Vectors

MATH 1511, BCIT

Technical Mathematics for Geomatics

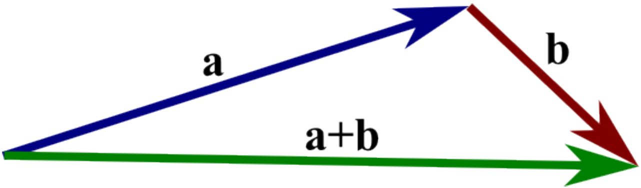
November 13, 2017

A vector is a special type of matrix: it has only one row or only one column. Here are two examples of a vector,

$$a = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \text{ and } b = \begin{pmatrix} \frac{1}{2} \\ 7 \end{pmatrix} \tag{1}$$

We add them as we would add matrices,

$$a + b = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 7 \end{pmatrix} = \begin{pmatrix} -1.5 \\ 10 \end{pmatrix} \tag{2}$$



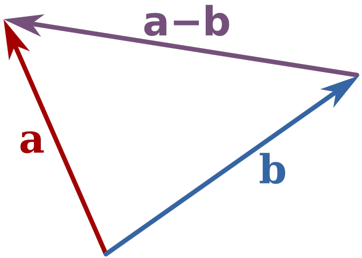
Vector Subtraction

We define the additive inverse $-a$ of a vector a to be the vector whose components are the additive inverses of a's components.

$$-a = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \tag{3}$$

Vector subtraction is defined as follows: $a - b = a + (-b)$.

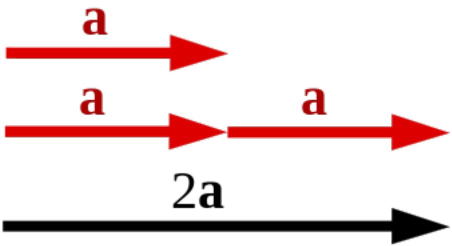
$$a - b = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ -7 \end{pmatrix} = \begin{pmatrix} -2.5 \\ -4 \end{pmatrix} \tag{4}$$



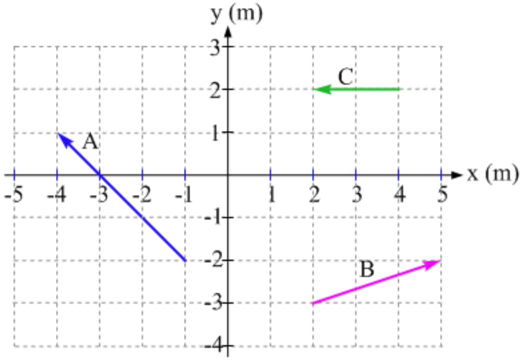
Scalar Multiplication

Scalar multiplication is defined for vectors as it was for matrices. A real number C and a vector a can be multiplied as follows,

$$C \cdot a = C \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} C \cdot a_1 \\ C \cdot a_2 \end{pmatrix} \tag{5}$$

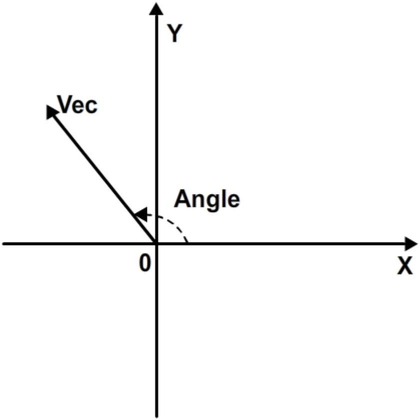


One way to interpret two-dimensional vectors is to have them describe movement in the plane. They have no home in the coordinate system, but only describe how to get from one point to another. By convention, unless we have reason to do otherwise, we let them start at the origin.



Vectors and Angles

Each vector (except the vector whose components are all zero) has an angle associated with it, which mathematicians conventionally measure counter-clockwise from the x-axis. You are welcome to represent the angle any way you like as long as it is clear what you mean.



