Fourier Transform

- Fourier transform converts a physical-space (or time series) representation of a function into frequency-space
  - Equivalent representation of the function, but gives a new window into its behavior
  - Inverse operation exists

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x k} dx$$

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{2\pi i x k} dk$$

- You can think of $F(k)$ as being the amount of the function f represented by a frequency k

Fourier Transform

- For discrete data, the discrete analog of the Fourier transform gives:
  - Amplitude and phase at discrete frequencies (wavenumbers)
  - Allows for an investigation into the periodicity of the discrete data
  - Allows for filtering in frequency-space
  - Can simplify the solution of PDEs: derivatives change into multiplications in frequency space.

Discrete Fourier Transform

- The discrete Fourier transform (DFT) operates on discrete data
  - Usually we have evenly-spaced, real data
    - E.g. a time-series from an experiment
    - Simulation data for a velocity field
- DFT transforms the N spatial/temporal points into N frequency points
  - Transform: $$F_k = \sum_{n=0}^{N-1} f_n e^{-2\pi i n k/N}$$
  - Inverse: $$f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{2\pi i n k/N}$$
  - This is the form used in NumPy, Garcia, and others. Pang splits the normalization

Notation

- Many different notations are used in various texts
  - Original function: $f(x)$ or $f(t)$
  - Transformed function: $F(k), \mathcal{F}(k), \hat{f}(k)$
- For the discrete version:
  - Original function: f_n
  - Transformed function: $F_k, \mathcal{F}_k, \hat{f}_k$

Real Space vs. Frequency Space

- What are the significance of the Re and Im parts?
  - Recall that we are integrating with $e^{-2\pi i k x}$
  - Euler's formula: $e^{i x} = \cos(x) + i \sin(x)$
  - Real part represents the cosine terms, symmetric functions
  - Imaginary part represents the sine terms, antisymmetric functions
  - Can also think in terms of an amplitude and a phase
  - If f_n is Real, then:

$$\mathrm{Re}(F_k) = \sum_{n=0}^{N-1} f_n \cos\left(\frac{2\pi n k}{N}\right)$$

$$\mathrm{Im}(F_k) = \sum_{n=0}^{N-1} f_n \sin\left(\frac{2\pi n k}{N}\right)$$

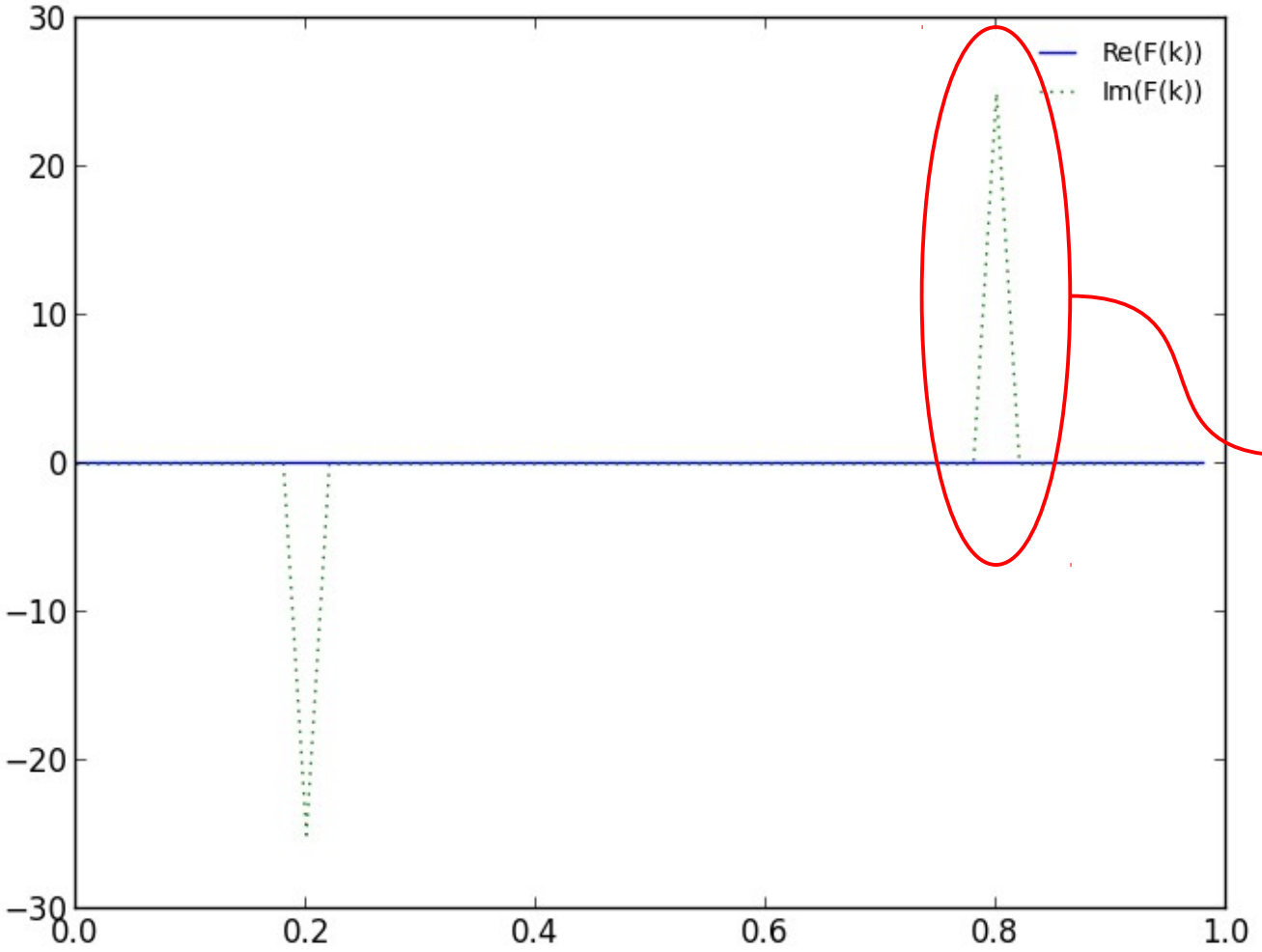
DFT Example

- Implementing the discrete Fourier transform is simple
  - Double sum: for each wavenumber, we sum over all the spatial points

```
def dft(f_n):

    N = len(f_n)
    f_k = numpy.zeros( (N), dtype=numpy.complex128)

    k = 0
    while k < N:

        n = 0
        while n < N:
            f_k[k] += f_n[n]*numpy.exp(-2.0*math.pi*1j*n*k/N)
            n += 1

        k += 1

    return f_k
```

DFT Example

- DFT of $\sin(2\,\pi\,f_0\,x)$ with $f_0 = 0.2$



We'll talk about this extra signal in a moment

DFT Example

- Note that in the DFT, nowhere does the physical coordinate value, x_n , enter—instead, we just look at the index n itself
  - This assumes that the data is regularly gridded
- In this index space, the smallest wavelength is from one cell to the next, and the smallest frequency is $1/N$
  - Note that this means that if we add points to our data, then we open up higher and higher frequencies (in terms of index space)
- Clearly there is a physical scale for the frequency

$$\nu_k = \frac{k}{N} \cdot \frac{1}{\Delta x} = \frac{k}{N} \cdot \frac{N}{L}$$

  - Lowest frequency: $1/L$
  - Highest frequency ~ $1/\Delta x$

DFT Example

- $k = 0$ is special:

$$\mathrm{Re}(F_0) = \sum_{n=0}^{N-1} f_n \cos\left(\frac{2\pi n 0}{N}\right) = \sum_{n=0}^{N-1} f_n$$

$$\mathrm{Im}(F_k) = \sum_{n=0}^{N-1} f_n \sin\left(\frac{2\pi n 0}{N}\right) = 0$$

  - This is sometimes termed the DC offset

FFT = DFT

- The Fast Fourier Transform (FFT) is equivalent to the discrete Fourier transform
  - Faster because of special symmetries exploited in performing the sums
  - O(N log N) instead of $O(N^2)$
- Both texts offer a reasonable discussion on how the FFT works—we'll defer it to those sources.

