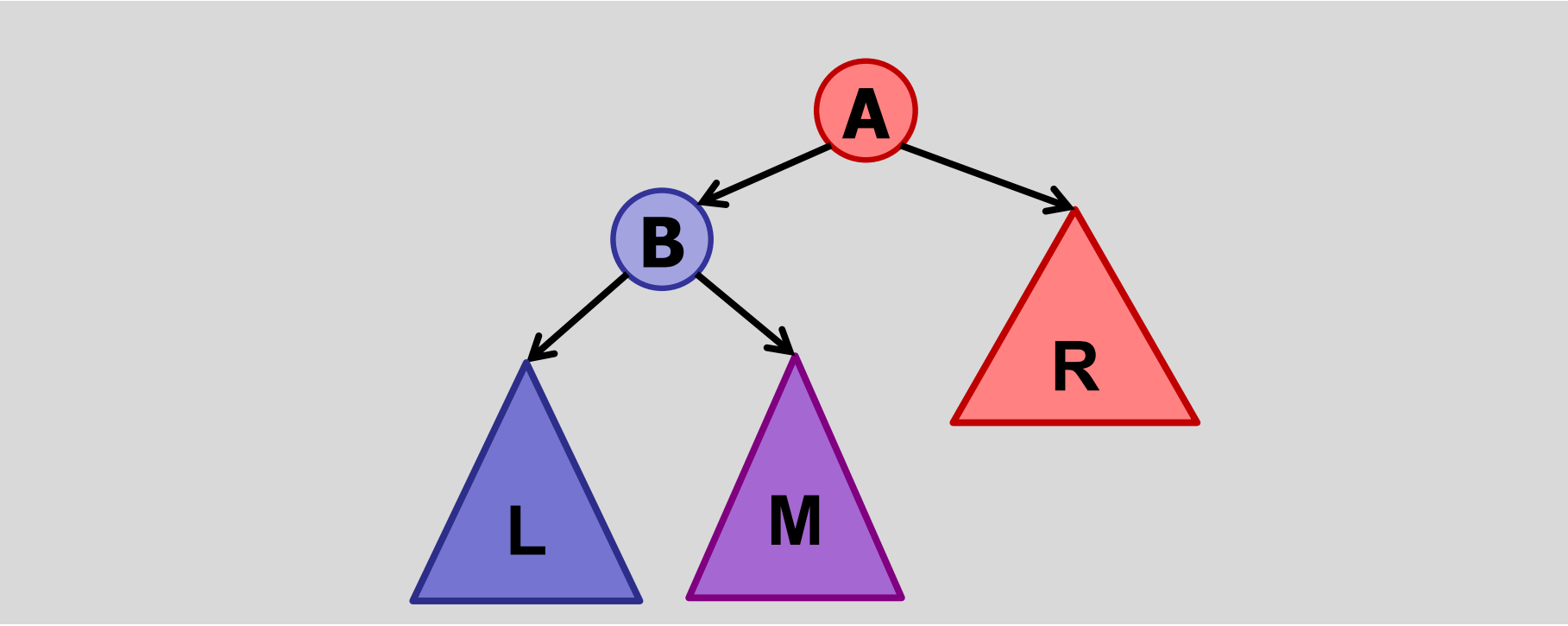
# Tree Rotations




Assume **A** is the lowest node in the tree violating balance property.

Assume A is **LEFT-heavy**.

# Tree Rotations (Left Heavy)

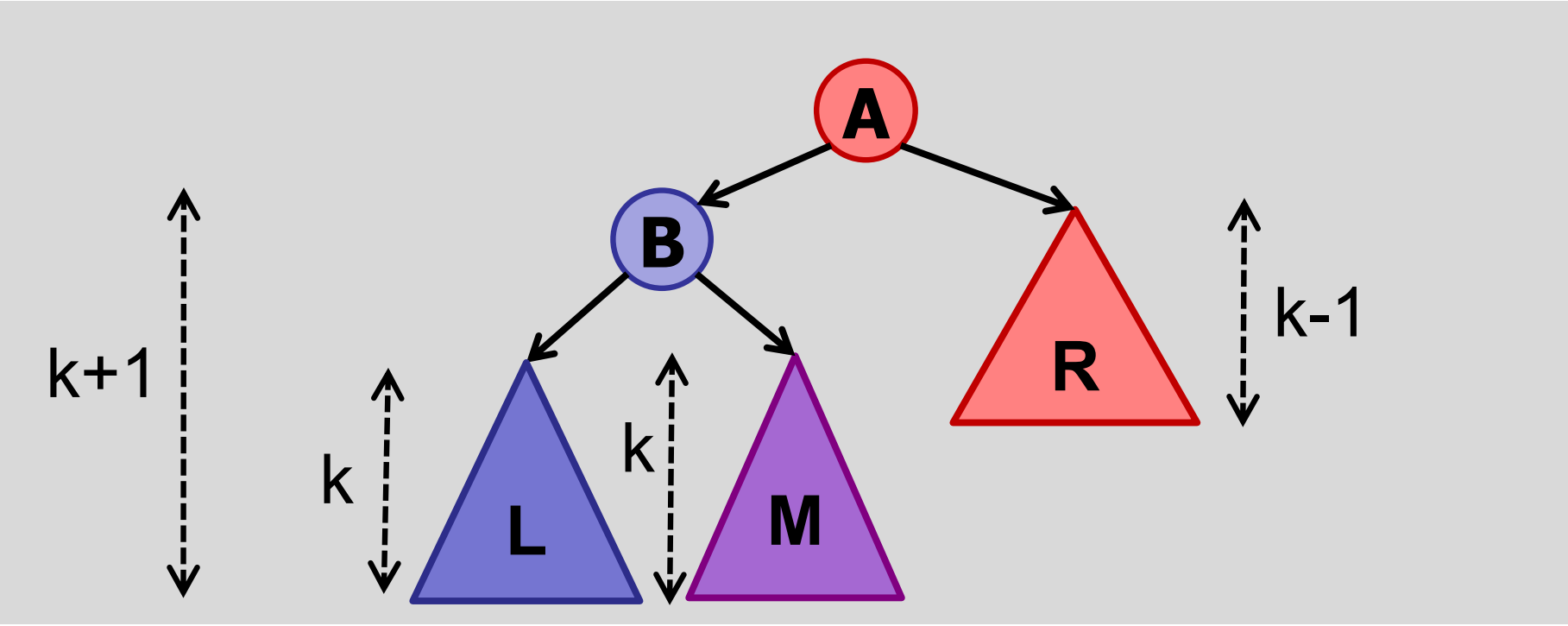



Assume **A** is the lowest node in the tree violating balance property.

Case 1: **B** is balanced : h(**L**) = h(**M**)

h(**R**) = h(**B**) – 2

# Tree Rotations (Left Heavy)

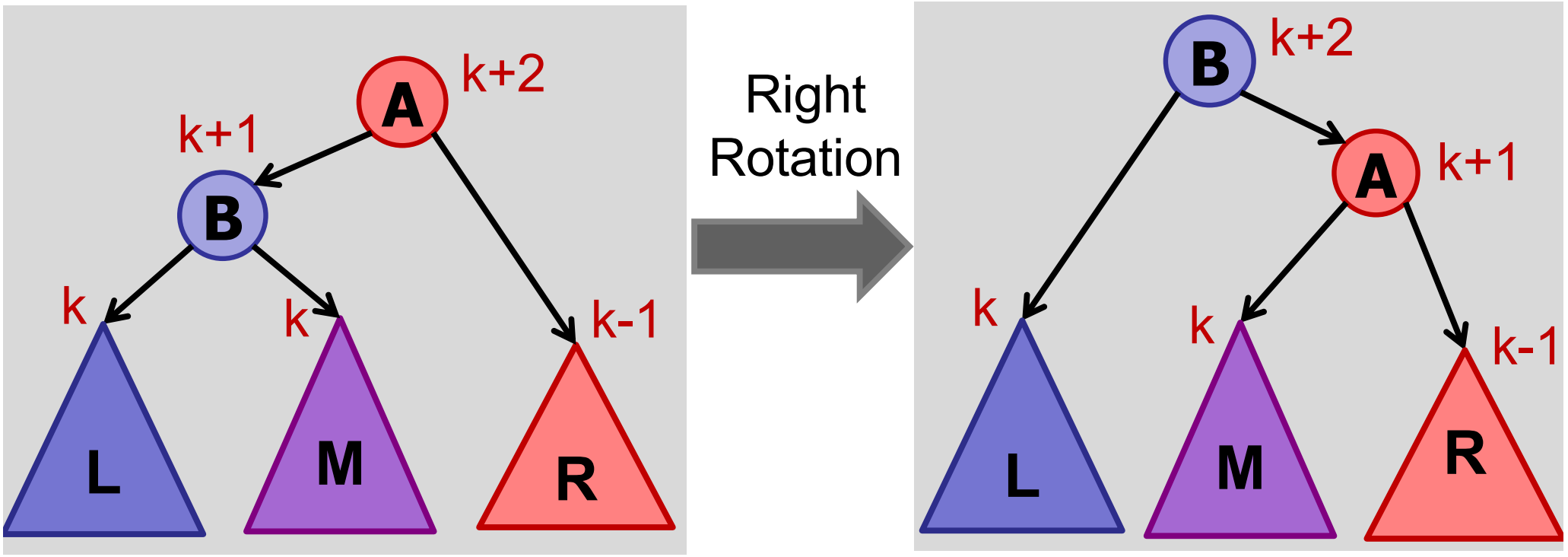



Assume **A** is the lowest node in the tree violating balance property.

