Right Triangles

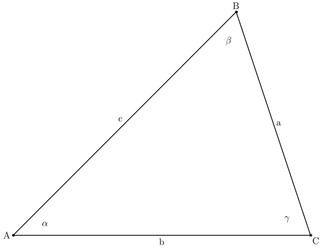
MATH 1511, BCIT

Technical Mathematics for Geomatics

September 20, 2017

What Is the Problem?

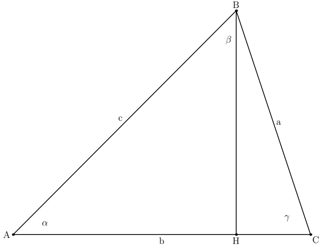
A triangle has three sides and three angles that we can measure.



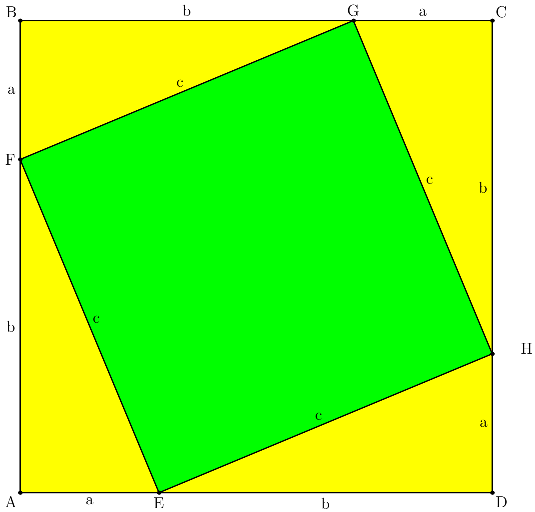
There are three vertices A, B, C, three sides a, b, c, and three angles α, β, γ. If three of them are given and three of them are unknown, then how can we calculate (instead of measure) the remaining three?

Towards a Solution: Right Triangles

We can always divide any triangle into two (or fewer) right triangles. Let's try to solve our problem first for right triangles.



Bhaskara's Square



Pythagorean Theorem

Calculate the area of the square in the diagram on the previous slide using two methods. Method 1: multiply the two sides of the large square (yellow and green). Method 2: add the area of the small square (green) to the four areas of the right triangles (yellow).

$$A_1 = (a+b)^2 = a^2 + 2ab + b^2 \tag{1}$$

$$A_2 = c^2 + 2ab \tag{2}$$

Since $A_1 = A_2$,

$$c^2 = a^2 + b^2 \tag{3}$$

Consequently, given two sides of a right triangle, we can calculate the length of the third side.

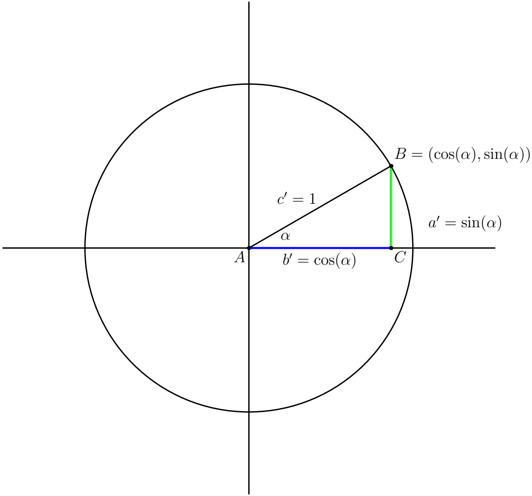
Definition of Sine and Cosine

Let a, b, c be the sides of an arbitrary right triangle. Then consider the similar triangle a', b', c', which has exactly the same angles but all sides are scaled by the factor $1/c$. This means that the side c' has length 1 (we use the following notation: $c' = 1$).

$$\begin{aligned} a' &= \tfrac{a}{c} \\ b' &= \tfrac{b}{c} \\ c' &= 1 \end{aligned} \tag{4}$$

Definition of Sine and Cosine

Because $c' = 1$, we can inscribe the right triangle in a unit circle as in the following diagram.



Definition of Sine and Cosine

Now define the cosine and sine of the angle α to be the coordinates of the point B.

$$\begin{aligned} \sin(\alpha) &= \frac{a}{c} \\ \cos(\alpha) &= \frac{b}{c} \end{aligned} \tag{5}$$

The Cosine Function

It is relatively easy to calculate the sine of an angle once you have the cosine and vice versa. We write $\cos^2\alpha$ as an abbreviation of $(\cos(\alpha))^2$.

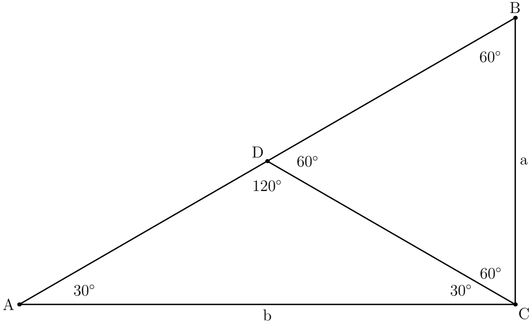
$$\cos^2\alpha + \sin^2\alpha = 1 \tag{6}$$

Consequently, it makes sense to focus on one of the two functions at first. We will focus on the cosine function.

