Shape Evolution of the Term Structure

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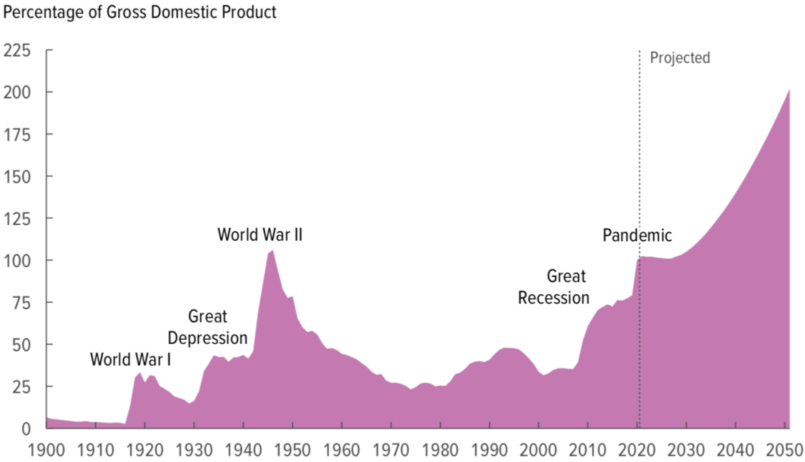
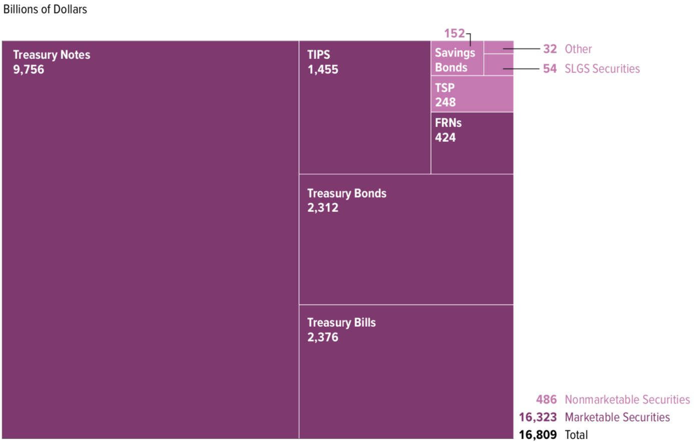
Federal Debt Held by the Public, 1900 to 2051



Source: Congressional Budget Office. It is projected to total 102% by the end of fiscal year 2021.

Components of Debt Held by the Public, FY 2019



Source: Congressional Budget Office, using data from the Department of the Treasury.

FRNs = floating-rate notes; SLGS = State and Local Government Series; TIPS = Treasury inflation-protected securities; TSP = Thrift Savings Plan.

Outline

Overview

Term structure of interest rate is a key concept in macroeconomics and finance. It plays a crucial role in econometric forecasting, monetary and fiscal policy analysis, asset pricing and risk management. Nevertheless, the shape evolution of the interest rate term structure has not yet been explored due to the challenge in measurement issues.

My dissertation and current research address

1. How to quantify the shapes of the yield curve?
2. How each shape is linked to the macroeconomy?
3. How transitions proceed from one shape to another?
4. How to apply the yield curve shapes in real time forecasting?

Contribution

My dissertation and current research contribute to the literature:

1. Perform statistical analysis on the interest rate term structure
2. Design an effective algorithm to classify the yield curve shapes
3. Examine the yield curve shapes over U.S. business cycles
4. Apply the yield curve shapes to forecasting U.S. recessions
5. Model, estimate, and forecast the shape transition dynamics
6. Apply the yield curve shapes to forecasting economic states

The term structure of interest rates is measured by yields on the U.S. Treasury securities. All data are downloaded from the Federal Reserve Board H.15 interest rate statistics. All graphs and statistics reported hereafter are based on author's research.

Highlights

1. The less frequent shapes tend to cluster before the recessions and the signals are time-varying in their timing and strength.
2. The severity of the recession is positively associated with the density of the less frequent shapes and the pre-recession shape signals become more monotonic and evident after the 1980s.
3. In forecasting one year ahead recessions based on a simple Probit model, the yield curve shapes outperform the spreads by various measures (rolling analysis with a 20-year window).
4. Surprisingly, the median-short spread trumps the long-short spread when forecasting recessions 12 months in the future.
5. Modeled in Markov chains, the shape transition displays significant momentum and asymmetry with some zero entries.
6. The shape transition converges to long run equilibrium states.

Basic Concepts

Definition

Term structure of interest rates: the relationship between interest rates of different maturities and their evolution over time, all else equal in the debt instruments (risk, liquidity, tax, etc..)

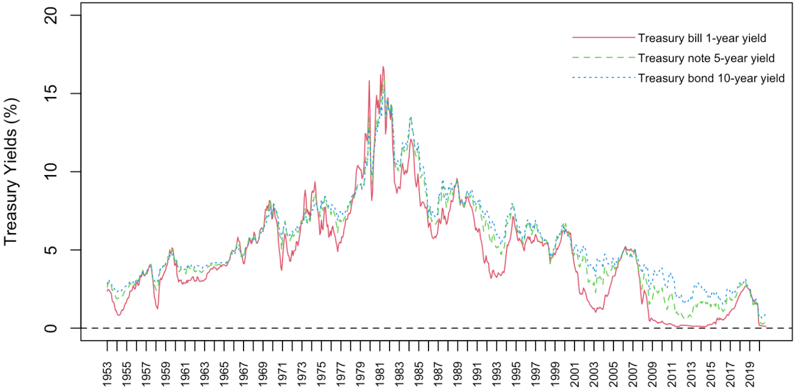
Definition

Yield to maturity (YTM): the rate of return to investors for holding the bond to maturity (zero-coupon). Or simply, yield.

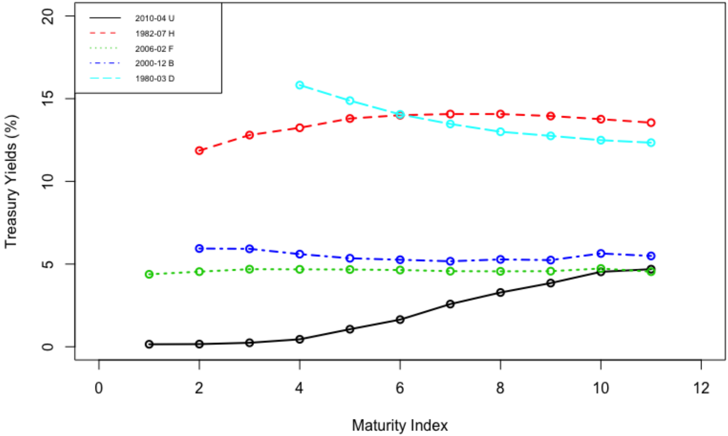
Definition

Yield curve (YC): a graphical representation of the interest rate term structure by plotting yields against corresponding maturities.