Example

Let

$$a = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{6}$$

Then

$$\tan \alpha' = \frac{3}{-2} \longrightarrow \alpha' = -56.31^\circ \tag{7}$$

This is not the correct angle, however. The correct angle is $\alpha = 180^\circ + \alpha' = 123.69^\circ$. You can also represent it as N33.69°W. The length of the vector is

$$|a| = \sqrt{(-2)^2 + 3^2} \approx 3.61 \tag{8}$$

Exercise 1: Calculate the length $|a_i|$ and the angle α_i for the following vectors,

$$a_1 = \begin{pmatrix} 7 \\ 3 \end{pmatrix} \tag{9}$$

$$a_2 = \begin{pmatrix} -6 \\ -11 \end{pmatrix} \tag{10}$$

$$a_3 = \begin{pmatrix} -\pi \\ 10.5 \end{pmatrix} \tag{11}$$

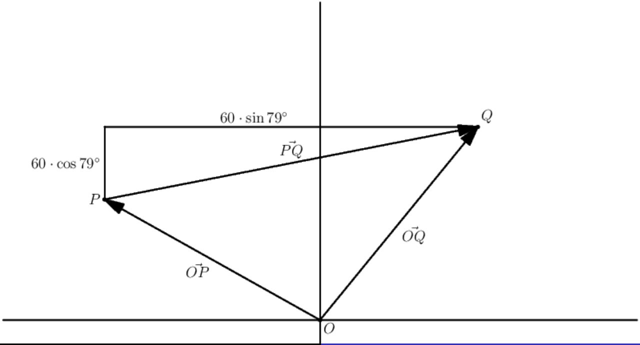
$$a_4 = \begin{pmatrix} \frac{7}{13} \\ -\frac{1}{5} \end{pmatrix} \tag{12}$$

Exercise 2: Jim starts at $P = (-34, 19)$ (units are metres) and walks 60 metres on a bearing of N79°E. Which vector describes Jim's movement? What are the coordinates of Jim's destination?

The vector $\vec{PQ}$ displaces P to Q. The coordinates of point P and the elements of vector $\vec{OP}$ match. To find the coordinates of Jim's destination in the problem on the last slide, note that

$$\vec{OQ} = \vec{OP} + \vec{PQ} \tag{13}$$

We can easily calculate the RHS (right-hand side), and the LHS provides us with the coordinates of point Q.



Exercise 3: Mr. X walks 5km east and 2km north from A to B. Ms. Y walks 6km west and 6km north from A to C. What angles could they have chosen to minimize the distance to their destination? What would be that minimal distance? What angle does Ms. Y need to choose and how far does she need to walk if she wants to rejoin Mr. X, going from C to B?

Vectors and Points

For the problem on the last slide, we are interested in the vector $\vec{CB}$. Note that

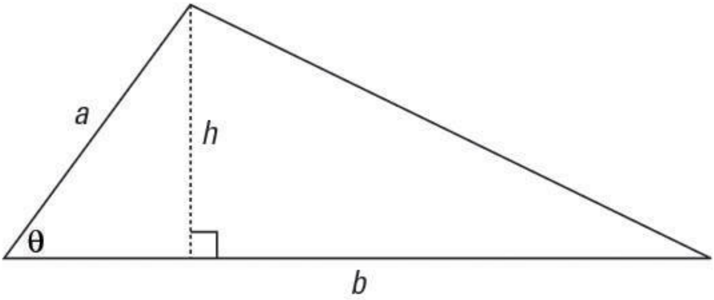
$$\vec{CB} = \vec{CO} + \vec{OB} = \vec{OB} - \vec{OC} \tag{14}$$

The solutions are

$$\begin{array}{rcccl}
|\vec{OB}| & = & \sqrt{5^2 + 2^2} & \approx & 5.39 \\
|\vec{OC}| & = & \sqrt{6^2 + 6^2} & \approx & 8.49 \\
|\vec{CB}| & = & \sqrt{11^2 + (-4)^2} & \approx & 11.70 \\
\arctan \frac{2}{5} & \approx & 21.80^\circ & & \text{bearing} \approx N68.20^\circ E \\
\arctan \frac{6}{6} & = & 45^\circ & & \text{bearing} = N45^\circ W \\
\arctan \frac{4}{11} & \approx & 19.98^\circ & & \text{bearing} \approx S70.02^\circ E
\end{array} \tag{15}$$

Area: Triangle

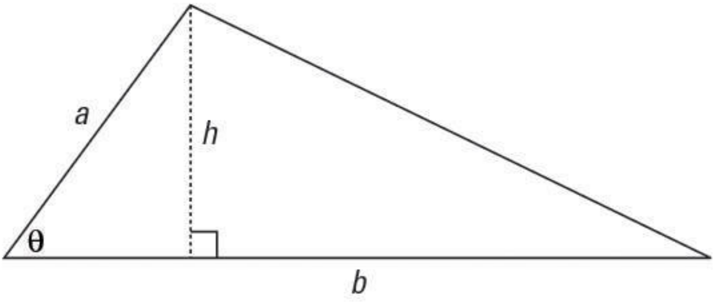
Consider the following triangle:



Start with the traditional formula for the area of this triangle,

$$A = \frac{1}{2}bh \tag{16}$$

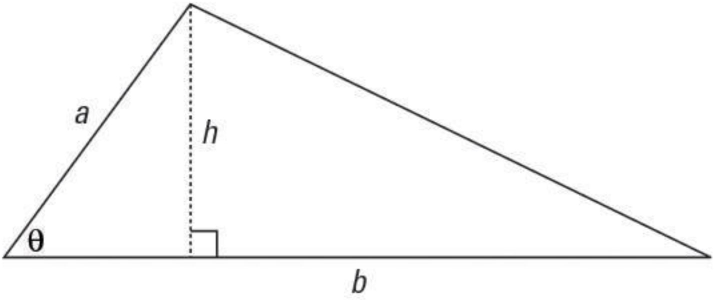
Area: Triangle



Then look at the smaller triangle to the left. Because the height is drawn perpendicular to the base, the sides and height form a right triangle. The acute angle θ has a sine equivalent to the following:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{h}{a} \tag{17}$$

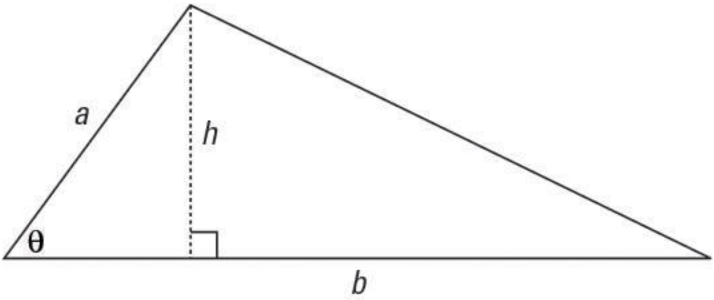
Area: Triangle



If you solve that equation for h by multiplying each side by a, you get

$$\sin\theta = \frac{h}{a} \tag{18}$$

$$a\sin\theta = h \tag{19}$$

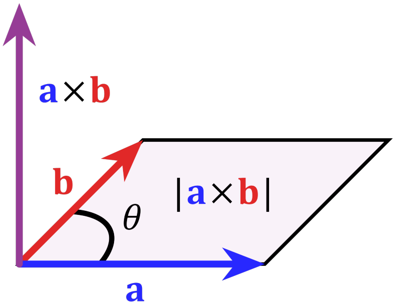


Replace the h in the traditional formula with its equivalent from the preceding equation, and you get

$$A = \frac{1}{2}bh = \frac{1}{2}b(a\sin\theta) = \frac{1}{2}ab\sin\theta \tag{20}$$

Area: Triangle

Two linearly independent vectors correspond to a triangle. Two vectors a and b are linearly dependent if and only if $a = C \cdot b$ for a real number C. The cross product $a \times b$ is the vector which is perpendicular to both a and b, follows the right-hand rule, and whose length is the area of the parallelogram generated by a and b. Knowing a and b, how would you calculate $|a \times b|$?

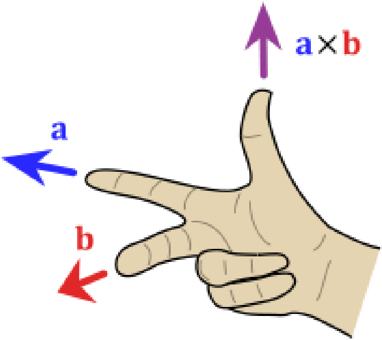


Area: Triangle

The length of the cross product is

$$|a \times b| = |a| \cdot |b| \cdot \sin\theta \tag{21}$$

where θ is the angle between a and b.



