# Systems of Linear Equations

MATH 1511, BCIT

Technical Mathematics for Geomatics

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Chaitali and Amulya go to a concession stand to buy fruit. Chaitali buys 5 bananas and 3 apples and spends \$13.50. Amulya buys 1 banana and 5 apples and spends 20 cents more than Chaitali. How much do bananas and apples cost at the concession stand?

# Systems of Linear Equations Introduced

Chaitali and Amulya go to a concession stand to buy fruit. Chaitali buys 5 bananas and 3 apples and spends \$13.50. Amulya buys 1 banana and 5 apples and spends 20 cents more than Chaitali. How much do bananas and apples cost at the concession stand?

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{1}$$

# What Is a System of Linear Equations?

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{2}$$

This system of linear equations is the rule for the following set $S \subset \mathbb{R} \times \mathbb{R}$:

$$S = \{(x, y) \in \mathbb{R} \times \mathbb{R} | 5x + 3y = 13.5 \text{ and } x + 5y = 13.7\} \tag{3}$$

# Solution Methods

$$
\begin{array}{rcrcl}
5x & + & 3y & = & 13.5 \\
x & + & 5y & = & 13.7
\end{array}
\tag{4}
$$

There are several ways to solve a system of equations like this.

- Graphing
- Substitution
- Elimination
- Using a Matrix

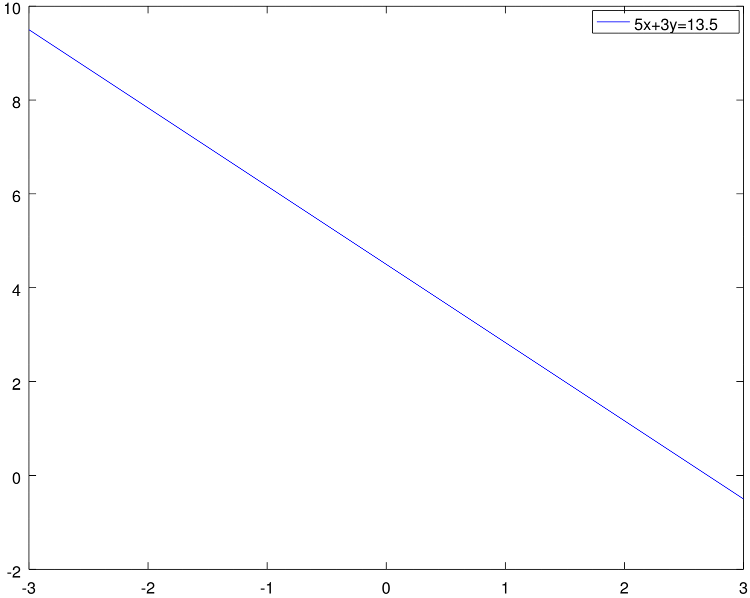
## Graphing Method I

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{5}$$

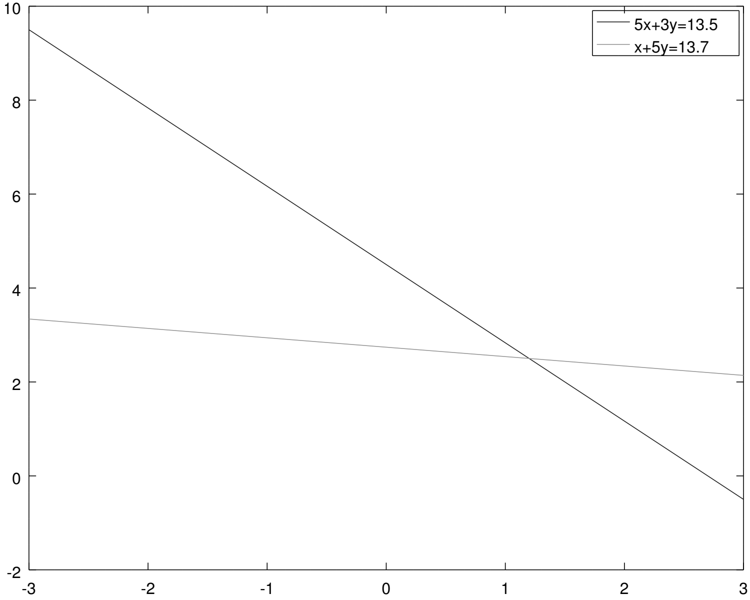
is equivalent to

$$\begin{array}{rcl} y & = & -\frac{5}{3}x + \frac{9}{2} \\ \\ y & = & -\frac{1}{5}x + \frac{137}{50} \end{array} \tag{6}$$