Treasury Yield Movements, 1953 to 2020



Treasury Yield Curves

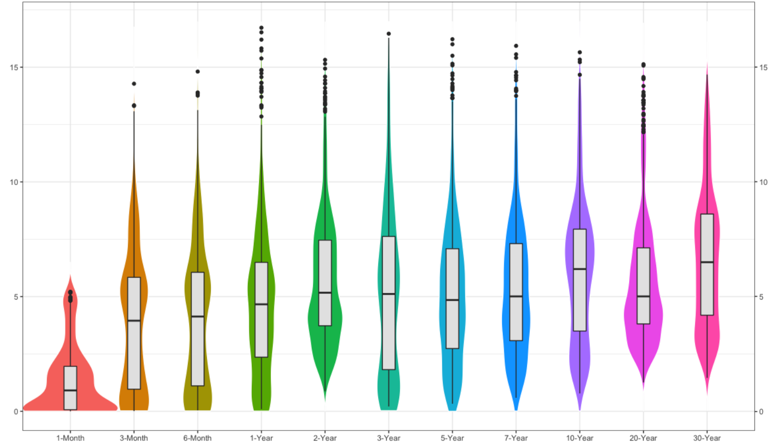


Treasury Yield Data, 1953.04 to 2020.12

Maturity	Instruments	Availability	Obs
1-month	Treasury bills	200107 : present	234
3-month	Treasury bills	198201 : present	468
6-month	Treasury bills	198201 : present	468
1-year	Treasury bills	195304 : present	813
2-year	Treasury notes	197606 : present	535
3-year	Treasury notes	195304 : present	813
5-year	Treasury notes	195304 : present	813
7-year	Treasury notes	196907 : present	618
10-year	Treasury notes	195304 : present	813
20-year	Treasury bonds	195304 : 198612	732
		199310 : present	
30-year	Treasury bonds	197702 : 200202	480
		200602 : present	

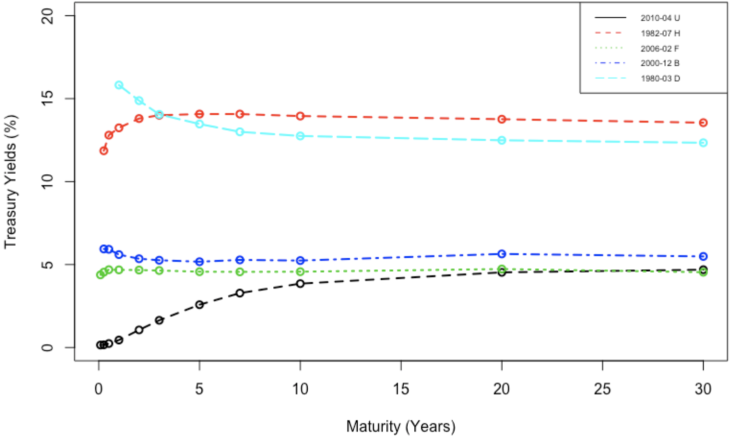
Source: Federal Reserve Board H.15Treasury yields.

Treasury Yield Probability Densities



Outline

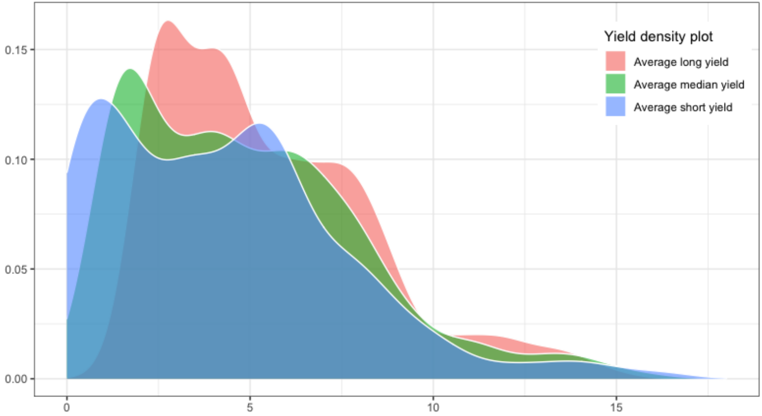
Treasury Yield Curves



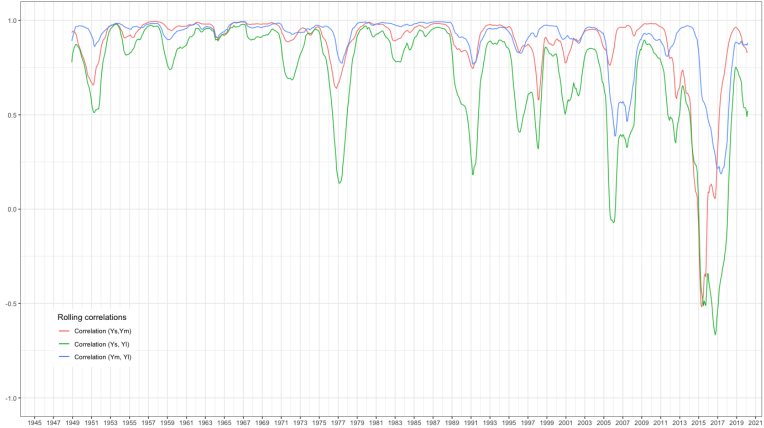
Yield Levels, Spreads, Curvature

Measure		Definition	Calculation
Level	Y_s	The average of short-term yields	$(Y_{1m} + Y_{3m} + Y_{6m} + Y_{1y})/4$
	Y_m	The average of median-term yields	$(Y_{2y} + Y_{3y} + Y_{5y} + Y_{7y})/4$
	Y_l	The average of long-term yields	$(Y_{10y} + Y_{20y} + Y_{30y})/3$
Spread	S_{ms}	The difference between Y_m and Y_s	$Y_m - Y_s$
	S_{lm}	The difference between Y_l and Y_m	$Y_l - Y_m$
	S_{ls}	The difference between Y_l and Y_s	$Y_l - Y_s$
Curvature	C_{urv}	The difference $2Y_m$ and $Y_s + Y_l$	$2Y_m - (Y_s + Y_l)$

Treasury Yield Levels: Distribution

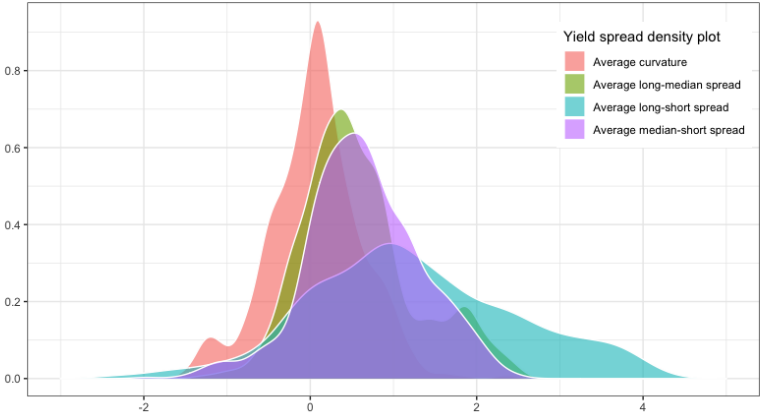


Treasury Yield Levels: Correlation

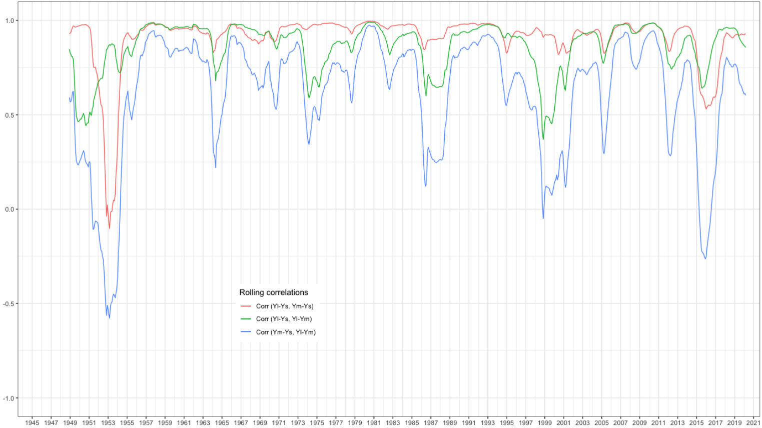


Note: Rolling correlation is computed for all yields with a window size of four years.