Normalization

- Normalization of the forward and inverse transforms follows from Parseval's theorem:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(k)|^2 dk$$

  - Discrete form(taking $\Delta x = 1$ and $\Delta k = 1/N$)

$$\sum_{n=0}^{N-1} |f_n|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |F_k|^2$$

- The definition of the inverse transformation compatible with this condition is

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{2\pi i n k/N}$$

Normalization

- To illustrate this, consider the transform of 1
  - Analytically: $f(x) = 1 \Longleftrightarrow F(k) = \delta(k)$
  - Discrete transform:

$$F_k = \sum_{n=0}^{N-1} 1 \cdot e^{-2\pi i k n/N} = \sum_{n=0}^{N-1} [\cos(2\pi k n/N) + i \sin(2\pi k n/N)]$$

    - For k = 0, cos(0) = 1, sin(0) = 0, so $F_0 = N$
    - For k > 0, we are essentially doing a discrete sum of the cosine and sine using equally-spaced points, and the sum is always over a multiple of a full wavelength, therefore, $F_k = 0$
  - Discrete inverse transform:

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} \underbrace{F_k} e^{2\pi i n k/N} = \frac{1}{N} N e^{2\pi i n 0/N} = 1$$

only non-zero for k = 0

Normalization

- When plotting, we want to put the physical scale back in, as well as make the correspondence to the continuous representation
  - Already saw how to put the wavenumbers into a physical frequency
  - Rewrite inverse expression:

This is what we plot

$$f_n = \frac{1}{N}\sum_{n=0}^{N-1} F_k e^{2\pi i n k/N} = \underbrace{\sum_{n=0}^{N-1} \left(\frac{F_k}{N}\right) e^{2\pi i n k/N}}$$

This looks like the continuous form

  - Another interpretation: $F_k \,/\, N \Leftrightarrow F_k\, dk$

Let's look at code that transforms 1...

Real Space vs. Frequency Space

- Imagine transforming a real-valued function, f , with N data points
  - The FFT of f will have $N/2$ complex data points
  - Same amount of information in each case, but each complex point corresponds to 2 real numbers.
    - This is a consequence of the analytic Fourier transform satisfying $F(-k) = F^{\star}(k)$ if $f(x)$ is real
  - Most FFT routines will return N complex points—half of them are duplicate, i.e. they don't add to the information content
    - Often, there are specific implementations optimized for the case where the function is real (e.g. rfft)
  - This affects normalization (note k=0 different)
- This is also referred to as aliasing. The maximum frequency, $1/(2\ \Delta x)$ is called the Nyquist frequency

Power Spectrum

- The power spectrum is simply defined as:

$$P(k) = |F(k)|^2 = F(k)F^\star(k)$$

  - Single number showing the importance/weight of a given wavenumber
- Also useful to look at the phase at each wavenumber

Python/NumPy's FFT

- `numpy.fft`: http://docs.scipy.org/doc/numpy/reference/routines.fft.html
- `fft`/`ifft`: 1-d data
  - By design, the k=0, ... N/2 data is first, followed by the negative frequencies. These later are not relevant for a real-valued f(x)
  - k's can be obtained from `fftfreq(n)`
  - `fftshift(x)` shifts the k=0 to the center of the spectrum
- `rfft`/`irfft`: for 1-d real-valued functions. Basically the same as `fft`/`ifft`, but doesn't return the negative frequencies
- 2-d and n-d routines analogously defined

Python's FFT

- It's always a good idea to run some simple tests to make sure the FFT is behaving the way you expect
  - $\sin(2\pi f_0 x)$ —should be purely imaginary at a single wavenumber
  - $\cos(2\pi f_0 x)$ —should be purely real at a single wavenumber
  - $\sin(2\pi f_0 x + \pi/4)$ —should have equal magnitude real and imaginary parts at a single wavenumber

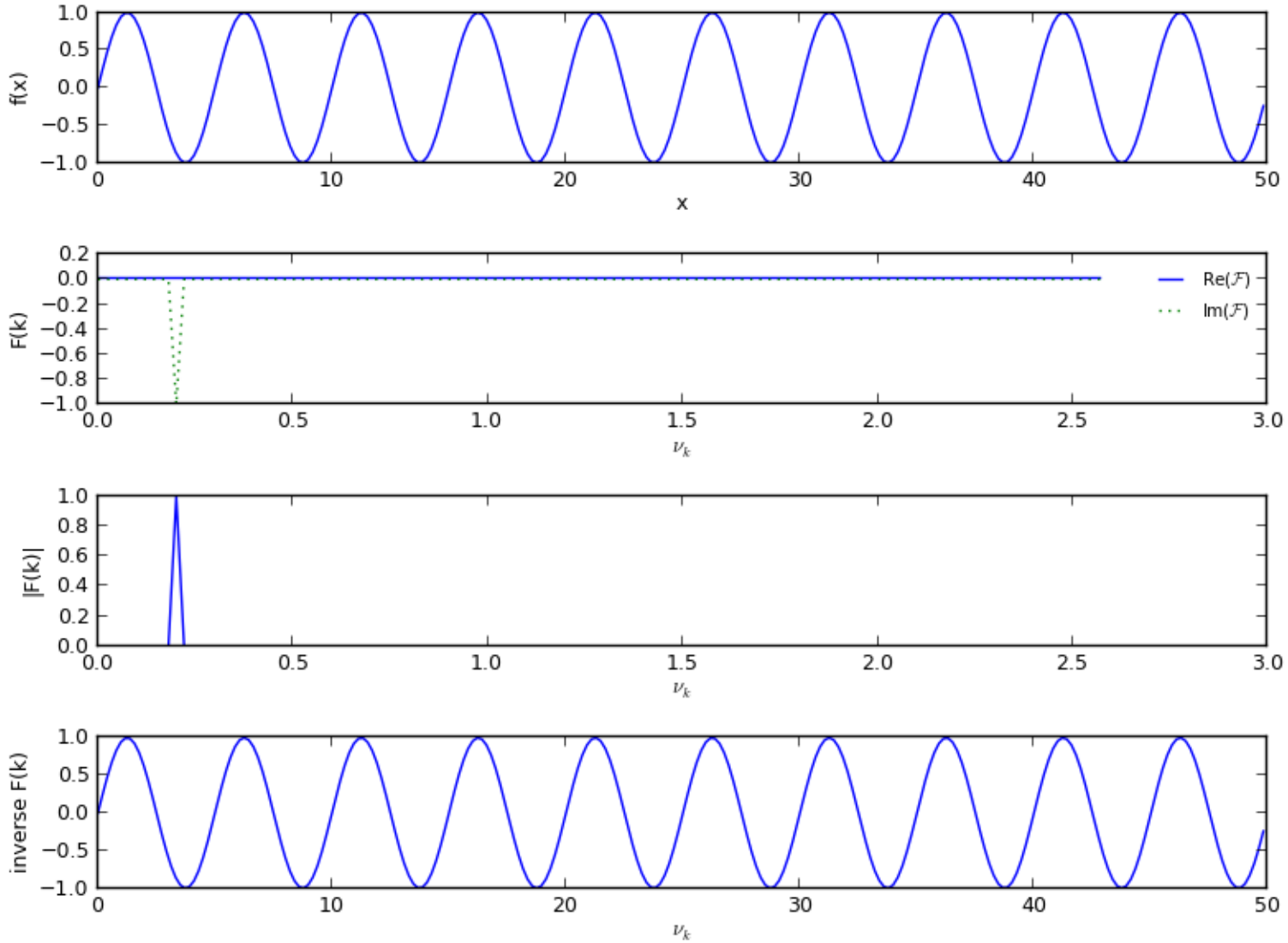
Ex: Single Frequency Sine

Transform a single mode sine ($f_0 = 0.2$) and inverse transform back—do you get the original result (to roundoff)?

Notice that the since sine is odd, the FFT is only non-zero in the imaginary component.

And... we get back our original component.

What happens when we increase the # of points?



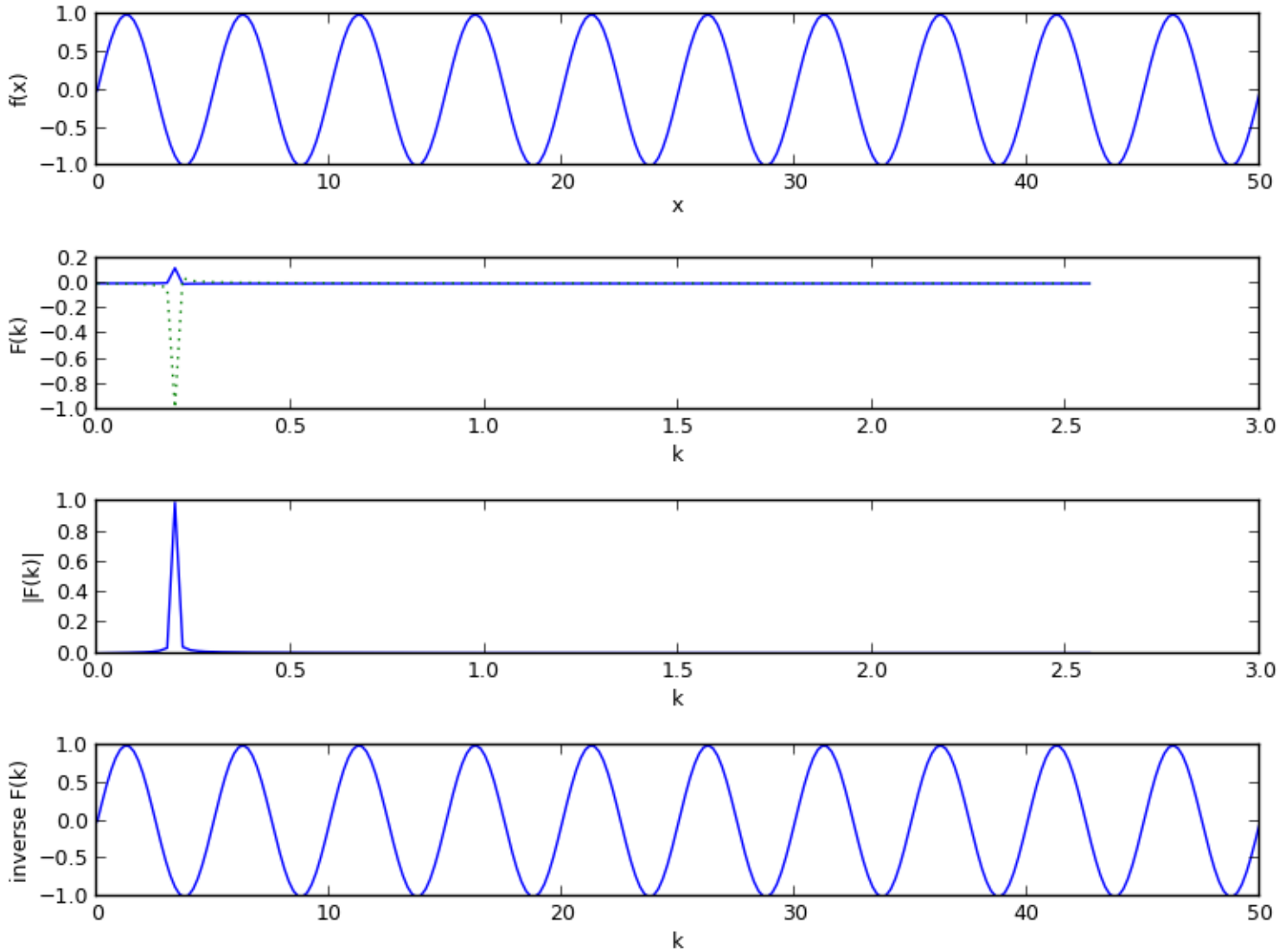
Gotyas...