How to calculate the cosine function for a given angle is still an open question. All we can see at this point is that it exists and that it is well-defined for angles between 0° and 90°.

The Cosine Function

The following triangle illustrates why the cosine of 30° is $\sqrt{3}/2$ and why the cosine of 60° is $1/2$.



The Cosine Function

If $\alpha = 45^\circ$, then $a = b$. From the theorem of Pythagoras it follows that $\cos(45^\circ) = \sqrt{2}/2$. Ptolemy's pentagram shows us that $\cos(36^\circ) = 0.80902$ (see Glen van Brummelen, "Heavenly Mathematics," page 9). We now have enough data points to see the shape of the cosine curve between 0° and 90°.

Ptolemy's Pentagram

Have a look at the diagram and find out why all angles in a pentagram add up to 540°. Notice that ΔABC and ΔBHA are similar. Therefore (the length of segment CH is 1),

$$\frac{1}{y+1} = \frac{y}{1} \tag{7}$$

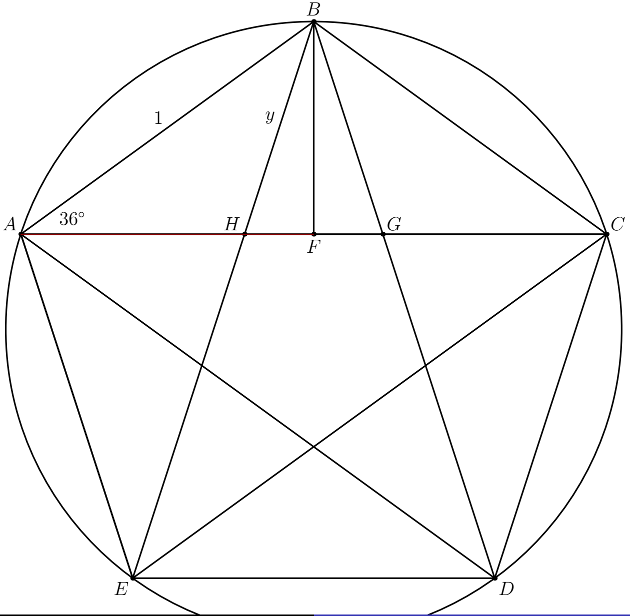
This is a quadratic equation with one positive solution,

$$y = \frac{1}{2}\left(\sqrt{5}-1\right) \tag{8}$$

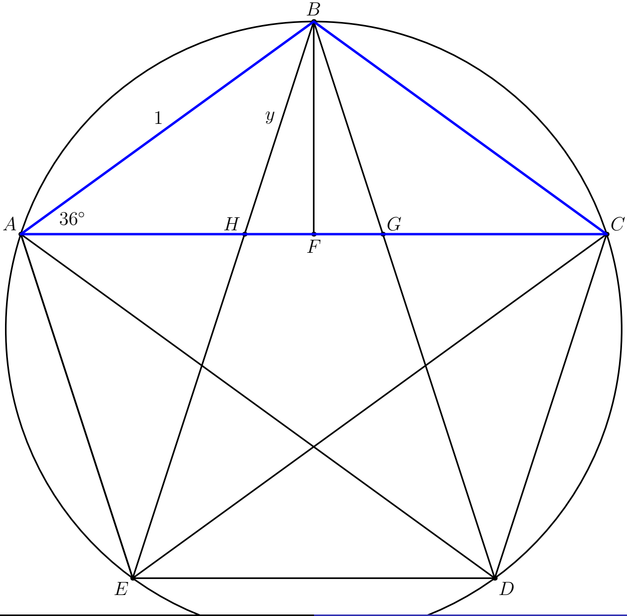
The length of segment AC is $y+1$, but it's also twice the length of segment AF. Therefore, the length of AF, which is $cos(36^\circ)$, is

$$cos(36^\circ) = \frac{1}{4}\left(\sqrt{5}-1\right) + \frac{1}{2} = 0.80902 \tag{9}$$

