Spherical Trigonometry

MATH 1511, BCIT

Technical Mathematics for Geomatics

November 29, 2017

Captain America I

Realizing that in our present national emergency trigonometry is used in practically every phase of our war effort, the objective of the authors in writing this book was to present a brief but mathematically accurate course in plane and spherical trigonometry with special emphasis on the computational or practical side of the subject. In those chapters dealing with computational trigonometry, thorough drill is first given through the use of many examples.

Captain America II

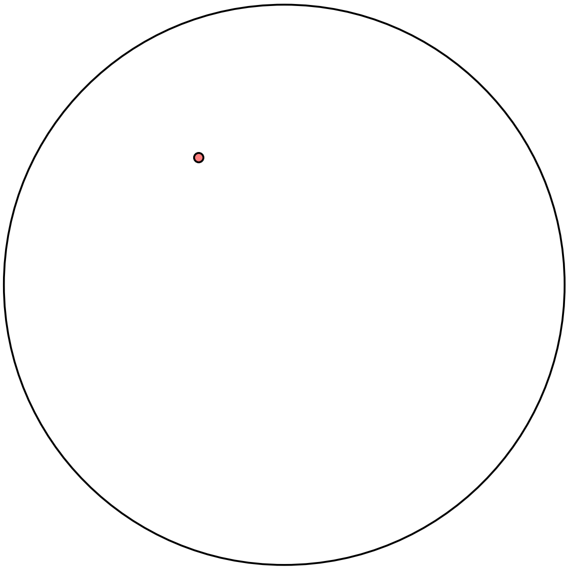
This is followed at the end of each of these chapters by practical applications introduced as problems, along with the necessary explanations and definitions, to secure conciseness of presentation. The applications deal with surveying, gun fire, course and position of airplanes, and navigation. This arrangement of theory and application in the book has permitted a sharp presentation of the underlying ideas which is necessary for rapid mastery of the subject. (Clifford Bell and Tracy Thomas, Essentials of Plane and Spherical Trigonometry, 1943, page iii)

The Halifax Problem I

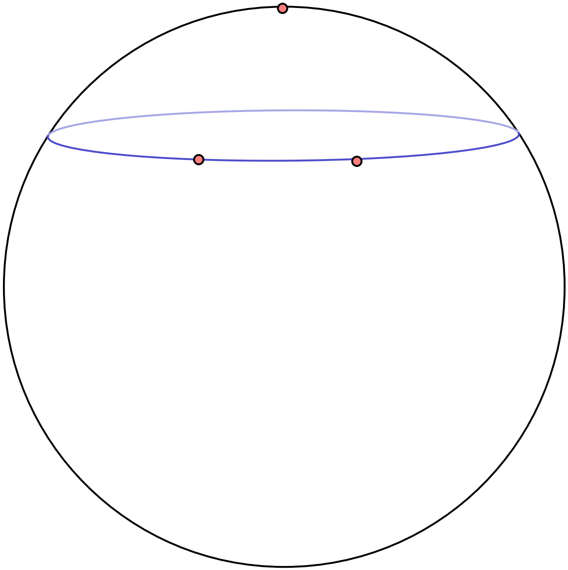
This is a problem from Raymond Brink, *Spherical Trigonometry*, 1942, page 17.

> *A ship leaves Halifax (position,* $44.67^\circ N, 63.58^\circ W$*), starting due east [...]. Find its position and direction after it has sailed 1000 nautical miles.*