- Be careful about how you define the points
  - The FFT considers the data without any spatial coordinates—it just considers distance in terms of the number of points
  - Using the NumPy `linspace()` routine puts a point at both the start and end of the interval
    - e.g., `np.linspace(0, 1, 5) = [ 0. 0.25 0.5 0.75 1.]`
    - The FFT routine treats the first and last point as distinct
    - If you define $\sin(2\pi\ x)$ on this data, the first and last point will be equivalent, and the FFT picks up an extra (non-periodic) signal
  - You should do `np.linspace(0, 1, 5, endpoint=False)` instead

Gotyas...

FFT with linspace putting a point at both endpoints.

Note the blip in the real portion of the FFT.

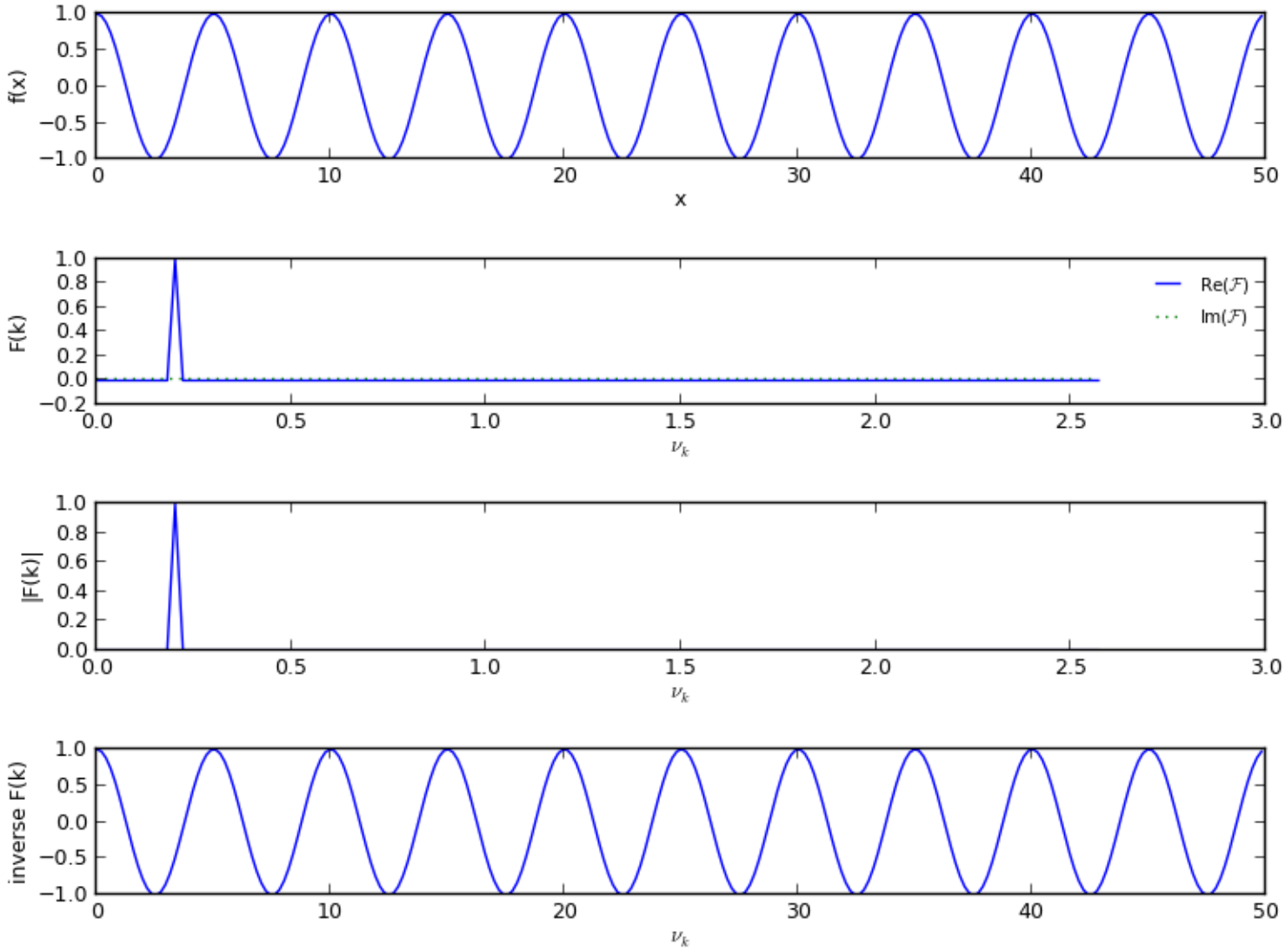


Ex: Single Frequency Cosine

- And a cosine is “opposite” of the sine

Notice that the since cosine is even, the FFT is only non-zero in the real component.