Case 1: **B** is balanced : h(**L**) = h(**M**)

h(**R**) = h(**M**) – 1

# Tree Rotations

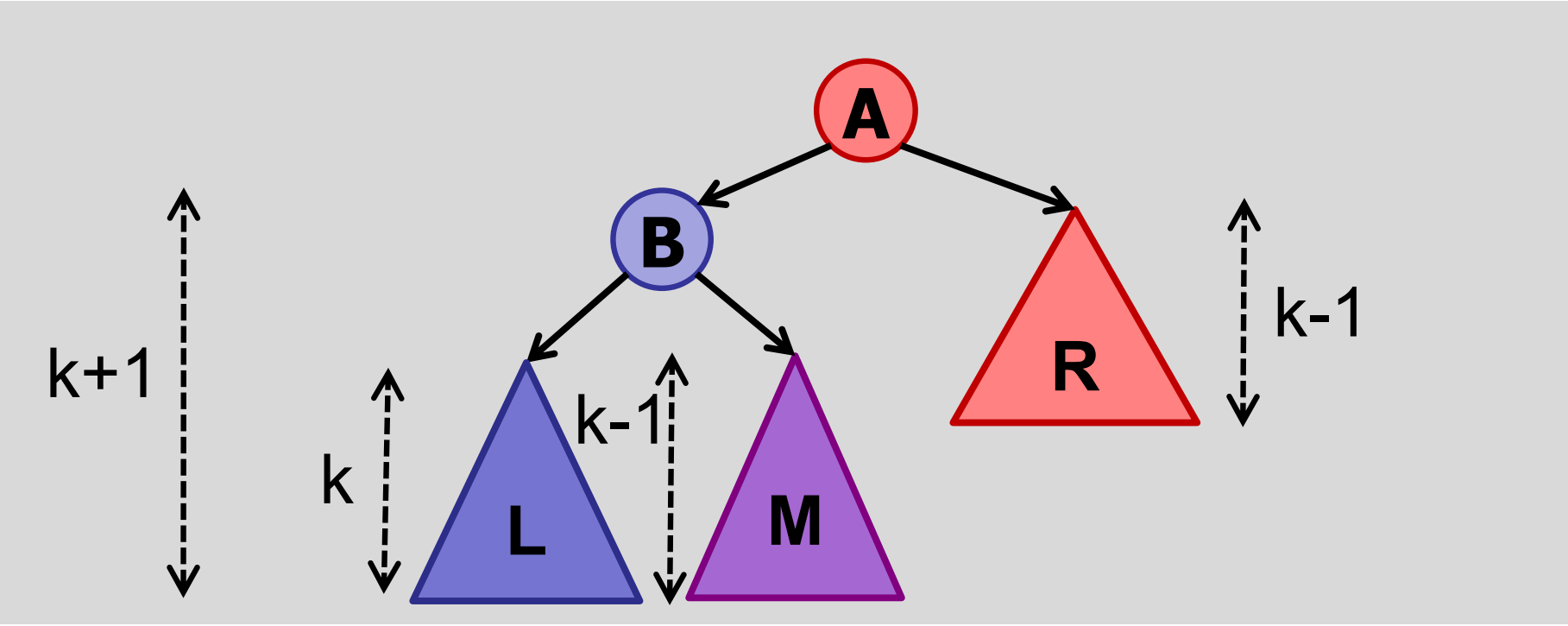



right-rotate:

Case 1: **B** is balanced : h(**L**) = h(**M**)

h(**R**) = h(**M**) – 1

# Tree Rotations (Left Heavy)

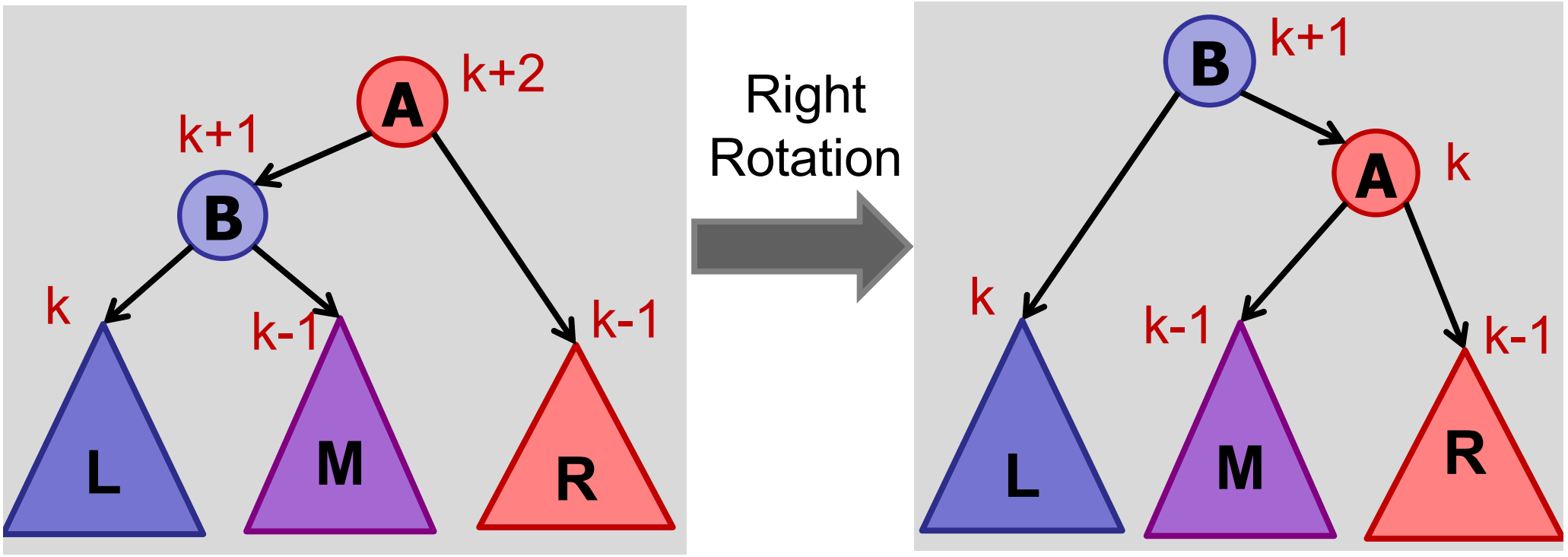



Assume **A** is the lowest node in the tree violating balance property.

Case 2: **B** is left-heavy : h(**L**) = h(**M**) + 1

h(**R**) = h(**M**)

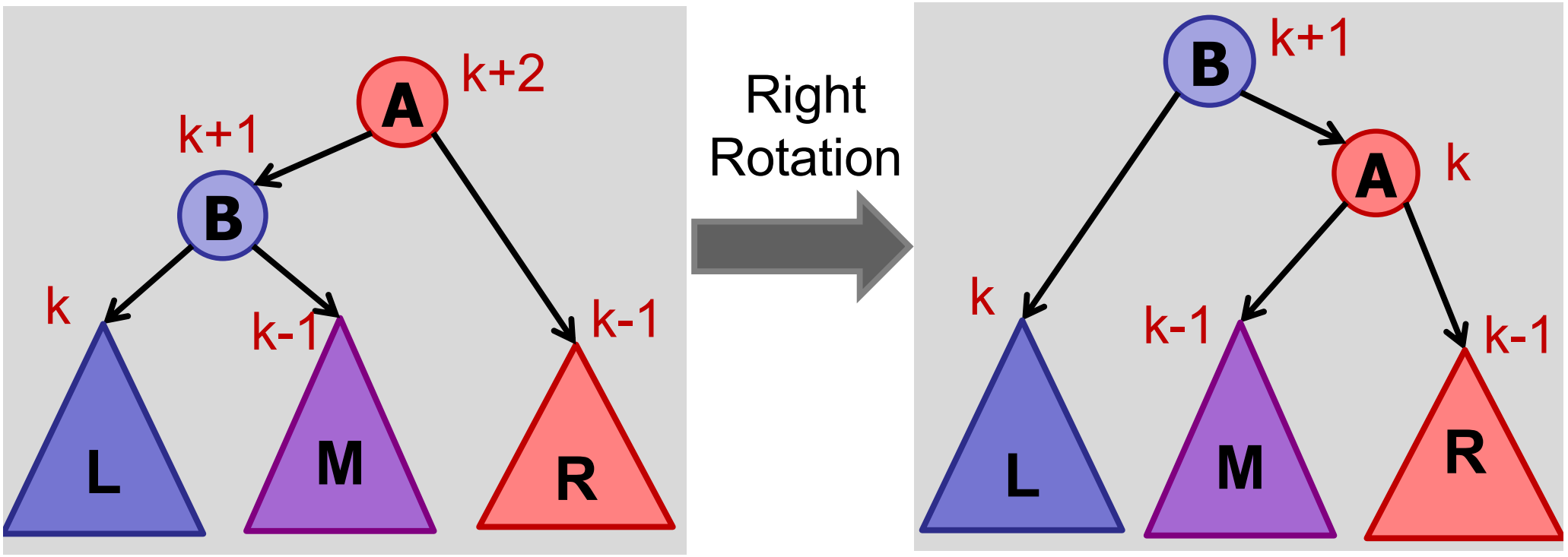
# Tree Rotations




right-rotate:

Case 2: **B** is left-heavy: h(**L**) = h(**M**) +1

h(**R**) = h(**M**)




Is it balanced?