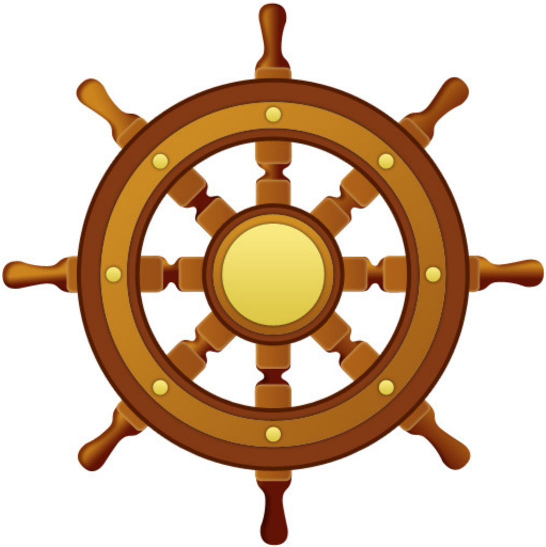
The Halifax Problem II



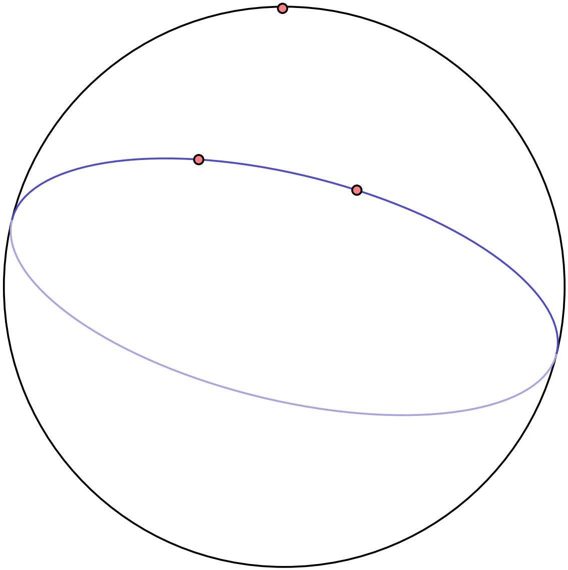
The Halifax Problem III



The Halifax Problem IV



The Halifax Problem V



The Halifax Problem IV

This is a problem from Raymond Brink, *Spherical Trigonometry*, 1942, page 17.

> *A ship leaves Halifax (position,* $44.67^\circ N, 63.58^\circ W$*), starting due east and continuing on the great circle. Find its position and direction after it has sailed 1000 nautical miles.*

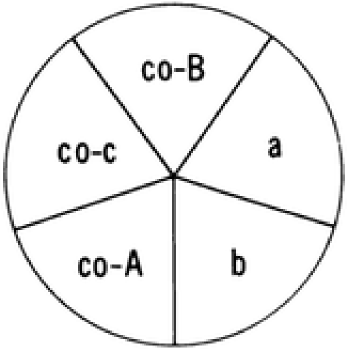
Great Circles

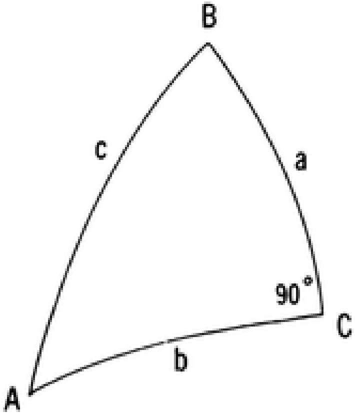
The intersection between a plane and a sphere is a circle. If the centre of the sphere is an element of the plane, then the intersection is a great circle. A spherical angle at point P is an arc length on a great circle for which P is the pole.

Triangle Sum

The sum of the angles of a spherical triangle is less than six right angles and greater than two right angles.

Napier's Pentagramma Mirificum

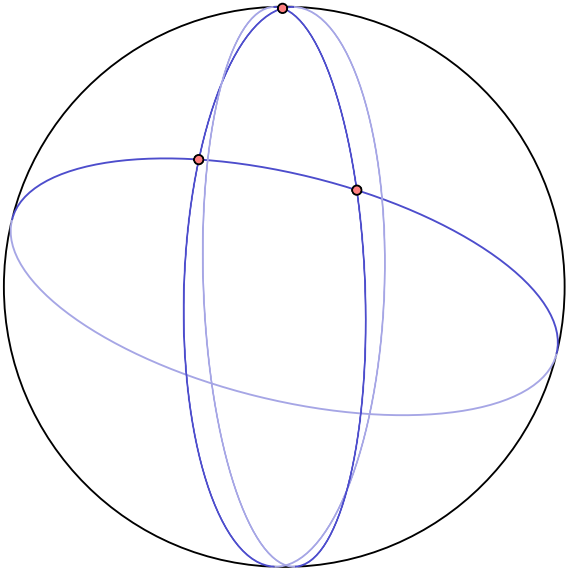




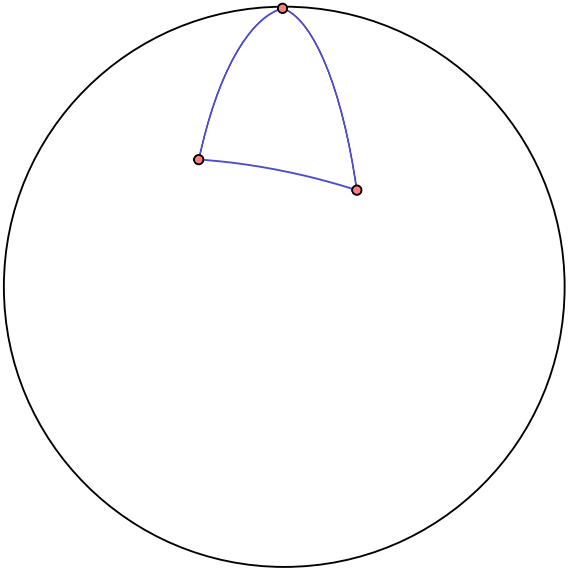
Napier's Rules

- *Rule I:* The sine of any circular part is equal to the product of the tangents of the two parts adjacent to it.
- *Rule II:* The sine of any circular part is equal to the product of the cosines of the two parts opposite to it.

