Correlation

MATH 2441, BCIT

Technical Mathematics for Food Technology

April 19, 2018

Correlation

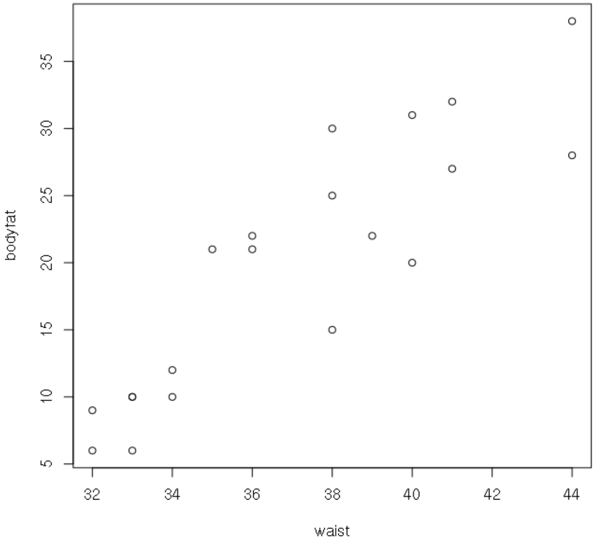
A correlation exists between two variables when the values of one variable are somehow associated with the values of the other variable.

A linear correlation exists between two variables when there is a correlation and the plotted points of paired data result in a pattern that can be approximated by a straight line.

Correlation

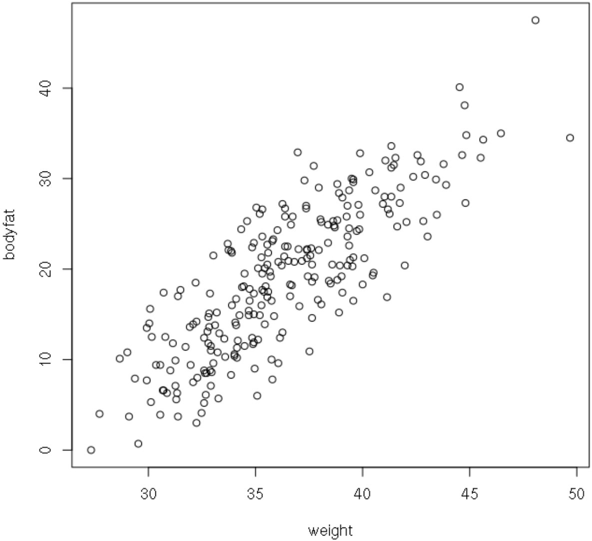
Here is the data for waist (in inches), weight (in pounds), and body fat (in percent) for 20 test subjects.

33	188	10	33	160	10	40	192	31	32	175	6
40	240	20	41	215	27	41	205	32	36	181	21
36	175	22	34	159	12	35	173	21	38	200	15
32	168	9	34	146	10	38	187	25	33	159	6
44	246	38	44	219	28	38	188	30	39	196	22

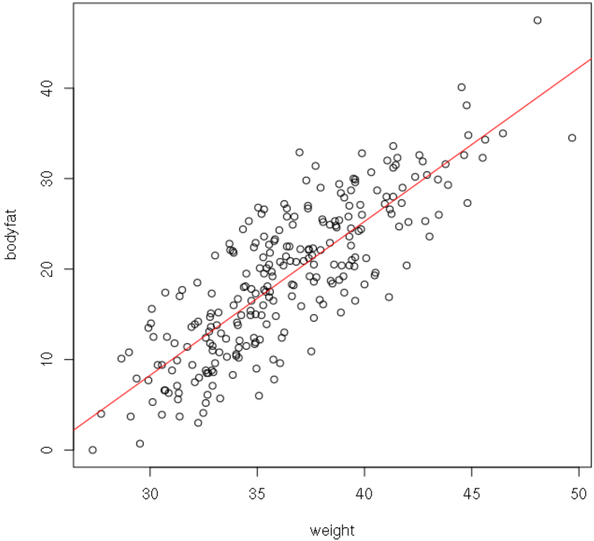

bodyfat
waist
5
10
15
20
25
30
35
32
34
36
38
40
42
44

In the previous slide, you can see the data from 20 test subjects. In the following slide, you can see the data from 250 test subjects. It appears that there is a relationship between waist and body fat.