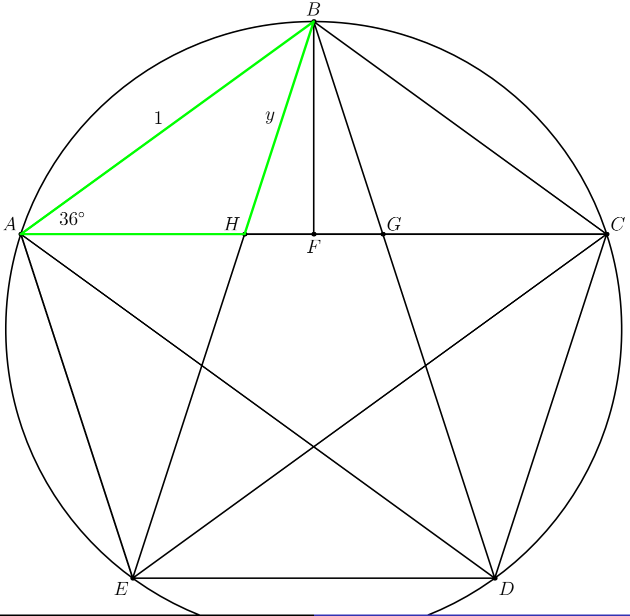
Ptolemy's Pentagram



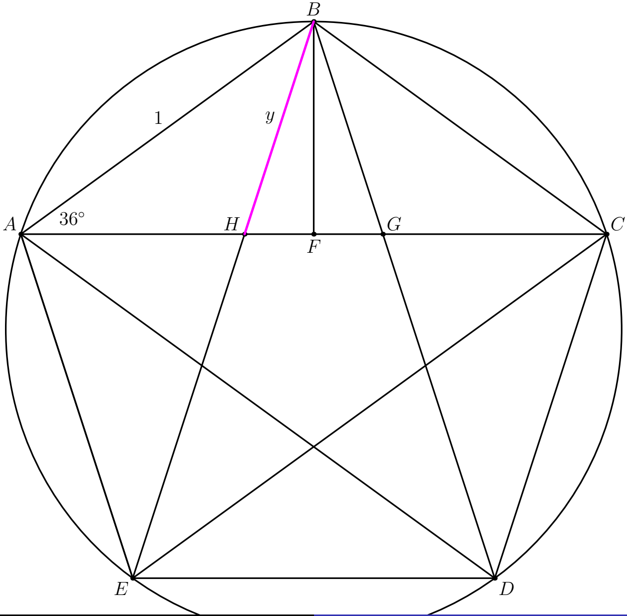
Ptolemy's Pentagram



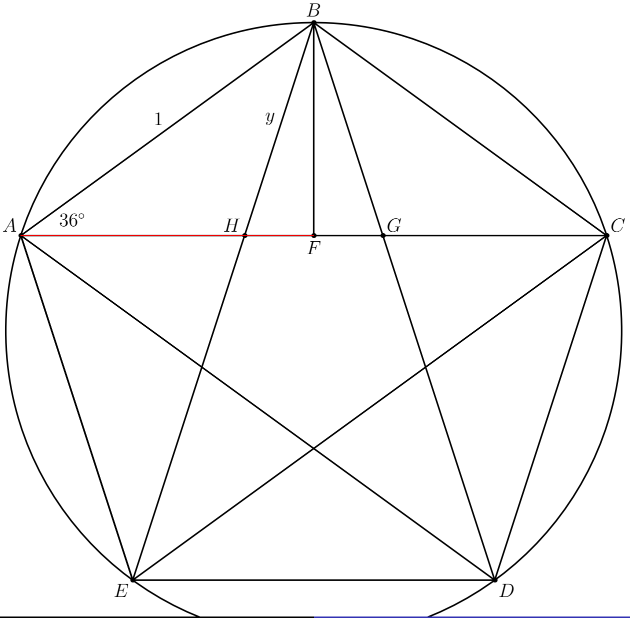
Ptolemy's Pentagram



Ptolemy's Pentagram

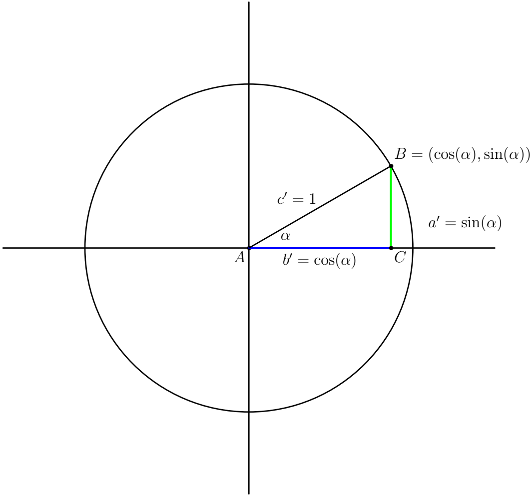


Ptolemy's Pentagram



Reference Triangle

Let's have another look at this diagram, called the reference triangle.

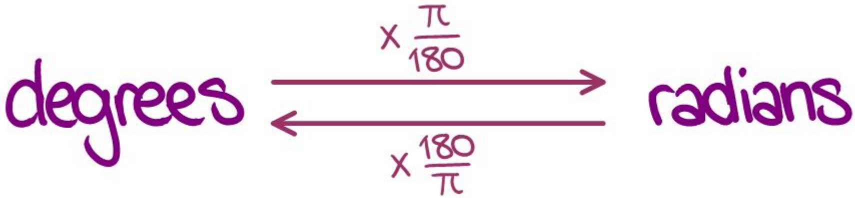


The Cosine Function

Now let us define the cosine function for all angles (even negative ones). Think of B going all the way around the circle, and think of A as the origin of a coordinate system. The cosine and sine of α are defined as the coordinates of the point B, and the angle α can be anything from $-\infty$ to ∞.

