1 Sound waves

sound eq initial $v_0 = 0, \rho_0, p_0$
perturbations $(p\prime, \rho\prime, v)$
$p = p_0 + p\prime$
$\rho = \rho_0 + \rho\prime$
Cont eq lin:
$\frac{\partial \rho\prime}{\partial t} + \rho_0 \nabla \cdot v = 0$
Euler lin:
$\frac{\partial v}{\partial t} + \frac{1}{\rho_0} \nabla p\prime = 0$
def:
$c_s = \left(\frac{\partial p}{\partial \rho}\right)_s$
energy eq: adiabatic process:
$p\prime = c_s^2 \rho\prime$
$\Longrightarrow$

1.1 sound equation in inhomogeneous

$\frac{\partial}{\partial t}\left(\frac{1}{c_s^2(x,t)} \frac{\partial p}{\partial t}\right) = \nabla^2 p$
Time independent:
$\frac{1}{c_s^2(x)} \frac{\partial^2 p}{\partial t^2} = \nabla^2 p$

Eikonal solution approx $p(x,t) = a(x,t) e^{i\phi(x,t)}$
Notations:
$\omega(x,t) = -\frac{\partial \phi}{\partial t}$
$k(x,t) = -\nabla \phi$
Reemplazano p en la ecuación y definiendo $c_g = \frac{\partial \omega}{\partial k}$:
$\omega^2 = c_s^2 k^2$
$\frac{\partial a}{\partial t} + c_g \cdot \nabla a = -\frac{1}{2} \frac{a}{|k| c_s} \left(\frac{\partial \omega}{\partial t} + c_s^2 \cdot \nabla k\right)$
$\Longrightarrow$
$\frac{\partial \omega}{\partial t} + c_g \cdot \nabla \omega = 0$
$\frac{\partial k}{\partial t} + c_g \cdot \nabla k = -k \cdot \nabla c_g$
Energy conservation:
$\frac{\partial E}{\partial t} + c_g \cdot \nabla E = -E \nabla \cdot c_g$
$E(x,t) = \frac{|a|^2}{\rho_0 c_s^2}$
1D $(c_s = c_g)$:
$\frac{\partial k}{\partial t} + c_s \frac{\partial k}{\partial x} = -c_s$
$\frac{\partial \omega}{\partial t} + c_s \frac{\partial \omega}{\partial x} = 0$
$\frac{\partial k}{\partial t} + c_s \frac{\partial k}{\partial x} = -k \frac{\partial c_s}{\partial x}$
$\frac{\partial E}{\partial t} + c_s \frac{\partial E}{\partial x} = -E \frac{\partial c_s}{\partial x}$
rays characterictics method:
para un rayo (trayectoria):
solución de la la ecuación
$\frac{dx}{dt} = c_s$
$x_p(t), x_p(0) = x_p$
def
$\omega_p(t) = \omega(x_p(t), t)$
$k_p(t) = k(x_p(t), t)$
$a_p(t) = a(x_p(t), t)$
$\frac{d\omega_p}{dt} = 0$
$\frac{dk_p}{dt} = -k_p \frac{\partial c_s}{\partial x}(x_p(t))$
$\frac{dc_s}{dt} = \frac{\partial c_s}{\partial x}(x_p(t)) c_s$
$\Longrightarrow$
$\frac{dk_p}{dt} = -k_p \frac{1}{c_s} \frac{dc_s}{dt}$
$\Longrightarrow$
$\frac{dln(k_p)}{dt} = -\frac{dln(c_s)}{dt}$
$\Longrightarrow$
$k(x_p(t), t) cs(x_p(t)) = constant$
$E(x_p(t), t) cs(x_p(t)) = constant$

$\omega(x_p(t), t) = constant$
v, p solutions in WKB approx (con amplitudes V y P) ($v = Ve^{i\phi}$ $p = Pe^{i\phi}$)
introducimos
$\rho\prime \frac{\partial v}{\partial t} = -\nabla p\prime$
$\frac{\rho_0 |V|^2}{2} = \frac{|P|^2}{2\rho_0 c_s^2}$
$\Longrightarrow$
$E = \frac{|A|^2}{\rho_0 c_s^2}$

2 Transf fourier

Salida de mathematica de la integral de la transf fourier:

```
$Assumptions = {Element[{k0,z0,zf,zc,W}, Reals], k0>0, z0>0, zf >0 , zf >0, zc >0, W>0 }
Print[FullSimplify[Integrate[Exp[- (z-zc)^2 / W^2] Cos[2 Pi  k0 (z - z0)/ (zf - z0)] Exp[- 2 Pi I m z
/ (zf - z0)],  {z, -Infinity, Infinity} ]]]
```

$$f1(m) = F(\frac{2\pi m}{z_f - z_0}) = \int_{-\infty}^{\infty} e^{-\frac{(z-z_c)^2}{W^2}} cos(\frac{2\pi k_0 (z - z_0)}{z_f - z_0}) e^{-\frac{2\pi i m z}{z_f - z_0}} dz =$$
$$\frac{1}{2} e^{-\frac{\pi\left(k_0{}^2 \pi W^2 + m\left(m\pi W^2 - 2 i z_c (z_0 - z_f)\right) + 2k_0\left(m\pi W^2 + i(z_0 + z_c)(z_0 + z_f)\right)\right)}{(z_0 - z_f)^2}} \left(e^{\frac{4 i k_0 \pi z_0 (z_c + z_f)}{(z_0 - z_f)^2}} + e^{\frac{4 k_0 \pi\left(m\pi W^2 + i\left(z_0{}^2 + z_c z_f\right)\right)}{(z_0 - z_f)^2}}\right) \sqrt{\pi} W$$

