3. Transport of energy: radiation

specific intensity, radiative flux

optical depth

absorption & emission

equation of transfer, source function

formal solution, limb darkening

temperature distribution

grey atmosphere, mean opacities

Energy flux conservation

No sinks and sources of energy in the atmosphere

→ all energy produced in stellar interior is transported through the atmosphere

→ at any given radius *r* in the atmosphere:

$$4\pi r^2 F(r) = const. = L$$

F is the energy flux per unit surface and per unit time. Dimensions: [erg/cm^2/sec]

The energy transport is sustained by the temperature gradient.

The steepness of this gradient is dependent on the effectiveness of the energy transport through the different atmospheric layers.

Transport of energy

Mechanisms of energy transport

a. radiation: F_{rad} (most important)

b. convection: F_{conv} (important especially in cool stars)

c. heat production: e.g. in the transition between solar cromosphere and corona

d. radial flow of matter: corona and stellar wind

e. sound waves: cromosphere and corona

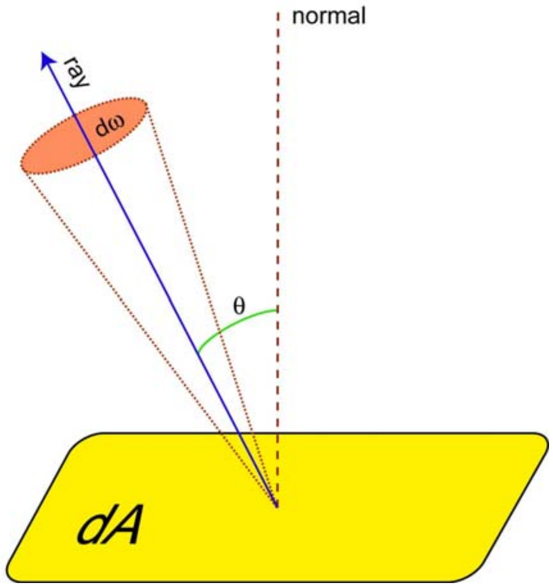
We will be mostly concerned with the first 2 mechanisms: $F(r)=F_{rad}(r) + F_{conv}(r)$. In the outer layers, we always have $F_{rad} >> F_{conv}$

The specific intensity

Measures of energy flow: **Specific Intensity** and **Flux**

The amount of energy ***dE_ν*** transported through a surface area ***dA*** is proportional to ***dt*** (length of time), ***$d\nu$*** (frequency width), ***$d\omega$*** (solid angle) and the projected unit surface area ***$\cos\theta\, dA$***.

The proportionality factor is the **specific Intensity *$I_\nu(\cos\theta)$***



$$dE_\nu = I_\nu(\cos\theta)\cos\theta\, dA\, d\omega\, d\nu\, dt$$

$$([\mathrm{I}_\nu]\mathrm{: erg\, cm^{-2}\, sr^{-1}\, Hz^{-1}\, s^{-1}})$$

$$I_\lambda = \tfrac{c}{\lambda^2} I_\nu$$

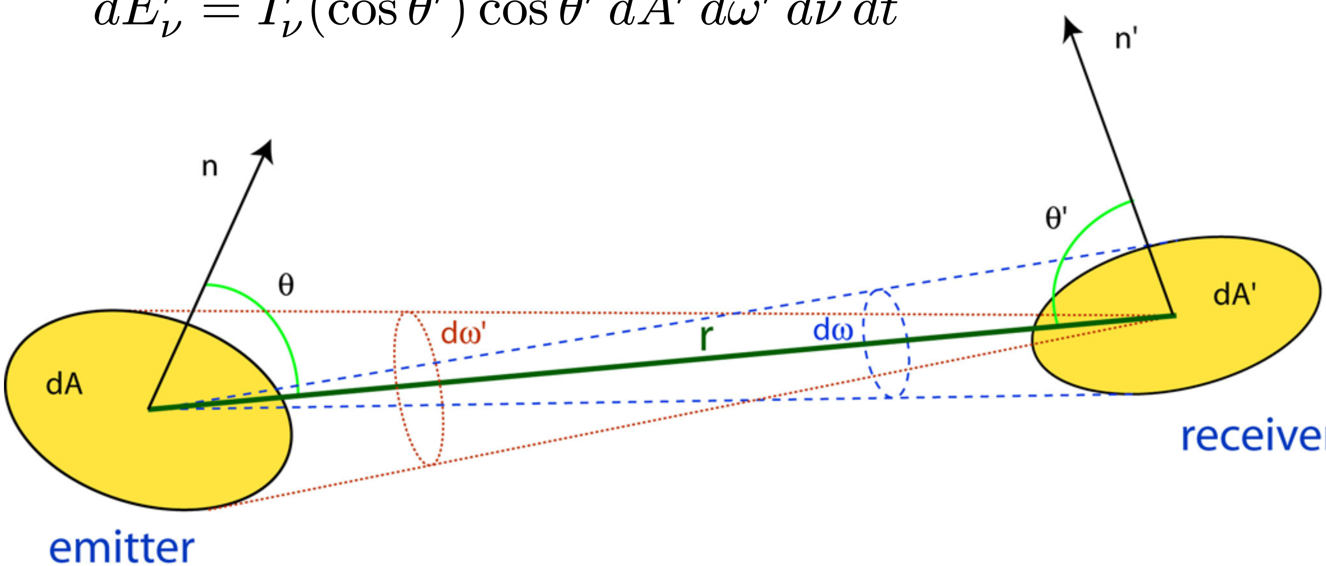
$$(\text{from } I_\lambda d\lambda = I_\nu d\nu \text{ and } \nu = c/\lambda)$$

Intensity depends on location in space, direction and frequency

Invariance of the specific intensity

The area element dA emits radiation towards dA'. In the absence of any matter between emitter and receiver (no absorption and emission on the light paths between the surface elements) the amount of energy emitted and received through each surface elements is:

$$dE_\nu = I_\nu(\cos\theta)\cos\theta\, dA\, d\omega\, d\nu\, dt$$
$$dE'_\nu = I'_\nu(\cos\theta')\cos\theta'\, dA'\, d\omega'\, d\nu\, dt$$



Invariance of the specific intensity

energy is conserved: $dE_\nu = dE'_\nu$ and

$$d\omega = \frac{\text{projected area}}{\text{distance}^2} = \frac{dA' \cos\theta'}{r^2}$$

$$\mathrm{d}\omega' = \frac{dA \cos\theta}{r^2}$$

and

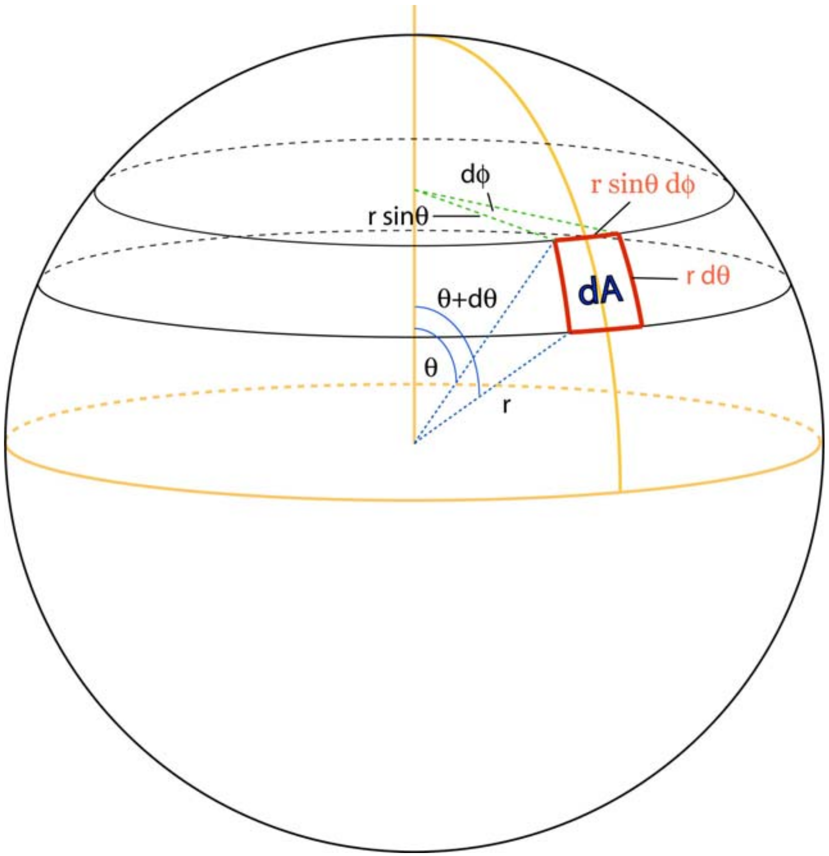
$$dE_\nu = I_\nu(\cos\theta)\cos\theta\, dA\, d\omega\, d\nu\, dt$$
$$dE'_\nu = I'_\nu(\cos\theta')\cos\theta'\, dA'\, d\omega'\, d\nu\, dt$$

$$\Longrightarrow \quad I_\nu = I'_\nu$$

Specific intensity is constant along rays - as long as there is no absorption and emission of matter between emitter and receiver

In TE: $I_\nu = B_\nu$

Spherical coordinate system and solid angle dω



$$\text{solid angle : } d\omega = \frac{dA}{r^2}$$

$$\text{Total solid angle} = \frac{4\pi r^2}{r^2} = 4\pi$$

$$dA = (r\,d\theta)(r\,\sin\theta\,d\phi)$$

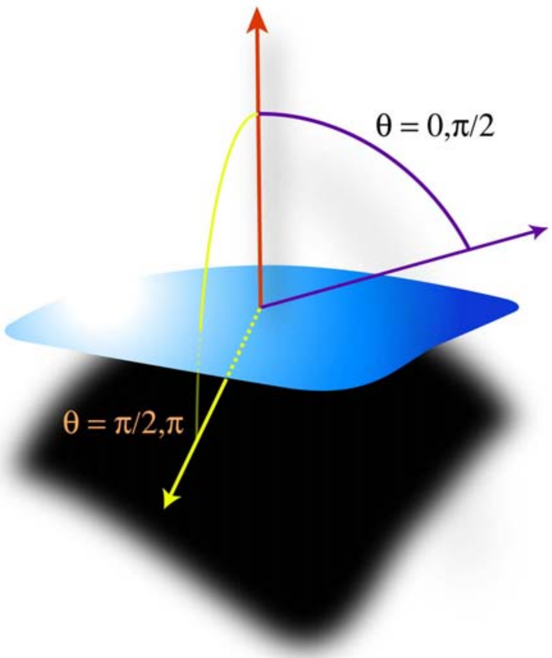
$$\rightarrow d\omega = \sin\theta\,d\theta\,d\phi$$

$$\text{define } \mu = \cos\theta$$

$$d\mu = -\sin\theta\,d\theta$$

$$d\omega = \sin\theta\,d\theta\,d\phi = -d\mu\,d\phi$$

Radiative flux



How much energy flows through surface element dA?

$dE_\nu \sim I_\nu \cos\theta \, d\omega$

→ integrate over the whole solid angle ($\Omega = 4\pi$):

$$\underbrace{\pi F_\nu}_{\text{“astrophysical flux”}} = \int_{4\pi} I_\nu(\cos\theta)\cos\theta \, d\omega = \int_0^{2\pi}\int_0^{\pi} I_\nu(\cos\theta)\cos\theta\sin\theta d\theta d\phi$$

F_ν is the **monochromatic radiative flux.**
The factor π in the definition is historical.

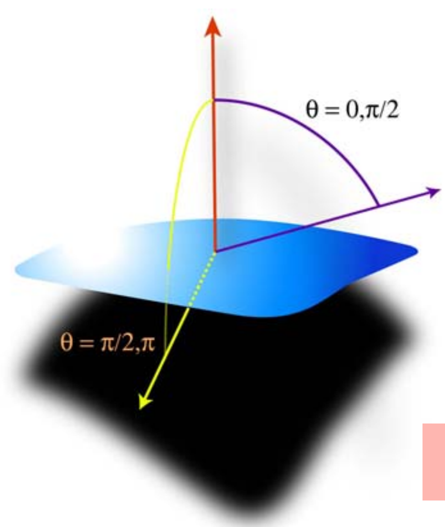
F_ν can also be interpreted as the net rate of energy flow through a surface element.

Radiative flux

The monochromatic **radiative flux** at frequency ν gives the net rate of energy flow through a surface element.

$dE_\nu \sim I_\nu \cos\theta \, d\omega$ → integrate over the whole solid angle ($\Omega = 4\pi$):

$$\underbrace{\pi F_\nu}_{\text{“astrophysical flux”}} = \int_{4\pi} I_\nu(\cos\theta)\cos\theta \, d\omega = \int_0^{2\pi}\int_0^{\pi} I_\nu(\cos\theta)\cos\theta\sin\theta d\theta d\phi$$



We distinguish between the outward direction ($0 < \theta < \pi/2$) and the inward direction ($\pi/2 < \theta < \pi$), so that the net flux is:

$$\pi F_\nu = \pi F_\nu^+ - \pi F_\nu^- =$$

$$= \int_0^{2\pi}\int_0^{\pi/2} I_\nu(\cos\theta)\cos\theta\sin\theta d\theta d\phi + \int_0^{2\pi}\int_{\pi/2}^{\pi} I_\nu(\cos\theta)\cos\theta\sin\theta d\theta d\phi$$

Note: for $\pi/2 < \theta < \pi \rightarrow \cos\theta < 0$ → second term negative !!

