Spherical Trigonometry

MATH 1511, BCIT

Technical Mathematics for Geomatics

November 29, 2017

Captain America I

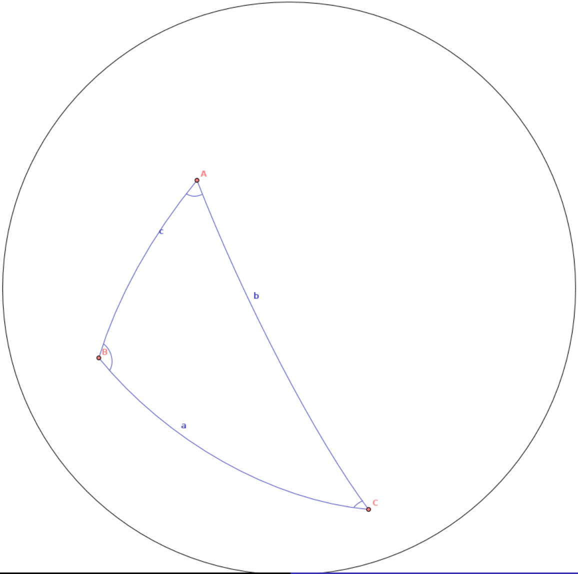
Realizing that in our present national emergency trigonometry is used in practically every phase of our war effort, the objective of the authors in writing this book was to present a brief but mathematically accurate course in plane and spherical trigonometry with special emphasis on the computational or practical side of the subject. In those chapters dealing with computational trigonometry, thorough drill is first given through the use of many examples.

Captain America II

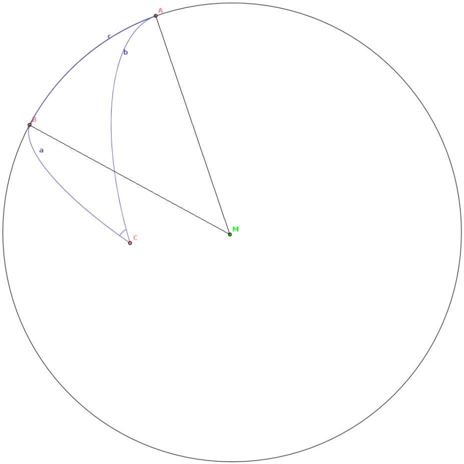
This is followed at the end of each of these chapters by practical applications introduced as problems, along with the necessary explanations and definitions, to secure conciseness of presentation. The applications deal with surveying, gun fire, course and position of airplanes, and navigation. This arrangement of theory and application in the book has permitted a sharp presentation of the underlying ideas which is necessary for rapid mastery of the subject. (Clifford Bell and Tracy Thomas, Essentials of Plane and Spherical Trigonometry, 1943, page iii)

Fundamentals of Spherical Trigonometry

Consider a spherical triangle.



Now notice that all sides of the spherical triangle correspond to angles at the centre of the sphere (in the diagram, the side c corresponds to the angle at M).



Fundamentals of Spherical Trigonometry

In spherical trigonometry, sides are always given as angles. If you need to find the length, multiply by the radius. A side $\pi/4$ (45°) on the surface of the Earth, for example, has length

$$\frac{\pi}{4} \cdot 6378.1km = 5009.3km \tag{1}$$

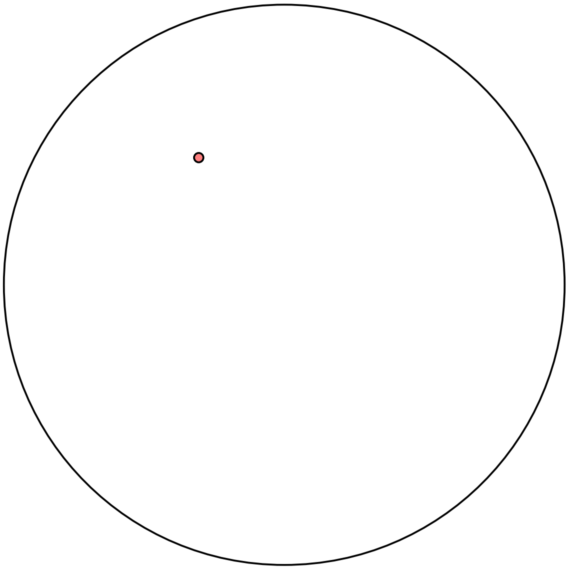
Three points on a sphere define eight different spherical triangles, depending on which way you loop around the sphere, the short way or the long way. We will always consider the triangle looping around the short way for all three sides. Therefore, all angles will be strictly between 0° and 180°.

