

# Conics

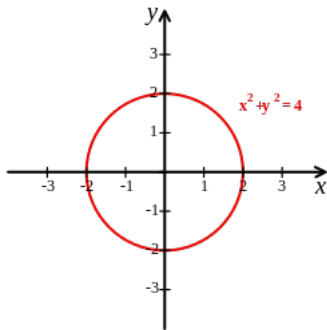
MATH 1511, BCIT

Technical Mathematics for Geomatics

October 30, 2017

# Curves

Equations have solution sets. These solution sets are subsets of spaces like the number line (one variable), the plane (two variables), or three-dimensional space (three variables). When there are two variables, the solution set may describe a curve. A curve is a one-dimensional space inside a higher-dimensional space, for example the plane. The most simple example is the circle. Consider the equation

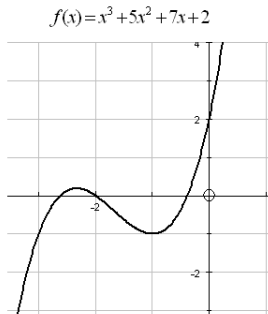


# Polynomials

The following expression is called a polynomial.

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 \quad (1)$$

For example,  $f(x) = x^3 + 5x^2 + 7x + 2$ .  $n \geq 0$  is a natural number called the **degree of the polynomial**, and the  $a_i$  are real numbers called **coefficients**. Here is what a curve described by a polynomial with degree 3 looks like.



# Equations and Curves

Curves in the plane may correspond to equations with two variables, as in the example with the circle. Sometimes, we push everything in the equation to the LHS (left-hand side) and leave 0 on the RHS (right-hand side). For the circle,

$$x^2 + y^2 - 4 = 0 \quad (2)$$

and for the polynomial in the last slide,

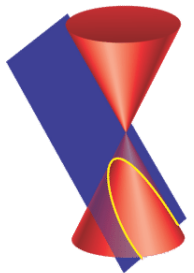
$$x^3 + 5x^2 + 7x + 2 - y = 0 \quad (3)$$

# Conics I

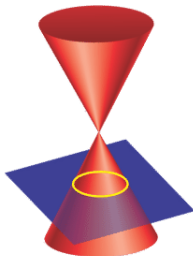
It turns out that all curves that correspond to polynomials of degree 2 or less in two-dimensional space are conic sections or **conics**. Equations such as

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (4)$$

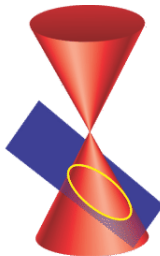
with  $A, B, C, D, E, F$  being real numbers (with  $|A| + |B| + |C| \neq 0$ ) correspond to curves that can be thought of as intersections of the plane with a **parabola**, an **ellipse**, or a **hyperbola**.



parabola



circle



ellipse

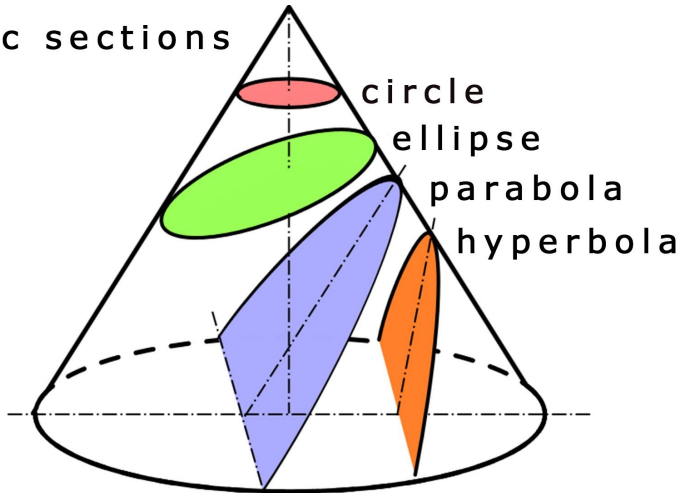


hyperbola

# Conics II

Circles are just special cases of ellipses. Is the polynomial curve with degree 3 from a couple of slides ago a conic? Why (not)?

## Conic sections

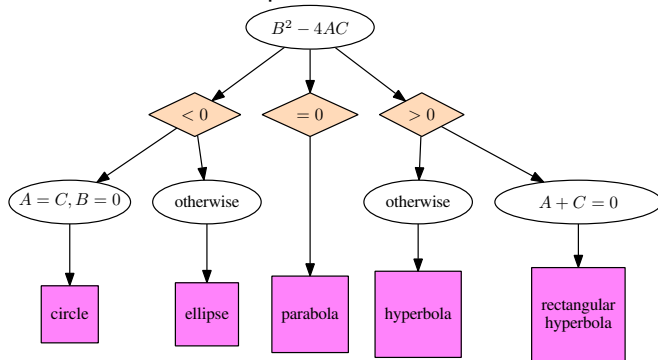


# Conics III

If (for now  $B$  is always zero in order to avoid conics whose symmetries do not line up with the coordinate system)

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad (5)$$

is the equation of the conic section, then  $B^2 - 4AC$  is the **discriminant** of the equation.