Total radiative flux

Integral over frequencies ν →

$$\int_0^\infty \pi F_\nu d\nu = \mathcal{F}_{rad}$$

F_{rad} is the **total radiative flux.**

It is the total net amount of energy going through the surface element per unit time and unit surface.

Stellar luminosity

At the outer boundary of atmosphere ($r = R_o$) there is no incident radiation

→ Integral interval over θ reduces from [0,π] to [0,π/2].

$$\pi F_\nu(R_o) = \pi F_\nu^+(R_o) = \int_0^{2\pi}\int_0^{\pi/2} I_\nu(\cos\theta)\cos\theta\sin\theta d\theta d\phi$$

This is the monochromatic energy that each surface element of the star radiates in all directions

If we multiply by the total stellar surface $4\pi R_0^2$

→ **monochromatic stellar luminosity** at frequency ν

$$4\pi R_o^2 \cdot \pi F_\nu(R_o) = L_\nu$$

and integrating over ν

→ **total stellar luminosity**

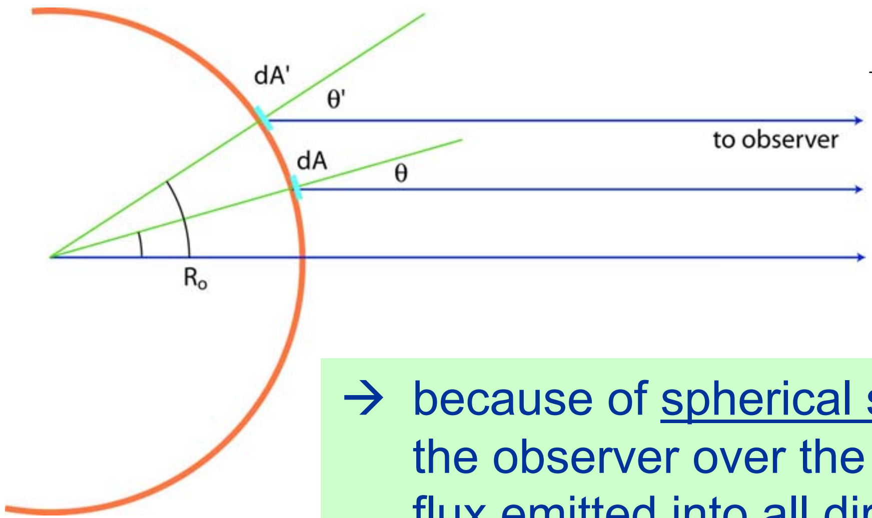
$$4\pi R_o^2 \cdot \int_0^\infty \pi F_\nu^+(R_o) d\nu = L \ \ (Luminosity)$$

Observed flux

What radiative flux is measured by an observer at distance d?
→ integrate specific intensity I_ν towards observer over all surface elements
note that only half sphere contributes

$$E_\nu = \int_{1/2\,sphere} dE = \Delta\omega\,\Delta\nu\,\Delta t \int_{1/2\,sphere} I_\nu(\cos\theta)\cos\theta\, dA$$

$$\text{in spherical symmetry: } dA = R_o^2 \sin\theta\, d\theta\, d\phi$$



$$\rightarrow E_\nu = \Delta\omega\,\Delta\nu\,\Delta t\, R_o^2 \underbrace{\int_o^{2\pi}\int_0^{\pi/2} I_\nu(\cos\theta)\cos\theta\sin\theta\, d\theta\, d\phi}_{\pi F_\nu^+}$$

→ because of <u>spherical symmetry</u> the integral of intensity towards the observer over the stellar surface is proportional to πF_ν^+, the flux emitted into all directions by one surface element !!

Observed flux

Solid angle of telescope at distance <u>*d*</u>:

$$\Delta\omega = \Delta A / d^2$$

$$+$$

$$E_\nu = \Delta\omega\, \Delta\nu\, \Delta t\, R_o^2\, \pi F_\nu^+(R_o)$$

$$\mathcal{F}_\nu^{obs} = \frac{\text{radiative energy}}{\text{area} \cdot \text{frequency} \cdot \text{time}} = \frac{R_o^2}{d^2}\, \pi F_\nu^+(R_o)$$

flux received = flux emitted x $(R/r)^2$

unlike I_ν, $\mathcal{F}_\nu$ decreases with increasing distance

This, and not I_ν, is the quantity generally measured for stars. For the Sun, whose disk is resolved, we can also measure I_ν (the variation of I_ν over the solar disk is called the <u>limb darkening</u>)

$\int_0^\infty \mathcal{F}_{\nu,\odot}^{obs}\, d\nu = 1.36\ KW/m^2$

Mean intensity, energy density & radiation pressure

Integrating over the solid angle and dividing by 4π:

$$J_\nu = \frac{1}{4\pi}\int_{4\pi} I_\nu \, d\omega$$

mean intensity

$$u_\nu = \frac{\text{radiation energy}}{\text{volume}} = \frac{1}{c}\int_{4\pi} I_\nu \, d\omega = \frac{4\pi}{c} J_\nu$$

energy density

$$p_\nu = \frac{1}{c}\int_{4\pi} I_\nu \cos^2\theta \, d\omega$$

radiation pressure
(important in hot stars)

$$\text{pressure} = \frac{\text{force}}{\text{area}} = \frac{d\ \text{momentum}(= \mathrm{E/c})}{dt}\frac{1}{\text{area}}$$

Moments of the specific intensity

$$J_\nu = \frac{1}{4\pi}\int I_\nu \, d\omega = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{-1}^{1} I_\nu \, d\mu = \frac{1}{2}\int_{-1}^{1} I_\nu \, d\mu$$

0th moment

for azimuthal symmetry

$$H_\nu = \frac{1}{4\pi}\int I_\nu \cos\theta \, d\omega = \frac{1}{2}\int_{-1}^{1} I_\nu \, \mu \, d\mu = \frac{F_\nu}{4}$$

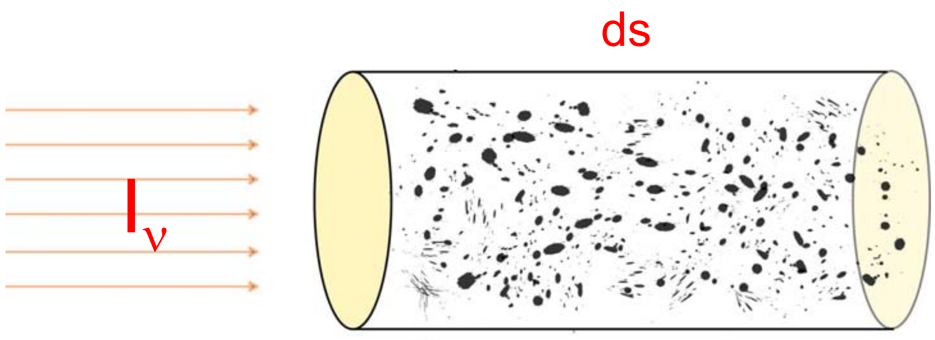
1st moment
(Eddington flux)

$$K_\nu = \frac{1}{4\pi}\int I_\nu \cos^2\theta \, d\omega = \frac{1}{2}\int_{-1}^{1} I_\nu \, \mu^2 \, d\mu = \frac{c}{4\pi} p_\nu$$

2nd moment

Interactions between photons and matter

absorption of radiation



$$dI_\nu = -\kappa_\nu I_\nu ds$$

κ_ν : absorption coefficient

$$[\kappa_\nu] = \mathrm{cm}^{-1}$$

microscopical view: κ_ν=n σ_ν

loss of intensity in the beam (true absorption/scattering)

Over a distance *s:*

$$I_\nu^o \xrightarrow{\quad s \quad} I_\nu(s)$$

$$I_\nu(s) = I_\nu^o \, e^{-\int_0^s \kappa_\nu \, ds}$$

$$\tau_\nu := \int_0^s \kappa_\nu \, ds$$

optical depth

(dimensionless)

or: $d\tau_\nu = \kappa_\nu$ ds

Convention: τ_ν = 0 at the outer edge of the atmosphere, increasing inwards

optical depth

$$I_\nu(s) = I_\nu^o \, e^{-\tau_\nu}$$

The optical thickness of a layer determines the fraction of the intensity passing through the layer

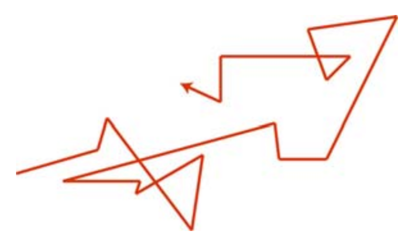
$$\text{if } \tau_\nu = 1 \rightarrow I_\nu = \frac{I_\nu^o}{e} \simeq 0.37 \, I_\nu^o$$

We can see through atmosphere until τ_ν ~ 1

optically **thick** (thin) medium: τ_ν **>** (<) 1

The quantity $\tau_\nu = 1$ has a geometrical interpretation in terms of ***mean free path of photons $\bar{s}$:***

$$\tau_\nu = 1 = \int_o^{\bar{s}} \kappa_\nu \, ds$$



photons travel on average
for a length $\bar{s}$
before absorption

photon mean free path

What is the average distance over which photons travel?

expectation value $$< \tau_\nu >= \int_0^\infty \tau_\nu \underbrace{p(\tau_\nu)\, d\tau_\nu}$$

probability of absorption in interval [τ_ν,τ_ν+dτ_ν]

= probability of non-absorption between 0 and τ_ν and absorption in dτ_ν

- probability that photon is absorbed: $p(0,\tau_\nu) = \dfrac{\Delta I(\tau)}{I_o} = \dfrac{I_o - I(\tau_\nu)}{I_o} = 1 - \dfrac{I(\tau_\nu)}{I_o}$

- probability that photon is not absorbed: $1 - p(0,\tau_\nu) = \dfrac{I(\tau_\nu)}{I_o} = e^{-\tau_\nu}$

- probability that photon is absorbed in $[\tau_\nu, \tau_\nu + d\tau_\nu]$: $p(\tau_\nu, \tau_\nu + d\tau_\nu) = \frac{dI_\nu}{I(\tau_\nu)} = d\tau_\nu$

total probability: $e^{-\tau_\nu}\, d\tau_\nu$

photon mean free path

$$\Rightarrow \quad < \tau_\nu >= \int_0^\infty \tau_\nu \, p(\tau_\nu) \, d\tau_\nu = \int_0^\infty \tau_\nu \, e^{-\tau_\nu} \, d\tau_\nu = 1$$

$$\int xe^{-x} \, dx = -(1+x) \, e^{-x}$$

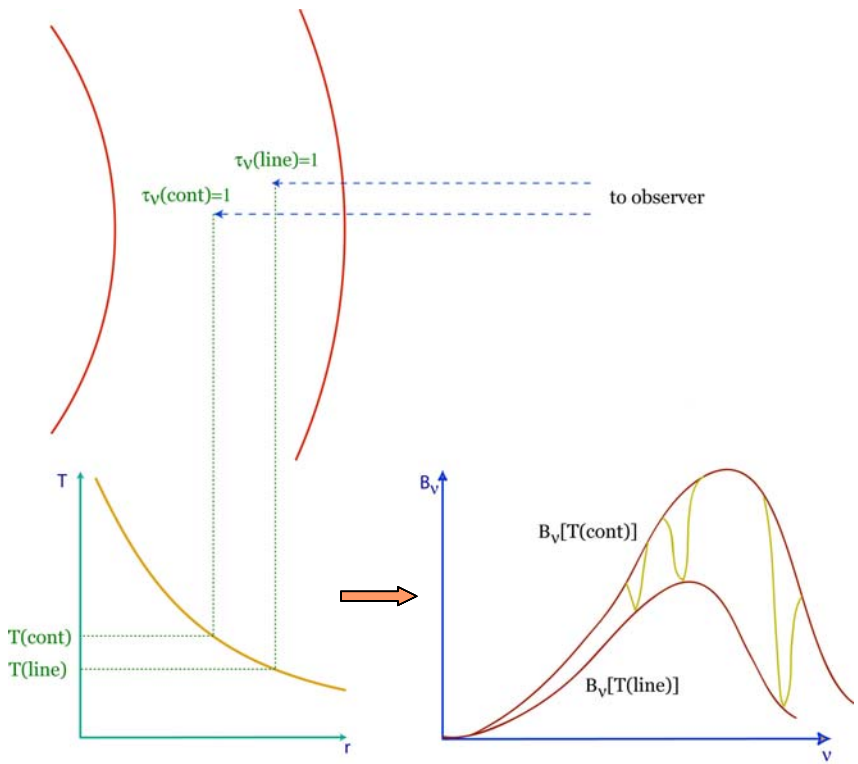
$$\Downarrow$$

mean free path corresponds to $<\tau_\nu>=1$

$$\text{if } \kappa_\nu(s) = \text{const}: \quad \Delta\tau_\nu = \kappa_\nu \, \Delta s \rightarrow \Delta s = \bar{s} = \frac{1}{\kappa_\nu}$$

(homogeneous material)