Correlation



Correlation



The red line is called the regression line. We will learn how to calculate it later. Here is its equation:

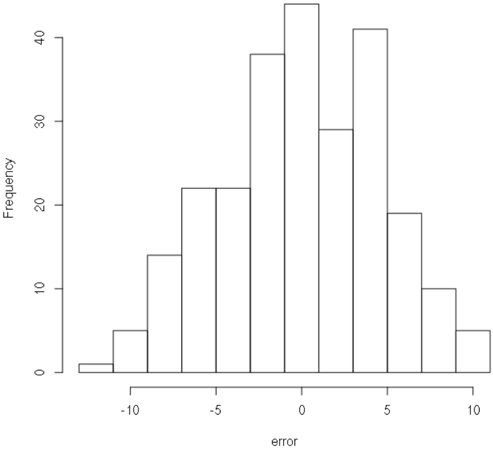
$$b = 1.7w - 42.73 \tag{1}$$

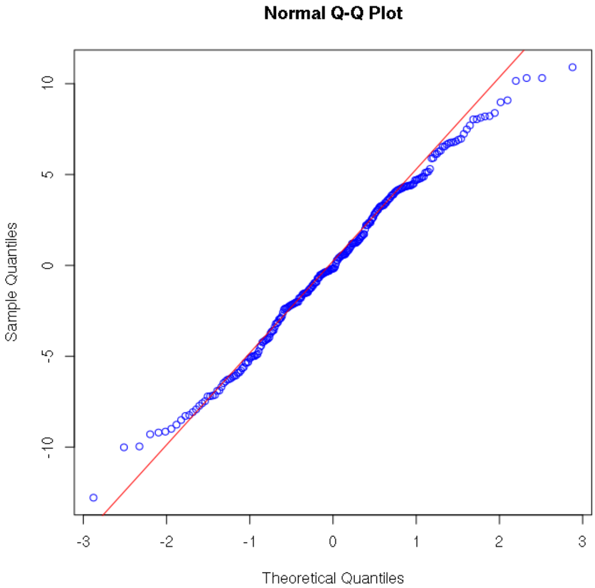
The regression line minimizes the mean square distance of the data points to the line (all other lines have a greater mean square distance from the data points). Have a look at three of the 250 test subjects.

	waist	bf (act)	bf (pred)	error
test subject 1	33.54	12.3	14.29	1.9936
test subject 2	32.68	6.1	12.82	6.7212
test subject 3	34.61	25.3	16.10	-9.1993

Correlation

It appears that the error is normally distributed (perhaps not quite on the margins).



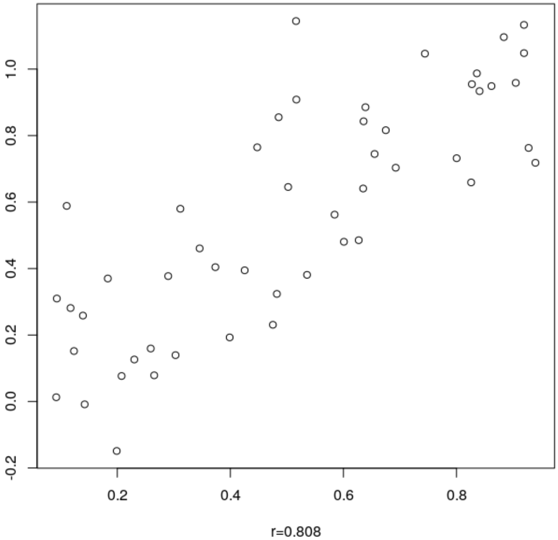

Normal Q-Q Plot
Sample Quantiles
Theoretical Quantiles
10
5
0
-5
-10
-3
-2
-1
0
1
2
3

Correlation

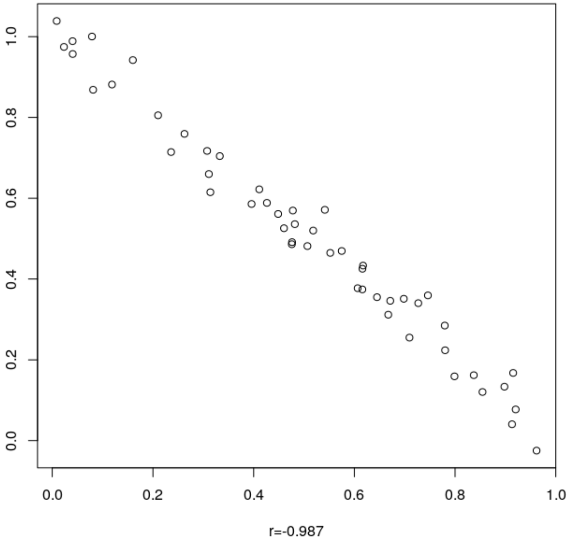
Let's have a look at a few scatterplots. Is there a correlation or not? Is there a linear correlation? The linear correlation coefficient r measures the strength of the linear correlation. It is a sample statistic. The linear correlation coefficient for the population is called ϱ ("rho" in the Greek alphabet).

The correlation coefficient for the sample of 250 test subjects measuring body fat and waist is $r = 0.8236847$.