The Halifax Problem I

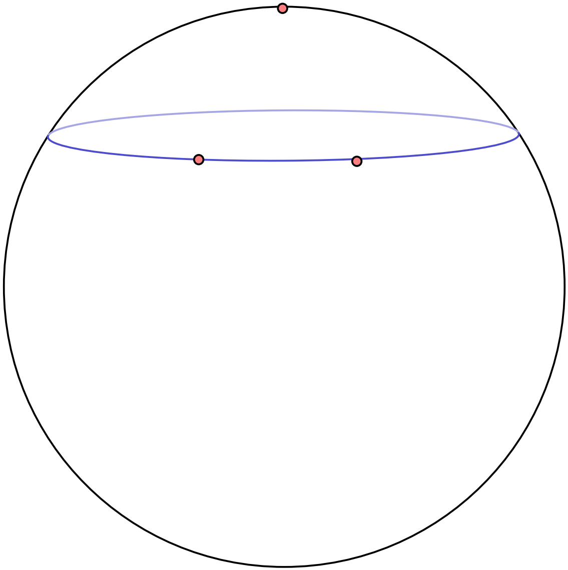
This is a problem from Raymond Brink, *Spherical Trigonometry*, 1942, page 17.

> *A ship leaves Halifax (position,* $44.67^\circ N, 63.58^\circ W$*), starting due east [...]. Find its position and direction after it has sailed 1000 nautical miles.*

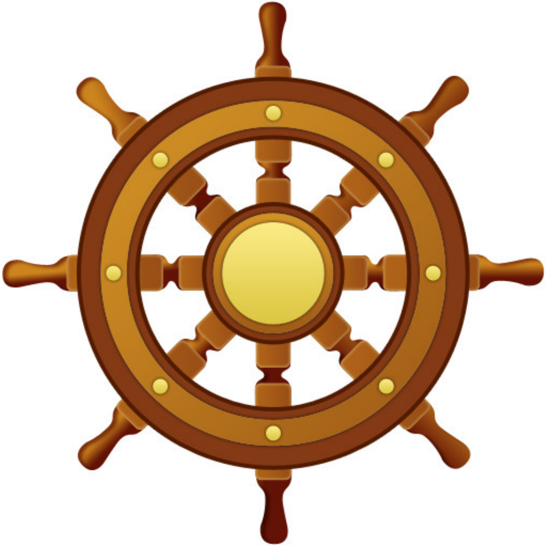
The Halifax Problem II



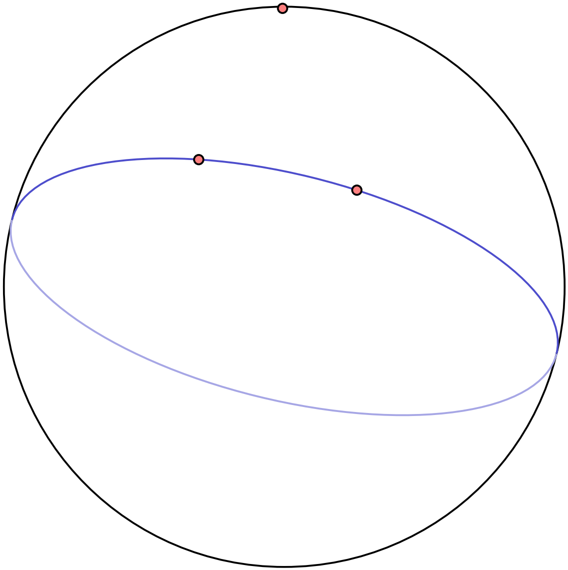
The Halifax Problem III



The Halifax Problem IV



The Halifax Problem V



The Halifax Problem IV

This is a problem from Raymond Brink, *Spherical Trigonometry*, 1942, page 17.