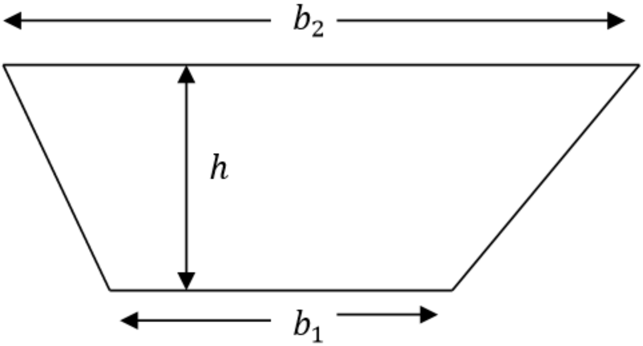
Exercise 4: Find the area for the triangle generated by

$$a = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \text{ and } b = \begin{pmatrix} 7 \\ -1 \end{pmatrix} \tag{22}$$

Exercise 5: Find the area for the parallelogram generated by

$$a = \begin{pmatrix} -301 \\ 754 \end{pmatrix} \text{ and } b = \begin{pmatrix} 590 \\ -538 \end{pmatrix} \tag{23}$$

Exercise 6: Find the formula for the area of a trapezoid.



Area: Units

Two common units of area are the hectare (ha) and the acre.

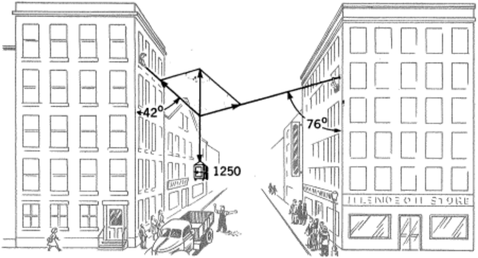
$$1\text{ha} = 100\text{m} \cdot 100\text{m} \tag{24}$$

$$1\text{acre} = 43560\text{ft}^2 \tag{25}$$

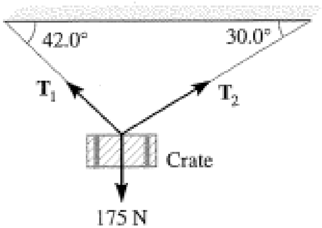
Note that 1ft$= 0.3048$m. Historically, an acre was the amount of land one man with one ox could plow in one day.

Vector Resultant

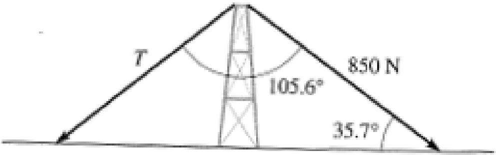
Exercise 7: An office safe was lowered from an office building by two cables attached to buildings on opposite sides of the street. At one time the cables formed angles of 42° and 76° with the buildings. Find the pull on each of the cables if the safe weighed 1250 pounds.



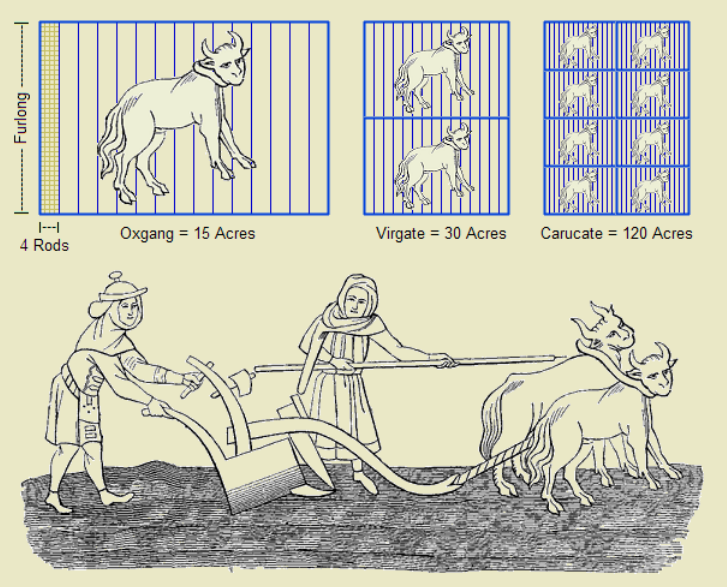
Exercise 8: Two ropes hold a 175 Newton crate as shown in the figure. Find the tensions T_1 and T_2 in the ropes. (Hint: move the vectors so that they are tail to head to form a triangle. The vector sum $T_1 + T_2$ must equal 175 Newton for equilibrium.)



Exercise 9: Find the tension T in the left guy wire attached to the top of the tower as shown in the figure. (Hint: the horizontal component of the tensions must be equal and opposite for equilibrium. Thus, move the tension vectors tail to head to form a triangle with a vertical resultant. This resultant equals the upward force at the top of the tower for equilibrium. This last force is not shown and does not have to be calculated.)