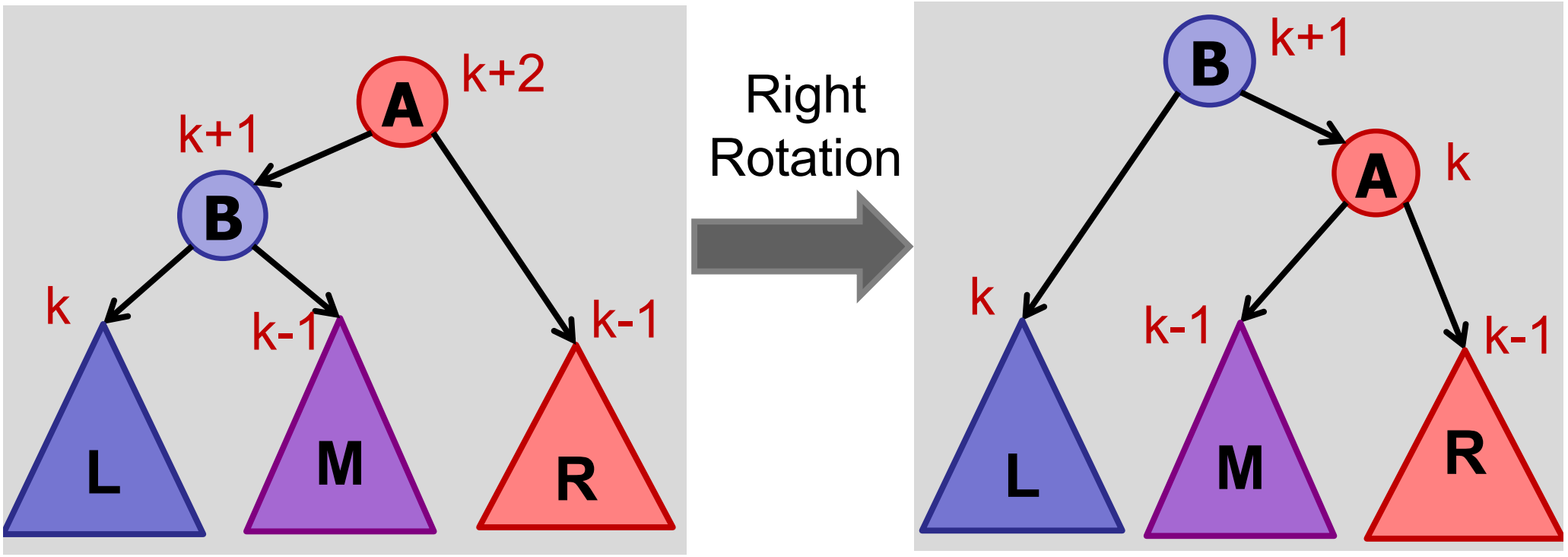
✓1. Yes.

2. No.

3. Maybe.

Yes or No on Zoom.

# Tree Rotations

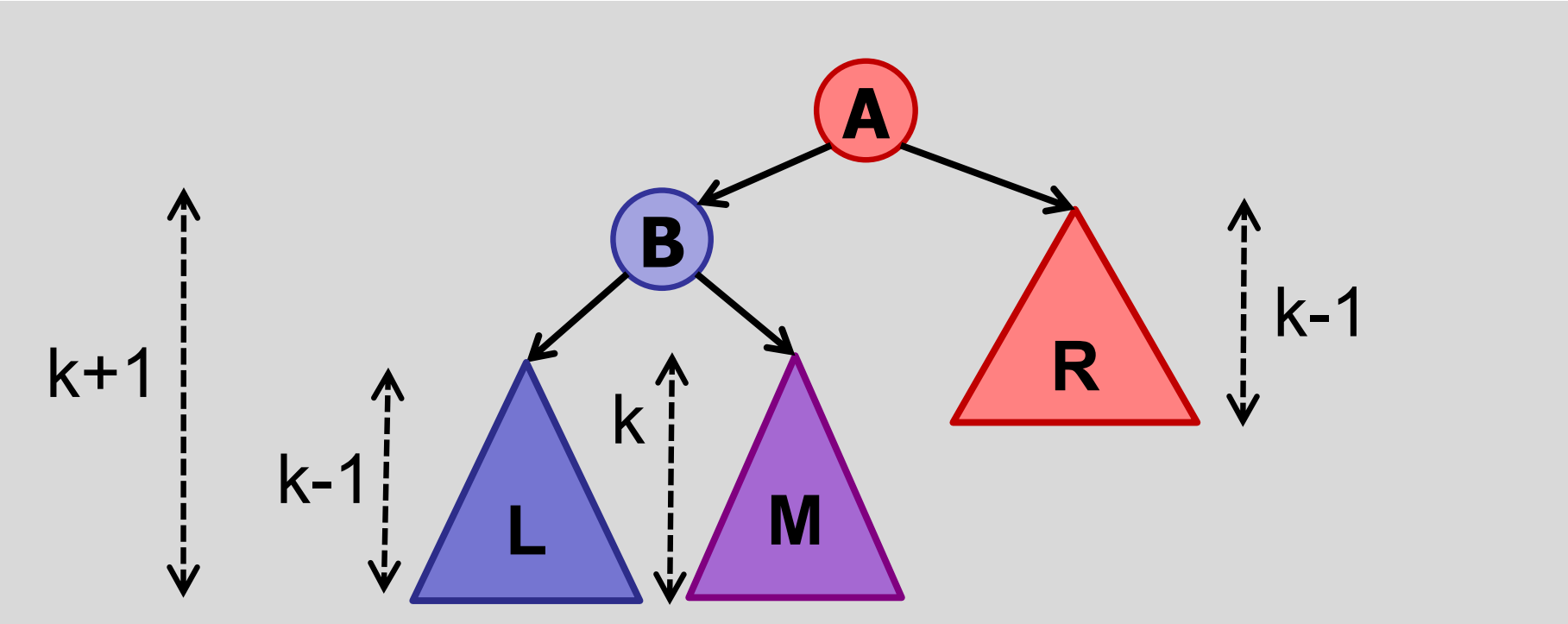



right-rotate:

Case 2: **B** is left-heavy: h(**L**) = h(**M**) +1

h(**R**) = h(**M**)

# Tree Rotations (Left Heavy)

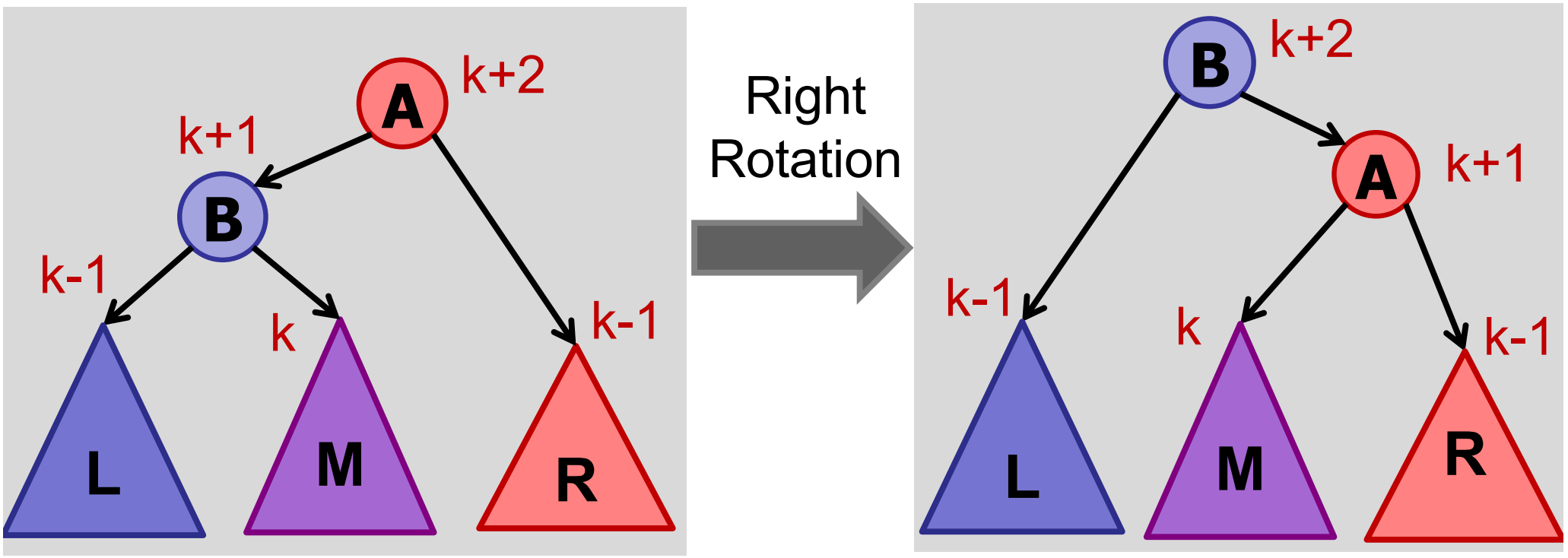



Assume **A** is the lowest node in the tree violating balance property.