> *A ship leaves Halifax (position,* $44.67^\circ N, 63.58^\circ W$*), starting due east and continuing on the great circle. Find its position and direction after it has sailed 1000 nautical miles.*

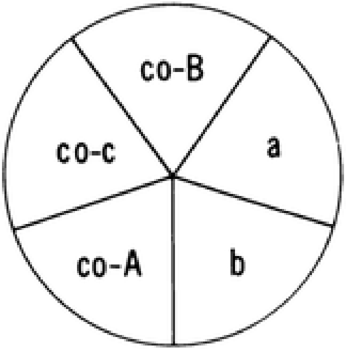
Great Circles

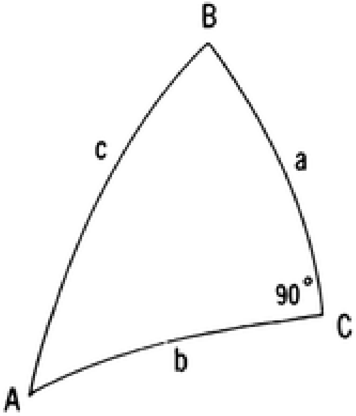
The intersection between a plane and a sphere is a circle. If the centre of the sphere is an element of the plane, then the intersection is a great circle. A spherical angle at point P is an arc length on a great circle for which P is the pole.

Triangle Sum

The sum of the angles of a spherical triangle is less than six right angles and greater than two right angles.

Napier's Pentagramma Mirificum

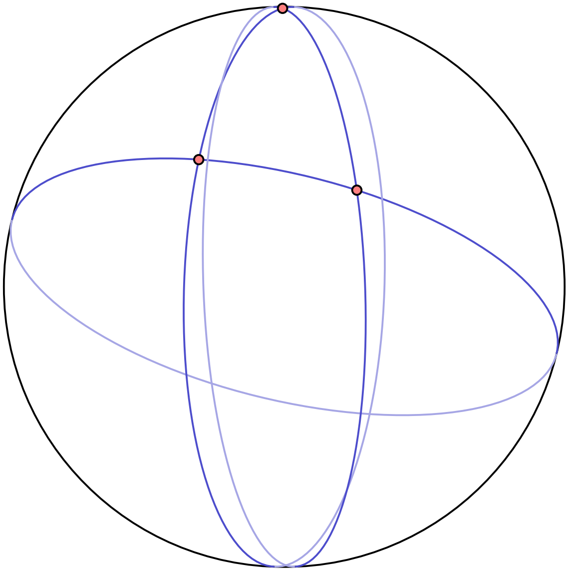




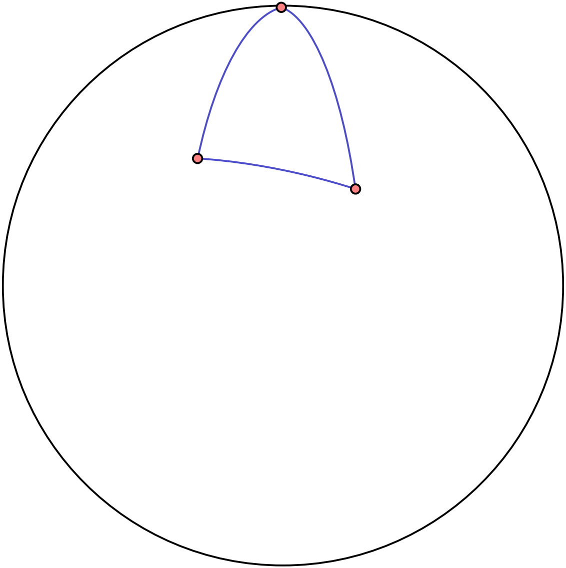
Napier's Rules

- *Rule I:* The sine of any circular part is equal to the product of the tangents of the two parts adjacent to it.
- *Rule II:* The sine of any circular part is equal to the product of the cosines of the two parts opposite to it.