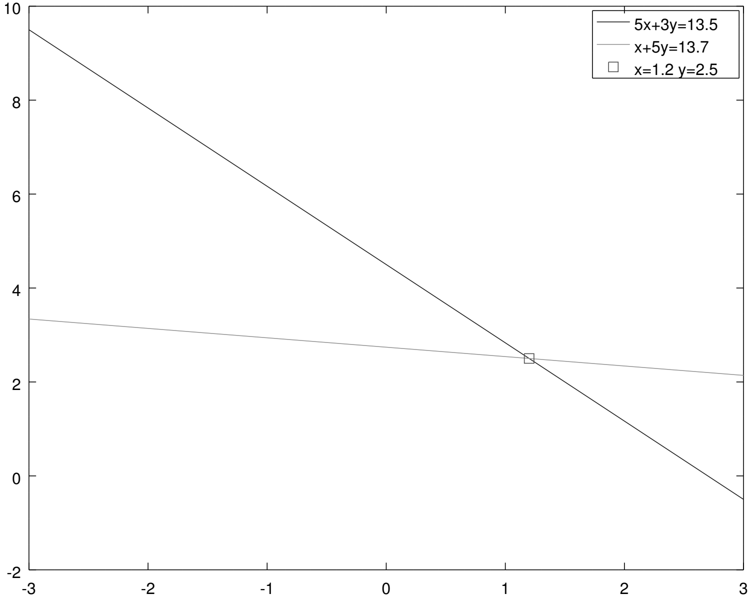
# Graphing Method II



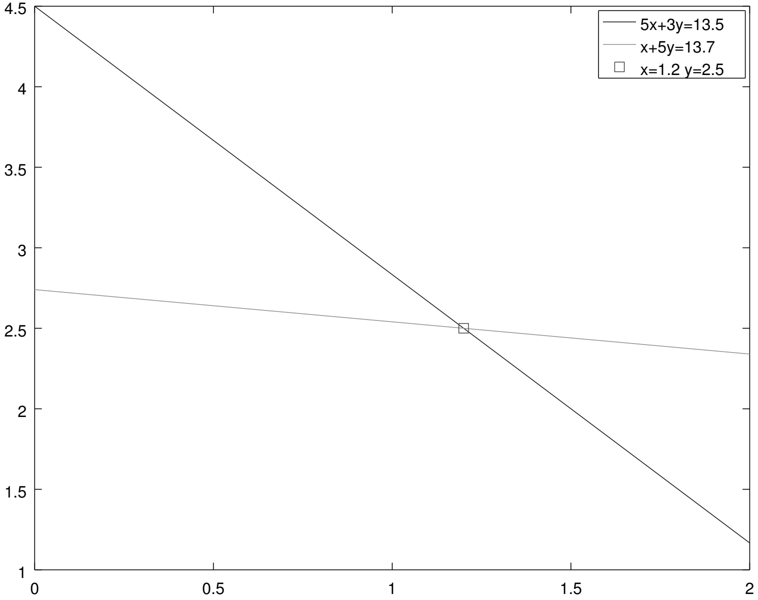
# Graphing Method III



# Graphing Method IV



# Graphing Method V



# Graphing Method Exercises

Find a solution to these systems of linear equations by graphing them and check your answer by substituting.

$$\begin{array}{rcrcr} 7x & - & 6y & = & 19 \\ -5x & + & 2y & = & -9 \end{array} \tag{7}$$

$$\begin{array}{rcrcr} x & + & 3y & = & 12 \\ 11x & - & 2y & = & 27 \end{array} \tag{8}$$

$$\begin{array}{rcrcr} \frac{1}{2}x & - & 2y & = & \frac{9}{2} \\ -\frac{5}{8}x & + & y & = & -\frac{15}{8} \end{array} \tag{9}$$

# Substitution Method I

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{10}$$

The second equation yields $x = 13.7 - 5y$. Use this to substitute in the first equation

$$5 \cdot (13.7 - 5y) + 3y = 13.5 \tag{11}$$

therefore, $-22y = -55$ and $y = 5/2$. Now substitute $y = 5/2$ in the first equation (you could just as well use the second equation), so

$$5x + 3 \cdot \frac{5}{2} = 13.5 \tag{12}$$

which implies $x = 1.2$. A banana costs \$1.20; an apple costs \$2.50.