Case 3: **B** is right-heavy : h(**L**) = h(**M**) - 1

h(**R**) = h(**L**)

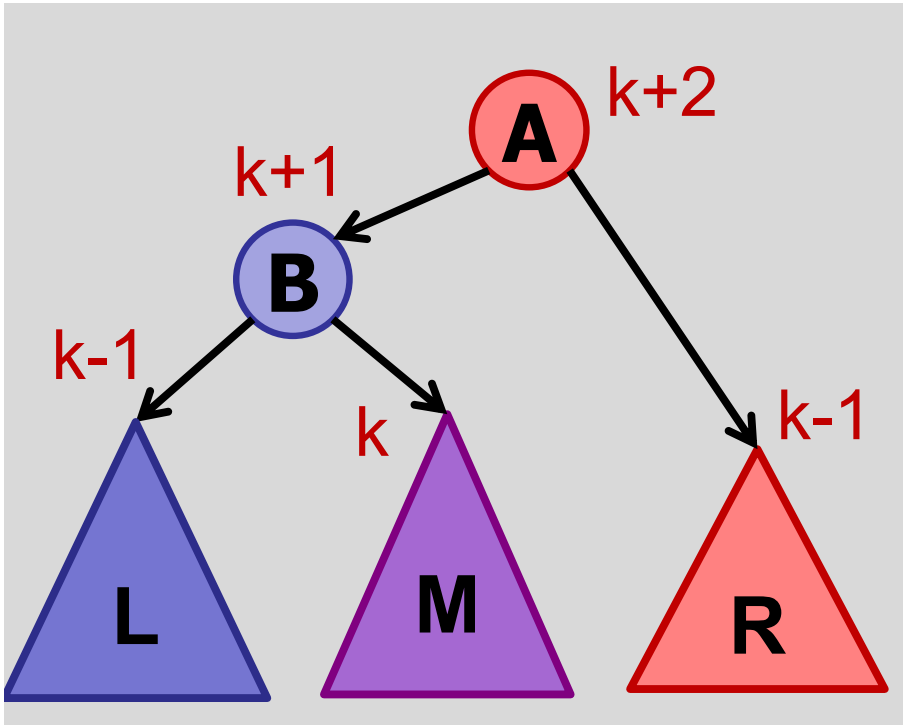
# Tree Rotations



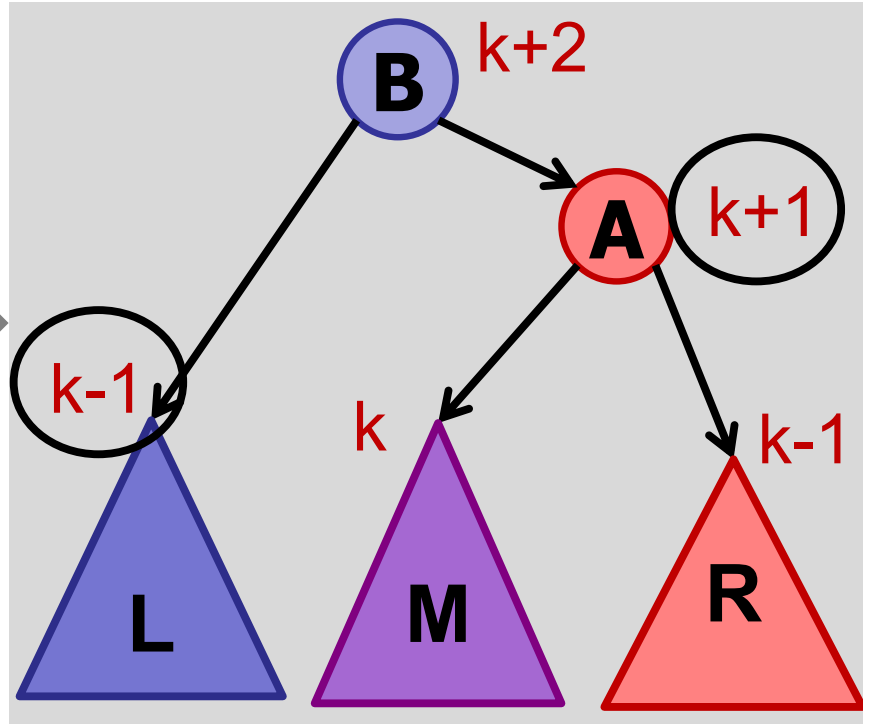

right-rotate:

Case 3: **B** is right-heavy: h(**L**) = h(**M**) – 1

h(**R**) = h(**L**)



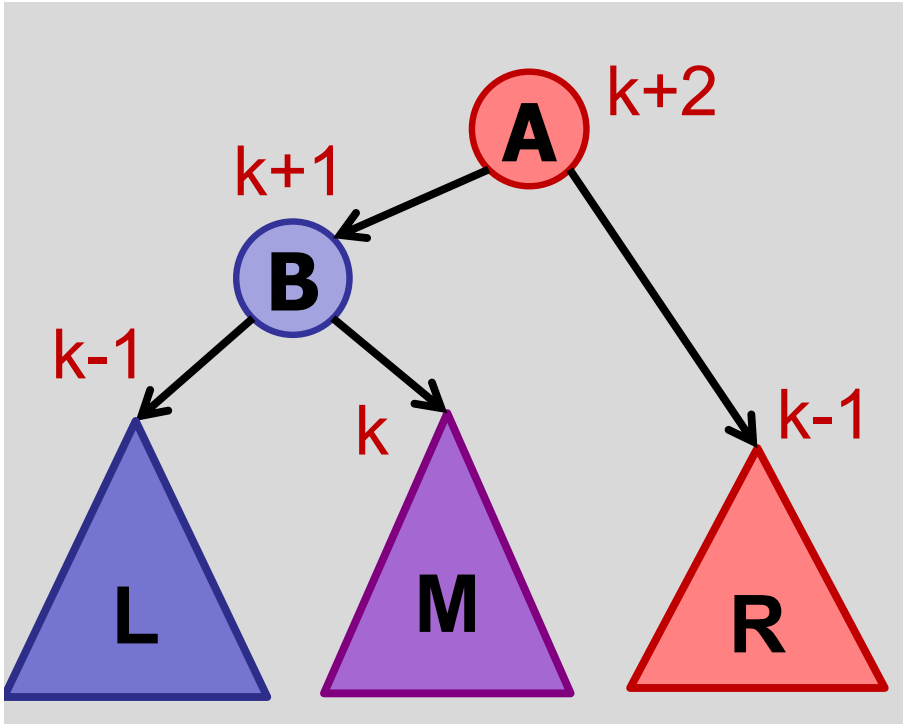

Right Rotation




Is it balanced?