Treasury Yield Spreads: Distribution



Treasury Yield Spreads: Correlation



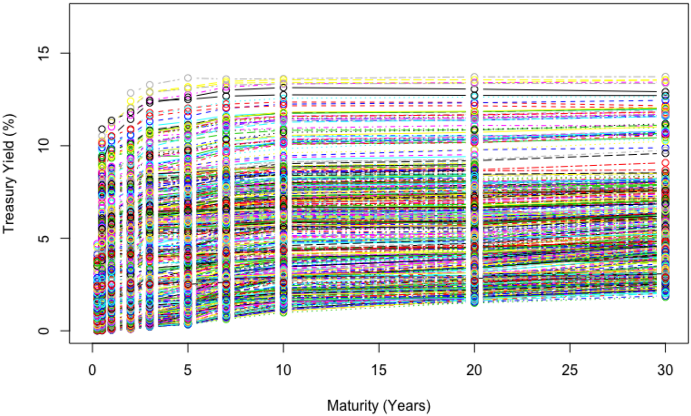
Note: Rolling correlation is computed for all yield spreads with a window size of four years.

Shape Classification Algorithm

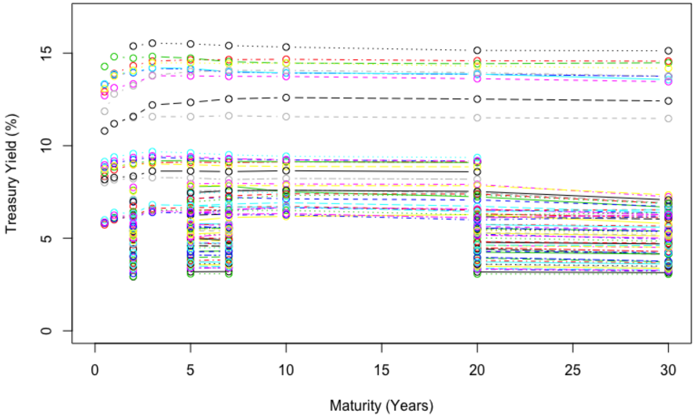
Yield curve shapes	Term structure relations with 0.1 percent threshold	Sample
Upward (U)	$(Y_m - Y_s > 0.1 \,\&\, Y_m \leqq Y_l)$ or $(Y_s \leqq Y_m \,\&\, Y_l - Y_m > 0.1)$	2010-04
Hump (H)	$(Y_m - Y_s > 0.1 \,\&\, Y_m > Y_l)$ or $(Y_s < Y_m \,\&\, Y_m - Y_l > 0.1)$	1982-07
Flat (F)	$\lvert Y_m - Y_s \rvert \leqq 0.1$ and $\lvert Y_l - Y_m \rvert \leqq 0.1$	2006-02
Bowl (B)	$(Y_s - Y_m > 0.1 \,\&\, Y_m < Y_l)$ or $(Y_s > Y_m \,\&\, Y_l - Y_m > 0.1)$	2000-12
Downward (D)	$(Y_s - Y_m > 0.1 \,\&\, Y_m \geqq Y_l)$ or $(Y_s \geqq Y_m \,\&\, Y_m - Y_l > 0.1)$	1980-03

Y_s is the average of Treasury bills (short yields, maturity $\leq$ 1 year)
Y_m is the average of Treasury notes (median yields, 2, 3, 5, 7 years)
Y_l is the average of Treasury bonds (long yields, 10, 20 & 30 years)

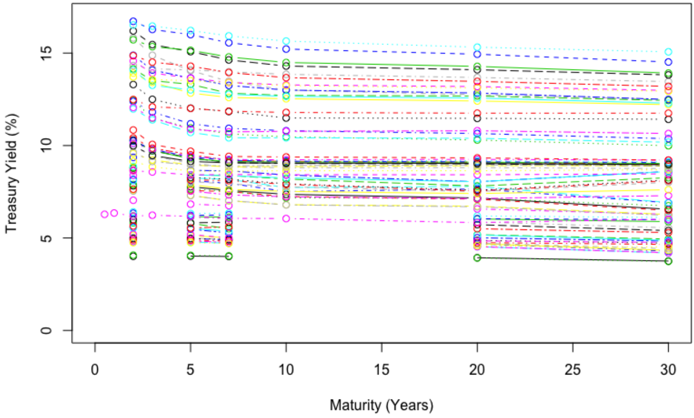
Treasury Yield Curve: Upward



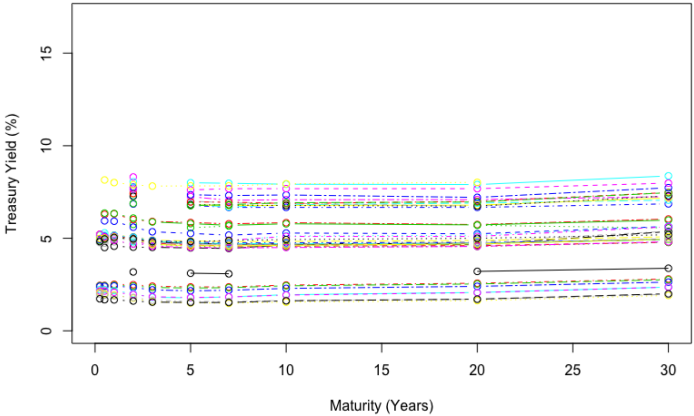
Treasury Yield Curve: Humped



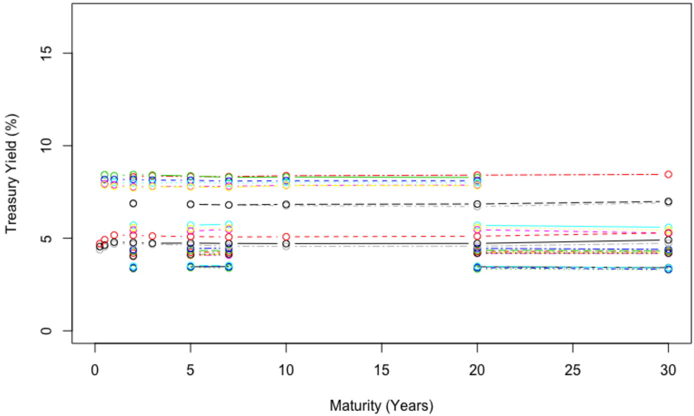
Treasury Yield Curve: Downward



Treasury Yield Curve: Bowl



Treasury Yield Curve: Flat

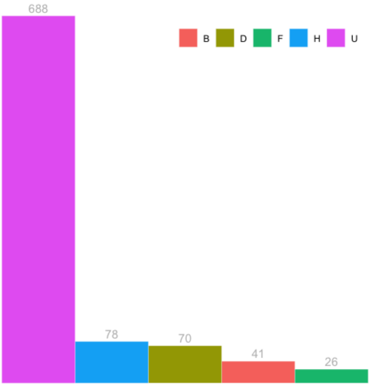


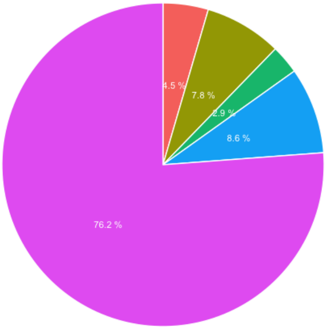
Treasury Yield Curve: Classification Statistics