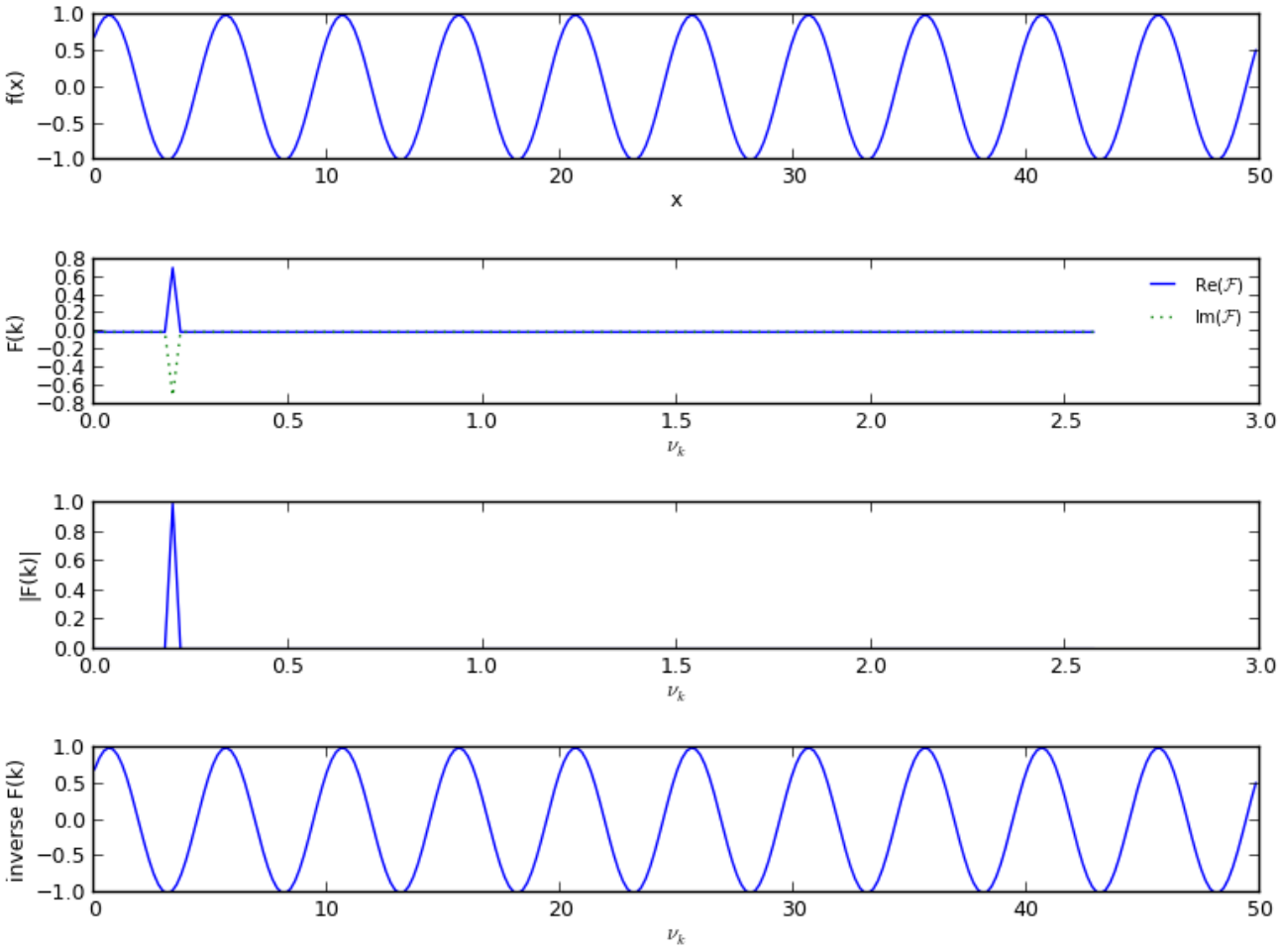
Also note the normalizations



Ex: Phase Shift

- Amplitude and phase. Consider: $\sin(2\pi\ f_0\ x + \pi/4)$
- Phase:

$$\phi = \tan^{-1}\left(\frac{\mathrm{Im}(F_k)}{\mathrm{Re}(F_k)}\right)$$



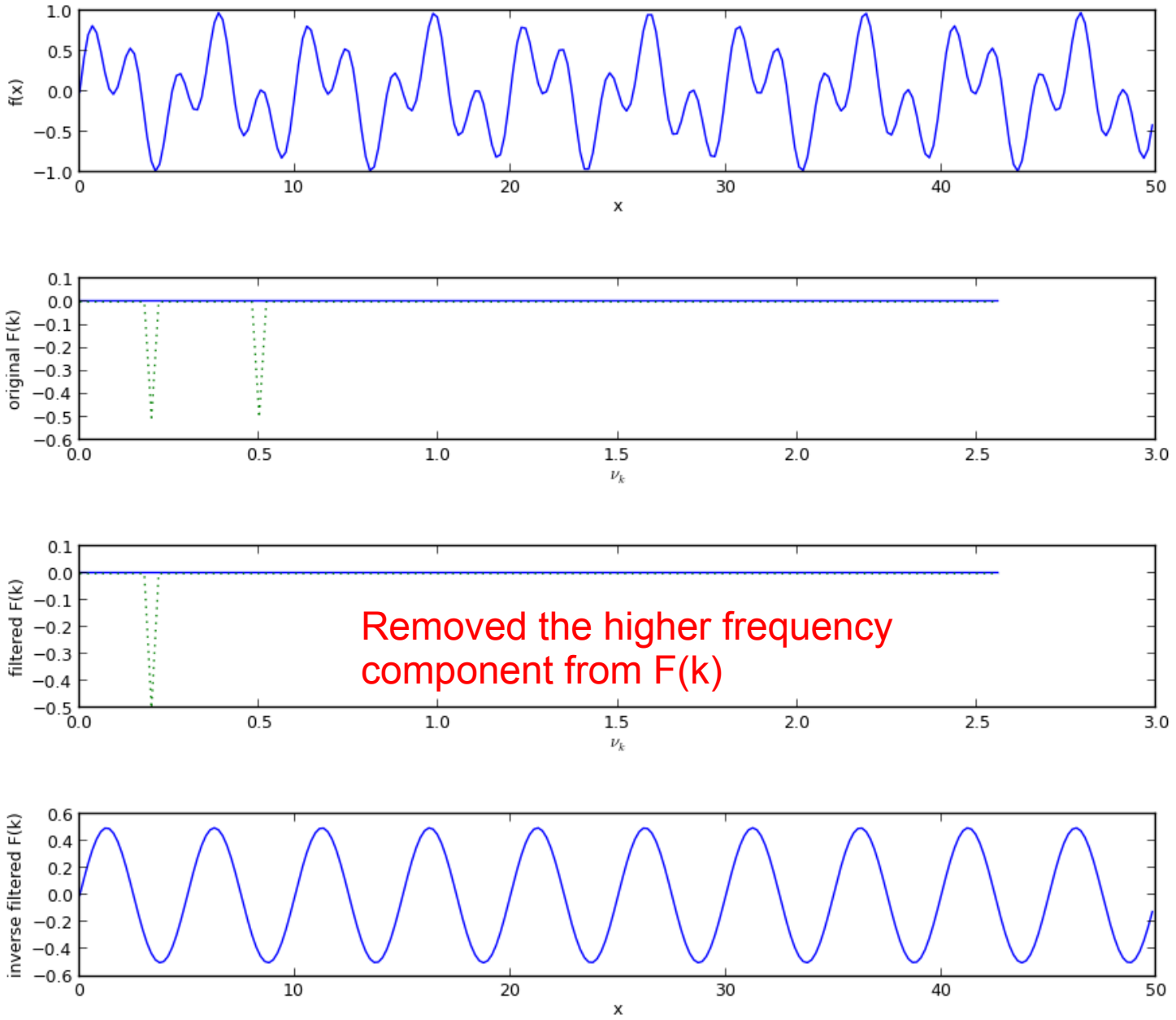
Example: Filtering

- In frequency-space, we can filter out high or low frequency components.
- Consider:

$$f(x) = \frac{1}{2}\left[\sin(2\pi f_0 x) + \sin(2\pi f_1 x)\right]$$

  - With $f_0 = 0.2, f_1 = 0.5$

Example: Filtering

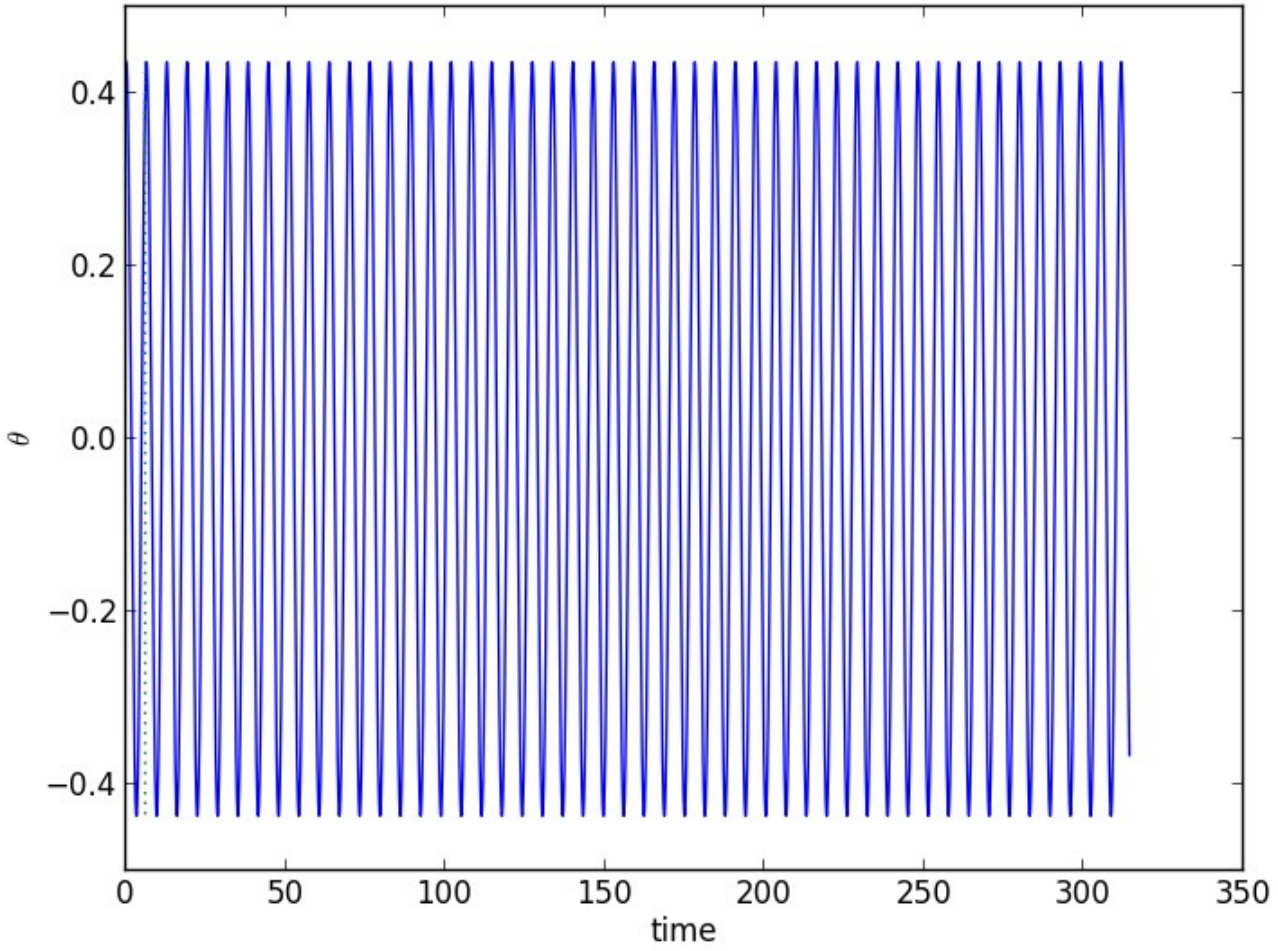


Example: Time-Series Analysis

- Remember our pendulum problem from homework #2
  - Finite-amplitude system
  - Here's the solution with $\theta_0 = 25°$

