Exponents

MATH 1511, BCIT

Technical Mathematics for Geomatics

October 23, 2017

Natural Exponents

Here is how exponents are defined for positive integers. For any $n \in \mathbb{N}$

$$\begin{aligned} a^0 &= 1 \\ a^{n+1} &= a^n \cdot a \end{aligned} \tag{1}$$

A more intuitive way to think about it is

$$a^n = \underbrace{a \cdot \ldots \cdot a}_{n \text{ times}} \tag{2}$$

a is called the base, n is called the exponent. But how will we define an exponential expression if the base is negative, a fraction, or irrational?

Integer Exponents

Notice that

$$\frac{a^n}{a^m} = a^{n-m} \tag{3}$$

if $n > m$. If we want (3) to be true when $n = m$ or $n < m$, then integer exponents must be defined as follows,

$$a^0 = 1 \tag{4}$$

$$a^{-n} = \frac{1}{a^n} \text{ for a positive integer } n \tag{5}$$

Integer Exponents Pattern

10^2	10^1	10^0	10^{-1}	10^{-2}
$10 \cdot 10$	10	1	$\frac{1}{10}$	$\frac{1}{10 \cdot 10}$
100	10	1	$\frac{1}{10} = 0.1$	$\frac{1}{100} = 0.01$

$\div 10$ $\div 10$ $\div 10$ $\div 10$

Look for a pattern in the table to extend what you know about exponents to include negative exponents.

Radicals

Let $x \in \mathbb{R}$. The following is defined to be the *m*-th root of x,

$$\sqrt[m]{x} = y \text{ if and only if } y^m = x \text{ and } y \geq 0 \tag{6}$$

$\sqrt[m]{x}$ is sometimes not defined, for example $\sqrt[2]{-1}$. The second root of a number is sometimes called the square root

$$\sqrt{x} = \sqrt[2]{x} \tag{7}$$

and the third root of a number is sometimes called the cube root.

Rational Exponents

Notice that

$$(a^n)^m = a^{n \cdot m} \tag{8}$$

for positive integers n and m. If we want (8) to be true for exponents of the form $\frac{1}{n}$ for an integer n, then the following must be true of them,

$$\left(a^{\frac{1}{n}}\right)^n = a^{\frac{1}{n} \cdot n} = a \tag{9}$$

and therefore

$$a^{\frac{1}{n}} = \sqrt[n]{a} \tag{10}$$

Rational and Real Exponents

In order to get

$$a^{\frac{1}{n}} = \sqrt[n]{a} \tag{11}$$

let us define expressions with a fraction as exponent as follows,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \tag{12}$$

Irrational numbers are the limit of a sequence of rational numbers, so for any real number c there is a sequence such that $\lim_{k\to\infty} c_k = c$, where all c_k are rational. Let us define expressions with an irrational number c as exponent as follows,

$$a^c = \lim_{k\to\infty} a^{c_k} \tag{13}$$

Laws of Exponents

Now we have the following laws,

$$a^x \cdot a^y = a^{x+y} \tag{14}$$

$$\frac{a^x}{a^y} = a^{x-y} \tag{15}$$

$$(a^x)^y = a^{x \cdot y} \tag{16}$$

$$(ab)^x = a^x b^x \tag{17}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \tag{18}$$

Radicals

Note that the laws of exponents also apply to expressions under a root sign, sometimes called radicals, because