Shapes	Counts	Y_s	Y_m	Y_l	S_{ls}	S_{ms}	S_{lm}	C_{urv}
Full sample	903 (100%)	4.28	4.92	5.51	1.23	0.64	0.59	0.06
		(3.29)	(3.15)	(2.84)	(1.25)	(0.68)	(0.69)	(0.55)
Upward (U)	688 (76.2%)	3.41	4.30	5.12	1.71	0.89	0.82	0.07
		(2.74)	(2.87)	(2.68)	(0.98)	(0.53)	(0.60)	(0.56)
Hump (H)	78 (8.64%)	6.60	6.89	6.69	0.09	0.29	-0.20	0.49
		(3.06)	(3.17)	(3.13)	(0.32)	(0.28)	(0.13)	(0.30)
Down (D)	70 (7.75%)	9.29	8.72	8.30	-0.99	-0.57	-0.42	-0.15
		(3.57)	(3.22)	(3.04)	(0.65)	(0.47)	(0.29)	(0.44)
Bowl (B)	41 (4.54%)	5.14	4.85	5.09	-0.06	-0.30	0.24	-0.54
		(2.03)	(1.96)	(1.88)	(0.34)	(0.23)	(0.16)	(0.21)
Flat (F)	26 (2.88%)	5.43	5.45	5.46	0.04	0.02	0.01	0.01
		(1.75)	(1.71)	(1.73)	(0.09)	(0.06)	(0.07)	(0.08)

Note: Reported statistics are the sample mean with the standard deviation in parenthesis. Y_s–average short yield, Y_m–average median yield, Y_l–average long yield, S_{ls}–long-short spread, S_{ms}–median-short spread, S_{lm}–long-median spread, C_{urv}–curvature. Sources: Federal Reserve Board H.15 Treasury nominal yield statistics (1953.04-2020.03) and NBER Macrohistory data on interest rates (1945.01-1953.03).

Treasury Yield Curve: Shape Frequency





Classification Sensitivity to Various Thresholds

Yield curve shapes	Term structure relations with 0.1 percent threshold	Sample
Upward (U)	$(Y_m - Y_s > 0.1 \,\&\, Y_m \leqq Y_l)$ or $(Y_s \leqq Y_m \,\&\, Y_l - Y_m > 0.1)$	2010-04
Hump (H)	$(Y_m - Y_s > 0.1 \,\&\, Y_m > Y_l)$ or $(Y_s < Y_m \,\&\, Y_m - Y_l > 0.1)$	1982-07
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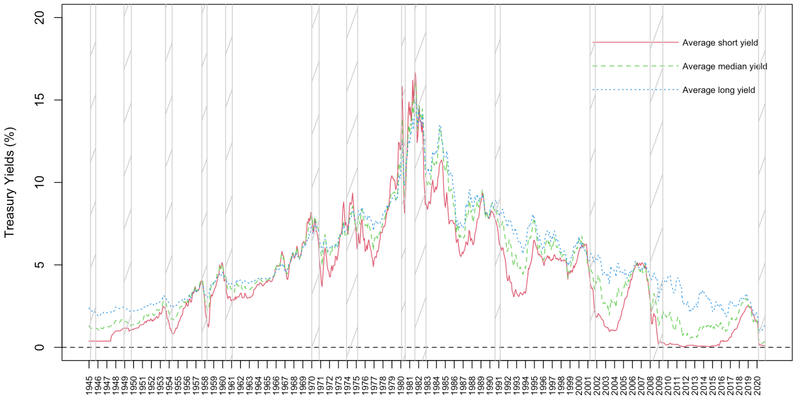
Downward (D)	$(Y_s - Y_m > 0.1 \,\&\, Y_m \geqq Y_l)$ or $(Y_s \geqq Y_m \,\&\, Y_m - Y_l > 0.1)$	1980-03

Threshold (%)	0.025	0.05	0.075	**0.1%**	0.125	0.15	0.175	0.2
Upward (U)	699	695	694	688	685	681	679	675
Hump (H)	84	84	84	78	73	63	57	53
Down (D)	74	73	70	70	70	70	67	61
Bowl (B)	46	45	44	41	41	39	37	35
Flat (F)	0	6	11	26	34	50	63	79

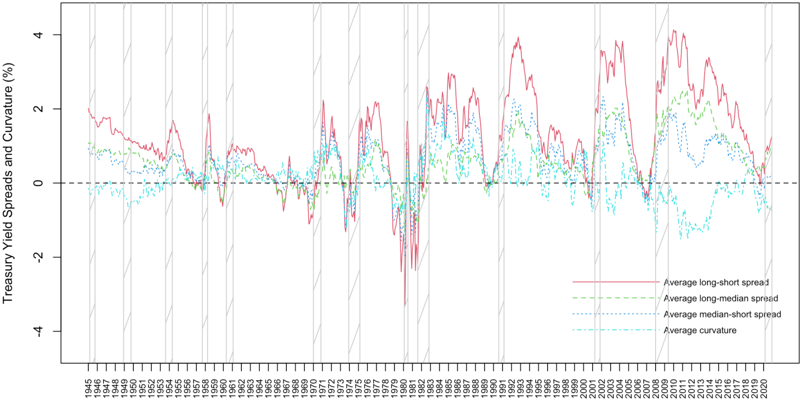
Outline

1. Introduction
2. Shape Classification
3. Recession Forecasts
4. Transition Dynamics

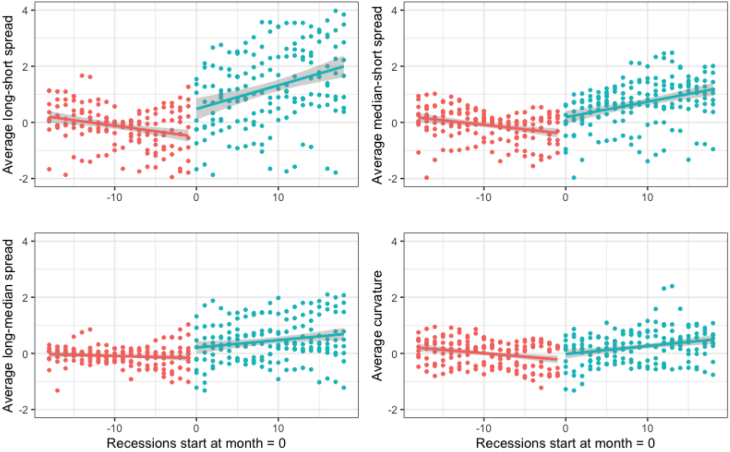
Treasury Yield Movements over Business Cycles



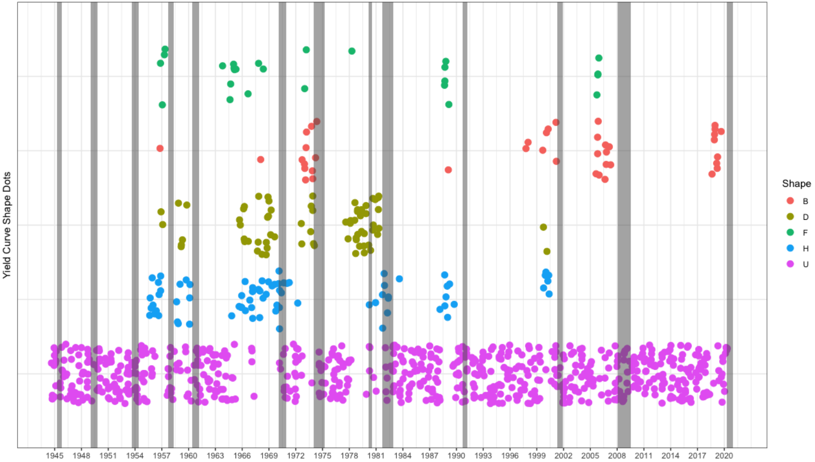
Treasury Yield Inversions over Business Cycles