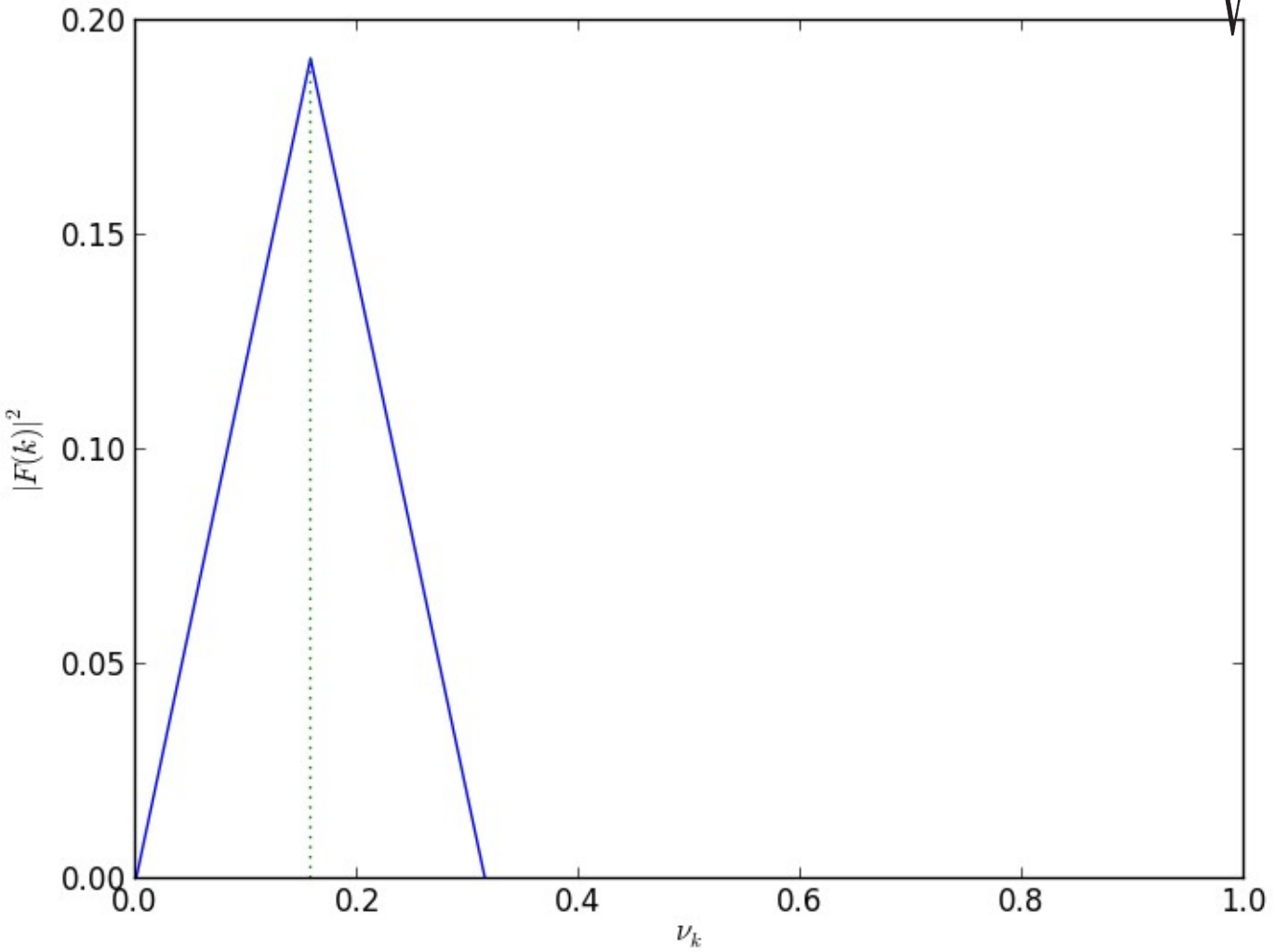
50 periods integrated (using small-angle estimate of period)

12,568 points from 4th-order Runge-Kutta



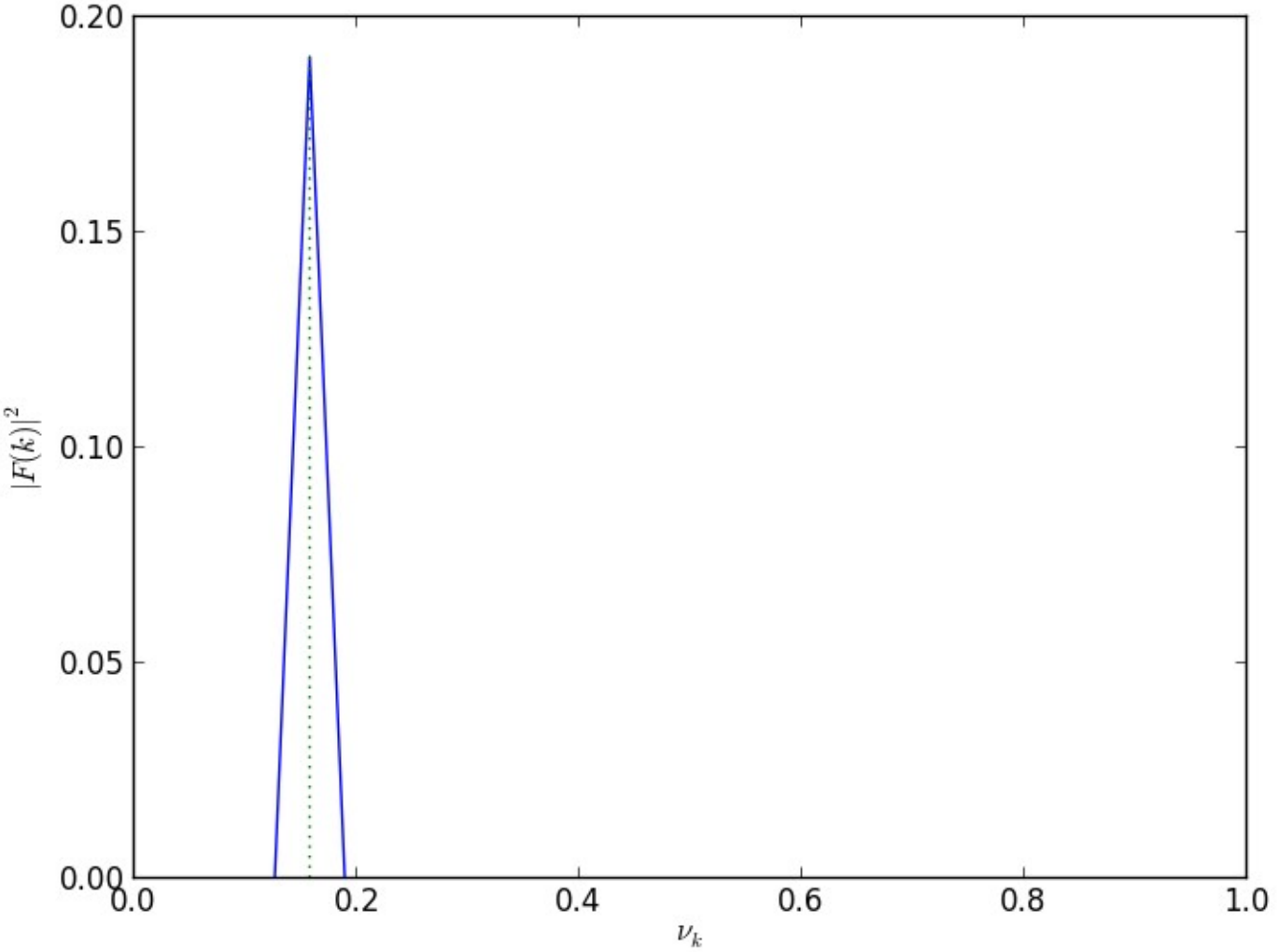
Example: Time-Series Analysis

- Power spectrum of the first period
  - Dotted line is analytic freq estimate: $f^{-1} = T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{\theta_0^2}{16}\right)$