# Substitution Method Exercises

Find a solution to these systems of linear equations by using the substitution method.

$$\begin{array}{rcrcr} 7x & - & 6y & = & 19 \\ -5x & + & 2y & = & -9 \end{array} \tag{13}$$

$$\begin{array}{rcrcr} x & + & 3y & = & 12 \\ 11x & - & 2y & = & 27 \end{array} \tag{14}$$

$$\begin{array}{rcrcr} \frac{1}{2}x & - & 2y & = & \frac{9}{2} \\ -\frac{5}{8}x & + & y & = & -\frac{15}{8} \end{array} \tag{15}$$

# Elimination Method I

$$
\begin{array}{rcrcl}
5x & + & 3y & = & 13.5 \\
x & + & 5y & = & 13.7
\end{array}
\tag{16}
$$

is equivalent to

$$
\begin{array}{rcrcl}
5x & + & 3y & = & 13.5 \\
5x & + & 25y & = & 68.5
\end{array}
\tag{17}
$$

# Elimination Method II

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ 5x & + & 25y & = & 68.5 \end{array} \tag{18}$$

implies

$$(5x + 3y) - (5x + 25y) = 13.5 - 68.5 \tag{19}$$

therefore, $-22y = -55$ and $y = 5/2$. Now substitute $y = 5/2$ in the first equation (you could just as well use the second equation), so

$$5x + 3 \cdot \frac{5}{2} = 13.5 \tag{20}$$

which implies $x = 1.2$. A banana costs \$1.20; an apple costs \$2.50.

# Elimination Method Exercises

Find a solution to these systems of linear equations by using the elimination method.

$$\begin{array}{rcrcr} 7x & - & 6y & = & 19 \\ -5x & + & 2y & = & -9 \end{array} \tag{21}$$

$$\begin{array}{rcrcr} x & + & 3y & = & 12 \\ 11x & - & 2y & = & 27 \end{array} \tag{22}$$

$$\begin{array}{rcrcr} \frac{1}{2}x & - & 2y & = & \frac{9}{2} \\ -\frac{5}{8}x & + & y & = & -\frac{15}{8} \end{array} \tag{23}$$

# Matrix Method

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{24}$$

is the system of linear equations that we are trying to solve. A matrix is a rectangular arrangement of numbers, for example

$$\begin{bmatrix} 5 & 3 & 13.5 \\ 1 & 5 & 13.7 \end{bmatrix} \tag{25}$$

There are many fascinating things you can do with matrices. The discipline that deals with matrices is called Linear Algebra.

# Matrix Addition

We can define operations on matrices just like we define operations on numbers. For example, we can add an $m \times n$ matrix to another one as follows,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \ddots & & \\ \vdots & & & \vdots \\ b_{m1} & & \cdots & b_{mn} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1}+b_{m1} & & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

# Matrix Scalar Multiplication

Next, we define what it means to multiply a matrix by a scalar, i.e. a real number (NOT a matrix).

$$k \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & \ddots & & \\ \vdots & & & \vdots \\ ka_{m1} & & \cdots & ka_{mn} \end{bmatrix}$$

# Matrix Product

Finally, we define matrix multiplication. You can multiply an $m \times j$ matrix by a $j \times n$ matrix, which will give you an $m \times n$ matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & \ddots & & \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mj} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \ddots & & \\ \vdots & & & \vdots \\ b_{j1} & & \cdots & b_{jn} \end{bmatrix} =$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & \ddots & & \\ \vdots & & & \vdots \\ c_{m1} & & \cdots & c_{mn} \end{bmatrix}$$

where $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{ij}b_{jk}$.