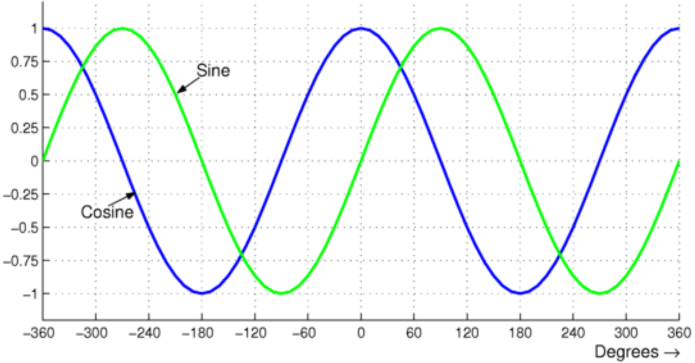
Radians

What kind of animal is 30°? It is a real number – not the one you might expect. Just as 25% is not the number 25, but the number 0.25; and 3 quarters is not the number 3, but the number 0.75. 30° is not the number 30, but the number $\pi/6$. The symbol $^\circ$ (pronounced "degrees") is equivalent to the factor $\pi/180$; just as the symbol % (pronounced "percent") is equivalent to the factor $1/100$, and a "quarter" is equivalent to the factor $1/4$.



Graph of Sine and Cosine Function

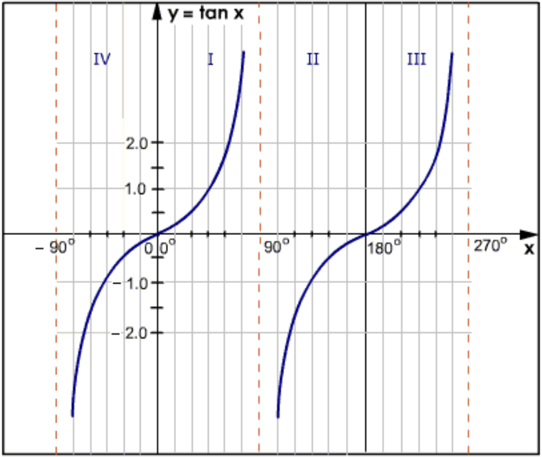
From our definitions it follows that the graph of the sine and cosine functions looks approximately like this,



Tangent Function

Eventually, we will need a third function, the tangent function, which is defined as

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha} \tag{10}$$



Problem of Three Missing Quantities

Let's see if we have made progress solving our problem of the three missing quantities. The way we have defined cosine and sine, it is apparent that for an arbitrary right triangle

$$\cos\alpha = \sin\beta = \frac{b}{c} \tag{11}$$

$$\sin\alpha = \cos\beta = \frac{a}{c} \tag{12}$$

$$\tan\alpha = \frac{a}{b} \tag{13}$$

Together with $c^2 = a^2 + b^2$, (11), (12), and (13) are all the tools we need to solve any right triangle, i.e. find the missing three quantities if the remaining three (one of which is the right angle) are given.

SohCahToa

A useful mnemonic device for the information on the last slide is SOH-CAH-TOA, which stands for

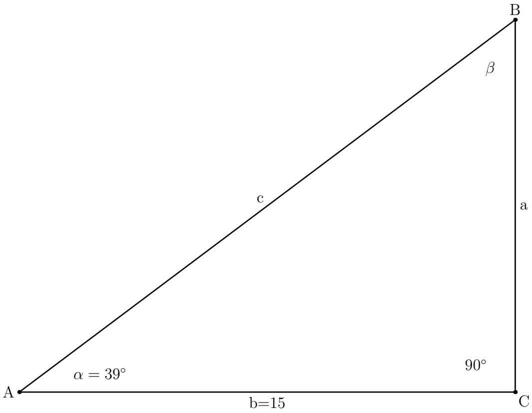
$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} \tag{14}$$

$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{15}$$

$$\tan\alpha = \frac{\text{opposite}}{\text{adjacent}} \tag{16}$$

Examples

Example 1: Finding Three Missing Measurements. Let ABC be a right triangle with a right angle at C, an angle $\alpha = 39°$, and a side $b = 15$. Find the length of the side a.



Examples

Use

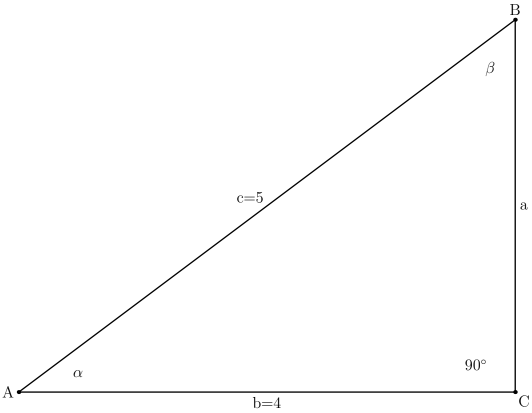
$$\tan(39^\circ) = \frac{a}{15} \tag{17}$$

and calculate

$$a = 15\tan(39^\circ) \approx 12.1468 \tag{18}$$

The length of the side $a \approx 12.1468$.

