Problem of Three Missing Quantities: Non-Right Triangles

MATH 1511, BCIT

Technical Mathematics for Geomatics

October 4, 2017

Law of Sines

Draw the circumcircle of a triangle and call its midpoint M. Why does M always uniquely exist? Draw the radius of the circumcircle from A to M, from B to M, and from C to M. Now note that the angles at the midpoint M are 2α, 2β, and 2γ. Draw the halfway point between B and C and call it D. The triangle MDC is a right triangle. The angle at M is α. Therefore,

$$a = 2r \sin \alpha \tag{1}$$

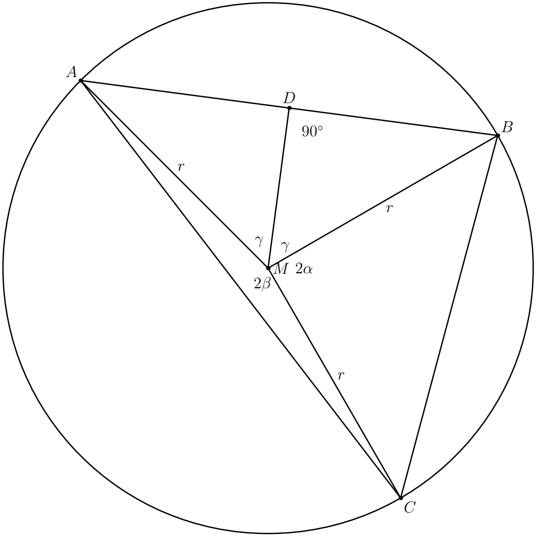
By symmetry,

$$b = 2r \sin \beta \text{ and } c = 2r \sin \gamma \tag{2}$$

and therefore (this is called the law of sines),

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} \tag{3}$$

Law of Sines



A circumcircle exists because there is a point M which is the intersection of the two lines going through the midpoints of A, B and B, C, respectively, and being perpendicular to AB and BC, respectively. $\overline{MA} = \overline{MB}$ and $\overline{MB} = \overline{MC}$ because of the way we have constructed M. Therefore $\overline{MA} = \overline{MC}$ so that the circle around M going through A and B also goes through C.

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{4}$$

It's inverse is

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \tag{5}$$

Law of Sines

Use the fact that ABM, BCM, ACM are isosceles triangles for

$$A \cdot \begin{bmatrix} \psi \\ \phi \\ \theta \end{bmatrix} = \begin{bmatrix} 360^\circ - 2\alpha \\ 360^\circ - 2\beta \\ 360^\circ - 2\gamma \end{bmatrix} \tag{6}$$

where $\theta = \angle BMC, \phi = \angle AMC, \psi = \angle AMB$. Then

$$\begin{bmatrix} \psi \\ \phi \\ \theta \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 360^\circ - 2\alpha \\ 360^\circ - 2\beta \\ 360^\circ - 2\gamma \end{bmatrix} \tag{7}$$

establishing that $\theta = 2\alpha, \phi = 2\beta, \psi = 2\gamma$.

Law of Cosines

Draw the height of a triangle (for example, at B) and call it h. Let its foot on the side b be called D. Let x be the segment from A to D. Note that

$$x^2 + h^2 = c^2 \tag{8}$$

and

$$x = c \cdot \cos\alpha \tag{9}$$

(8) and (9) substituted in

$$a^2 = (b - x)^2 + h^2 \tag{10}$$

give us the law of cosines

$$a^2 = b^2 - 2bx + (x^2 + h^2) = b^2 + c^2 - 2bc \cdot \cos\alpha \tag{11}$$

Definition of Inverse Trigonometric Functions

Here is a formal definition of the arcsin function.

Arcsin Function

$\arcsin : [-1, 1] \to \mathbb{R}$ such that

$$\arcsin(x) = \vartheta \text{ if and only if } \sin\vartheta = x \text{ and } -90^\circ \leq \vartheta \leq 90^\circ$$

- $\arcsin : [-1, 1] \to \mathbb{R}$ has a range of $[-90^\circ, 90^\circ]$.
- $\arccos : [-1, 1] \to \mathbb{R}$ is defined similarly with a range of $[0^\circ, 180^\circ]$.
- $\arctan : \mathbb{R} \to \mathbb{R}$ is defined similarly with a range of $[-90^\circ, 90^\circ]$.
- $\arctan : \mathbb{R} \to \mathbb{R}$ is defined similarly with a range of $[0^\circ, 180^\circ]$.

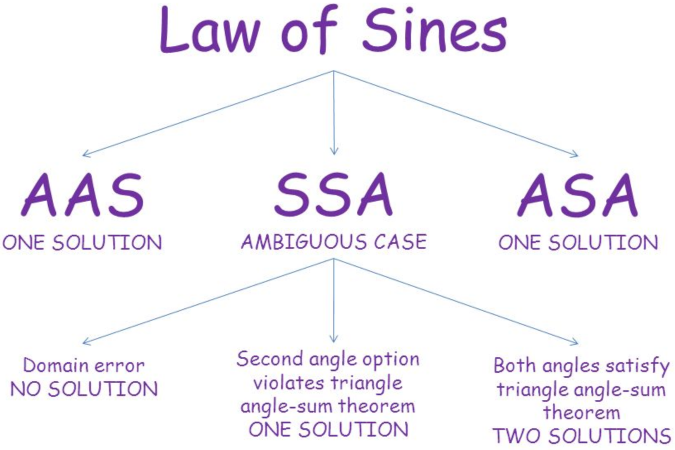
Applying the Laws of Sines/Cosines

SSS Use the law of cosines to find an angle. Make sure to solve for the largest angle (across from the longest side) first, or else a subsequent application of the law of sines may give you an incorrect result!

