Conics

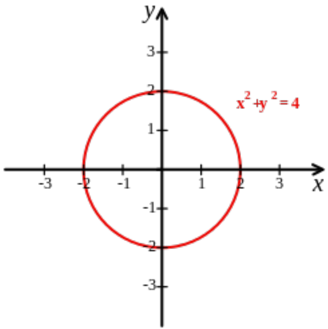
MATH 1511, BCIT

Technical Mathematics for Geomatics

October 30, 2017

Curves

Equations have solution sets. These solution sets are subsets of spaces like the number line (one variable), the plane (two variables), or three-dimensional space (three variables). When there are two variables, the solution set may describe a curve. A curve is a one-dimensional space inside a higher-dimensional space, for example the plane. The most simple example is the circle. Consider the equation

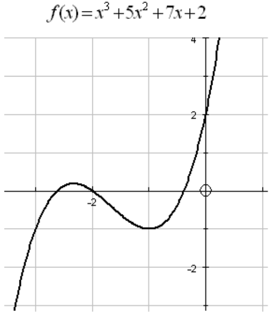


Polynomials

The following expression is called a polynomial.

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \qquad (1)$$

For example, $f(x) = x^3 + 5x^2 + 7x + 2$. $n \geq 0$ is a natural number called the degree of the polynomial, and the a_i are real numbers called coefficients. Here is what a curve described by a polynomial with degree 3 looks like.



Equations and Curves

Curves in the plane may correspond to equations with two variables, as in the example with the circle. Sometimes, we push everything in the equation to the LHS (left-hand side) and leave 0 on the RHS (right-hand side). For the circle,

$$x^2 + y^2 - 4 = 0 \tag{2}$$

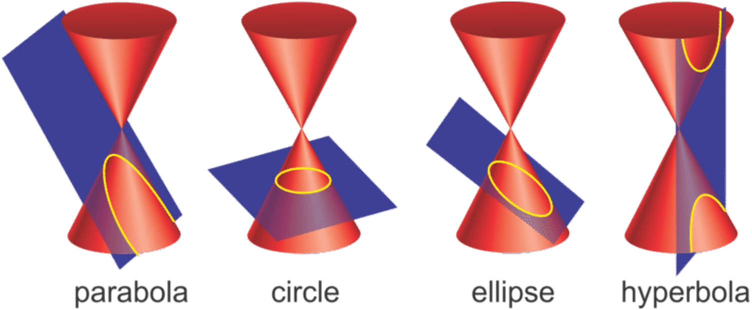
and for the polynomial in the last slide,

$$x^3 + 5x^2 + 7x + 2 - y = 0 \tag{3}$$

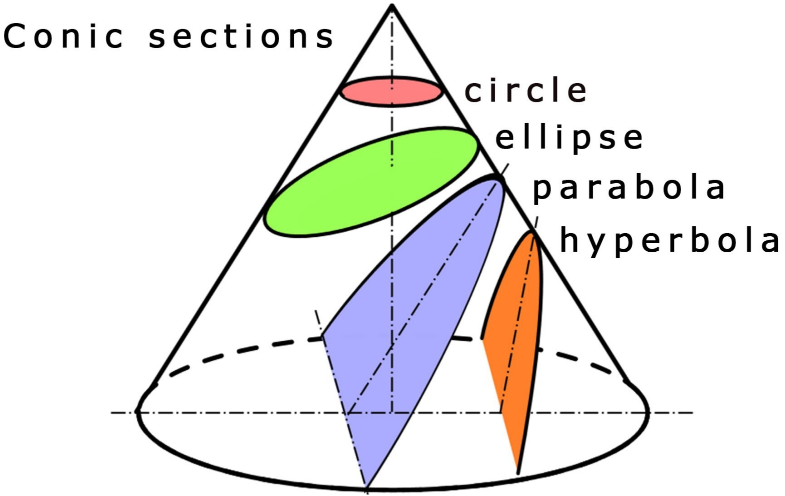
It turns out that all curves that correspond to polynomials of degree 2 or less in two-dimensional space are conic sections or conics. Equations such as

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{4}$$

with A, B, C, D, E, F being real numbers (with $|A| + |B| + |C| \neq 0$) correspond to curves that can be thought of as intersections of the plane with a parabola, an ellipse, or a hyperbola.



Circles are just special cases of ellipses. Is the polynomial curve with degree 3 from a couple of slides ago a conic? Why (not)?