$$\sqrt{a} = a^{\frac{1}{2}} \tag{19}$$

$$\sqrt[m]{a} = a^{\frac{1}{m}} \tag{20}$$

Therefore,

$$\sqrt[m]{a \cdot b} = \sqrt[m]{a} \cdot \sqrt[m]{b} \tag{21}$$

$$\sqrt[m]{\frac{a}{b}} = \frac{\sqrt[m]{a}}{\sqrt[m]{b}} \tag{22}$$

but

$$\sqrt[m]{a+b} \neq \sqrt[m]{a} + \sqrt[m]{b} \tag{23}$$

$$\sqrt[m]{a-b} \neq \sqrt[m]{a} - \sqrt[m]{b} \tag{24}$$

Exercise 1: Simplify the following expression,

$$16^{\frac{7}{4}} \cdot 16^{-\frac{1}{2}} \tag{25}$$

Exercise 2: Simplify the following expression,

$$\frac{8^{\frac{5}{3}}}{8^{-\frac{1}{3}}} \tag{26}$$

Exercise 3: Simplify the following expression,

$$\left(64^{\frac{4}{3}}\right)^{-\frac{1}{2}} \tag{27}$$

Exercise 4: Simplify the following expression,

$$(16 \cdot 81)^{-\frac{1}{4}} \tag{28}$$

Exercises

Exercise 5: Simplify the following expression,

$$\left(\frac{3^{\frac{1}{2}}}{2^{\frac{1}{3}}}\right)^4 \tag{29}$$

Exercise 6: Simplify the following expression,

$$\left[\left(-\frac{5ax^2}{3b^2y}\right)^4 \div \left(\frac{5ax}{12b^3y^2}\right)^3\right] \cdot \left(\frac{by}{2ax}\right)^4 \tag{30}$$

Exercise 7: Simplify the following expression,

$$\sqrt[3]{108} - \sqrt[3]{32} \tag{31}$$

Exercise 8: Simplify the following expression,

$$\left(2s^3t^{-1}\right)\left(\frac{1}{4}s^6\right)\left(16t^4\right) \tag{32}$$

Exercises

Exercise 9: Simplify the following expression,

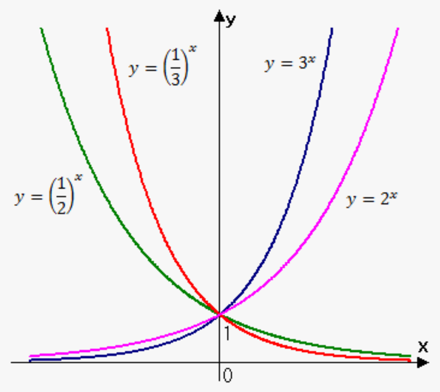
$$(3ab^2c)\left(\frac{2a^2b}{c^3}\right)^{-2} \tag{33}$$

Exercise 10: Simplify the following expression,

$$\frac{2v + 3w}{\sqrt{4v^2 - 9w^2}} \tag{34}$$

The Exponential Function: Graph

Let's have a look at the graph for the exponential function.



The Exponential Function: Properties

Here are some properties for the following exponential function ($a > 0$),

$$f(x) = a^x \tag{35}$$

The Exponential Function: Properties

- if $a = 1$ then the exponential function is the constant function $f(x) = 1$
- $f(0) = 1$ and $f(1) = a$
- the domain of f is the real numbers, the range of f is all positive real numbers, and f is injective (one-to-one)
- if $a > 1$ then $f(x)$ tends to 0 as $x \to -\infty$, and $f(x)$ goes very fast to $+\infty$ as $x \to \infty$
- if $a < 1$ then $f(x)$ tends to 0 as $x \to \infty$, and $f(x)$ goes very fast to $+\infty$ as $x \to -\infty$
- how fast the graph rises to $+\infty$ on the left or the right depends on how large a is (if $a > 1$) or how small a is (if $a < 1$). The closer a is to 1, the flatter the graph. 'Flat,' of course, is a relative term here: no matter how close a is to 1, the function graph will still rise faster than any polynomial.

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Euler's Number

It is convenient to agree on a base that we will use most of the time (just as 10 is the base for our decimal system, even though it could be any other natural number greater than 1). 2 or 10 are obvious candidates, but it turns out that another number, which is irrational, has special properties in calculus. We call this number e (Euler's number). It is the limit of the following series (infinite addition),

$$e = \frac{1}{1} + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \dots \tag{36}$$

It is also the limit of a sequence (infinite sequence of numbers),

$$e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m \tag{37}$$

Exercise 11: Simplify the following expression,

$$\left(\frac{x^3}{-27y^{-6}}\right)^{-\frac{2}{3}} \tag{38}$$

Exercise 12: Simplify the following expression,

$$\left(\frac{x^{-3}}{y^{-2}}\right)^2 \left(\frac{y}{x}\right)^4 \tag{39}$$

Exercise 13: Simplify the following expression,

$$\sqrt[3]{x^{-2}} \cdot \sqrt{4x^5} \tag{40}$$

Exercise 14: Evaluate the following expression,

$$\left(\frac{7^{-5} \cdot 7^{2}}{7^{-2}}\right)^{-1} \tag{41}$$

Exercise 15: Evaluate the following expression,

$$\sqrt[3]{\frac{-8}{27}} \tag{42}$$

Exercise 16: Evaluate the following expression,

$$\left(\frac{1}{\sqrt{3}}\right)^0 \tag{43}$$

End of Lesson

Next Lesson: Logarithms.