Treasury Yield Spreads across Recessions



Shape Evolution over Business Cycles



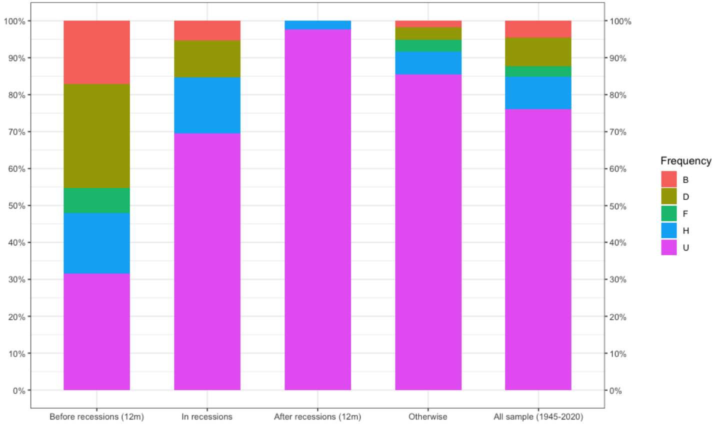
Shape Counts over Business Cycles

	I	II	III	IV	All
Upward (U)	46 (31.5%)	91 (69.5%)	129 (97.7%)	422 (85.4%)	688 (76.2%)
Down (D)	41 (28.1%)	13 (10.0%)	0 (0.00%)	16 (3.24%)	70 (7.75%)
Bowl (B)	25 (17.1%)	7 (5.34%)	0 (0.00%)	9 (1.82%)	41 (4.54%)
Hump (H)	24 (16.4%)	20 (15.3%)	3 (2.27%)	31 (6.27%)	78 (8.64%)
Flat (F)	10 (6.85%)	0 (0.00%)	0 (0.00%)	16 (3.24%)	26 (2.88%)
All shapes	146 (16.2 %)	131 (14.5%)	132 (14.6%)	469 (54.7%)	903 (100%)

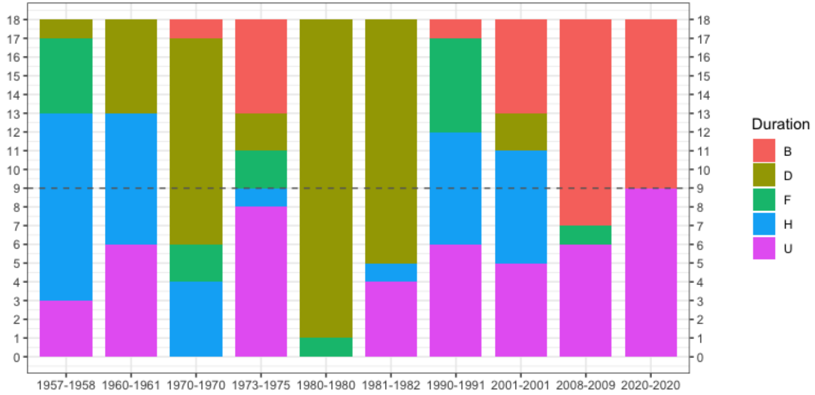
I: all pre-recession 12-month periods (146 months), II: all in-recession periods (131 months)

III: all post-recession 12-month periods (132 months), IV: all otherwise periods (496 months)

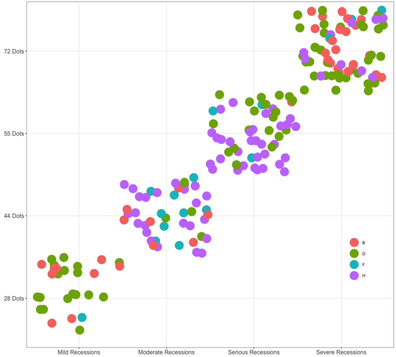
Shape Frequency over U.S. Business Cycles

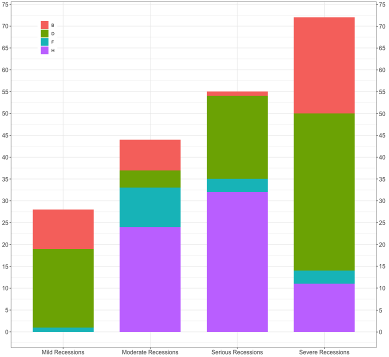


Shape Duration before U.S. Recessions

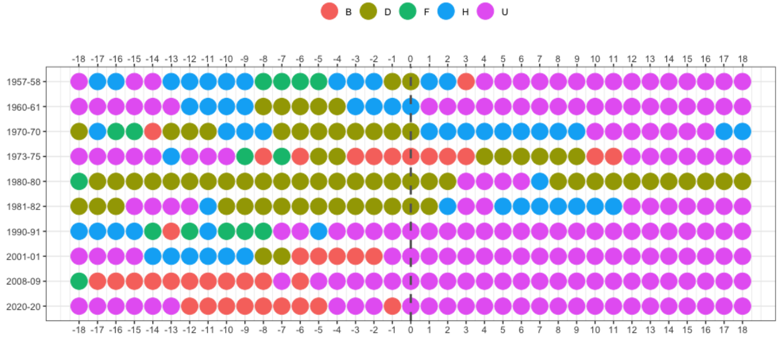


Shape Signal Strength and Recession Severity



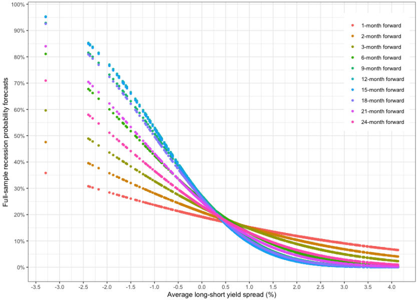


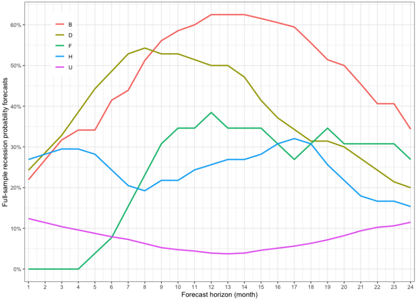
Shape Timing over U.S. Recessions



Recession Forecast: Probit Models

$$P(Recession_{t+h}|Spread_t) = P(Z \leq \alpha + \beta * Spread_t) = \Phi(\alpha + \beta * Spread_t)$$
$$P(Recession_{t+h}|Shape_t) = P(Z \leq \gamma_B B_t + \gamma_D D_t + \gamma_F F_t + \gamma_H H_t + \gamma_U U_t)$$
$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-0.5u^2} du$$





Note: $Recession_{t+h}$ is a binary indicator, so are the shapes. At time t, the model predicts h-month ahead.