1. Yes.
2. No.
3. Maybe.

# Tree Rotations



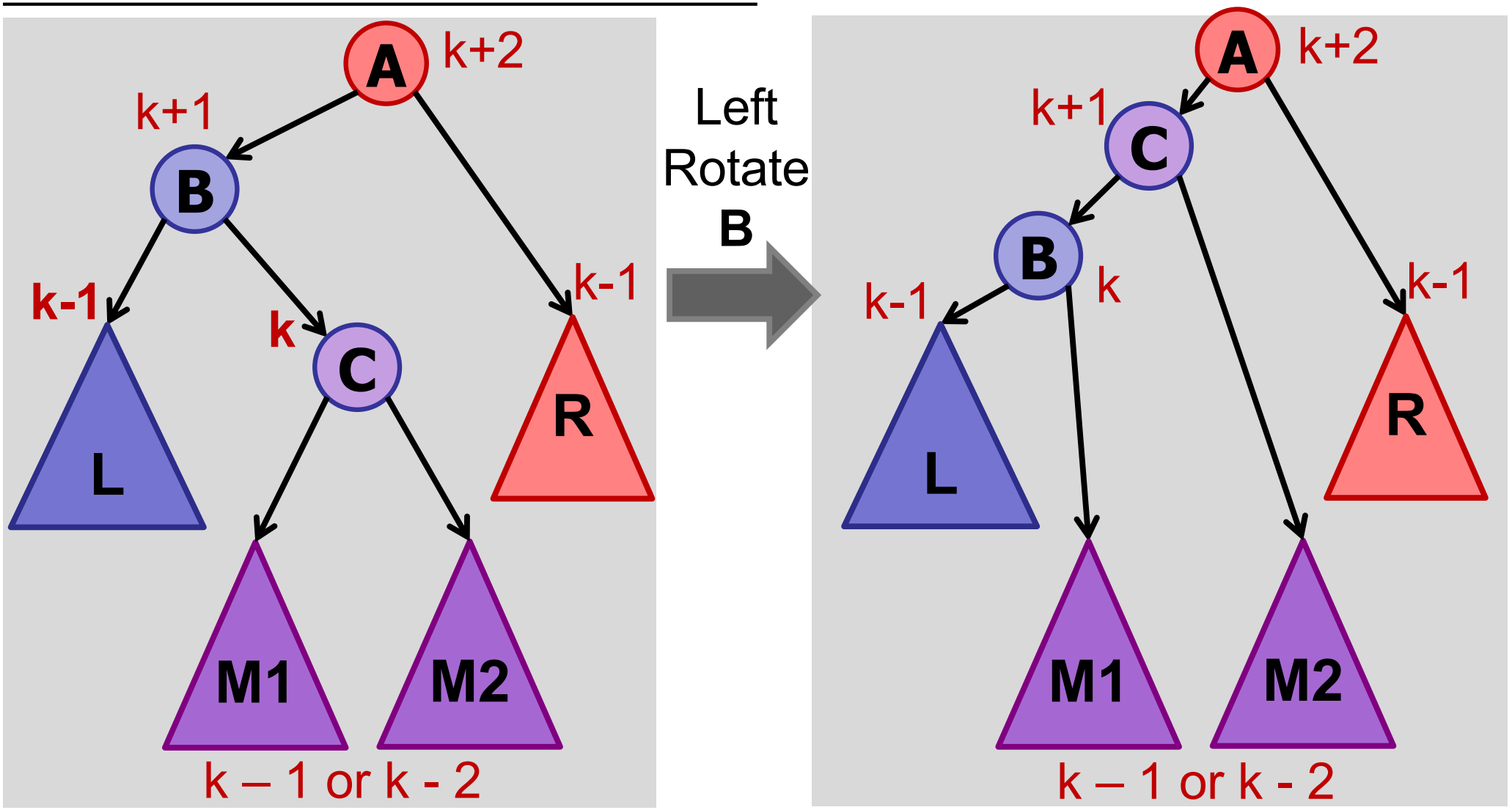

Let's do something first before we right-rotate(A)

right-rotate:

Case 3: **B** is right-heavy: h(**L**) = h(**M**) – 1

h(**R**) = h(**L**)

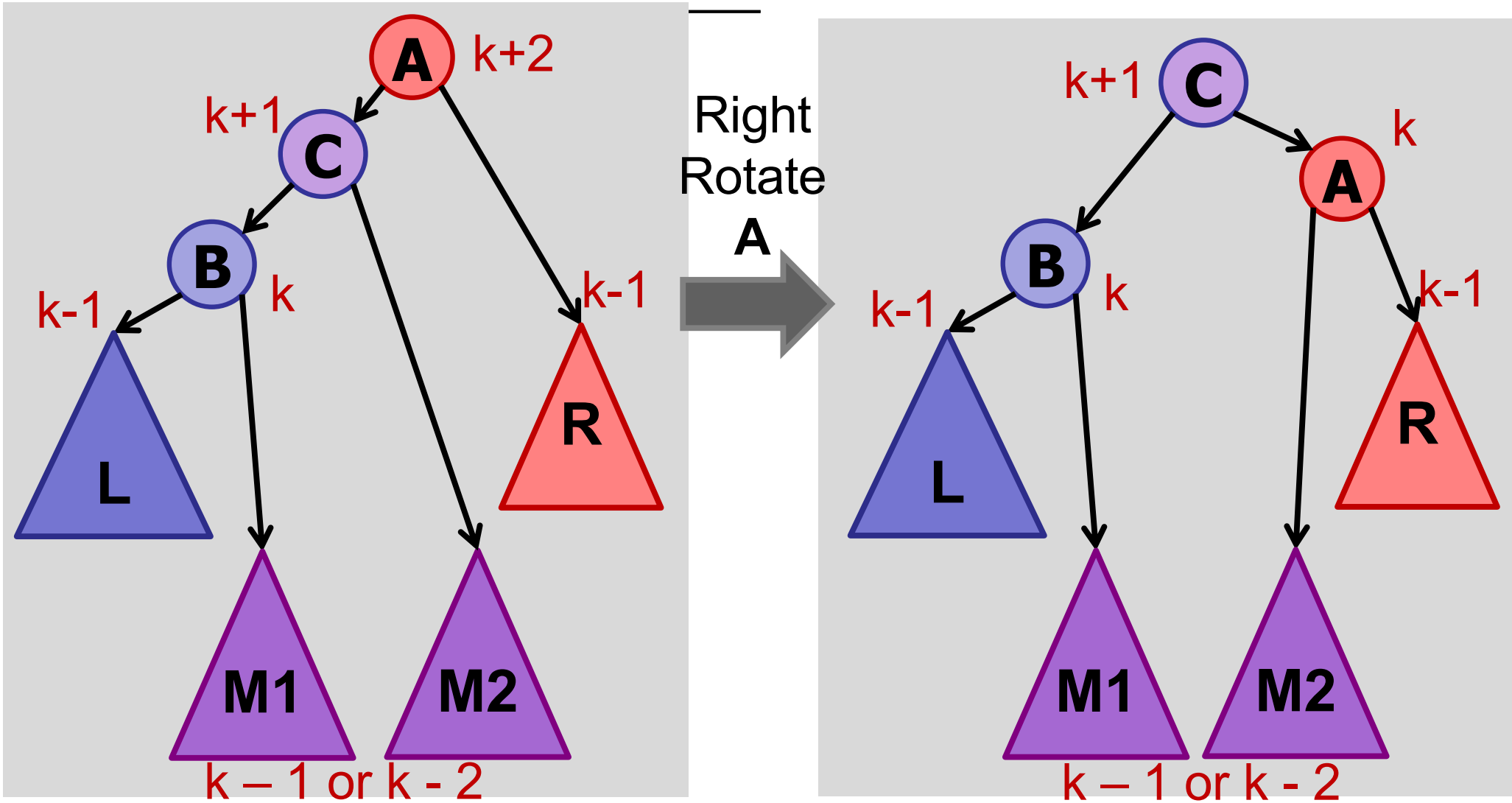
# Tree Rotations




Left-rotate B

After left-rotate B: **A** and **C** still out of balance.

# Tree Rotations




After right-rotate A: all in balance.

# Rotations

Summary:

If v is out of balance and left heavy:

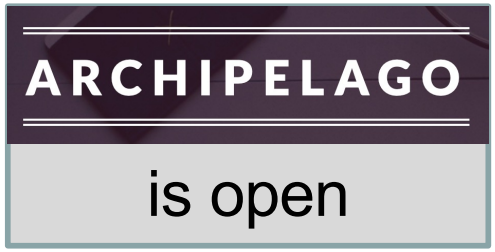
1. v.left is balanced: right-rotate(v)
2. v.left is left-heavy: right-rotate(v)
3. v.left is right-heavy: left-rotate(v.left)
   right-rotate(v)

If v is out of balance and right heavy:

Symmetric three cases….

How many rotations do you need after an insertion (in the worst case)?

1. 1
2. 2
3. 4
4. log(n)
5. 2log(n)
6. n



How many rotations do you need after an insertion (in the worst case)?

1. 1
2. ✓ 2
3. 4
4. log(n)
5. 2log(n)
6. n

Question:
Why isn't it 2log(n)?

# Insert in AVL Tree

Summary:

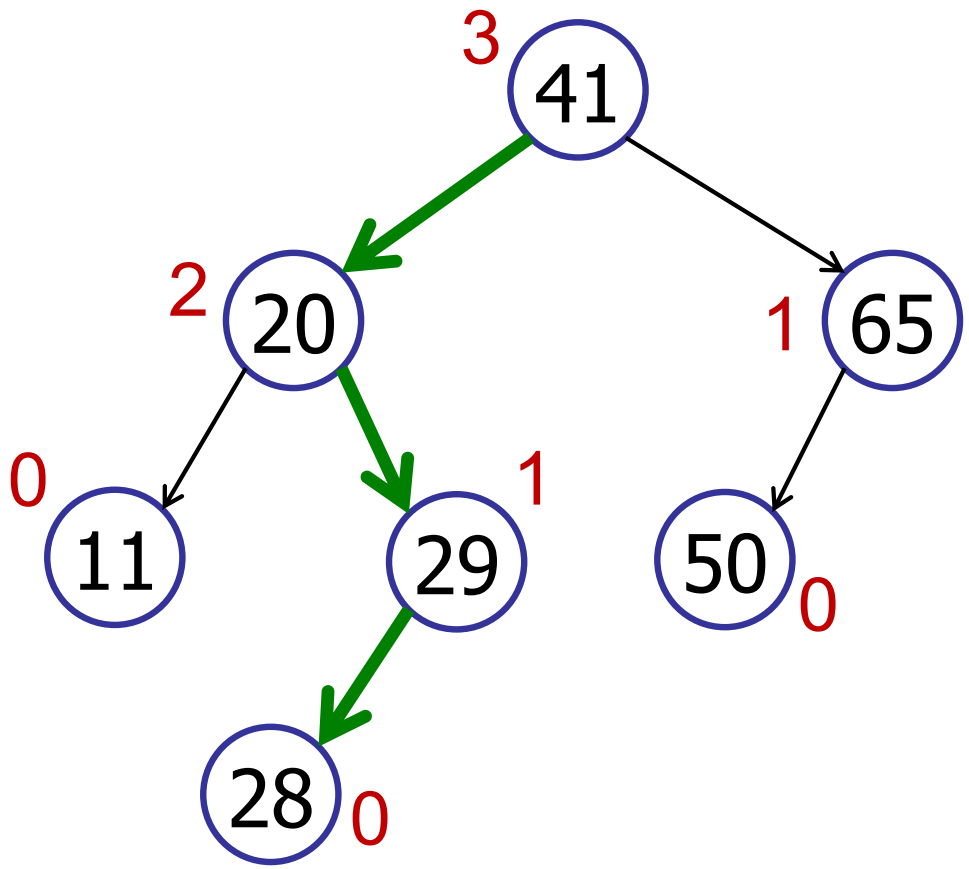
- Insert key in BST.
- Walk up tree:
  - At every step, check for balance.
  - If out-of-balance, use rotations to rebalance.

