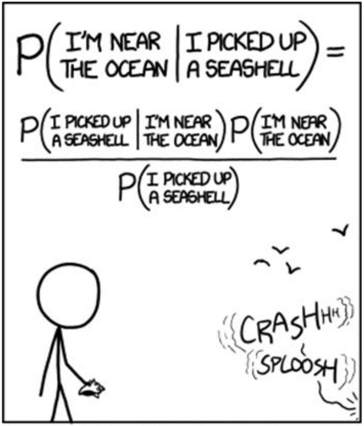
Bayes and Conditional Probability

MATH 2441, BCIT

Technical Mathematics for Food Technology

January 22, 2018

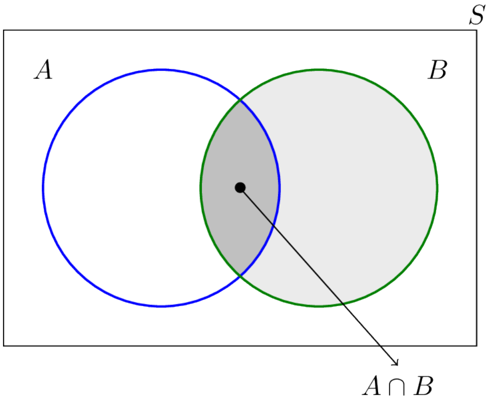
xkcd on Bayes' Formula



STATISTICALLY SPEAKING, IF YOU PICK UP A SEASHELL AND *DON'T* HOLD IT TO YOUR EAR, YOU CAN PROBABLY HEAR THE OCEAN.

Conditional Probability

Let's remember what conditional probability means.



$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Multiplication Rule

Remember the thief who wants to crack the four-digit PIN of a bank card. Let A be the event that she successfully cracks the PIN. If A_1 is the event that she succeeds on her first attempt (and so on for A_2 and A_3), then

$$P(A) = P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) = 0.003 \quad (1)$$

because A_1, A_2, A_3 are disjoint. We are assuming that her attempts happen without replacement. Therefore, A_1, A_2, A_3 are not independent, and the correct application of the multiplication rule is

$$P(A) = 1 - P(\neg A) =$$

$$1 - P(\neg A_1) \cdot P(\neg A_2 | \neg A_1) \cdot P(\neg A_3 | \neg A_1 \cap \neg A_2) = 0.003 \qquad (2)$$

Law of Total Probability

It is often easier to calculate conditional probabilities than unconditional probabilities. To express one by the other use the law of total probability,

$$P(A) = P(A|B)P(B) + P(A|\neg B)P(\neg B) \tag{3}$$

This formula also applies when you split up B into three or more disjoint subsets that exhaust B. It follows from set theory.

Example: Suppose that two factories supply light bulbs to the market. Factory X's bulbs work for over 5000 hours in 99% of cases, whereas factory Y's bulbs work for over 5000 hours in 95% of cases. It is known that factory X supplies 60% of the total bulbs available. What is the chance that a purchased bulb will work for longer than 5000 hours?

Law of Total Probability

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Let X be the event that the light bulb is from factory X. Let F be the event that the bulb will work for longer than 5000 hours. Then

$$P(F) = P(F|X)P(X) + P(F|\neg X)P(\neg X) =$$

$$0.99 \cdot 0.60 + 0.95 \cdot 0.40 = 0.974 \tag{4}$$

Law of Total Probability Exercises I

What is the probability that the second card in a conventional deck of cards is an ace?

Law of Total Probability Exercises II

Suppose we have two hats: one has 4 red balls and 7 green balls, the other has 11 red and 5 green. We toss an unfair coin (60/40 for heads), if heads, pick a random ball from the first hat, if tails from the second. What is the probability of getting a red ball?

Law of Total Probability Exercises III

You have three bags that each contain 100 marbles:

- Bag 1 has 75 red and 25 blue marbles
- Bag 2 has 60 red and 40 blue marbles
- Bag 3 has 45 red and 55 blue marbles

I choose one of the bags at random and then pick a marble from the chosen bag, also at random. What is the probability that the chosen marble is red?