Model Estimation: Spreads vs Shapes

Panel A: Spread model $P(Recession_{t+12}|Spread_t) = P(Z \leq \alpha + \beta * Spread_t) = \Phi(\alpha + \beta * Spread_t)$

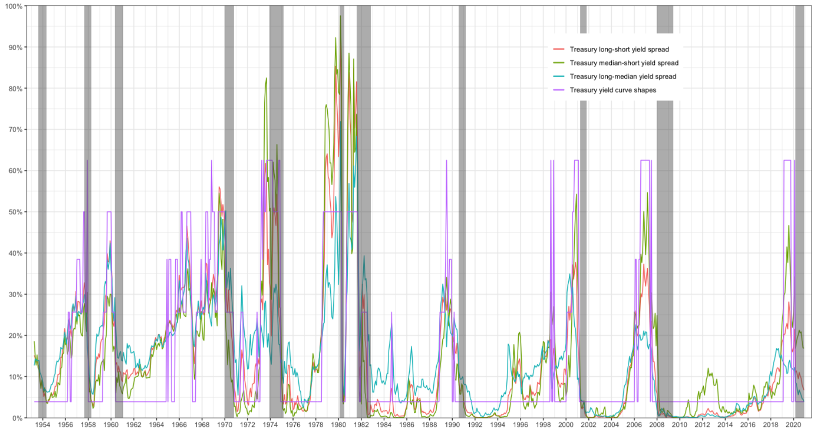
Predictor	α	β	MF R^2	ML R^2	α	β	MF R^2	ML R^2	α	β	MF R^2	ML R^2
	1953.04:2020.12				1953.04:1986.12				1987.01:2020.12			
$Spread_{l-s}$	-0.61*	-0.70*	0.25	0.19	-0.80*	-0.82*	**0.26**	**0.21**	-0.23	-0.80*	0.28	0.18
	(0.07)	(0.07)			(0.09)	(0.10)			(0.14)	(0.12)		
$Spread_{m-s}$	-0.62*	-1.32*	**0.29**	**0.21**	-0.74*	-0.19*	0.24	0.20	-0.25	-1.86*	**0.36**	**0.22**
	(0.07)	(0.12)			(0.09)	(0.15)			(0.13)	(0.25)		
$Spread_{l-m}$	-0.75*	-1.00*	0.15	0.12	-0.92*	-1.74*	0.20	0.17	-0.56*	0.90*	0.14	0.10
	(0.06)	(0.12)			(0.08)	(0.23)			(0.13)	(0.17)		
$Curvature$	-1.07*	-0.48*	0.04	0.03	-0.75*	1.03*	0.10	0.09	-1.29*	-0.40*	0.03	0.02
	(0.06)	(0.10)			(0.08)	(0.19)			(0.09)	(0.14)		

Panel B: Shape model $P(Recession_{t+12}|Shape_t) = P(Z \leq \gamma_B B_t + \gamma_D D_t + \gamma_F F_t + \gamma_H H_t + \gamma_U U_t)$

Sample	γ_B	γ_D	γ_F	γ_H	γ_U	MF R^2	ML R^2
1953.04:2020.12	0.31	0.00	-0.29	-0.65*	-1.76*	0.28	0.21
	(0.20)	(0.15)	(0.25)	(0.15)	(0.09)		
1953.04:1986.12	0.50	-0.04	-0.38	-0.78*	-1.88*	0.28	0.23
	(0.36)	(0.15)	(0.31)	(0.18)	(0.16)		
1987.01:2020.12	0.23	5.02	-0.14	-0.18	-1.68*	0.29	0.18
	(0.24)	(166.2)	(0.41)	(0.33)	(0.11)		

Note: Models are estimated via maximum likelihood method. The intercept term is excluded from the shape model to avoid multicolinearity problem. Coefficient estimates with an asterisk are significant at less than 0.1% . Standard errors are shown in parentheses. MF R^2 is the McFadden's pseudo R-squared and ML R^2 is the maximum likelihood pseudo R-squared. Both are a measure of goodness of fit for the Probit model regression. The pseudo R^2 is defined as $1 - ln\hat{L}(M_u)/ln\hat{L}(M_r)$, where M_u is the unrestricted model with intercept and slope predictors, M_r is the restricted model with only an intercept term, and $\hat{L}$ is the estimated likelihood.

Recession Forecast: Spreads vs Shapes



Note: Full-sample estimates of 12-month ahead recession probabilities based on the Probit Models.

Recession Forecast: Evaluation Metrics

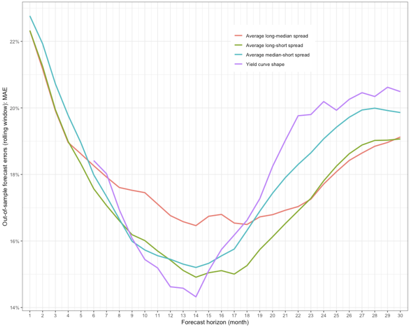
$$MAE(\hat{P}_i) = \frac{1}{T-h} \sum_{i=1}^{T-h} \mid \hat{P}_i - I_i \mid \tag{1}$$

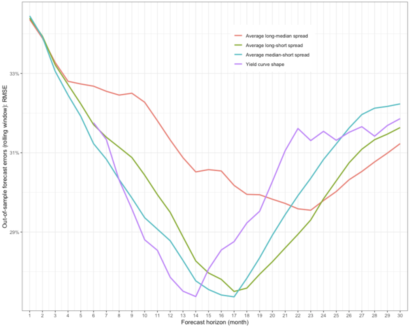
$$RMSE(\hat{P}_i) = \sqrt{\frac{1}{T-h} \sum_{i=1}^{T-h} (\hat{P}_i - I_i)^2} \tag{2}$$

$$LPS(\hat{P}_i) = \frac{-1}{T-h} \sum_{i=1}^{T-h} [(1-I_i)ln(1-\hat{P}_i) + I_i ln(\hat{P}_i)] \tag{3}$$

$$CER(\hat{P}_i) = E(\hat{I}_i) = \frac{1}{T-h} \sum_{i=1}^{T-h} |I_i - \hat{I}_i|, \text{where } \hat{I}_i = \begin{cases} 1, & \text{if } \hat{P}_i > \text{cutoff} \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Forecast Evaluation: MAE and RMSE

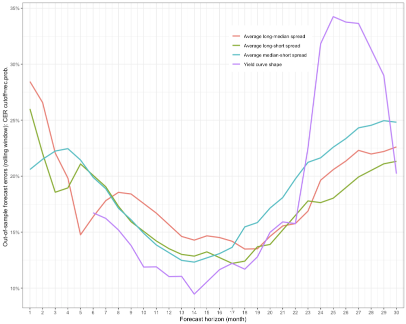




Method: Out-of-sample rolling window forecasts (window size = 20 years).

Note: Mean Absolute Error (MAE) and Root Mean Square Error (RMSE).

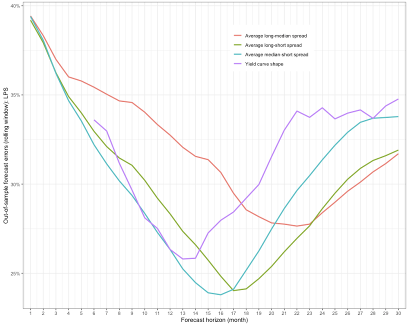
Forecast Evaluation: CER (Cutoff = 15% and 50%)

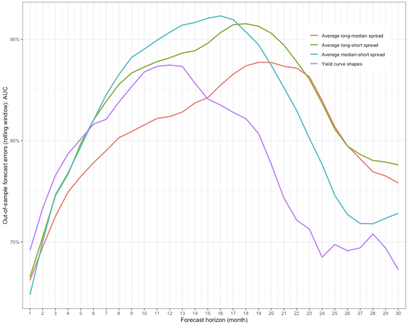


Method: Out-of-sample rolling window forecasts (window size = 20 years).