# Matrix Inverse I

Matrix multiplication for an an $m \times j$ matrix by a $k \times n$ matrix is not defined when $j \neq k$. An inverse matrix $A^{-1}$ of a square matrix $A$ is defined to be the matrix

$$A \cdot A^{-1} = A^{-1} \cdot A = E \tag{26}$$

where

$$E = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \cdots & 1 & 0 \\ 0 & & \cdots & 0 & 1 \end{bmatrix}$$

# Matrices and Systems of Linear Equations I

Remember our system of linear equations.

$$\begin{array}{rcrcl} 5x & + & 3y & = & 13.5 \\ x & + & 5y & = & 13.7 \end{array} \tag{27}$$

In matrix notation, we can write

$$\begin{bmatrix} 5 & 3 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13.5 \\ 13.7 \end{bmatrix}$$

Let's call these three matrices $A, v, b$ respectively. $A$ and $b$ are provided, and we are looking for $v$. If we had $A^{-1}$, we could go from

$$Av = b \tag{28}$$

to

$$A^{-1}Av = A^{-1}b \tag{29}$$

which is the same as

$$v = A^{-1}b \tag{30}$$

The challenge is therefore to find $A^{-1}$. Scientific calculators and computers can find $A^{-1}$ for you.

# Matrix Inverse and Determinants

If you want to know how to find the inverse yourself, one method to use is calculating the determinant of a matrix. It takes a bit of time to understand determinants, and then it's still a complicated (and not very transparent) procedure to get to the inverse. For $2 \times 2$ matrices, however, the inverse is

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{31}$$

for

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{32}$$

and the determinant is $\det A = ad - bc$.

# Matrix Row Operations

Another method to find the inverse of a matrix is using matrix row operations. There are three matrix row operations.

- Row Switching means you are allowed to switch two rows, for example $R_1 \leftrightarrow R_2$
- Row Multiplication means you are allowed to multiply all elements of a row by a real non-zero number, for example $\frac{2}{5}R_2 \rightarrow R_2$
- Row Addition means you are allowed to add one row to another and then replace one of the original rows by the sum of the two rows, for example $R_1 + R_2 \rightarrow R_1$

Row multiplication and row addition are often used together, for example $\frac{7}{8}R_1 - R_3 \rightarrow R_3$.

# Matrix Row Operations

To find the inverse of a square matrix, we combine $A$ and $E$

$$\begin{bmatrix} 5 & 3 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{bmatrix}$$

and apply matrix row operations until we get

$$\begin{bmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \end{bmatrix}$$

where

$$A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

# Inverse Example

For our example,

$$\left[\begin{array}{cccc} 5 & 3 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 25/3 & 5 & 5/3 & 0 \\ 1 & 5 & 0 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cccc} 22/3 & 0 & 5/3 & -1 \\ 1 & 5 & 0 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cccc} 22/3 & 0 & 5/3 & -1 \\ 22/3 & 110/3 & 0 & 22/3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc} 22/3 & 0 & 5/3 & -1 \\ 0 & 110/3 & -5/3 & 25/3 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cccc} 1 & 0 & 5/22 & -3/22 \\ 0 & 1 & -1/22 & 5/22 \end{array}\right]$$

# Inverse Example

For step 1, we multiplied the first row by $5/3$ (row multiplication). For step 2, we subtracted the second row from the first row and replaced the first row by the result (row addition). For step 3, we multiplied the second row by $22/3$ (row multiplication). For step 4, we subtracted the first row from the second row and replaced the second row by the result (row addition). For the last step, we multiplied the first row by $3/22$ and the second row by $3/110$ (row multiplication applied twice).

Thus,

$$A^{-1} = \begin{bmatrix} 5/22 & -3/22 \\ -1/22 & 5/22 \end{bmatrix} = \frac{1}{22} \cdot \begin{bmatrix} 5 & -3 \\ -1 & 5 \end{bmatrix}$$

and

$$v = A^{-1}b = \begin{bmatrix} 5/22 & -3/22 \\ -1/22 & 5/22 \end{bmatrix} \cdot \begin{bmatrix} 13.5 \\ 13.7 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 2.5 \end{bmatrix}$$

# Professor van Shmowhawk

# Which Method to Use I

The elimination method is usually the fastest. The substitution method is most transparent, which means it's easy to see what is going on. The substitution method is also helpful for systems of equations that are not linear. The matrix method is very powerful for systems of equations that have more than two equations. Consider this example.

$$\begin{array}{rcrcrcrcr} 5x & + & 9y & + & 7z & - & 6w & = & -42 \\ -7x & + & y & - & 9z & + & 9w & = & 38 \\ -2x & - & 8y & + & 3z & - & 7w & = & -6 \\ 10x & + & 9y & - & 2z & - & 2w & = & -117 \end{array} \tag{33}$$

# Which Method to Use II

A computer program tells us that

$$\begin{bmatrix} 5 & 9 & 7 & -6 \\ -7 & -1 & -9 & 9 \\ -2 & -8 & 3 & -7 \\ 10 & 9 & -2 & -2 \end{bmatrix}^{-1} = \frac{1}{4477}\begin{bmatrix} -632 & -667 & -379 & 221 \\ 572 & 462 & 88 & 55 \\ -79 & -643 & -607 & -532 \\ -507 & -613 & -892 & -354 \end{bmatrix}$$

and the inverse matrix multiplied by the right-hand side of the system of equations gives us the solution $x = -5, y = -3, z = 10, w = 10$.