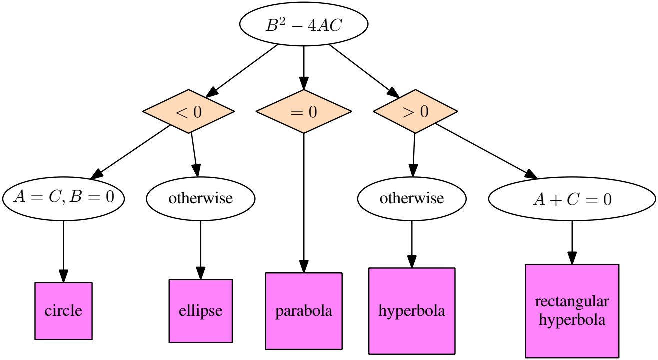


Conics III

If (for now B is always zero in order to avoid conics whose symmetries do not line up with the coordinate system)

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{5}$$

is the equation of the conic section, then $B^2 - 4AC$ is the discriminant of the equation.



Conics Exercises

Classify the following equations according to the type of conic each represents.

1. $3x^2 + 3y^2 - 6x + 9y - 14 = 0$
2. $6x^2 + 12x - y + 15 = 0$
3. $x^2 + 2y^2 + 4x + 2y - 27 = 0$
4. $x^2 - y^2 + 3x - 2y - 43 = 0$

Conics Exercises

Classify the following equations according to the type of conic each represents.

1. $3x^2 + 3y^2 - 6x + 9y - 14 = 0$
2. $6x^2 + 12x - y + 15 = 0$
3. $x^2 + 2y^2 + 4x + 2y - 27 = 0$
4. $x^2 - y^2 + 3x - 2y - 43 = 0$

Conics Exercises

Classify the following equations according to the type of conic each represents.

1. $3x^2 + 3y^2 - 6x + 9y - 14 = 0$
2. $6x^2 + 12x - y + 15 = 0$
3. $x^2 + 2y^2 + 4x + 2y - 27 = 0$
4. $x^2 - y^2 + 3x - 2y - 43 = 0$

Conics Exercises

Classify the following equations according to the type of conic each represents.

1. $3x^2 + 3y^2 - 6x + 9y - 14 = 0$
2. $6x^2 + 12x - y + 15 = 0$
3. $x^2 + 2y^2 + 4x + 2y - 27 = 0$
4. $x^2 - y^2 + 3x - 2y - 43 = 0$

Parabolas

Here is one form of a parabola, which tells us where the vertex is. The vertex is the point where the line of symmetry intersects with the parabola.

$$y = a(x-p)^2 + q \text{ or } x = a(y-q)^2 + p \tag{6}$$

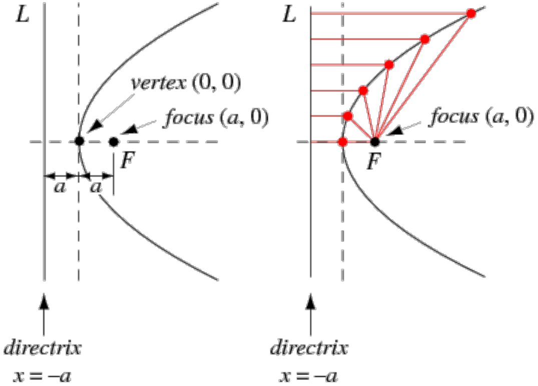
$a \neq 0, p, q$ are real numbers. The vertex is at (p, q).

- If $a > 0$ and y is not squared, then the parabola opens upward. The parabola is convex.
- If $a < 0$ and y is not squared, then the parabola opens downward. The parabola is concave.
- If $a > 0$ and x is not squared, then the parabola opens to the right.
- If $a < 0$ and x is not squared, then the parabola opens to the left.

Exercise: determine whether the functions $f(x) = e^x$ and $g(x) = \ln x$ are convex or concave.

Vertex, Focus, Directrix

Besides a vertex, a parabola also has a focus and a directrix (two out of these three uniquely determine a parabola). You may learn later how to calculate them given the equation of the parabola; or to determine the equation for the parabola, given the focus and the directrix. All points on a parabola are equidistant (equally far away) from the directrix and the focus.



Parabola Exercises

Consider the following curve.

$$2y^2 - \frac{1}{2}x - 12y + 19 = 0 \tag{7}$$

Use the discriminant to show that the curve is a parabola. Calculate the position of the vertex and determine whether the parabola opens upward, downward, to the right, or to the left.