Note: Classification Error Rate (CER) and cutoff is the threshold for a recession.

Forecast Evaluation: LPS and AUC



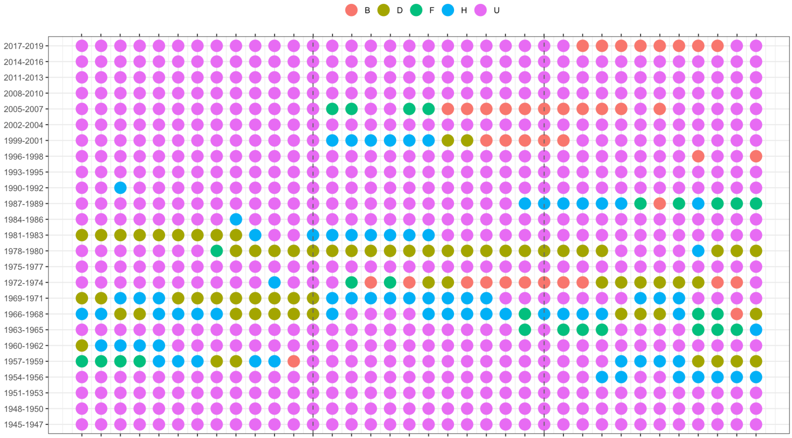


Method: Out-of-sample rolling window forecasts (window size = 20 years).

Note: Logarithm Probability Score (LPS) and Area Under the Curve (AUC).

Outline

Shape Sequence, 1945 to 2019



Why Interested in Shape Transitions?

- From the shape classification, decision-makers can estimate the probability of observing any shape? $P(Shape_t)$
- The unconditional probability of a particular shape is just its sample frequency, e.g., what is the probability of having a downward yield curve in any given month? $P(Shape_t = D)$
- Macroeconomists, investors, and policy-makers may be more interested in forecasting the shape in the future given today's shape. It is a conditional probability: $P(Shape_{t+1}|Shape_t)$.
- Given that we observe an upward yield curve today, what are the probabilities of seeing other shapes next month, next quarter, or next year? $P(Shape_{t+1}|U_t = 1)$
- Modeling and estimating these transition probabilities can be useful in macroeconomic forecasting and bond investment.

Shape Transition: Markov Chain Model

- A stochastic process $\{X_t, t \geq 0\}$ on state space S is said to be a discrete-time Markov chain (DTMC) if, for all i and j in S, the conditional probability satisfies

$$P(X_{t+1} = j | X_t = i, X_{t-1}, ..., X_0) = P(X_{t+1} = j | X_t = i) \quad (5)$$

- Simply put, future state depends only on present state.
- A DTMC $\{X_t, t \geq 0\}$ is said to be time homogenous if, for all $t = 0, 1, 2, ...,$

$$P_{ij} = P(X_{t+1} = j | X_t = i) = P(X_1 = j | X_0 = i). \quad (6)$$

- All state transitions are independent of time index.

Markov Chain Model Elements

- Initial state distribution: the starting probabilities for each state at t=0.
- Transition probabilities matrix $\boldsymbol{P}$: putting all transition probabilities into a $N * N$ matrix.
- Stationary distribution: a state distribution that satisfies the balance equation $\pi^* * \boldsymbol{P} = \pi^*$

$$\pi^{(\mathbf{0})} = (\pi_B^{(0)}, \pi_D^{(0)}, \pi_F^{(0)}, \pi_H^{(0)}, \pi_U^{(0)})$$

$$\begin{array}{c c} & \begin{array}{ccccc} B & D & F & H & U \end{array} \\ \begin{array}{c} B \\ D \\ F \\ H \\ U \end{array} & \begin{pmatrix} P_{BB} & P_{BD} & P_{BF} & P_{BH} & P_{BU} \\ P_{DB} & P_{DD} & P_{DF} & P_{FH} & P_{DU} \\ P_{FB} & P_{FD} & P_{FF} & P_{FH} & P_{FU} \\ P_{HB} & P_{HD} & P_{HF} & P_{HH} & P_{HU} \\ P_{UB} & P_{UD} & P_{UF} & P_{UH} & P_{UU} \end{pmatrix} \end{array}$$

$$\boldsymbol{\pi^* = (\pi_B^*, \pi_D^*, \pi_F^*, \pi_H^*, \pi_U^*)}$$

Transition Probability Estimation Techniques

Lee, Judge, and Zellner (1968), Dent and Ballintine (1971), Athreya and Fuh (1992), Guerra (1997), Teodorescu (2009)

- Maximum likelihood estimation: simple derivation and analytical solution, more efficient than least square.
- MLE bootstrap: offers a simple way to obtain a good approximate sample distribution of the MLE estimator, conditional on the observed data
- Laplace smoothing: solution to the sparse estimation problem, particularly useful when large sample dataset is not feasible.
- Bayesian method: superior to MLE and weighted least squares in terms of smaller root mean square error in simulation.

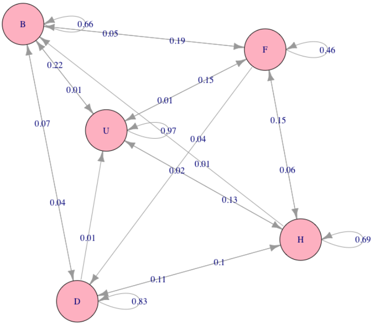
R statistical package: Spedicato (2020) Markovchain

Shape Transition Probability Matrix

	B	D	F	H	U
B	0.659 (0.123)	0.073 (0.042)	0.049 (0.034)	0.000 (0.000)	0.220 (0.073)
D	0.043 (0.024)	0.829 (0.109)	0.000 (0.000)	0.114 (0.040)	0.014 (0.014)
F	0.192 (0.086)	0.038 (0.038)	0.462 (0.133)	0.154 (0.077)	0.154 (0.077)
H	0.013 (0.013)	0.103 (0.036)	0.064 (0.029)	0.692 (0.094)	0.128 (0.041)
U	0.007 (0.003)	0.000 (0.000)	0.010 (0.004)	0.017 (0.005)	0.965 (0.037)

Note: Maximum likelihood estimates with standard errors in parentheses.

Shape Transition Diagram and Matrix



$$\begin{array}{cc} & \begin{array}{ccccc} B & D & F & H & U \end{array} \\ \begin{array}{c} B \\ D \\ F \\ H \\ U \end{array} & \begin{pmatrix} \mathbf{.66} & .07 & .05 & .00 & .21 \\ .04 & \mathbf{.83} & .00 & .11 & .01 \\ .19 & .04 & \mathbf{.46} & .15 & .15 \\ .01 & .10 & .06 & \mathbf{.69} & .13 \\ .01 & .00 & .01 & .02 & \mathbf{.97} \end{pmatrix} \end{array}$$

Note: Yield curve shape from 1945.1 to 2020.3

Shape Transition Dynamics: Summary

- Diagonal elements: significant transition momentum: $P_{UU} = 0.97, P_{DD} = 0.83, P_{HH} = 0.69, P_{BB} = 0.66, P_{FF} = 0.46$
- Off-diagonal elements: highly asymmetric transitions: $P_{BU} = 0.21 >> P_{UB} = 0.01$ and $P_{FB} = 0.19 >> P_{BF} = 0.05$
- Never happen in one step: $P_{BH} = 0$, $P_{DF} = 0$, $P_{UD} = 0$
- Almost unlikely in one step: $P_{DU} \approx P_{HB} \approx P_{UF} \approx 0.01$
- Long run convergence to its stationary distribution.
- A unique stationary distribution: $\pi^*_B = 0.045, \pi^*_D = 0.078, \pi^*_F = 0.029, \pi^*_H = 0.086, \pi^*_U = 0.762$
- Close to its sample relative frequency: P_B, P_D, P_F, P_H, P_U.
- Convergence takes about 5-6 years without external shocks.