Example 2: Finding Three Missing Measurements. Let ABC be a right triangle with a right angle at C, and sides $b = 4, c = 5$. Find the angle β.



Examples

Use

$$\sin \beta = \frac{4}{5} \tag{19}$$

and calculate

$$\beta = \arcsin\left(\frac{4}{5}\right) \approx 53.130 \tag{20}$$

The angle $\beta \approx 53.130$. Note that the arc sine function is the inverse of the sine function, defined as

$$\arcsin(x) = \alpha \text{ only if } \sin \alpha = x \tag{21}$$

There is also a corresponding arc cosine and arc tangent function.

We ought not to be satisfied with what we have learned so far about solving the problem of the three missing quantities. The reason is that we have only defined the cosine function without giving any indication on how to calculate it. We have cosine values for $\alpha = 0°, \alpha = 30°, \alpha = 36°, \alpha = 45°, \alpha = 60°, \alpha = 90°$, but what about other angles? A more generally satisfying answer to this question will need to come from the mathematical discipline of calculus, but in the meantime we can find more values for the cosine function by considering certain trigonometric identities. Here are some that are easy to derive from what we have already done,

$$\cos^2\alpha + \sin^2\alpha = 1 \tag{22}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \tag{23}$$

Trigonometric Identities II

Other trigonometric identities follow from careful consideration of the reference triangle,

$$\sin(-\alpha) = -\sin\alpha \tag{24}$$

$$\cos(-\alpha) = \cos\alpha \tag{25}$$

$$\tan(-\alpha) = -\tan\alpha \tag{26}$$

and also,

$$\sin(90^\circ - \alpha) = \cos\alpha \tag{27}$$

$$\cos(90^\circ - \alpha) = \sin\alpha \tag{28}$$

$$\tan(90^\circ - \alpha) = \frac{1}{\tan\alpha} \tag{29}$$

The last function, $\cot\alpha = \frac{1}{\tan\alpha}$ is also called the cotangent function. The identities on this page are reflection identities.

Trigonometric Identities III

Here are some identities related to shifts and periodicity,

$$\sin(\alpha + 90^\circ) = \cos\alpha \tag{30}$$

$$\cos(\alpha + 90^\circ) = -\sin\alpha \tag{31}$$

$$\tan(\alpha + 90^\circ) = -\cot\alpha \tag{32}$$

and also,

$$\sin(\alpha + 180^\circ) = -\sin\alpha \tag{33}$$

$$\cos(\alpha + 180^\circ) = -\cos\alpha \tag{34}$$

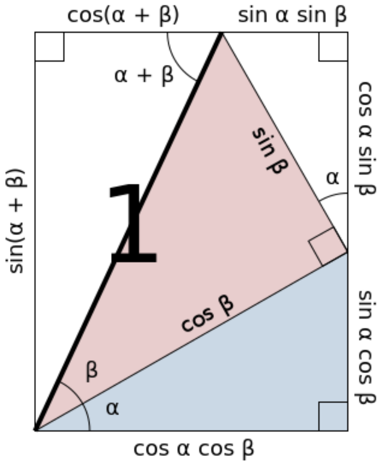
$$\tan(\alpha + 180^\circ) = \tan\alpha \tag{35}$$

$$\cot(\alpha + 180^\circ) = \cot\alpha \tag{36}$$

All trigonometric functions repeat when an integer multiple of 360° is added, i.e. $\text{trigfun}(\alpha + k \cdot 360^\circ) = \text{trigfun}(\alpha)$.

Angle Sum Identities I

Here is a trigonometric identity that will help us find more cosine values, using what we have learned so far. Consider the following diagram.



Angle Sum Identities II

From the diagram, we deduce

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{37}$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{38}$$

This means we can calculate cosine values of all angles between 0° and 90° that are divisible by three, for example

$$\cos(6^\circ) = \sin(84^\circ) = \sin(30^\circ + 54^\circ) =$$

$$\cos(30^\circ)\sin(54^\circ) + \cos(54^\circ)\sin(30^\circ) \tag{39}$$

where

$$\sin(54^\circ) = \cos(36^\circ) \tag{40}$$

and

$$\cos(54^\circ) = \sqrt{1 - \sin^2(54^\circ)} \tag{41}$$

Double-Angle Formula

If $\alpha = \beta$, the angle sum identities give us the double-angle formulas

$$\sin(2\alpha) = 2\cos\alpha\sin\alpha \tag{42}$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha \tag{43}$$