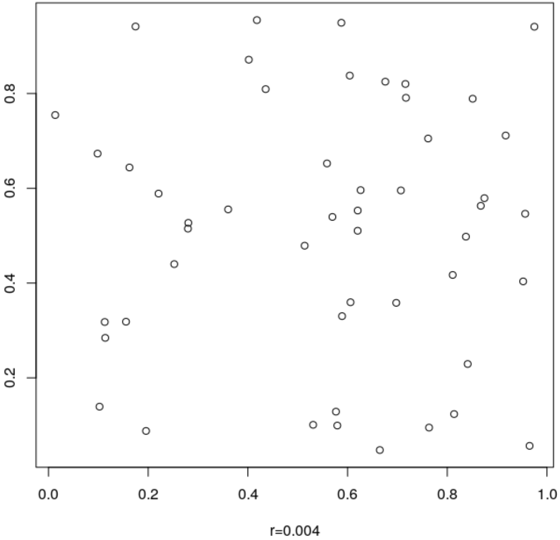
Scatterplot Examples



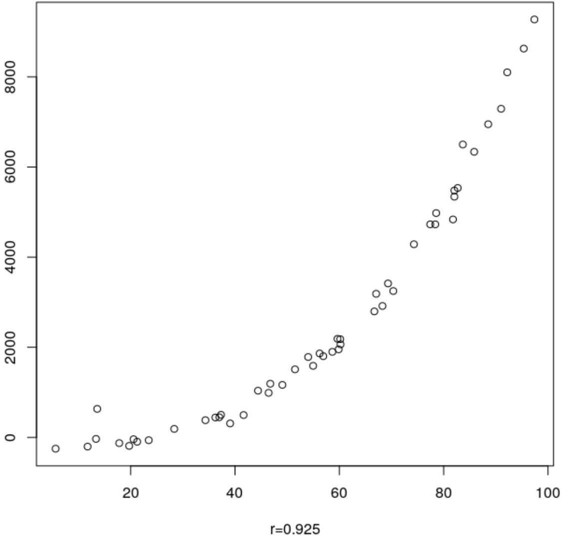
Scatterplot Examples



Scatterplot Examples



Scatterplot Examples



Notation

To determine whether there is a linear correlation between two variables, first take note of the following notation.

n	number of pairs of sample data
$\sum$	denotes addition of items indicated
$\sum x$	sum of all x-values
$\sum x^2$	sum of all x^2-values
$(\sum x)^2$	sum of all x-values squared
$\sum xy$	sum of all $x \cdot y$-values
r	linear correlation coefficient for sample data
ϱ	linear correlation coefficient for population of paired data

Requirements

Here are the requirements for the procedure that follows.

1. The sample of paired (x, y) data is a simple random sample of quantitative data.
2. Visual examination of the scatterplot confirms that the points approximate a straight-line pattern.
3. Outliers must be removed if they are known to be errors. The procedure is not robust with respect to erroneous outliers.

Requirements 2 and 3 are an intuitive summary of a more stringent requirement: the pairs of (x, y) data must have (or approximate) a bivariate normal distribution, which means that for a fixed value x, the corresponding y-values have a normal distribution, and vice versa. Think of the deviation of the actual y-value from a perfectly linear corresponding y-value as a normally distributed error.

Calculating r

Here is a simple formula for r that is difficult to calculate. Let z_x be the z-score of an individual x-value and z_y be the z-score of an individual y-value. Then

$$r = \frac{\sum (z_x z_y)}{n-1} \tag{2}$$

Here is a more difficult formula that makes calculation much easier.

$$r = \frac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{\sqrt{n\left(\sum x^2\right) - \left(\sum x\right)^2}\sqrt{n\left(\sum y^2\right) - \left(\sum y\right)^2}} \tag{3}$$

Calculating r Example

Here are the data for females, shoe prints, and heights.

24.8	165.1	28.1	179.1
28.6	166.4	27.6	175.9
25.4	177.8	26.5	166.4
26.7	167.6	26.5	167.6
26.7	168.3	28.4	162.6
27.9	165.7	26.5	167.6
27.9	165.1	26.0	165.1
28.9	165.1	27.0	172.7
27.9	165.1	25.1	157.5
25.9	152.4	27.9	167.6
25.4	162.6		

Calculating r Example

Now calculate the following ...

$\sum x$	565.7	$\sum y$	3503.3
$\sum x^2$	15268.45	$\sum y^2$	585173.7
$(\sum x)^2$	320016.5	$(\sum y)^2$	12273111
$\sum xy$	94404.95		

... and fill in the formula

$$r = \frac{21 \cdot 94404.95 - 565.7 \cdot 3503.3}{\sqrt{21 \cdot 15268.45 - 320016.5}\sqrt{21 \cdot 585173.7 - 12273111}} \tag{4}$$

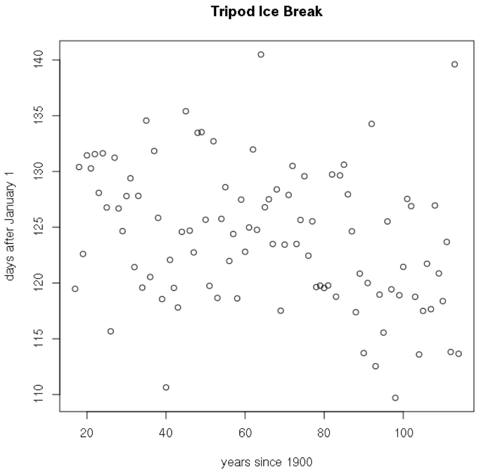
There are many opportunities here to make an error. It is better to use statistics software. In R Statistics, for example, the relevant command is `cor(x,y)`. The result, in either case, is $r = 0.22122$.

