Trigonometric Equations

MATH 1511, BCIT

Technical Mathematics for Geomatics

October 17, 2017

Example 1: Trigonometric Equation with Sine and Cosine.
Solve the following trigonometric equation,

$$1 + \sin\vartheta = 2\cos^2\vartheta \tag{1}$$

It is often a good strategy to convert everything to either sine or cosine and then solve for the unknown $\sin\vartheta$ or $\cos\vartheta$. In this case, for example, write

$$1 + \sin\vartheta = 2(1 - \sin^2\vartheta) \tag{2}$$

and then replace $\sin\vartheta = x$ (this method is sometimes called substitution). Therefore,

$$2x^2 + x - 1 = 0 \tag{3}$$

The solutions for equation (3) are

$$x = -1 \text{ or } x = \frac{1}{2} \tag{4}$$

Now revert to the meaning of $x = \sin\vartheta$. Therefore,

$$\sin\vartheta = -1 \text{ or } \sin\vartheta = \frac{1}{2} \tag{5}$$

Then take the multiple solutions into account that exist within one period of the sine, for example $[0, 2\pi)$.

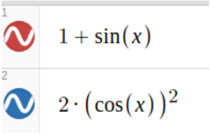
$$\vartheta = 30^\circ \text{ or } \vartheta = 150^\circ \text{ or } \vartheta = 270^\circ \tag{6}$$

Trigonometric Equations

Consider the period of the trigonometric function (sine, in this case) to finalize the solution set.

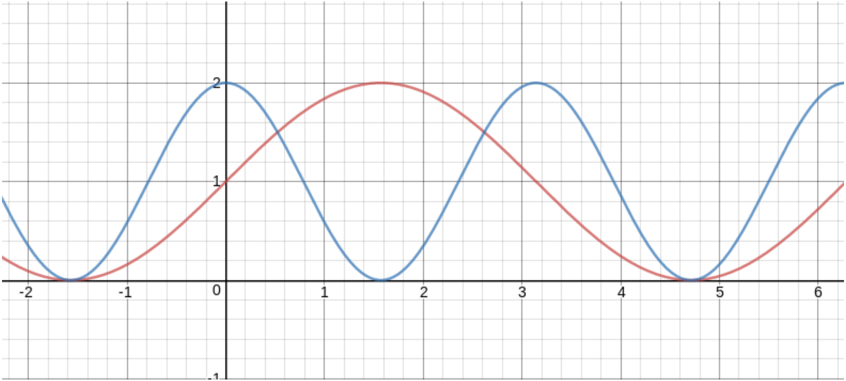
$$S = \{\vartheta \in \mathbb{R} | \vartheta = 30^\circ + k \cdot 360^\circ, \vartheta = 150^\circ + k \cdot 360^\circ,$$
$$\vartheta = 270^\circ + k \cdot 360^\circ, k \in \mathbb{Z}\} \tag{7}$$

It is easy to miss solutions, so it is a good idea to check that you have them all in a function graph. Here is my input on `www.desmos.com`



Look at the function graph on the next slide and compare the intersections to the solution set.

Trigonometric Equations



Example 2: Trigonometric Equation with Tangent and Cotangent. Solve the following trigonometric equation,

$$\tan 2\vartheta = \cot \vartheta \tag{8}$$

It may be helpful to turn these trigonometric functions immediately into sines and cosines.

$$\frac{\sin 2\vartheta}{\cos 2\vartheta} = \frac{cos\vartheta}{\sin \vartheta} \tag{9}$$

Now use the double-angle formula

$$\frac{2\sin\vartheta\cos\vartheta}{\cos^2\vartheta - \sin^2\vartheta} = \frac{cos\vartheta}{\sin \vartheta} \tag{10}$$

Use cross-multiplication for

$$2\sin^2\vartheta\cos\vartheta = (\cos^2\vartheta - \sin^2\vartheta)\cos\vartheta \tag{11}$$

Shift everything to the left-hand side and factor out $\cos\vartheta$ for

$$\cos\vartheta \cdot (3\sin^2\vartheta\cos\vartheta - cos^2\vartheta) = 0 \tag{12}$$

There are two factors here that multiply to give us zero. Look at them individually when they might turn zero.

$$\cos\vartheta = 0 \text{ or } 3\sin^2\vartheta - \cos^2\vartheta = 0 \tag{13}$$

Replace $\cos^2\vartheta$ by $1 - \sin^2\vartheta$ in the second factor for $\sin\vartheta = 1/2$ or $\sin\vartheta = 1/2 = -1/2$.