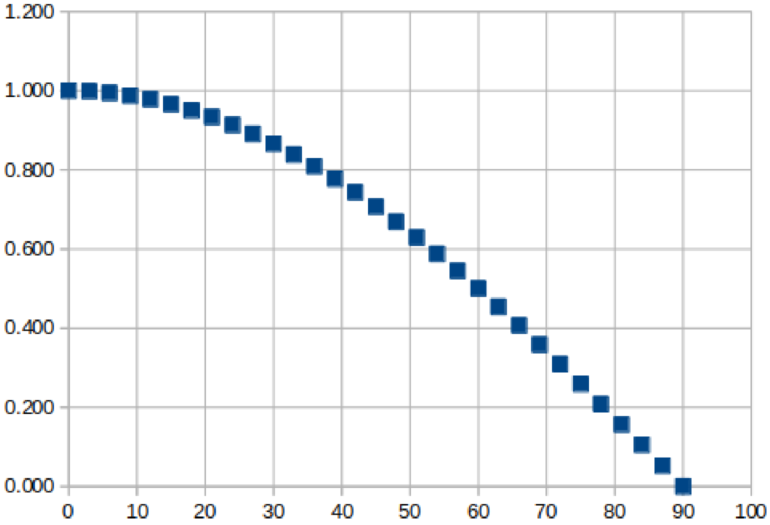
Now we can calculate cosine values of angles that are not divisible by three and create a much more fine-grained web of values, even though we have no general formula how to calculate cosine values (this will have to wait until we learn calculus). For example

$$\cos(10.5^\circ) = \sqrt{\frac{\cos(21^\circ) + 1}{2}} \tag{44}$$

List of Cosines (and Sines)

cos	**0** sin	**90**	1.000	one							
cos	**3** sin	**87**	0.999	"87"	cos	**48** sin	**42**	0.669	"42"		
cos	**6** sin	**84**	0.995	"84"	cos	**51** sin	**39**	0.629	"39"		
cos	**9** sin	**81**	0.988	"81"	cos	**54** sin	**36**	0.588	"36"		
cos	**12** sin	**78**	0.978	"78"	cos	**57** sin	**33**	0.545	"33"		
cos	**15** sin	**75**	0.966	"75"	cos	**60** sin	**30**	0.500	one half		
cos	**18** sin	**72**	0.951	"72"	cos	**63** sin	**27**	0.454	"27"		
cos	**21** sin	**69**	0.934	"6,15"	cos	**66** sin	**24**	0.407	"30,36"		
cos	**24** sin	**66**	0.914	"66"	cos	**69** sin	**21**	0.358	"21"		
cos	**27** sin	**63**	0.891	"6,21"	cos	**72** sin	**18**	0.309	"36"		
cos	**30** sin	**60**	0.866	square root of three divided by two	cos	**75** sin	**15**	0.259	"30,45"		
cos	**33** sin	**57**	0.839	"15,18"	cos	**78** sin	**12**	0.208	"39"		
cos	**36** sin	**54**	0.809	Ptolemy's pentagram	cos	**81** sin	**9**	0.156	"39,42"		
cos	**39** sin	**51**	0.777	"15,24"	cos	**84** sin	**6**	0.105	"30,54"		
cos	**42** sin	**48**	0.743	"21"	cos	**87** sin	**3**	0.052	"42,45"		
cos	**45** sin	**45**	0.707	one divided by the square root of two	cos	**90** sin	**0**	0.000	zero		

List of Cosines (and Sines)



Solving Trigonometric Equations

Exercise 1: Solve the following trigonometric equations:

$$\sin\vartheta + 7 = 8 \tag{45}$$

$$\tan^2\vartheta - 3 = 0 \tag{46}$$

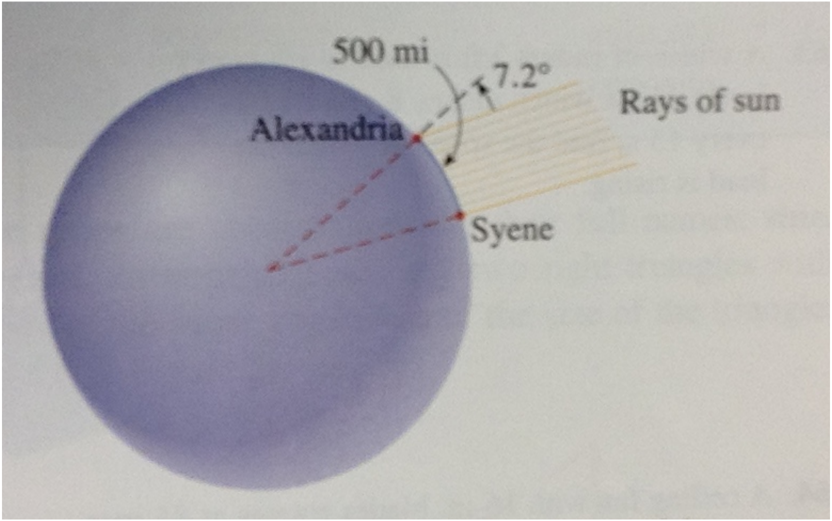
$$2\cos^2(\vartheta) - \sqrt{3}\cos\vartheta = 0 \tag{47}$$