Hypothesis Testing for Correlation

Use the critical values for the Pearson Correlation Coefficient r to determine whether there is a correlation between the two variables or not. In the case of female shoe prints and heights, $n = 20$, and therefore, at a significance level $\alpha = 0.05$, the critical value is $r = 0.444$. The null hypothesis is $\varrho = 0$.

Since our test statistic is only $r^* = 0.221$, we fail to reject the null hypothesis that there is no correlation. There is not enough evidence to show that there is a linear correlation (remember to check the requirements first).

Example 1: Nenana Tripod Ice Break. Have a look at `http://www.nenanaakiceclassic.com/`. Is there a correlation? (Might it support the theory that the Earth is warming?)



Hypothesis Testing in R

Step 1 The null hypothesis is $\varrho = 0$. The alternative hypothesis is $\varrho < 0$ (the Earth is warming, therefore the tripod will break up the ice earlier in the year the more recently we measure). We will test the null hypothesis at a significance level of $\alpha = 0.01$.

Step 2 The test statistic is $r = -0.3130899$ (calculated using R Statistics).

Step 3 The critical value of the Pearson Correlation Coefficient r is approximately 0.256 at $\alpha = 0.01$ and $n = 98$ (consult the table).

Decision: reject the null hypothesis. The data supports the hypothesis that there is a linear correlation between years after 1900 and the days after January 1 when the tripod breaks the ice (assuming that the data have a bivariate normal distribution and a linear rather than some other correlation).

Hypothesis Testing in R

Here is how you can do the hypothesis testing in R Statistics. Let y be the years after 1900 and d be the days after January 1 when the tripod breaks the ice. Try the command `summary(lm(d~y))`. The output is on the next slide. Notice the p-value. It clearly suggests that we should reject H_0.

Hypothesis Testing in R

```
Residuals:
     Min       1Q   Median       3Q      Max 
-15.2035  -3.6805  -0.2056   4.0684  18.7533 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) 128.58108    1.50954   85.18   <2e-16 ***
y            -0.06834    0.02116   -3.23   0.0017 ** 
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 5.925 on 96 degrees of freedom
Multiple R-squared:  0.09803,Adjusted R-squared:  0.08863
F-statistic: 10.43 on 1 and 96 DF,  p-value: 0.001695
```

Hypothesis Testing in R

Let's have a closer look at the summary on the last slide. The residuals are the errors of our predictions using the regression line. For each x-value (independent variable), there is a y-value (dependent variable) and a $\hat{y}$-value (prediction using the regression line. The residual is $\hat{y} - y$ (in R Statistics, the residuals are in `z[[3]]` for `z<-summary(lm(d~y))`). The residual standard deviation s_e is a measure how much the data scatters along the regression line:

$$s_e = \sqrt{\frac{\sum(\hat{y} - y)^2}{n - 2}} \tag{5}$$

The residual standard deviation for the Nenana data is large, 5.925 days, because even if you know the regression line it's hard to predict the date when the ice will be broken in a particular year.

Hypothesis Testing in R

s_e is one measure of the relationship between x-values and y-values. The correlation coefficient r is another one. In the R summary it is called "Multiple R-squared" and equals r^2. The reason why it is squared is because one could say that the correlation accounts for r^2 of the variation in the y-values. In the Nenana example, which year it is accounts for 9.8% of the variation in the number of days it takes for the ice to break.

Some statisticians prefer "Adjusted R-squared" which penalizes larger numbers of parameters.

Hypothesis Testing in R

A theorem in statistics tells us that

$$\frac{b_1 - \beta_1}{\frac{s_e}{s_x\sqrt{n-1}}} \tag{6}$$

is distributed according to a t-distribution with degree of freedom $df = n - 2$.

s_x is the standard deviation of the x-values. s_e is the residual standard deviation. b_1 is the slope of the regression line calculated from the sample; β_1 is the slope of the regression line hypothesized for the population.

The R summary tells us that the slope of the regression line for the sample is $b_1 = -0.06834$ and the p-value for the hypothesis that $\beta_1 = 0$ is 0.0017 (two-tailed) (you can check this by looking at the t-distribution with degree of freedom $df = 96$ and the result of the formula on the last slide, which is $t^* = -3.23$). We reject the hypothesis that $\beta_1 = 0$, which is similar to rejecting the hypothesis that $r = 0$. It is usually not interesting to investigate the hypothesis that the y-intercept is zero.