**Exercise 1:** A woman rows a boat upstream from one point on a river to another point 4 miles away in 1.5 hours. The return trip, traveling with the current, takes only 45 minutes. How fast does she row relative to the water, and at what speed is the current flowing?

**Exercise 2:** A vintner fortifies wine that contains 10% alcohol by adding 70% alcohol solution to it. The resulting mixture has an alcoholic strength of 16% and fills 1000 one-litre bottles. How many litres of the wine and of the alcoholic solution does she use?

**Exercise 3:** Isabella and Wei-Shen leave their house at the same time and drive in opposite directions. Isabella drives at 60 kilometres an hour and travels 35 kilometres farther than Wei-Shen, who drives at 40 kilometres an hour. Wei-Shen's trip takes 15 minutes longer than Isabella's. For what length of time does each one of them drive?

# Three Variables

Solve the following system of linear equations:

$$\begin{array}{rcrcrcr} -6x & + & & & 6z & = & 48 \\ 2x & + & 5y & - & 6z & = & -44 \\ -3x & + & y & + & 2z & = & 18 \end{array} \tag{34}$$

# Three Variables

Solve the following system of linear equations:

$$\begin{aligned} 4y &= 32 + 5x + 6z \\ 7z + 5x &= 6y - 35 \\ 7y + 3z &= 3x - \tfrac{19}{2} \end{aligned} \tag{35}$$

# Solution

$$\left[\begin{array}{cccccc} -5 & 4 & -6 & 1 & 0 & 0 \\ 5 & -6 & 7 & 0 & 1 & 0 \\ -3 & 7 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 + 2 \cdot R_3 \longrightarrow R_1$$

$$\left[\begin{array}{cccccc} -11 & 18 & 0 & 1 & 0 & 2 \\ 5 & -6 & 7 & 0 & 1 & 0 \\ -3 & 7 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 - \frac{7}{3} \cdot R_3 \longrightarrow R_2$$

$$\left[\begin{array}{cccccc} -11 & 18 & 0 & 1 & 0 & 2 \\ 12 & -\frac{67}{3} & 0 & 0 & 1 & -\frac{7}{3} \\ -3 & 7 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 + \frac{11}{12} \cdot R_2 \longrightarrow R_2$$

# Solution

$$\left[\begin{array}{cccccc} -11 & 18 & 0 & 1 & 0 & 2 \\ 12 & -\frac{67}{3} & 0 & 0 & 1 & -\frac{7}{3} \\ -3 & 7 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 + \frac{11}{12} \cdot R_2 \longrightarrow R_2$$

$$\left[\begin{array}{cccccc} -11 & 18 & 0 & 1 & 0 & 2 \\ 0 & -\frac{89}{36} & 0 & 1 & \frac{11}{12} & -\frac{5}{36} \\ -3 & 7 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 - \frac{11}{3} \cdot R_3 \longrightarrow R_3$$

$$\left[\begin{array}{cccccc} -11 & 18 & 0 & 1 & 0 & 2 \\ 0 & -\frac{89}{36} & 0 & 1 & \frac{11}{12} & -\frac{5}{36} \\ 0 & -\frac{23}{3} & -11 & 1 & 0 & -\frac{5}{3} \end{array}\right]$$

$$36 \cdot R_2 \longrightarrow R_2 \qquad 3 \cdot R_3 \longrightarrow R_3$$

# Solution

$$\left[\begin{array}{cccccc} -11 & 18 & 0 & 1 & 0 & 2 \\ 0 & -\frac{89}{36} & 0 & 1 & \frac{11}{12} & -\frac{5}{36} \\ 0 & -\frac{23}{3} & -11 & 1 & 0 & -\frac{5}{3} \end{array}\right]$$