$$\sin^2\vartheta + 2\cos\vartheta = 2 \tag{48}$$

$$\sin\vartheta = \sin(2\vartheta) \tag{49}$$

$$\sin\vartheta + \cos\vartheta = 1 \tag{50}$$

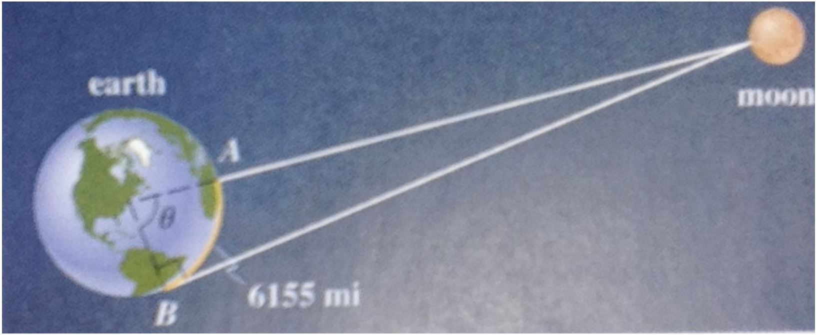
The Greek mathematician Eratosthenes measured the circumference of the earth from the following observations. He noticed that on a certain day the sun shone directly down a deep well in Syene (modern Aswan). At the same time in Alexandria, 500 miles north (on the same meridian), the rays of the sun shone at an angle of 7.2° to the zenith. Use this information and the figure to find the radius and circumference of the earth. (The data used in this problem are more accurate than those available to Eratosthenes.)


500 mi
7.2°
Rays of sun
Alexandria
Syene

Astronomical Distances

Here is a way to estimate the distance from the earth to the moon: When the moon is seen at its zenith at a point A on the earth, it is observed to be at the horizon from point B (see the figure). Point A and B are 6155 miles apart, and the radius of the earth is 3960 miles. Find the angle θ in degrees. Estimate the distance from point A to the moon.

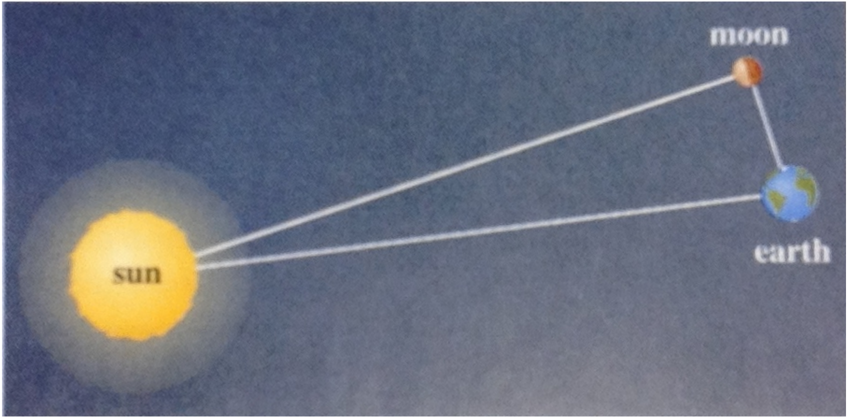
Astronomical Distances



Astronomical Distances

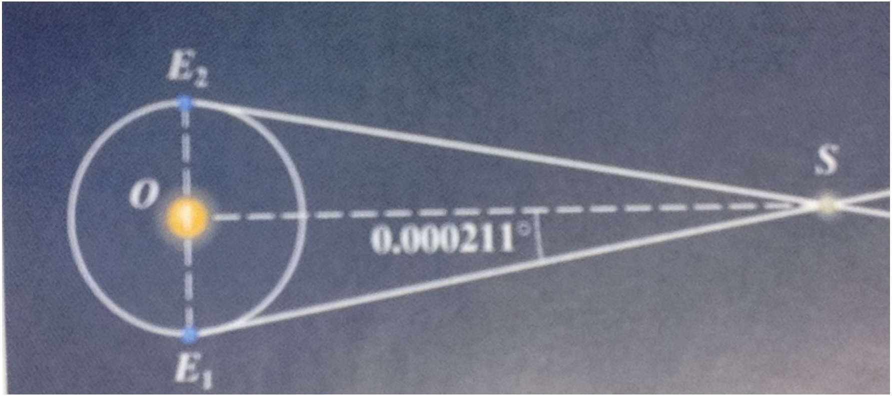
When the moon is exactly half full, the earth, moon, and sun form a right angle (see the figure). At that time the angle formed by the sun, earth, and the moon is measured to be 89.85°. If the distance from the earth to the moon is 240,000 miles, estimate the distance from the earth to the sun.

Astronomical Distances

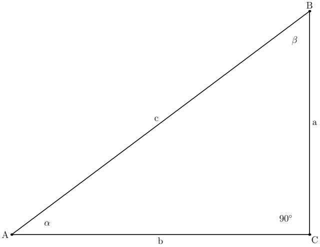


To find the distance to nearby stars, the method of parallax is used. The idea is to find a triangle with a star at one vertex and with the base as large as possible. To do this, the star is observed at two different times exactly 6 months apart, and its apparent change in position is recorded. From these two observations, $\angle E_1SE_2$ can be calculated. (The times are chosen so that $\angle E_1SE_2$ is as large as possible, which guarantees that $\angle E_1OS$ is 90.) The angle $\angle E_1SO$ is called the *parallax* of the star. Alpha Centauri, the star nearest the earth, has a parallax of 0.000211°. Estimate the distance to this star. (Take the distance from the earth to the sun to be $9.3 \cdot 10^7$ miles.)