Hypothesis Testing in R

Here is yet another hypothesis test whether there is a linear correlation or not. A relatively complicated formula gives us the F-statistic of the regression analysis. The F-distribution (named after Ronald Fisher) looks similar to the chi-squared distribution. It has two degrees of freedom. In R Statistics, `qf(0.95,5,2)` gives you the F-statistic for which 95% of the area under the curve is to the left of the F-statistic, with degrees of freedom 5 and 2. `pf(19.29641,5,2)` is the reverse procedure which gives you the area under the curve to the left of the F-statistic 19.29641. We will meet this distribution again when we cover ANOVA.

Hypothesis Testing for Correlation Exercise

Exercise 1: The table below lists measured amounts of redshift and the distances (billions of light-years) to randomly selected clusters of galaxies. Is there sufficient evidence to conclude that there is a linear correlation between amounts of redshift and distances to clusters of galaxies?

Redshift	Distance
0.0233	0.32
0.0539	0.75
0.0718	1.00
0.0395	0.55
0.0438	0.61
0.0103	0.14

The Regression Line

To find the regression line $\hat{y} = b_0 + b_1 x$, use the following formula for the slope b_1 and the y-intercept b_0:

$$b_1 = \frac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{n\left(\sum x^2\right) - \left(\sum x\right)^2} \tag{7}$$

$$b_0 = \frac{\left(\sum y\right)\left(\sum x^2\right) - \left(\sum x\right)\left(\sum xy\right)}{n\left(\sum x^2\right) - \left(\sum x\right)^2} \tag{8}$$

You may wonder where these equations come from. They identify the line equation which best fits the data using the least squares method. The least squares method identifies the line that best fits the data by measuring the distance that each data point is away from the line, squaring it, and then adding all of those numbers. The line that scores lowest on this fitness test is the regression line.

Regression Line Example

Example 2: Galaxy Distances. It is clear in the hypothesis test that there is a linear correlation between redshift and galaxy distances, even with a small sample size. What is the regression line? How could we predict the distance of a galaxy, knowing its redshift?

$$\begin{aligned} b_0 &= -0.004396 \\ b_1 &= 13.999899 \end{aligned} \tag{9}$$

Each thousandth unit of redshift adds fourteen million light-years to the distance.

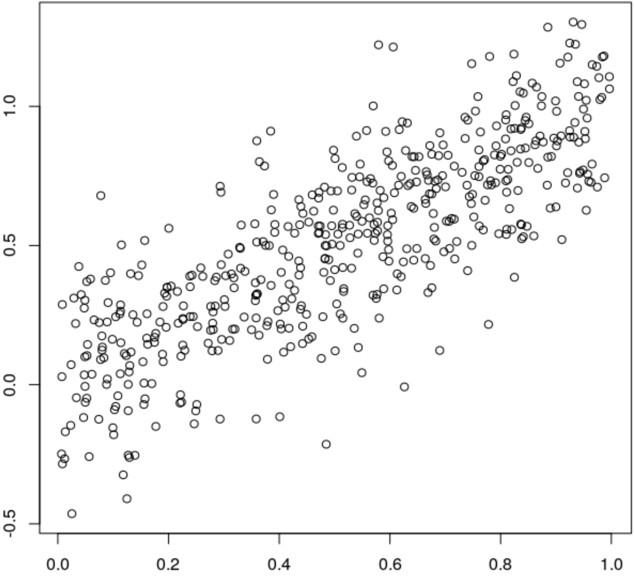
In R Statistics, you can create a scatterplot of data sets `x` and `y` using the command `plot(x,y)`. Adding the command `abline(lm(y~x),col="red")` will add the regression line to the plot (in red colour). `lm(y~x)` will give you both the *y*-intercept and the slope of the regression line.

Regression Line Example

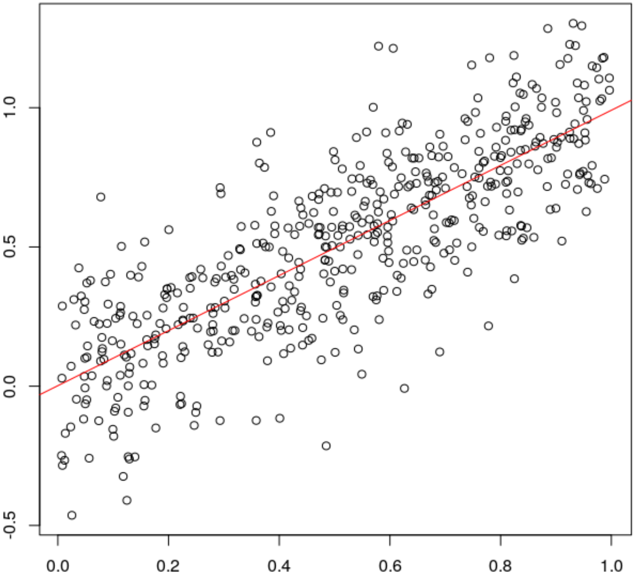
Here is some R Statistics code:

```
a<-runif(500,0,1)

b<-rnorm(500,a,0.2)

plot(a,b)

abline(lm(b~a),col="red")
```