Area: Acre



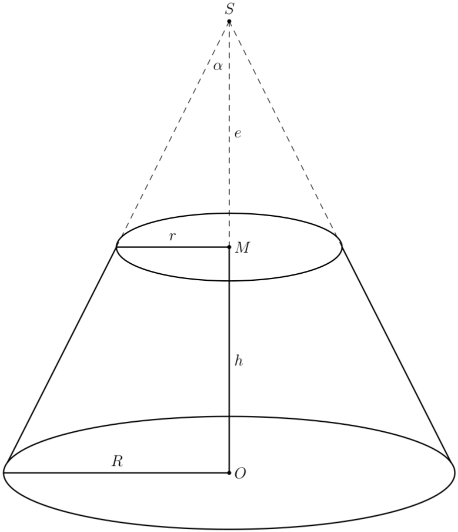
Furlong
4 Rods
Oxgang = 15 Acres
Virgate = 30 Acres
Carucate = 120 Acres

Exercise 10: Find the answer as a complete sentence.

1. A plot has an area of 68 acres. Determine the area in hectares.
2. A quarter section of land is one quarter of a mile by mile plot. Determine the area of a quarter section.
   a. Expressed in acres (recall that 1 mile= 1760 yards and 1 yard= 3 feet).
   b. Expressed in hectares.

Volume: Cone

Consider a frustum.



Volume: Cone

First off, verify that

$$(a-b)^3 = (a^2+ab+b^2)(a-b) \tag{26}$$

Let A be the base area of the frustum and a the base area of the cone on top of the frustum. Let α be the angle between the revolution axis of the cone and the generating side. Then

$$\frac{\sqrt{a}}{e} = \frac{\sqrt{r^2\pi}}{e} = \frac{r\sqrt{\pi}}{e} = \sqrt{\pi}\tan\alpha = \frac{R\sqrt{\pi}}{e+h} =$$

$$\frac{\sqrt{R^2\pi}}{e+h} = \frac{\sqrt{A}}{e+h} \tag{27}$$

(27) gives us

$$e = h \cdot \frac{\sqrt{a}}{\sqrt{A} - \sqrt{a}} \tag{28}$$

and

$$e + h = h \cdot \frac{\sqrt{A}}{\sqrt{A} - \sqrt{a}} \tag{29}$$

Now we make an assumption that we leave to intuition for justification: the ratio of the volumes of a cone and a cylinder with the same dimensions (radius and height) is constant, i.e.

$$V_{\text{cone}} = c \cdot V_{\text{cylinder}} \tag{30}$$

As we know the formula for the volume of a cylinder (base times height), finding c will give us the formula for the cone.

The volume of the frustum is

$$V_{\text{frustum}} = cA(e+h) - cae \tag{31}$$

Use (26), (28) and (29) for the following equation,

$$V_{\text{frustum}} = c(A + \sqrt{Aa} + a)h \tag{32}$$

Now consider what happens as a tends to A. The frustum becomes a cylinder, and we find that

$$V = 3cAh \tag{33}$$

But we know that, for a cylinder, $V = Ah$, so $c = 1/3$, and we conclude that the volume of a cone is

$$V_{\text{cone}} = \frac{1}{3} r^2 \pi h \tag{34}$$

Volume: Pyramid

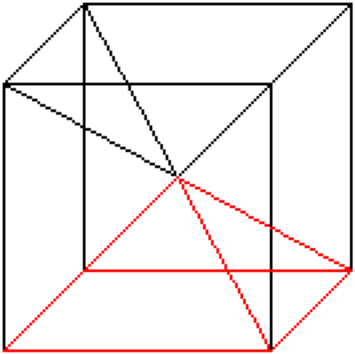
Let the pyramid have the following dimensions: a (side), b (side), h (height).

Again, we make the assumption that the ratio of the volumes of pyramids and rectangular cuboids (the right rectangular prism that looks like a cube except that all the side lengths may be different) are constant if the dimensions are the same, i.e.

$$c \cdot V_{\text{cuboid}} = c \cdot a \cdot b \cdot h = V_{\text{pyramid}} \tag{35}$$

Volume: Pyramid

Now consider a cube, where all the sides are the same. It can be divided up into 6 pyramids, whose tops are at the centre of the cube.