Note: only need to perform two rotations

- Why?
- In each case, reduce height of sub-tree by 1
- What about Case 1, above?

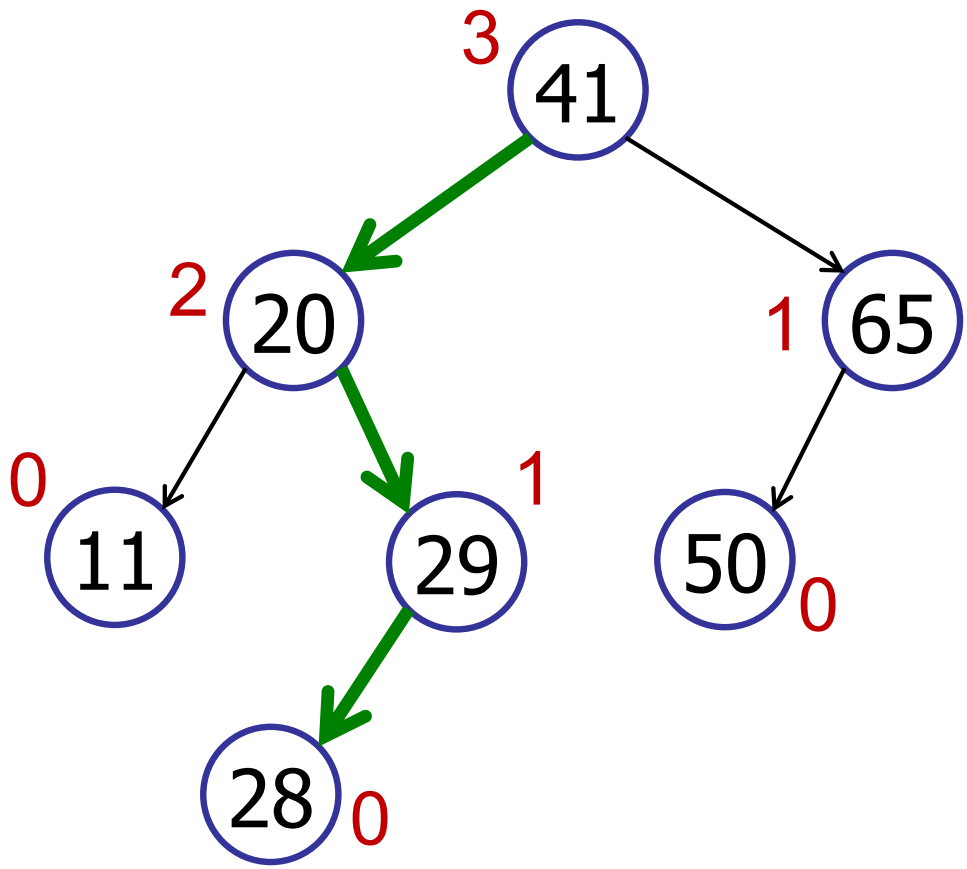
# Example

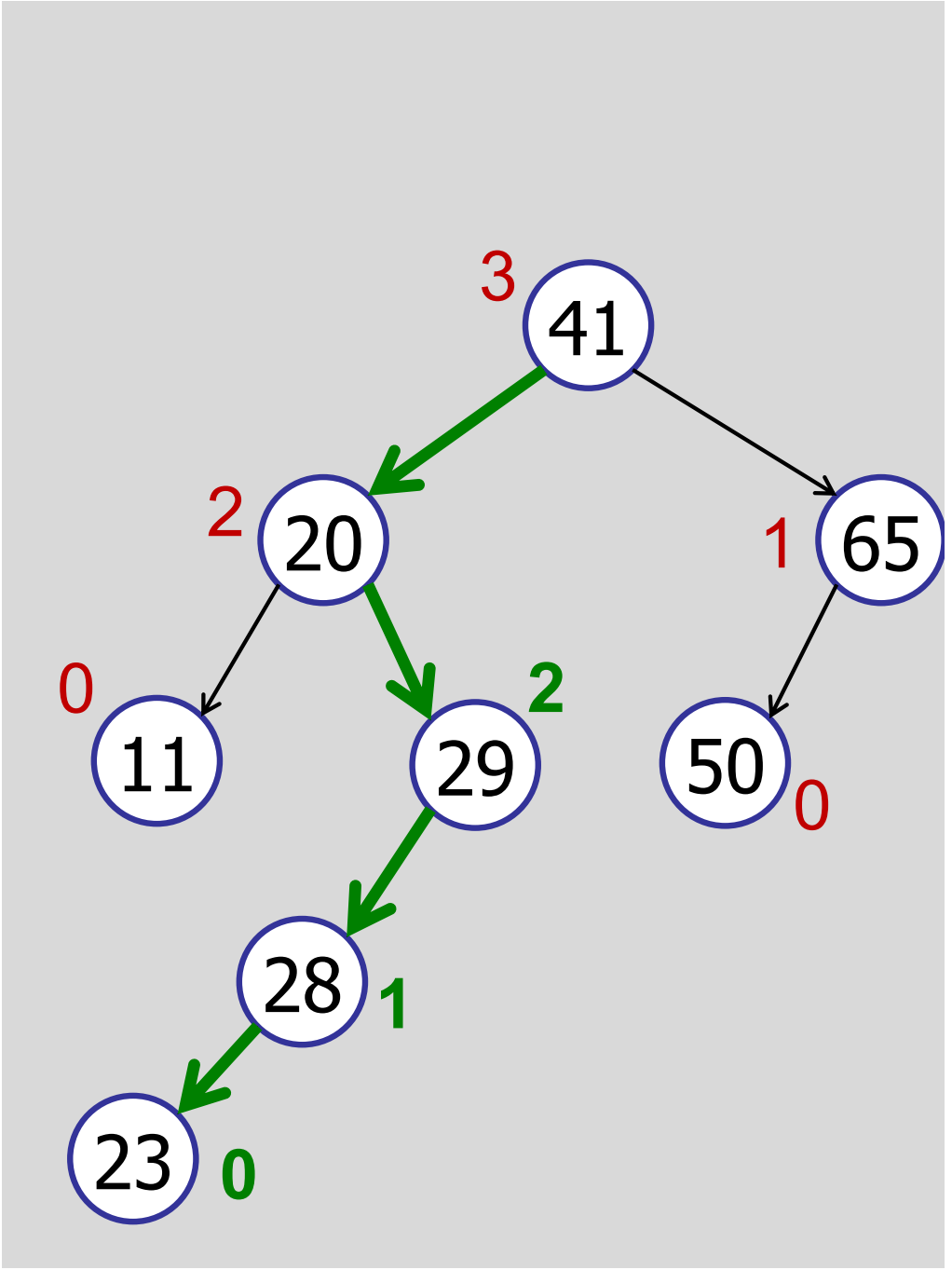
insert(23)



# Example

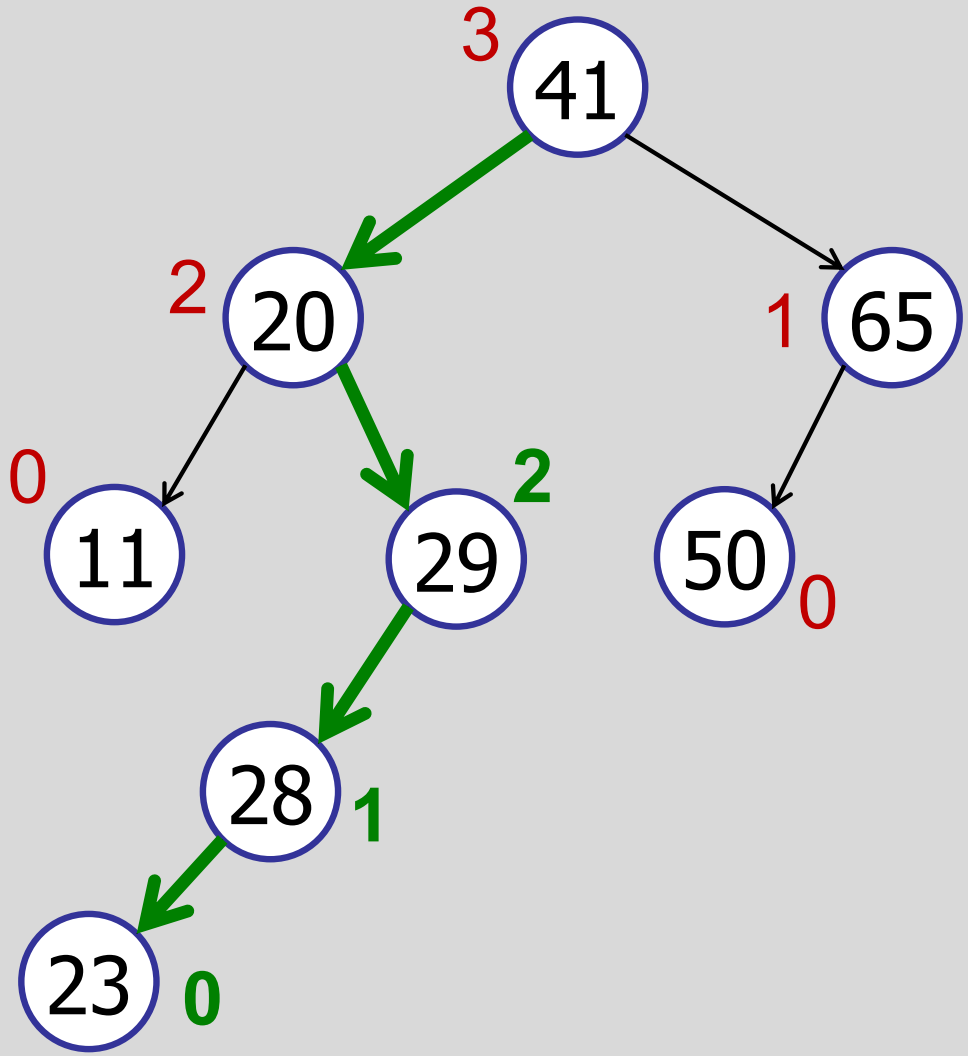
insert(23)