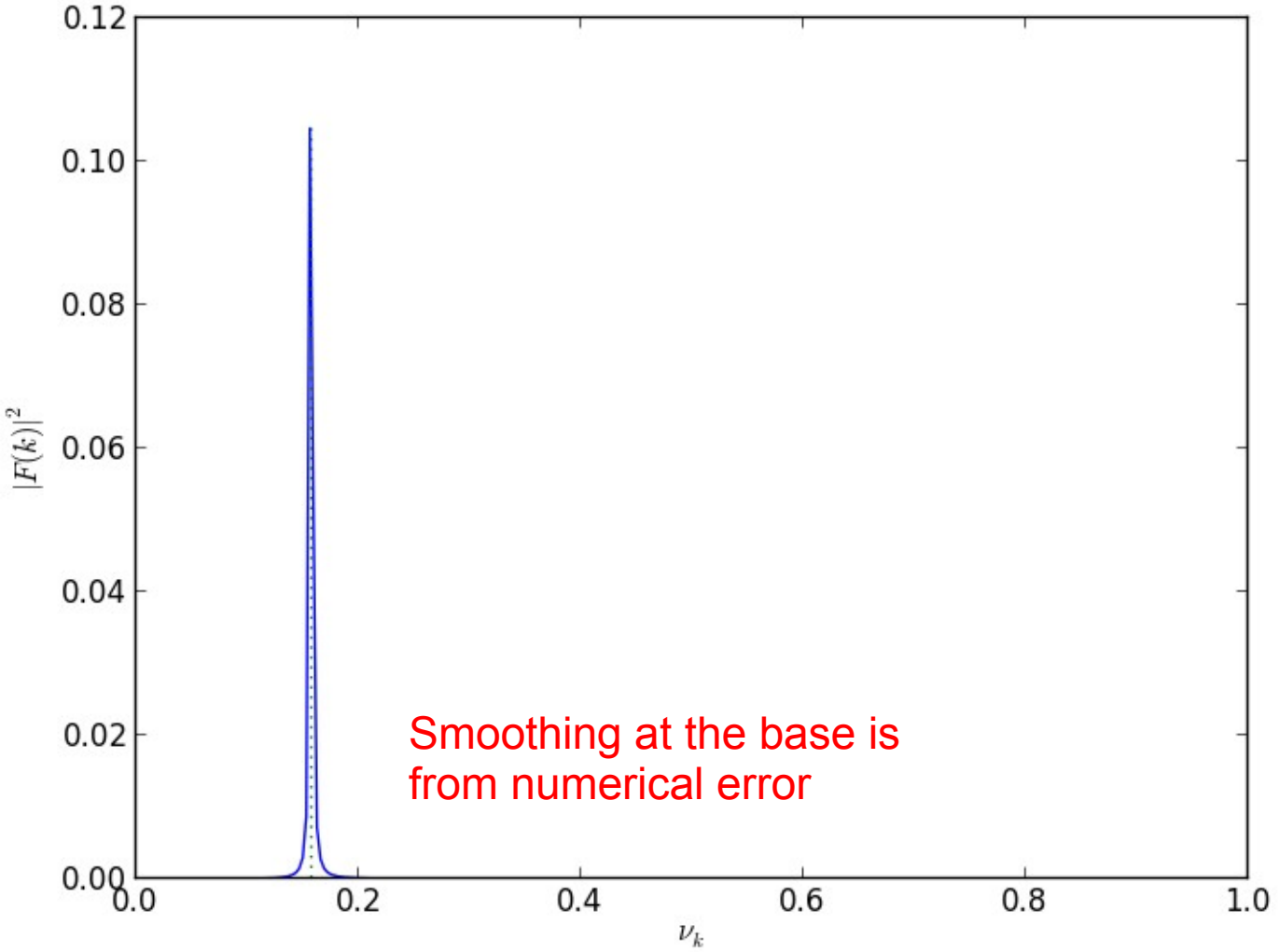
Example: Time-Series Analysis

- Power spectrum of the first 5 periods



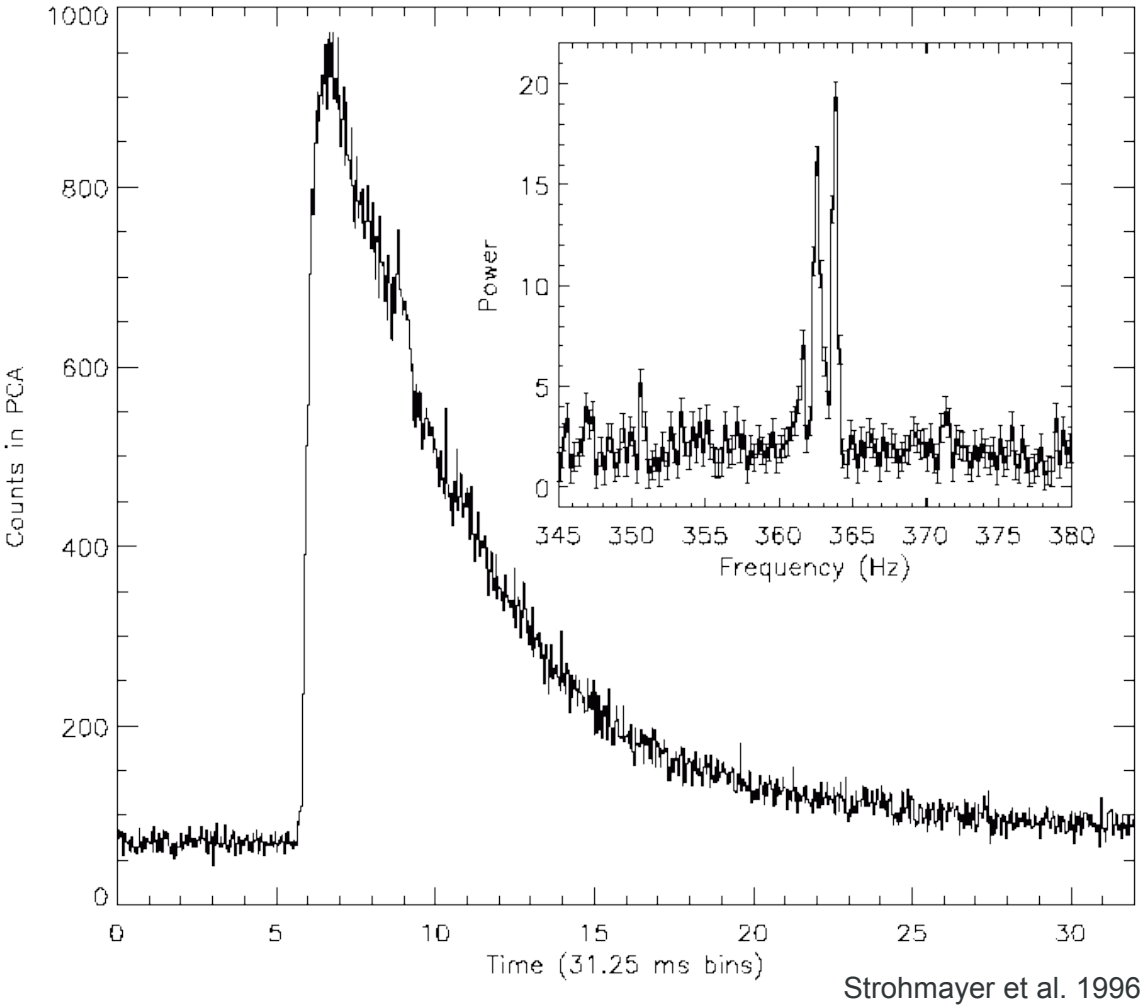
Example: Time-Series Analysis

- Power spectrum of the all 50 periods



Example: Time-Series Analysis

- Here's an astronomical example from an X-ray burst time-series



Strohmayer et al. 1996

Multidimensional FFT

- Multidimensional FFTs decompose into a sequence of one-dimensional FFTs

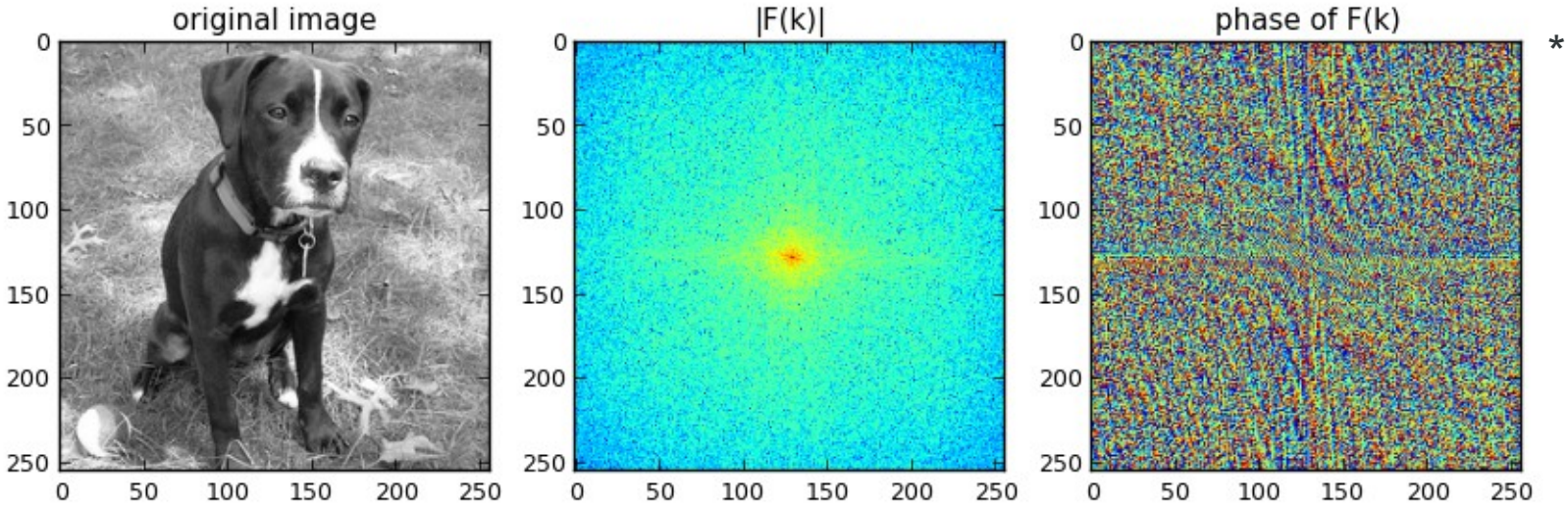
$$F_{k_x,k_y} = \sum_{m=0}^{N_x-1} \sum_{n=0}^{N_y-1} f_{mn} e^{-2\pi i(k_x m/N_x + k_y n/N_y)}$$