Volume: Pyramid

It follows that

$$c = \frac{\frac{1}{6}a^3}{a^2 \cdot \frac{a}{2}} = \frac{1}{3} \tag{36}$$

We conclude that the volume of a pyramid is

$$V_{\text{pyramid}} = \frac{1}{3}abh \tag{37}$$

Cavalieri's Principle

Suppose two regions in three-dimensional space (solids) are included between two parallel planes. If every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes.

Volume: Sphere

Consider a sphere with radius R, together with a cylinder of radius R and height $2R$. Cones are drilled out from each end of the cylinder to its centre (the bases of the cones are the bases of the cylinder; the height of the cones is R). If we slice each object (the sphere and what is left over of the cylinder after drilling out the two cones) at a distance x from its centre, the area of the slice is, in each case,

$$\pi(R^2 - x^2) \tag{38}$$

Thus the two solids have the same volume, and we conclude that

$$V_{\text{sphere}} = \pi R^2 \cdot 2R - 2 \cdot \frac{1}{3}\pi R^2 \cdot R = \frac{4}{3}\pi R^3 \tag{39}$$

Surface Area: Sphere

Given a sphere, we divide the surface into very many small (flat) pieces of area $A_i, i = 1, \ldots, n$. We join each to the centre, forming sharp cones. The volume of a typical cone is

$$V_{\text{cone}} = \frac{1}{3} A_i R \tag{40}$$

and the total volume of all the cones is

$$V_n = \frac{1}{3} R(A_1 + \ldots + A_n) = \frac{1}{3} R S_n \tag{41}$$

where S_n will approach the surface area S of the sphere, and V_n will approach the volume V of the sphere, as $n \to \infty$.

$$S = 4\pi R^2 \tag{42}$$

Volume: Pyramids, Cones, and Spheres

Note the following formulas:

$$V_{\text{sphere}} = \frac{4}{3} r^3 \pi \tag{43}$$

$$V_{\text{cone}} = \frac{1}{3} r^2 \pi h \tag{44}$$

$$V_{\text{pyramid}} = \frac{1}{3} abh \tag{45}$$

Note also the surface of the sphere:

$$S = 4r^2\pi \tag{46}$$

31. The Great Pyramid of Egypt has a square base approximately 230 m on a side. The height of the pyramid is about 150 m. What is its volume? See Fig. 2.114.

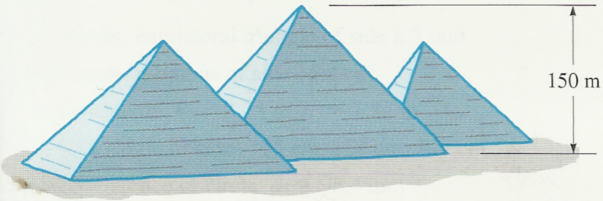


Fig. 2.114

Volume: Word Problems II

32. A paper cup is in the shape of a cone, as shown in Fig. 2.115. What is the surface area of the cup?

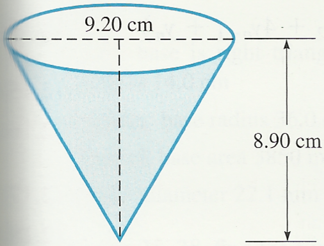


Fig. 2.115

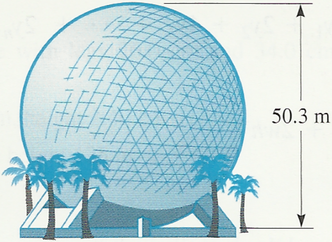


Fig. 2.116

33. *Spaceship Earth* (shown in Fig. 2.116) at Epcot Center in Florida is a sphere 50.3 m in diameter. What is the volume of *Spaceship Earth*?

34. A propane tank is constructed in the shape of a cylinder with a hemisphere at each end, as shown in Fig. 2.117. Find the volume of the tank.

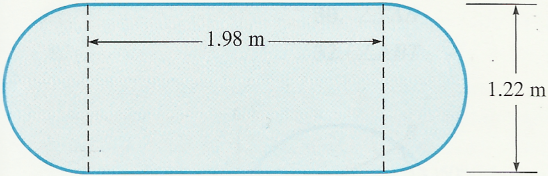


Fig. 2.117

End of Lesson

Next Lesson: Normal Distribution.