The Right-Angled Euler Triangle I



The Right-Angled Euler Triangle II



Halifax Problem: Latitude

Use the relevant three slices of the Pentagramma Mirificum for the following formula:

$$\cos c = \cos a \cdot \cos b \tag{1}$$

a is 1000 nautical miles. One nautical mile is one minute of arc, or $\frac{1}{60}^{\circ}$, on the Earth's surface. Therefore, $a = 16.667^{\circ}$ and $b = 45.33^{\circ}$. Using the inverse function of cosine on $\cos a \cdot \cos b$ and subtracting the result from 90°, the result for the latitude of E is $42.337^{\circ} N$.

Halifax Problem: Longitude

Use the relevant three slices of the Pentagramma Mirificum for the following formula:

$$\cos A = \cot c \cdot \tan b \tag{2}$$

Using the inverse function of cosine on $\cot c \cdot \tan b$ and subtracting the result from 63.58°, the result for the longitude of E is $40.75^\circ W$.

Halifax Problem: Direction

Use the relevant three slices of the Pentagramma Mirificum for the following formula:

$$\cos B = \cot c \cdot \tan a \tag{3}$$

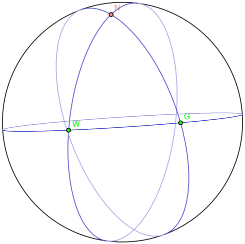
Using the inverse function of cosine on $\cot c \cdot \tan a$, the result for the direction at E is 74.171° east of south.

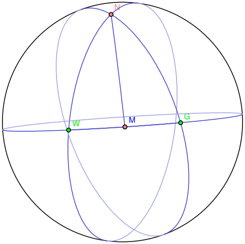
Remember this problem a while ago:

Windsor to Grenoble

Consider two towns, Windsor, Nova Scotia, at $(45^\circ N, 65^\circ W)$ and Grenoble, France at $(45^\circ N, 5^\circ E)$. If you follow a line of latitude, how far are the two towns apart?

We can now calculate the distance along the great circle.



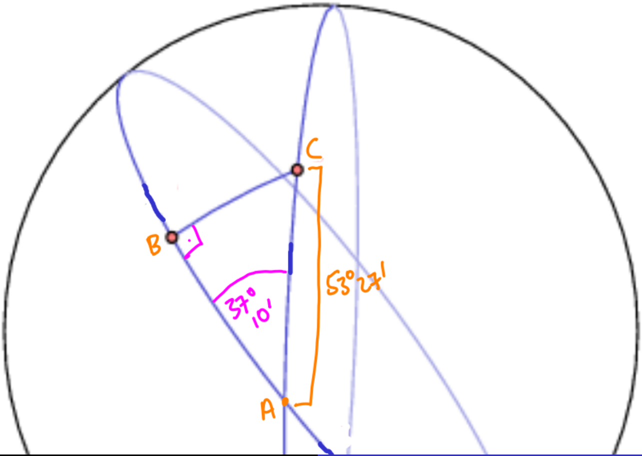


Notice that NWG is an isosceles triangle. Consequently, NWM is a right triangle. Let $a = NM, b = MW, c = WN = 45^\circ$. The angle $\angle WNM$ is 35°. Napier's miraculous pentagram gives us

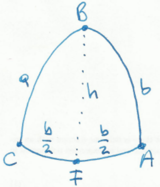
$$\sin b = \sin 35^\circ \cdot \sin 45^\circ \text{ therefore } b = 0.41761 \tag{4}$$

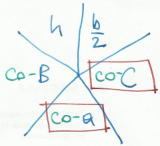
b is in radians in equation (4). Multiply twice this number by the radius of the Earth (6378.1km), and the correct solution is approximately 5327.2km, compared to approximately 5510.0km along the circle of latitude.

One of the angles formed by the intersection of two great circles is $37^\circ 10'$. A point on one of the circles is $53^\circ 27'$ from the intersection point of the circles. Find the shortest distance from this point to the other circle.