Classify the following equations according to the type of conic each represents.

①  $3x^2 + 3y^2 - 6x + 9y - 14 = 0$

②  $6x^2 + 12x - y + 15 = 0$

③  $x^2 + 2y^2 + 4x + 2y - 27 = 0$

④  $x^2 - y^2 + 3x - 2y - 43 = 0$



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# Parabolas

Here is one form of a parabola, which tells us where the vertex is. The vertex is the point where the line of symmetry intersects with the parabola.

$$y = a(x - p)^2 + q \text{ or } x = a(y - q)^2 + p \quad (6)$$

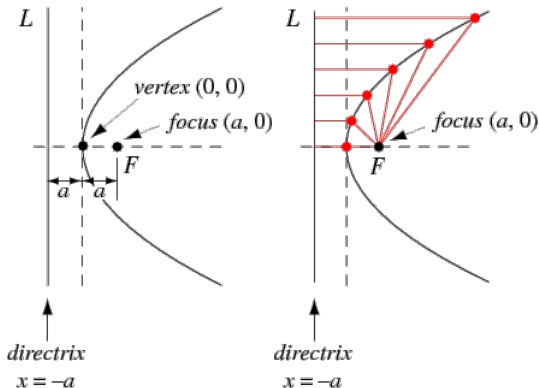
$a \neq 0$ ,  $p, q$  are real numbers. The vertex is at  $(p, q)$ .

- If  $a > 0$  and  $y$  is not squared, then the parabola opens upward. The parabola is **convex**.
- If  $a < 0$  and  $y$  is not squared, then the parabola opens downward. The parabola is **concave**.
- If  $a > 0$  and  $x$  is not squared, then the parabola opens to the right.
- If  $a < 0$  and  $x$  is not squared, then the parabola opens to the left.

Exercise: determine whether the functions  $f(x) = e^x$  and  $g(x) = \ln x$  are convex or concave.

# Vertex, Focus, Directrix

Besides a **vertex**, a parabola also has a **focus** and a **directrix** (two out of these three uniquely determine a parabola). You may learn later how to calculate them given the equation of the parabola; or to determine the equation for the parabola, given the focus and the directrix. All points on a parabola are equidistant (equally far away) from the directrix and the focus.



Consider the following curve.

$$2y^2 - \frac{1}{2}x - 12y + 19 = 0 \quad (7)$$

Use the discriminant to show that the curve is a parabola.  
Calculate the position of the vertex and determine whether the parabola opens upward, downward, to the right, or to the left.

Try again with the following equation.

$$-63x^2 + 84x + 315y = 253 \quad (8)$$

# Parabola Exercise Solution

$$2y^2 - \frac{1}{2}x - 12y + 19 = 0$$

Step 1: isolate the variable that is not squared.

$$\frac{1}{2}x = 2y^2 - 12y + 19$$

$$x = 4y^2 - 24y + 38$$

Step 2: complete the square for the terms involving the squared variable.

$$x = 4(y^2 - 6y) + 38$$

$$x = 4(y^2 - 6y + 9 - 9) + 38$$

$$x = 4(y^2 - 6y + 9) - 36 + 38$$

$$x = 4(y - 3)^2 + 2$$

Step 3: read off the coordinates of the vertex.

$$V = (2, 3)$$

# Parabola Exercise Solutions

$$2y^2 - \frac{1}{2}x - 12y + 19 \longrightarrow x = 4(y - 3)^2 + 2 \quad (9)$$

The vertex is  $V = (2, 3)$ .

$$-63x^2 + 84x + 315y = 253 \longrightarrow y = \frac{1}{5} \left( x - \frac{2}{3} \right)^2 + \frac{5}{7} \quad (10)$$

The vertex is  $V = \left( \frac{2}{3}, \frac{5}{7} \right)$ .



Here is one form of a circle, which tells us where the **centre** of the circle and the **radius** are.

$$(x - p)^2 + (y - q)^2 = r^2 \quad (11)$$

Find centre and radius for the following two circle equations.