Principle of line formation



observer sees through the atmospheric layers up to $\tau_\nu \approx 1$

In the continuum κ_ν is smaller than in the line → see deeper into the atmosphere

T(cont) > T(line)

radiative acceleration

In the absorption process photons release momentum E/c to the atoms, and the corresponding force is:

$$\text{force} = \mathrm{df}_{\text{phot}} = \frac{\text{momentum}(=\mathrm{E/c})}{dt}$$

The infinitesimal energy absorbed is:

$$dE_\nu^{\text{abs}} = dI_\nu \, \cos\theta \, dA \, d\omega \, dt \, d\nu = \kappa_\nu \, I_\nu \, \, cos\theta \, dA \, d\omega \, dt \, d\nu \, ds$$

The total energy absorbed is (assuming that κ_ν does not depend on ω):

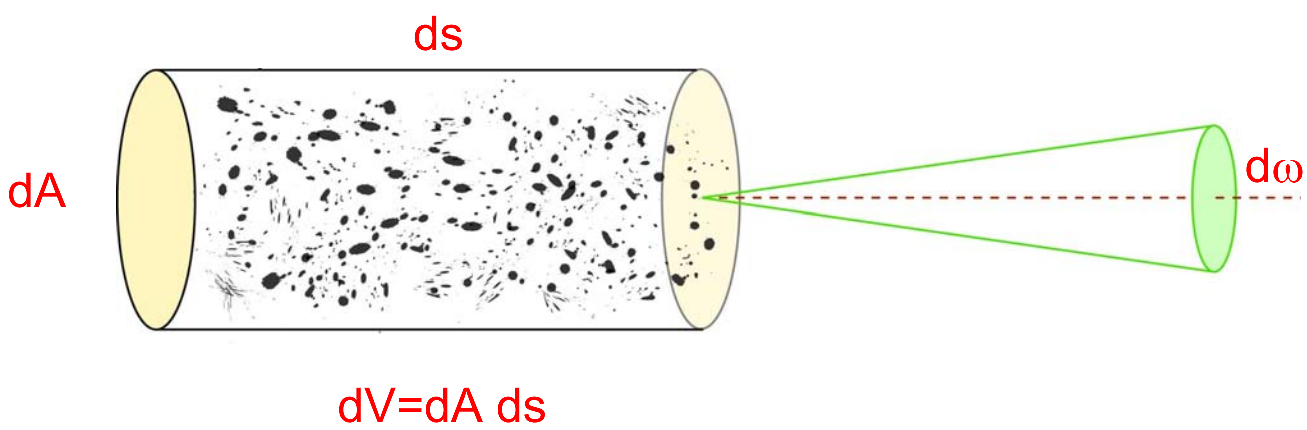
$$E^{\text{abs}} = \int_0^\infty \kappa_\nu \underbrace{\int_{4\pi} I_\nu \, \cos\theta \, d\omega}_{\pi F_\nu} \, d\nu \quad dA \, dt \, ds = \pi \int_0^\infty \kappa_\nu \, F_\nu \, d\nu \quad dA \, dt \, ds$$

radiative acceleration

$$\Rightarrow \quad df_{\mathrm{phot}} = \frac{\pi}{c}\frac{\int\limits_0^\infty \kappa_\nu\, F_\nu\, d\nu}{dt}\quad dA\, dt\, ds = g_{\mathrm{rad}}\, dm \qquad (dm = \rho\, dA\, ds)$$

$$g_{\mathrm{rad}} = \frac{\pi}{c\rho}\int\limits_0^\infty \kappa_\nu\, F_\nu\, d\nu$$

emission of radiation



energy added by emission processes within dV

$$dE_\nu^{\rm em} = \epsilon_\nu \, dV \, d\omega \, d\nu \, dt$$

ϵ_ν : emission coefficient

$$[\epsilon_\nu] = {\rm erg\,cm^{-3}\,sr^{-1}\,Hz^{-1}\,s^{-1}}$$

The equation of radiative tranfer

If we combine absorption and emission together:

$$dE_\nu^{\rm abs} = dI_\nu^{\rm abs}\, dA\, \cos\theta\, d\omega\, d\nu\, dt = -\kappa_\nu\, I_\nu\, dA\, \cos\theta\, d\omega\, dt\, d\nu\, ds$$

$$dE_\nu^{\rm em} = dI_\nu^{\rm em}\, dA\, \cos\theta\, d\omega\, d\nu\, dt = \epsilon_\nu\, dA\, \cos\theta\, d\omega\, d\nu\, dt\, ds$$

$$dE_\nu^{\rm abs} + dE_\nu^{\rm em} = (dI_\nu^{abs} + dI_\nu^{\rm em})\, dA\, \cos\theta\, d\omega\, d\nu\, dt = (-\kappa_\nu\, I_\nu + \epsilon_\nu)\, dA\, \cos\theta\, d\omega\, d\nu\, dt\, ds$$

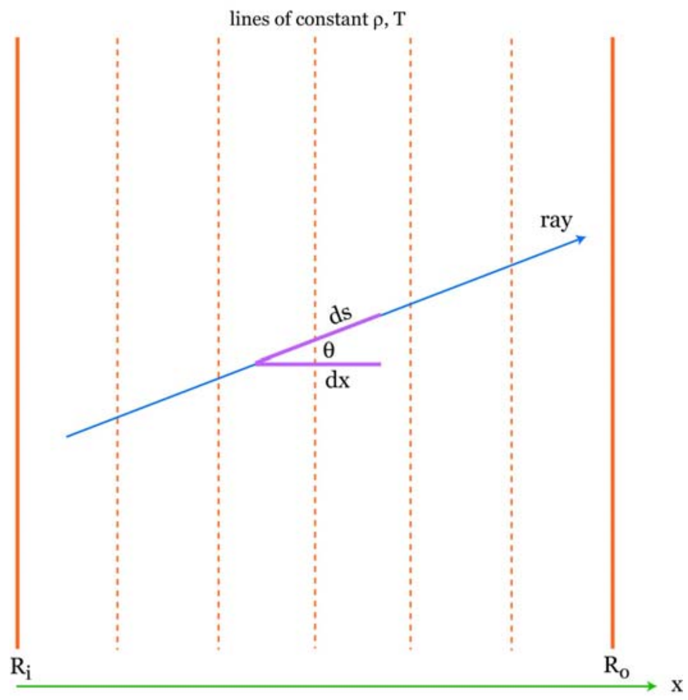
$$dI_\nu = dI_\nu^{abs} + dI_\nu^{\rm em} = (-\kappa_\nu\, I_\nu + \epsilon_\nu)\, ds$$

$$\boxed{\frac{dI_\nu}{ds} = -\kappa_\nu\, I_\nu + \epsilon_\nu}$$

differential equation describing the flow of radiation through matter

The equation of radiative tranfer

Plane-parallel symmetry



$$dx = \cos\theta \, ds = \mu \, ds$$

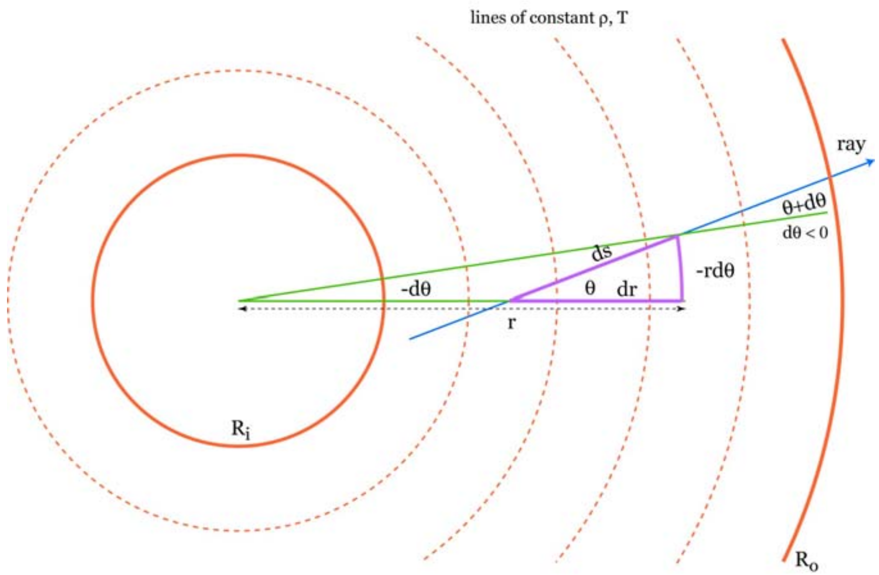
$$\frac{d}{ds} = \mu \frac{d}{dx}$$

$$\boxed{\mu \frac{dI_\nu(\mu,x)}{dx} = -\kappa_\nu \, I_\nu(\mu,x) + \epsilon_\nu}$$

The equation of radiative tranfer

Spherical symmetry

$$\frac{d}{ds} = \frac{dr}{ds}\frac{\partial}{\partial r} + \frac{d\theta}{ds}\frac{\partial}{\partial \theta}$$



$$dr = ds\cos\theta \rightarrow \frac{dr}{ds} = \cos\theta \quad (as\,in\,plane{-}parallel)$$

$$-r\,d\theta = \sin\theta\,ds\ (d\theta < 0) \rightarrow \frac{d\theta}{ds} = -\frac{\sin\theta}{r}$$

$$\frac{\partial}{\partial\theta} = \frac{\partial\mu}{\partial\theta}\frac{\partial}{\partial\mu} = -\sin\theta\frac{\partial}{\partial\mu}$$

$$\Longrightarrow \frac{d}{ds} = \mu\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial\mu} = \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}$$

$$\mu\,\tfrac{\partial}{\partial r} I_\nu(\mu,r) + \tfrac{1-\mu^2}{r}\tfrac{\partial}{\partial\mu} I_\nu(\mu,r) = -\kappa_\nu\,I_\nu(\mu,r) + \epsilon_\nu$$

angle θ between ray and radial direction
is not constant

The equation of radiative tranfer

Optical depth and source function

In plane-parallel symmetry:

$$\mu \frac{dI_\nu(\mu,x)}{dx} = -\kappa_\nu(x)\, I_\nu(\mu,x) + \epsilon_\nu(x)$$

$$\boxed{\mu \frac{dI_\nu(\mu,\tau_\nu)}{d\tau_\nu} = I_\nu(\mu,\tau_\nu) - S_\nu(\tau_\nu)}$$

$$\boxed{S_\nu = \frac{\epsilon_\nu}{\kappa_\nu}}$$

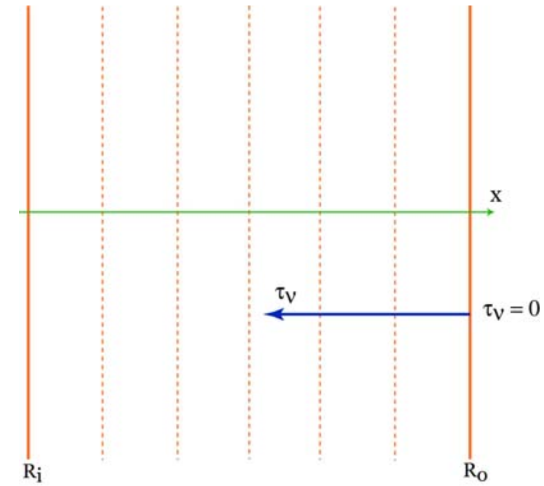
source function

dim [S_ν] = [I_ν]

optical depth increasing towards interior:

$$-\kappa_\nu\, dx = d\tau_\nu$$

$$\tau_\nu = -\int_{R_o}^{x} \kappa_\nu\, dx$$



$$\kappa_\nu = \frac{d\tau_\nu}{ds} \approx \frac{\Delta\tau_\nu}{\Delta s} \approx \frac{1}{\bar{s}}$$

$\tau = 1$ corresponds to free mean path of photons

$$S_\nu = \frac{\epsilon_\nu}{\kappa_\nu} \approx \epsilon_\nu \cdot \bar{s}$$

source function S_ν corresponds to intensity emitted over the free mean path of photons

Observed emerging intensity $I_\nu(\cos\theta, \tau_\nu = 0)$ depends on $\mu = \cos\theta$, $\tau_\nu(R_i)$ and S_ν

The physics of S_ν is crucial for radiative transfer

The equation of radiative tranfer

Source function: simple cases

a. LTE (thermal absorption/emission)

$$S_\nu = \frac{\epsilon_\nu}{\kappa_\nu} = B_\nu(T)$$

independent of radiation field

Kirchhoff's law

photons are absorbed and re-emitted at the local temperature T

Knowledge of T stratification T=T(x) or T(τ)
→ solution of transfer equation $I_\nu(\mu,\tau_\nu)$

The equation of radiative tranfer

Source function: simple cases

b. coherent isotropic scattering (e.g. Thomson scattering)

the absorption process is characterized by the scattering coefficient σ_ν, analogous to κ_ν:

$\nu = \nu'$

incident = scattered

$$dI_\nu = -\sigma_\nu I_\nu ds$$

$$dE_\nu^{em} = \int\limits_{4\pi} \epsilon_\nu^{sc} d\omega$$

$$dE_\nu^{abs} = \int\limits_{4\pi} \sigma_\nu \, I_\nu d\omega$$

and at each frequency ν: $dE_\nu^{em} = dE_\nu^{abs}$

$$\int\limits_{4\pi} \epsilon_\nu^{sc} d\omega = \int\limits_{4\pi} \sigma_\nu \, I_\nu d\omega$$

$$\epsilon_\nu^{sc} \int\limits_{4\pi} d\omega = \sigma_\nu \int\limits_{4\pi} I_\nu d\omega \qquad \frac{\epsilon_\nu^{sc}}{\sigma_\nu} = \frac{1}{4\pi} \int\limits_{4\pi} I_\nu d\omega$$

$$S_\nu = J_\nu$$

completely dependent on radiation field

not dependent on temperature T

The equation of radiative tranfer

Source function: simple cases

c. mixed case

$$S_\nu = \frac{\epsilon_\nu + \epsilon_\nu^{sc}}{\kappa_\nu + \sigma_\nu} = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu}\frac{\epsilon_\nu}{\kappa_\nu} + \frac{\sigma_\nu}{\kappa_\nu + \sigma_\nu}\frac{\epsilon_\nu^{sc}}{\sigma_\nu}$$

$$\Rightarrow \quad S_\nu = \frac{\epsilon_\nu + \epsilon_\nu^{sc}}{\kappa_\nu + \sigma_\nu} = \frac{\kappa_\nu}{\kappa_\nu + \sigma_\nu} B_\nu + \frac{\sigma_\nu}{\kappa_\nu + \sigma_\nu} J_\nu$$

Formal solution of the equation of radiative tranfer

linear 1st order differential equation

we want to solve the equation of RT with a known source function and in plane-parallel geometry

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

$$e^{-\tau_\nu/\mu} \mu \frac{dI_\nu}{d\tau_\nu} = I_\nu e^{-\tau_\nu/\mu} - S_\nu e^{-\tau_\nu/\mu}$$

multiply by $e^{-\tau_\nu/\mu}$ and integrate between τ_1 (outside) and τ_2 (> τ_1, inside)

$$\frac{d}{d\tau_\nu}(I_\nu \, e^{-\tau_\nu/\mu}) = -\frac{S_\nu \, e^{-\tau_\nu/\mu}}{\mu}$$

check, whether this really yields transfer equation above

$$\left[I_\nu \, e^{-\frac{\tau_\nu}{\mu}} \right]_{\tau_1}^{\tau_2} = -\int_{\tau_1}^{\tau_2} S_\nu \, e^{-\frac{\tau_\nu}{\mu}} \, \frac{dt_\nu}{\mu}$$

$$\left[I_\nu\, e^{-\frac{\tau_\nu}{\mu}}\right]_{\tau_1}^{\tau_2} = -\int_{\tau_1}^{\tau_2} S_\nu\, e^{-\frac{\tau_\nu}{\mu}}\,\frac{dt_\nu}{\mu}$$

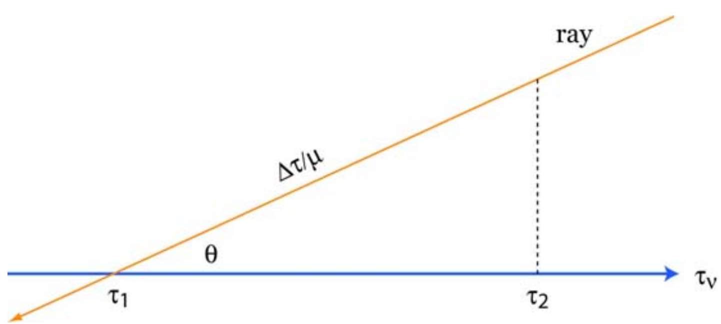
Formal solution of the equation of radiative tranfer

integral form of equation of radiation transfer

$$I_\nu(\tau_1,\mu) = I_\nu(\tau_2,\mu)\, e^{-\frac{\tau_2-\tau_1}{\mu}} + \int_{\tau_1}^{\tau_2} S_\nu(t)\, e^{-\frac{t-\tau_1}{\mu}}\,\frac{dt}{\mu}$$

intensity originating at τ_2 decreased by exponential factor to τ_1

contribution to the intensity by emission along the path from τ_2 to τ_1 (at each point decreased by the exponential factor)



Formal solution! actual solution can be complex, since S_ν can depend on I_ν

$$I_\nu(\tau_1,\mu) = I_\nu(\tau_2,\mu)\, e^{-\frac{\tau_2-\tau_1}{\mu}} + \int_{\tau_1}^{\tau_2} S_\nu(t)\, e^{-\frac{t-\tau_1}{\mu}}\, \frac{dt}{\mu}$$

Boundary conditions

solution of RT equation requires boundary conditions, which are different for incoming and outgoing radiation

a. incoming radiation: $\mu < 0$ at $\tau_2 = 0$

usually we can neglect irradiation from outside: $I_\nu(\tau_2 = \mathbf{0}, \mu < \mathbf{0}) = 0$

$$I_\nu^{in}(\tau_\nu,\mu) = \int_{\tau_\nu}^{0} S_\nu(t)\, e^{-\frac{t-\tau_\nu}{\mu}}\, \frac{dt}{\mu}$$

$$I_\nu(\tau_1,\mu) = I_\nu(\tau_2,\mu)\, e^{-\frac{\tau_2-\tau_1}{\mu}} + \int_{\tau_1}^{\tau_2} S_\nu(t)\, e^{-\frac{t-\tau_1}{\mu}}\, \frac{dt}{\mu}$$

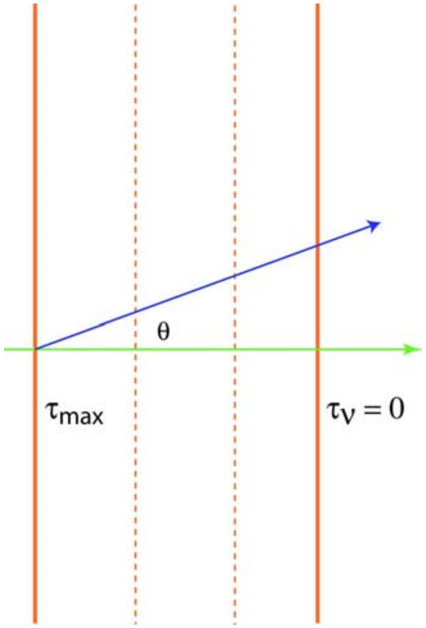
Boundary conditions

b. outgoing radiation: $\mu > 0$ at $\tau_2 = \tau_{max} \rightarrow \infty$

We have either $$I_\nu(\tau_{max},\mu) = I_\nu^+(\mu)$$ finite slab or shell

or $$\lim_{\tau\to\infty} I_\nu(\tau,\mu)\, e^{-\tau/\mu} = 0$$ semi-infinite case (planar or spherical)

I_ν increases less rapidly than the exponential



$$I_\nu^{out}(\tau_\nu,\mu) = \int_{\tau_\nu}^{\infty} S_\nu(t)\, e^{-\frac{t-\tau_\nu}{\mu}}\, \frac{dt}{\mu}$$

and at a given position τ_ν in the atmosphere:

$$I_\nu(\tau_\nu) = I_\nu^{out}(\tau_\nu) + I_\nu^{in}(\tau_\nu)$$

Emergent intensity

from the latter → emergent intensity

$\tau_\nu = 0$, $\mu > 0$

$$I_\nu(0,\mu) = \int_0^\infty S_\nu(t)\, e^{-\frac{t}{\mu}}\, \frac{dt}{\mu}$$

intensity observed is a weighted average of the source function along the line of sight. The contribution to the emerging intensity comes mostly from each depths with $\tau/\mu < 1$.

Emergent intensity

suppose that S_ν is linear in τ_ν (Taylor expansion around $\tau_\nu = 0$):

$$S_\nu(\tau_\nu) = S_{0\nu} + S_{1\nu}\tau_\nu$$

$$\int xe^{-x}\,dx = -(1+x)\,e^{-x}$$

emergent intensity

$$I_\nu(0,\mu) = \int_0^\infty (S_{0\nu} + S_{1\nu}t)e^{-\frac{t}{\mu}}\,\frac{dt}{\mu} = S_{0\nu} + S_{1\nu}\mu$$

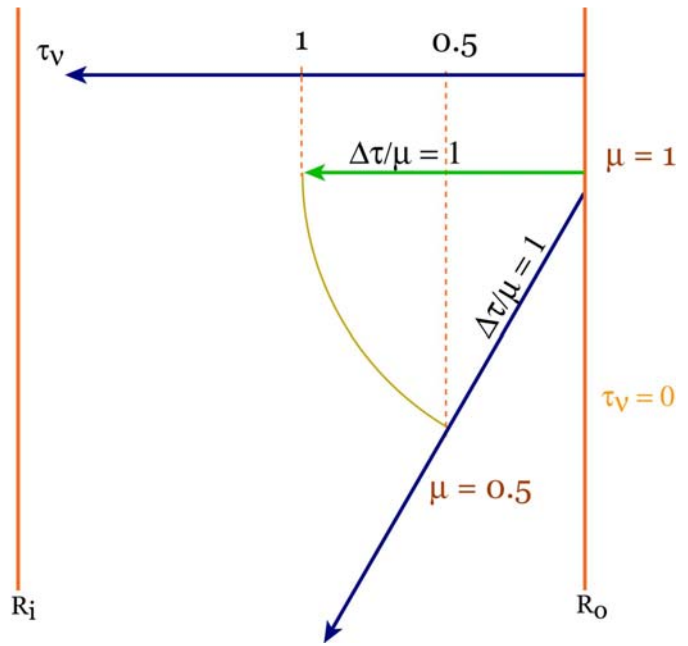
$$I_\nu(0,\mu) = S_\nu(\tau_\nu = \mu)$$

Eddington-Barbier relation

we see the source function at location $\tau_\nu = \mu$

the emergent intensity corresponds to the source function at τ_ν = 1 along the line of sight

Emergent intensity



μ = 1 (normal direction):

$$I_\nu(0,1) = S_\nu(\tau_\nu = 1)$$

μ = 0.5 (slanted direction):

$$I_\nu(0,0.5) = S_\nu(\tau_\nu = 0.5)$$

in both cases: $\Delta\tau/\mu \approx 1$

spectral lines: compared to continuum $\tau_\nu/\mu = 1$ is reached at higher layer in the atmosphere

→ $S_\nu^{line} < S_\nu^{cont}$

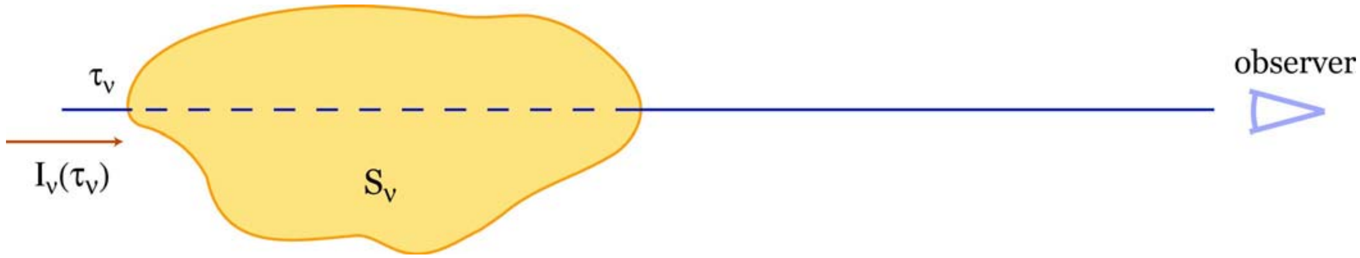
→ a dip is created in the spectrum

Line formation

simplify: $\mu = 1$, $\tau_1 = 0$ (emergent intensity), $\tau_2 = \tau$

S_ν independent of location

$$I_\nu(0) = I_\nu(\tau_\nu)\, e^{-\tau_\nu} + S_\nu \int_0^{\tau_\nu} e^{-t}\, dt = I_\nu(\tau_\nu)\, e^{-\tau_\nu} + S_\nu\,(1 - e^{-\tau_\nu})$$



Line formation

Optically thick object:

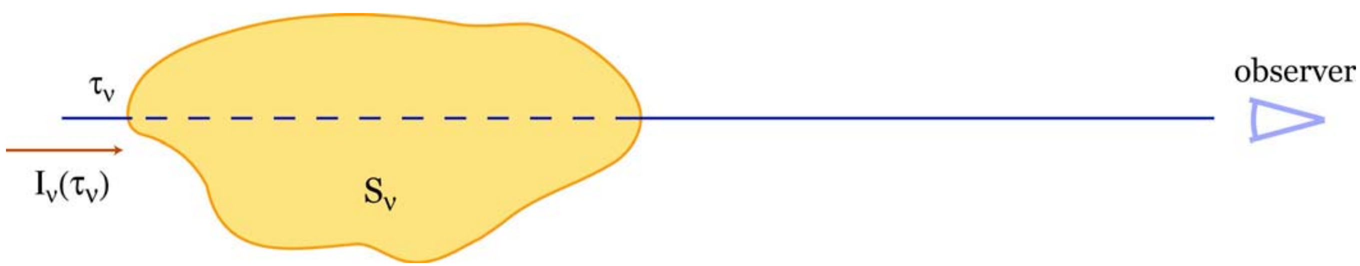
$\tau \rightarrow \infty$

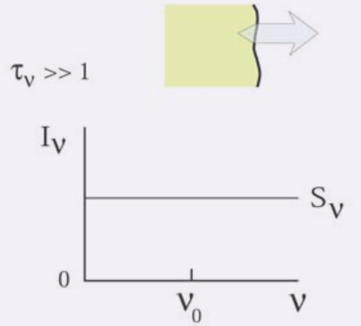
$$I_\nu(0) = I_\nu(\tau_\nu)\, \cancel{e^{-\tau_\nu}} + S_\nu\,(1 - \cancel{e^{-\tau_\nu}}) = S_\nu$$