Solve the isosceles triangle ABC, where $a = c = 79^\circ 17'$ and $A = C = 59^\circ 37'$.





$$a = c = 79^\circ 17' \approx 1.3838$$
$$C = A = 59^\circ 37' \approx 1.0405$$

$$\sin h = \sin C \sin a$$
$$h = 1.0115 = 57.953^\circ$$

$$\cos B = \tan h \cdot \cot a$$
$$B = 1.2637 = 72.403^\circ = 72^\circ 24'$$

$$\sin \frac{b}{2} = \sin B \cdot \sin a$$
$$\frac{b}{2} = 1.2127$$
$$b = 2.4255 \cong 138^\circ 58'$$

Oblique Triangles Law of Cosines

To solve triangles that do not have a right angle, use the cosine law for sides,

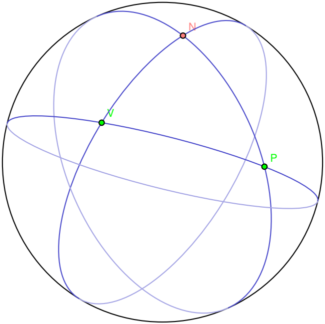
$$\cos a = \cos b \cos c + \sin b \sin c \cos A \tag{5}$$

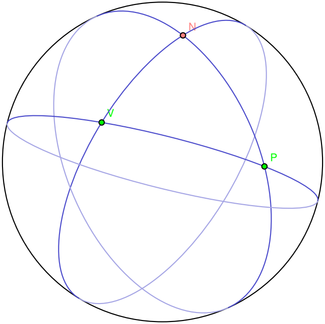
and the corresponding cosine law for angles derived from the polar triangle,

$$\cos A = \cos B \cos C + \sin B \sin C \cos a \tag{6}$$

Law of Cosines Example

Calculate the distance along the great circle between Vancouver ($49°15'$N, $123°6'$W) and Palma de Mallorca ($39°34'$N, $2°39'$E).





Let N be the angle $\angle VNP$. Let $a = VN, b = NP, c = VP$. According to the law of cosines,

$$\cos c = \cos a \cos b + \sin a \sin b cos N \tag{7}$$

Consequently, $c = 1.3811$ in radians, which translates to 8808.8km.

Oblique Triangles Napier's Analogies

Here are Napier's Analogies for oblique triangles. There is nothing special about the labels a, b, c and A, B, C. These analogies are true for any permutation of a, b, c and A, B, C, as long as it is consistently applied. Note that $\sin \frac{1}{2}(a-b)$ means $\sin[\frac{1}{2}(a-b)]$, not $\sin[\frac{1}{2}] \cdot (a-b)$.

$$
\begin{aligned}
\tan \tfrac{1}{2}(A-B) &= \cot \tfrac{C}{2} \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \\
\tan \tfrac{1}{2}(A+B) &= \cot \tfrac{C}{2} \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \\
\tan \tfrac{c}{2} &= \tan \left(\tfrac{1}{2}(a-b)\right) \frac{\sin \frac{1}{2}(A+B)}{\sin \frac{1}{2}(A-B)} \\
\tan \tfrac{c}{2} &= \tan \left(\tfrac{1}{2}(a+b)\right) \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)}
\end{aligned}
\tag{8}
$$

Oblique Triangles Law of Sines

Sometimes, the law of sines is needed to calculate a missing side,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \tag{9}$$

The law of sines, however, is a fickle friend. The calculator will often give you the incorrect solution if you use the inverse sine function, which is ambiguous. It is usually safer to use Napier's Analogies, which also solve the problem that in plane trigonometry is solved by the angle-sum formula $\alpha + \beta + \gamma = 180^\circ$.

Laws of Quadrants

When you take the arcsine of a number on a calculator, the calculator will always give you an angle less than 90°, for example the arcsine of $\pi/4$ is 45°. However, the angle that you want may be 135°, for $\sin 135^\circ = \pi/4$ as well. The laws of quadrants help you identify which angle (the acute or the obtuse) you should accept as your solution.

Angles between 0° and 90° are considered to be in the first quadrant. Angles between 90° and 180° are considered to be in the second quadrant.

Laws of Quadrants

1. An angle and its opposite side are in the same quadrant.
2. If any two of the three sides are in the same quadrant, the third side is in the first quadrant.
3. If any two sides are in different quadrants, the third side is in the second quadrant.

ABC and Non-ABC

There are two types of oblique triangles.

ABC Type
: If the knowns are "ABC", "ABc", "AbC", "abC", etc., then the triangle is of the ABC type. Use the law of cosines first and then either the law of sines or Napier's Analogies.

Non-ABC Type
: If the knowns are "Aac","ABb","aCc", etc., the triangle is of the non-ABC type. Use the law of sines first and then Napier's analogies (in plane trigonometry, you would use the law of sines and then the angle-sum formula).

Things to Remember

- In all the oblique triangle formulas (law of cosines, law of sines, Napier's Analogies), sides and angles are interchangable. The labels "a", "b", and "c" are arbitrary, so they are also interchangable. You must apply these changes consistently within the formula.
- Using the law of cosines, law of sines, and Napier's Analogies can saddle you with an incorrect solution. Always run all the sine laws and all the cosine laws to check your solution.

End of Lesson

Next Lesson: No Such Thing. Enjoy Your Holidays.