The Right-Angled Euler Triangle I



The Right-Angled Euler Triangle II



Halifax Problem: Latitude

Use the relevant three slices of the Pentagramma Mirificum for the following formula:

$$\cos c = \cos a \cdot \cos b \tag{2}$$

a is 1000 nautical miles. One nautical mile is one minute of arc, or $\frac{1}{60}^\circ$, on the Earth's surface. Therefore, $a = 16.667^\circ$ and $b = 45.33^\circ$. Using the inverse function of cosine on $\cos a \cdot \cos b$ and subtracting the result from 90°, the result for the latitude of E is $42.337^\circ N$.

Halifax Problem: Longitude

Use the relevant three slices of the Pentagramma Mirificum for the following formula:

$$\cos A = \cot c \cdot \tan b \tag{3}$$

Using the inverse function of cosine on $\cot c \cdot \tan b$ and subtracting the result from 63.58°, the result for the longitude of E is $40.75^\circ W$.

Halifax Problem: Direction

Use the relevant three slices of the Pentagramma Mirificum for the following formula:

$$\cos B = \cot c \cdot \tan a \tag{4}$$

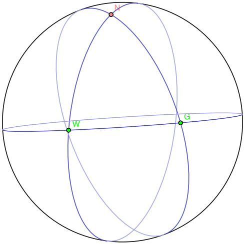
Using the inverse function of cosine on $\cot c \cdot \tan a$, the result for the direction at E is 74.171° east of south.

Exercise 1: Remember this problem a while ago:

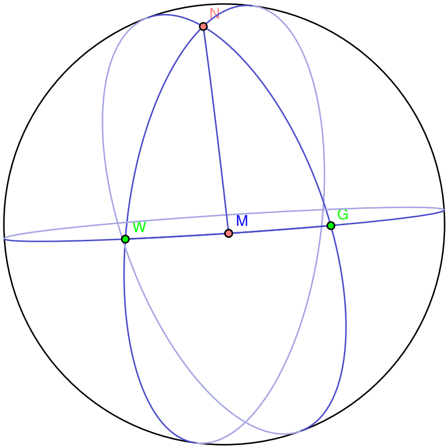
Windsor to Grenoble

Consider two towns, Windsor, Nova Scotia, at $(45^\circ N, 65^\circ W)$ and Grenoble, France at $(45^\circ N, 5^\circ E)$. If you follow a line of latitude, how far are the two towns apart?

We can now calculate the distance along the great circle.



Solution



Solution

NWG is an isosceles triangle. Consequently, *NWM* is a right triangle. Let $a = NM, b = MW, c = WN = 45^\circ$. The angle $\angle WNM$ is 35°. Napier's miraculous pentagram gives us

$$\sin b = \sin 35^\circ \cdot \sin 45^\circ \text{ therefore } b = 0.41761 \tag{5}$$

b is in radians in equation (5). Multiply twice this number by the radius of the Earth (6378.1km), and the correct solution is approximately 5327.2km, compared to approximately 5510.0km along the circle of latitude.

Note that the arcsine only gives us the shorter of two solutions: we could also loop around the Pacific Ocean instead of the Atlantic Ocean and get a much longer distance. The two solutions add up to the circumference of the Earth.

Laws of Quadrants

When you take the arcsine of a number on a calculator, the calculator will always give you an angle less than 90°, for example the arcsine of $\pi/4$ is 45°. However, the angle that you want may be 135°, for $\sin 135^\circ = \pi/4$ as well. The law of quadrants helps you identify which angle (the acute or the obtuse) you should accept as your solution.