Optically thin object:

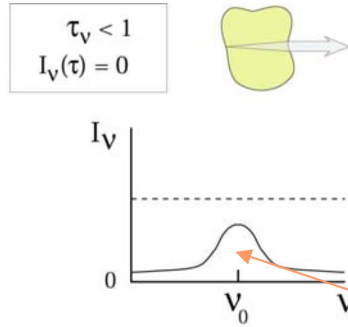
$\exp(-\tau_\nu) \approx 1 - \tau_\nu$

$$I_\nu(0) = I_\nu(\tau_\nu) + [S_\nu - I_\nu(\tau_\nu)]\,\tau_\nu$$





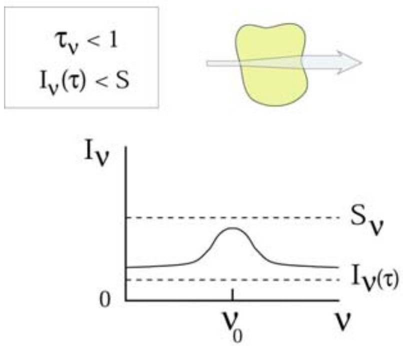
independent of κ_ν, no line
(e.g. black body B_ν)

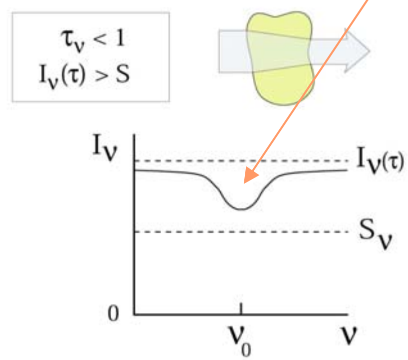


$I_\nu = \tau_\nu S_\nu = \kappa_\nu ds_\nu S_\nu$

e.g. HII region, solar corona

enhanced κ_ν

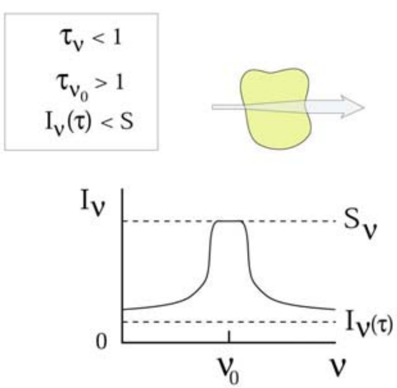


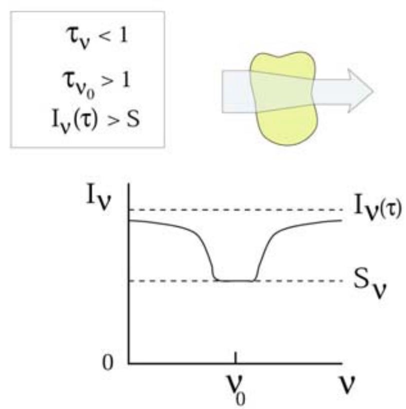


$$I_\nu(0) = I_\nu(\tau_\nu) + [S_\nu - I_\nu(\tau_\nu)]\,\tau_\nu$$

e.g. stellar absorption spectrum
(temperature decreasing outwards)

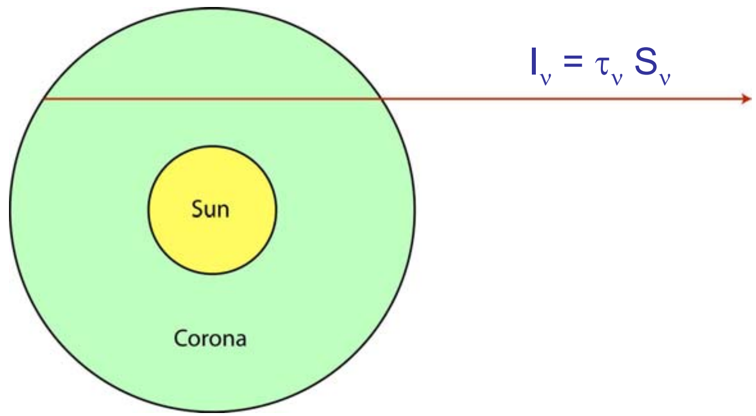
e.g. stellar spectrum with temperature increasing outwards (e.g. Sun in the UV)

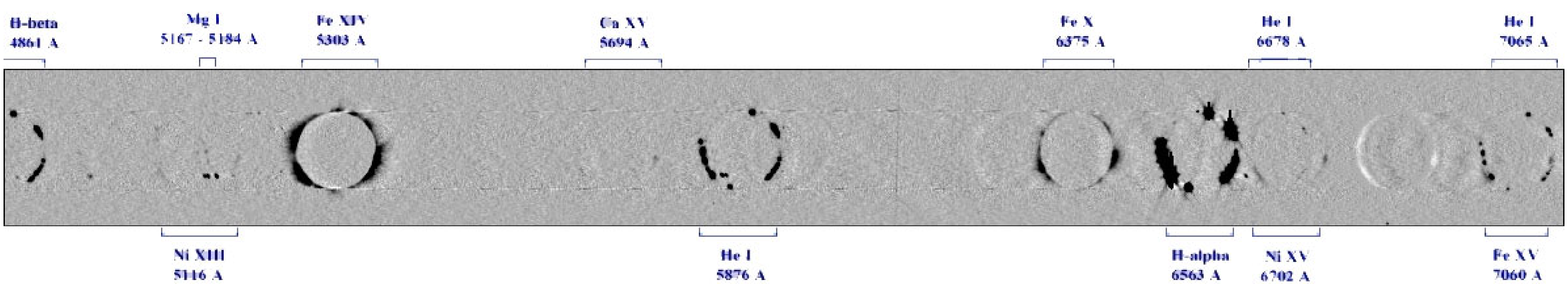




From Rutten's web notes

Line formation example: solar corona





The diffusion approximation

At large optical depth in stellar atmosphere photons are local: $S_\nu \rightarrow B_\nu$

Expand S_ν ($= B_\nu$) as a power-series:

$$S_\nu(t) = \sum_{n=0}^{\infty} \frac{d^n B_\nu}{d\tau_\nu^n}(t - \tau_\nu)^n / n!$$

In the diffusion approximation (τ_ν >>1) we retain only first order terms:

$$B_\nu(t) = B_\nu(\tau_\nu) + \frac{dB_\nu}{d\tau_\nu}(t - \tau_\nu)$$

$$I_\nu^{out}(\tau_\nu, \mu) = \int_{\tau_\nu}^{\infty} [B_\nu(\tau_\nu) + \frac{dB_\nu}{d\tau_\nu}(t - \tau_\nu)] e^{-(t-\tau_\nu)/\mu} \frac{dt}{\mu}$$

$$I_\nu^{out}(\tau_\nu,\mu) = \int_{\tau_\nu}^{\infty} [B_\nu(\tau_\nu) + \frac{dB_\nu}{d\tau_\nu}(t-\tau_\nu)] e^{-(t-\tau_\nu)/\mu} \frac{dt}{\mu}$$

The diffusion approximation

Substituting: $t \to u = \dfrac{t-\tau_\nu}{\mu} \to dt = \mu\, du$

$$\int_0^\infty u^k e^{-u} du = k!$$

$$I_\nu^{out}(\tau_\nu,\mu) = \int_0^{\infty} [B_\nu(\tau_\nu) + \frac{dB_\nu}{d\tau_\nu}\mu u] e^{-u}\, du = B_\nu(\tau_\nu) + \mu \frac{dB_\nu}{d\tau_\nu}$$

$$I_\nu^{in}(\tau_\nu,\mu) = -\int_0^{\tau_\nu/\mu} [B_\nu(\tau_\nu) + \frac{dB_\nu}{d\tau_\nu}\mu u] e^{-u}\, du$$

At $\tau_\nu = 0$ we obtain the Eddington-Barbier relation for the observed emergent intensity.

It is given by the Planck-function and its gradient at $\tau_\nu = 0$.

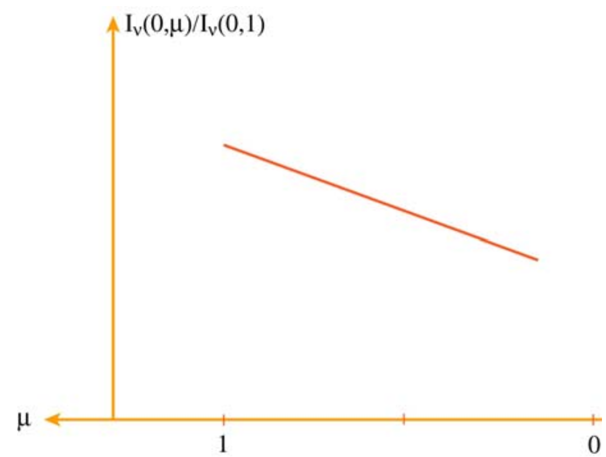
It depends linearly on $\mu = \cos\theta$.

diffusion approximation: $I_\nu^{out}(0,\mu) = B_\nu(0) + \mu \frac{dB_\nu}{d\tau_\nu}(0)$

Solar limb darkening

$$\frac{I_\nu(0,\mu)}{I_\nu(0,1)} = \frac{B_\nu(0) + \frac{dB_\nu}{d\tau_\nu}\mu}{B_\nu(0) + \frac{dB_\nu}{d\tau_\nu}}$$

center-to-limb variation of intensity

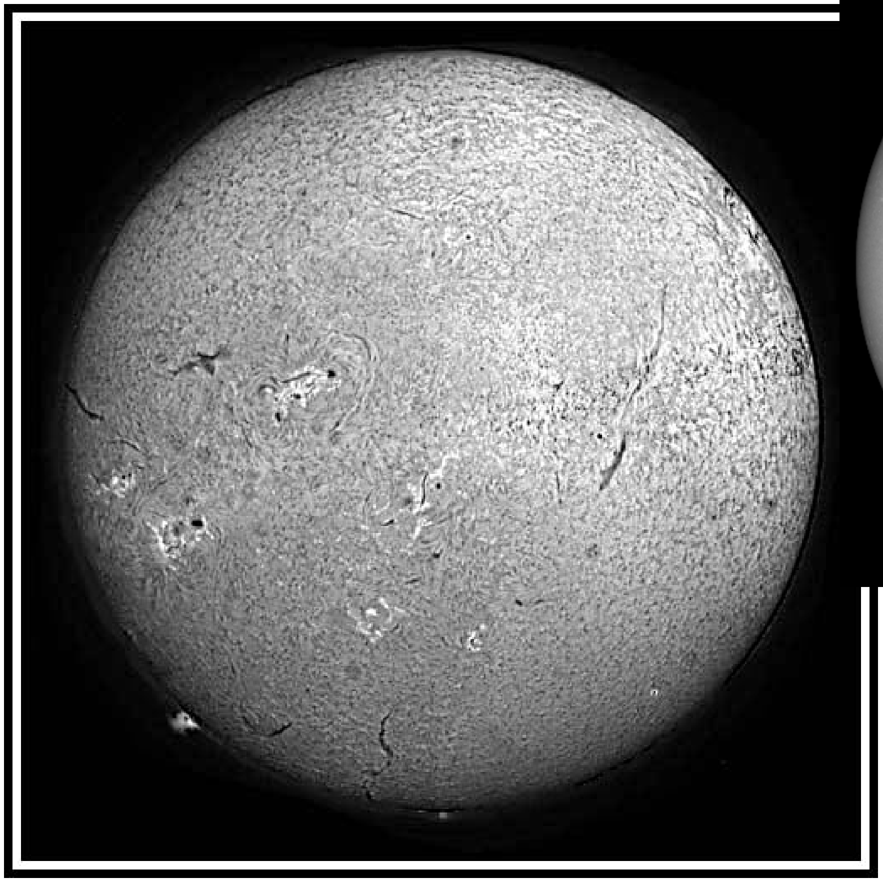


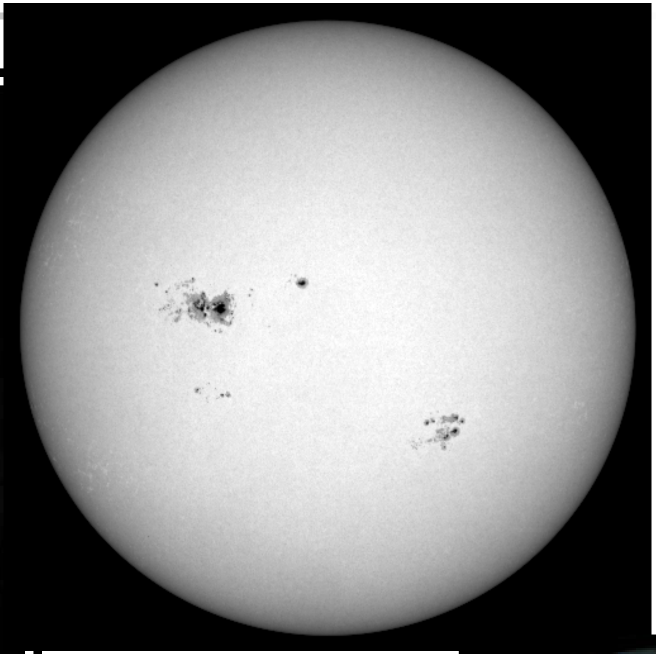
from the intensity measurements
→ $B_\nu(0)$, $dB_\nu/d\tau_\nu$

$$B_\nu(t) = B_\nu(0) + \frac{dB_\nu}{d\tau_\nu} t = a + b \cdot t = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT(t)} - 1}$$