Shape Transition : Multi-Step Estimation

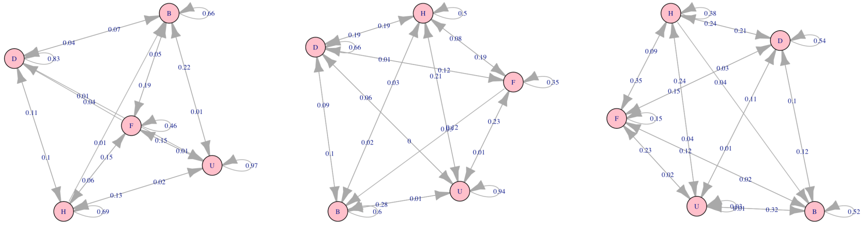
$$\pi^{(t+N+1)} = \sum_{n=1}^{N} \lambda_n \pi^{(t+N+1-n)} \boldsymbol{P_n} \text{ for } N = 3.$$

	P1 transition matrix					P2 transition matrix					P3 transition matrix				
	U	H	F	B	D	U	H	F	B	D	U	H	F	B	D
U	.960	.026	.007	.007	.000	.936	.039	.011	.011	.004	.921	.046	.013	.015	.006
H	.141	.692	.026	.013	.128	.218	.564	.051	.013	.154	.244	.474	.051	.038	.192
F	.188	.063	.500	.188	.063	.313	.125	.250	.188	.125	.375	.186	.186	.186	.063
B	.175	.000	.025	.750	.050	.225	.000	.025	.650	.100	.275	.025	.025	.576	.100
D	.013	.118	.013	.026	.829	.053	.145	.013	.053	.737	.092	.158	.013	.066	.671

First order MC $\lambda_1 = 1$; Second-order MC $\lambda_1 = \lambda_2 = 0.5$; Third-order MC $\lambda_1 = \lambda_2 = \lambda_3 = 0.3333$.

Note: Yield curve Markov chain (1953.4 to 2016.3.) in estimation is a categorical five-state sequence classified by my algorithm. Data Source: Federal Reserve Board H.15 interest rate statistics.

Shape Transition: Multi-Step Probabilities



Model Evaluation: K-Fold Cross-Validation

$$Ave.(I(S_t \neq \hat{S}_t)) = \frac{1}{T}\sum_{t=1}^{T} I(S_t \neq \hat{S}_t)$$

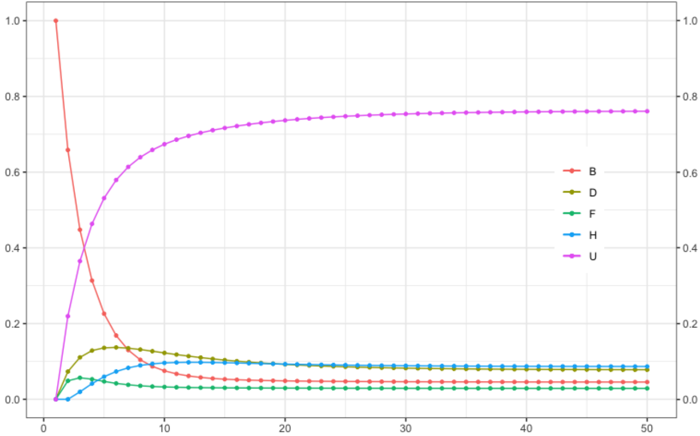
	1st-order MC	2nd-order MC	3rd-order MC
k=1	0.1020	0.1154	0.1261
k=2	0.1257	0.1984	0.2169
k=5	0.1020	0.1271	0.1444

Data: monthly yield curve sequence 1953.4 to 2016.3.

Forecast Future Shapes: $\pi^{(t)} = \pi^{(0)} * P^t$

Forecast	$\pi^{(0)} = (1,0,0,0,0)$					$\pi^{(0)} = (0,1,0,0,0)$				
Horizon	U	H	F	B	D	U	H	F	B	D
1-month	0.9596	0.0257	0.0073	0.0073	0.0000	0.1410	0.6923	0.0256	0.0128	0.1282
2-month	0.9272	0.0429	0.0116	0.0143	0.0041	0.2417	0.4997	0.0336	0.0277	0.1973
3-month	0.9005	0.0547	0.0141	0.0203	0.0103	0.3162	0.3776	0.0347	0.0405	0.2311
6-month	0.8431	0.0741	0.0175	0.0333	0.0320	0.4557	0.2132	0.0299	0.0612	0.2399
1-year	0.7810	0.0893	0.0196	0.0450	0.0650	0.5907	0.1389	0.0246	0.0649	0.1809
2-year	0.7367	0.0997	0.0208	0.0512	0.0916	0.6883	0.1115	0.0220	0.0568	0.1213
5-year	0.7221	0.1033	0.0212	0.0530	0.1005	0.7213	0.1034	0.0212	0.0530	0.1010
	$\pi^{(0)} = (0,0,1,0,0)$					$\pi^{(0)} = (0,0,0,1,0)$				
1-month	0.1875	0.0625	0.5000	0.1875	0.0625	0.1750	0.0000	0.0250	0.7500	0.0500
2-month	0.3161	0.0867	0.2585	0.2382	0.1004	0.3045	0.0120	0.0332	0.5698	0.0805
3-month	0.4071	0.0962	0.1411	0.2332	0.1225	0.4009	0.0277	0.0344	0.4381	0.0988
6-month	0.5592	0.1045	0.0402	0.1545	0.1416	0.5686	0.0688	0.0291	0.2151	0.1184
1-year	0.6635	0.1094	0.0233	0.0756	0.1281	0.6778	0.1005	0.0231	0.0833	0.1155
2-year	0.7095	0.1060	0.0215	0.0549	0.1080	0.7144	0.1047	0.0214	0.0546	0.1049
5-year	0.7217	0.1034	0.0212	0.0530	0.1008	0.7217	0.1033	0.0212	0.0530	0.1007
	$\pi^{(0)} = (0,0,0,0,1)$					$\pi^{(0)} = \pi^* = \pi(\infty)$				
1-month	0.0132	0.1184	0.0132	0.0263	0.8289	0.7219	0.1033	0.0212	0.0530	0.1007
2-month	0.0473	0.1813	0.0213	0.0456	0.7045	0.7219	0.1033	0.0212	0.0530	0.1007
3-month	0.0922	0.2115	0.0260	0.0594	0.6105	0.7219	0.1033	0.0212	0.0530	0.1007
6-month	0.2407	0.2182	0.0303	0.0788	0.4320	0.7219	0.1033	0.0212	0.0530	0.1007
1-year	0.4681	0.1672	0.0273	0.0771	0.2603	0.7219	0.1033	0.0212	0.0530	0.1007
2-year	0.6565	0.1193	0.0228	0.0605	0.1408	0.7219	0.1033	0.0212	0.0530	0.1007
5-year	0.7208	0.1036	0.0212	0.0531	0.1013	0.7219	0.1033	0.0212	0.0530	0.1007

Shape Transition: Long-Run Equilibrium



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