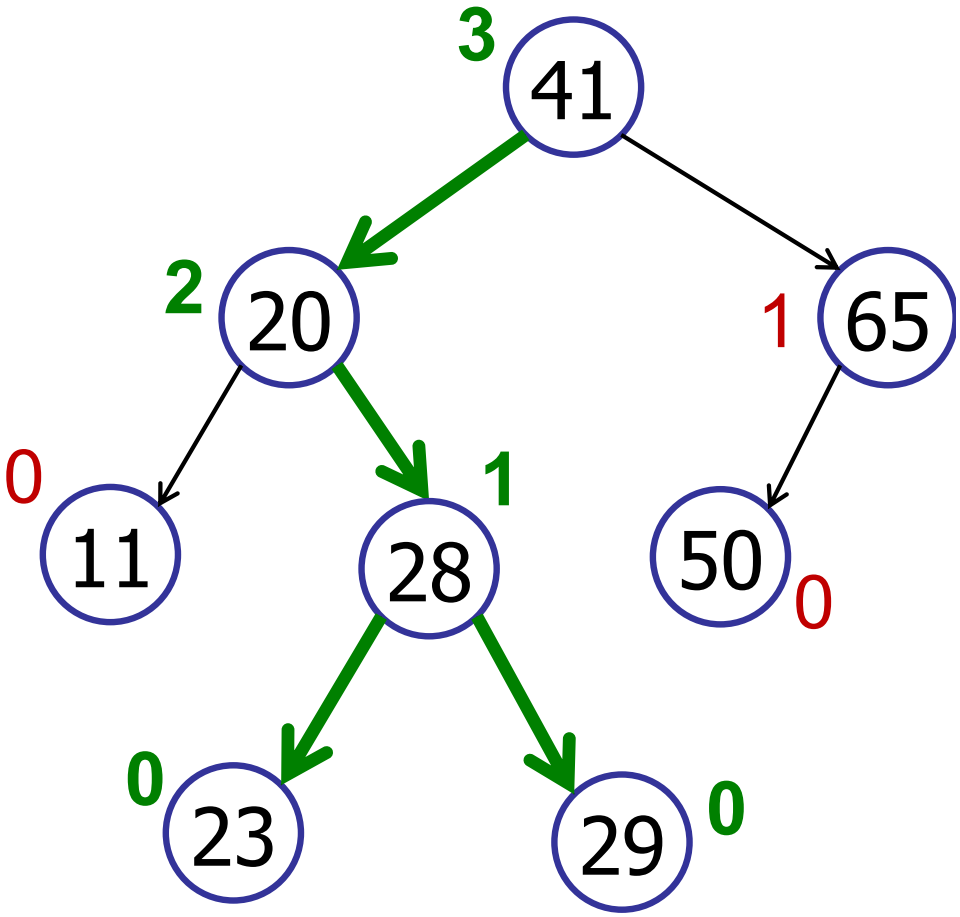


# Example

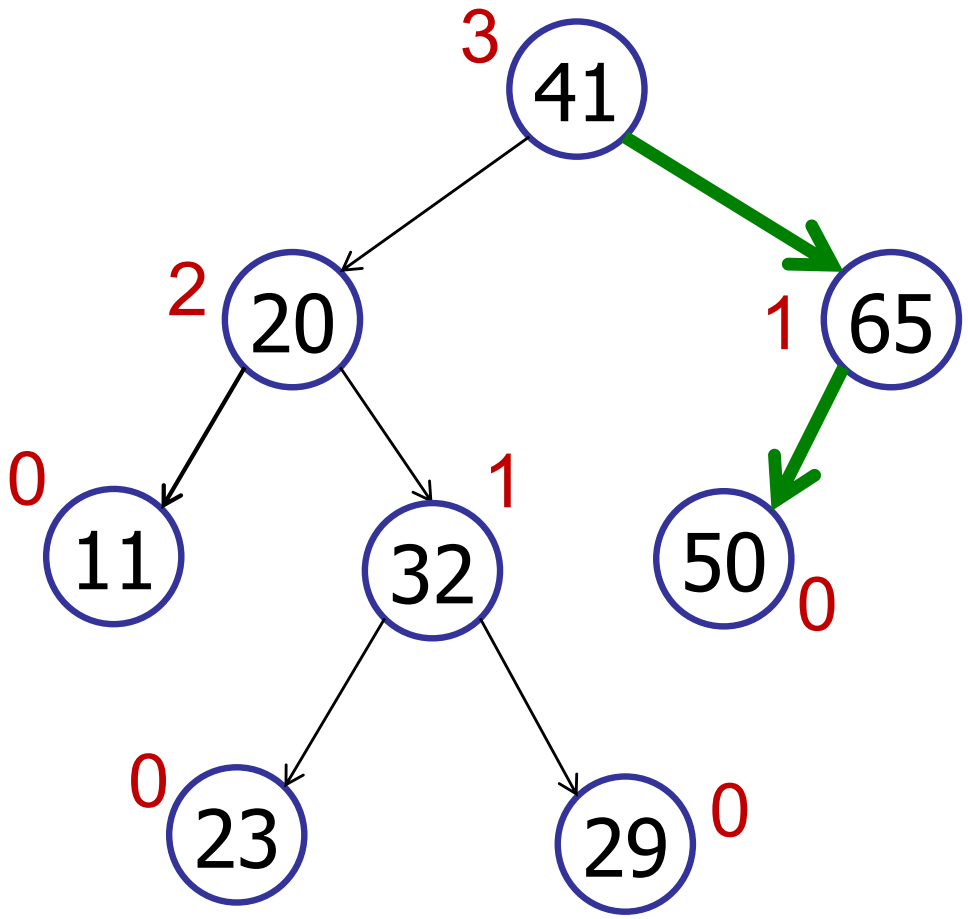
right-rotate(29)