Try again with the following equation.

$$-63x^2 + 84x + 315y = 253 \tag{8}$$

Parabola Exercise Solution

$$2y^2 - \frac{1}{2}x - 12y + 19 = 0$$

Step 1: isolate the variable that is not squared.

$$\frac{1}{2}x = 2y^2 - 12y + 19$$

$$x = \boxed{4y^2 - 24y} + 38$$

Step 2: complete the square for the terms involving the squared variable.

$$x = 4(y^2 - 6y) + 38$$

$$x = 4(y^2 - 6y + 9 - 9) + 38$$

$$x = 4(y^2 - 6y + 9) - 36 + 38$$

$$x = 4(y - \boxed{3})^2 + \boxed{2}$$

Step 3: read off the coordinates of the vertex.

$$V = (2, 3)$$

Parabola Exercise Solutions

$$2y^2 - \frac{1}{2}x - 12y + 19 \longrightarrow x = 4(y-3)^2 + 2 \tag{9}$$

The vertex is $V = (2, 3)$.

$$-63x^2 + 84x + 315y = 253 \longrightarrow y = \frac{1}{5}\left(x - \frac{2}{3}\right)^2 + \frac{5}{7} \tag{10}$$

The vertex is $V = \left(\frac{2}{3}, \frac{5}{7}\right)$.

Here is one form of a circle, which tells us where the centre of the circle and the radius are.

$$(x-p)^2+(y-q)^2=r^2 \tag{11}$$

Find centre and radius for the following two circle equations.

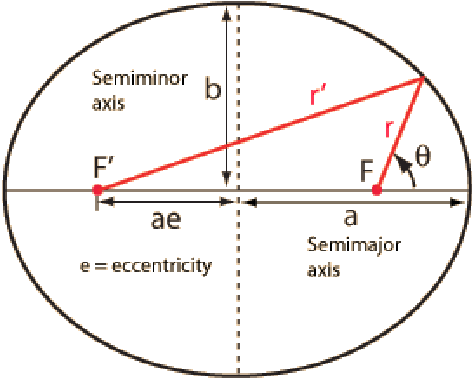
$$x^2+y^2-4x-21=0 \tag{12}$$

$$x^2+y^2-4x-6y-3=0 \tag{13}$$

Now, find the equation of the circle for which the centre is $M=(4,-2)$ and the radius is $r=10$.

Ellipse I

There is a lot going on when we consider an ellipse. Consider the following diagram.



Ellipse II

One way to define an ellipse is to say that for all points P on an ellipse, the distance of P from the two points of focus sum to a constant. This constant is $2a$, two times the semimajor axis.
Here is how we will define an ellipse. Points on an ellipse fulfill the equation

$$\frac{(x-p)^2}{a^2}+\frac{(y-q)^2}{b^2}=1 \tag{14}$$

The centre of the ellipse is at $M=(p,q)$. a is the length of the semimajor axis, b is the length of the seminor axis.

Ellipse Exercises

Determine centre and dimensions of the following ellipse,

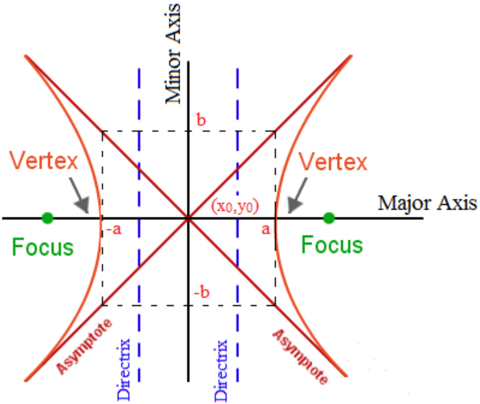
$$x^2 + 2y^2 = 2 \tag{15}$$

Do the same for the following ellipse,

$$9x^2 - 18x + 4y^2 + 40y = -73 \tag{16}$$

Hyperbola I

A hyperbola, just like an ellipse, has two points of focus. The hyperbola is the set of points for which the difference of the distance to the two foci is constant. This difference, again analogous to an ellipse, is $2a$, which is also the distance between the two vertices of the hyperbola.



Hyperbola II

Hyperbolas have two branches which approach lines called asymptotes. There is also a major axis and a minor axis. The major axis is sometimes called the transverse axis. If the two asymptotes are perpendicular to each other, the hyperbola is a rectangular hyperbola. The function $f(x) = 1/x$ is the graph of a rectangular hyperbola.