Salida de FourierTransform mathematica con FourierParameters 0, -2π

```
$Assumptions = {Element[{k0,z0,zf,zc,W}, Reals], k0>0, z0>0, zf >0 , zf >0, zc >0, W>0 }
h[z_, k0_, z0_, zf_, zc_, W_]:= Exp[-(z-zc)^2/W^2] Cos[2 Pi k0 (z - z0) / (zf - z0) ]
Print[FullSimplify[FourierTransform[h[z, k0, z0, zf, zc, W], z, k, FourierParameters->{0,-2 Pi}]] ]
```

$$f2(k) = \frac{1}{2}\left(e^{-\frac{\pi\left(k_0{}^2\pi W^2 - 2k_0\left(k\pi W^2 - i z_0 + i z_c\right)(z_0 - z_f) + k\left(k\pi W^2 + 2 i z_c\right)(z_0 - z_f)^2\right)}{(z_0 - z_f)^2}} + e^{-\frac{\pi\left(k_0{}^2\pi W^2 + 2k_0\left(k\pi W^2 - i z_0 + i z_c\right)(z_0 - z_f) + k\left(k\pi W^2 + 2 i z_c\right)(z_0 - z_f)^2\right)}{(z_0 - z_f)^2}}\right)\sqrt{\pi} W$$

En esta reemplazo k = k / (zf - z0) y los graficos salen iguales ($f1(k) = f2(\frac{k}{z_f - z0})$)
Ademas la salida de:

```
$Assumptions = {Element[{k0,z0,zf,zc,W}, Reals], k0>0, z0>0, zf >0 , zf >0, zc >0, W>0 }

f1[m_]:=((E^(((4*I)*k0*Pi*z0*(zc + zf))/(z0 - zf)^2) + E^((4*k0*Pi*(m*Pi*W^2 + I*(z0^2 + zc*zf)))/(z0
- zf)^2))*Sqrt[Pi]*W)/
  (2*E^((Pi*(k0^2*Pi*W^2 + m*(m*Pi*W^2 - (2*I)*zc*(z0 - zf)) + 2*k0*(m*Pi*W^2 + I*(z0 + zc)*(z0 + zf))))/(
- zf)^2))
f2[k_]:=((E^(-((Pi*(k0^2*Pi*W^2 - 2*k0*(k*Pi*W^2 - I*z0 + I*zc)*(z0 - zf) + k*(k*Pi*W^2 + (2*I)*zc)*(z0
- zf)^2))/(z0 - zf)^2)) +
      E^(-((Pi*(k0^2*Pi*W^2 + 2*k0*(k*Pi*W^2 - I*z0 + I*zc)*(z0 - zf) + k*(k*Pi*W^2 + (2*I)*zc)*(z0 -
zf)^2))/(z0 - zf)^2)))*Sqrt[Pi]*W)/2

Print[FullSimplify[f1[k]-f2[k/(zf-z0)]]]
```

es 0
Elijo f2 forma para simplificar (después de hacer los gráficos de los modulos de los valores de la función , tal como imaginaba la primera exponencial corresponde a la gaussiana de las frecuancias negativas y la segunda de las frequencias positivas):

$$f1(k) = f2(\frac{k}{(z_f - z_0)}) =$$
$$\frac{W\sqrt{\pi}}{2}(e^{-\frac{\pi(k_0^2\pi W^2 + 2k_0 k\pi W^2 + 2k_0 i(z_c - z_0)(z_f - z_0) + k^2\pi W^2 + 2ikz_c(z_f - z_0))}{(z_0 - z_f)^2}} + e^{-\frac{\pi(k_0^2\pi W^2 - 2k_0 k\pi W^2 - 2k_0 i(z_c - z_0)(z_f - z_0) + k^2\pi W^2 + 2ikz_c(z_f - z_0))}{(z_0 - z_f)^2}}) =$$
$$\frac{W\sqrt{\pi}}{2}(e^{-\frac{\pi\left[\pi W^2(k + k_0)^2 + 2k_0 i z_c(z_f - z_0) - 2k_0 i z_0(z_f - z_0) + 2ikz_c(z_f - z_0)\right]}{(z_0 - z_f)^2}} + e^{-\frac{\pi\left[\pi W^2(k - k_0)^2 - 2k_0 i z_c(z_f - z_0) + 2k_0 i z_0(z_f - z_0) + 2ikz_c(z_f - z_0)\right]}{z_0 - z_f)^2}}) =$$
$$\frac{W\sqrt{\pi}}{2}(e^{-\frac{\pi\left[\pi W^2(k + k_0)^2 + 2iz_c(z_f - z_0)(k_0 + k) - 2k_0 i z_0(z_f - z_0)\right]}{(z_0 - z_f)^2}} + e^{-\frac{\pi\left[\pi W^2(k - k_0)^2 + 2iz_c(z_f - z_0)(k - k_0) + 2k_0 i z_0(z_f - z_0)\right]}{(z_0 - z_f)^2}}) =$$

Considerando $z_c = 0$ las exponenciales son gaussianas con w $= \frac{z_f - z_0}{\pi W}$ la primera centrada en $-k_0$ y la segunda en k_0 y cuando calculamos el modulo las constantes $abs(exp(-2k_0 i z_0(z_f - z_0))) = abs(exp(2k_0 i z_0(z_f - z_0))) = 1$ y la amplitud queda $\frac{W\sqrt{\pi}}{2}$ igual que se ve en el gráfico (con valores : z0=3.100, zf = 7.400, k0 = 60, zc = 3.745, W = 0.050) : con rojo había hecho el plot de la función entera y con verde y azul de las 2 partes al principio para estar segura que correspondían a las 2 partes