T(t): empirical temperature stratification of solar photosphere

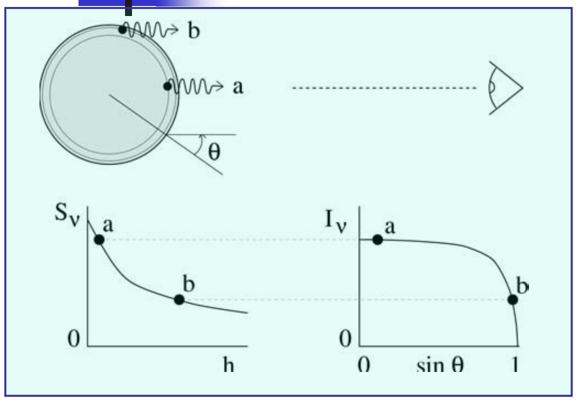
Solar limb darkening





...and also giant planets

Solar limb darkening: temperature stratification



$$I_\nu(0,\mu) = \int_0^\infty S_\nu(t)\, e^{-\frac{t}{\mu}} \,\frac{dt}{\mu}$$

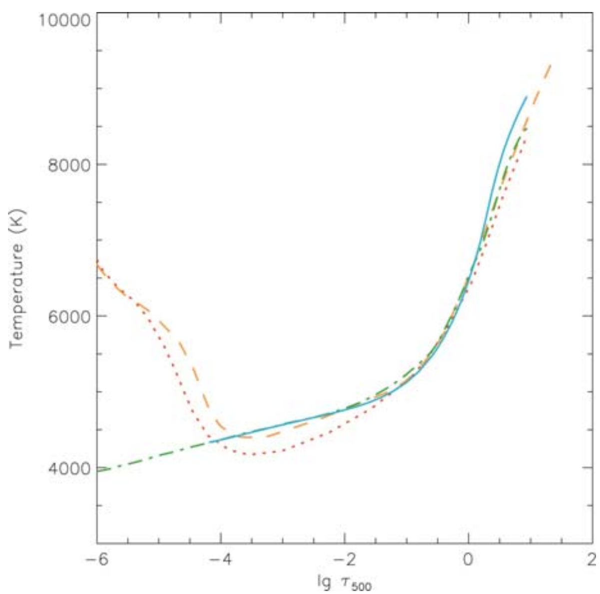
exponential extinction varies as $-\tau_\nu/\cos\theta$

From $S_\nu = a + b\tau_\nu$:

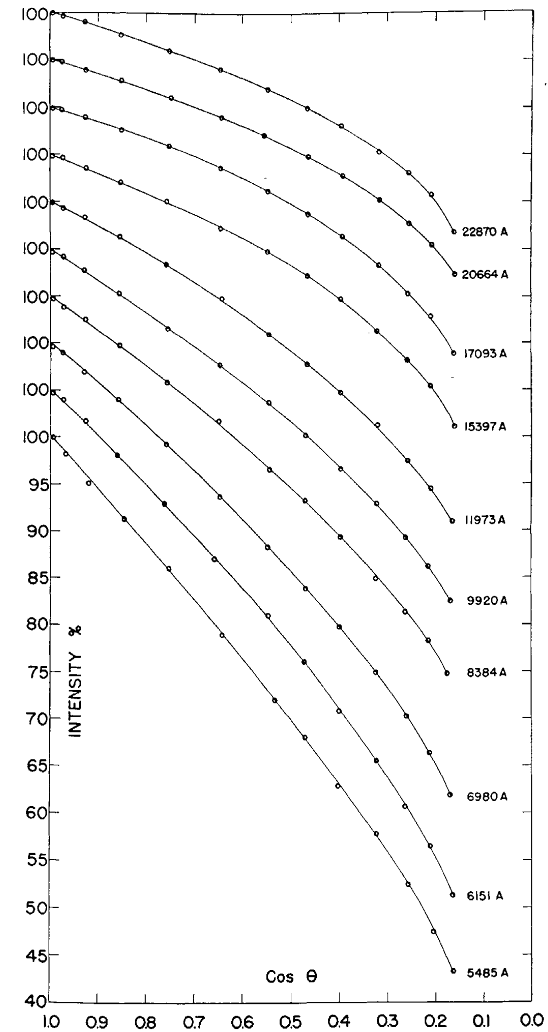
$$I_\nu(0,\cos\theta) = a_\nu + b_\nu \cos\theta$$

$$I_\nu(0,\mu) = S_\nu(\tau_\nu = \mu)$$ → S_ν

R. Rutten, web notes



Unsoeld, 68



Eddington approximation

we want to obtain an approximation for the radiation field – both inward and outward radiation - at large optical depth

→ stellar interior, inner boundary of atmosphere

In the diffusion approximation we had:

$$B_\nu(t) = B_\nu(\tau_\nu) + \frac{dB_\nu}{d\tau_\nu}(t - \tau_\nu)$$

$$I_\nu^{out}(\tau_\nu, \mu) = B_\nu(\tau_\nu) + \mu \frac{dB_\nu}{d\tau_\nu}$$ $0 < \mu < 1$

$$I_\nu^{in}(\tau_\nu, \mu) = -\int_0^{\tau_\nu/\mu} [B_\nu(\tau_\nu) + \frac{dB_\nu}{d\tau_\nu}\mu u] e^{-u}\, du$$ $-1 < \mu < 0$

$\tau >> 1$

$$I_\nu^-(\tau_\nu, \mu) = B_\nu(\tau_\nu) + \mu \frac{dB_\nu}{d\tau_\nu}$$

Eddington approximation

With this approximation for I_ν we can calculate the angle averaged momenta of the intensity

→simple approximation for photon flux and a relationship between mean intensity J_ν and K_ν

→ very important for analytical estimates

$$J_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu \, d\mu = B_\nu(\tau_\nu)$$

$$H_\nu = \frac{F_\nu}{4} = \frac{1}{2}\int_{-1}^{1} \mu \, I_\nu \, d\mu = \frac{1}{3}\frac{dB_\nu}{d\tau_\nu} = -\frac{1}{3}\frac{1}{\kappa_\nu}\frac{dB_\nu}{dx} = -\frac{1}{3\kappa_\nu}\frac{dB_\nu}{dT}\frac{dT}{dx}$$

flux F_ν ~ dT/dx

diffusion: flux ~ gradient (e.g. heat conduction)

$$K_\nu = \frac{1}{2}\int_{-1}^{1} \mu^2 \, I_\nu \, d\mu = \frac{1}{3} B_\nu(\tau_\nu)$$

$$K_\nu = \tfrac{1}{3}\, J_\nu$$

Eddington approximation

Schwarzschild-Milne equations

After the previous approximations, we now want to calculate **exact solutions for tha radiative momenta J_ν, H_ν, K_ν.** Those are important to calculate spectra and atmospheric structure

$$I_\nu^{out}(\tau_\nu,\mu) = \int_{\tau_\nu}^{\infty} S_\nu(t)\, e^{-\frac{t-\tau_\nu}{\mu}}\, \frac{dt}{\mu}$$

$$J_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu\, d\mu = \frac{1}{2}\int_{0}^{1} I_\nu^{out}\, d\mu + \frac{1}{2}\int_{-1}^{0} I_\nu^{in}\, d\mu$$

$$I_\nu^{in}(\tau_\nu,\mu) = \int_{\tau_\nu}^{0} S_\nu(t)\, e^{-\frac{t-\tau_\nu}{\mu}}\, \frac{dt}{\mu}$$

$\mu > 0$ $\mu < 0$

$$J_\nu = \frac{1}{2}\left[\int_0^1\int_{\tau_\nu}^{\infty} S_\nu(t)e^{-(t-\tau_\nu)/\mu}\frac{dt}{\mu}d\mu - \int_{-1}^{0}\int_{0}^{\tau_\nu} S_\nu(t)e^{-(t-\tau_\nu)/\mu}\frac{dt}{\mu}d\mu\right]$$

$$\text{substitute } w = \frac{1}{\mu} \Rightarrow \frac{dw}{w} = -\frac{1}{\mu}d\mu \qquad w = -\frac{1}{\mu} \Rightarrow \frac{dw}{w} = -\frac{1}{\mu}d\mu$$

$$J_\nu = \frac{1}{2}\left[\int_1^{\infty}\int_{\tau_\nu}^{\infty} S_\nu(t)e^{-(t-\tau_\nu)w}dt\frac{dw}{w} + \int_1^{\infty}\int_0^{\tau_\nu} S_\nu(t)e^{-(\tau_\nu-t)w}dt\frac{dw}{w}\right]$$

Schwarzschild-Milne equations

$$J_\nu = \frac{1}{2}\left[\int_{\tau_\nu}^{\infty} S_\nu(t) \int_1^{\infty} e^{-\underbrace{(t-\tau_\nu)}_{>0} w} \frac{dw}{w} dt + \int_0^{\tau_\nu} S_\nu(t) \int_1^{\infty} e^{-\underbrace{(\tau_\nu - t)}_{>0} w} \frac{dw}{w} dt\right]$$

$$\Rightarrow J_\nu = \frac{1}{2}\int_0^{\infty} S_\nu(t) \int_1^{\infty} e^{-w|t-\tau_\nu|} \frac{dw}{w} dt = \frac{1}{2}\int_0^{\infty} S_\nu(t) E_1(|t-\tau_\nu|) dt$$

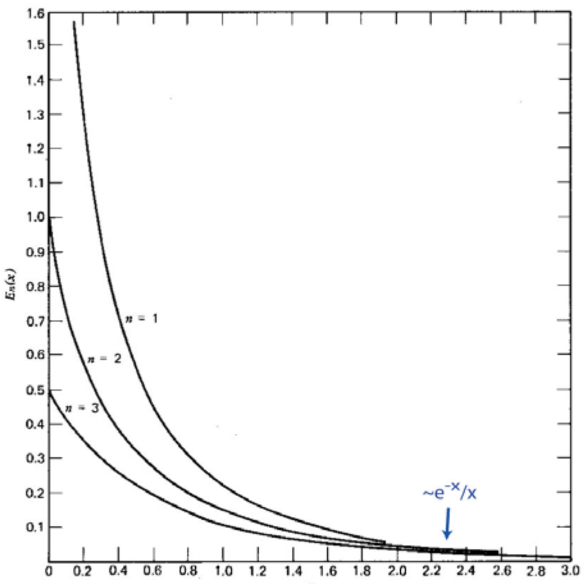
Schwarzschild's equation

Schwarzschild-Milne equations

where

$$E_1(t) = \int_1^\infty e^{-tx} \frac{dx}{x} = \int_t^\infty \frac{e^{-x}}{x} dx$$

is the first exponential integral (singularity at t=0)



Exponential integrals

$$E_n(t) = t^{n-1} \int_t^\infty x^{-n} e^{-x} dx$$

$$E_n(0) = 1/(n-1), E_n(t \to \infty) = e^{-t}/t \to 0$$

$$\frac{dE_n}{dt} = -E_{n-1}, \int E_n(t) = -E_{n+1}(t)$$

$$E_1(0) = \infty \quad E_2(0) = 1 \quad E_3(0) = 1/2 \quad E_n(\infty) = 0$$

Gray, 92

Schwarzschild-Milne equations

Introducing the Λ operator:

$$\Lambda_{\tau_\nu}[f(t)] = \frac{1}{2}\int_0^\infty f(t)E_1(|t-\tau_\nu|)\,dt$$

$$J_\nu(\tau_\nu) = \Lambda_{\tau_\nu}[S_\nu(t)]$$

Similarly for the other 2 moments of Intensity:

$$H_\nu(\tau_\nu) = \frac{1}{2}\int_{\tau\nu}^\infty S_\nu(t)E_2(t-\tau_\nu)dt - \frac{1}{2}\int_0^{\tau_\nu} S_\nu(t)E_2(\tau_\nu - t)dt = \Phi_{\tau_\nu}(S_\nu(t))$$