Hyperbola III

If the transverse axis is horizontal, one form of an equation for a hyperbola identifying the centre M is

$$\frac{(x-p)^2}{a^2} - \frac{(y-q)^2}{b^2} = 1 \tag{17}$$

If the transverse axis is vertical, one form of an equation for a hyperbola identifying the centre M is

$$\frac{(y-q)^2}{a^2} - \frac{(x-p)^2}{b^2} = 1 \tag{18}$$

Find the centre, vertices, foci, eccentricity, and asymptotes of the hyperbola with the given equation, and sketch,

$$\frac{y^2}{25} - \frac{x^2}{144} = 1 \tag{19}$$

Give the center, vertices, foci, and asymptotes for the hyperbola with equation,

$$\frac{(x+3)^2}{16} - \frac{(y-2)^2}{9} = 1 \tag{20}$$

Find the center, vertices, and asymptotes of the hyperbola with equation,

$$4x^2 - 5y^2 + 40x - 30y - 45 = 0 \tag{21}$$

Hyperbola Exercise Solution

$$\frac{(x+3)^2}{16} - \frac{(y-2)^2}{9} = 1 \tag{22}$$

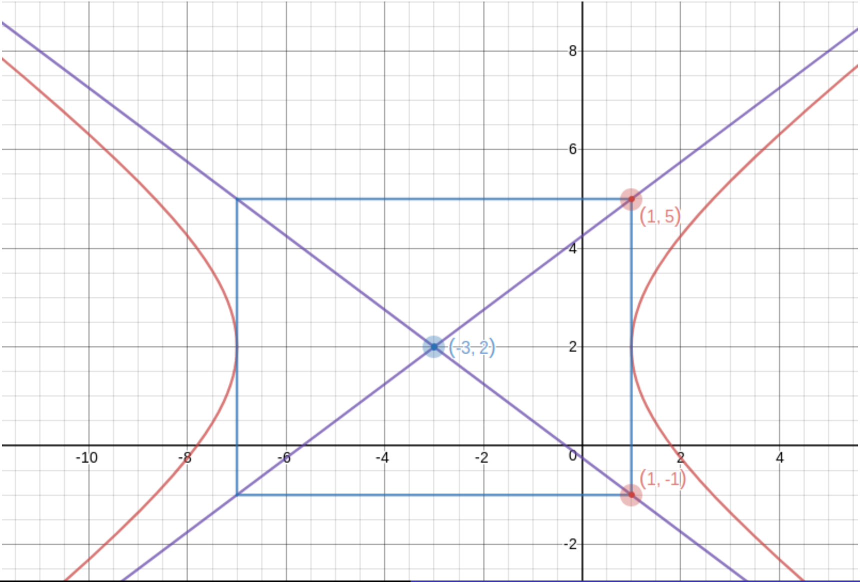
The centre is $M = (-3, 2)$. The dimensions are $a = 4, b = 3$. Because the coefficient for x^2 is positive and the coefficient for y^2 is negative, the transverse axis is horizontal. Considering that $P_1 = (1, 5)$ and M are on the upward-sloping asymptote A_1, the equation for A_1 is

$$y = \frac{3}{4}x + \frac{17}{4} \tag{23}$$

Considering that $P_2 = (1, -1)$ and M are on the downward-sloping asymptote A_2, the equation for A_2 is

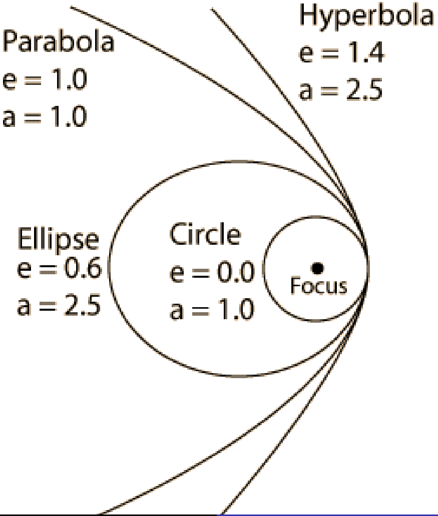
$$y = -\frac{3}{4}x - \frac{1}{4} \tag{24}$$

Hyperbola Exercise Solution Diagram



Eccentricity

The concept of eccentricity is another unifying feature for different types of conics.



End of Lesson

Next Lesson: Vectors.