Regression Line Example



Regression Line Example



Exercise 2: Consider the data on the next slide. These are the results for two successive term tests (the names are randomly made up by a computer program, but the grades are real). Answer the following questions:

1. Is there a linear correlation between the first and the second term test? Answer the question for a significance level of $\alpha = 0.05$. If you were doing this problem with a significance level $\alpha = 0.01$, what would be the decision and what type of error (type I or type II) would it make less likely compared to using the higher significance level?
2. What is the equation of the regression line?
3. If William Jones (again, fake name but real grade) had a score of 85 on the first term test, what score is the point estimate for the second term test given the linear correlation? His true score for the second term test was 77.

Regression Line Exercise

Nancy Rogge	83	60		Arnold Murray	73	82
Elizabeth Rushing	98	95		Ann Coburn	60	52
Katy Nunez	68	62		Kim Lazzari	63	62
Michael Preuss	68	90		Valentina Martinez	68	88
Edna Phipps	68	90		Eric Mumford	78	91
George Thompson	45	85		Alyssa Warner	98	90
James Newman	42	20		Kevin Ellis	65	72
Nathan Stowman	108	90		Susan Ervin	90	80
Kimberly Gaitor	83	90		Albert Gutierrez	55	52
Leland Garner	60	65		Robin Calderon	95	100
Bryan Veilleux	53	75		Jennifer Blackburn	60	57
Mary Watts	73	86		Doris Larkin	83	85
Jerry Brown	58	60		James Miller	63	42
Jacob Ludwick	47	52		Gregory Myklebust	70	87
Wayne Vega	100	85		Rita Swinton	90	90
Kathryn Wilson	55	80		Barbara Richardson	63	82
Tony Bateman	20	11		Ora Tidmore	108	87

Regression Line Exercise

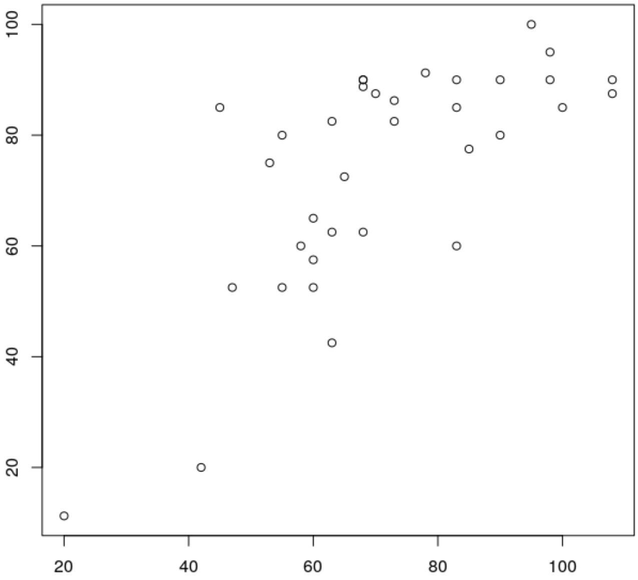
$\sum x$	2496	$\sum y$	2580
$\sum x^2$	191344	$\sum y^2$	204700
$(\sum x)^2$	6230016	$(\sum y)^2$	6656400
$\sum xy$	193904	n	34

Regression Line Exercise

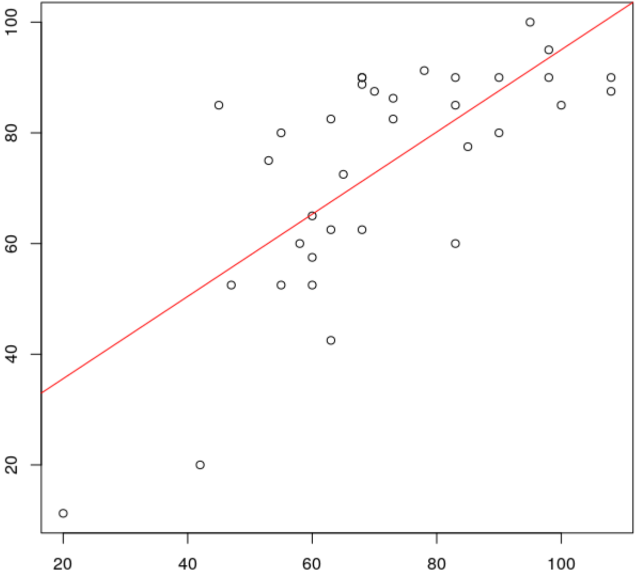
The solution for the linear correlation coefficient is $r = 0.7122366$.

The solution for the y-intercept and slope of the regression line is $b_0 = 20.7350$ and $b_1 = 0.7429$. The point estimate for William Jones' grade is 83.88.