Some Interesting Cases

A group of police officers have breathalyzers displaying false drunkenness in 5% of the cases in which the driver is sober. However, the breathalyzers never fail to detect a truly drunk person. One in a thousand drivers is driving drunk. Suppose the police officers then stop a driver at random, and force the driver to take a breathalyzer test. It indicates that the driver is drunk. We assume you don't know anything else about him or her. How high is the probability he or she really is drunk?

Some Interesting Cases

A room is full of engineers and lawyers (most of them are lawyers, 90%). The probability that an engineer enjoyed physics in school is 80%. The probability that a lawyer enjoyed physics in school is 30%. You ask someone in the room whether they enjoyed physics, and the answer is yes. Should you bet that this person is a lawyer, or should you bet that she is an engineer?

Some Interesting Cases

You have a million food items, of which 1 in 1000 is contaminated. You have a contamination test with a 2% false positive rate and a 0.5% false negative rate. A food item tests positive for contamination. What is the probability that it is contaminated?

Some Interesting Cases

In a city of 1 million inhabitants let there be 100 terrorists and 999,900 non-terrorists. To simplify the example, it is assumed that all people present in the city are inhabitants. Thus, the base rate probability of a randomly selected inhabitant of the city being a terrorist is 0.0001, and the base rate probability of that same inhabitant being a non-terrorist is 0.9999. In an attempt to catch the terrorists, the city installs an alarm system with a surveillance camera and automatic facial recognition software.

The software has two failure rates of 1%:

- The false negative rate: If the camera scans a terrorist, a bell will ring 99% of the time, and it will fail to ring 1% of the time.
- The false positive rate: If the camera scans a non-terrorist, a bell will not ring 99% of the time, but it will ring 1% of the time.

Suppose now that an inhabitant triggers the alarm. What is the chance that the person is a terrorist?

Bayes' Formula

Consider the definition of conditional probability,

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \tag{5}$$

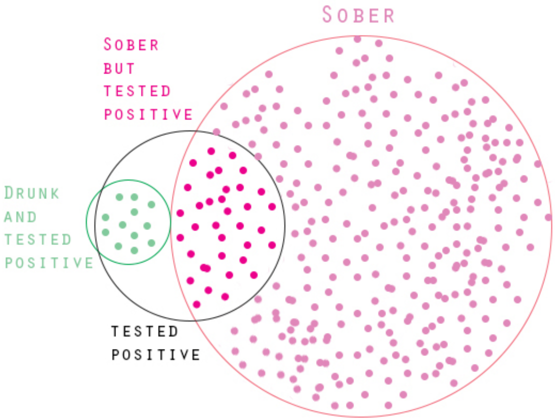
Now notice that $P(B \cap A) = P(A \cap B) = P(B)P(A|B)$. That means that

$$P(B|A) = \frac{P(B)P(A|B)}{P(A)} \tag{6}$$

By the law of total probability we can replace the denominator to give us Bayes' Formula

$$P(B|A) = \frac{P(B)P(A|B)}{P(A|B)P(B) + P(A|\neg B)P(\neg B)} \tag{7}$$

Base Rate Fallacy Diagram



Base Rate Fallacy Example

Let 100 out of 100,000 people have a disease. The test for this disease has a 5% false positive rate and a 5% false negative rate. If you test positive for this disease, what is your probability of actually having the disease. Consider the following contingency table and then apply Bayes' formula.

		Have Disease	
		Yes	No
Test Results	Positive	95	4,995
	Negative	5	94,905

Contingency Tables

Event	Event		Total
	B_1	B_2	
A_1	$P(A_1 \text{ and } B_1)$	$P(A_1 \text{ and } B_2)$	$P(A_1)$
A_2	$P(A_2 \text{ and } B_1)$	$P(A_2 \text{ and } B_2)$	$P(A_2)$
Total	$P(B_1)$	$P(B_2)$	1

Joint Probabilities

Marginal (Simple) Probabilities

Prison and Plea

Here is a contingency table:

	Guilty Plea	Plea of Not Guilty
Sentenced to Prison	392	58
Not Sentenced to Prison	564	14