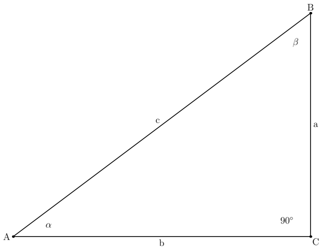
Astronomical Distances



Exercise 2: Calculate the remaining side/angle values.
$a = 23.453mm$, $b = 15.791mm$.

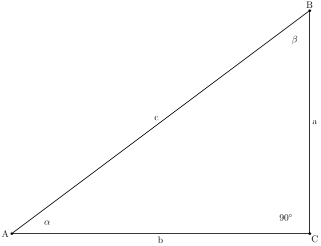


Exercise 3: Calculate the remaining side/angle values.
$\beta = 78^\circ 34' 12''$, $b = 15.791m$.

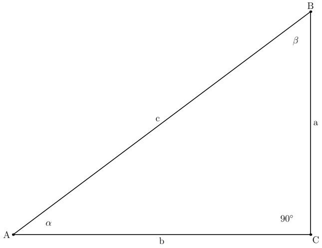


Exercises

Exercise 4: Calculate the remaining side/angle values.
$c = 89.221m, a = 75.791m.$



Exercise 5: Calculate the remaining side/angle values.
$\alpha = 24°44'05'', c = 12.998mm.$



Exercise 6: Assume you know all the values of the cosine function for the interval between 0° and 90°. Evaluate the following expressions,

$$\sin\left(214^\circ\right) \tag{51}$$

$$\cot\left(146^\circ\right) \tag{52}$$

$$\sin\left(248^\circ\right) \tag{53}$$

$$\cos\left(216^\circ\right) \tag{54}$$

$$\tan\left(280^\circ\right) \tag{55}$$

$$\cos\left(307^\circ\right) \tag{56}$$

$$\sin\left(49^\circ\right) \tag{57}$$

$$\tan\left(147^\circ\right) \tag{58}$$

$$\tan\left(316^\circ\right) \tag{59}$$

Exercises

Exercise 7: Solve the following equations for the given angle,

$$\sin\vartheta = \frac{1}{2}, 0^\circ \leq \vartheta \leq 90^\circ \tag{60}$$

$$\sin\vartheta = \frac{1}{2}, 90^\circ \leq \vartheta \leq 180^\circ \tag{61}$$

$$\sin\vartheta = -\frac{1}{2}, 180^\circ \leq \vartheta \leq 270^\circ \tag{62}$$

$$\sin\vartheta = -\frac{1}{2}, 270^\circ \leq \vartheta \leq 360^\circ \tag{63}$$

$$\cos\vartheta = -\frac{1}{2}, 0^\circ \leq \vartheta \leq 360^\circ \tag{64}$$

$$\tan\vartheta = 1.9, 0^\circ \leq \vartheta \leq 360^\circ \tag{65}$$

Exercise 8: Consider a right triangle with the length of the hypotenuse $c = 10\sqrt{2}$ and the length of one side $a = 8\sqrt{3}$. How long is the other side b? (Simplify the radical as in this example: $\sqrt{18} = 3\sqrt{2}$.)

Exercise 9: Solve the following trigonometric equations,

$$2\sin\vartheta - 1 = 0 \tag{66}$$

$$(\cot^2\vartheta)(3+\sqrt{5}) + \sqrt{5} = 5 \tag{67}$$

$$2\cos^2\vartheta - 7\cos\vartheta + 3 = 0 \tag{68}$$

$$1 + \sin\vartheta = 2\cos^2\vartheta \tag{69}$$

$$\cos 2\vartheta = \sqrt{3} - \sin\vartheta \tag{70}$$

$$2\sin 3\vartheta - 1 = 0 \tag{71}$$

$$\sqrt{3}\tan\frac{\vartheta}{2} - 1 = 0 \tag{72}$$

$$\tan^2\vartheta - \tan\vartheta - 2 = 0 \tag{73}$$

Exercise 10: Express the following in terms of cosines of arguments between 0° and 90°.

$$\text{(a)}\ \sin(136^\circ) \tag{74}$$

$$\text{(a)}\ \tan(229^\circ) \tag{75}$$

$$\text{(a)}\ \cot(-171^\circ) \tag{76}$$

Exercises

Exercise 11: The sine of 54° is $\frac{\sqrt{5}+1}{4}$. The sine of 30° is $\frac{1}{2}$. Use this information to find the sine of 84°.

Exercise 12: What is the solution set for the following equation, $\cos\theta = \frac{\sqrt{3}}{2}$?

End of Lesson

Next Lesson: Non-Right Triangles