Regression Line Exercise



Regression Line Exercise



Linear Correlation Exercise

Costs listed below are repair costs (in dollars) for cars crashed at 6 mi/h in full-front crash tests and the same cars crashed at 6 mi/h in full-rear crash tests (based on data from the Insurance Institute for Highway Safety). The cars are the Toyota Camry, Mazda 6, Volvo S40, Saturn Aura, Subaru Legacy, Hyundai Sonata, and Honda Accord. Is there sufficient evidence to conclude that there is a linear correlation between the repair costs from full-front crashes and full-rear crashes?

Front	936	978	2252	1032	3911	4312	3469
Rear	1480	1202	802	3191	1122	739	2767

Linear Correlation Exercise

Listed below are systolic blood pressure measurements (in mm Hg) obtained from the same woman (based on data from "Consistency of Blood Pressure Differences Between the Left and Right Arms," by Eguchi et al., Archives of Internal Medicine, Vol. 167). Is there sufficient evidence to conclude that there is a linear correlation between right and left arm systolic blood pressure measurements?

Right Arm	102	101	94	79	79
Left Arm	175	169	182	146	144

Linear Correlation Exercise

One classic application of correlation involves the association between the temperature and the number of times a cricket chirps in a minute. Listed below are the numbers of chirps in one minute and the corresponding temperatures in $^\circ$F (based on data from "The Song of Insects" by George W. Pierce, Harvard University Press). Is there sufficient evidence to conclude that there is a linear correlation between the number of chirps in one minute and the temperature?

Chirps	882	1188	1104	864	1200	1032	960	900
$^\circ$F	69.7	93.3	84.3	76.3	88 6	82.6	71.6	79.6

Linear Correlation Exercise

Lemons and Car Crashes. Find the best predicted crash fatality rate for a year in which there are 500 metric tons of lemon imports.

Lemon Imports	230	265	358	480	530
Crash Fatality Rate	15.9	15.7	15.4	15.3	14.9

Linear Correlation Exercise

Altitude and Temperature. At 6327 ft (or 6.327 thousand feet), Mario Triola, the author of many of these exercises, recorded the temperature. Find the best predicted temperature at that altitude. How does the result compare to the actual recorded value of 48°F?

Altitude	3	10	14	22	28	31	33
Temperature	57	37	24	-5	-30	-41	-54

Confidence Interval for Regression Line Slope

If the regression line slope for a sample of size n is b_1, then a confidence interval for the regression line slope β_1 of the population is (the confidence level being $1 - \alpha$)

$$b_1 - E < \beta_1 < b_1 + E \tag{10}$$

with

$$E = t_{\frac{\alpha}{2}} \frac{s_e}{s_x \sqrt{n-1}} \tag{11}$$

The degree of freedom for $t_{\frac{\alpha}{2}}$ is $n - 2$.

Confidence Interval for Regression Line Slope

The data on the next slide shows observations of the Old Faithful geyser in the USA Yellowstone National Park. There are two observation variables in the data set. The first one, called eruptions, is the duration of the geyser eruptions. The second one, called waiting, is the length of waiting period until the next eruption (all in minutes). It turns out there is a correlation between the two variables. The data is available on R Statistics in the dataframe called `faithful`.

Confidence Interval for Regression Line Slope

3.600	79	3.833	74	2.067	65	2.100	49	1.883	51	1.917	49	4.600	78	3.950	79
1.800	54	2.017	52	4.700	73	4.500	83	4.933	86	2.083	57	1.783	46	2.333	64
3.333	74	1.867	48	4.033	82	4.050	81	2.033	53	4.583	77	4.367	77	4.150	75
2.283	62	4.833	80	1.967	56	1.867	47	3.733	79	3.333	68	3.850	84	2.350	47
4.533	85	1.833	59	4.500	79	4.700	84	4.233	81	4.167	81	1.933	49	4.933	86
2.883	55	4.783	90	4.000	71	1.783	52	2.233	60	4.333	81	4.500	83	2.900	63
4.700	88	4.350	80	1.983	62	4.850	86	4.533	82	4.500	73	2.383	71	4.583	85
3.600	85	1.883	58	5.067	76	3.683	81	4.817	77	2.417	50	4.700	80	3.833	82
1.950	51	4.567	84	2.017	60	4.733	75	4.333	76	4.000	85	1.867	49	2.083	57
4.350	85	1.750	58	4.567	78	2.300	59	1.983	59	4.167	74	3.833	75	4.367	82
1.833	54	4.533	73	3.883	76	4.900	89	4.633	80	1.883	55	3.417	64	2.133	67
3.917	84	3.317	83	3.600	83	4.417	79	2.017	49	4.583	77	4.233	76	4.350	74
4.200	78	3.833	64	4.133	75	1.700	59	5.100	96	4.250	83	2.400	53	2.200	54
1.750	47	2.100	53	4.333	82	4.633	81	1.800	53	3.767	83	4.800	94	4.450	83
4.700	83	4.633	82	4.100	70	2.317	50	5.033	77	2.033	51	2.000	55	3.567	73
2.167	52	2.000	59	2.633	65	4.600	85	4.000	77	4.433	78	4.150	76	4.500	73
1.750	62	4.800	75	4.067	73	1.817	59	2.400	65	4.083	84	1.867	50	4.150	88
4.800	84	4.716	90	4.933	88	4.417	87	4.600	81	1.833	46	4.267	82	3.817	80
1.600	52	1.833	54	3.950	76	2.617	53	3.567	71	4.417	83	1.750	54	3.917	71
4.250	79	4.833	80	4.517	80	4.067	69	4.000	70	2.183	55	4.483	75	4.450	83
1.800	51	1.733	54	2.167	48	4.250	77	4.500	81	4.800	81	4.000	78	2.000	56
1.750	47	4.883	83	4.000	86	1.967	56	4.083	93	1.833	57	4.117	79	4.283	79
3.450	78	3.717	71	2.200	60	4.600	88	1.800	53	4.800	76	4.083	78	4.767	78
3.067	69	1.667	64	4.333	90	3.767	81	3.967	89	4.100	84	4.267	78	4.533	84
4.533	74	4.567	77	1.867	50	1.917	45	2.200	45	3.966	77	3.917	70	1.850	58
3.600	83	4.317	81	4.817	78	4.500	82	4.150	86	4.233	81	4.550	79	4.250	83
1.967	55	2.233	59	1.833	63	2.267	55	2.000	58	3.500	87	4.083	70	1.983	43
4.083	76	4.500	84	4.300	72	4.650	90	3.833	78	4.366	77	2.417	54	2.250	60
3.850	78	1.750	48	4.667	84	1.867	45	3.500	66	2.250	51	4.183	86	4.750	75