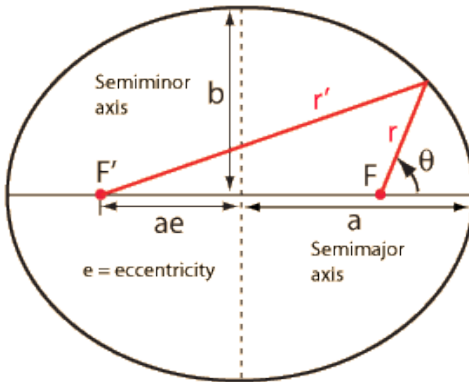
$$x^2 + y^2 - 4x - 21 = 0 \quad (12)$$

$$x^2 + y^2 - 4x - 6y - 3 = 0 \quad (13)$$

Now, find the equation of the circle for which the centre is  $M = (4, -2)$  and the radius is  $r = 10$ .

# Ellipse I

There is a lot going on when we consider an ellipse. Consider the following diagram.



One way to define an ellipse is to say that for all points  $P$  on an ellipse, the distance of  $P$  from the two points of **focus** sum to a constant. This constant is  $2a$ , two times the **semimajor axis**. Here is how we will define an ellipse. Points on an ellipse fulfill the equation

$$\frac{(x - p)^2}{a^2} + \frac{(y - q)^2}{b^2} = 1 \quad (14)$$

The **centre of the ellipse** is at  $M = (p, q)$ .  $a$  is the length of the semimajor axis,  $b$  is the length of the **semiminor axis**.

Determine centre and dimensions of the following ellipse,

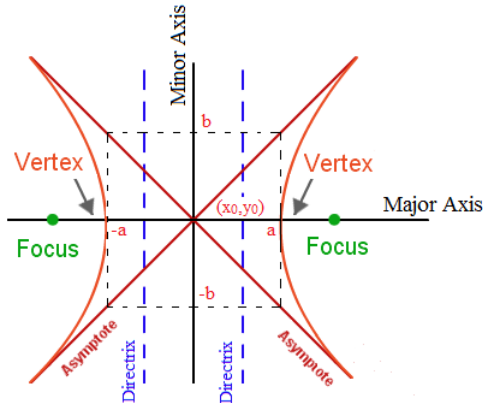
$$x^2 + 2y^2 = 2 \quad (15)$$

Do the same for the following ellipse,

$$9x^2 - 18x + 4y^2 + 40y = -73 \quad (16)$$

# Hyperbola I

A hyperbola, just like an ellipse, has two points of **focus**. The hyperbola is the set of points for which the difference of the distance to the two foci is constant. This difference, again analogous to an ellipse, is  $2a$ , which is also the distance between the two **vertices** of the hyperbola.



# Hyperbola II

Hyperbolas have two branches which approach lines called **asymptotes**. There is also a **major axis** and a **minor axis**. The major axis is sometimes called the **transverse axis**. If the two asymptotes are perpendicular to each other, the hyperbola is a **rectangular hyperbola**. The function  $f(x) = 1/x$  is the graph of a rectangular hyperbola.

# Hyperbola III

If the transverse axis is horizontal, one form of an equation for a hyperbola identifying the centre  $M$  is

$$\frac{(x - p)^2}{a^2} - \frac{(y - q)^2}{b^2} = 1 \quad (17)$$

If the transverse axis is vertical, one form of an equation for a hyperbola identifying the centre  $M$  is

$$\frac{(y - q)^2}{a^2} - \frac{(x - p)^2}{b^2} = 1 \quad (18)$$

# Hyperbola Exercises

Find the centre, vertices, foci, eccentricity, and asymptotes of the hyperbola with the given equation, and sketch,