$$= \sum_{m=0}^{N_x-1} e^{-2\pi i k_x m/N_x} \underbrace{\sum_{n=0}^{N_y-1} f_{mn} e^{-2\pi i k_y n/N_y}}$$

This is the transform in the y-direction

Multidimensional FFT

- To visualize what is going on, we need to look at both the amplitude and the phase
  - Note that only 1 quadrant is significant, because our input was real-valued.



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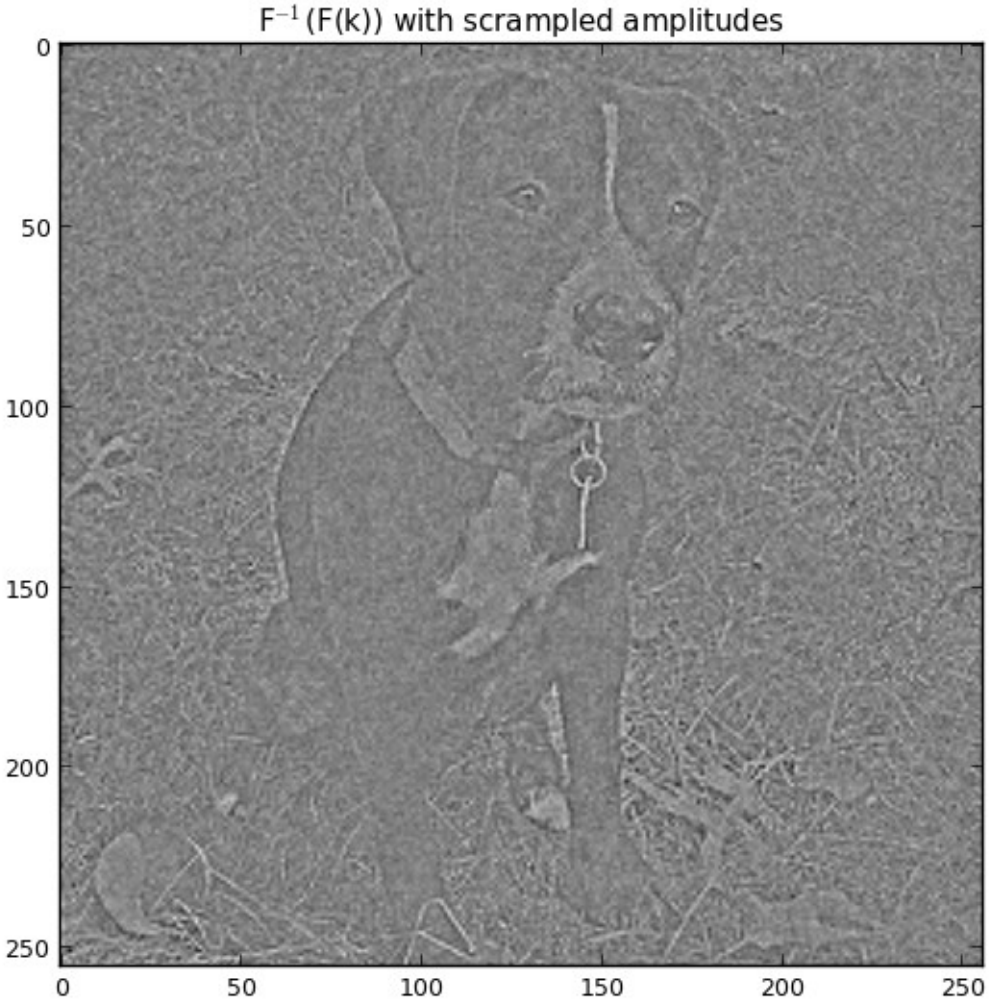
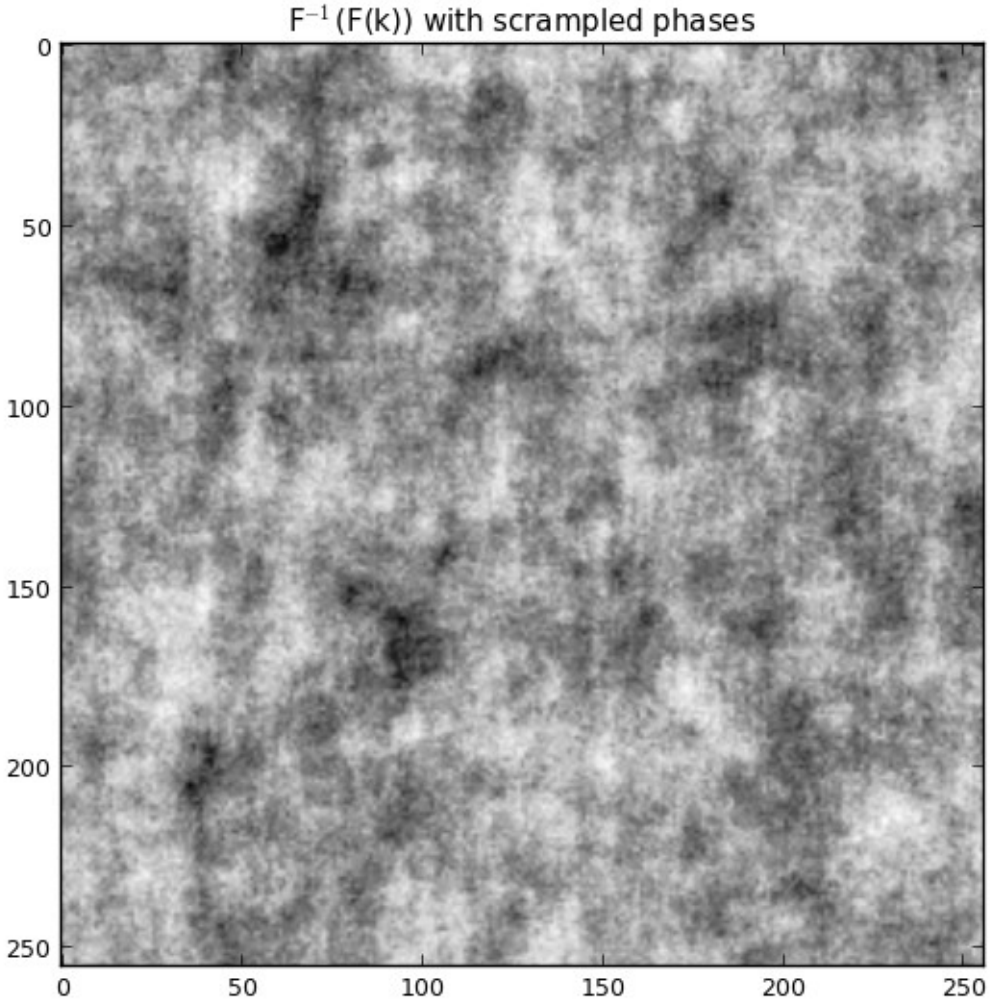
*No animals were harmed in making these slides

Multidimensional FFT

- To understand what the magnitude and phase influence, here we scramble each of these in turn