Prison and Plea

Answer the following questions:

1. Find the probability of a randomly selected subject being sentenced to prison.
2. Find the probability of being sentenced to prison, given that the subject entered a plea of guilty.
3. Find the probability of being sentenced to prison, given that the subject entered a plea of not guilty.
4. Find the probability of a randomly selected subject being sentenced to prison or entering a plea of guilty.
5. If two subjects are randomly selected, find the probability that they were both sentenced to prison.
6. If two subjects are randomly selected, find the probability that they both entered pleas of not guilty.
7. Find the probability of a randomly selected subject being entering a plea of not guilty or not being sentenced to prison.
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Exercises I

Three urns contain respectively 1 white and 2 black balls; 3 white and 1 black ball; 2 white and 3 black balls. One ball is taken from each urn. What is the probability that among the balls drawn there are 2 white and 1 black?

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A student has a box containing 25 computer disks, of which 15 are blank and 10 are not. She randomly selects disks one by one and examines each one, terminating the process only when she finds a blank disk. What is the probability that she must examine at least two disks?

Exercises II

A student has a box containing 25 computer disks, of which 15 are blank and 10 are not. She randomly selects disks one by one and examines each one, terminating the process only when she finds a blank disk. What is the probability that she must examine at least two disks? Answer: 40%

Exercises III

There are five faculty members in a certain academic department. These individuals have 3, 6, 7, 10, and 14 years of teaching experience, respectively. Two of these individuals are randomly selected to serve on a committee. What is the probability that they have at least 15 years of teaching experience?

Exercises III

There are five faculty members in a certain academic department. These individuals have 3, 6, 7, 10, and 14 years of teaching experience, respectively. Two of these individuals are randomly selected to serve on a committee. What is the probability that they have at least 15 years of teaching experience? Answer: 60%

Exercises IV

Suppose three cards are selected from a well-mixed deck without replacement.

1. What is the probability that all three are hearts?
2. What is the probability that all three are from the same suit?
3. If five cards are dealt from a randomized deck, determine the probability that they are all hearts.

Exercises IV

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Exercises IV

Suppose three cards are selected from a well-mixed deck without replacement.

1. What is the probability that all three are hearts? Answer: 1.29%
2. What is the probability that all three are from the same suit? Answer: 5.18%
3. If five cards are dealt from a randomized deck, determine the probability that they are all hearts. Answer: 0.0495%

Exercises V

A tennis coach has brought out 12 tubes of Penn balls and 8 tubes of Wilson balls for his class. If 5 tubes are randomly selected, what is the probability that all 5 are of the same brand?

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Exercises VI

In Orange County, 51% of the adults are males. (It doesn't take too much advanced mathematics to deduce that the other 49% are females.) One adult is randomly selected for a survey involving credit card usage.

a. Find the prior probability that the selected person is a male.

b. It is later learned that the selected survey subject was smoking a cigar. Also, 9.5% of males smoke cigars, whereas 1.7% of females smoke cigars (based on data from the Substance Abuse and Mental Health Services Administration). Use this additional information to find the probability that the selected subject is a male.

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Answer: (a.) 51% (b.) 85.3%

7. Pleas and Sentences In a study of pleas and prison sentences, it is found that 45% of the subjects studied were sent to prison. Among those sent to prison, 40% chose to plead guilty. Among those not sent to prison, 55% chose to plead guilty.

a. If one of the study subjects is randomly selected, find the probability of getting someone who was not sent to prison.

b. If a study subject is randomly selected and it is then found that the subject entered a guilty plea, find the probability that this person was not sent to prison.

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b. If a study subject is randomly selected and it is then found that the subject entered a guilty plea, find the probability that this person was not sent to prison.

Answer: (a.) 55% (b.) 62.7%

13. Biased Coin In an article about confusion of eyewitnesses, John Allen Paulos cites the problem of three coins, one of which is biased so that it turns up heads 75% of the time. If you randomly select one of the coins, toss it three times, and obtain three heads, what is the probability that this is the biased coin?

Exercises VIII

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Answer: 62.8%

End of Lesson

Next Lesson: Uniform and Binomial Distribution