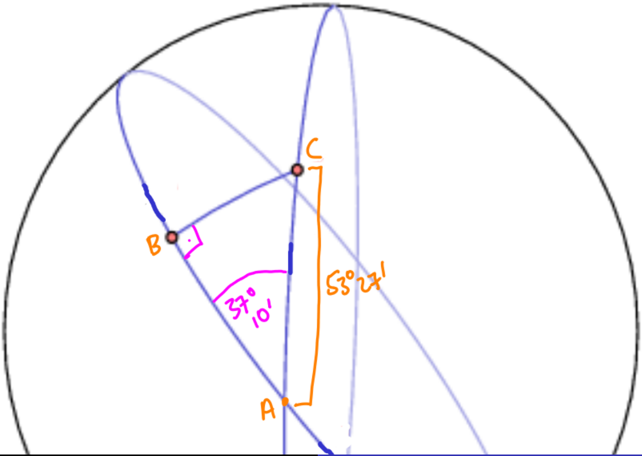
Angles between 0° and 90° are considered to be in the first quadrant. Angles between 90° and 180° are considered to be in the second quadrant.

Law of Quadrants for Right Triangles

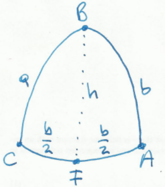
For right spherical triangles,

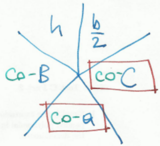
- LoQ I An angle and its opposite side are in the same quadrant.
- LoQ II If any two of the three sides are in the same quadrant, the third side is in the first quadrant.
- LoQ III If any two sides are in different quadrants, the third side is in the second quadrant.

Exercise 2: One of the angles formed by the intersection of two great circles is $37°10'$. A point on one of the circles is $53°27'$ from the intersection point of the circles. Find the shortest distance from this point to the other circle.



Exercise 3: Solve the isosceles triangle ABC, where $a = c = 79^\circ 17'$ and $A = C = 59^\circ 37'$.





$$a = c = 79^\circ 17' \approx 1.3838$$

$$C = A = 59^\circ 37' \approx 1.0405$$

$$\sin h = \sin C \sin a$$

$$h = 1.0115 = 57.953^\circ$$

$$\cos B = \tan h \cdot \cot a$$

$$B = 1.2637 = 72.403^\circ = 72^\circ 24'$$

$$\sin \frac{b}{2} = \sin B \cdot \sin a$$

$$\frac{b}{2} = 1.2127$$

$$b = 2.4255 \cong 138^\circ 58'$$

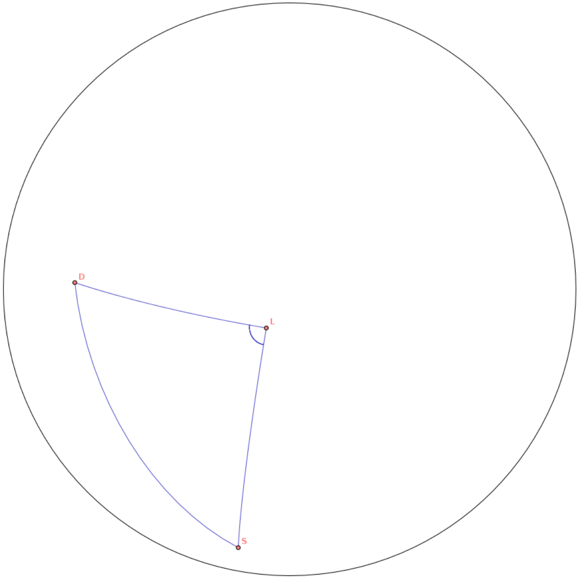
Ambiguous Case

Consider the following problem.