Confidence Interval for Regression Line Slope

For the data on the last slide, the residual standard error $s_e = 5.914$. The regression line for the sample is

$$\hat{y} = 33.4744 + 10.7296x \tag{12}$$

These numbers were gathered from the R Statistics command `summary(lm(faithful[[2]]~faithful[[1]]))`.

Confidence Interval for Regression Line Slope

The error for the 95% confidence interval is of

$$E = t_{\frac{\alpha}{2}} \frac{s_e}{s_x\sqrt{n-1}} =$$

$$1.968789 \cdot \frac{5.914}{3.4878\sqrt{272-1}} = 0.61968 \tag{13}$$

Consequently, the confidence interval

$$b_1 - E < \beta_1 < b_1 + E \tag{14}$$

is $(10.10996, 11.34932)$ The R Statistics command `confint(lm(y~x),'x',level=0.95)` will give you the same result.

Prediction Interval for Linear Regression

We already know how to predict the dependent variable (in this case, the waiting time for the next geyser) if we know the independent variable (in this case, the eruption time). For example, if the eruption time is four minutes (which happens to be the median of the data set), then the point estimate for the waiting time is

$$\hat{y} = 33.4744 + 10.7296 \cdot 4 = 76.3928 \tag{15}$$

What is the confidence interval around this point estimate, again at a confidence level of $1 - \alpha = 0.95$? The error in this case is

$$E = t_{\frac{\alpha}{2}} s_e \sqrt{1 + \frac{1}{n} + \frac{(x - \bar{x})^2}{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}} \tag{16}$$

Prediction Interval for Linear Regression

Plugging in the numbers from the faithful example, the confidence interval

$$\bar{y} - E < y < \bar{y} + E \tag{17}$$

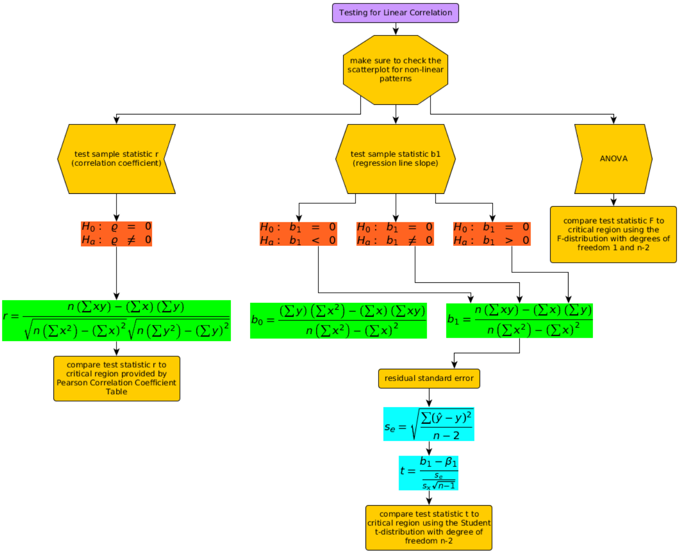
is $(64.72368, 88.06192)$.

The R Statistics command

```
predict(lm(y~x),data.frame(x=4.5),level=0.95,interval="predict")
```

will give you the same result.

Flow Chart for Linear Regression Hypothesis Testing



End of Lesson

Next Lesson: Goodness of Fit