Trigonometric Equations

Consequently,

$$\vartheta = 90^\circ \text{ or } \vartheta = 270^\circ \text{ or}$$
$$\vartheta = 30^\circ \text{ or } \vartheta = 150^\circ \text{ or}$$
$$\vartheta = 210^\circ \text{ or } \vartheta = 330^\circ \tag{14}$$

Therefore, the solution set is

$$S = \{\vartheta \in \mathbb{R} | \vartheta = 30^\circ + k \cdot 180^\circ, \vartheta = 90^\circ + k \cdot 180^\circ,$$
$$\vartheta = 150^\circ + k \cdot 180^\circ, k \in \mathbb{Z}\} \tag{15}$$

Notice that the period is now 180°, which is what we would expect from a trigonometric equation with tangents and cotangents.

Trigonometric Equations

Example 3: Trigonometric Equation with Modified Angles.
Solve the following trigonometric equation,

$$2\sin 3\vartheta - 1 = 0 \tag{16}$$

We could use a formula for the triple angle here,

$$\sin 3\vartheta = \sin(2\vartheta + \vartheta) = \sin 2\vartheta \cos\vartheta + \sin\vartheta \cos 2\vartheta =$$

$$2\sin\vartheta\cos\vartheta \cdot \cos\vartheta + \sin\vartheta(\cos^2\vartheta - \sin^2\vartheta) \tag{17}$$

but this expression looks daunting.

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Instead, substitute $\alpha = 3\vartheta$ for

$$2\sin\alpha - 1 = 0 \tag{18}$$

The solutions are

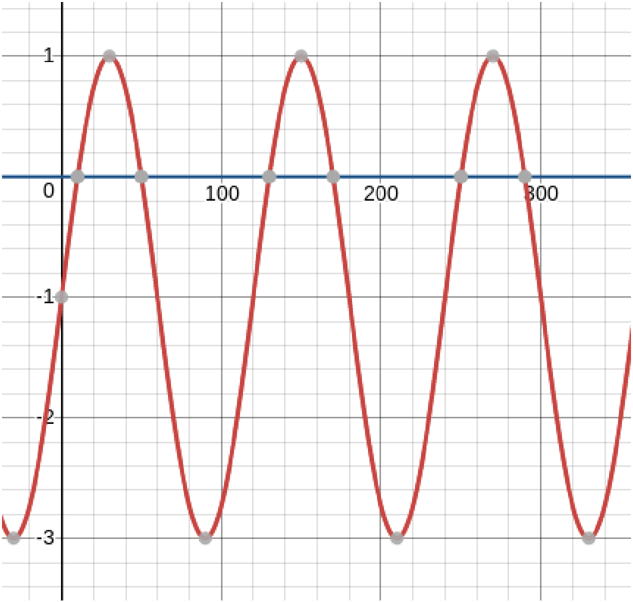
$$\alpha = \ldots, 30^\circ, 150^\circ, 390^\circ, 510^\circ, \ldots \tag{19}$$

Therefore,

$$\vartheta = \ldots, 10^\circ, 50^\circ, 130^\circ, 170^\circ, \ldots \tag{20}$$

Notice how the period has changed from 360° to 120°.

$$S = \{\vartheta \in \mathbb{R} | \vartheta = 10^\circ + k \cdot 120^\circ, \vartheta = 50^\circ + k \cdot 120^\circ, k \in \mathbb{Z}\} \tag{21}$$


1
0
-1
-2
-3
100
200
300

Trigonometric Equations

Exercise 1: Solve the following trigonometric equation,

$$2\sin\vartheta - 1 = 0 \tag{22}$$

Exercise 2: Solve the following trigonometric equation,

$$(\cot^2 \vartheta)(3 + \sqrt{5}) + \sqrt{5} = 5 \tag{23}$$

Exercise 3: Solve the following trigonometric equation,

$$2\cos^2\vartheta - 7\cos\vartheta + 3 = 0 \tag{24}$$

Exercise 4: Solve the following trigonometric equation,

$$\cos 2\vartheta = \sqrt{3} - \sin\vartheta \tag{25}$$

Exercise 5: Solve the following trigonometric equation,

$$\sqrt{3}\tan\frac{\vartheta}{2} - 1 = 0 \tag{26}$$

Exercise 6: Solve the following trigonometric equation,

$$\tan^2\vartheta - \tan\vartheta - 2 = 0 \tag{27}$$

Exercise 7: Solve the following trigonometric equation,

$$\sin 3\vartheta + \sin\vartheta = 0 \tag{28}$$

Exercise 8: Solve the following trigonometric equation,

$$2\sin^2\frac{1}{2}\vartheta - \cos\vartheta = 2 \tag{29}$$

Exercise 9: Solve the following trigonometric equation,

$$\sec 2\vartheta = 2\cos\vartheta - 1 \tag{30}$$

Exercise 10: Solve the following trigonometric equation,

$$\sin^2 \vartheta - \cos^2 \vartheta - \cos 2\vartheta = 1 \tag{31}$$

Exercise 11: Solve the following trigonometric equation,

$$2\sin\left(\vartheta - \frac{\pi}{6}\right) = \sqrt{3}\sin\vartheta \tag{32}$$

End of Lesson

Next Lesson: Conics.