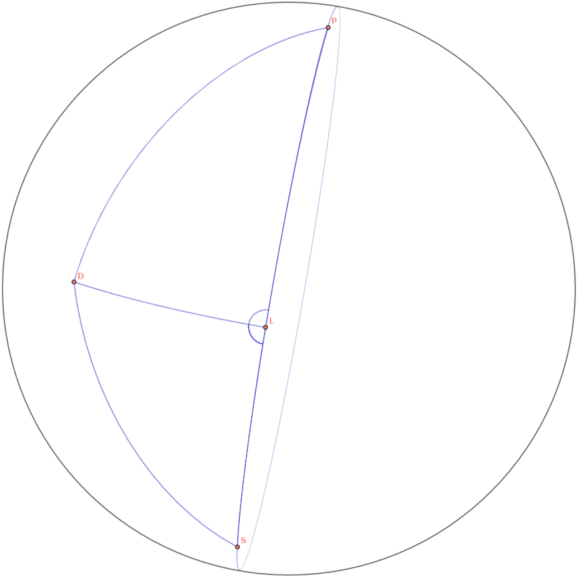
Sailing from Lima

Your GPS is broken. You can only read your longitudes, but not your latitudes. You start from the Peruvian coast going west on a great circle at $77^\circ 1' 42'' W$ and are now in the middle of the Pacific, having sailed 1750 nautical miles. Your longitude is $106^\circ 44' 27'' W$. Where in Peru did you start (which town or city)?

Ambiguous Case



Ambiguous Case



Ambiguous Case Answer

Let the right angle be at L and label it $C = 90^\circ$. Label sides a, b, c and angles A, B accordingly. We know $b = 29.167^\circ$ and $B = 29.712^\circ$. The relevant law from Napier's miraculous pentagram is

$$\sin a = \tan b \cdot \cot B \tag{6}$$

It follows that $\sin a = 0.97799$. There are two solutions for a.

$$\begin{aligned} a &= 77.957^\circ \\ a &= 102.04^\circ \end{aligned}$$

In the context of the word problem, only the first solution makes sense. The latitude of the origin is 12.043°, which on the coast of Peru is the latitude for Lima.

Non-Right (Oblique) Spherical Triangles

Use the following three laws to solve non-right (oblique) spherical triangles.

1. Law of Sines
2. Law of Cosines
3. Napier's Analogies

Theorems of Spherical Trigonometry

The following theorems help to figure out which (if any) of the two arcsine solutions need to be rejected. I will call them oblique spherical triangle laws or OSTL in contrast to the Law of Quadrants for right spherical triangles. OSTL III is the law that does most of the heavy lifting.

- **OSTL I** The sum of the sides of a spherical triangle is less than 360°.
- **OSTL II** If two sides of a spherical triangle are equal, the angles opposite these sides are equal; and conversely.
- **OSTL III** If two sides of a spherical triangle are unequal, the angle opposite the greater side is the greater; and conversely.
- **OSTL IV** Each side of a spherical triangle is less than the sum of the other two sides.

Polar Triangles

The three sides a, b, c of a spherical triangle D are arcs on a great circle. These three great circles have two poles each. Now choose three poles (one for each side) which determine a triangle such that all interior angles are strictly between 0° and 180°. This new triangle D' with sides a', b', c' is called the polar triangle of D. The polar triangle of D' is D (in other words, $D'' = D$). If A, B, C are the angles of D and A', B', C' are the angles of D', then

$$\boxed{\begin{aligned} a + A' &= 180^\circ & A + a' &= 180^\circ \\ b + B' &= 180^\circ & B + b' &= 180^\circ \\ c + C' &= 180^\circ & C + c' &= 180^\circ \end{aligned}}$$

Oblique Triangles Law of Cosines

Here is the cosine law for spherical triangles.

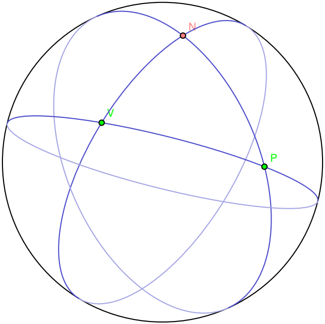
$$\cos a = \cos b \cos c + \sin b \sin c \cos A \tag{7}$$

The corresponding cosine law for angles derived from the polar triangle is

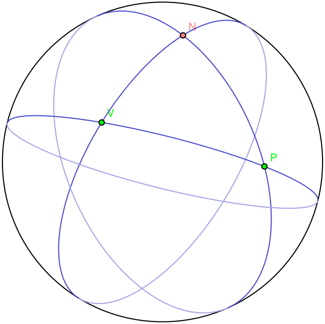
$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \tag{8}$$

Law of Cosines Example

Calculate the distance along the great circle between Vancouver (49°15′N, 123°6′W) and Palma de Mallorca (39°34′N, 2°39′E).