$$\frac{y^2}{25} - \frac{x^2}{144} = 1 \quad (19)$$

Give the center, vertices, foci, and asymptotes for the hyperbola with equation,

$$\frac{(x+3)^2}{16} - \frac{(y-2)^2}{9} = 1 \quad (20)$$

Find the center, vertices, and asymptotes of the hyperbola with equation,

$$4x^2 - 5y^2 + 40x - 30y - 45 = 0 \quad (21)$$



# Hyperbola Exercise Solution

$$\frac{(x+3)^2}{16} - \frac{(y-2)^2}{9} = 1 \quad (22)$$

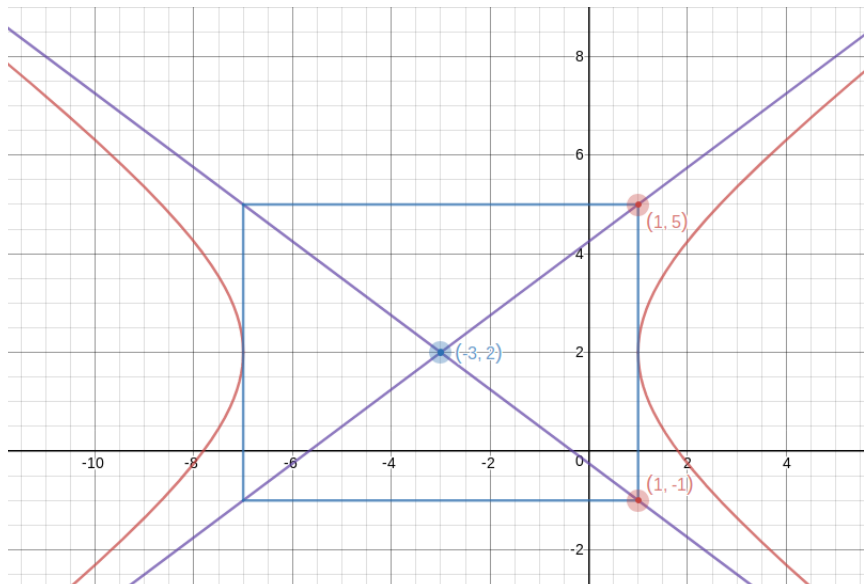
The centre is  $M = (-3, 2)$ . The dimensions are  $a = 4$ ,  $b = 3$ . Because the coefficient for  $x^2$  is positive and the coefficient for  $y^2$  is negative, the transverse axis is horizontal. Considering that  $P_1 = (1, 5)$  and  $M$  are on the upward-sloping asymptote  $A_1$ , the equation for  $A_1$  is

$$y = \frac{3}{4}x + \frac{17}{4} \quad (23)$$

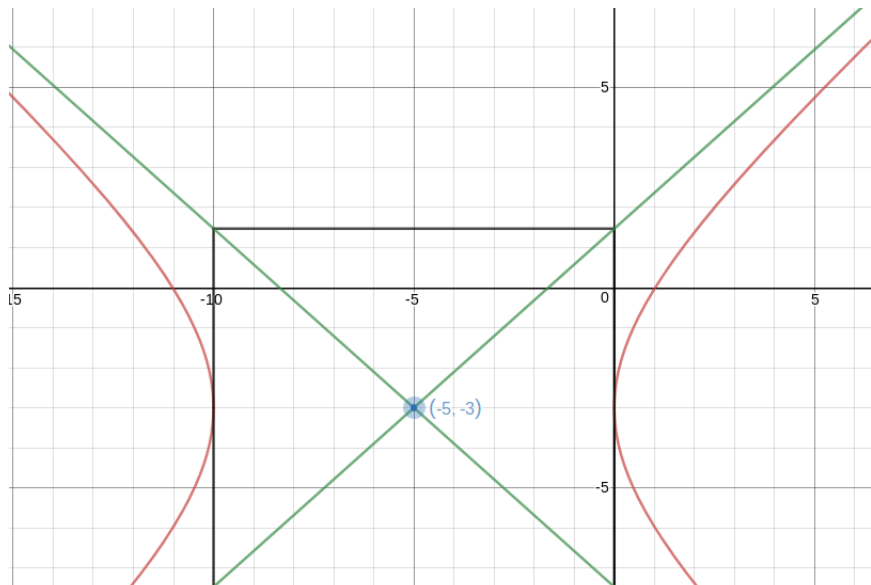
Considering that  $P_2 = (1, -1)$  and  $M$  are on the downward-sloping asymptote  $A_2$ , the equation for  $A_2$  is

$$y = -\frac{3}{4}x - \frac{1}{4} \quad (24)$$

# Hyperbola Exercise Solution Diagram

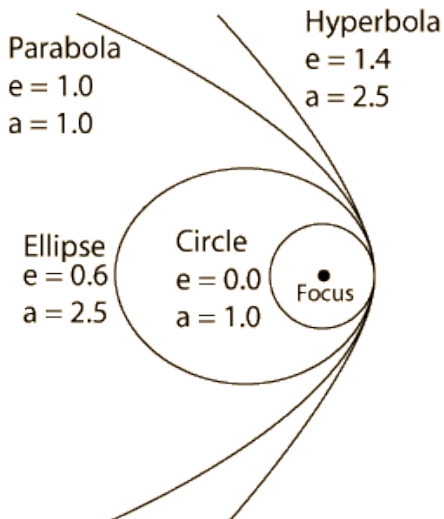


# Hyperbola Exercise Solution Diagram



# Eccentricity

The concept of eccentricity is another unifying feature for different types of conics.



# End of Lesson

Next Lesson: Vectors.