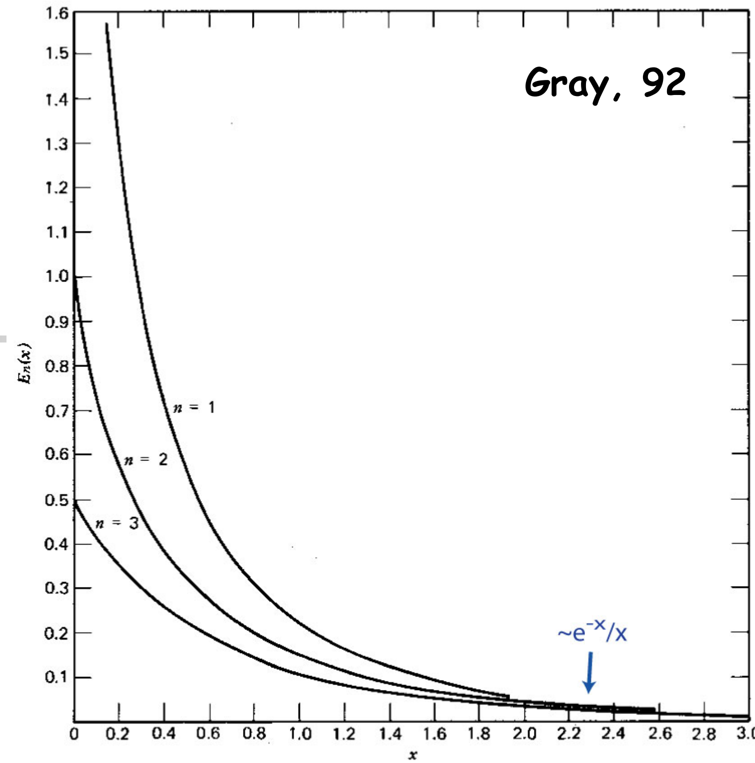
Milne's equations

$$K_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t)E_3(|\ t-\tau_\nu\ |)dt = X_{\tau_\nu}(S_\nu(t))$$

J_ν, H_ν and K_ν are all depth-weighted means of S_ν

Schwarzschild-Milne equations

the 3 moments of Intensity:



$$J_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t)E_1(|\, t-\tau_\nu \,|)dt = \Lambda_{\tau_\nu}(S_\nu(t))$$

$$H_\nu(\tau_\nu) = \frac{1}{2}\int_{\tau\nu}^\infty S_\nu(t)E_2(t-\tau_\nu)dt - \frac{1}{2}\int_0^{\tau_\nu} S_\nu(t)E_2(\tau_\nu - t)dt = \Phi_{\tau_\nu}(S_\nu(t))$$

$$K_\nu(\tau_\nu) = \frac{1}{2}\int_0^\infty S_\nu(t)E_3(|\, t-\tau_\nu \,|)dt = X_{\tau_\nu}(S_\nu(t))$$

J_ν, H_ν and K_ν are all depth-weighted means of S_ν

→ the strongest contribution comes from the depth, where the argument of the exponential integrals is zero, i.e. $t=\tau_\nu$

The temperature-optical depth relation

Radiative equilibrium

The condition of radiative equilibrium (expressing conservation of energy) requires that the flux at any given depth remains constant:

$$\mathcal{F}(r) = \pi F = \int_0^\infty \int_{4\pi} I_\nu \cos\theta \, d\omega \, d\nu = \pi \int_0^\infty F_\nu \, d\nu = 4\pi \int_0^\infty H_\nu \, d\nu$$

$$\Rightarrow \quad 4\pi r^2 \mathcal{F}(r) = 4\pi r^2 \cdot 4\pi \int_0^\infty H_\nu \, d\nu = \text{const} = L$$

In plane-parallel geometry r ≈ R = const

$$4\pi \int_0^\infty H_\nu \, d\nu = \text{const}$$

and in analogy to the black body radiation, from the Stefan-Boltzmann law we define the effective temperature:

$$4\pi \int_0^\infty H_\nu \, d\nu = \sigma T_{\text{eff}}^4$$

The **effective temperature**

The effective temperature is defined by:

$$4\pi \int_0^\infty H_\nu \, d\nu = \sigma T_{\rm eff}^4$$

It characterizes the total radiative flux transported through the atmosphere.

It can be regarded as an average of the temperature over depth in the atmosphere.

A blackbody radiating the same amount of total energy would have a temperature $T = T_{\mathbf{eff}}$.

Radiative equilibrium

Let us now combine the condition of radiative equilibrium with the equation of radiative transfer in plane-parallel geometry:

$$\mu \frac{dI_\nu}{dx} = -(\kappa_\nu + \sigma_\nu)\,(I_\nu - S_\nu)$$

$$\frac{1}{2}\int_{-1}^{1} \mu \frac{dI_\nu}{dx}\,d\mu = -\frac{1}{2}\int_{-1}^{1} (\kappa_\nu + \sigma_\nu)\,(I_\nu - S_\nu)\,d\mu$$

$$\frac{d}{dx}\underbrace{\left[\frac{1}{2}\int_{-1}^{1} \mu\, I_\nu\, d\mu\right]}_{H_\nu} = -(\kappa_\nu + \sigma_\nu)\,(J_\nu - S_\nu)$$

Radiative equilibrium

Integrate over frequency:

$$\underbrace{\frac{d}{dx}\int_0^\infty H_\nu \, d\nu}_{\text{const}} = -\int_0^\infty (\kappa_\nu + \sigma_\nu)\,(J_\nu - S_\nu)\, d\nu$$

$$\Rightarrow \quad \int_0^\infty (\kappa_\nu + \sigma_\nu)\,(J_\nu - S_\nu)\, d\nu = 0$$

substitute $S_\nu = \dfrac{\kappa_\nu}{\kappa_\nu + \sigma_\nu} B_\nu + \dfrac{\sigma_\nu}{\kappa_\nu + \sigma_\nu} J_\nu$

$$\left(\int_0^\infty \kappa_\nu J_\nu \, d\nu = \text{absorbed energy}\right)$$

$$\left(\int_0^\infty \kappa_\nu B_\nu \, d\nu = \text{emitted energy}\right)$$

at each depth: $\displaystyle\int_0^\infty \kappa_\nu \left[J_\nu - B_\nu(T)\right] d\nu = 0$

in addition: $\displaystyle 4\pi \int_0^\infty H_\nu \, d\nu = \sigma T_{\text{eff}}^4 \quad \Rightarrow \quad \boxed{T(x) \text{ or } T(\tau)}$

Radiative equilibrium

$$\int_0^\infty \kappa_\nu \left[J_\nu - B_\nu(T)\right] d\nu = 0$$

$$4\pi \int_0^\infty H_\nu \, d\nu = \sigma T_{\mathrm{eff}}^4$$

→ T(x) or T(τ)

The temperature T(r) at every depth has to assume the value for which the left integral over all frequencies becomes zero.

→ This determines the local temperature.

Iterative method for calculation of a stellar atmosphere: the major parameters are **T_{eff} and g**

T(x), κ_ν(x), B_ν[T(x)],P(x), ρ(x)

$\xrightarrow{\text{equation of transfer}}$

J_ν(x), H_ν(x)

$$\int_0^\infty \kappa_\nu (J_\nu - B_\nu)\, d\nu = 0 \text{ ?}$$

$$4\pi \int_0^\infty H_\nu \, d\nu = \sigma\, T_{\rm eff}^4 \text{ ?}$$

ΔT(x), $\Delta\kappa_\nu$(x), ΔB_ν[T(x)], Δρ(x)

a. hydrostatic equilibrium

$$\frac{dP}{dx} = -g\rho(x)$$

b. equation of radiation transfer

$$\mu \frac{dI_\nu}{dx} = -(\kappa_\nu + \sigma_\nu)\,(I_\nu - S_\nu)$$

c. radiative equilibrium

$$\int_0^\infty \kappa_\nu \left[J_\nu - B_\nu(T)\right] d\nu = 0$$

d. flux conservation

$$4\pi \int_0^\infty H_\nu \, d\nu = \sigma T_{\rm eff}^4$$

e. equation of state

$$P = \frac{\rho\, k\, T}{\mu\, m_H}$$

Grey atmosphere - an approximation for the temperature structure

We derive a simple analytical approximation for the temperature structure. We assume that we can approximate the radiative equilibrium integral by using a frequency-averaged absorption coefficient, which we can put in front of the integral.

$$\int_0^\infty \kappa_\nu \left[J_\nu - B_\nu(T)\right] d\nu = 0 \quad \Longrightarrow \quad \bar{\kappa} \int_0^\infty \left[J_\nu - B_\nu(T)\right] d\nu = 0$$

$$\text{With: } J = \int_0^\infty J_\nu \, d\nu \quad H = \int_0^\infty H_\nu \, d\nu \quad K = \int_0^\infty K_\nu \, d\nu \quad B = \int_0^\infty B_\nu \, d\nu = \frac{\sigma T^4}{\pi}$$

$$J = B$$
$$4\pi H = \sigma T_{\rm eff}^4$$

Grey atmosphere

We then assume LTE: S = B.

From

$$J_\nu(\tau_\nu) = \Lambda_{\tau_\nu}[S_\nu(t)] = \frac{1}{2}\int_0^\infty S_\nu(t) E_1(|t-\tau_\nu|)dt$$

and a similar expression for frequency-integrated quantities

$$J(\bar{\tau}) = \Lambda_{\bar{\tau}}[S(t)], \qquad d\bar{\tau} = \bar{\kappa} dx$$

and with the approximations **S = B, B = J**:

$$J(\bar{\tau}) = \Lambda_{\bar{\tau}}[J(t)] = \frac{1}{2}\int_0^\infty J(t) E_1(|t-\bar{\tau}|)\, dt$$

Milne's equation

!!! this is an integral equation for J(τ) !!!

The exact solution of the Hopf integral equation

Milne's equation $J(\tau) = \Lambda_\tau[J(t)]$ → exact solution (see Mihalas, "Stellar Atmospheres")

$$J(\tau) = \text{const.}\ [\ \tau + q(\tau)], \qquad \text{with } q(\tau) \text{ monotonic}$$

$$\frac{1}{\sqrt{3}} = 0.577 = q(0) \le q(\bar{\tau}) \le q(\infty) = 0.710$$

Radiative equilibrium - grey approximation

⇒ $J(\tau) = B(\tau) = \sigma/\pi\ T^4(\tau) = \text{const.}\ [\ \tau + q(\tau)]$

with boundary conditions →

$$\mathbf{T^4(\tau) = \tfrac{3}{4}\ T^4_{eff}\ [\tau + q(\tau)]}$$

A simple approximation for T(τ)

0th moment of equation of transfer (integrate both sides in dμ from -1 to 1)

$$\mu \frac{dI}{dx} = -\bar{\kappa}(I - B) \quad \Longrightarrow \quad \frac{dH}{d\bar{\tau}} = J - B = 0 \quad (J = B) \quad \Longrightarrow \quad H = \mathrm{const} = \frac{\sigma T_{\mathrm{eff}}^4}{4\pi}$$

1st moment of equation of transfer (integrate both sides in μdμ from -1 to 1)

$$\mu \frac{dI}{dx} = -\bar{\kappa}(I - B) \quad \Longrightarrow \quad \frac{dK}{d\bar{\tau}} = H = \frac{\sigma T_{\mathrm{eff}}^4}{4\pi} \quad \Longrightarrow \quad K(\bar{\tau}) = H\bar{\tau} + constant$$

From Eddington's approximation at large depth: K = 1/3 J

$$\Downarrow$$

$$J(\bar{\tau}) = 3H(\bar{\tau} + c) = \frac{\sigma T^4}{\pi}$$

$$J(\bar{\tau}) = 3H(\bar{\tau} + c) = \frac{\sigma T^4}{\pi}$$

Grey atmosphere – temperature distribution

$$T^4(\bar{\tau}) = \frac{3\pi H}{\sigma}(\bar{\tau} + c) \qquad H = \frac{\sigma}{4\pi} T_{\text{eff}}^4$$

$$T^4(\bar{\tau}) = \frac{3}{4} T_{\text{eff}}^4 \, (\bar{\tau} + c)$$

T^4 is linear in τ

Estimation of c

$$H(\bar{\tau} = 0) = \frac{1}{2}\int_0^\infty J(t)E_2(t)dt = \frac{1}{2}3H\int_0^\infty (t + c)E_2(t)dt$$

$$H(\bar{\tau} = 0) = \frac{1}{2}3H\left[\underbrace{\int_0^\infty tE_2(t)\,dt}_{1/3} + c\underbrace{\int_0^\infty E_2(t)\,dt}_{1/2}\right]$$

$$\int_0^\infty t^s E_n(t)\,dt = \frac{s!}{s+n}$$

Grey atmosphere – Hopf function

$$H(0) = H = \frac{1}{2} H (1 + \frac{3}{2} c) \rightarrow c = \frac{2}{3}$$

$$\boxed{T^4(\bar{\tau}) = \frac{3}{4} T_{\text{eff}}^4 \, (\bar{\tau} + \frac{2}{3})}$$

based on approximation K/J = 1/3
$T = T_{eff}$ at $\tau = 2/3$, $T(0) = 0.84\ T_{eff}$

Remember: More in general J is obtained from $J(\bar{\tau}) = \Lambda_{\bar{\tau}}[S(t)]$

$$\boxed{T^4(\bar{\tau}) = \frac{3}{4} T_{\text{eff}}^4 \, [\bar{\tau} + q(\bar{\tau})]}$$

$q(\bar{\tau})$: Hopf function

Once Hopf function is specified → solution of the grey atmosphere (temperature distribution)

$$\frac{1}{\sqrt{3}} = 0.577 = q(0) \leq q(\bar{\tau}) \leq q(\infty) = 0.710$$

Selection of the appropriate $\kappa_\nu \Rightarrow \bar{\kappa}$

In the grey case we define a 'suitable' mean opacity (absorption coefficient).

$$\kappa_\nu \Rightarrow \bar{\kappa} \qquad I = \int_0^\infty I_\nu \, d\nu \quad J = \int_0^\infty J_\nu \, d\nu \quad \ldots$$

	non-grey	grey
Equation of transfer	$\mu \frac{dI_\nu}{dx} = -\kappa_\nu (I_\nu - S_\nu)$	$\mu \frac{dI}{dx} = -\tilde{\kappa}\,(I - S)$
1st moment	$\frac{dH_\nu}{dx} = -\kappa_\nu (J_\nu - S_\nu)$	$\frac{dH}{dx} = -\hat{\kappa}\,(J - S)$
2nd moment	$\frac{dK_\nu}{dx} = -\kappa_\nu H_\nu$	$\frac{dK}{dx} = -\bar{\kappa}\, H$

Selection of the appropriate $\kappa_\nu \Rightarrow \bar{\kappa}$

	non-grey	grey
Equation of transfer	$\mu \frac{dI_\nu}{dx} = -\kappa_\nu (I_\nu - S_\nu)$	$\mu \frac{dI}{dx} = -\tilde{\kappa}\,(I - S)$
1st moment	$\frac{dH_\nu}{dx} = -\kappa_\nu (J_\nu - S_\nu)$	$\frac{dH}{dx} = -\widehat{\kappa}\,(J - S)$
2nd moment	$\frac{dK_\nu}{dx} = -\kappa_\nu H_\nu$	$\frac{dK}{dx} = -\bar{\kappa}\,H$

For each equation there is one opacity average that fits “grey equations”, however, all averages are different. Which one to select?