SAS Use the law of cosines to find the missing side. Then (if you are using the law of sines) make sure to solve for the smaller angle (across from the shorter side) first, or else the law of sines may give you an incorrect result!

SSA Use the law of sines. Consider both the acute and the obtuse solution suggested by the arcsin function.

AAS (Any two angles and one side.) Calculate the third angle using the triangle postulate (the three angles add up to 180°). Then use the law of sines.


Law of Sines
AAS
ONE SOLUTION
SSA
AMBIGUOUS CASE
ASA
ONE SOLUTION
Domain error
NO SOLUTION
Second angle option
violates triangle
angle-sum theorem
ONE SOLUTION
Both angles satisfy
triangle angle-sum
theorem
TWO SOLUTIONS

Law of Cosines Scenarios

Law of Cosines

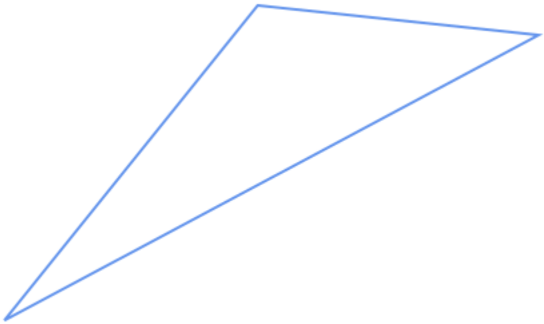
SAS

Use LoC to solve for side corresponding to included angle first. Then use LoS to solve for smallest of remaining two angles. Find third through subtraction.

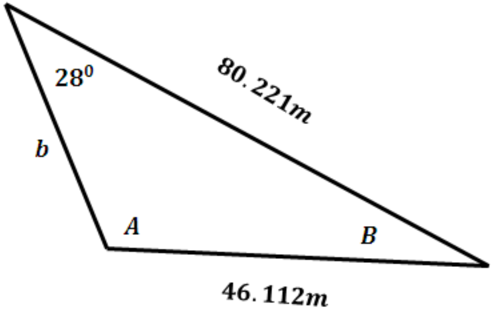
SSS

Use LoC to solve for largest angle first. Then use LoS to find either of remaining two angles. Find third through subtraction.

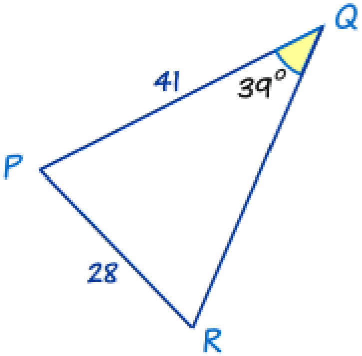
Exercise 1: Solve the following triangle.
$\alpha = 23^\circ, a = 7in, c = 10in$. ($\alpha$ is the acute angle on the bottom. Vertices are labeled clockwise.)



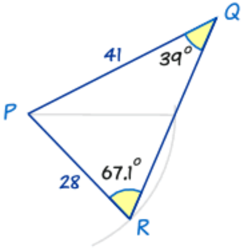
Exercise 2: Solve the following triangle.
$\gamma = 28^\circ, a = 80.221m, c = 46.112m.$

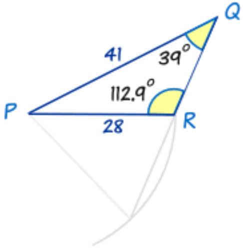


Exercise 3: Solve the triangle in the diagram.

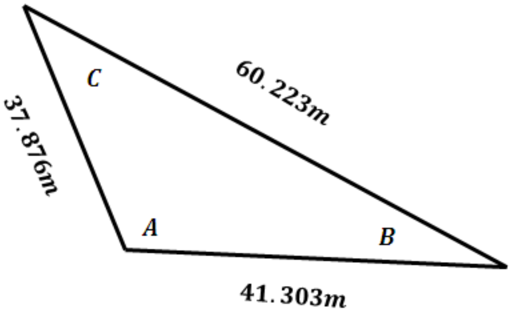


Note that there are two possible answers in this case. There are only two possible answers in the scenario where you are given two sides and an angle that is not between the two sides.





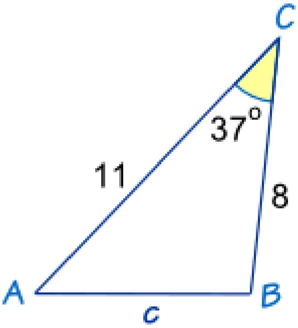
Exercise 4: Solve the following triangle.
$a = 60.223m, b = 37.876m, c = 41.303m.$



Law of Cosines Nota Bene

Solve for the largest angle first! The cosine is one-to-one on the domain $[0^\circ, 180^\circ]$, so the arccosine will give you a unique result and correctly identify an obtuse angle. If you solve for a smaller angle first and then use the law of sines to find the obtuse angle, the arcsine will give you the acute angle instead of the obtuse one. Alternatively, you can always use the law of cosines and not get any ambiguities.

Exercise 5: Solve the triangle in the diagram.



Exercise 6: Solve the following triangles.

1. $a = 11, b = 6, c = 7$
2. $b = 0.49, c = 0.98, \alpha = 19^\circ$
3. $b = 8, c = 7, \beta = 48^\circ$
4. $a = 211, \beta = 96^\circ 14' 51'', \gamma = 31^\circ 1' 40''$

End of Lesson

Next Lesson: Quadratic Equations