$$36 \cdot R_2 \longrightarrow R_2 \qquad 3 \cdot R_3 \longrightarrow R_3$$

$$\left[\begin{array}{cccccc} -11 & 18 & 0 & 1 & 0 & 2 \\ 0 & -89 & 0 & 36 & 33 & -5 \\ 0 & -23 & -33 & 3 & 0 & -5 \end{array}\right]$$

$$R_2 + \frac{89}{18} \cdot R_1 \longrightarrow R_1$$

$$\left[\begin{array}{cccccc} -\frac{979}{18} & 0 & 0 & \frac{737}{18} & 33 & \frac{88}{18} \\ 0 & -89 & 0 & 36 & 33 & -5 \\ 0 & -23 & -33 & 3 & 0 & -5 \end{array}\right]$$

$$R_2 - \frac{89}{23} \cdot R_3 \longrightarrow R_3 \qquad 18 \cdot R_1 \longrightarrow R_1$$

# Solution

$$\left[\begin{array}{cccccc} -\frac{979}{18} & 0 & 0 & \frac{737}{18} & 33 & \frac{88}{18} \\ 0 & -89 & 0 & 36 & 33 & -5 \\ 0 & -23 & -33 & 3 & 0 & -5 \end{array}\right]$$

$$R_2 - \frac{89}{23} \cdot R_3 \longrightarrow R_3 \qquad 18 \cdot R_1 \longrightarrow R_1$$

$$\left[\begin{array}{cccccc} -979 & 0 & 0 & 737 & 594 & 88 \\ 0 & -89 & 0 & 36 & 33 & -5 \\ 0 & 0 & \frac{2937}{23} & \frac{561}{23} & 33 & \frac{330}{23} \end{array}\right]$$

clean up

$$A^{-1} = \frac{1}{89}\left[\begin{array}{ccc} -67 & -54 & -8 \\ -36 & -33 & 5 \\ 17 & 23 & 10 \end{array}\right]$$

# Solution

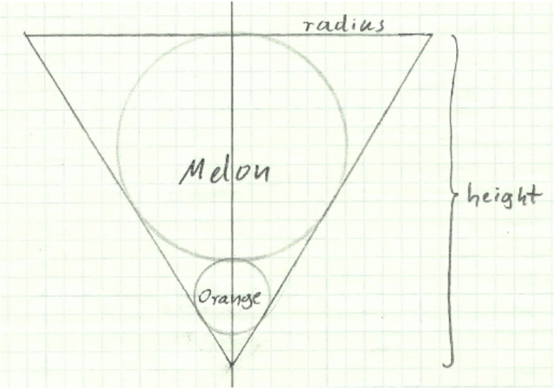
The solution vector for the system of linear equations is

$$v = A^{-1} \cdot b = \frac{1}{89} \cdot \begin{bmatrix} -67 & -54 & -8 \\ -36 & -33 & 5 \\ 17 & 23 & 10 \end{bmatrix} \begin{bmatrix} 32 \\ -35 \\ -\frac{19}{2} \end{bmatrix} = \begin{bmatrix} -2 \\ -\frac{1}{2} \\ -4 \end{bmatrix}$$

Therefore, $S = \{(-2, -\frac{1}{2}, -4)\}$. Note that it is sometimes difficult to keep zeros you already have in place when you do elementary row operations. The trick is that when you are trying to get a second zero in a row, you must do this with another row that shares the first row's first zero. For example, use row 1 and row 3 to get a zero in row 1's third place. Then use row 2 and row 3 to get a zero in row 2's third place. Now you have a zero both in row 1's third place and in row 2's third place, so you can use row 1 and row 2 to get another zero in row 1's second place without losing the zero that is already in third place.

# Fruit in a Bag

You have a cone shaped bag. At the bottom of the bag is an orange with a radius of 2 inches. On top of the orange is a melon with a radius of 6 inches. It touches the orange and fits snugly in the bag, touching it in a ring around the orange. Its top is at the same level as the top of the bag. Calculate the height of the cone.



**Exercise 4:** Solve the following system of equations,

$$\begin{aligned} \frac{x}{3} + \frac{y}{2} &= \frac{4}{3} \\ \frac{x}{2} + \frac{y}{3} &= \frac{7}{6} \end{aligned} \tag{36}$$

**Exercise 5:** Find the inverse of the following matrix. Show your work (i.e. the elementary row operations).

$$\begin{bmatrix} 3 & 6 \\ -7 & 1 \end{bmatrix} \tag{37}$$

# End of Lesson

Next Lesson: The Right Triangle.