→ For flux constant models with $H(\tau)$ = const. 2nd moment equation is relevant →

Mean opacities: flux-weighted

1st possibility: Flux-weighted mean

$$\boxed{\bar{\kappa} = \frac{\int\limits_0^\infty \kappa_\nu H_\nu \, d\nu}{H}}$$

allows the preservation of the K-integral (radiation pressure)

Problem: H_ν not known a priory (requires iteration of model atmospheres)

$$\frac{dK_\nu}{dx} = -\kappa_\nu H_\nu \quad \Rightarrow \quad \int_0^\infty \frac{1}{\kappa_\nu}\frac{dK_\nu}{dx} d\nu = -\int_0^\infty H_\nu d\nu$$

Mean opacities: Rosseland

2nd possibility: Rosseland mean

to obtain correct integrated energy flux and use local T

$$\int_0^\infty \frac{1}{\kappa_\nu}\frac{dK_\nu}{dx} d\nu = -H \Rightarrow (grey) \Rightarrow \frac{1}{\bar\kappa}\frac{dK}{dx} = -H$$

$$\frac{1}{\bar\kappa} = \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{dK_\nu}{dx} d\nu}{\frac{dK}{dx}}$$

$$K_\nu \rightarrow \frac{1}{3}J_\nu, \quad J_\nu \rightarrow B_\nu \ \text{ as } \tau \rightarrow \infty$$

$$\frac{dK_\nu}{dx} \rightarrow \frac{1}{3}\frac{dB_\nu}{dx} = \frac{1}{3}\frac{dB_\nu}{dT}\frac{dT}{dx}$$

$$\boxed{\frac{1}{\bar\kappa_{Ross}} = \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu(T)}{dT} d\nu}{\int_0^\infty \frac{dB_\nu(T)}{dT} d\nu}}$$

large weight for low-opacity (more transparent to radiation) regions

Mean opacities: Rosseland

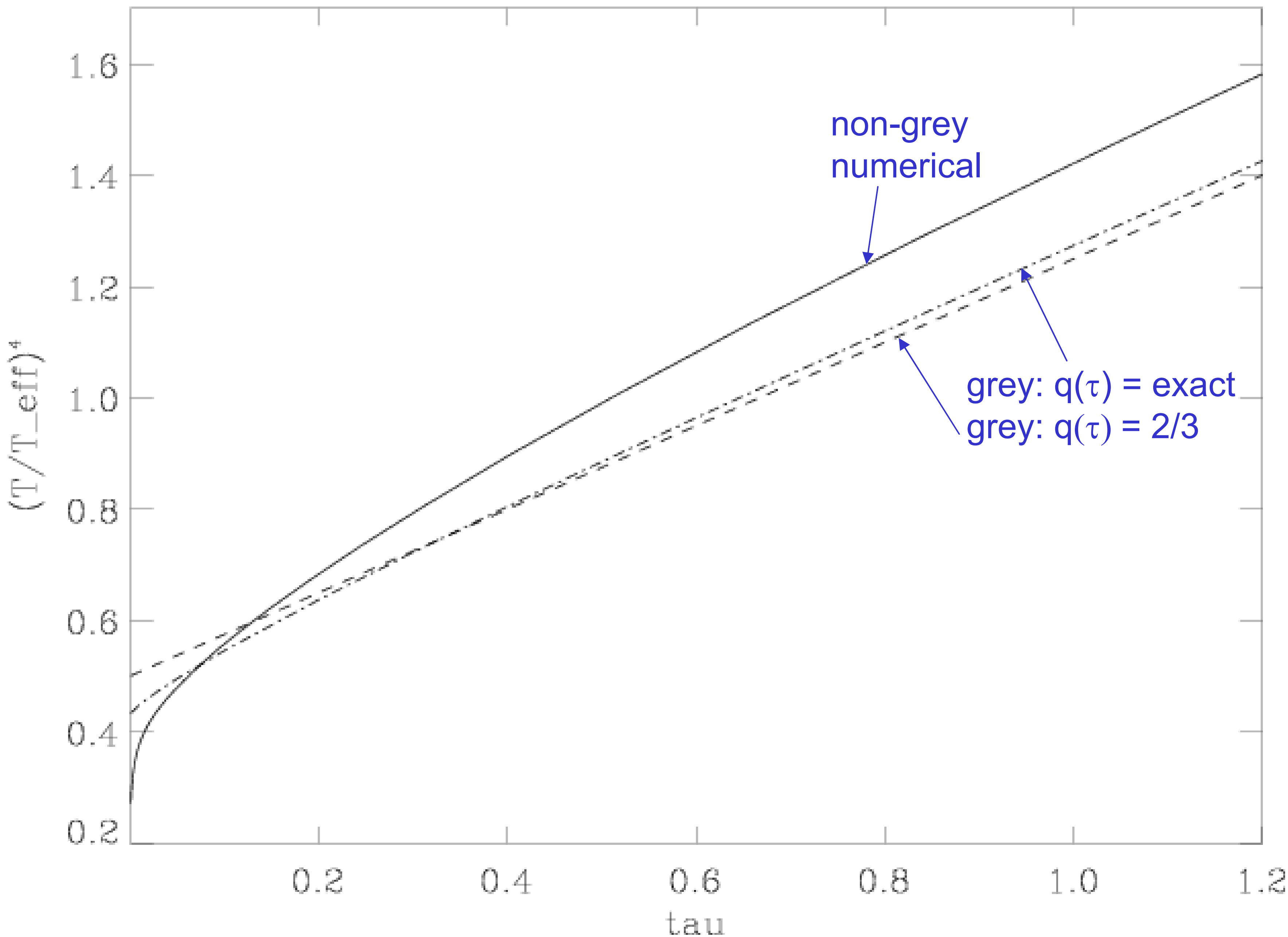
at large τ the T structure is accurately given by

$$T^4 = \frac{3}{4} T_{\rm eff}^4 \left[\tau_{\rm Ross} + q(\tau_{\rm Ross})\right]$$

Rosseland opacities used in stellar interiors

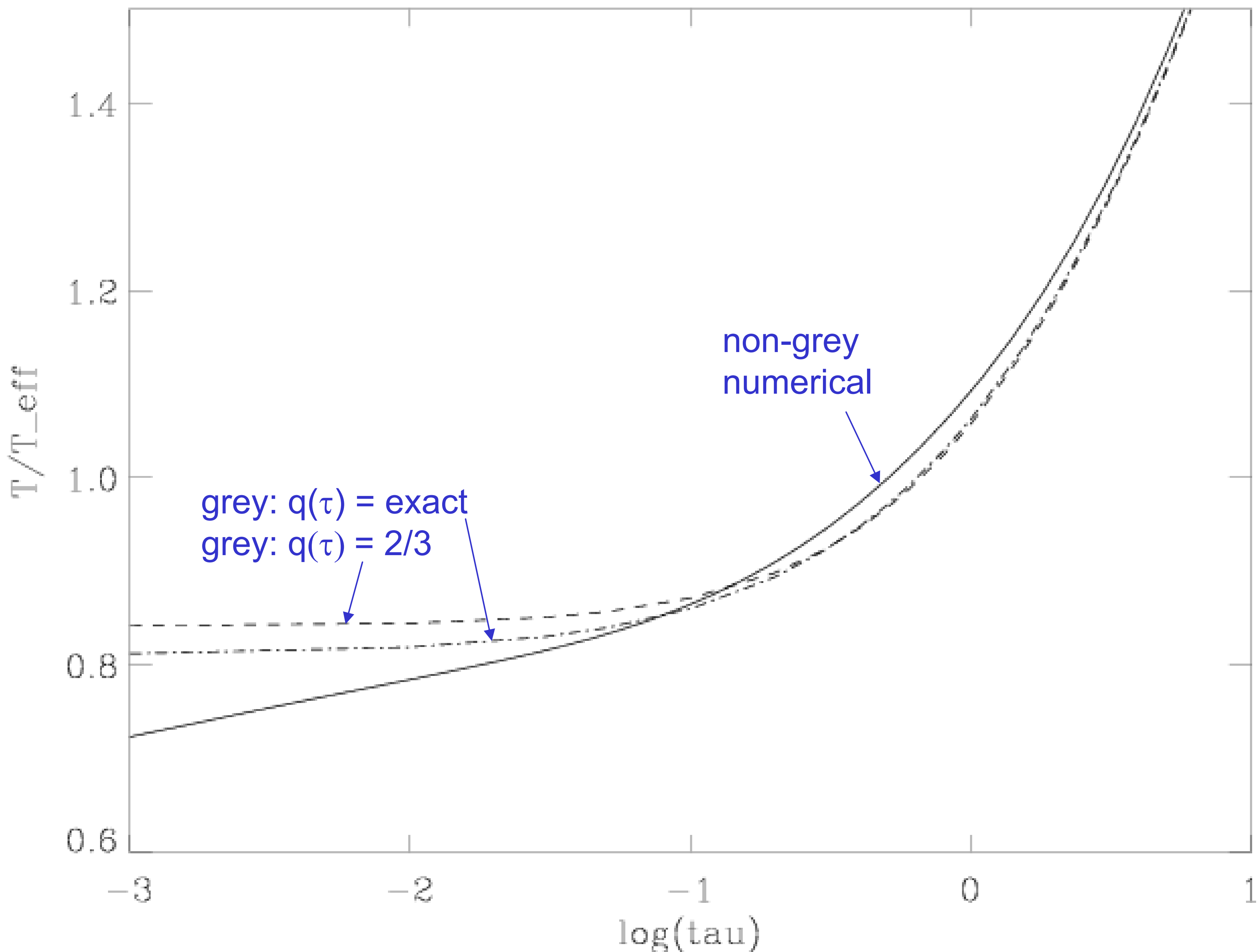
For stellar atmospheres Rosseland opacities allow us to obtain initial approximate values for the Temperature stratification (used for further iterations).

T^4 vs. τ



non-grey
numerical
grey: q(τ) = exact
grey: q(τ) = 2/3
(T/T_eff)⁴
tau
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.0
1.2

T vs. log(τ)



non-grey
numerical
grey: q(τ) = exact
grey: q(τ) = 2/3
T/T_eff
log(tau)
1.4
1.2
1.0
0.8
0.6
−3
−2
−1
0
1

Iterative method for calculation of a stellar atmosphere:

the major parameters are T_{eff} and g

$T(x)$, $\kappa_\nu(x)$, $B_\nu[T(x)]$, $P(x)$, $\rho(x)$

equation of transfer →

$J_\nu(x)$, $H_\nu(x)$

$$\int_0^\infty \kappa_\nu (J_\nu - B_\nu)\, d\nu = 0\ ?$$

$$4\pi \int_0^\infty H_\nu\, d\nu = \sigma\, T_{\rm eff}^4\ ?$$

$\Delta T(x)$, $\Delta\kappa_\nu(x)$, $\Delta B_\nu[T(x)]$, $\Delta\rho(x)$

a. hydrostatic equilibrium

$$\frac{dP}{dx} = -g\rho(x)$$

b. equation of radiation transfer

$$\mu \frac{dI_\nu}{dx} = -(\kappa_\nu + \sigma_\nu)\,(I_\nu - S_\nu)$$

c. radiative equilibrium

$$\int_0^\infty \kappa_\nu \,[J_\nu - B_\nu(T)]\, d\nu = 0$$

d. flux conservation

$$4\pi \int_0^\infty H_\nu\, d\nu = \sigma T_{\rm eff}^4$$

e. equation of state

$$P = \frac{\rho\, k\, T}{\mu\, m_H}$$