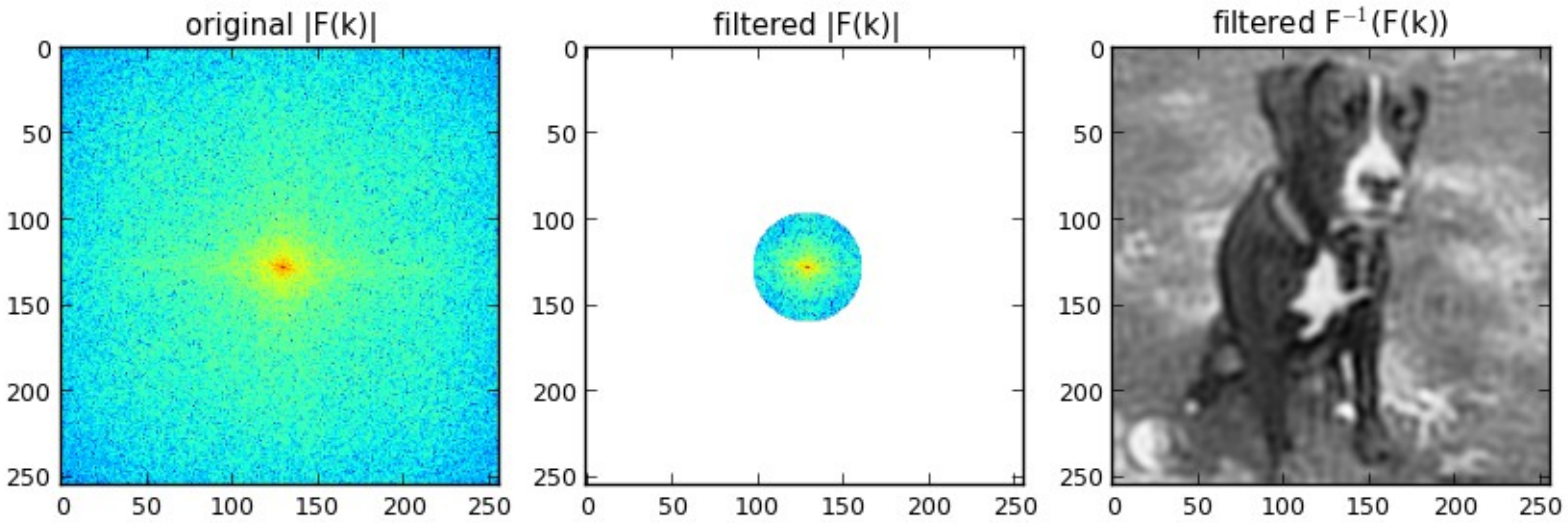




Example based on :http://matlabgeeks.com/tips-tutorials/how-to-do-a-2-d-fourier-transform-in-matlab/

Multidimensional FFT

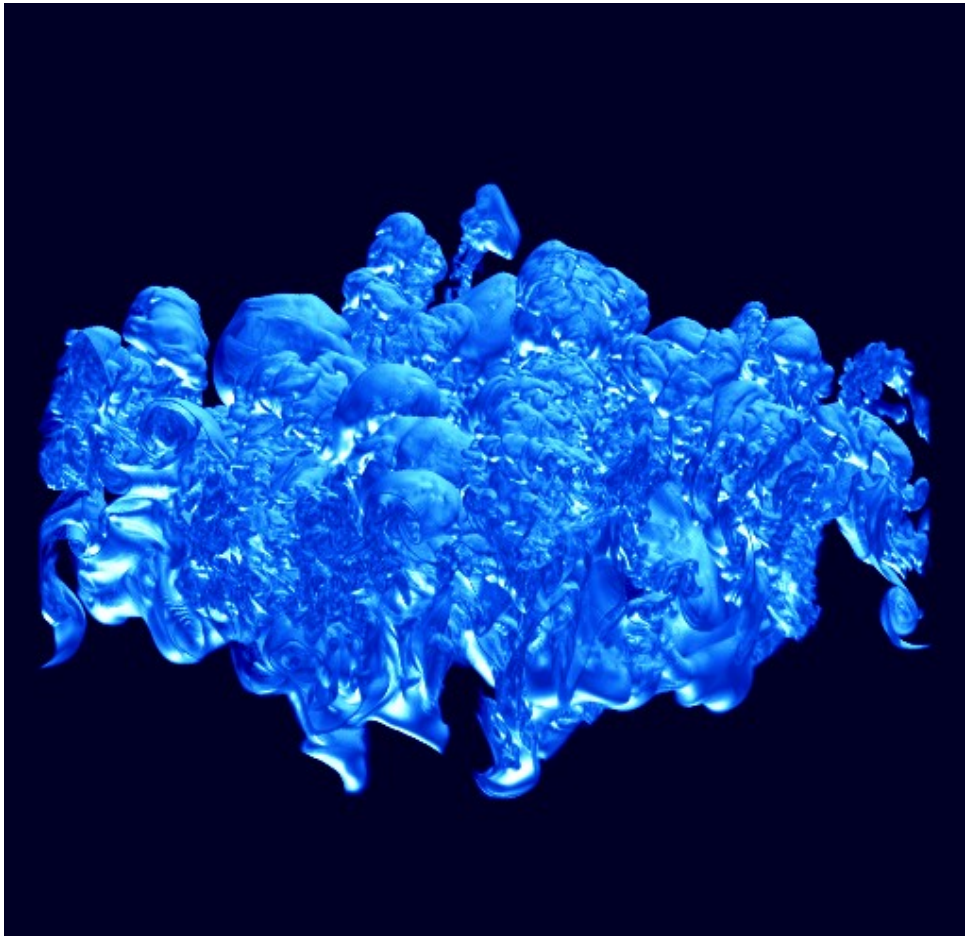
- In phase space we can filter out frequency components to do image processing



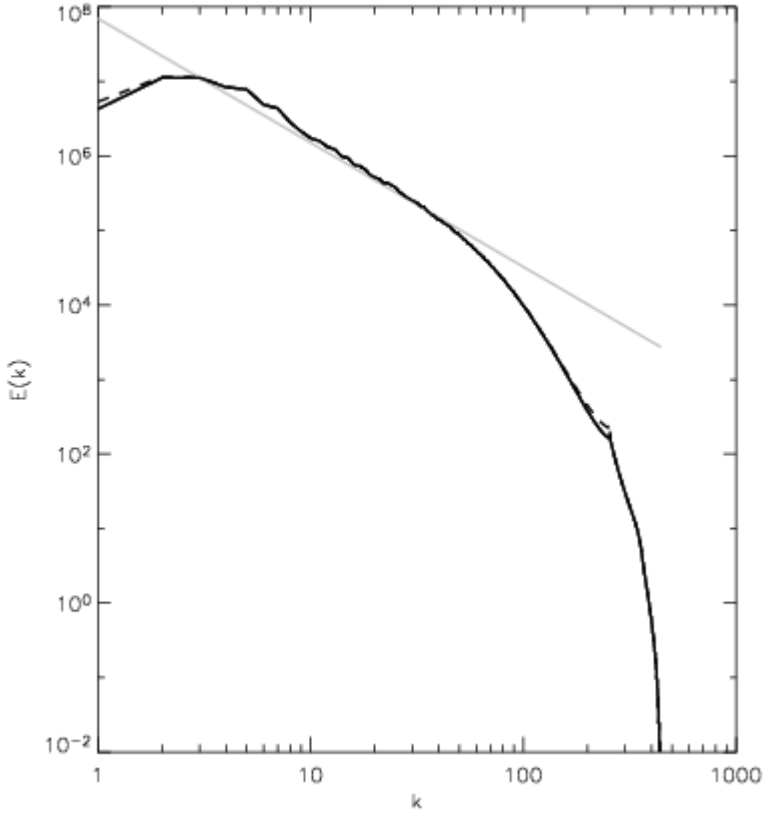
Example base on http://glowingpython.blogspot.com/2011/08/fourier-transforms-and-image-filtering.html

Example: Turbulence

- Power spectrum of the velocity field is used to understand the turbulent energy cascade



(Zingale et al. 2004)



$$E(K) = \int_{K=|k|} dk[\hat{u}(k)^2 + \hat{v}(k)^2 + \hat{w}(k)^2]$$

$$k^2 = k_x^2 + k_y^2 + k_z^2$$

Beyond Data Analysis

- So far we've focused on using the FFT for data analysis
- We can also use it directly for solving (some) PDEs.
- Consider the Poisson equation:

$$\frac{d^2\phi_n}{dx^2} = f_n$$

  - Express things in terms of the transforms

$$\phi(x) = \int \Phi(k) e^{-2\pi i k x} dk$$

$$f(x) = \int F(k) e^{-2\pi i k x} dk$$

Beyond Data Analysis

- Easy to differentiate:

$$\frac{d^2\phi(x)}{dx^2} = -4\pi^2 k^2 \int \Phi(k) e^{2\pi i k x} dk$$

- Then:

$$-4\pi^2 k^2 \Phi(k) = F(k)$$

  - Easy to solve:

$$\Phi(k) = -\frac{F(k)}{4\pi^2 k^2}$$

- Solve algebraically in Fourier space and then transform back
  - Only works for certain boundary conditions