Law of Cosines Example Solution



Let N be the angle $\angle VNP$. Let $a = VN, b = NP, c = VP$.
According to the law of cosines,

$$\cos c = \cos a \cos b + \sin a \sin b cos N \tag{9}$$

Consequently, $c = 1.3811$ in radians, which translates to 8808.8km.

Here are Napier's Analogies for oblique triangles. There is nothing special about the labels a, b, c and A, B, C. These analogies are true for any permutation of a, b, c and A, B, C, as long as it is consistently applied. Note that $\sin \frac{1}{2}(a-b)$ means $\sin[\frac{1}{2}(a-b)]$, not $\sin[\frac{1}{2}] \cdot (a-b)$.

$$\frac{\tan \frac{c}{2}}{\tan\left(\frac{1}{2}(a-b)\right)} = \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \tag{10}$$

$$\frac{\tan \frac{c}{2}}{\tan\left(\frac{1}{2}(a+b)\right)} = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tag{11}$$

Napier's Analogies are helpful in roughly the same way the angle-sum formula $\alpha + \beta + \gamma = 180^\circ$ is helpful in plane trigonometry.

Oblique Triangles Law of Sines

Here is the sine law for spherical triangles.

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \tag{12}$$

The sine law usually gives you two solutions. Use OSTL III or the cosine law to reject one of the two solutions.

ABC and Non-ABC

There are two types of oblique triangles.

ABC Type: If the knowns are “ABC”, “ABc”, “AbC”, “abC”, etc., then the triangle is of the ABC type. Use the law of cosines first and then either the law of sines or Napier’s Analogies.

Non-ABC Type: If the knowns are “Aac”,“ABb”,“aCc”, etc., the triangle is of the non-ABC type. Use the law of sines first and then Napier’s analogies (in plane trigonometry, you would use the law of sines and then the angle-sum formula).

Spherical Trigonometry Decision Tree

There is a decision tree for solving spherical triangles in D2L.

Exercises

Exercise 4: Solve the following right spherical triangles:

$$A = 29^\circ 11' \qquad a = 23^\circ 56' \tag{13}$$

$$A = 33^\circ 20' \qquad B = 72^\circ 40' \tag{14}$$

$$a = 126^\circ 5' 20'' \qquad A = 105^\circ 55' 30'' \tag{15}$$

Exercise 5: Solve the following right spherical triangles:

$$a = 128^\circ 12' 10'' \qquad b = 48^\circ 56' 20'' \tag{16}$$

$$A = 79^\circ 2' \qquad a = 72^\circ 3' \tag{17}$$

$$a = 75^\circ 16' \qquad b = 130^\circ 6' \tag{18}$$

Exercises

Exercise 6: Solve the following right spherical triangles:

$$B = 84^\circ 14' 12'' \qquad b = 78^\circ 20' 36'' \tag{19}$$

$$b = 96^\circ 20' 45'' \qquad A = 52^\circ 8' 30'' \tag{20}$$

$$B = 111^\circ 42' \qquad b = 127^\circ 35' \tag{21}$$

Exercise 7: Solve the following isosceles spherical triangles:

$$A = B = 80^\circ \qquad C = 110^\circ \tag{22}$$

$$a = b = 83^\circ 12' 50'' \qquad c = 42^\circ 24' 10'' \tag{23}$$

$$B = C = 56^\circ 56' \qquad b = 82^\circ 12' \tag{24}$$

Exercise 8: Solve the following spherical triangles:

$$a = 122^\circ 18' \qquad b = 88^\circ 21' \qquad C = 100^\circ 16' \tag{25}$$

$$a = 44^\circ 10' \qquad b = 18^\circ 20' \qquad A = 64^\circ 30' \tag{26}$$

$$A = 127^\circ 20' \qquad B = 105^\circ 40' \qquad c = 124^\circ 30' \tag{27}$$

Exercise 9: Solve the following spherical triangles:

$$a = 30^\circ 40' \qquad b = 32^\circ 30' \qquad A = 88^\circ 2' \tag{28}$$

$$b = 80^\circ 5' \qquad c = 82^\circ 55' \qquad B = 85^\circ 12' \tag{29}$$

$$a = 54^\circ 20' \qquad b = 96^\circ 40' \qquad c = 122^\circ 18' \tag{30}$$

Exercises

Exercise 10: Solve the following spherical triangles:

$$A = 98^\circ 16' \qquad B = 82^\circ 24' \qquad C = 38^\circ 48' \tag{31}$$

$$a = 148^\circ 12' \qquad c = 140^\circ 33' \qquad A = 152^\circ 45' \tag{32}$$

$$a = 96^\circ 54' \qquad B = 82^\circ 6' \qquad c = 104^\circ 36' \tag{33}$$

Exercise 11: Find the length of the great-circle track from the Brooklyn Navy Yard $(40^\circ 42' N, 73^\circ 59' W)$ to the point where the prime meridian (the longitudinal circle that goes through Greenwich near London) crosses the equator.

Exercise 12: Find the length and the initial bearing of the great-circle track from Pearl Harbor $(21^\circ 27' N, 157^\circ 57' W)$ to San Francisco $(37^\circ 32' N, 122^\circ 13' W)$.

Exercise 13: A ship sails from Boston $(42^\circ 20' N, 70^\circ 53' W)$ to Lisbon $(38^\circ 40' N, 9^\circ 18' W)$. Find the distance of the great-circle track. Also find the bearing of the track when leaving Boston and when approaching Lisbon.

Exercise 14: Find the length of the shortest air route between Cape Town $(33^\circ 56' S, 18^\circ 28' E)$ and Dakar $(14^\circ 40' N, 17^\circ 25' W)$. What is the bearing of this track as the plane leaves Cape Town?

Exercise 15: Find the shortest distance between Greenwich ($51^\circ 29' N$) and New York City ($40^\circ 46' N, 73^\circ 51' W$). Also find the initial direction of the track.

Exercise 16: An airplane flies the great-circle track from Tokyo $(35^\circ 39' N, 139^\circ 45' E)$ to Wellington $(41^\circ 17' S, 174^\circ 47' E)$.

1. Find the length and initial direction of the great-circle track.
2. How far from Tokyo does the plane cross the equator and what is the longitude at the point of crossing?
3. What is the bearing of the great-circle track as the plane crosses the equator?

Exercise 17: A plane leaves Chicago $(41^\circ 50' N, 87^\circ 36' W)$ on a great-circle track with an initial bearing of $63^\circ 30'$ (counter-clockwise from east). What is the latitude and longitude of the plane after traveling 1000 nautical miles?

Exercise 18: If you were to make a nonstop flight from the city of Dayton, Ohio $(39^\circ 46' N, 84^\circ 12' W)$ to Tokyo $(35^\circ 39' N, 139^\circ 45' E)$ by the shortest route, in which direction would you start your flight?

Exercise 19: A plane leaves Los Angeles $(34^\circ 3' N, 118^\circ 14' W)$ on a great-circle track with an initial bearing of 65°. Find the latitude and longitude of the plane when it has flown 1000 nautical miles.

Exercise 20: What is the shortest distance from Moscow $(55^\circ 43' N, 37^\circ 34' E)$ to London $(51^\circ 31' N, 0^\circ 6' W)$? Find the initial bearing of the track.

End of Lesson

Next Lesson: No Such Thing. Enjoy Your Holidays.