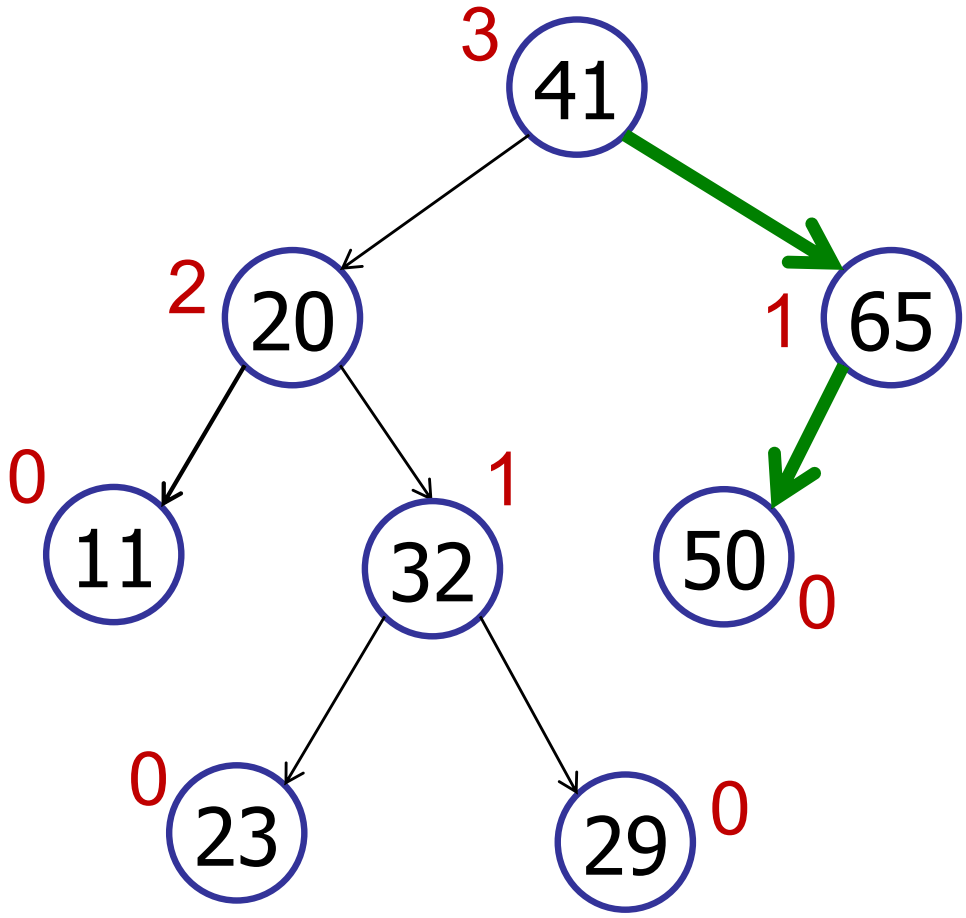
# Example

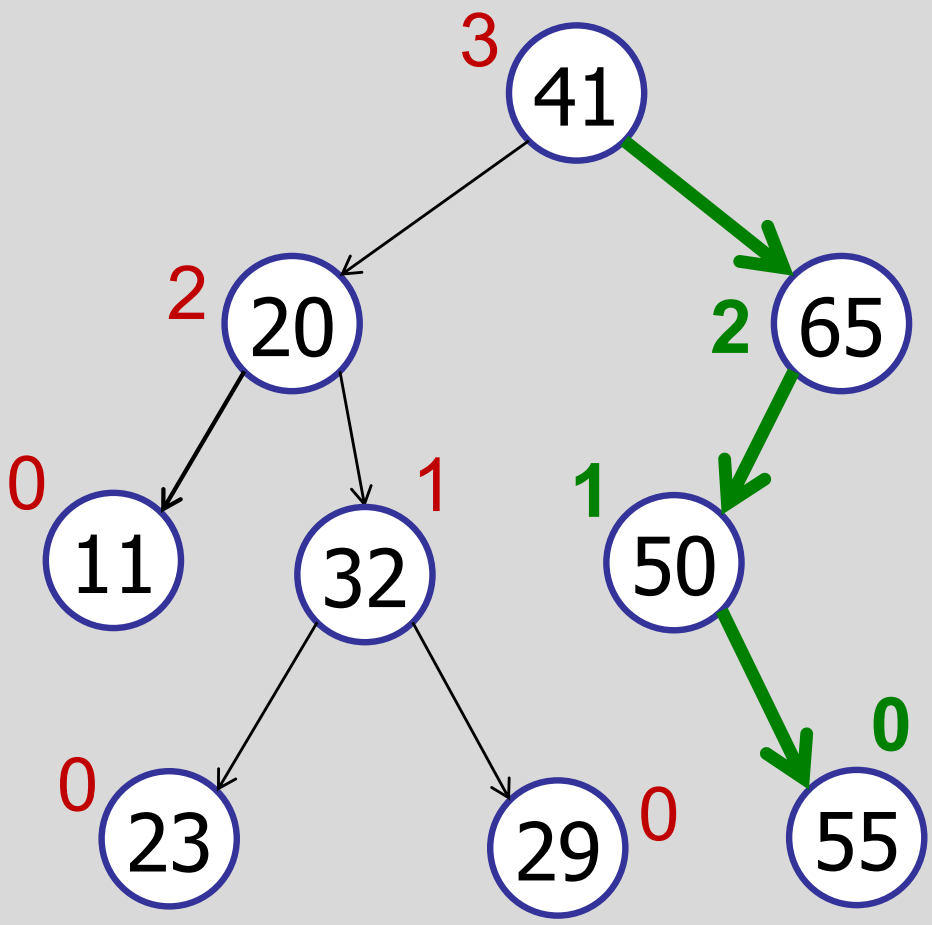
insert(55)



# Example

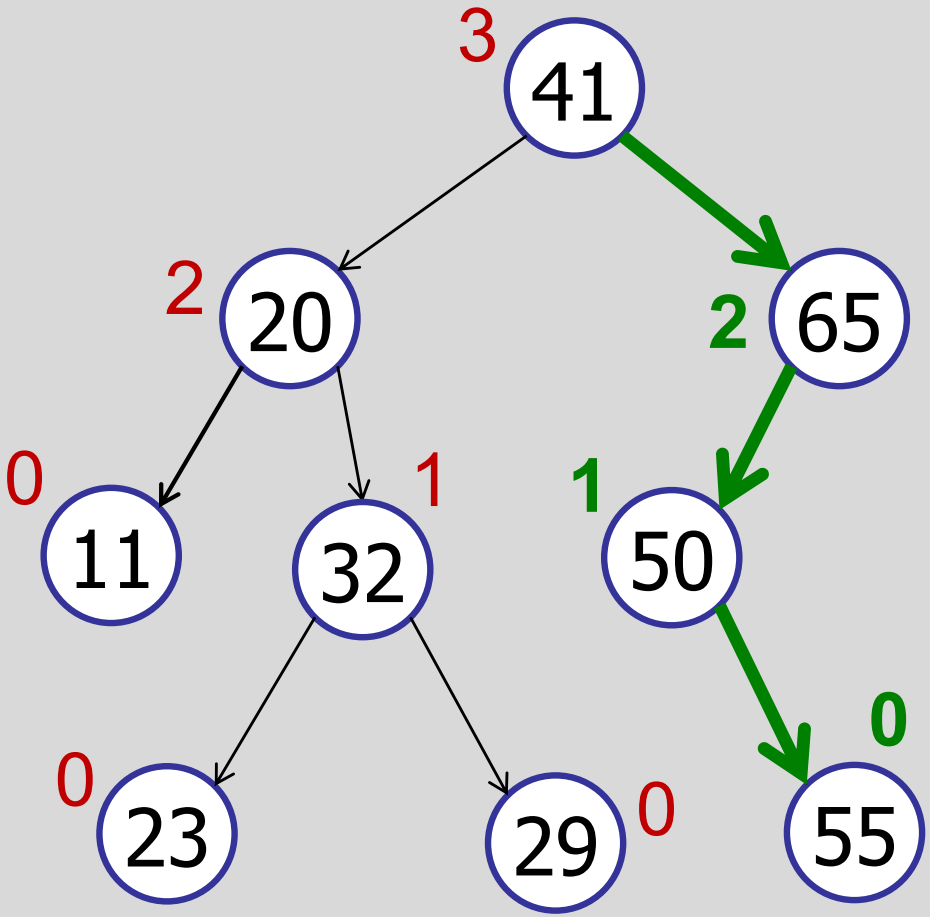
insert(55)

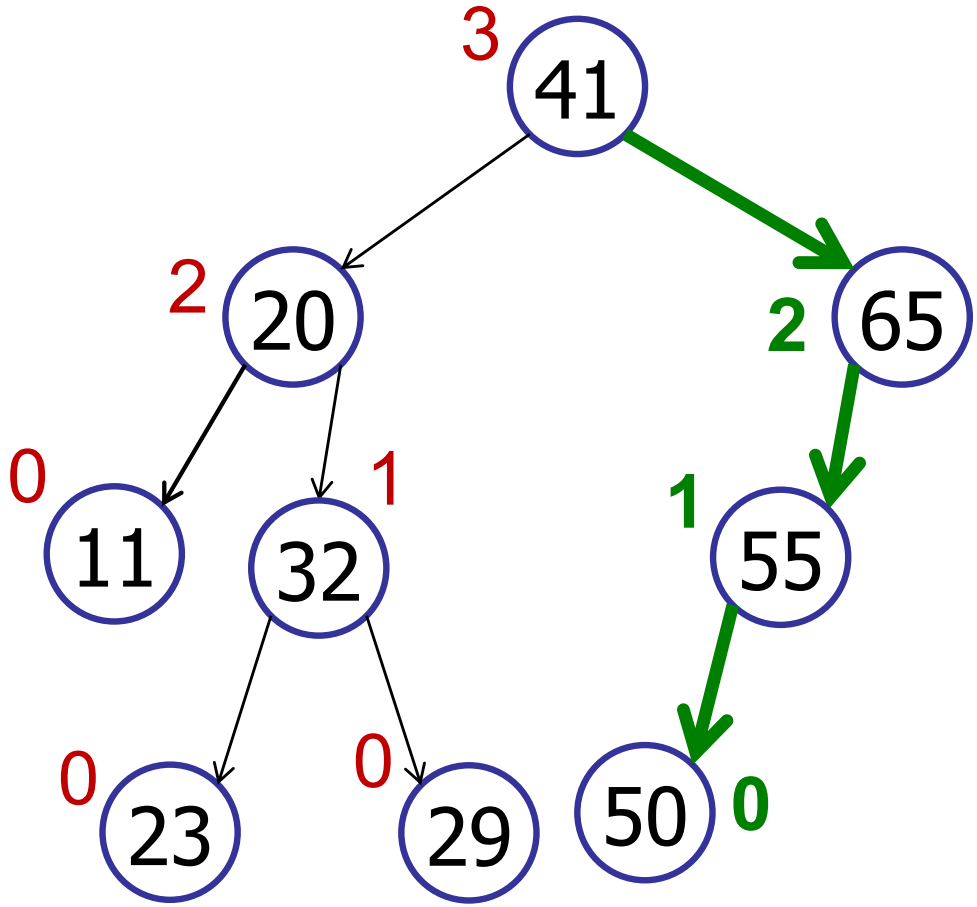


# Example

## left-rotate(50)



# Example

right-rotate(65)